UNITED STATES DEPARTMENT OF COMMERCE • Luther H. Hodges, *Secretary*
NATIONAL BUREAU OF STANDARDS • A. V. Astin, *Director*

Handbook of Mathematical Functions

With

Formulas, Graphs, and Mathematical Tables

Edited by
Milton Abramowitz and Irene A. Stegun

National Bureau of Standards
Applied Mathematics Series • 55

Issued June 1964
Tenth Printing, December 1972, with corrections

For sale by the Superintendent of Documents, U.S. Government Printing Office
Washington, D.C. 20402 – Price $11.35 domestic postpaid, or $10.50 GPO Bookstore

The text relating to physical constants and conversion factors (page 6) has been modified to take into account the newly adopted Système International d'Unites (SI).

ERRATA NOTICE

The original printing of this Handbook (June 1964) contained errors that have been corrected in the reprinted editions. These corrections are marked with an asterisk (*) for identification. The errors occurred on the following pages: 2–3, 6–8, 10, 15, 19–20, 25, 76, 85, 91, 102, 187, 189–197, 218, 223, 225, 233, 250, 255, 260–263, 268, 271–273, 292, 302, 328, 332, 333–337, 362, 365, 415, 423, 438–440, 443, 445, 447, 449, 451, 484, 498, 505–506, 509–510, 543, 556, 558, 562, 571, 595, 599, 600, 722–723, 739, 742, 744, 746, 752, 756, 760–765, 774, 777–785, 790, 797, 801, 822–823, 832, 835, 844, 886–889, 897, 914, 915, 920, 930–931, 936, 940–941, 944–950, 953, 960, 963, 989–990, 1010, 1026.

Originally issued June 1964. Second printing, November 1964. Third printing, March 1965. Fourth printing, December 1965. Fifth printing, August 1966. Sixth printing, November 1967. Seventh printing, May 1968. Eighth printing, 1969. Ninth printing, November 1970.

Library of Congress Catalog Card Number: 64–60036

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Preface

The present volume is an outgrowth of a Conference on Mathematical Tables held at Cambridge, Mass., on September 15-16, 1954, under the auspices of the National Science Foundation and the Massachusetts Institute of Technology. The purpose of the meeting was to evaluate the need for mathematical tables in the light of the availability of large scale computing machines. It was the consensus of opinion that in spite of the increasing use of the new machines the basic need for tables would continue to exist.

Numerical tables of mathematical functions are in continual demand by scientists and engineers. A greater variety of functions and higher accuracy of tabulation are now required as a result of scientific advances and, especially, of the increasing use of automatic computers. In the latter connection, the tables serve mainly for preliminary surveys of problems before programming for machine operation. For those without easy access to machines, such tables are, of course, indispensable.

Consequently, the Conference recognized that there was a pressing need for a modernized version of the classical tables of functions of Jahnke-Emde. To implement the project, the National Science Foundation requested the National Bureau of Standards to prepare such a volume and established an Ad Hoc Advisory Committee, with Professor Philip M. Morse of the Massachusetts Institute of Technology as chairman, to advise the staff of the National Bureau of Standards during the course of its preparation. In addition to the Chairman, the Committee consisted of A. Erdelyi, M. C. Gray, N. Metropolis, J. B. Rosser, H. C. Thacher, Jr., John Todd, C. B. Tompkins, and J. W. Tukey.

The primary aim has been to include a maximum of useful information within the limits of a moderately large volume, with particular attention to the needs of scientists in all fields. An attempt has been made to cover the entire field of special functions. To carry out the goal set forth by the Ad Hoc Committee, it has been necessary to supplement the tables by including the mathematical properties that are important in computation work, as well as by providing numerical methods which demonstrate the use and extension of the tables.

The Handbook was prepared under the direction of the late Milton Abramowitz, and Irene A. Stegun. Its success has depended greatly upon the cooperation of many mathematicians. Their efforts together with the cooperation of the Ad Hoc Committee are greatly appreciated. The particular contributions of these and other individuals are acknowledged at appropriate places in the text. The sponsorship of the National Science Foundation for the preparation of the material is gratefully recognized.

It is hoped that this volume will not only meet the needs of all table users but will in many cases acquaint its users with new functions.

ALLEN V. ASTIN, *Director*

June 1964
Washington, D.C.

Preface to the Ninth Printing

The enthusiastic reception accorded the "Handbook of Mathematical Functions" is little short of unprecedented in the long history of mathematical tables that began when John Napier published his tables of logarithms in 1614. Only four and one-half years after the first copy came from the press in 1964, Myron Tribus, the Assistant Secretary of Commerce for Science and Technology, presented the 100,000th copy of the Handbook to Lee A. DuBridge, then Science Advisor to the President. Today, total distribution is approaching the 150,000 mark at a scarcely diminished rate.

The success of the Handbook has not ended our interest in the subject. On the contrary, we continue our close watch over the growing and changing world of computation and to discuss with outside experts and among ourselves the various proposals for possible extension or supplementation of the formulas, methods and tables that make up the Handbook.

In keeping with previous policy, a number of errors discovered since the last printing have been corrected. Aside from this, the mathematical tables and accompanying text are unaltered. However, some noteworthy changes have been made in Chapter 2: Physical Constants and Conversion Factors, pp. 6-8. The table on page 7 has been revised to give the values of physical constants obtained in a recent reevaluation; and pages 6 and 8 have been modified to reflect changes in definition and nomenclature of physical units and in the values adopted for the acceleration due to gravity in the revised Potsdam system.

The record of continuing acceptance of the Handbook, the praise that has come from all quarters, and the fact that it is one of the most-quoted scientific publications in recent years are evidence that the hope expressed by Dr. Astin in his Preface is being amply fulfilled.

LEWIS M. BRANSCOMB, *Director*
National Bureau of Standards

November 1970

Foreword

This volume is the result of the cooperative effort of many persons and a number of organizations. The National Bureau of Standards has long been turning out mathematical tables and has had under consideration, for at least 10 years, the production of a compendium like the present one. During a Conference on Tables, called by the NBS Applied Mathematics Division on May 15, 1952, Dr. Abramowitz of that Division mentioned preliminary plans for such an undertaking, but indicated the need for technical advice and financial support.

The Mathematics Division of the National Research Council has also had an active interest in tables; since 1943 it has published the quarterly journal, "Mathematical Tables and Aids to Computation" (MTAC), editorial supervision being exercised by a Committee of the Division.

Subsequent to the NBS Conference on Tables in 1952 the attention of the National Science Foundation was drawn to the desirability of financing activity in table production. With its support a 2-day Conference on Tables was called at the Massachusetts Institute of Technology on September 15–16, 1954, to discuss the needs for tables of various kinds. Twenty-eight persons attended, representing scientists and engineers using tables as well as table producers. This conference reached consensus on several conclusions and recommendations, which were set forth in the published Report of the Conference. There was general agreement, for example, "that the advent of high-speed computing equipment changed the task of table making but definitely did not remove the need for tables". It was also agreed that "an outstanding need is for a Handbook of Tables for the Occasional Computer, with tables of usually encountered functions and a set of formulas and tables for interpolation and other techniques useful to the occasional computer". The Report suggested that the NBS undertake the production of such a Handbook and that the NSF contribute financial assistance. The Conference elected, from its participants, the following Committee: P. M. Morse (Chairman), M. Abramowitz, J. H. Curtiss, R. W. Hamming, D. H. Lehmer, C. B. Tompkins, J. W. Tukey, to help implement these and other recommendations.

The Bureau of Standards undertook to produce the recommended tables and the National Science Foundation made funds available. To provide technical guidance to the Mathematics Division of the Bureau, which carried out the work, and to provide the NSF with independent judgments on grants for the work, the Conference Committee was reconstituted as the Committee on Revision of Mathematical Tables of the Mathematics Division of the National Research Council. This, after some changes of membership, became the Committee which is signing this Foreword. The present volume is evidence that Conferences can sometimes reach conclusions and that their recommendations sometimes get acted on.

Active work was started at the Bureau in 1956. The overall plan, the selection of authors for the various chapters, and the enthusiasm required to begin the task were contributions of Dr. Abramowitz. Since his untimely death, the effort has continued under the general direction of Irene A. Stegun. The workers at the Bureau and the members of the Committee have had many discussions about content, style and layout. Though many details have had to be argued out as they came up, the basic specifications of the volume have remained the same as were outlined by the Massachusetts Institute of Technology Conference of 1954.

The Committee wishes here to register its commendation of the magnitude and quality of the task carried out by the staff of the NBS Computing Section and their expert collaborators in planning, collecting and editing these Tables, and its appreciation of the willingness with which its various suggestions were incorporated into the plans. We hope this resulting volume will be judged by its users to be a worthy memorial to the vision and industry of its chief architect, Milton Abramowitz. We regret he did not live to see its publication.

P. M. MORSE, *Chairman.*
A. ERDÉLYI
M. C. GRAY
N. C. METROPOLIS
J. B. ROSSER
H. C. THACHER, Jr.
JOHN TODD
C. B. TOMPKINS
J. W. TUKEY.

Contents

Handbook of Mathematical Functions

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1. Introduction

The present Handbook has been designed to provide scientific investigators with a comprehensive and self-contained summary of the mathematical functions that arise in physical and engineering problems. The well-known Tables of Functions by E. Jahnke and F. Emde has been invaluable to workers in these fields in its many editions[1] during the past half-century. The present volume extends the work of these authors by giving more extensive and more accurate numerical tables, and by giving larger collections of mathematical properties of the tabulated functions. The number of functions covered has also been increased.

The classification of functions and organization of the chapters in this Handbook is similar to that of An Index of Mathematical Tables by A. Fletcher, J. C. P. Miller, and L. Rosenhead.[2] In general, the chapters contain numerical tables, graphs, polynomial or rational approximations for automatic computers, and statements of the principal mathematical properties of the tabulated functions, particularly those of computational importance. Many numerical examples are given to illustrate the use of the tables and also the computation of function values which lie outside their range. At the end of the text in each chapter there is a short bibliography giving books and papers in which proofs of the mathematical properties stated in the chapter may be found. Also listed in the bibliographies are the more important numerical tables. Comprehensive lists of tables are given in the Index mentioned above, and current information on new tables is to be found in the National Research Council quarterly Mathematics of Computation (formerly Mathematical Tables and Other Aids to Computation).

The mathematical notations used in this Handbook are those commonly adopted in standard texts, particularly Higher Transcendental Functions, Volumes 1–3, by A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi (McGraw-Hill, 1953–55). Some alternative notations have also been listed. The introduction of new symbols has been kept to a minimum, and an effort has been made to avoid the use of conflicting notation.

2. Accuracy of the Tables

The number of significant figures given in each table has depended to some extent on the number available in existing tabulations. There has been no attempt to make it uniform throughout the Handbook, which would have been a costly and laborious undertaking. In most tables at least five significant figures have been provided, and the tabular intervals have generally been chosen to ensure that linear interpolation will yield four- or five-figure accuracy, which suffices in most physical applications. Users requiring higher precision in their interpolates may obtain them by use of higher-order interpolation procedures, described below.

In certain tables many-figured function values are given at irregular intervals in the argument. An example is provided by Table 9.4. The purpose of these tables is to furnish "key values" for the checking of programs for automatic computers; no question of interpolation arises.

The maximum end-figure error, or "tolerance" in the tables in this Handbook is $^{6}/_{10}$ of 1 unit everywhere in the case of the elementary functions, and 1 unit in the case of the higher functions except in a few cases where it has been permitted to rise to 2 units.

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[1] The most recent, the sixth, with F. Loesch added as co-author, was published in 1960 by McGraw-Hill, U.S.A., and Teubner, Germany.

[2] The second edition, with L. J. Comrie added as co-author, was published in two volumes in 1962 by Addison-Wesley, U.S.A., and Scientific Computing Service Ltd., Great Britain.

3. Auxiliary Functions and Arguments

One of the objects of this Handbook is to provide tables or computing methods which enable the user to evaluate the tabulated functions over complete ranges of real values of their parameters. In order to achieve this object, frequent use has been made of auxiliary functions to remove the infinite part of the original functions at their singularities, and auxiliary arguments to cope with infinite ranges. An example will make the procedure clear.

The exponential integral of positive argument is given by

$$\begin{aligned}\mathrm{Ei}(x)&=\int_{-\infty}^{x}\frac{e^u}{u}du\\&=\gamma+\ln x+\frac{x}{1\cdot 1!}+\frac{x^2}{2\cdot 2!}+\frac{x^3}{3\cdot 3!}+\ .\ .\ .\\&\sim\frac{e^x}{x}\left[1+\frac{1!}{x}+\frac{2!}{x^2}+\frac{3!}{x^3}+\ .\ .\ .\ \right](x\rightarrow\infty)\end{aligned}$$

The logarithmic singularity precludes direct interpolation near $x=0$. The functions $\mathrm{Ei}(x)-\ln x$ and $x^{-1}[\mathrm{Ei}(x)-\ln x-\gamma]$, however, are well-behaved and readily interpolable in this region. Either will do as an auxiliary function; the latter was in fact selected as it yields slightly higher accuracy when $\mathrm{Ei}(x)$ is recovered. The function $x^{-1}[\mathrm{Ei}(x)-\ln x-\gamma]$ has been tabulated to nine decimals for the range $0\leq x\leq\frac{1}{2}$. For $\frac{1}{2}\leq x\leq 2$, $\mathrm{Ei}(x)$ is sufficiently well-behaved to admit direct tabulation, but for larger values of x, its exponential character predominates. A smoother and more readily interpolable function for large x is $xe^{-x}\mathrm{Ei}(x)$; this has been tabulated for $2\leq x\leq 10$. Finally, the range $10\leq x\leq\infty$ is covered by use of the inverse argument x^{-1}. Twenty-one entries of $xe^{-x}\mathrm{Ei}(x)$, corresponding to $x^{-1}=.1(-.005)0$, suffice to produce an interpolable table.

4. Interpolation

The tables in this Handbook are not provided with differences or other aids to interpolation, because it was felt that the space they require could be better employed by the tabulation of additional functions. Admittedly aids could have been given without consuming extra space by increasing the intervals of tabulation, but this would have conflicted with the requirement that linear interpolation is accurate to four or five figures.

For applications in which linear interpolation is insufficiently accurate it is intended that Lagrange's formula or Aitken's method of iterative linear interpolation[3] be used. To help the user, there is a statement at the foot of most tables of the maximum error in a linear interpolate, and the number of function values needed in Lagrange's formula or Aitken's method to interpolate to full tabular accuracy.

As an example, consider the following extract from Table 5.1.

x	$xe^xE_1(x)$	x	$xe^xE_1(x)$
7.5	.89268 7854	8.0	.89823 7113
7.6	.89384 6312	8.1	.89927 7888
7.7	.89497 9666	8.2	.90029 7306
7.8	.89608 8737	8.3	.90129 6073
7.9	.89717 4302	8.4	.90227 4695

$$\left[\begin{matrix}(-6)3\\ 5\end{matrix}\right]$$

The numbers in the square brackets mean that the maximum error in a linear interpolate is 3×10^{-6}, and that to interpolate to the full tabular accuracy five points must be used in Lagrange's and Aitken's methods.

Let us suppose that we wish to compute the value of $xe^xE_1(x)$ for $x=7.9527$ from this table. We describe in turn the application of the methods of linear interpolation, Lagrange and Aitken, and of alternative methods based on differences and Taylor's series.

(1) Linear interpolation. The formula for this process is given by

$$f_p=(1-p)f_0+pf_1$$

where f_0, f_1 are consecutive tabular values of the function, corresponding to arguments x_0, x_1, respectively; p is the given fraction of the argument interval

$$p=(x-x_0)/(x_1-x_0)$$

and f_p the required interpolate. In the present instance, we have

$$f_0=.89717\ 4302\qquad f_1=.89823\ 7113\qquad p=.527$$

The most convenient way to evaluate the formula on a desk calculating machine is to set f_0 and f_1 in turn on the keyboard, and carry out the multiplications by $1-p$ and p cumulatively; a partial check is then provided by the multiplier dial reading unity. We obtain

$$\begin{aligned}f_{.527}&=(1-.527)(.89717\ 4302)+.527(.89823\ 7113)\\&=.89773\ 4403.\end{aligned}$$

Since it is known that there is a possible error of 3×10^{-6} in the linear formula, we round off this result to .89773. The maximum possible error in this answer is composed of the error committed

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[3] A. C. Aitken, On interpolation by iteration of proportional parts, without the use of differences, Proc. Edinburgh Math. Soc. 3, 56–76 (1932).

by the last rounding, that is, $.4403\times10^{-5}$, plus 3×10^{-6}, and so certainly cannot exceed $.8\times10^{-5}$.

(2) Lagrange's formula. In this example, the relevant formula is the 5-point one, given by

$$f=A_{-2}(p)f_{-2}+A_{-1}(p)f_{-1}+A_0(p)f_0+A_1(p)f_1 +A_2(p)f_2$$

Tables of the coefficients $A_k(p)$ are given in chapter 25 for the range $p=0(.01)1$. We evaluate the formula for $p=.52$, $.53$ and $.54$ in turn. Again, in each evaluation we accumulate the $A_k(p)$ in the multiplier register since their sum is unity. We now have the following subtable.

x	$xe^xE_1(x)$		
7.952	.89772 9757		
		10622	
7.953	.89774 0379		−2
		10620	
7.954	.89775 0999		

The numbers in the third and fourth columns are the first and second differences of the values of $xe^xE_1(x)$ (see below); the smallness of the second difference provides a check on the three interpolations. The required value is now obtained by linear interpolation:

$$f_p=.3(.89772\ 9757)+.7(.89774\ 0379)$$
$$=.89773\ 7192.$$

In cases where the correct order of the Lagrange polynomial is not known, one of the preliminary interpolations may have to be performed with polynomials of two or more different orders as a check on their adequacy.

(3) Aitken's method of iterative linear interpolation. The scheme for carrying out this process in the present example is as follows:

n	x_n	$y_n=xe^xE_1(x)$	$y_{0,n}$	$y_{0,1,n}$	$y_{0,1,2,n}$	$y_{0,1,2,3,n}$	x_n-x
0	8.0	.89823 7113					.0473
1	7.9	.89717 4302	.89773 44034				−.0527
2	8.1	.89927 7888	.89774 48264	.89773 71499			.1473
3	7.8	.89608 8737	2 90220	2394	.89773 71938		−.1527
4	8.2	.90029 7306	4 98773	1216	16	89773 71930	.2473
5	7.7	.89497 9666	2 35221	2706	43	30	−.2527

Here

$$y_{0,n}=\frac{1}{x_n-x_0}\begin{vmatrix} y_0 & x_0-x \\ y_n & x_n-x \end{vmatrix}$$

$$y_{0,1,n}=\frac{1}{x_n-x_1}\begin{vmatrix} y_{0,1} & x_1-x \\ y_{0,n} & x_n-x \end{vmatrix}$$

$$y_{0,1,\ldots,m-1,m,n}=\frac{1}{x_n-x_m}\begin{vmatrix} y_{0,1,\ldots,m-1,m} & x_m-x \\ y_{0,1,\ldots,m-1,n} & x_n-x \end{vmatrix}$$

If the quantities x_n-x and x_m-x are used as multipliers when forming the cross-product on a desk machine, their accumulation $(x_n-x)-(x_m-x)$ in the multiplier register is the divisor to be used at that stage. An extra decimal place is usually carried in the intermediate interpolates to safeguard against accumulation of rounding errors.

The order in which the tabular values are used is immaterial to some extent, but to achieve the maximum rate of convergence and at the same time minimize accumulation of rounding errors, we begin, as in this example, with the tabular argument nearest to the given argument, then take the nearest of the remaining tabular arguments, and so on.

The number of tabular values required to achieve a given precision emerges naturally in the course of the iterations. Thus in the present example six values were used, even though it was known in advance that five would suffice. The extra row confirms the convergence and provides a valuable check.

(4) Difference formulas. We use the central difference notation (chapter 25),

$$\begin{array}{cccccc}
x_0 & f_0 & & & & \\
 & & \delta f_{1/2} & & & \\
x_1 & f_1 & & \delta^2 f_1 & & \\
 & & \delta f_{3/2} & & \delta^3 f_{3/2} & \\
x_2 & f_2 & & \delta^2 f_2 & & \delta^4 f_2 \\
 & & \delta f_{5/2} & & \delta^3 f_{5/2} & \\
x_3 & f_3 & & \delta^2 f_3 & & \\
 & & \delta f_{7/2} & & & \\
x_4 & f_4 & & & &
\end{array}$$

Here

$$\delta f_{1/2}=f_1-f_0,\ \delta f_{3/2}=f_2-f_1,\ \ldots,$$
$$\delta^2 f_1=\delta f_{3/2}-\delta f_{1/2}=f_2-2f_1+f_0$$
$$\delta^3 f_{3/2}=\delta^2 f_2-\delta^2 f_1=f_3-3f_2+3f_1-f_0$$
$$\delta^4 f_2=\delta^3 f_{5/2}-\delta^3 f_{3/2}=f_4-4f_3+6f_2-4f_1+f_0$$

and so on.

In the present example the relevant part of the difference table is as follows, the differences being written in units of the last decimal place of the function, as is customary. The smallness of the high differences provides a check on the function values

x	$xe^xE_1(x)$	$\delta^2 f$	$\delta^4 f$
7.9	.89717 4302	−2 2754	−34
8.0	.89823 7113	−2 2036	−39

Applying, for example, Everett's interpolation formula

$$f_p=(1-p)f_0+E_2(p)\delta^2 f_0+E_4(p)\delta^4 f_0+\ \ldots$$
$$+pf_1+F_2(p)\delta^2 f_1+F_4(p)\delta^4 f_1+\ \ldots$$

and taking the numerical values of the interpolation coefficients $E_2(p)$, $E_4(p)$, $F_2(p)$ and $F_4(p)$ from Table 25.1, we find that

$$10^9 f_{.527} = .473(89717\ 4302) + .061196(2\ 2754) - .012(34)$$
$$+ .527(89823\ 7113) + .063439(2\ 2036) - .012(39)$$
$$= 89773\ 7193.$$

We may notice in passing that Everett's formula shows that the error in a linear interpolate is approximately

$$E_2(p)\delta^2 f_0 + F_2(p)\delta^2 f_1 \approx \tfrac{1}{2}[E_2(p) + F_2(p)][\delta^2 f_0 + \delta^2 f_1]$$

Since the maximum value of $|E_2(p)+F_2(p)|$ in the range $0 < p < 1$ is ⅛, the maximum error in a linear interpolate is approximately

$$\frac{1}{16}|\delta^2 f_0 + \delta^2 f_1|,\text{ that is, }\frac{1}{16}|f_2 - f_1 - f_0 + f_{-1}|.$$

(5) Taylor's series. In cases where the successive derivatives of the tabulated function can be computed fairly easily, Taylor's expansion

$$f(x) = f(x_0) + (x-x_0)\frac{f'(x_0)}{1!} + (x-x_0)^2\frac{f''(x_0)}{2!} + (x-x_0)^3\frac{f'''(x_0)}{3!} + \ldots$$

can be used. We first compute as many of the derivatives $f^{(n)}(x_0)$ as are significant, and then evaluate the series for the given value of x. An advisable check on the computed values of the derivatives is to reproduce the adjacent tabular values by evaluating the series for $x = x_{-1}$ and x_1.

In the present example, we have

$$\begin{aligned} f(x) &= xe^x E_1(x) \\ f'(x) &= (1+x^{-1})f(x) - 1 \\ f''(x) &= (1+x^{-1})f'(x) - x^{-2}f(x) \\ f'''(x) &= (1+x^{-1})f''(x) - 2x^{-2}f'(x) + 2x^{-3}f(x). \end{aligned}$$

With $x_0 = 7.9$ and $x - x_0 = .0527$ our computations are as follows; an extra decimal has been retained in the values of the terms in the series to safeguard against accumulation of rounding errors.

k	$f^{(k)}(x_0)/k!$	$(x-x_0)^k f^{(k)}(x_0)/k!$
0	.89717 4302	.89717 4302
1	.01074 0669	.00056 6033 3
2	−.00113 7621	−.00000 3159 5
3	.00012 1987	.00000 0017 9
		.89773 7194

5. Inverse Interpolation

With linear interpolation there is no difference in principle between direct and inverse interpolation. In cases where the linear formula provides an insufficiently accurate answer, two methods are available. We may interpolate directly, for example, by Lagrange's formula to prepare a new table at a fine interval in the neighborhood of the approximate value, and then apply accurate inverse linear interpolation to the subtabulated values. Alternatively, we may use Aitken's method or even possibly the Taylor's series method, with the roles of function and argument interchanged.

It is important to realize that the accuracy of an inverse interpolate may be very different from that of a direct interpolate. This is particularly true in regions where the function is slowly varying, for example, near a maximum or minimum. The maximum precision attainable in an inverse interpolate can be estimated with the aid of the formula

$$\Delta x \approx \Delta f \Big/ \frac{df}{dx}$$

in which Δf is the maximum possible error in the function values.

Example. Given $xe^x E_1(x) = .9$, find x from the table on page X.

(i) Inverse linear interpolation. The formula for p is

$$p = (f_p - f_0)/(f_1 - f_0).$$

In the present example, we have

$$p = \frac{.9 - .89927\ 7888}{.90029\ 7306 - .89927\ 7888} = \frac{72\ 2112}{101\ 9418} = .708357.$$

The desired x is therefore

$$x = x_0 + p(x_1 - x_0) = 8.1 + .708357(.1) = 8.17083\ 57$$

To estimate the possible error in this answer, we recall that the maximum error of direct linear interpolation in this table is $\Delta f = 3\times10^{-6}$. An approximate value for df/dx is the ratio of the first difference to the argument interval (chapter 25), in this case .010. Hence the maximum error in x is approximately $3\times10^{-6}/(.010)$, that is, .0003.

(ii) Subtabulation method. To improve the approximate value of x just obtained, we interpolate directly for $p = .70$, .71 and .72 with the aid of Lagrange's 5-point formula,

x	$xe^x E_1(x)$	δ	δ^2
8.170	.89999 3683		
		1 0151	
8.171	.90000 3834		−2
		1 0149	
8.172	.90001 3983		

Inverse linear interpolation in the new table gives

$$p = \frac{.9 - .89999\ 3683}{.00001\ 0151} = .6223$$

Hence $x = 8.17062\ 23$.

An estimate of the maximum error in this result is

$$\Delta f \Big/ \frac{df}{dx} \approx \frac{1\times10^{-9}}{.010} = 1\times10^{-7}$$

(iii) Aitken's method. This is carried out in the same manner as in direct interpolation.

n	$y_n = xe^x E_1(x)$	x_n	$x_{0,n}$	$x_{0,1,n}$	$x_{0,1,2,n}$	$x_{0,1,2,3,n}$	$y_n - y$
0	.90029 7306	8.2					.00029 7306
1	.89927 7888	8.1	8.17083 5712				−.00072 2112
2	.90129 6033	8.3	8.17023 1505	8.17061 9521			.00129 6033
3	.89823 7113	8.0	8.17113 8043	2 5948	8.17062 2244		−.00176 2887
4	.90227 4695	8.4	8.16992 9437	1 7335	415	8.17062 2318	.00227 4695
5	.89717 4302	7.9	8.17144 0382	2 8142	231	265	−.00282 5098

The estimate of the maximum error in this result is the same as in the subtabulation method. An indication of the error is also provided by the discrepancy in the highest interpolates, in this case $x_{0,1,2,3,4}$, and $x_{0,1,2,3,5}$.

6. Bivariate Interpolation

Bivariate interpolation is generally most simply performed as a sequence of univariate interpolations. We carry out the interpolation in one direction, by one of the methods already described, for several tabular values of the second argument in the neighborhood of its given value. The interpolates are differenced as a check, and interpolation is then carried out in the second direction.

An alternative procedure in the case of functions of a complex variable is to use the Taylor's series expansion, provided that successive derivatives of the function can be computed without much difficulty.

7. Generation of Functions from Recurrence Relations

Many of the special mathematical functions which depend on a parameter, called their index, order or degree, satisfy a linear difference equation (or recurrence relation) with respect to this parameter. Examples are furnished by the Legendre function $P_n(x)$, the Bessel function $J_n(x)$ and the exponential integral $E_n(x)$, for which we have the respective recurrence relations

$$(n+1)P_{n+1}-(2n+1)xP_n+nP_{n-1}=0$$

$$J_{n+1}-\frac{2n}{x}J_n+J_{n-1}=0$$

$$nE_{n+1}+xE_n=e^{-x}.$$

Particularly for automatic work, recurrence relations provide an important and powerful computing tool. If the values of $P_n(x)$ or $J_n(x)$ are known for two consecutive values of n, or $E_n(x)$ is known for one value of n, then the function may be computed for other values of n by successive applications of the relation. Since generation is carried out perforce with rounded values, it is vital to know how errors may be propagated in the recurrence process. If the errors do not grow relative to the size of the wanted function, the process is said to be stable. If, however, the relative errors grow and will eventually overwhelm the wanted function, the process is unstable.

It is important to realize that stability may depend on (i) the particular solution of the difference equation being computed; (ii) the values of x or other parameters in the difference equation; (iii) the direction in which the recurrence is being applied. Examples are as follows.

Stability—increasing n

$P_n(x)$, $P_n^m(x)$

$Q_n(x)$, $Q_n^m(x)$ $(x<1)$

$Y_n(x)$, $K_n(x)$

$J_{-n-\frac{1}{2}}(x)$, $I_{-n-\frac{1}{2}}(x)$

$E_n(x)$ $(n<x)$

Stability—decreasing n

$P_n(x)$, $P_n^m(x)$ $(x<1)$

$Q_n(x)$, $Q_n^m(x)$

$J_n(x)$, $I_n(x)$

$J_{n+\frac{1}{2}}(x)$, $I_{n+\frac{1}{2}}(x)$

$E_n(x)$ $(n>x)$

$F_n(\eta, \rho)$ (Coulomb wave function)

Illustrations of the generation of functions from their recurrence relations are given in the pertinent chapters. It is also shown that even in cases where the recurrence process is unstable, it may still be used when the starting values are known to sufficient accuracy.

Mention must also be made here of a refinement, due to J. C. P. Miller, which enables a recurrence process which is stable for decreasing n to be applied without any knowledge of starting values for large n. Miller's algorithm, which is well-suited to automatic work, is described in **19.28, Example 1.**

8. Acknowledgments

The production of this volume has been the result of the unrelenting efforts of many persons, all of whose contributions have been instrumental in accomplishing the task. The Editor expresses his thanks to each and every one.

The Ad Hoc Advisory Committee individually and together were instrumental in establishing the basic tenets that served as a guide in the formation of the entire work. In particular, special thanks are due to Professor Philip M. Morse for his continuous encouragement and support. Professors J. Todd and A. Erdélyi, panel members of the Conferences on Tables and members of the Advisory Committee have maintained an undiminished interest, offered many suggestions and carefully read all the chapters.

Irene A. Stegun has served effectively as associate editor, sharing in each stage of the planning of the volume. Without her untiring efforts, completion would never have been possible.

Appreciation is expressed for the generous cooperation of publishers and authors in granting permission for the use of their source material. Acknowledgments for tabular material taken wholly or in part from published works are given on the first page of each table. Myrtle R. Kellington corresponded with authors and publishers to obtain formal permission for including their material, maintained uniformity throughout the bibliographic references and assisted in preparing the introductory material.

Valuable assistance in the preparation, checking and editing of the tabular material was received from Ruth E. Capuano, Elizabeth F. Godefroy, David S. Liepman, Kermit Nelson, Bertha H. Walter and Ruth Zucker.

Equally important has been the untiring cooperation, assistance, and patience of the members of the NBS staff in handling the myriad of detail necessarily attending the publication of a volume of this magnitude. Especially appreciated have been the helpful discussions and services from the members of the Office of Technical Information in the areas of editorial format, graphic art layout, printing detail, preprinting reproduction needs, as well as attention to promotional detail and financial support. In addition, the clerical and typing staff of the Applied Mathematics Division merit commendation for their efficient and patient production of manuscript copy involving complicated technical notation.

Finally, the continued support of Dr. E. W. Cannon, chief of the Applied Mathematics Division, and the advice of Dr. F. L. Alt, assistant chief, as well as of the many mathematicians in the Division, is gratefully acknowledged.

M. ABRAMOWITZ.

2. Physical Constants and Conversion Factors

A. G. McNish [1]

Contents

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[1] National Bureau of Standards.

2. Physical Constants and Conversion Factors

The tables in this chapter supply some of the more commonly needed physical constants and conversion factors.*

The International System of Units (SI) established in 1960 by the General Conference of Weights and Measures under the Treaty of the Meter is based upon: the meter (m) for length, defined as 1 650 763.73 wave-lengths in vacuum corresponding to the transition $2p_{10}-5d_5$ of krypton 86; the kilogram (kg) for mass, defined as the mass of the prototype kilogram at Sevres, France; the second (s) for time, defined as the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of cesium 133; the kelvin (K) for temperature, defined as 1/273.16 of the thermodynamic temperature of the triple point of water; the ampere (A) for electric current, defined as the current which, if flowing in two infinitely long parallel wires *in vacuo* separated by one meter, would produce a force of 2×10^{-7} newtons per meter of length between the wires; and the candela (cd) for luminous intensity, defined as the luminous intensity of 1/600 000 square meter of a perfect radiator at the temperature of freezing platinum.

All other units of SI are derived from these base units by assigning the value unity to the proportionality constants in the defining equations (official symbols for other SI units appear in Tables 2.1 and 2.2). Taking 1/100 of the meter as the unit for length and 1/1000 of the kilogram as the unit for mass gives rise similarly to the cgs system, often used in physics and chemistry.

SI, as it is ordinarily used in electromagnetism, is a rationalized system, i.e., the electromagnetic units of SI relate to the quantities appearing in the so-called rationalized electromagnetic equations. Thus, the force per unit length between two current-carrying parallel wires of infinite length separated by unit distance *in vacuo* is $2f = \mu_0 i_1 i_2/4\pi$, where μ_0 has the value $4\pi \times 10^{-7}$H/m. The force between two electric charges *in vacuo* is correspondingly given by $f = q_1 q_2/4\pi\epsilon_0 r^2$, ϵ_0 having the value $1/\mu_0 c^2$, where c is the speed of light in meters per second. ($\epsilon_0 \sim 8.854 \times 10^{-12}$F/m)

Setting μ_0 equal to unity and deleting 4π from the denominator in the first equation above defines the cgs-emu system. Setting ϵ_0 equal to unity and deleting 4π from the denominator in the second equation correspondingly defines the cgs-esu system. The cgs-emu and the cgs-esu systems are most frequently used in the unrationalized forms.

Table 2.1. Common Units and Conversion Factors, CGS System and SI

Quantity	SI Name	CGS Name	Factor
Force	newton (N)	dyne	10^5
Energy	joule (J)	erg	10^7
Power	watt (W)	....	10^7

---

*See also "Preface to Ninth Printing," page IIIa and page II.

Table 2.2. Names and Conversion Factors for Electric and Magnetic Units

Quantity	SI name	emu name	esu name	emu-SI factors	esu-SI factors
Current	ampere (A)	abampere	statampere	10^{-1}	$\sim 3 \times 10^9$
Charge	coulomb (C)	abcoulomb	statcoulomb	10^{-1}	$\sim 3 \times 10^9$
Potential	volt (V)	abvolt	statvolt	10^8	$\sim (1/3) \times 10^{-2}$
Resistance	ohm (Ω)	abohm	statohm	10^9	$\sim (1/9) \times 10^{-11}$
Inductance	henry (H)	centimeter	....	10^9	$\sim (1/9) \times 10^{-11}$
Capacitance	farad (F)	....	centimeter	10^{-9}	$\sim 9 \times 10^{11}$
Magnetizing force	$A \cdot m^{-1}$	oersted	....	$4\pi \times 10^{-3}$	$\sim 3 \times 10^9$
Magnetomotive force	A	gilbert	....	$4\pi \times 10^{-1}$	$\sim 3/10^6$
Magnetic flux	weber (Wb)	maxwell	....	10^8	$\sim (1/3) \times 10^{-2}$
Magnetic flux density	tesla (T)	gauss (G)	....	10^4	$\sim (1/3) \times 10^{-6}$
Electric displacement	....	....	....	10^{-5}	$\sim 3 \times 10^5$

Example: If the value assigned to a current is 100 amperes its value in abamperes is $100 \times 10^{-1} = 10$.

The values of constants given in Table 2.3 are based on an adjustment by Taylor, Parker, and Langenberg, Rev. Mod. Phys. 41, p.375 (1969). They are being considered for adoption by the Task Group on Fundamental Constants of the Committee on Data for Science and Technology, International Council of Scientific Unions. The uncertainties given are standard errors estimated from the experimental data included in the adjustment. Where applicable, values are based on the unified scale of atomic masses in which the atomic mass unit (u) is defined as 1/12 of the mass of the atom of the ^{12}C nuclide.

Table 2.3. Adjusted Values of Constants

Constant	Symbol	Value	Uncertainty ‡	Unit			
				Systeme International (SI)		Centimeter-gram-second (CGS)	
Speed of light in vacuum	c	2.997 925 0	±10	$\times 10^{8}$	m/s	$\times 10^{10}$	cm/s
Elementary charge	e	1.602 191 7	70	10^{-19}	C	10^{-20}	$\mathrm{cm^{1/2}g^{1/2}}$ *
		4.803 250	21	....	....	10^{-10}	$\mathrm{cm^{3/2}g^{1/2}s^{-1}}$ †
Avogadro constant	N_A	6.022 169	40	10^{23}	$\mathrm{mol^{-1}}$	10^{23}	$\mathrm{mol^{-1}}$
Atomic mass unit	u	1.660 531	11	10^{-27}	kg	10^{-24}	g
Electron rest mass	m_e	9.109 558	54	10^{-31}	kg	10^{-28}	g
		5.485 930	34	10^{-4}	u	10^{-4}	u
Proton rest mass	m_p	1.672 614	11	10^{-27}	kg	10^{-24}	g
		1.007 276 61	8	10^{0}	u	10^{0}	u
Neutron rest mass	m_n	1.674 920	11	10^{-27}	kg	10^{-24}	g
		1.008 665 20	10	10^{0}	u	10^{0}	u
Faraday constant	F	9.648 670	54	10^{4}	C/mol	10^{3}	$\mathrm{cm^{1/2}g^{1/2}mol^{-1}}$*
		2.892 599	16	....	....	10^{14}	$\mathrm{cm^{3/2}g^{1/2}s^{-1}mol^{-1}}$ †
Planck constant	h	6.626 196	50	10^{-34}	$\mathrm{J \cdot s}$	10^{-27}	$\mathrm{erg \cdot s}$
	$\hbar$	1.054 591 9	80	10^{-34}	$\mathrm{J \cdot s}$	10^{-27}	$\mathrm{erg \cdot s}$
Fine structure constant	α	7.297 351	11	10^{-3}	....	10^{-3}	....
	$1/\alpha$	1.370 360 2	21	10^{2}	....	10^{2}	....
Charge to mass ratio for electron	e/m_e	1.758 802 8	54	10^{11}	C/kg	10^{7}	$\mathrm{cm^{1/2}/g^{1/2}}$ *
		5.272 759	16	....	....	10^{17}	$\mathrm{cm^{3/2}g^{-1/2}s^{-1}}$ †
Quantum-charge ratio	h/e	4.135 708	14	10^{-15}	$\mathrm{J \cdot s/C}$	10^{-7}	$\mathrm{cm^{3/2}g^{1/2}s^{-1}}$ *
		1.379 523 4	46	....	....	10^{-17}	$\mathrm{cm^{1/2}g^{1/2}}$ †
Compton wavelength of electron	λ_C	2.426 309 6	74	10^{-12}	m	10^{-10}	cm
	$\lambda_C/2\pi$	3.861 592	12	10^{-13}	m	10^{-11}	cm
Compton wavelength of proton	$\lambda_{C,p}$	1.321 440 9	90	10^{-15}	m	10^{-13}	cm
	$\lambda_{C,p}/2\pi$	2.103 139	14	10^{-16}	m	10^{-14}	cm
Rydberg constant	$R\infty$	1.097 373 12	11	10^{7}	$\mathrm{m^{-1}}$	10^{5}	$\mathrm{cm^{-1}}$
Bohr radius	a_0	5.291 771 5	81	10^{-11}	m	10^{-9}	cm
Electron radius	r_e	2.817 939	13	10^{-15}	m	10^{-13}	cm
Gyromagnetic ratio of proton	γ	2.675 196 5	82	10^{8}	$\mathrm{rad \cdot s^{-1}T^{-1}}$	10^{4}	$\mathrm{rad \cdot s^{-1}G^{-1}}$ *
	$\gamma/2\pi$	4.257 707	13	10^{7}	Hz/T	10^{3}	$\mathrm{s^{-1}G^{-1}}$ *
(uncorrected for diamagnetism, H$_2$O)	γ'	2.675 127 0	82	10^{8}	$\mathrm{rad \cdot s^{-1}T^{-1}}$	10^{4}	$\mathrm{rad \cdot s^{-1}G^{-1}}$ *
	$\gamma'/2\pi$	4.257 597	13	10^{7}	Hz/T	10^{3}	$\mathrm{s^{-1}G^{-1}}$ *
Bohr magneton	μ_B	9.274 096	65	10^{-24}	J/T	10^{-21}	erg/G *
Nuclear magneton	μ_N	5.050 951	50	10^{-27}	J/T	10^{-24}	erg/G *
Proton moment	μ_p	1.410 620 3	99	10^{-26}	J/T	10^{-23}	erg/G *
	μ_p/μ_N	2.792 782	17	10^{0}	....	10^{0}	....
(uncorrected for diamagnetism, H$_2$O)	μ'_p/μ_N	2.792 709	17	10^{0}	....	10^{0}	....
Gas constant	R	8.314 34	35	10^{0}	$\mathrm{J \cdot K^{-1}\,mol^{-1}}$	10^{7}	$\mathrm{erg \cdot K^{-1}\,mol^{-1}}$
Normal volume perfect gas	V_0	2.241 36	39	10^{-2}	$\mathrm{m^3/mol}$	10^{4}	$\mathrm{cm^3/mol}$
Boltzmann constant	k	1.380 622	59	10^{-23}	J/K	10^{-16}	erg/K
First radiation constant ($8\pi hc$)	c_1	4.992 579	38	10^{-24}	$\mathrm{J \cdot m}$	10^{-15}	$\mathrm{erg \cdot cm}$
Second radiation constant	c_2	1.438 833	61	10^{-2}	$\mathrm{m \cdot K}$	10^{0}	$\mathrm{cm \cdot K}$
Stefan-Boltzmann constant	σ	5.669 61	96	10^{-8}	$\mathrm{W \cdot m^{-2}K^{-4}}$	10^{-5}	$\mathrm{erg \cdot cm^{-2}s^{-1}K^{-4}}$
Gravitational constant	G	6.673 2	31	10^{-11}	$\mathrm{N \cdot m^2/kg^2}$	10^{-8}	$\mathrm{dyn \cdot cm^2/g^2}$

‡Based on 1 std. dev; applies to last digits in preceding column. *Electromagnetic system. †Electrostatic system.

Table 2.4. Miscellaneous Conversion Factors

Standard gravity, g_0 = 9.806 65 meters per second per second*

Standard atmospheric pressure, P_0 = $1.013\ 25 \times 10^5$ newtons per square meter*

= $1.013\ 25 \times 10^6$ dynes per square centimeter*

1 thermodynamic calorie,[1] cal$_c$ = 4.1840 joules*

1 IT calorie[2], cal$_s$ = 4.1868 joules*

1 liter, l = 10^{-3} cubic meter*

1 angstrom unit, Å = 10^{-10} meter*

1 bar = 10^5 newtons per square meter*

= 10^6 dynes per square centimeter*

1 gal = 10^{-2} meter per second per second*

= 1 centimeter per second per second*

1 astronomical unit, AU = 1.496×10^{11} meters

1 light year = 9.46×10^{15} meters

1 parsec = 3.08×10^{16} meters

= 3.26 light years

1 curie, the quantity of radioactive material undergoing 3.7×10^{10} disintegrations per second*.

1 roentgen, the exposure of x- or gamma radiation which produces together with its secondaries 2.082×10^9 electron-ion pairs in 0.001 293 gram of air.

The index of refraction of the atmosphere for radio waves of frequency less than 3×10^{10} Hz is given by $(n - 1)10^6 = (77.6/t)\ (p + 4810e/t)$, where n is the refractive index; t, temperature in kelvins; p, total pressure in millibars; e, water vapor partial pressure in millibars.

Factors for converting the customary United States units to units of the metric system are given in Table 2.5.

Table 2.5. Factors for Converting Customary U.S. Units to SI Units

1 yard	0.914 4 meter*
1 foot	0.304 8 meter*
1 inch	0.025 4 meter*
1 statute mile	1 609.344 meters*
1 nautical mile (international)	1 852 meters*
1 pound (avdp.)	0.453 592 37 kilogram*
1 oz. (avdp.)	0.028 349 52 kilogram
1 pound force	4.448 22 newtons
1 slug	14.593 9 kilograms
1 poundal	0.138 255 newtons
1 foot pound	1.355 82 joules
Temperature (Fahrenheit)	32 + (9/5) Celsius temperature*
1 British thermal unit[3]	1055 joules

Geodetic constants for the international (Hayford) spheroid are given in Table 2.6. The gravity values are on the basis of the revised Potsdam value. They are about 14 parts per million smaller than previous values. They are calculated for the surface of the geoid by the international formula.

Table 2.6. Geodetic Constants

a = 6 378 388 m; f = 1/297; b = 6 356 912 m

Latitude	Length of 1′ of longitude	Length of 1′ of latitude	g
	Meters	*Meters*	*m/s²*
0°	1 855.398	1 842.925	9.780 350
15	1 792.580	1 844.170	9.783 800
30	1 608.174	1 847.580	9.793 238
45	1 314.175	1 852.256	9.806 154
60	930.047	1 856.951	9.819 099
75	481.725	1 860.401	9.828 593
90	0	1 861.666	9.832 072

[1] Used principally by chemists.

[2] Used principally by engineers.

[3] Various definitions are given for the British thermal unit. This represents a rounded mean value differing from none of the more important definitions by more than 3 in 10^4.

* Exact value.

3. Elementary Analytical Methods

MILTON ABRAMOWITZ [1]

Contents

The author acknowledges the assistance of Peter J. O'Hara and Kermit C. Nelson in the preparation and checking of the table of powers and roots.

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[1] National Bureau of Standards. (Deceased.)

3. Elementary Analytical Methods

3.1. Binomial Theorem and Binomial Coefficients; Arithmetic and Geometric Progressions; Arithmetic, Geometric, Harmonic and Generalized Means

Binomial Theorem

3.1.1

$$(a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\binom{n}{3}a^{n-3}b^3+\ldots+b^n$$

(n a positive integer)

Binomial Coefficients (see chapter 24)

3.1.2

$$*\quad\binom{n}{k}={}_nC_k=\frac{n(n-1)\ldots(n-k+1)}{k!}=\frac{n!}{(n-k)!k!}$$

3.1.3 $$\binom{n}{k}=\binom{n}{n-k}=(-1)^k\binom{k-n-1}{k}$$

3.1.4 $$\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$$

3.1.5 $$\binom{n}{0}=\binom{n}{n}=1$$

3.1.6 $$1+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}=2^n$$

3.1.7 $$1-\binom{n}{1}+\binom{n}{2}-\ldots+(-1)^n\binom{n}{n}=0$$

Table of Binomial Coefficients $\binom{n}{k}$

3.1.8

n \ k	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1	1											
2	1	2	1										
3	1	3	3	1									
4	1	4	6	4	1								
5	1	5	10	10	5	1							
6	1	6	15	20	15	6	1						
7	1	7	21	35	35	21	7	1					
8	1	8	28	56	70	56	28	8	1				
9	1	9	36	84	126	126	84	36	9	1			
10	1	10	45	120	210	252	210	120	45	10	1		
11	1	11	55	165	330	462	462	330	165	55	11	1	
12	1	12	66	220	495	792	924	792	495	220	66	12	1

For a more extensive table see chapter **24**.

3.1.9

Sum of Arithmetic Progression to n Terms

$$a+(a+d)+(a+2d)+\ldots+(a+(n-1)d)=na+\frac{1}{2}n(n-1)d=\frac{n}{2}(a+l),$$

last term in series $=l=a+(n-1)d$

Sum of Geometric Progression to n Terms

3.1.10

$$s_n=a+ar+ar^2+\ldots+ar^{n-1}=\frac{a(1-r^n)}{1-r}$$

$$\lim_{n\to\infty}s_n=a/(1-r)\qquad(-1<r<1)$$

Arithmetic Mean of n Quantities A

3.1.11 $$A=\frac{a_1+a_2+\ldots+a_n}{n}$$

Geometric Mean of n Quantities G

3.1.12 $$G=(a_1a_2\ldots a_n)^{1/n}\qquad(a_k>0, k=1, 2, \ldots, n)$$

Harmonic Mean of n Quantities H

3.1.13

$$\frac{1}{H}=\frac{1}{n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)\qquad(a_k>0, k=1, 2, \ldots, n)$$

Generalized Mean

3.1.14 $$M(t)=\left(\frac{1}{n}\sum_{k=1}^{n}a_k^t\right)^{1/t}$$

3.1.15 $M(t)=0(t<0$, some a_k zero)

3.1.16 $$\lim_{t\to\infty}M(t)=\text{max.}\qquad(a_1, a_2, \ldots, a_n)=\text{max. }a$$

3.1.17 $$\lim_{t\to-\infty}M(t)=\text{min.}\qquad(a_1, a_2, \ldots, a_n)=\text{min. }a$$

3.1.18 $$\lim_{t\to0}M(t)=G$$

3.1.19 $$M(1)=A$$

3.1.20 $$M(-1)=H$$

3.2. Inequalities

Relation Between Arithmetic, Geometric, Harmonic and Generalized Means

3.2.1

$A\geq G\geq H$, equality if and only if $a_1=a_2=\ldots=a_n$

3.2.2 $$\text{min. }a<M(t)<\text{max. }a$$

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*See page II.

3.2.3 $$\text{min. } a < G < \text{max. } a$$

equality holds if all a_k are equal, or $t<0$ and an a_k is zero

3.2.4 $M(t)<M(s)$ if $t<s$ unless all a_k are equal, or $s<0$ and an a_k is zero.

Triangle Inequalities

3.2.5 $$|a_1|-|a_2|\leq|a_1+a_2|\leq|a_1|+|a_2|$$

3.2.6 $$\left|\sum_{k=1}^{n} a_k\right|\leq\sum_{k=1}^{n}|a_k|$$

Chebyshev's Inequality

If $a_1\geq a_2\geq a_3\geq \ . \ . \ . \ \geq a_n$

$b_1\geq b_2\geq b_3\geq \ . \ . \ . \ \geq b_n$

3.2.7 $$n\sum_{k=1}^{n} a_k b_k\geq\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right)$$

Hölder's Inequality for Sums

If $\frac{1}{p}+\frac{1}{q}=1,\ p>1,\ q>1$

3.2.8 $$\sum_{k=1}^{n}|a_k b_k|\leq\left(\sum_{k=1}^{n}|a_k|^p\right)^{1/p}\left(\sum_{k=1}^{n}|b_k|^q\right)^{1/q};$$

equality holds if and only if $|b_k|=c|a_k|^{p-1}$ (c=constant>0). If $p=q=2$ we get

Cauchy's Inequality

3.2.9

$$\left[\sum_{k=1}^{n} a_k b_k\right]^2\leq\sum_{k=1}^{n} a_k^2\sum_{k=1}^{n} b_k^2$$ (equality for $a_k=cb_k$, c constant).

Hölder's Inequality for Integrals

If $\frac{1}{p}+\frac{1}{q}=1,\ p>1,\ q>1$

3.2.10

$$\int_a^b|f(x)g(x)|dx\leq\left[\int_a^b|f(x)|^p dx\right]^{1/p}\left[\int_a^b|g(x)|^q dx\right]^{1/q}$$

equality holds if and only if $|g(x)|=c|f(x)|^{p-1}$ (c=constant>0).

If $p=q=2$ we get

Schwarz's Inequality

3.2.11

$$\left[\int_a^b f(x)g(x)dx\right]^2\leq\int_a^b[f(x)]^2dx\int_a^b[g(x)]^2dx$$

Minkowski's Inequality for Sums

If $p>1$ and $a_k,\ b_k>0$ for all k,

3.2.12

$$\left(\sum_{k=1}^{n}(a_k+b_k)^p\right)^{1/p}\leq\left(\sum_{k=1}^{n}a_k^p\right)^{1/p}+\left(\sum_{k=1}^{n}b_k^p\right)^{1/p},$$

equality holds if and only if $b_k=ca_k$ (c=constant>0).

Minkowski's Inequality for Integrals

If $p>1$,

3.2.13

$$\left(\int_a^b|f(x)+g(x)|^p dx\right)^{1/p}\leq\left(\int_a^b|f(x)|^p dx\right)^{1/p}+\left(\int_a^b|g(x)|^p dx\right)^{1/p}$$

equality holds if and only if $g(x)=cf(x)$ (c=constant>0).

3.3. Rules for Differentiation and Integration

Derivatives

3.3.1 $$\frac{d}{dx}(cu)=c\frac{du}{dx},\ c\text{ constant}$$

3.3.2 $$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$$

3.3.3 $$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

3.3.4 $$\frac{d}{dx}(u/v)=\frac{vdu/dx-udv/dx}{v^2}$$

3.3.5 $$\frac{d}{dx}u(v)=\frac{du}{dv}\frac{dv}{dx}$$

3.3.6 $$\frac{d}{dx}(u^v)=u^v\left(\frac{v}{u}\frac{du}{dx}+\ln u\frac{dv}{dx}\right)$$

Leibniz's Theorem for Differentiation of an Integral

3.3.7

$$\frac{d}{dc}\int_{a(c)}^{b(c)}f(x,c)dx=\int_{a(c)}^{b(c)}\frac{\partial}{\partial c}f(x,c)dx+f(b,c)\frac{db}{dc}-f(a,c)\frac{da}{dc}$$

Leibniz's Theorem for Differentiation of a Product

3.3.8

$$\frac{d^n}{dx^n}(uv)=\frac{d^n u}{dx^n}v+\binom{n}{1}\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx}+\binom{n}{2}\frac{d^{n-2}u}{dx^{n-2}}\frac{d^2v}{dx^2}+\ldots+\binom{n}{r}\frac{d^{n-r}u}{dx^{n-r}}\frac{d^rv}{dx^r}+\ldots+u\frac{d^nv}{dx^n}$$

3.3.9 $$\frac{dx}{dy}=1/\frac{dy}{dx}$$

3.3.10 $$\frac{d^2x}{dy^2}=\frac{-d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-3}$$

3.3.11 $$\frac{d^3x}{dy^3}=-\left[\frac{d^3y}{dx^3}\frac{dy}{dx}-3\left(\frac{d^2y}{dx^2}\right)^2\right]\left(\frac{dy}{dx}\right)^{-5}$$

Integration by Parts

3.3.12 $$\int u\,dv=uv-\int v\,du$$

3.3.13 $$\int uv\,dx=\left(\int u\,dx\right)v-\int\left(\int u\,dx\right)\frac{dv}{dx}\,dx$$

Integrals of Rational Algebraic Functions

(Integration constants are omitted)

3.3.14 $$\int(ax+b)^n dx=\frac{(ax+b)^{n+1}}{a(n+1)}\qquad(n\neq-1)$$

3.3.15 $$\int\frac{dx}{ax+b}=\frac{1}{a}\ln|ax+b|$$

The following formulas are useful for evaluating $\int\frac{P(x)dx}{(ax^2+bx+c)^n}$ where $P(x)$ is a polynomial and $n>1$ is an integer.

3.3.16

$$\int\frac{dx}{(ax^2+bx+c)}=\frac{2}{(4ac-b^2)^{\frac{1}{2}}}\arctan\frac{2ax+b}{(4ac-b^2)^{\frac{1}{2}}}\qquad(b^2-4ac<0)$$

3.3.17 $$=\frac{1}{(b^2-4ac)^{\frac{1}{2}}}\ln\left|\frac{2ax+b-(b^2-4ac)^{\frac{1}{2}}}{2ax+b+(b^2-4ac)^{\frac{1}{2}}}\right|\qquad(b^2-4ac>0)$$

3.3.18 $$=\frac{-2}{2ax+b}\qquad(b^2-4ac=0)$$

3.3.19

$$\int\frac{x\,dx}{ax^2+bx+c}=\frac{1}{2a}\ln|ax^2+bx+c|-\frac{b}{2a}\int\frac{dx}{ax^2+bx+c}$$

3.3.20

$$\int\frac{dx}{(a+bx)(c+dx)}=\frac{1}{ad-bc}\ln\left|\frac{c+dx}{a+bx}\right|\qquad(ad\neq bc)$$

3.3.21 $$\int\frac{dx}{a^2+b^2x^2}=\frac{1}{ab}\arctan\frac{bx}{a}$$

3.3.22 $$\int\frac{x\,dx}{a^2+b^2x^2}=\frac{1}{2b^2}\ln|a^2+b^2x^2|$$

3.3.23 $$\int\frac{dx}{a^2-b^2x^2}=\frac{1}{2ab}\ln\left|\frac{a+bx}{a-bx}\right|$$

3.3.24 $$\int\frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^3}\arctan\frac{x}{a}+\frac{x}{2a^2(x^2+a^2)}$$

3.3.25 $$\int\frac{dx}{(x^2-a^2)^2}=\frac{-x}{2a^2(x^2-a^2)}+\frac{1}{4a^3}\ln\left|\frac{a+x}{a-x}\right|$$

Integrals of Irrational Algebraic Functions

3.3.26 $$\int\frac{dx}{[(a+bx)(c+dx)]^{1/2}}=\frac{2}{(-bd)^{1/2}}\arctan\left[\frac{-d(a+bx)}{b(c+dx)}\right]^{1/2}\qquad(bd<0)$$

3.3.27 $$=\frac{-1}{(-bd)^{1/2}}\arcsin\left(\frac{2bdx+ad+bc}{bc-ad}\right)\qquad(b>0,\ d<0)$$

3.3.28 $$=\frac{2}{(bd)^{1/2}}\ln|[bd(a+bx)]^{1/2}+b(c+dx)^{1/2}|\qquad(bd>0)$$

3.3.29 $$\int\frac{dx}{(a+bx)^{1/2}(c+dx)}=\frac{2}{[d(bc-ad)]^{1/2}}\arctan\left[\frac{d(a+bx)}{(bc-ad)}\right]^{1/2}\qquad(d(ad-bc)<0)$$

3.3.30 $$=\frac{1}{[d(ad-bc)]^{1/2}}\ln\left|\frac{d(a+bx)^{1/2}-[d(ad-bc)]^{1/2}}{d(a+bx)^{1/2}+[d(ad-bc)]^{1/2}}\right|\qquad(d(ad-bc)>0)$$

3.3.31

$$\int [(a+bx)(c+dx)]^{1/2}dx$$
$$=\frac{(ad-bc)+2b(c+dx)}{4bd}[(a+bx)(c+dx)]^{1/2}$$
$$-\frac{(ad-bc)^2}{8bd}\int\frac{dx}{[(a+bx)(c+dx)]^{1/2}}$$

3.3.32

$$\int\left[\frac{c+dx}{a+bx}\right]^{1/2}dx=\frac{1}{b}[(a+bx)(c+dx)]^{1/2}$$
$$-\frac{(ad-bc)}{2b}\int\frac{dx}{[(a+bx)(c+dx)]^{1/2}}$$

3.3.33

$$\int\frac{dx}{(ax^2+bx+c)^{1/2}}$$
$$=a^{-1/2}\ln|2a^{1/2}(ax^2+bx+c)^{1/2}+2ax+b|\,(a>0)$$

3.3.34 $$=a^{-1/2}\operatorname{arcsinh}\frac{(2ax+b)}{(4ac-b^2)^{1/2}}\qquad(a>0,\ 4ac>b^2)$$

3.3.35 $$=a^{-1/2}\ln|2ax+b|\,(a>0,\ b^2=4ac)$$

3.3.36 $$=-(-a)^{-1/2}\arcsin\frac{(2ax+b)}{(b^2-4ac)^{1/2}}$$
$$(a<0,\ b^2>4ac,\ |2ax+b|<(b^2-4ac)^{1/2})$$

3.3.37

$$\int(ax^2+bx+c)^{1/2}dx=\frac{2ax+b}{4a}(ax^2+bx+c)^{1/2}$$
$$+\frac{4ac-b^2}{8a}\int\frac{dx}{(ax^2+bx+c)^{1/2}}$$

3.3.38

$$\int\frac{dx}{x(ax^2+bx+c)^{1/2}}=-\int\frac{dt}{(a+bt+ct^2)^{1/2}}\text{ where }t=1/x$$

3.3.39

$$\int\frac{xdx}{(ax^2+bx+c)^{1/2}}$$
$$=\frac{1}{a}(ax^2+bx+c)^{1/2}-\frac{b}{2a}\int\frac{dx}{(ax^2+bx+c)^{1/2}}$$

3.3.40 $$\int\frac{dx}{(x^2\pm a^2)^{\frac{1}{2}}}=\ln|x+(x^2\pm a^2)^{\frac{1}{2}}|$$

3.3.41

$$\int(x^2\pm a^2)^{\frac{1}{2}}dx=\frac{x}{2}(x^2\pm a^2)^{\frac{1}{2}}\pm\frac{a^2}{2}\ln|x+(x^2\pm a^2)^{\frac{1}{2}}|$$

3.3.42 $$\int\frac{dx}{x(x^2+a^2)^{\frac{1}{2}}}=-\frac{1}{a}\ln\left|\frac{a+(x^2+a^2)^{\frac{1}{2}}}{x}\right|$$

3.3.43 $$\int\frac{dx}{x(x^2-a^2)^{\frac{1}{2}}}=\frac{1}{a}\arccos\frac{a}{x}$$

3.3.44 $$\int\frac{dx}{(a^2-x^2)^{\frac{1}{2}}}=\arcsin\frac{x}{a}$$

3.3.45 $$\int(a^2-x^2)^{\frac{1}{2}}dx=\frac{x}{2}(a^2-x^2)^{\frac{1}{2}}+\frac{a^2}{2}\arcsin\frac{x}{a}$$

3.3.46 $$\int\frac{dx}{x(a^2-x^2)^{\frac{1}{2}}}=-\frac{1}{a}\ln\left|\frac{a+(a^2-x^2)^{\frac{1}{2}}}{x}\right|$$

3.3.47 $$\int\frac{dx}{(2ax-x^2)^{\frac{1}{2}}}=\arcsin\frac{x-a}{a}$$

3.3.48

$$\int(2ax-x^2)^{\frac{1}{2}}dx=\frac{(x-a)}{2}(2ax-x^2)^{\frac{1}{2}}+\frac{a^2}{2}\arcsin\frac{x-a}{a}$$

3.3.49

$$\int\frac{dx}{(ax^2+b)(cx^2+d)^{\frac{1}{2}}}$$
$$=\frac{1}{[b(ad-bc)]^{\frac{1}{2}}}\arctan\frac{x(ad-bc)^{\frac{1}{2}}}{[b(cx^2+d)]^{\frac{1}{2}}}\qquad(ad>bc)$$

3.3.50

$$=\frac{1}{2[b(bc-ad)]^{\frac{1}{2}}}\ln\left|\frac{[b(cx^2+d)]^{\frac{1}{2}}+x(bc-ad)^{\frac{1}{2}}}{[b(cx^2+d)]^{\frac{1}{2}}-x(bc-ad)^{\frac{1}{2}}}\right|$$
$$(bc>ad)$$

3.4. Limits, Maxima and Minima

Indeterminate Forms (L'Hospital's Rule)

3.4.1 Let $f(x)$ and $g(x)$ be differentiable on an interval $a\leq x<b$ for which $g'(x)\neq 0$.

If

$$\lim_{x\to b-}f(x)=0\text{ and }\lim_{x\to b-}g(x)=0$$

or if

$$\lim_{x\to b-}f(x)=\infty\text{ and }\lim_{x\to b-}g(x)=\infty$$

and if

$$\lim_{x\to b-}\frac{f'(x)}{g'(x)}=l\text{ then }\lim_{x\to b-}\frac{f(x)}{g(x)}=l.$$

Both b and l may be finite or infinite.

Maxima and Minima

3.4.2 (1) *Functions of One Variable*

The function $y=f(x)$ has a maximum at $x=x_0$ if $f'(x_0)=0$ and $f''(x_0)<0$, and a minimum at $x=x_0$ if $f'(x_0)=0$ and $f''(x_0)>0$. Points x_0 for which $f'(x_0)=0$ are called stationary points.

3.4.3 (2) *Functions of Two Variables*

The function $f(x, y)$ has a maximum or minimum for those values of (x_0, y_0) for which

$$\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0,$$

and for which $\begin{vmatrix} \partial^2 f/\partial x \partial y & \partial^2 f/\partial x^2 \\ \partial^2 f/\partial y^2 & \partial^2 f/\partial x \partial y \end{vmatrix} < 0;$

(a) $f(x,y)$ has a maximum

$$\text{if } \frac{\partial^2 f}{\partial x^2}<0 \text{ and } \frac{\partial^2 f}{\partial y^2}<0 \text{ at } (x_0, y_0),$$

(b) $f(x,y)$ has a minimum

$$\text{if } \frac{\partial^2 f}{\partial x^2}>0 \text{ and } \frac{\partial^2 f}{\partial y^2}>0 \text{ at } (x_0, y_0).$$

3.5. Absolute and Relative Errors

(1) If x_0 is an approximation to the true value of x, then

3.5.1 (a) the *absolute error* of x_0 is $\Delta x=x_0-x$, $x-x_0$ is the correction to x.

3.5.2 (b) the *relative error* of x_0 is $\delta x=\frac{\Delta x}{x}\approx\frac{\Delta x}{x_0}$

3.5.3 (c) the *percentage error* is 100 times the relative error.

3.5.4 (2) The absolute error of the sum or difference of several numbers is at most equal to the sum of the absolute errors of the individual numbers.

3.5.5 (3) If $f(x_1, x_2, \ldots, x_n)$ is a function of $x_1, x_2, \ldots, x_n$ and the absolute error in x_i $(i=1, 2, \ldots n)$ is Δx_i, then the absolute error in f is

$$\Delta f \approx \frac{\partial f}{\partial x_1}\Delta x_1+\frac{\partial f}{\partial x_2}\Delta x_2+\ldots+\frac{\partial f}{\partial x_n}\Delta x_n$$

3.5.6 (4) The relative error of the product or quotient of several factors is at most equal to the sum of the relative errors of the individual factors.

3.5.7

(5) If $y=f(x)$, the relative error $\delta y=\frac{\Delta y}{y}\approx\frac{f'(x)}{f(x)}\Delta x$

Approximate Values

If $|\epsilon|<<1, |\eta|<<1, b<<a,$

3.5.8 $$(a+b)^k \approx a^k+ka^{k-1}b$$

3.5.9 $$(1+\epsilon)(1+\eta)\approx 1+\epsilon+\eta$$

3.5.10 $$\frac{1+\epsilon}{1+\eta}\approx 1+\epsilon-\eta$$

3.6. Infinite Series

Taylor's Formula for a Single Variable

3.6.1

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x)+R_n$$

3.6.2

$$R_n=\frac{h^n}{n!}f^{(n)}(x+\theta_1 h)=\frac{h^n}{(n-1)!}(1-\theta_2)^{n-1}f^{(n)}(x+\theta_2 h) \qquad (0<\theta_{1,2}(x)<1)$$

3.6.3

$$=\frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1}f^{(n)}(x+th)dt$$

3.6.4

$$f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+ \ldots+\frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+R_n$$

3.6.5 $$R_n=\frac{(x-a)^n}{n!}f^{(n)}(\xi) \qquad (a<\xi<x)$$

Lagrange's Expansion

If $y=f(x)$, $y_0=f(x_0)$, $f'(x_0)\neq 0$, then

3.6.6

$$x=x_0+\sum_{k=1}^{\infty}\frac{(y-y_0)^k}{k!}[\frac{d^{k-1}}{dx^{k-1}}\{\frac{x-x_0}{f(x)-y_0}\}^k]_{x=x_0}$$

3.6.7

$$g(x)=g(x_0) +\sum_{k=1}^{\infty}\frac{(y-y_0)^k}{k!}\left[\frac{d^{k-1}}{dx^{k-1}}\left(g'(x)\left\{\frac{x-x_0}{f(x)-y_0}\right\}^k\right)\right]_{x=x_0}$$

where $g(x)$ is any function indefinitely differentiable.

Binomial Series

3.6.8

$$(1+x)^\alpha=\sum^{\infty}\binom{\alpha}{k}x^k \qquad (-1<x<1)$$

3.6.9

$$(1+x)^{\alpha}=1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\ldots,$$

3.6.10

$$(1+x)^{-1}=1-x+x^2-x^3+x^4-\ldots \qquad (-1<x<1)$$

3.6.11

$$(1+x)^{\frac{1}{2}}=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\frac{7x^5}{256}-\frac{21x^6}{1024}+\ldots \qquad (-1<x<1)$$

3.6.12

$$(1+x)^{-\frac{1}{2}}=1-\frac{x}{2}+\frac{3x^2}{8}-\frac{5x^3}{16}+\frac{35x^4}{128}-\frac{63x^5}{256}+\frac{231x^6}{1024}-\ldots \qquad (-1<x<1)$$

3.6.13

$$(1+x)^{\frac{1}{3}}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+\frac{22}{729}x^5-\frac{154}{6561}x^6+\ldots \qquad (-1<x<1)$$

3.6.14

$$(1+x)^{-\frac{1}{3}}=1-\frac{1}{3}x+\frac{2}{9}x^2-\frac{14}{81}x^3+\frac{35}{243}x^4-\frac{91}{729}x^5+\frac{728}{6561}x^6-\ldots \qquad (-1<x<1)$$

Asymptotic Expansions

3.6.15 A series $\sum_{k=0}^{\infty} a_k x^{-k}$ is said to be an asymptotic expansion of a function $f(x)$ if

$$f(x)-\sum_{k=0}^{n-1} a_k x^{-k}=O(x^{-n}) \text{ as } x\to\infty$$

for every $n=1, 2, \ldots$. We write

$$f(x)\sim\sum_{k=0}^{\infty} a_k x^{-k}.$$

The series itself may be either convergent or divergent.

Operations With Series

Let $s_1=1+a_1x+a_2x^2+a_3x^3+a_4x^4+\ \ldots$

$s_2=1+b_1x+b_2x^2+b_3x^3+b_4x^4+\ \ldots$

$s_3=1+c_1x+c_2x^2+c_3x^3+c_4x^4+\ \ldots$

	Operation	c_1	c_2	c_3	c_4
3.6.16	$s_3=s_1^{-1}$	$-a_1$	$a_1^2-a_2$	$2a_1a_2-a_3-a_1^3$	$2a_1a_3-3a_1^2a_2-a_4+a_2^2+a_1^4$
3.6.17	$s_3=s_1^{-2}$	$-2a_1$	$3a_1^2-2a_2$	$6a_1a_2-2a_3-4a_1^3$	$6a_1a_3+3a_2^2-2a_4-12a_1^2a_2+5a_1^4$
3.6.18	$s_3=s_1^{\frac{1}{2}}$	$\frac{1}{2}a_1$	$\frac{1}{2}a_2-\frac{1}{8}a_1^2$	$\frac{1}{2}a_3-\frac{1}{4}a_1a_2+\frac{1}{16}a_1^3$	$\frac{1}{2}a_4-\frac{1}{4}a_1a_3-\frac{1}{8}a_2^2+\frac{3}{16}a_1^2a_2-\frac{5}{128}a_1^4$
3.6.19	$s_3=s_1^{-\frac{1}{2}}$	$-\frac{1}{2}a_1$	$\frac{3}{8}a_1^2-\frac{1}{2}a_2$	$\frac{3}{4}a_1a_2-\frac{1}{2}a_3-\frac{5}{16}a_1^3$	$\frac{3}{4}a_1a_3+\frac{3}{8}a_2^2-\frac{1}{2}a_4-\frac{15}{16}a_1^2a_2+\frac{35}{128}a_1^4$
3.6.20	$s_3=s_1^n$	na_1	$\frac{1}{2}(n-1)c_1a_1+na_2$ *	$c_1a_2(n-1)$ $+\frac{1}{6}c_1a_1^2(n-1)(n-2)$ $+na_3$ *	$na_4+c_1a_3(n-1)+\frac{1}{2}n(n-1)a_2^2$ $+\frac{1}{2}(n-1)(n-2)c_1a_1a_2$ $+\frac{1}{24}(n-1)(n-2)(n-3)c_1a_1^3$
3.6.21	$s_3=s_1s_2$	a_1+b_1	$b_2+a_1b_1+a_2$	$b_3+a_1b_2+a_2b_1+a_3$	$b_4+a_1b_3+a_2b_2+a_3b_1+a_4$
3.6.22	$s_3=s_1/s_2$	a_1-b_1	$a_2-(b_1c_1+b_2)$	$a_3-(b_1c_2+b_2c_1+b_3)$	$a_4-(b_1c_3+b_2c_2+b_3c_1+b_4)$
3.6.23	$s_3=\exp\ (s_1-1)$	a_1	$a_2+\frac{1}{2}a_1^2$	$a_3+a_1a_2+\frac{1}{6}a_1^3$	$a_4+a_1a_3+\frac{1}{2}a_2^2+\frac{1}{2}a_2a_1^2+\frac{1}{24}a_1^4$
3.6.24	$s_3=1+\ln s_1$	a_1	$a_2-\frac{1}{2}a_1c_1$	$a_3-\frac{1}{3}(a_2c_1+2a_1c_2)$	$a_4-\frac{1}{4}(a_3c_1+2a_2c_2+3a_1c_3)$ *

Reversion of Series

3.6.25 Given

$$y=ax+bx^2+cx^3+dx^4+ex^5+fx^6+gx^7+\ .\ .\ .$$

then

$$x=Ay+By^2+Cy^3+Dy^4+Ey^5+Fy^6+Gy^7+\ .\ .\ .$$

where

$$aA=1$$

$$a^3B=-b$$

$$a^5C=2b^2-ac$$

$$a^7D=5abc-a^2d-5b^3$$

$$a^9E=6a^2bd+3a^2c^2+14b^4-a^3e-21ab^2c$$

$$a^{11}F=7a^3be+7a^3cd+84ab^3c-a^4f-28a^2bc^2-42b^5-28a^2b^2d$$

$$a^{13}G=8a^4bf+8a^4ce+4a^4d^2+120a^2b^3d+180a^2b^2c^2+132b^6-a^5g-36a^3b^2e-72a^3bcd-12a^3c^3-330ab^4c$$

Kummer's Transformation of Series

3.6.26 Let $\sum\limits_{k=0}^{\infty} a_k=s$ be a given convergent series and $\sum\limits_{k=0}^{\infty} c_k=c$ be a given convergent series with known sum c such that $\lim\limits_{k\to\infty} \frac{a_k}{c_k}=\lambda\neq 0$.

Then

$$s=\lambda c+\sum_{k=0}^{\infty}\left(1-\lambda\frac{c_k}{a_k}\right)a_k.$$

Euler's Transformation of Series

3.6.27 If $\sum\limits_{k=0}^{\infty}(-1)^k a_k=a_0-a_1+a_2-\ .\ .\ .$ is a convergent series with sum s then

$$s=\sum_{k=0}^{\infty}\frac{(-1)^k\Delta^k a_0}{2^{k+1}},\ \Delta^k a_0=\sum_{m=0}^{k}(-1)^m\binom{k}{m}a_{k-m}$$

Euler-Maclaurin Summation Formula

3.6.28

$$\sum_{k=1}^{n-1}f_k=\int_0^n f(k)dk-\frac{1}{2}[f(0)+f(n)]+\frac{1}{12}[f'(n)-f'(0)]-\frac{1}{720}[f'''(n)-f'''(0)]+\frac{1}{30240}[f^{(V)}(n)-f^{(V)}(0)]-\frac{1}{1209600}[f^{(VII)}(n)-f^{(VII)}(0)]+\ .\ .\ .$$

3.7. Complex Numbers and Functions

Cartesian Form

3.7.1 $z=x+iy$

Polar Form

3.7.2 $z=re^{i\theta}=r(\cos\theta+i\sin\theta)$

3.7.3 *Modulus:* $|z|=(x^2+y^2)^{\frac{1}{2}}=r$

3.7.4 *Argument:* $\arg z=\arctan(y/x)=\theta$ (other notations for arg z are am z and ph z).

3.7.5 Real Part: $x=\mathscr{R}z=r\cos\theta$

3.7.6 Imaginary Part: $y=\mathscr{I}z=r\sin\theta$

Complex Conjugate of z

3.7.7 $\bar{z}=x-iy$

3.7.8 $|\bar{z}|=|z|$

3.7.9 $\arg\bar{z}=-\arg z$

Multiplication and Division

If $z_1=x_1+iy_1$, $z_2=x_2+iy_2$, then

3.7.10 $z_1z_2=x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)$

3.7.11 $|z_1z_2|=|z_1||z_2|$

3.7.12 $\arg(z_1z_2)=\arg z_1+\arg z_2$

3.7.13 $\frac{z_1}{z_2}=\frac{z_1\bar{z}_2}{|z_2|^2}=\frac{x_1x_2+y_1y_2+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2}$

3.7.14 $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$

3.7.15 $\arg\left(\frac{z_1}{z_2}\right)=\arg z_1-\arg z_2$

Powers

3.7.16 $z^n=r^ne^{in\theta}$

3.7.17 $=r^n\cos n\theta+ir^n\sin n\theta$ $\quad(n=0,\pm1,\pm2,\ .\ .\ .)$

3.7.18 $z^2=x^2-y^2+i(2xy)$

3.7.19 $z^3=x^3-3xy^2+i(3x^2y-y^3)$

3.7.20 $z^4=x^4-6x^2y^2+y^4+i(4x^3y-4xy^3)$

3.7.21 $z^5=x^5-10x^3y^2+5xy^4+i(5x^4y-10x^2y^3+y^5)$

3.7.22

$$z^n=\left[x^n-\binom{n}{2}x^{n-2}y^2+\binom{n}{4}x^{n-4}y^4-\ .\ .\ .\right]+i\left[\binom{n}{1}x^{n-1}y-\binom{n}{3}x^{n-3}y^3+\ .\ .\ .\right],$$

$$(n=1,2,\ .\ .\ .)$$

If $z^n=u_n+iv_n$, then $z^{n+1}=u_{n+1}+iv_{n+1}$ where

3.7.23 $u_{n+1}=xu_n-yv_n;\ v_{n+1}=xv_n+yu_n$

$\mathscr{R}z^n$ and $\mathscr{I}z^n$ are called harmonic polynomials.

3.7.24
$$\frac{1}{z}=\frac{\bar{z}}{|z|^2}=\frac{x-iy}{x^2+y^2}$$

3.7.25
$$\frac{1}{z^n}=\frac{\bar{z}^n}{|z|^{2n}}=(z^{-1})^n$$

Roots

3.7.26 $z^{\frac12}=\sqrt{z}=r^{\frac12}e^{\frac12 i\theta}=r^{\frac12}\cos\frac12\theta+ir^{\frac12}\sin\frac12\theta$

If $-\pi<\theta\leq\pi$ this is the principal root. The other root has the opposite sign. The principal root is given by

3.7.27 $z^{\frac12}=[\frac12(r+x)]^{\frac12}\pm i[\frac12(r-x)]^{\frac12}=u\pm iv$ where $2uv=y$ and where the ambiguous sign is taken to be the same as the sign of y.

3.7.28 $z^{1/n}=r^{1/n}e^{i\theta/n}$, (principal root if $-\pi<\theta\leq\pi$). Other roots are $r^{1/n}e^{i(\theta+2\pi k)/n}$ $(k=1, 2, 3, \ldots, n-1)$.

Inequalities

3.7.29
$$\left||z_1|-|z_2|\right|\leq|z_1\pm z_2|\leq|z_1|+|z_2|$$

Complex Functions, Cauchy-Riemann Equations

$f(z)=f(x+iy)=u(x,y)+iv(x,y)$ where $u(x,y)$, $v(x,y)$ are real, is *analytic* at those points $z=x+iy$ at which

3.7.30
$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

If $z=re^{i\theta}$,

3.7.31
$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{1}{r}\frac{\partial u}{\partial \theta}=-\frac{\partial v}{\partial r}$$

Laplace's Equation

The functions $u(x,y)$ and $v(x,y)$ are called harmonic functions and satisfy Laplace's equation:

Cartesian Coordinates

3.7.32
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

Polar Coordinates

3.7.33
$$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial \theta^2}=r\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{\partial^2 v}{\partial \theta^2}=0$$

3.8. Algebraic Equations

Solution of Quadratic Equations

3.8.1 Given $az^2+bz+c=0$,

$$z_{1,2}=-\left(\frac{b}{2a}\right)\pm\frac{1}{2a}q^{\frac12},\ q=b^2-4ac,$$

$$z_1+z_2=-b/a,\ z_1z_2=c/a$$

If $q>0$, two real roots,
$q=0$, two equal roots,
$q<0$, pair of complex conjugate roots.

Solution of Cubic Equations

3.8.2 Given $z^3+a_2z^2+a_1z+a_0=0$, let

$$q=\frac{1}{3}a_1-\frac{1}{9}a_2^2;\ r=\frac{1}{6}(a_1a_2-3a_0)-\frac{1}{27}a_2^3.$$

If $q^3+r^2>0$, one real root and a pair of complex conjugate roots,
$q^3+r^2=0$, all roots real and at least two are equal,
$q^3+r^2<0$, all roots real (irreducible case).

Let

$$s_1=[r+(q^3+r^2)^{\frac12}]^{\frac13},\ s_2=[r-(q^3+r^2)^{\frac12}]^{\frac13}$$

then

$$z_1=(s_1+s_2)-\frac{a_2}{3}$$

$$z_2=-\frac{1}{2}(s_1+s_2)-\frac{a_2}{3}+\frac{i\sqrt{3}}{2}(s_1-s_2)$$

$$z_3=-\frac{1}{2}(s_1+s_2)-\frac{a_2}{3}-\frac{i\sqrt{3}}{2}(s_1-s_2).$$

If z_1, z_2, z_3 are the roots of the cubic equation

$$z_1+z_2+z_3=-a_2$$

$$z_1z_2+z_1z_3+z_2z_3=a_1$$

$$z_1z_2z_3=-a_0$$

Solution of Quartic Equations

3.8.3 Given $z^4+a_3z^3+a_2z^2+a_1z+a_0=0$, find the real root u_1 of the cubic equation

$$u^3-a_2u^2+(a_1a_3-4a_0)u-(a_1^2+a_0a_3^2-4a_0a_2)=0$$

and determine the four roots of the quartic as solutions of the two quadratic equations

$$v^2+\left[\frac{a_3}{2}\mp\left(\frac{a_3^2}{4}+u_1-a_2\right)^{\frac12}\right]v+\frac{u_1}{2}\mp\left[\left(\frac{u_1}{2}\right)^2-a_0\right]^{\frac12}=0$$

If all roots of the cubic equation are real, use the value of u_1 which gives real coefficients in the *quadratic equation and select signs so that if

$$z^4+a_3z^3+a_2z^2+a_1z+a_0=(z^2+p_1z+q_1)(z^2+p_2z+q_2),$$

then

$$p_1+p_2=a_3, p_1p_2+q_1+q_2=a_2, p_1q_2+p_2q_1=a_1, q_1q_2=a_0.$$

If z_1, z_2, z_3, z_4 are the roots,

$$\Sigma z_i=-a_3, \Sigma z_iz_jz_k=-a_1,$$

$$\Sigma z_iz_j=a_2, z_1z_2z_3z_4=a_0.$$

3.9. Successive Approximation Methods

General Comments

3.9.1 Let $x=x_1$ be an approximation to $x=\xi$ where $f(\xi)=0$ and both x_1 and ξ are in the interval $a\leq x\leq b$. We define

$$x_{n+1}=x_n+c_nf(x_n) \qquad (n=1, 2, \ldots).$$

Then, if $f'(x)\geq 0$ and the constants c_n are negative and bounded, the sequence x_n converges monotonically to the root ξ.

If $c_n=c=\text{constant}<0$ and $f'(x)>0$, then the process converges but not necessarily monotonically.

Degree of Convergence of an Approximation Process

3.9.2 Let x_1, x_2, x_3, . . . be an infinite sequence of approximations to a number ξ. Then, if

$$|x_{n+1}-\xi|<A|x_n-\xi|^k, \qquad (n=1, 2, \ldots)$$

where A and k are independent of n, the sequence is said to have convergence of at most the kth degree (or order or index) to ξ. If $k=1$ and $A<1$ the convergence is linear; if $k=2$ the convergence is quadratic.

Regula Falsi (False Position)

3.9.3 Given $y=f(x)$ to find ξ such that $f(\xi)=0$, choose x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ have opposite signs and compute

$$x_2=x_1-\frac{(x_1-x_0)}{(f_1-f_0)}\,f_1=\frac{f_1x_0-f_0x_1}{f_1-f_0}.$$

Then continue with x_2 and either of x_0 or x_1 for which $f(x_0)$ or $f(x_1)$ is of opposite sign to $f(x_2)$.

Regula falsi is equivalent to inverse linear interpolation.

Method of Iteration (Successive Substitution)

3.9.4 The iteration scheme $x_{k+1}=F(x_k)$ will converge to a zero of $x=F(x)$ if

$$(1) \quad |F'(x)|\leq q<1 \text{ for } a\leq x\leq b,$$

$$(2) \quad a\leq x_0\pm\frac{|F(x_0)-x_0|}{1-q}\leq b.$$

Newton's Method of Successive Approximations

3.9.5

Newton's Rule

If $x=x_k$ is an approximation to the solution $x=\xi$ of $f(x)=0$ then the sequence

$$x_{k+1}=x_k-\frac{f(x_k)}{f'(x_k)}$$

will converge quadratically to $x=\xi$: (if instead of the condition (2) above),

(1) *Monotonic convergence*, $f(x_0)f''(x_0)>0$ and $f'(x)$, $f''(x)$ do not change sign in the interval (x_0, ξ), or

(2) *Oscillatory convergence*, $f(x_0)f''(x_0)<0$ and $f'(x)$, $f''(x)$ do not change sign in the interval (x_0, x_1), $x_0\leq\xi\leq x_1$.

Newton's Method Applied to Real *n*th Roots

3.9.6 Given $x^n=N$, if x_k is an approximation $x=N^{1/n}$ then the sequence

$$x_{k+1}=\frac{1}{n}\left[\frac{N}{x_k^{n-1}}+(n-1)x_k\right]$$

will converge quadratically to x.

$$\text{If } n=2,\ x_{k+1}=\frac{1}{2}\left(\frac{N}{x_k}+x_k\right),$$

$$\text{If } n=3,\ x_{k+1}=\frac{1}{3}\left(\frac{N}{x_k^2}+2x_k\right).$$

Aitken's δ^2-Process for Acceleration of Sequences

3.9.7 If x_k, x_{k+1}, x_{k+2} are three successive iterates in a sequence converging with an error which is approximately in geometric progression, then

$$\bar{x}_k=x_k-\frac{(x_k-x_{k+1})^2}{\Delta^2x_k}=\frac{x_kx_{k+2}-x_{k+1}^2}{\Delta^2x_k};$$

$$\Delta^2x_k=x_k-2x_{k+1}+x_{k+2}$$

is an improved estimate of x. In fact, if $x_k=x+$ * $O(\lambda^k)$ then $\bar{x}=x+O(\lambda^k)$, $|\lambda|<1$.

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*See page II

3.10. Theorems on Continued Fractions

Definitions

3.10.1

(1) Let
$$f=b_0+\cfrac{a_1}{b_1+\cfrac{a_2}{b_2+\cfrac{a_3}{b_3+}}}\cdots$$
$$=b_0+\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\,\frac{a_3}{b_3+}\cdots$$

If the number of terms is finite, f is called a terminating continued fraction. If the number of terms is infinite, f is called an infinite continued fraction and the terminating fraction

$$f_n=\frac{A_n}{B_n}=b_0+\frac{a_1}{b_1+}\,\frac{a_2}{b_2+}\cdots\frac{a_n}{b_n}$$

is called the nth convergent of f.

(2) If $\lim\limits_{n\to\infty}\frac{A_n}{B_n}$ exists, the infinite continued fraction f is said to be convergent. If $a_i=1$ and the b_i are integers there is always convergence.

Theorems

(1) If a_i and b_i are positive then $f_{2n}<f_{2n+2}$, $f_{2n-1}>f_{2n+1}$.

(2) If $f_n=\frac{A_n}{B_n}$,
$$A_n=b_nA_{n-1}+a_nA_{n-2}$$
$$B_n=b_nB_{n-1}+a_nB_{n-2}$$
where $A_{-1}=1$, $A_0=b_0$, $B_{-1}=0$, $B_0=1$.

(3)
$$\begin{bmatrix}A_n\\B_n\end{bmatrix}=\begin{bmatrix}A_{n-1}&A_{n-2}\\B_{n-1}&B_{n-2}\end{bmatrix}\begin{bmatrix}b_n\\a_n\end{bmatrix}$$

(4)
$$A_nB_{n-1}-A_{n-1}B_n=(-1)^{n-1}\prod_{k=1}^{n}a_k$$

(5) For every $n\geq 0$,
$$f_n=b_0+\frac{c_1a_1}{c_1b_1+}\,\frac{c_1c_2a_2}{c_2b_2+}\,\frac{c_2c_3a_3}{c_3b_3+}\cdots\frac{c_{n-1}c_na_n}{c_nb_n}.$$

(6)
$$1+b_2+b_2b_3+\ldots+b_2b_3\ldots b_n$$
$$=\frac{1}{1-}\,\frac{b_2}{b_2+1-}\,\frac{b_3}{b_3+1-}\cdots\frac{b_n}{-b_n+1}$$
$$\frac{1}{u_1}+\frac{1}{u_2}+\ldots+\frac{1}{u_n}=\frac{1}{u_1-}\,\frac{u_1^2}{u_1+u_2-}\cdots\frac{u_{n-1}^2}{-u_{n-1}+u_n}$$
$$\frac{1}{a_0}-\frac{x}{a_0a_1}+\frac{x^2}{a_0a_1a_2}\ldots+(-1)^n\frac{x^n}{a_0a_1a_2\ldots a_n}$$
$$=\frac{1}{a_0+}\,\frac{a_0x}{a_1-x+}\,\frac{a_1x}{a_2-x+}\cdots\frac{a_{n-1}x}{+a_n-x}$$

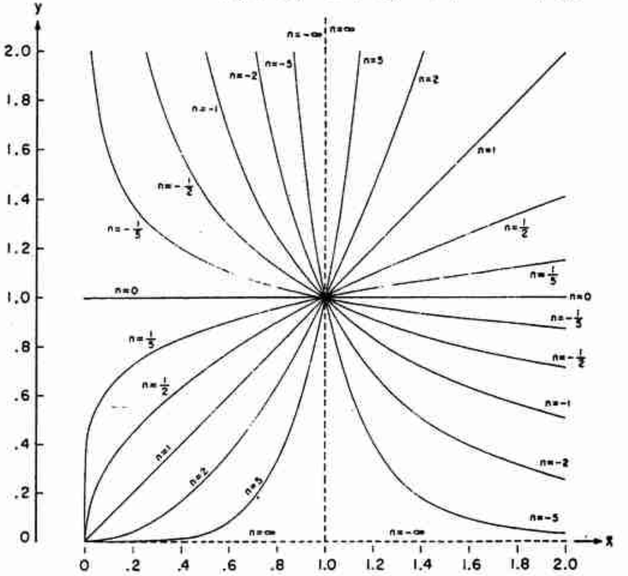

FIGURE 3.1. $y=x^n$.
$\pm n=0, \frac{1}{5}, \frac{1}{2}, 1, 2, 5.$

Numerical Methods

3.11. Use and Extension of the Tables

Example 1. Compute x^{19} and x^{47} for $x=29$ using **Table 3.1.**

$$x^{19}=x^9\cdot x^{10}$$
$$=(1.45071\ 4598\cdot 10^{13})(4.20707\ 2333\cdot 10^{14})$$
$$=6.10326\ 1248\cdot 10^{27}$$
$$x^{47}=(x^{24})^2/x$$
$$=(1.25184\ 9008\cdot 10^{35})^2/29$$
$$=5.40388\ 2547\cdot 10^{68}$$

Example 2. Compute $x^{-3/4}$ for $x=9.19826$.

$(9.19826)^{1/4}=(919.826/100)^{1/4}=(919.826)^{1/4}/10^{\frac{1}{2}}$

Linear interpolation in **Table 3.1** gives $(919.826)^{1/4}\approx 5.507144$.

By Newton's method for fourth roots with $N=919.826$,

$$\frac{1}{4}\left[\frac{919.826}{(5.507144)^3}+3(5.507144)\right]=5.50714\ 3845$$

Repetition yields the same result. Thus,

$$x^{1/4}=5.50714\ 3845/10^{\frac{1}{2}}=1.74151\ 1796,$$
$$x^{-3/4}=x^{\frac{1}{4}}/x=.18933\ 05683.$$

3.12. Computing Techniques

Example 3. Solve the quadratic equation $x^2-18.2x+.056$ given the coefficients as $18.2\pm.1$,

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*See page II.

$.056\pm.001$. From **3.8.1** the solution is

$$x=\tfrac{1}{2}(18.\underline{2}\pm[(18.\underline{2})^2-4(.05\underline{6})]^{\frac{1}{2}})$$
$$=\tfrac{1}{2}(18.\underline{2}\pm[33\underline{1.016}]^{\frac{1}{2}})=\tfrac{1}{2}(18.\underline{2}\pm18.\underline{1939})$$
$$=18.\underline{1969},\ .00\underline{3}$$

The smaller root may be obtained more accurately from

* $$.05\underline{6}/18.\underline{1969}=.003\underline{1}\pm.0001.$$

Example 4. Compute $(-3+.0076i)^{\frac{1}{2}}$.

From **3.7.26,** $(-3+.0076i)^{\frac{1}{2}}=u+iv$ where

$$u=\frac{y}{2v},\ v=\left(\frac{r-x}{2}\right)^{\frac{1}{2}},\ r=(x^2+y^2)^{\frac{1}{2}}$$

Thus

$$r=[(-3)^2+(.0076)^2]^{\frac{1}{2}}=(9.00005776)^{\frac{1}{2}}=3.00000\ 9627$$

$$v=\left[\frac{3.00000\ 9627-(-3)}{2}\right]^{\frac{1}{2}}=1.73205\ 2196$$

$$u=\frac{y}{2v}=\frac{.0076}{2(1.73205\ 2196)}=.00219\ 392926$$

We note that the principal square root has been computed.

Example 5. Solve the cubic equation $x^3-18.1x-34.8=0$.

To use Newton's method we first form the table of $f(x)=x^3-18.1x-34.8$

x	$f(x)$
4	-43.2
5	$-\ \ .3$
6	72.6
7	181.5

We obtain by linear inverse interpolation:

$$x_0=5+\frac{0-(-.3)}{72.6-(-.3)}=5.004.$$

Using Newton's method, $f'(x)=3x^2-18.1$ we get

$$x_1\approx x_0-f(x_0)/f'(x_0)$$
$$\approx5.004-\frac{(-.07215\ 9936)}{57.020048}\approx5.00526.$$

Repetition yields $x_1=5.00526\ 5097$. Dividing $f(x)$ by $x-5.00526\ 5097$ gives $x^2+5.00526\ 5097x+6.95267\ 869$ the zeros of which are $-2.50263\ 2549\pm.83036\ 800i$.

Example 6. Solve the quartic equation

$$x^4-2.37752\ 4922x^3+6.07350\ 5741x^2$$
$$-11.17938\ 023x+9.05265\ 5259=0.$$

Resolution Into Quadratic Factors
$(x^2+p_1x+q_1)(x^2+p_2x+q_2)$
by Inverse Interpolation

Starting with the trial value $q_1=1$ we compute successively

q_1	$q_2=\frac{a_0}{q_1}$	$p_1=\frac{a_1-a_3q_1}{q_2-q_1}$	$p_2=a_3-p_1$	$y(q_1)=q_1+q_2+p_1p_2-a_2$
1	9. 053	$-1.$ 093	$-1.$ 284	5. 383
2	4. 526	$-2.$ 543	. 165	. 032
2. 2	4. 115	$-3.$ 106	. 729	$-2.$ 023

We seek that value of q_1 for which $y(q_1)=0$. Inverse interpolation in $y(q_1)$ gives $y(q_1)\approx0$ for $q_1\approx2.003$. Then,

q_1	q_2	p_1	p_2	$y(q_1)$
2. 003	4. 520	$-2.$ 550	. 172	. 011

Inverse interpolation between $q_1=2.2$ and $q_1=2.003$ gives $q_1=2.0041$, and thus,

q_1	q_2	p_1	p_2	$y(q_1)$
2. 0041	4. 51706 7640	$-2.$ 55259 257	. 17506 765	. 00078 552
2. 0042	4. 51684 2260	$-2.$ 55282 851	. 17530 358	. 00001 655
2. 0043	4. 51661 6903	$-2.$ 55306 447	. 17553 955	$-.$ 00075 263

Inverse interpolation gives $q_1=2.00420\ 2152$, and we get finally,

q_1	q_2	p_1	p_2	$y(q_1)$
2. 00420 2152	4. 51683 7410	$-2.$ 55283 358	. 17530 8659	$-.$ 00000 0011

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4. Elementary Transcendental Functions

Logarithmic, Exponential, Circular and Hyperbolic Functions

RUTH ZUCKER [1]

Contents

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4. Elementary Transcendental Functions

Logarithmic, Exponential, Circular and Hyperbolic Functions

Mathematical Properties

4.1. Logarithmic Function

Integral Representation

4.1.1
$$\ln z=\int_1^z \frac{dt}{t}$$

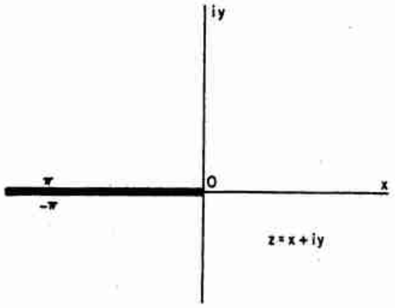

FIGURE 4.1. *Branch cut for* ln *z and* z^a.
(a not an integer or zero.)

where the path of integration does not pass through the origin or cross the negative real axis. ln z is a single-valued function, regular in the z-plane cut along the negative real axis, real when z is positive.

$$z=x+iy=re^{i\theta}.$$

4.1.2
$$\ln z=\ln r+i\theta \quad (-\pi<\theta\leq\pi).$$

4.1.3
$$r=(x^2+y^2)^{\frac{1}{2}}, \quad x=r\cos\theta, \quad y=r\sin\theta,$$
$$\theta=\arctan\frac{y}{x}.$$

The general logarithmic function is the many-valued function Ln z defined by

4.1.4
$$\operatorname{Ln} z=\int_1^z \frac{dt}{t}$$

where the path does not pass through the origin.

4.1.5
$$\operatorname{Ln}(re^{i\theta})=\ln(re^{i\theta})+2k\pi i=\ln r+i(\theta+2k\pi),$$

k being an arbitrary integer. ln z is said to be the *principal branch* of Ln z.

Logarithmic Identities

4.1.6
$$\operatorname{Ln}(z_1z_2)=\operatorname{Ln} z_1+\operatorname{Ln} z_2.$$

(i.e., every value of Ln (z_1z_2) is one of the values of Ln z_1+Ln z_2.)

4.1.7
$$\ln(z_1z_2)=\ln z_1+\ln z_2$$
$$(-\pi<\arg z_1+\arg z_2\leq\pi)$$

4.1.8
$$\operatorname{Ln}\frac{z_1}{z_2}=\operatorname{Ln} z_1-\operatorname{Ln} z_2$$

4.1.9
$$\ln\frac{z_1}{z_2}=\ln z_1-\ln z_2$$
$$(-\pi<\arg z_1-\arg z_2\leq\pi)$$

4.1.10
$$\operatorname{Ln} z^n=n\operatorname{Ln} z \qquad (n \text{ integer})$$

4.1.11
$$\ln z^n=n\ln z$$
$$(n \text{ integer}, \quad -\pi<n\arg z\leq\pi)$$

Special Values (see chapter 1)

4.1.12
$$\ln 1=0$$

4.1.13
$$\ln 0=-\infty$$

4.1.14
$$\ln(-1)=\pi i$$

4.1.15
$$\ln(\pm i)=\pm\tfrac{1}{2}\pi i$$

4.1.16 $\ln e=1$, e is the real number such that
$$\int_1^e \frac{dt}{t}=1$$

4.1.17
$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=2.71828\ 18284\ldots$$
(see **4.2.21**)

Logarithms to General Base

4.1.18
$$\log_a z=\ln z/\ln a$$

4.1.19
$$\log_a z=\frac{\log_b z}{\log_b a}$$

4.1.20
$$\log_a b=\frac{1}{\log_b a}$$

4.1.21
$$\log_e z=\ln z$$

4.1.22
$$\log_{10} z=\ln z/\ln 10=\log_{10} e\ln z$$
$$=(.43429\quad 44819\ldots)\ln z$$

4.1.23 $\ln z = \ln 10 \log_{10} z = (2.30258\ 50929\ldots) \log_{10} z$

($\log_e x = \ln x$, called natural, Napierian, or hyperbolic logarithms; $\log_{10} x$, called common or Briggs logarithms.)

Series Expansions

4.1.24 $$\ln(1+z) = z - \tfrac{1}{2}z^2 + \tfrac{1}{3}z^3 - \ldots \qquad (|z| \leq 1 \text{ and } z \neq -1)$$

4.1.25 $$\ln z = \left(\frac{z-1}{z}\right) + \frac{1}{2}\left(\frac{z-1}{z}\right)^2 + \frac{1}{3}\left(\frac{z-1}{z}\right)^3 + \ldots \qquad (\mathscr{R}z \geq \tfrac{1}{2})$$

4.1.26 $$\ln z = (z-1) - \tfrac{1}{2}(z-1)^2 + \tfrac{1}{3}(z-1)^3 - \ldots \qquad (|z-1| \leq 1, \quad z \neq 0)$$

4.1.27 $$\ln z = 2\left[\left(\frac{z-1}{z+1}\right) + \frac{1}{3}\left(\frac{z-1}{z+1}\right)^3 + \frac{1}{5}\left(\frac{z-1}{z+1}\right)^5 + \ldots\right] \qquad (\mathscr{R}z \geq 0, \quad z \neq 0)$$

4.1.28 $$\ln\left(\frac{z+1}{z-1}\right) = 2\left(\frac{1}{z} + \frac{1}{3z^3} + \frac{1}{5z^5} + \ldots\right) \qquad (|z| \geq 1, \quad z \neq \pm 1)$$

4.1.29 $$\ln(z+a) = \ln a + 2\left[\left(\frac{z}{2a+z}\right) + \frac{1}{3}\left(\frac{z}{2a+z}\right)^3 + \frac{1}{5}\left(\frac{z}{2a+z}\right)^5 + \ldots\right] \qquad (a > 0, \quad \mathscr{R}z \geq -a \neq z)$$

Limiting Values

4.1.30 $$\lim_{x \to \infty} x^{-\alpha} \ln x = 0 \qquad (\alpha \text{ constant}, \quad \mathscr{R}\alpha > 0)$$

4.1.31 $$\lim_{x \to 0} x^{\alpha} \ln x = 0 \qquad (\alpha \text{ constant}, \quad \mathscr{R}\alpha > 0)$$

4.1.32 $$\lim_{m \to \infty}\left(\sum_{k=1}^{m} \frac{1}{k} - \ln m\right) = \gamma \text{ (Euler's constant)} = .57721\ 56649\ldots$$

(see chapters **1**, **6** and **23**)

Inequalities

4.1.33 $$\frac{x}{1+x} < \ln(1+x) < x \qquad (x > -1, \quad x \neq 0)$$

4.1.34 $$x < -\ln(1-x) < \frac{x}{1-x} \qquad (x < 1, \quad x \neq 0)$$

4.1.35 $$|\ln(1-x)| < \frac{3x}{2} \qquad (0 < x \leq .5828)$$

4.1.36 $$\ln x \leq x - 1 \qquad (x > 0)$$

4.1.37 $\ln x \leq n(x^{1/n} - 1)$ for any positive n $\quad (x > 0)$

4.1.38 $$|\ln(1+z)| \leq -\ln(1-|z|) \qquad (|z| < 1)$$

Continued Fractions

4.1.39 $$\ln(1+z) = \frac{z}{1+}\,\frac{z}{2+}\,\frac{z}{3+}\,\frac{4z}{4+}\,\frac{4z}{5+}\,\frac{9z}{6+}\cdots$$

(z in the plane cut from -1 to $-\infty$)

4.1.40 $$\ln\left(\frac{1+z}{1-z}\right) = \frac{2z}{1-}\,\frac{z^2}{3-}\,\frac{4z^2}{5-}\,\frac{9z^2}{7-}\cdots$$

(z in the cut plane of Figure 4.7.)

Polynomial Approximations [2]

4.1.41 $$\frac{1}{\sqrt{10}} \leq x \leq \sqrt{10}$$

$$\log_{10} x = a_1 t + a_3 t^3 + \epsilon(x), \quad t = (x-1)/(x+1)$$

$$|\epsilon(x)| \leq 6 \times 10^{-4}$$

$$a_1 = .86304 \qquad a_3 = .36415$$

4.1.42 $$\frac{1}{\sqrt{10}} \leq x \leq \sqrt{10}$$

$$\log_{10} x = a_1 t + a_3 t^3 + a_5 t^5 + a_7 t^7 + a_9 t^9 + \epsilon(x)$$

$$t = (x-1)/(x+1)$$

$$|\epsilon(x)| \leq 10^{-7}$$

$$a_1 = .86859\ 1718 \qquad a_7 = .09437\ 6476$$
$$a_3 = .28933\ 5524 \qquad a_9 = .19133\ 7714$$
$$a_5 = .17752\ 2071$$

4.1.43 $$0 \leq x \leq 1$$

$$\ln(1+x) = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \epsilon(x)$$

$$|\epsilon(x)| \leq 1 \times 10^{-5}$$

$$a_1 = .99949\ 556 \qquad a_4 = -.13606\ 275$$
$$a_2 = -.49190\ 896 \qquad a_5 = .03215\ 845$$
$$a_3 = .28947\ 478$$

---

[2] The approximations **4.1.41** to **4.1.44** are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

4.1.44 $0 \leq x \leq 1$

$$\ln(1+x)=a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6 +a_7x^7+a_8x^8+\epsilon(x)$$

$$|\epsilon(x)| \leq 3\times 10^{-8}$$

$a_1=$	.99999 64239	$a_5=$	.16765 40711
$a_2=$	−.49987 41238	$a_6=$	−.09532 93897
$a_3=$	.33179 90258	$a_7=$	.03608 84937
$a_4=$	−.24073 38084	$a_8=$	−.00645 35442

Approximation in Terms of Chebyshev Polynomials[3]

4.1.45 $0 \leq x \leq 1$

$T_n^*(x)=\cos n\theta$, $\cos\theta=2x-1$ (see chapter **22**)

$$\ln(1+x)=\sum_{n=0}^{\infty} A_nT_n^*(x)$$

n	A_n	n	A_n
0	.37645 2813	6	−.00000 8503
1	.34314 5750	7	.00000 1250
2	−.02943 7252	8	−.00000 0188
3	.00336 7089	9	.00000 0029
4	−.00043 3276	10	−.00000 0004
5	.00005 9471	11	.00000 0001

Differentiation Formulas

4.1.46 $$\frac{d}{dz}\ln z=\frac{1}{z}$$

4.1.47 $$\frac{d^n}{dz^n}\ln z=(-1)^{n-1}(n-1)!z^{-n}$$

Integration Formulas

4.1.48 $$\int\frac{dz}{z}=\ln z$$

4.1.49 $$\int \ln z\,dz=z\ln z-z$$

4.1.50

$$\int z^n\ln z\,dz=\frac{z^{n+1}}{n+1}\ln z-\frac{z^{n+1}}{(n+1)^2} \qquad (n\neq -1,\quad n \text{ integer})$$

4.1.51

$$\int z^n(\ln z)^m\,dz=\frac{z^{n+1}(\ln z)^m}{n+1}-\frac{m}{n+1}\int z^n(\ln z)^{m-1}dz \qquad (n\neq -1)$$

4.1.52 $$\int\frac{dz}{z\ln z}=\ln\ln z$$

4.1.53

$$\int\ln[z+(z^2\pm1)^{\frac{1}{2}}]dz=z\ln[z+(z^2\pm1)^{\frac{1}{2}}]-(z^2\pm1)^{\frac{1}{2}}$$

4.1.54

$$\int z^n\ln[z+(z^2\pm1)^{\frac{1}{2}}]dz=\frac{z^{n+1}}{n+1}\ln[z+(z^2\pm1)^{\frac{1}{2}}] -\frac{1}{n+1}\int\frac{z^{n+1}}{(z^2\pm1)^{\frac{1}{2}}}dz \quad (n\neq -1)$$

Definite Integrals

4.1.55 $$\int_0^1\frac{\ln t}{1-t}dt=-\pi^2/6$$

4.1.56 $$\int_0^1\frac{\ln t}{1+t}dt=-\pi^2/12$$

4.1.57 $$\int_0^x\frac{dt}{\ln t}=li(x) \quad (\text{see } \mathbf{5.1.3})$$

4.2. Exponential Function

Series Expansion

4.2.1

$$e^z=\exp z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots \quad (z=x+iy)$$

where e is the real number defined in **4.1.16**

Fundamental Properties

4.2.2 $\mathrm{Ln}(\exp z)=z+2k\pi i$ (k any integer)

4.2.3 $\ln(\exp z)=z \quad (-\pi<\mathscr{I}z\leq\pi)$

4.2.4 $\exp(\ln z)=\exp(\mathrm{Ln}\, z)=z$

4.2.5 $$\frac{d}{dz}\exp z=\exp z$$

Definition of General Powers

4.2.6 If $N=a^z$, then $z=\mathrm{Log}_a N$

4.2.7 $a^z=\exp(z\ln a)$

4.2.8 If $a=|a|\exp(i\arg a)\quad(-\pi<\arg a\leq\pi)$

4.2.9 $|a^z|=|a|^xe^{-y\arg a}$

4.2.10 $\arg(a^z)=y\ln|a|+x\arg a$

4.2.11

$\mathrm{Ln}\, a^z=z\ln a$ for one of the values of $\mathrm{Ln}\, a^z$

4.2.12 $\ln a^x=x\ln a$ (a real and positive)

4.2.13 $|e^z|=e^x$

---

[3] The approximation **4.1.45** is from C. W. Clenshaw, Polynomial approximations to elementary functions, Math. Tables Aids Comp. **8**, 143–147 (1954) (with permission).

4.2.14 $$\arg(e^z)=y$$

4.2.15 $$a^{z_1}a^{z_2}=a^{z_1+z_2}$$

4.2.16 $$a^zb^z=(ab)^z \qquad (-\pi<\arg a+\arg b\leq\pi)$$

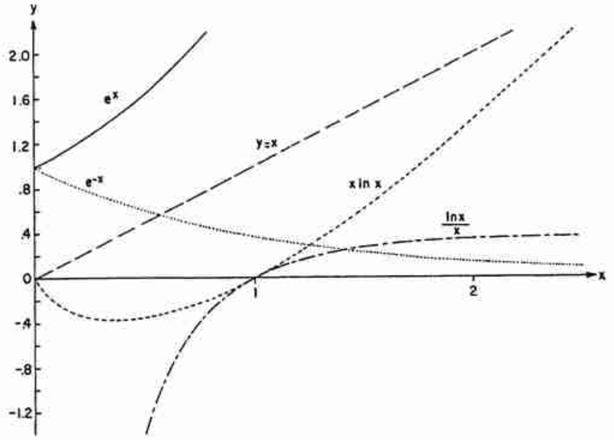

FIGURE 4.2. *Logarithmic and exponential functions.*

Periodic Property

4.2.17 $$e^{z+2\pi ki}=e^z \qquad (k \text{ any integer})$$

Exponential Identities

4.2.18 $$e^{z_1}e^{z_2}=e^{z_1+z_2}$$

4.2.19 $$(e^{z_1})^{z_2}=e^{z_1z_2} \qquad (-\pi<\mathscr{I}z_1\leq\pi)$$

The restriction $(-\pi<\mathscr{I}z_1\leq\pi)$ can be removed if z_2 is an integer.

Limiting Values

4.2.20
$$\lim_{|z|\to\infty} z^\alpha e^{-z}=0 \quad (|\arg z|\leq\tfrac{1}{2}\pi-\epsilon<\tfrac{1}{2}\pi, \quad \alpha \text{ constant})$$

4.2.21 $$\lim_{m\to\infty}\left(1+\frac{z}{m}\right)^m=e^z$$

Special Values (see chapter 1)

4.2.22 $$e=2.71828\ 18284\ldots$$

4.2.23 $$e^0=1$$

4.2.24 $$e^\infty=\infty$$

4.2.25 $$e^{-\infty}=0$$

4.2.26 $$e^{\pm\pi i}=-1$$

4.2.27 $$e^{\pm\frac{\pi i}{2}}=\pm i$$

4.2.28 $$e^{2\pi ki}=1 \quad (k \text{ any integer})$$

Exponential Inequalities

If x is real and different from zero

4.2.29 $$e^{-\frac{x}{1-x}}<1-x<e^{-x} \quad (x<1)$$

4.2.30 $$e^x>1+x$$

4.2.31 $$e^x<\frac{1}{1-x} \quad (x<1)$$

4.2.32 $$\frac{x}{1+x}<(1-e^{-x})<x \quad (x>-1)$$

4.2.33 $$x<(e^x-1)<\frac{x}{1-x} \quad (x<1)$$

4.2.34 $$1+x>e^{\frac{x}{1+x}} \quad (x>-1)$$

4.2.35 $$e^x>1+\frac{x^n}{n!} \quad (n>0, \quad x>0)$$

4.2.36 $$e^x>\left(1+\frac{x}{y}\right)^y>e^{\frac{xy}{x+y}} \quad (x>0, \quad y>0)$$

4.2.37 $$e^{-x}<1-\frac{x}{2} \quad (0<x\leq 1.5936)$$

4.2.38 $$\frac{1}{4}|z|<|e^z-1|<\frac{7}{4}|z| \quad (0<|z|<1)$$

4.2.39 $$|e^z-1|\leq e^{|z|}-1\leq|z|e^{|z|} \quad (\text{all } z)$$

Continued Fractions

4.2.40
$$e^z=\frac{1}{1-}\,\frac{z}{1+}\,\frac{z}{2-}\,\frac{z}{3+}\,\frac{z}{2-}\,\frac{z}{5+}\,\frac{z}{2-}\cdots \qquad (|z|<\infty)$$

$$=1+\frac{z}{1-}\,\frac{z}{2+}\,\frac{z}{3-}\,\frac{z}{2+}\,\frac{z}{5-}\,\frac{z}{2+}\,\frac{z}{7-}\cdots \qquad (|z|<\infty)$$

$$=1+\frac{z}{(1-z/2)+}\,\frac{z^2/4\cdot 3}{1+}\,\frac{z^2/4\cdot 15}{1+}\,\frac{z^2/4\cdot 35}{1+}\cdots\frac{z^2/4(4n^2-1)}{1+}\cdots (|z|<\infty)$$

4.2.41
$$e^z-e_{n-1}(z)=\frac{z^n}{n!-}\,\frac{n!z}{(n+1)+}\,\frac{z}{(n+2)-}\,\frac{(n+1)z}{(n+3)+}\,\frac{2z}{(n+4)-}\,\frac{(n+2)z}{(n+5)+}\,\frac{3z}{(n+6)-}\cdots (|z|<\infty)$$

(For $e_n(z)$ see **6.5.11**)

4.2.42

$$e^{2a \arctan \frac{1}{z}}=1+\frac{2a}{z-a+}\,\frac{a^2+1}{3z+}\,\frac{a^2+4}{5z+}\,\frac{a^2+9}{7z+}\cdots$$

(z in the cut plane of Figure 4.4.)

Polynomial Approximations[4]

4.2.43 $0\leq x\leq \ln 2=.693\ldots$

$$e^{-x}=1+a_1x+a_2x^2+\epsilon(x)$$

$$|\epsilon(x)|\leq 3\times 10^{-3}$$

$a_1=-.9664 \qquad a_2=.3536$

4.2.44 $0\leq x\leq \ln 2$

$$e^{-x}=1+a_1x+a_2x^2+a_3x^3+a_4x^4+\epsilon(x)$$

$$|\epsilon(x)|\leq 3\times 10^{-5}$$

$a_1=-.99986\,84 \qquad a_3=-.15953\,32$
$a_2=.49829\,26 \qquad a_4=.02936\,41$

4.2.45 $0\leq x\leq \ln 2$

$$e^{-x}=1+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7+\epsilon(x)$$

$$|\epsilon(x)|\leq 2\times 10^{-10}$$

$a_1=-.99999\,99995 \qquad a_5=-.00830\,13598$
$a_2=.49999\,99206 \qquad a_6=.00132\,98820$
$a_3=-.16666\,53019 \qquad a_7=-.00014\,13161$
$a_4=.04165\,73475$

4.2.46[5] $0\leq x\leq 1$

$$10^x=(1+a_1x+a_2x^2+a_3x^3+a_4x^4)^2+\epsilon(x)$$

$$|\epsilon(x)|\leq 7\times 10^{-4}$$

$a_1=1.14991\,96 \qquad a_3=.20800\,30$
$a_2=.67743\,23 \qquad a_4=.12680\,89$

4.2.47 $0\leq x\leq 1$

$$10^x=(1+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7)^2+\epsilon(x)$$

$$|\epsilon(x)|<5\times 10^{-8}$$

$a_1=1.\,15129\,277603 \qquad a_5=.\,01742\,111988$
$a_2=.\,66273\,088429 \qquad a_6=.\,00255\,491796$
$a_3=.\,25439\,357484 \qquad a_7=.\,00093\,264267$
$a_4=.\,07295\,173666$

Approximations in Terms of Chebyshev Polynomials[6]

4.2.48 $0\leq x\leq 1$

$$T_n^*(x)=\cos n\theta, \quad \cos\theta=2x-1 \text{ (see chapter } \mathbf{22}\text{)}$$

$$e^x=\sum_{n=0}^{\infty} A_nT_n^*(x) \qquad\qquad e^{-x}=\sum_{n=0}^{\infty} A_nT_n^*(x)$$

n	A_n	n	A_n
0	1. 75338 7654	0	. 64503 5270
1	. 85039 1654	1	−. 31284 1606
2	. 10520 8694	2	. 03870 4116
3	. 00872 2105	3	−. 00320 8683
4	. 00054 3437	4	. 00019 9919
5	. 00002 7115	5	−. 00000 9975
6	. 00000 1128	6	. 00000 0415
7	. 00000 0040	7	−. 00000 0015
8	. 00000 0001		

Differentiation Formulas

4.2.49 $$\frac{d}{dz}\,e^z=e^z$$

4.2.50 $$\frac{d^n}{dz^n}\,e^{az}=a^ne^{az}$$

4.2.51 $$\frac{d}{dz}\,a^z=a^z\ln a$$

4.2.52 $$\frac{d}{dz}\,z^a=az^{a-1}$$

4.2.53 $$\frac{d}{dz}\,z^z=(1+\ln z)z^z$$

Integration Formulas

4.2.54 $$\int e^{az}dz=e^{az}/a$$

4.2.55

$$\int z^ne^{az}dz=\frac{e^{az}}{a^{n+1}}\,[(az)^n-n(az)^{n-1}+n(n-1)(az)^{n-2}+\ldots+(-1)^{n-1}n!(az)+(-1)^nn!] \quad (n\geq 0)$$

4.2.56

$$\int\frac{e^{az}}{z^n}\,dz=-\frac{e^{az}}{(n-1)z^{n-1}}+\frac{a}{n-1}\int\frac{e^{az}}{z^{n-1}}\,dz \quad (n>1)$$

(See chapters **5**, **7** and **29** for other integrals involving exponential functions.)

4.3. Circular Functions

Definitions

4.3.1 $$\sin z=\frac{e^{iz}-e^{-iz}}{2i} \qquad (z=x+iy)$$

4.3.2 $$\cos z=\frac{e^{iz}+e^{-iz}}{2}$$

---

[4] The approximations **4.2.43** to **4.2.45** are from B. Carlson, M. Goldstein, Rational approximation of functions, Los Alamos Scientific Laboratory LA–1943, Los Alamos, N. Mex., 1955 (with permission).

[5] The approximations **4.2.46** to **4.2.47** are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

[6] The approximations **4.2.48** are from C. W. Clenshaw, Polynomial approximations to elementary functions, Math. Tables Aids Comp. **8**, 143–147 (1954) (with permission).

4.3.3 $$\tan z = \frac{\sin z}{\cos z}$$

4.3.4 $$\csc z = \frac{1}{\sin z}$$

4.3.5 $$\sec z = \frac{1}{\cos z}$$

4.3.6 $$\cot z = \frac{1}{\tan z}$$

Periodic Properties

4.3.7 $\sin(z+2k\pi) = \sin z$ (k any integer)

4.3.8 $\cos(z+2k\pi) = \cos z$

4.3.9 $\tan(z+k\pi) = \tan z$

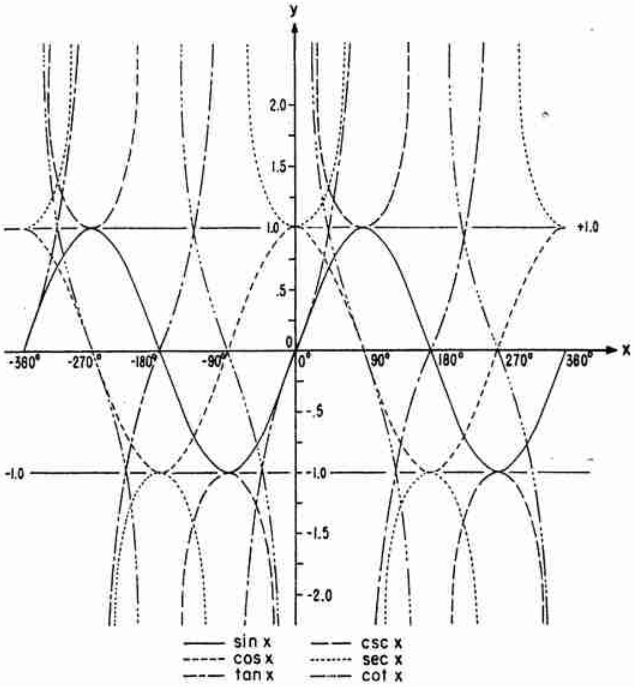

FIGURE 4.3. *Circular functions.*

Relations Between Circular Functions

4.3.10 $\sin^2 z + \cos^2 z = 1$

4.3.11 $\sec^2 z - \tan^2 z = 1$

4.3.12 $\csc^2 z - \cot^2 z = 1$

Negative Angle Formulas

4.3.13 $\sin(-z) = -\sin z$

4.3.14 $\cos(-z) = \cos z$

4.3.15 $\tan(-z) = -\tan z$

Addition Formulas

4.3.16 $\sin(z_1+z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

4.3.17 $\cos(z_1+z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

4.3.18 $$\tan(z_1+z_2) = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$$

4.3.19 $$\cot(z_1+z_2) = \frac{\cot z_1 \cot z_2 - 1}{\cot z_2 + \cot z_1}$$

Half-Angle Formulas

4.3.20 $$\sin\frac{z}{2} = \pm\left(\frac{1-\cos z}{2}\right)^{\frac{1}{2}}$$

4.3.21 $$\cos\frac{z}{2} = \pm\left(\frac{1+\cos z}{2}\right)^{\frac{1}{2}}$$

4.3.22 $$\tan\frac{z}{2} = \pm\left(\frac{1-\cos z}{1+\cos z}\right)^{\frac{1}{2}} = \frac{1-\cos z}{\sin z} = \frac{\sin z}{1+\cos z}$$

The ambiguity in sign may be resolved with the aid of a diagram.

Transformation of Trigonometric Integrals

If $\tan\frac{u}{2} = z$ then

4.3.23 $$\sin u = \frac{2z}{1+z^2}, \quad \cos u = \frac{1-z^2}{1+z^2}, \quad du = \frac{2}{1+z^2}\,dz$$

Multiple-Angle Formulas

4.3.24 $$\sin 2z = 2\sin z \cos z = \frac{2\tan z}{1+\tan^2 z}$$

4.3.25 $$\cos 2z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$$
$$= \cos^2 z - \sin^2 z = \frac{1-\tan^2 z}{1+\tan^2 z}$$

4.3.26 $$\tan 2z = \frac{2\tan z}{1-\tan^2 z} = \frac{2\cot z}{\cot^2 z - 1} = \frac{2}{\cot z - \tan z}$$

4.3.27 $\sin 3z = 3\sin z - 4\sin^3 z$

4.3.28 $\cos 3z = -3\cos z + 4\cos^3 z$

4.3.29 $\sin 4z = 8\cos^3 z \sin z - 4\cos z \sin z$

4.3.30 $\cos 4z = 8\cos^4 z - 8\cos^2 z + 1$

Products of Sines and Cosines

4.3.31 $2\sin z_1 \sin z_2 = \cos(z_1-z_2) - \cos(z_1+z_2)$

4.3.32 $2\cos z_1 \cos z_2 = \cos(z_1-z_2) + \cos(z_1+z_2)$

4.3.33 $2\sin z_1 \cos z_2 = \sin(z_1-z_2) + \sin(z_1+z_2)$

Addition and Subtraction of Two Circular Functions

4.3.34
$$\sin z_1 + \sin z_2 = 2\sin\left(\frac{z_1+z_2}{2}\right)\cos\left(\frac{z_1-z_2}{2}\right)$$

4.3.35

$$\sin z_1-\sin z_2=2\cos\left(\frac{z_1+z_2}{2}\right)\sin\left(\frac{z_1-z_2}{2}\right)$$

4.3.36

$$\cos z_1+\cos z_2=2\cos\left(\frac{z_1+z_2}{2}\right)\cos\left(\frac{z_1-z_2}{2}\right)$$

4.3.37

$$\cos z_1-\cos z_2=-2\sin\left(\frac{z_1+z_2}{2}\right)\sin\left(\frac{z_1-z_2}{2}\right)$$

4.3.38

$$\tan z_1\pm\tan z_2=\frac{\sin(z_1\pm z_2)}{\cos z_1\cos z_2}$$

4.3.39

$$\cot z_1\pm\cot z_2=\frac{\sin(z_2\pm z_1)}{\sin z_1\sin z_2}$$

Relations Between Squares of Sines and Cosines

4.3.40

$$\sin^2 z_1-\sin^2 z_2=\sin(z_1+z_2)\sin(z_1-z_2)$$

4.3.41

$$\cos^2 z_1-\cos^2 z_2=-\sin(z_1+z_2)\sin(z_1-z_2)$$

4.3.42

$$\cos^2 z_1-\sin^2 z_2=\cos(z_1+z_2)\cos(z_1-z_2)$$

4.3.43

Signs of the Circular Functions in the Four Quadrants

Quadrant	sin csc	cos sec	tan cot
I	+	+	+
II	+	−	−
III	−	−	+
IV	−	+	−

4.3.44

Functions of Angles in Any Quadrant in Terms of Angles in the First Quadrant. $(0\leq\theta\leq\frac{\pi}{2},\quad k$ any integer)

	$-\theta$	$\frac{\pi}{2}\pm\theta$	$\pi\pm\theta$	$\frac{3\pi}{2}\pm\theta$	$2k\pi\pm\theta$
sin	$-\sin\theta$	$\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$
cos	$\cos\theta$	$\mp\sin\theta$	$-\cos\theta$	$\pm\sin\theta$	$+\cos\theta$
tan	$-\tan\theta$	$\mp\cot\theta$	$\pm\tan\theta$	$\mp\cot\theta$	$\pm\tan\theta$
csc	$-\csc\theta$	$+\sec\theta$	$\mp\csc\theta$	$-\sec\theta$	$\pm\csc\theta$
sec	$\sec\theta$	$\mp\csc\theta$	$-\sec\theta$	$\pm\csc\theta$	$+\sec\theta$
cot	$-\cot\theta$	$\mp\tan\theta$	$\pm\cot\theta$	$\mp\tan\theta$	$\pm\cot\theta$

4.3.45 Relations Between Circular (or Inverse Circular) Functions

	$\sin x=a$	$\cos x=a$	$\tan x=a$	$\csc x=a$	$\sec x=a$	$\cot x=a$
$\sin x$	a	$(1-a^2)^{\frac12}$	$a(1+a^2)^{-\frac12}$	a^{-1}	$a^{-1}(a^2-1)^{\frac12}$	$(1+a^2)^{-\frac12}$
$\cos x$	$(1-a^2)^{\frac12}$	a	$(1+a^2)^{-\frac12}$	$a^{-1}(a^2-1)^{\frac12}$	a^{-1}	$a(1+a^2)^{-\frac12}$
$\tan x$	$a(1-a^2)^{-\frac12}$	$a^{-1}(1-a^2)^{\frac12}$	a	$(a^2-1)^{-\frac12}$	$(a^2-1)^{\frac12}$	a^{-1}
$\csc x$	a^{-1}	$(1-a^2)^{-\frac12}$	$a^{-1}(1+a^2)^{\frac12}$	a	$a(a^2-1)^{-\frac12}$	$(1+a^2)^{\frac12}$
$\sec x$	$(1-a^2)^{-\frac12}$	a^{-1}	$(1+a^2)^{\frac12}$	$a(a^2-1)^{-\frac12}$	a	$a^{-1}(1+a^2)^{\frac12}$
$\cot x$	$a^{-1}(1-a^2)^{\frac12}$	$a(1-a^2)^{-\frac12}$	a^{-1}	$(a^2-1)^{\frac12}$	$(a^2-1)^{-\frac12}$	a

$\left(0\leq x\leq\frac{\pi}{2}\right)$ Illustration: If $\sin x=a$, $\cot x=a^{-1}(1-a^2)^{\frac12}$

$\operatorname{arcsec} a=\operatorname{arccot}(a^2-1)^{-\frac12}$

4.3.46 Circular Functions for Certain Angles

	0 $0°$	$\pi/12$ $15°$	$\pi/6$ $30°$	$\pi/4$ $45°$	$\pi/3$ $60°$
sin	0	$\frac{\sqrt{2}}{4}(\sqrt{3}-1)$	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$
cos	1	$\frac{\sqrt{2}}{4}(\sqrt{3}+1)$	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$
tan	0	$2-\sqrt{3}$	$\sqrt{3}/3$	1	$\sqrt{3}$
csc	∞	$\sqrt{2}(\sqrt{3}+1)$	2	$\sqrt{2}$	$2\sqrt{3}/3$
sec	1	$\sqrt{2}(\sqrt{3}-1)$	$2\sqrt{3}/3$	$\sqrt{2}$	2
cot	∞	$2+\sqrt{3}$	$\sqrt{3}$	1	$\sqrt{3}/3$

	$5\pi/12$ $75°$	$\pi/2$ $90°$	$7\pi/12$ $105°$	$2\pi/3$ $120°$
sin	$\frac{\sqrt{2}}{4}(\sqrt{3}+1)$	1	$\frac{\sqrt{2}}{4}(\sqrt{3}+1)$	$\sqrt{3}/2$
cos	$\frac{\sqrt{2}}{4}(\sqrt{3}-1)$	0	$\frac{-\sqrt{2}}{4}(\sqrt{3}-1)$	$-1/2$
tan	$2+\sqrt{3}$	∞	$-(2+\sqrt{3})$	$-\sqrt{3}$
csc	$\sqrt{2}(\sqrt{3}-1)$	1	$\sqrt{2}(\sqrt{3}-1)$	$2\sqrt{3}/3$
sec	$\sqrt{2}(\sqrt{3}+1)$	∞	$-\sqrt{2}(\sqrt{3}+1)$	-2
cot	$2-\sqrt{3}$	0	$-(2-\sqrt{3})$	$-\sqrt{3}/3$

	$3\pi/4$ $135°$	$5\pi/6$ $150°$	$11\pi/12$ $165°$	π $180°$
sin	$\sqrt{2}/2$	$1/2$	$\frac{\sqrt{2}}{4}(\sqrt{3}-1)$	0
cos	$-\sqrt{2}/2$	$-\sqrt{3}/2$	$\frac{-\sqrt{2}}{4}(\sqrt{3}+1)$	-1
tan	-1	$-\sqrt{3}/3$	$-(2-\sqrt{3})$	0
csc	$\sqrt{2}$	2	$\sqrt{2}(\sqrt{3}+1)$	∞
sec	$-\sqrt{2}$	$-2\sqrt{3}/3$	$-\sqrt{2}(\sqrt{3}-1)$	-1
cot	-1	$-\sqrt{3}$	$-(2+\sqrt{3})$	∞

Euler's Formula

4.3.47 $e^z=e^{x+iy}=e^x\,(\cos y+i\sin y)$

De Moivre's Theorem

4.3.48 $(\cos z+i\sin z)^\nu=\cos \nu z+i\sin \nu z$

$(-\pi<\mathscr{R}z\leq\pi$ unless ν is an integer$)$

Relation to Hyperbolic Functions (see 4.5.7 to 4.5.12)

4.3.49 $\sin z=-i\sinh iz$

4.3.50 $\cos z=\cosh iz$

4.3.51 $\tan z=-i\tanh iz$

4.3.52 $\csc z=i\,\mathrm{csch}\, iz$

4.3.53 $\sec z=\mathrm{sech}\, iz$

4.3.54 $\cot z=i\coth iz$

Circular Functions in Terms of Real and Imaginary Parts

4.3.55 $\sin z=\sin x\cosh y+i\cos x\sinh y$

4.3.56 $\cos z=\cos x\cosh y-i\sin x\sinh y$

4.3.57 $\tan z=\dfrac{\sin 2x+i\sinh 2y}{\cos 2x+\cosh 2y}$

4.3.58 $\cot z=\dfrac{\sin 2x-i\sinh 2y}{\cosh 2y-\cos 2x}$

Modulus and Phase (Argument) of Circular Functions

4.3.59 $|\sin z|=(\sin^2 x+\sinh^2 y)^{\frac{1}{2}}$

$=[\frac{1}{2}(\cosh 2y-\cos 2x)]^{\frac{1}{2}}$

4.3.60 $\arg\sin z=\arctan(\cot x\tanh y)$

4.3.61 $|\cos z|=(\cos^2 x+\sinh^2 y)^{\frac{1}{2}}$

$=[\frac{1}{2}(\cosh 2y+\cos 2x)]^{\frac{1}{2}}$

4.3.62 $\arg\cos z=-\arctan(\tan x\tanh y)$

4.3.63 $|\tan z|=\left(\dfrac{\cosh 2y-\cos 2x}{\cosh 2y+\cos 2x}\right)^{\frac{1}{2}}$

4.3.64 $\arg\tan z=\arctan\left(\dfrac{\sinh 2y}{\sin 2x}\right)$

Series Expansions

4.3.65

$$\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\ldots \qquad (|z|<\infty)$$

4.3.66

$$\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\ldots \qquad (|z|<\infty)$$

4.3.67

$$\tan z = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \dots + \frac{(-1)^{n-1}2^{2n}(2^{2n}-1)B_{2n}}{(2n)!} z^{2n-1} + \dots \qquad \left(|z| < \frac{\pi}{2}\right)$$

4.3.68

$$\csc z = \frac{1}{z} + \frac{z}{6} + \frac{7}{360} z^3 + \frac{31}{15120} z^5 + \dots + \frac{(-1)^{n-1}2(2^{2n-1}-1)B_{2n}}{(2n)!} z^{2n-1} + \dots \qquad (|z| < \pi)$$

4.3.69

$$\sec z = 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \frac{61z^6}{720} + \dots + \frac{(-1)^n E_{2n}}{(2n)!} z^{2n} + \dots \qquad \left(|z| < \frac{\pi}{2}\right)$$

4.3.70

$$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \dots - \frac{(-1)^{n-1}2^{2n}B_{2n}}{(2n)!} z^{2n-1} - \dots \qquad (|z| < \pi)$$

4.3.71

$$\ln \frac{\sin z}{z} = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} B_{2n}}{n(2n)!} z^{2n} \qquad (|z| < \pi)$$

4.3.72

$$\ln \cos z = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1}(2^{2n}-1) B_{2n}}{n(2n)!} z^{2n} \qquad (|z| < \tfrac{1}{2}\pi)$$

4.3.73

$$\ln \frac{\tan z}{z} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 2^{2n}(2^{2n-1}-1) B_{2n}}{n(2n)!} z^{2n} \qquad (|z| < \tfrac{1}{2}\pi)$$

where B_n and E_n are the Bernoulli and Euler numbers (see chapter **23**).

Limiting Values

4.3.74 $$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

4.3.75 $$\lim_{x \to 0} \frac{\tan x}{x} = 1$$

4.3.76 $$\lim_{n \to \infty} n \sin \frac{x}{n} = x$$

4.3.77 $$\lim_{n \to \infty} n \tan \frac{x}{n} = x$$

4.3.78 $$\lim_{n \to \infty} \cos \frac{x}{n} = 1$$

Inequalities

4.3.79 $$\frac{\sin x}{x} > \frac{2}{\pi} \qquad \left(-\frac{\pi}{2} < x < \frac{\pi}{2}\right)$$

4.3.80 $$\sin x \le x \le \tan x \qquad \left(0 \le x \le \frac{\pi}{2}\right)$$

4.3.81 $$\cos x \le \frac{\sin x}{x} \le 1 \qquad (0 \le x \le \pi)$$

4.3.82 $$\pi < \frac{\sin \pi x}{x(1-x)} \le 4 \qquad (0 < x < 1)$$

4.3.83 $$|\sinh y| \le |\sin z| \le \cosh y$$

4.3.84 $$|\sinh y| \le |\cos z| \le \cosh y$$

4.3.85 $$|\csc z| \le \operatorname{csch}|y|$$

4.3.86 $$|\cos z| \le \cosh|z|$$

4.3.87 $$|\sin z| \le \sinh|z|$$

4.3.88 $$|\cos z| < 2, \quad |\sin z| \le \frac{6}{5}|z| \qquad (|z| < 1)$$

Infinite Products

4.3.89 $$\sin z = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2\pi^2}\right)$$

4.3.90 $$\cos z = \prod_{k=1}^{\infty} \left(1 - \frac{4z^2}{(2k-1)^2\pi^2}\right)$$

Expansion in Partial Fractions

4.3.91 $$\cot z = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{1}{z^2 - k^2\pi^2} \qquad (z \ne 0, \pm\pi, \pm 2\pi, \dots)$$

4.3.92 $$\csc^2 z = \sum_{k=-\infty}^{\infty} \frac{1}{(z-k\pi)^2} \qquad (z \ne 0, \pm\pi, \pm 2\pi, \dots)$$

4.3.93 $$\csc z = \frac{1}{z} + 2z \sum_{k=1}^{\infty} \frac{(-1)^k}{z^2 - k^2\pi^2} \qquad (z \ne 0, \pm\pi, \pm 2\pi, \dots)$$

Continued Fractions

4.3.94 $$\tan z = \frac{z}{1-} \frac{z^2}{3-} \frac{z^2}{5-} \frac{z^2}{7-} \dots \qquad \left(z \ne \frac{\pi}{2} \pm n\pi\right)$$

4.3.95

$$\tan az = \frac{a \tan z}{1+} \frac{(1-a^2)\tan^2 z}{3+} \frac{(4-a^2)\tan^2 z}{5+} \frac{(9-a^2)\tan^2 z}{7+} \dots \left(-\frac{\pi}{2} < \mathscr{R} z < \frac{\pi}{2}, \quad az \ne \frac{\pi}{2} \pm n\pi\right)$$

Polynomial Approximations [7]

4.3.96 $\qquad 0\leq x\leq\frac{\pi}{2}$

$$\frac{\sin x}{x}=1+a_2x^2+a_4x^4+\epsilon(x)$$

$$|\epsilon(x)|\leq 2\times 10^{-4}$$

$a_2=-.16605 \qquad a_4=.00761$

4.3.97 $\qquad 0\leq x\leq\frac{\pi}{2}$

$$\frac{\sin x}{x}=1+a_2x^2+a_4x^4+a_6x^6+a_8x^8+a_{10}x^{10}+\epsilon(x)$$

$$|\epsilon(x)|\leq 2\times 10^{-9}$$

$a_2=-.16666\ 66664 \qquad a_8=\ \ .00000\ 27526$

$a_4=\ \ .00833\ 33315 \qquad a_{10}=-.00000\ 00239$

$a_6=-.00019\ 84090$

4.3.98 $\qquad 0\leq x\leq\frac{\pi}{2}$

$$\cos x=1+a_2x^2+a_4x^4+\epsilon(x)$$

$$|\epsilon(x)|\leq 9\times 10^{-4}$$

$a_2=-.49670 \qquad a_4=.03705$

4.3.99 $\qquad 0\leq x\leq\frac{\pi}{2}$

$$\cos x=1+a_2x^2+a_4x^4+a_6x^6+a_8x^8+a_{10}x^{10}+\epsilon(x)$$

$$|\epsilon(x)|\leq 2\times 10^{-9}$$

$a_2=-.49999\ 99963 \qquad a_8=\ \ .00002\ 47609$

$a_4=\ \ .04166\ 66418 \qquad a_{10}=-.00000\ 02605$

$a_6=-.00138\ 88397$

4.3.100 $\qquad 0\leq x\leq\frac{\pi}{4}$

$$\frac{\tan x}{x}=1+a_2x^2+a_4x^4+\epsilon(x)$$

$$|\epsilon(x)|\leq 1\times 10^{-3}$$

$a_2=.31755 \qquad a_4=.20330$

4.3.101 $\qquad 0\leq x\leq\frac{\pi}{4}$

$$\frac{\tan x}{x}=1+a_2x^2+a_4x^4+a_6x^6+a_8x^8+a_{10}x^{10}+a_{12}x^{12}+\epsilon(x)$$

$$|\epsilon(x)|\leq 2\times 10^{-8}$$

$a_2=.33333\ 14036 \qquad a_8=.02456\ 50893$

$a_4=.13339\ 23995 \qquad a_{10}=.00290\ 05250$

$a_6=.05337\ 40603 \qquad a_{12}=.00951\ 68091$

4.3.102 $\qquad 0\leq x\leq\frac{\pi}{4}$

$$*\qquad x\cot x=1+a_2x^2+a_4x^4+\epsilon(x)$$

$$|\epsilon(x)|\leq 3\times 10^{-5}$$

$a_2=-.332867 \qquad a_4=-.024369$

4.3.103 $\qquad 0\leq x\leq\frac{\pi}{4}$

$$x\cot x=1+a_2x^2+a_4x^4+a_6x^6+a_8x^8+a_{10}x^{10}+\epsilon(x)$$

$$|\epsilon(x)|\leq 4\times 10^{-10}$$

$a_2=-.33333\ 33410 \qquad a_8=-.00020\ 78504$

$a_4=-.02222\ 20287 \qquad a_{10}=-.00002\ 62619$

$a_6=-.00211\ 77168$

Approximations in Terms of Chebyshev Polynomials [8]

4.3.104 $\qquad -1\leq x\leq 1$

$$T_n^*(x)=\cos n\theta,\ \cos\theta=2x-1 \quad \text{(see chapter 22)}$$

$$\sin\tfrac{1}{2}\pi x=x\sum_{n=0}^{\infty}A_nT_n^*(x^2) \qquad \cos\tfrac{1}{2}\pi x=\sum_{n=0}^{\infty}A_nT_n^*(x^2)$$

n	A_n	n	A_n
0	1.27627 8962	0	.47200 1216
1	−.28526 1569	1	−.49940 3258
2	.00911 8016	2	.02799 2080
3	−.00013 6587	3	−.00059 6695
4	.00000 1185	4	.00000 6704
5	−.00000 0007	5	−.00000 0047

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[7] The approximations **4.3.96** to **4.3.103** are from B. Carlson, M. Goldstein, Rational approximation of functions, Los Alamos Scientific Laboratory LA–1943, Los Alamos, N. Mex., 1955 (with permission).

[8] The approximations **4.3.104** are from C. W. Clenshaw, Polynomial approximations to elementary functions, Math. Tables Aids Comp. **8**, 143–147 (1954) (with permission).

*See page II.

Differentiation Formulas

4.3.105 $$\frac{d}{dz}\sin z=\cos z$$

4.3.106 $$\frac{d}{dz}\cos z=-\sin z$$

4.3.107 $$\frac{d}{dz}\tan z=\sec^2 z$$

4.3.108 $$\frac{d}{dz}\csc z=-\csc z\cot z$$

4.3.109 $$\frac{d}{dz}\sec z=\sec z\tan z$$

4.3.110 $$\frac{d}{dz}\cot z=-\csc^2 z$$

4.3.111 $$\frac{d^n}{dz^n}\sin z=\sin\left(z+\frac{1}{2}n\pi\right)$$

4.3.112 $$\frac{d^n}{dz^n}\cos z=\cos\left(z+\frac{1}{2}n\pi\right)$$

Integration Formulas

4.3.113 $$\int\sin z\,dz=-\cos z$$

4.3.114 $$\int\cos z\,dz=\sin z$$

4.3.115 $$\int\tan z\,dz=-\ln\cos z=\ln\sec z$$

4.3.116

$$\int\csc z\,dz=\ln\tan\frac{z}{2}=\ln(\csc z-\cot z)=\frac{1}{2}\ln\frac{1-\cos z}{1+\cos z}$$

4.3.117

$$\int\sec z\,dz=\ln(\sec z+\tan z)=\ln\tan\left(\frac{\pi}{4}+\frac{z}{2}\right)=\mathrm{gd}^{-1}(z)$$

=Inverse Gudermannian Function

$$\mathrm{gd}\,z=2\arctan e^z-\frac{\pi}{2}$$

4.3.118 $$\int\cot z\,dz=\ln\sin z=-\ln\csc z$$

4.3.119

$$\int z^n\sin z\,dz=-z^n\cos z+n\int z^{n-1}\cos z\,dz$$

4.3.120

$$\int\frac{\sin z}{z^n}dz=\frac{-\sin z}{(n-1)z^{n-1}}+\frac{1}{n-1}\int\frac{\cos z}{z^{n-1}}dz\qquad(n>1)$$

4.3.121 $$\int\frac{z}{\sin^2 z}dz=-z\cot z+\ln\sin z$$

4.3.122

$$\int\frac{z\,dz}{\sin^n z}=\frac{-z\cos z}{(n-1)\sin^{n-1}z}-\frac{1}{(n-1)(n-2)\sin^{n-2}z}+\frac{(n-2)}{(n-1)}\int\frac{z\,dz}{\sin^{n-2}z}\ (n>2)$$

4.3.123

$$\int z^n\cos z\,dz=z^n\sin z-n\int z^{n-1}\sin z\,dz$$

4.3.124

$$\int\frac{\cos z}{z^n}dz=-\frac{\cos z}{(n-1)z^{n-1}}-\frac{1}{n-1}\int\frac{\sin z}{z^{n-1}}dz\quad(n>1)$$

4.3.125 $$\int\frac{z}{\cos^2 z}dz=z\tan z+\ln\cos z$$

4.3.126

$$\int\frac{z\,dz}{\cos^n z}=\frac{z\sin z}{(n-1)\cos^{n-1}z}-\frac{1}{(n-1)(n-2)\cos^{n-2}z}+\frac{(n-2)}{(n-1)}\int\frac{z\,dz}{\cos^{n-2}z}\ (n>2)$$

4.3.127

$$\begin{aligned}\int\sin^m z\cos^n z\,dz&=\frac{\sin^{m+1}z\cos^{n-1}z}{m+n}+\frac{(n-1)}{(m+n)}\int\sin^m z\cos^{n-2}z\,dz\\&=-\frac{\sin^{m-1}z\cos^{n+1}z}{m+n}+\frac{(m-1)}{(m+n)}\int\sin^{m-2}z\cos^n z\,dz\\&\qquad(m\neq-n)\end{aligned}$$

4.3.128

$$\begin{aligned}\int\frac{dz}{\sin^m z\cos^n z}&=\frac{1}{(n-1)\sin^{m-1}z\cos^{n-1}z}+\frac{m+n-2}{n-1}\int\frac{dz}{\sin^m z\cos^{n-2}z}\\&\qquad(n>1)\\&=\frac{-1}{(m-1)\sin^{m-1}z\cos^{n-1}z}+\frac{m+n-2}{m-1}\int\frac{dz}{\sin^{m-2}z\cos^n z}\\&\qquad(m>1)\end{aligned}$$

4.3.129 $$\int\tan^n z\,dz=\frac{\tan^{n-1}z}{n-1}-\int\tan^{n-2}z\,dz\quad(n\neq1)$$

4.3.130 $$\int\cot^n z\,dz=-\frac{\cot^{n-1}z}{n-1}-\int\cot^{n-2}z\,dz\ (n\neq1)$$

4.3.131

$$\int\frac{dz}{a+b\sin z}=\frac{2}{(a^2-b^2)^{\frac{1}{2}}}\arctan\frac{a\tan\left(\frac{z}{2}\right)+b}{(a^2-b^2)^{\frac{1}{2}}}\qquad(a^2>b^2)$$

$$=\frac{1}{(b^2-a^2)^{\frac{1}{2}}}\ln\left[\frac{a\tan\left(\frac{z}{2}\right)+b-(b^2-a^2)^{\frac{1}{2}}}{a\tan\left(\frac{z}{2}\right)+b+(b^2-a^2)^{\frac{1}{2}}}\right]\qquad(b^2>a^2)$$

4.3.132 $$\int\frac{dz}{1\pm\sin z}=\mp\tan\left(\frac{\pi}{4}\mp\frac{z}{2}\right)$$

4.3.133

$$\int\frac{dz}{a+b\cos z}=\frac{2}{(a^2-b^2)^{\frac{1}{2}}}\arctan\frac{(a-b)\tan\frac{z}{2}}{(a^2-b^2)^{\frac{1}{2}}}\qquad(a^2>b^2)$$

$$=\frac{1}{(b^2-a^2)^{\frac{1}{2}}}\ln\left[\frac{(b-a)\tan\frac{z}{2}+(b^2-a^2)^{\frac{1}{2}}}{(b-a)\tan\frac{z}{2}-(b^2-a^2)^{\frac{1}{2}}}\right]\qquad(b^2>a^2)$$

4.3.134 $$\int\frac{dz}{1+\cos z}=\tan\frac{z}{2}$$

4.3.135 $$\int\frac{dz}{1-\cos z}=-\cot\frac{z}{2}$$

4.3.136

$$\int e^{az}\sin bz\,dz=\frac{e^{az}}{a^2+b^2}(a\sin bz-b\cos bz)$$

4.3.137

$$\int e^{az}\cos bz\,dz=\frac{e^{az}}{a^2+b^2}(a\cos bz+b\sin bz)$$

4.3.138

$$\int e^{az}\sin^n bz\,dz=\frac{e^{az}\sin^{n-1}bz}{a^2+n^2b^2}(a\sin bz-nb\cos bz)+\frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{az}\sin^{n-2}bz\,dz$$

4.3.139

$$\int e^{az}\cos^n bz\,dz=\frac{e^{az}\cos^{n-1}bz}{a^2+n^2b^2}(a\cos bz+nb\sin bz)+\frac{n(n-1)b^2}{a^2+n^2b^2}\int e^{az}\cos^{n-2}bz\,dz$$

Definite Integrals

4.3.140 $$\int_0^\pi\sin mt\sin nt\,dt=0$$

$(m\neq n,\quad m$ and n integers)

$$\int_0^\pi\cos mt\cos nt\,dt=0$$

4.3.141 $$\int_0^\pi\sin^2 nt\,dt=\int_0^\pi\cos^2 nt\,dt=\frac{\pi}{2}$$

(n an integer, $n\neq0$)

4.3.142 $$\int_0^\infty\frac{\sin mt}{t}\,dt=\frac{\pi}{2}\qquad(m>0)$$

$$=0\qquad(m=0)$$

$$=-\frac{\pi}{2}\qquad(m<0)$$

4.3.143 $$\int_0^\infty\frac{\cos at-\cos bt}{t}\,dt=\ln(b/a)$$

4.3.144 $$\int_0^\infty\sin t^2\,dt=\int_0^\infty\cos t^2\,dt=\frac{1}{2}\sqrt{\frac{\pi}{2}}$$

4.3.145

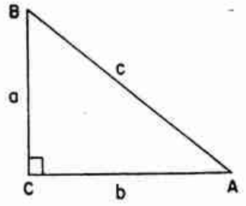
$$\int_0^{\pi/2}\ln\sin t\,dt=\int_0^{\pi/2}\ln\cos t\,dt=-\frac{\pi}{2}\ln 2$$

4.3.146 $$\int_0^\infty\frac{\cos mt}{1+t^2}\,dt=\frac{\pi}{2}e^{-m}$$

(See chapters **5** and **7** for other integrals involving circular functions.)
(See [5.3] for Fourier transforms.)

4.3.147

Formulas for Solution of Plane Right Triangles



If A, B and C are the vertices (C the right angle), and a, b and c the sides opposite respectively,

$$\sin A=\frac{a}{c}=\frac{1}{\csc A}$$

$$\cos A=\frac{b}{c}=\frac{1}{\sec A}$$

$$\tan A=\frac{a}{b}=\frac{1}{\cot A}$$

versine $A=\text{vers }A=1-\cos A$

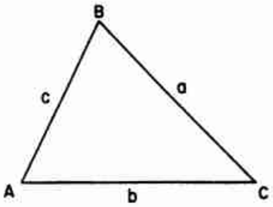
coversine $A=\text{covers }A=1-\sin A$

haversine $A=\text{hav }A=\frac{1}{2}\text{ vers }A$

exsecant $A=\text{exsec }A=\sec A-1$

4.3.148

Formulas for Solution of Plane Triangles



In a triangle with angles A, B and C and sides opposite a, b and c respectively,

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$\cos A=\frac{c^2+b^2-a^2}{2bc}$$

$$a=b\cos C+c\cos B$$

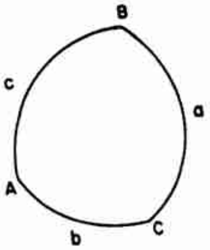
$$\frac{a+b}{a-b}=\frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}$$

$$\text{area}=\frac{bc\sin A}{2}=[s(s-a)(s-b)(s-c)]^{\frac{1}{2}}$$

$$s=\tfrac{1}{2}(a+b+c)$$

4.3.149

Formulas for Solution of Spherical Triangles



If A, B and C are the three angles and a, b and c the opposite sides,

$$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}$$

$$\cos a=\cos b\cos c+\sin b\sin c\cos A$$

$$=\frac{\cos b\cos(c\pm\theta)}{\cos\theta}$$

where $\tan\theta=\tan b\cos A$

$$\cos A=-\cos B\cos C+\sin B\sin C\cos a$$

4.4. Inverse Circular Functions

Definitions

4.4.1

$$\arcsin z=\int_0^z \frac{dt}{(1-t^2)^{\frac{1}{2}}} \qquad (z=x+iy)$$

4.4.2

$$\arccos z=\int_z^1 \frac{dt}{(1-t^2)^{\frac{1}{2}}}=\frac{\pi}{2}-\arcsin z$$

4.4.3

$$\arctan z=\int_0^z \frac{dt}{1+t^2} \qquad *$$

The path of integration must not cross the real axis in the case of **4.4.1** and **4.4.2** and the imaginary axis in the case of **4.4.3** except possibly inside the unit circle. Each function is single-valued and regular in the z-plane cut along the real axis from $-\infty$ to -1 and $+1$ to $+\infty$ in the case of **4.4.1** and **4.4.2** and along the imaginary axis from i to $i\infty$ and $-i$ to $-i\infty$ in the case of **4.4.3.**

Inverse circular functions are also written $\arcsin z=\sin^{-1} z$, $\arccos z=\cos^{-1} z$, $\arctan z=\tan^{-1} z$, ...

When $-1\leq x\leq 1$, $\arcsin x$ and $\arccos x$ are real and

4.4.4 $$-\tfrac{1}{2}\pi\leq\arcsin x\leq\tfrac{1}{2}\pi, \qquad 0\leq\arccos x\leq\pi$$

4.4.5 $$\arctan z+\operatorname{arccot} z=\pm\frac{\pi}{2} \quad \begin{matrix}\mathscr{R}z\geq 0\\ \mathscr{R}z<0^*\end{matrix}$$

4.4.6 $$\operatorname{arccsc} z=\arcsin 1/z$$

4.4.7 $$\operatorname{arcsec} z=\arccos 1/z$$

4.4.8 $$\operatorname{arccot} z=\arctan 1/z$$

4.4.9 $$\operatorname{arcsec} z+\operatorname{arccsc} z=\tfrac{1}{2}\pi$$

(see **4.3.45**)

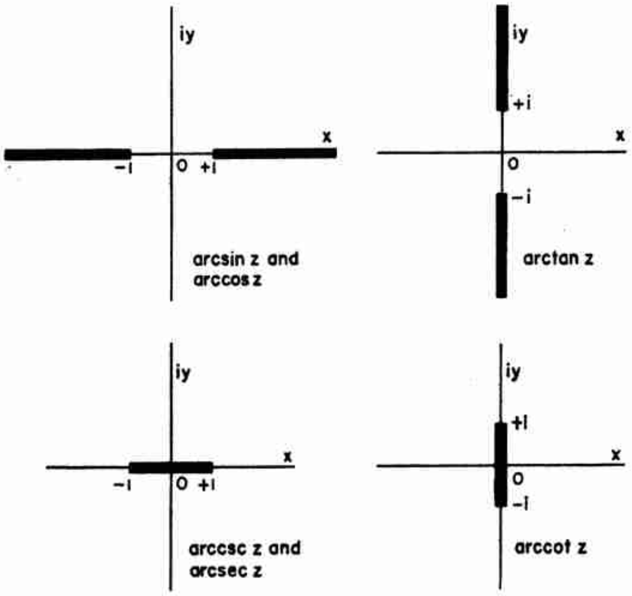


FIGURE 4.4. *Branch cuts for inverse circular functions.*

Fundamental Property

The general solutions of the equations

$$\sin t = z$$

$$\cos t = z$$

$$\tan t = z$$

are respectively

4.4.10 $t = \text{Arcsin } z = (-1)^k \arcsin z + k\pi$

4.4.11 $t = \text{Arccos } z = \pm \arccos z + 2k\pi$

4.4.12
$t = \text{Arctan } z = \arctan z + k\pi \qquad (z^2 \neq -1)$

where k is an arbitrary integer.

4.4.13 Interval containing principal value

y	x positive or zero	x negative
arcsin x and arctan x	$0 \leq y \leq \pi/2$	$-\pi/2 \leq y < 0$
*arccos x and arcsec x	$0 \leq y \leq \pi/2$	$\pi/2 < y \leq \pi$
*arccot x and arccsc x	$0 \leq y \leq \pi/2$	$-\pi/2 \leq y < 0$

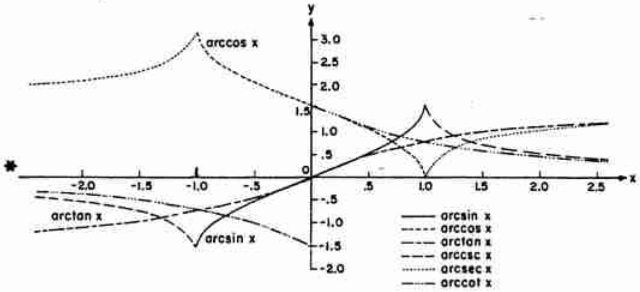


FIGURE 4.5. *Inverse circular functions.*

Functions of Negative Arguments

4.4.14 $\arcsin(-z) = -\arcsin z$

4.4.15 $\arccos(-z) = \pi - \arccos z$

4.4.16 $\arctan(-z) = -\arctan z$

4.4.17 $\text{arccsc}(-z) = -\text{arccsc } z$

4.4.18 $\text{arcsec}(-z) = \pi - \text{arcsec } z$

***4.4.19** $\text{arccot}(-z) = -\text{arccot } z$

Relation to Inverse Hyperbolic Functions (see 4.6.14 to 4.6.19)

4.4.20 $\text{Arcsin } z = -i \text{ Arcsinh } iz$

4.4.21 $\text{Arccos } z = \pm i \text{ Arccosh } z$

4.4.22 $\text{Arctan } z = -i \text{ Arctanh } iz \qquad (z^2 \neq -1)$

4.4.23 $\text{Arccsc } z = i \text{ Arccsch } iz$

4.4.24 $\text{Arcsec } z = \pm i \text{ Arcsech } z$

4.4.25 $\text{Arccot } z = i \text{ Arccoth } iz$

Logarithmic Representations

4.4.26 $\text{Arcsin } x = -i \text{ Ln}\left[(1-x^2)^{\frac{1}{2}} + ix\right] \quad (x^2 \leq 1)$

4.4.27 $\text{Arccos } x = -i \text{ Ln}\left[x + i(1-x^2)^{\frac{1}{2}}\right] \quad (x^2 \leq 1)$

4.4.28 $\text{Arctan } x = \frac{i}{2} \text{ Ln } \frac{1-i\,x}{1+i\,x} = \frac{i}{2} \text{ Ln } \frac{i+x}{i-x}$

(x real)

4.4.29 $\text{Arccsc } x = -i \text{ Ln}\left[\frac{(x^2-1)^{\frac{1}{2}}+i}{x}\right] \quad (x^2 \geq 1)$

4.4.30 $\text{Arcsec } x = -i \text{ Ln}\left[\frac{1+i(x^2-1)^{\frac{1}{2}}}{x}\right] \quad (x^2 \geq 1)$

4.4.31 $\text{Arccot } x = \frac{i}{2} \text{Ln}\left(\frac{ix+1}{ix-1}\right) = \frac{i}{2} \text{Ln}\left(\frac{x-i}{x+i}\right)$

(x real)

Addition and Subtraction of Two Inverse Circular Functions

4.4.32

$$\text{Arcsin } z_1 \pm \text{Arcsin } z_2 = \text{Arcsin}\left[z_1(1-z_2^2)^{\frac{1}{2}} \pm z_2(1-z_1^2)^{\frac{1}{2}}\right]$$

4.4.33

$$\text{Arccos } z_1 \pm \text{Arccos } z_2 = \text{Arccos}\left\{z_1 z_2 \mp \left[(1-z_1^2)(1-z_2^2)\right]^{\frac{1}{2}}\right\}$$

4.4.34

$$\text{Arctan } z_1 \pm \text{Arctan } z_2 = \text{Arctan}\left(\frac{z_1 \pm z_2}{1 \mp z_1 z_2}\right)$$

4.4.35

$$\text{Arcsin } z_1 \pm \text{Arccos } z_2 = \text{Arcsin}\left\{z_1 z_2 \pm \left[(1-z_1^2)(1-z_2^2)\right]^{\frac{1}{2}}\right\} = \text{Arccos}\left[z_2(1-z_1^2)^{\frac{1}{2}} \mp z_1(1-z_2^2)^{\frac{1}{2}}\right]$$

4.4.36

$$\text{Arctan } z_1 \pm \text{Arccot } z_2 = \text{Arctan}\left(\frac{z_1 z_2 \pm 1}{z_2 \mp z_1}\right) = \text{Arccot}\left(\frac{z_2 \mp z_1}{z_1 z_2 \pm 1}\right)$$

Inverse Circular Functions in Terms of Real and Imaginary Parts

4.4.37

$$\text{Arcsin } z = k\pi + (-1)^k \arcsin \beta + (-1)^k i \ln\left[\alpha + (\alpha^2-1)^{\frac{1}{2}}\right]$$

4.4.38

$$\text{Arccos } z = 2k\pi \pm \left\{\arccos \beta - i \ln\left[\alpha + (\alpha^2-1)^{\frac{1}{2}}\right]\right\}$$

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*See page II.

4.4.39

$$\text{Arctan } z = k\pi + \tfrac{1}{2} \arctan \left(\frac{2x}{1-x^2-y^2} \right) + \frac{i}{4} \ln \left[\frac{x^2+(y+1)^2}{x^2+(y-1)^2} \right] \quad (z^2 \neq -1)$$

where k is an integer or zero and

$$\alpha = \tfrac{1}{2}[(x+1)^2+y^2]^{\frac{1}{2}} + \tfrac{1}{2}[(x-1)^2+y^2]^{\frac{1}{2}}$$

$$\beta = \tfrac{1}{2}[(x+1)^2+y^2]^{\frac{1}{2}} - \tfrac{1}{2}[(x-1)^2+y^2]^{\frac{1}{2}}$$

Series Expansions

4.4.40

$$\arcsin z = z + \frac{z^3}{2 \cdot 3} + \frac{1 \cdot 3 z^5}{2 \cdot 4 \cdot 5} + \frac{1 \cdot 3 \cdot 5 z^7}{2 \cdot 4 \cdot 6 \cdot 7} + \ldots \quad (|z|<1)$$

4.4.41

$$\arcsin (1-z) = \frac{\pi}{2} - (2z)^{\frac{1}{2}} \left[1 + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \ldots (2k-1)}{2^{2k}(2k+1)k!} z^k \right] \quad (|z|<2)$$

4.4.42

$$\arctan z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \ldots \quad (|z| \leq 1 \text{ and } z^2 \neq -1)$$

$$= \frac{\pi}{2} - \frac{1}{z} + \frac{1}{3z^3} - \frac{1}{5z^5} + \ldots \quad (|z|>1 \text{ and } z^2 \neq -1)$$

$$= \frac{z}{1+z^2} \left[1 + \frac{2}{3} \frac{z^2}{1+z^2} + \frac{2 \cdot 4}{3 \cdot 5} \left(\frac{z^2}{1+z^2} \right)^2 + \ldots \right] \quad (z^2 \neq -1)$$

Continued Fractions

4.4.43 $\arctan z = \frac{z}{1+} \frac{z^2}{3+} \frac{4z^2}{5+} \frac{9z^2}{7+} \frac{16z^2}{9+} \ldots$

(z in the cut plane of Figure 4.4.)

4.4.44 $\frac{\arcsin z}{\sqrt{1-z^2}} = \frac{z}{1-} \frac{1 \cdot 2z^2}{3-} \frac{1 \cdot 2z^2}{5-} \frac{3 \cdot 4z^2}{7-} \frac{3 \cdot 4z^2}{9-} \ldots$

(z in the cut plane of Figure 4.4.)

Polynomial Approximations[9]

4.4.45 $0 \leq x \leq 1$

$$\arcsin x = \frac{\pi}{2} - (1-x)^{\frac{1}{2}} (a_0 + a_1 x + a_2 x^2 + a_3 x^3) + \epsilon(x)$$

$$|\epsilon(x)| \leq 5 \times 10^{-5}$$

$a_0 = 1.57072\ 88$

$a_1 = -.21211\ 44$

$a_2 = .07426\ 10$

$a_3 = -.01872\ 93$

4.4.46 $0 \leq x \leq 1$

$$\arcsin x = \frac{\pi}{2} - (1-x)^{\frac{1}{2}} (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + a_7 x^7) + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-8}$$

$a_0 = 1.57079\ 63050$

$a_1 = -.21459\ 88016$

$a_2 = .08897\ 89874$

$a_3 = -.05017\ 43046$

$a_4 = .03089\ 18810$

$a_5 = -.01708\ 81256$

$a_6 = .00667\ 00901$

$a_7 = -.00126\ 24911$

4.4.47 $-1 \leq x \leq 1$

$$\arctan x = a_1 x + a_3 x^3 + a_5 x^5 + a_7 x^7 + a_9 x^9 + \epsilon(x)$$

$$|\epsilon(x)| \leq 10^{-5}$$

$a_1 = .99986\ 60$

$a_3 = -.33029\ 95$

$a_5 = .18014\ 10$

$a_7 = -.08513\ 30$

$a_9 = .02083\ 51$

4.4.48[10] $-1 \leq x \leq 1$

$$\arctan x = \frac{x}{1+.28x^2} + \epsilon(x)$$

$$|\epsilon(x)| \leq 5 \times 10^{-3}$$

4.4.49[11] $0 \leq x \leq 1$

$$\frac{\arctan x}{x} = 1 + \sum_{k=1}^{8} a_{2k} x^{2k} + \epsilon(x)$$

$$|\epsilon(x)| \leq 2 \times 10^{-8}$$

$a_2 = -.33333\ 14528$

$a_4 = .19993\ 55085$

$a_6 = -.14208\ 89944$

$a_8 = .10656\ 26393$

$a_{10} = -.07528\ 96400$

$a_{12} = .04290\ 96138$

$a_{14} = -.01616\ 57367$

$a_{16} = .00286\ 62257$

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[9] The approximations **4.4.45** to **4.4.47** are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

[10] The approximation **4.4.48** is from C. Hastings, Jr., Note 143, Math. Tables Aids Comp. **6**, 68 (1953) (with permission).

[11] The approximation **4.4.49** is from B. Carlson, M. Goldstein, Rational approximation of functions, Los Alamos Scientific Laboratory LA-1943, Los Alamos, N. Mex., 1955 (with permission).

Approximations in Terms of Chebyshev Polynomials[12]

4.4.50 $$-1\leq x\leq 1$$

$T^*_n(x)=\cos n\theta,\qquad \cos\theta=2x-1\qquad$ (see chapter **22**)

$$\arctan x=x\sum_{n=0}^{\infty}A_nT^*_n(x^2)$$

n	A_n	n	A_n
0	.88137 3587	6	.00000 3821
1	−.10589 2925	7	−.00000 0570
2	.01113 5843	8	.00000 0086
3	−.00138 1195	9	−.00000 0013
4	.00018 5743	10	.00000 0002
5	−.00002 6215		

*For $x>1$, use $\arctan x=\frac{1}{2}\pi-\arctan(1/x)$

4.4.51 $$-\tfrac{1}{2}\sqrt{2}\leq x\leq\tfrac{1}{2}\sqrt{2}$$

$$\arcsin x=x\sum_{n=0}^{\infty}A_nT^*_n(2x^2)$$

$$0\leq x\leq\tfrac{1}{2}\sqrt{2}$$

$$\arccos x=\tfrac{1}{2}\pi-x\sum_{n=0}^{\infty}A_nT^*_n(2x^2)$$

n	A_n	n	A_n
0	1.05123 1959	5	.00000 5881
1	.05494 6487	6	.00000 0777
2	.00408 0631	7	.00000 0107
3	.00040 7890	8	.00000 0015
4	.00004 6985	9	.00000 0002

For $\frac{1}{2}\sqrt{2}\leq x\leq 1$, use $\arcsin x=\arccos(1-x^2)^{\frac{1}{2}}$, $\arccos x=\arcsin(1-x^2)^{\frac{1}{2}}$.

Differentiation Formulas

4.4.52 $$\frac{d}{dz}\arcsin z=(1-z^2)^{-\frac{1}{2}}$$

4.4.53 $$\frac{d}{dz}\arccos z=-(1-z^2)^{-\frac{1}{2}}$$

4.4.54 $$\frac{d}{dz}\arctan z=\frac{1}{1+z^2}$$

4.4.55 $$\frac{d}{dz}\operatorname{arccot} z=\frac{-1}{1+z^2}$$

4.4.56 $$\frac{d}{dz}\operatorname{arcsec} z=\frac{1}{z(z^2-1)^{\frac{1}{2}}}$$

4.4.57 $$\frac{d}{dz}\operatorname{arccsc} z=-\frac{1}{z(z^2-1)^{\frac{1}{2}}}$$

Integration Formulas

4.4.58 $$\int\arcsin z\,dz=z\arcsin z+(1-z^2)^{\frac{1}{2}}$$

4.4.59 $$\int\arccos z\,dz=z\arccos z-(1-z^2)^{\frac{1}{2}}$$

4.4.60 $$\int\arctan z\,dz=z\arctan z-\tfrac{1}{2}\ln(1+z^2)$$

4.4.61

$$\int\operatorname{arccsc} z\,dz=z\operatorname{arccsc} z\pm\ln[z+(z^2-1)^{\frac{1}{2}}]$$

$$\left(\begin{array}{c}0<\operatorname{arccsc} z<\frac{\pi}{2}\\ -\frac{\pi}{2}<\operatorname{arccsc} z<0\end{array}\right)$$

4.4.62

$$\int\operatorname{arcsec} z\,dz=z\operatorname{arcsec} z\mp\ln[z+(z^2-1)^{\frac{1}{2}}]$$

$$\left(\begin{array}{c}0<\operatorname{arcsec} z<\frac{\pi}{2}\\ \frac{\pi}{2}<\operatorname{arcsec} z<\pi\end{array}\right)$$

4.4.63

$$\int\operatorname{arccot} z\,dz=z\operatorname{arccot} z+\tfrac{1}{2}\ln(1+z^2)$$

4.4.64

$$\int z\arcsin z\,dz=\left(\frac{z^2}{2}-\frac{1}{4}\right)\arcsin z+\frac{z}{4}(1-z^2)^{\frac{1}{2}}$$

4.4.65

$$\int z^n\arcsin z\,dz=\frac{z^{n+1}}{n+1}\arcsin z-\frac{1}{n+1}\int\frac{z^{n+1}}{(1-z^2)^{\frac{1}{2}}}dz$$

$$(n\neq-1)$$

4.4.66

$$\int z\arccos z\,dz=\left(\frac{z^2}{2}-\frac{1}{4}\right)\arccos z-\frac{z}{4}(1-z^2)^{\frac{1}{2}}$$

4.4.67

$$\int z^n\arccos z\,dz=\frac{z^{n+1}}{n+1}\arccos z+\frac{1}{n+1}\int\frac{z^{n+1}}{(1-z^2)^{\frac{1}{2}}}dz$$

$$(n\neq-1)$$

4.4.68

$$\int z\arctan z\,dz=\frac{1}{2}(1+z^2)\arctan z-\frac{z}{2}$$

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[12] The approximations **4.4.50** to **4.4.51** are from C. W. Clenshaw, Polynomial approximations to elementary functions, Math. Tables Aids Comp. **8**, 143–147 (1954) (with permission).

*See page II.

4.4.69

$$\int z^n \arctan z\,dz=\frac{z^{n+1}}{n+1}\arctan z-\frac{1}{n+1}\int\frac{z^{n+1}}{1+z^2}dz$$

$$(n\neq -1)$$

4.4.70

$$\int z \operatorname{arccot} z\,dz=\frac{1}{2}(1+z^2)\operatorname{arccot} z+\frac{z}{2}$$

4.4.71

$$\int z^n \operatorname{arccot} z\,dz=\frac{z^{n+1}}{n+1}\operatorname{arccot} z+\frac{1}{n+1}\int\frac{z^{n+1}}{1+z^2}dz$$

$$(n\neq -1)$$

4.5. Hyperbolic Functions

Definitions

4.5.1 $\sinh z=\dfrac{e^z-e^{-z}}{2}$ $(z=x+iy)$

4.5.2 $\cosh z=\dfrac{e^z+e^{-z}}{2}$

4.5.3 $\tanh z=\sinh z/\cosh z$

4.5.4 $\operatorname{csch} z=1/\sinh z$

4.5.5 $\operatorname{sech} z=1/\cosh z$

4.5.6 $\coth z=1/\tanh z$

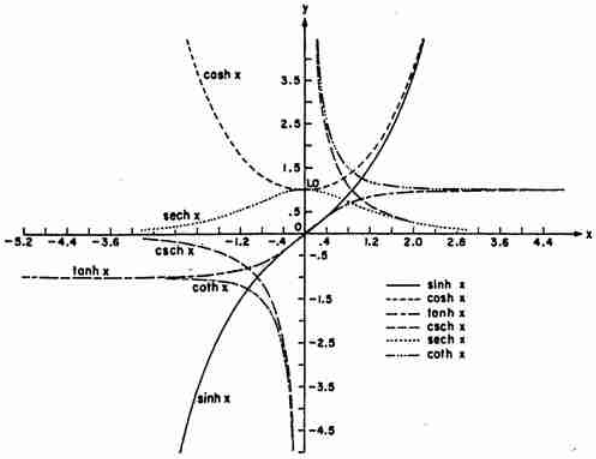

FIGURE 4.6. *Hyperbolic functions.*

Relation to Circular Functions (see 4.3.49 to 4.3.54)

Hyperbolic formulas can be derived from trigonometric identities by replacing z by iz

4.5.7 $\sinh z=-i\sin iz$

4.5.8 $\cosh z=\cos iz$

4.5.9 $\tanh z=-i\tan iz$

4.5.10 $\operatorname{csch} z=i\csc iz$

4.5.11 $\operatorname{sech} z=\sec iz$

4.5.12 $\coth z=i\cot iz$

Periodic Properties

4.5.13 $\sinh(z+2k\pi i)=\sinh z$

(k any integer)

4.5.14 $\cosh(z+2k\pi i)=\cosh z$

4.5.15 $\tanh(z+k\pi i)=\tanh z$

Relations Between Hyperbolic Functions

4.5.16 $\cosh^2 z-\sinh^2 z=1$

4.5.17 $\tanh^2 z+\operatorname{sech}^2 z=1$

4.5.18 $\coth^2 z-\operatorname{csch}^2 z=1$

4.5.19 $\cosh z+\sinh z=e^z$

4.5.20 $\cosh z-\sinh z=e^{-z}$

Negative Angle Formulas

4.5.21 $\sinh(-z)=-\sinh z$

4.5.22 $\cosh(-z)=\cosh z$

4.5.23 $\tanh(-z)=-\tanh z$

Addition Formulas

4.5.24 $\sinh(z_1+z_2)=\sinh z_1\cosh z_2$
$+\cosh z_1\sinh z_2$

4.5.25 $\cosh(z_1+z_2)=\cosh z_1\cosh z_2$
$+\sinh z_1\sinh z_2$

4.5.26 $\tanh(z_1+z_2)=(\tanh z_1+\tanh z_2)/$
$(1+\tanh z_1\tanh z_2)$

4.5.27 $\coth(z_1+z_2)=(\coth z_1\coth z_2+1)/$
$(\coth z_2+\coth z_1)$

Half-Angle Formulas

4.5.28

$$\sinh\frac{z}{2}=\left(\frac{\cosh z-1}{2}\right)^{\frac{1}{2}}$$

4.5.29

$$\cosh\frac{z}{2}=\left(\frac{\cosh z+1}{2}\right)^{\frac{1}{2}}$$

4.5.30

$$\tanh\frac{z}{2}=\left(\frac{\cosh z-1}{\cosh z+1}\right)^{\frac{1}{2}}=\frac{\cosh z-1}{\sinh z}=\frac{\sinh z}{\cosh z+1}$$

Multiple-Angle Formulas

4.5.31 $$\sinh 2z=2\sinh z\cosh z=\frac{2\tanh z}{1-\tanh^2 z}$$

4.5.32 $$\cosh 2z=2\cosh^2 z-1=2\sinh^2 z+1$$
$$=\cosh^2 z+\sinh^2 z$$

4.5.33 $$\tanh 2z=\frac{2\tanh z}{1+\tanh^2 z}$$

4.5.34 $$\sinh 3z=3\sinh z+4\sinh^3 z$$

4.5.35 $$\cosh 3z=-3\cosh z+4\cosh^3 z$$

4.5.36 $$\sinh 4z=4\sinh^3 z\cosh z+4\cosh^3 z\sinh z$$

4.5.37 $$\cosh 4z=\cosh^4 z+6\sinh^2 z\cosh^2 z+\sinh^4 z$$

Products of Hyperbolic Sines and Cosines

4.5.38 $$2\sinh z_1\sinh z_2=\cosh(z_1+z_2)-\cosh(z_1-z_2)$$

4.5.39 $$2\cosh z_1\cosh z_2=\cosh(z_1+z_2)+\cosh(z_1-z_2)$$

4.5.40 $$2\sinh z_1\cosh z_2=\sinh(z_1+z_2)+\sinh(z_1-z_2)$$

Addition and Subtraction of Two Hyperbolic Functions

4.5.41

$$\sinh z_1+\sinh z_2=2\sinh\left(\frac{z_1+z_2}{2}\right)\cosh\left(\frac{z_1-z_2}{2}\right)$$

4.5.42

$$\sinh z_1-\sinh z_2=2\cosh\left(\frac{z_1+z_2}{2}\right)\sinh\left(\frac{z_1-z_2}{2}\right)$$

4.5.43

$$\cosh z_1+\cosh z_2=2\cosh\left(\frac{z_1+z_2}{2}\right)\cosh\left(\frac{z_1-z_2}{2}\right)$$

4.5.44

$$\cosh z_1-\cosh z_2=2\sinh\left(\frac{z_1+z_2}{2}\right)\sinh\left(\frac{z_1-z_2}{2}\right)$$

4.5.45

$$\tanh z_1+\tanh z_2=\frac{\sinh(z_1+z_2)}{\cosh z_1\cosh z_2}$$

4.5.46

$$\coth z_1+\coth z_2=\frac{\sinh(z_1+z_2)}{\sinh z_1\sinh z_2}$$

Relations Between Squares of Hyperbolic Sines and Cosines

4.5.47

$$\sinh^2 z_1-\sinh^2 z_2=\sinh(z_1+z_2)\sinh(z_1-z_2)$$
$$=\cosh^2 z_1-\cosh^2 z_2$$

4.5.48

$$\sinh^2 z_1+\cosh^2 z_2=\cosh(z_1+z_2)\cosh(z_1-z_2)$$
$$=\cosh^2 z_1+\sinh^2 z_2$$

Hyperbolic Functions in Terms of Real and Imaginary Parts

$$(z=x+iy)$$

4.5.49 $$\sinh z=\sinh x\cos y+i\cosh x\sin y$$

4.5.50 $$\cosh z=\cosh x\cos y+i\sinh x\sin y$$

4.5.51 $$\tanh z=\frac{\sinh 2x+i\sin 2y}{\cosh 2x+\cos 2y}$$

4.5.52 $$\coth z=\frac{\sinh 2x-i\sin 2y}{\cosh 2x-\cos 2y}$$

De Moivre's Theorem

4.5.53 $$(\cosh z+\sinh z)^n=\cosh nz+\sinh nz$$

Modulus and Phase (Argument) of Hyperbolic Functions

4.5.54 $$|\sinh z|=(\sinh^2 x+\sin^2 y)^{\frac{1}{2}}$$
$$=[\tfrac{1}{2}(\cosh 2x-\cos 2y)]^{\frac{1}{2}}$$

4.5.55 $$\arg\sinh z=\arctan(\coth x\tan y)$$

4.5.56 $$|\cosh z|=(\sinh^2 x+\cos^2 y)^{\frac{1}{2}}$$
$$=[\tfrac{1}{2}(\cosh 2x+\cos 2y)]^{\frac{1}{2}}$$

4.5.57 $$\arg\cosh z=\arctan(\tanh x\tan y)$$

4.5.58 $$|\tanh z|=\left(\frac{\cosh 2x-\cos 2y}{\cosh 2x+\cos 2y}\right)^{\frac{1}{2}}$$

4.5.59 $$\arg\tanh z=\arctan\left(\frac{\sin 2y}{\sinh 2x}\right)$$

4.5.60 Relations Between Hyperbolic (or Inverse Hyperbolic) Functions

	$\sinh x=a$	$\cosh x=a$	$\tanh x=a$	$\operatorname{csch} x=a$	$\operatorname{sech} x=a$	$\coth x=a$
$\sinh x$	a	$(a^2-1)^{\frac{1}{2}}$	$a(1-a^2)^{-\frac{1}{2}}$	a^{-1}	$a^{-1}(1-a^2)^{\frac{1}{2}}$	$(a^2-1)^{-\frac{1}{2}}$
$\cosh x$	$(1+a^2)^{\frac{1}{2}}$	a	$(1-a^2)^{-\frac{1}{2}}$	$a^{-1}(1+a^2)^{\frac{1}{2}}$	a^{-1}	$a(a^2-1)^{-\frac{1}{2}}$
$\tanh x$	$a(1+a^2)^{-\frac{1}{2}}$	$a^{-1}(a^2-1)^{\frac{1}{2}}$	a	$(1+a^2)^{-\frac{1}{2}}$	$(1-a^2)^{\frac{1}{2}}$	a^{-1}
$\operatorname{csch} x$	a^{-1}	$(a^2-1)^{-\frac{1}{2}}$	$a^{-1}(1-a^2)^{\frac{1}{2}}$	a	$a(1-a^2)^{-\frac{1}{2}}$	$(a^2-1)^{\frac{1}{2}}$
$\operatorname{sech} x$	$(1+a^2)^{-\frac{1}{2}}$	a^{-1}	$(1-a^2)^{\frac{1}{2}}$	$a(1+a^2)^{-\frac{1}{2}}$	a	$a^{-1}(a^2-1)^{\frac{1}{2}}$
$\coth x$	$a^{-1}(a^2+1)^{\frac{1}{2}}$	$a(a^2-1)^{-\frac{1}{2}}$	a^{-1}	$(1+a^2)^{\frac{1}{2}}$	$(1-a^2)^{-\frac{1}{2}}$	a

Illustration: If $\sinh x=a$, $\coth x=a^{-1}(a^2+1)^{\frac{1}{2}}$

$\operatorname{arcsech} a=\operatorname{arccoth}(1-a^2)^{-\frac{1}{2}}$

4.5.61 Special Values of the Hyperbolic Functions

z	0	$\frac{\pi}{2}i$	πi	$\frac{3\pi}{2}i$	∞
$\sinh z$	0	i	0	$-i$	∞
$\cosh z$	1	0	-1	0	∞
$\tanh z$	0	∞i	0	$-\infty i$	1
$\operatorname{csch} z$	∞	$-i$	∞	i	0
$\operatorname{sech} z$	1	∞	-1	∞	0
$\coth z$	∞	0	∞	0	1

Series Expansions

4.5.62 $$\sinh z=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\frac{z^7}{7!}+\dots \qquad (|z|<\infty)$$

4.5.63 $$\cosh z=1+\frac{z^2}{2!}+\frac{z^4}{4!}+\frac{z^6}{6!}+\dots \qquad (|z|<\infty)$$

4.5.64 $$\tanh z=z-\frac{z^3}{3}+\frac{2}{15}z^5-\frac{17}{315}z^7$$

* $$+\dots+\frac{2^{2n}(2^{2n}-1)B_{2n}}{(2n)!}z^{2n-1}+\dots \qquad \left(|z|<\frac{\pi}{2}\right)$$

4.5.65

$$\operatorname{csch} z=\frac{1}{z}-\frac{z}{6}+\frac{7}{360}z^3-\frac{31}{15120}z^5+\dots$$
$$-\frac{2(2^{2n-1}-1)B_{2n}}{(2n)!}z^{2n-1}+\dots \qquad (|z|<\pi)$$

4.5.66

$$\operatorname{sech} z=1-\frac{z^2}{2}+\frac{5}{24}z^4-\frac{61}{720}z^6+\dots+\frac{E_{2n}}{(2n)!}z^{2n}+\dots \qquad \left(|z|<\frac{\pi}{2}\right)$$

4.5.67

$$\coth z=\frac{1}{z}+\frac{z}{3}-\frac{z^3}{45}+\frac{2}{945}z^5-\dots+\frac{2^{2n}B_{2n}}{(2n)!}z^{2n-1}+\dots \qquad (|z|<\pi)$$

where B_n and E_n are the nth Bernoulli and Euler numbers, see chapter **23.**

Infinite Products

4.5.68 $$\sinh z=z\prod_{k=1}^{\infty}\left(1+\frac{z^2}{k^2\pi^2}\right)$$

4.5.69 $$\cosh z=\prod_{k=1}^{\infty}\left[1+\frac{4z^2}{(2k-1)^2\pi^2}\right]$$

Continued Fraction

4.5.70 $$\tanh z=\frac{z}{1+}\,\frac{z^2}{3+}\,\frac{z^2}{5+}\,\frac{z^2}{7+}\cdots \qquad \left(z\neq\frac{\pi}{2}i\pm n\pi i\right)$$

Differentiation Formulas

4.5.71 $$\frac{d}{dz}\sinh z=\cosh z$$

4.5.72 $$\frac{d}{dz}\cosh z=\sinh z$$

4.5.73 $$\frac{d}{dz}\tanh z=\operatorname{sech}^2 z$$

4.5.74 $$\frac{d}{dz}\operatorname{csch} z=-\operatorname{csch} z\coth z$$

4.5.75 $$\frac{d}{dz}\operatorname{sech} z=-\operatorname{sech} z \tanh z$$

4.5.76 $$\frac{d}{dz}\coth z=-\operatorname{csch}^2 z$$

Integration Formulas

4.5.77 $$\int \sinh z\,dz=\cosh z$$

4.5.78 $$\int \cosh z\,dz=\sinh z$$

4.5.79 $$\int \tanh z\,dz=\ln \cosh z$$

4.5.80 $$\int \operatorname{csch} z\,dz=\ln \tanh \frac{z}{2}$$

4.5.81 $$\int \operatorname{sech} z\,dz=\arctan (\sinh z)$$

4.5.82 $$\int \coth z\,dz=\ln \sinh z$$

4.5.83
$$\int z^n \sinh z\,dz=z^n \cosh z-n\int z^{n-1} \cosh z\,dz$$

4.5.84
$$\int z^n \cosh z\,dz=z^n \sinh z-n\int z^{n-1} \sinh z\,dz$$

4.5.85
$$\begin{aligned}\int \sinh^m z \cosh^n z\,dz&=\frac{1}{m+n} \sinh^{m+1} z \cosh^{n-1} z\\&\quad+\frac{n-1}{m+n}\int \sinh^m z \cosh^{n-2} z\,dz\\&=\frac{1}{m+n} \sinh^{m-1} z \cosh^{n+1} z\\&\quad-\frac{m-1}{m+n}\int \sinh^{m-2} z \cosh^n z\,dz\\&\qquad(m+n\neq 0)\end{aligned}$$

4.5.86 $$\begin{aligned}\int\frac{dz}{\sinh^m z \cosh^n z}&=\frac{-1}{m-1}\,\frac{1}{\sinh^{m-1} z \cosh^{n-1} z}\\&\quad-\frac{m+n-2}{m-1}\int\frac{dz}{\sinh^{m-2} z \cosh^n z}\qquad(m\neq 1)\\&=\frac{1}{n-1}\,\frac{1}{\sinh^{m-1} z \cosh^{n-1} z}\\&\quad+\frac{m+n-2}{n-1}\int\frac{dz}{\sinh^m z \cosh^{n-2} z}\qquad(n\neq 1)\end{aligned}$$

4.5.87
$$\int \tanh^n z\,dz=-\frac{\tanh^{n-1} z}{n-1}+\int \tanh^{n-2} z\,dz \qquad (n\neq 1)$$

4.5.88
$$\int \coth^n z\,dz=-\frac{\coth^{n-1} z}{n-1}+\int \coth^{n-2} z\,dz \qquad (n\neq 1)$$

(See chapters **5** and **7** for other integrals involving hyperbolic functions.)

4.6. Inverse Hyperbolic Functions

Definitions

4.6.1 $$\operatorname{arcsinh} z=\int_0^z \frac{dt}{(1+t^2)^{\frac{1}{2}}} \qquad (z=x+iy)$$

4.6.2 $$\operatorname{arccosh} z=\int_1^z \frac{dt}{(t^2-1)^{\frac{1}{2}}}$$

4.6.3 $$\operatorname{arctanh} z=\int_0^z \frac{dt}{1-t^2}$$

The paths of integration must not cross the following cuts.

4.6.1 imaginary axis from $-i\infty$ to $-i$ and i to $i\infty$

4.6.2 real axis from $-\infty$ to $+1$

4.6.3 real axis from $-\infty$ to -1 and $+1$ to $+\infty$

Inverse hyperbolic functions are also written $\sinh^{-1} z$, $\operatorname{arsinh} z$, $\mathscr{A}r \sinh z$, etc.

4.6.4 $\operatorname{arccsch} z=\operatorname{arcsinh} 1/z$

4.6.5 $\operatorname{arcsech} z=\operatorname{arccosh} 1/z$

4.6.6 $\operatorname{arccoth} z=\operatorname{arctanh} 1/z$

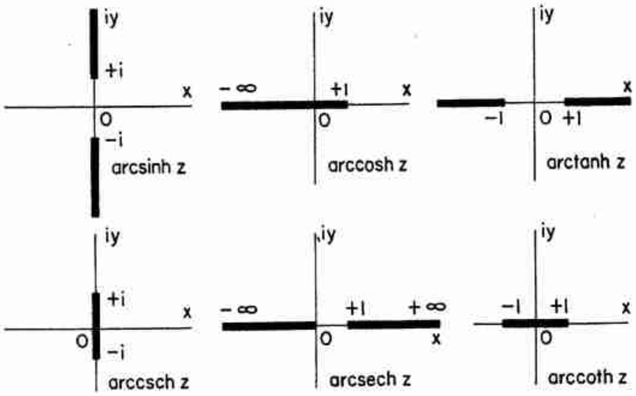

FIGURE 4.7. *Branch cuts for inverse hyperbolic functions.*

4.6.7 $\quad \operatorname{arctanh} z = \operatorname{arccoth} z \pm \frac{1}{2}\pi i$

(see **4.5.60**) $\quad$ (according as $\mathscr{I}z \gtrless 0$)

Fundamental Property

The general solutions of the equations

$$z = \sinh t$$
$$z = \cosh t$$
$$z = \tanh t$$

are respectively

4.6.8 $\quad t = \operatorname{Arcsinh} z = (-1)^k \operatorname{arcsinh} z + k\pi i$

4.6.9 $\quad t = \operatorname{Arccosh} z = \pm \operatorname{arccosh} z + 2k\pi i$

4.6.10 $\quad t = \operatorname{Arctanh} z = \operatorname{arctanh} z + k\pi i$

(k, integer)

Functions of Negative Arguments

4.6.11 $\quad \operatorname{arcsinh}(-z) = -\operatorname{arcsinh} z$

***4.6.12** $\quad \operatorname{arccosh}(-z) = \pi i - \operatorname{arccosh} z$

4.6.13 $\quad \operatorname{arctanh}(-z) = -\operatorname{arctanh} z$

Relation to Inverse Circular Functions (see 4.4.20 to 4.4.25)

Hyperbolic identities can be derived from trigonometric identities by replacing z by iz.

4.6.14 $\quad \operatorname{Arcsinh} z = -i \operatorname{Arcsin} iz$

4.6.15 $\quad \operatorname{Arccosh} z = \pm i \operatorname{Arccos} z$

4.6.16 $\quad \operatorname{Arctanh} z = -i \operatorname{Arctan} iz$

4.6.17 $\quad \operatorname{Arccsch} z = i \operatorname{Arccsc} iz$

4.6.18 $\quad \operatorname{Arcsech} z = \pm i \operatorname{Arcsec} z$

4.6.19 $\quad \operatorname{Arccoth} z = i \operatorname{Arccot} iz$

Logarithmic Representations

4.6.20 $\quad \operatorname{arcsinh} x = \ln [x + (x^2+1)^{\frac{1}{2}}]$

4.6.21 $\quad \operatorname{arccosh} x = \ln [x + (x^2-1)^{\frac{1}{2}}] \qquad (x \geq 1)$

4.6.22 $\quad \operatorname{arctanh} x = \frac{1}{2} \ln \dfrac{1+x}{1-x} \qquad (0 \leq x^2 < 1)$

4.6.23 $\quad \operatorname{arccsch} x = \ln \left[\dfrac{1}{x} + \left(\dfrac{1}{x^2}+1\right)^{\frac{1}{2}}\right] \qquad (x \neq 0)$

4.6.24 $\quad \operatorname{arcsech} x = \ln \left[\dfrac{1}{x} + \left(\dfrac{1}{x^2}-1\right)^{\frac{1}{2}}\right] \qquad (0 < x \leq 1)$

4.6.25 $\quad \operatorname{arccoth} x = \dfrac{1}{2} \ln \dfrac{x+1}{x-1} \qquad (x^2 > 1)$

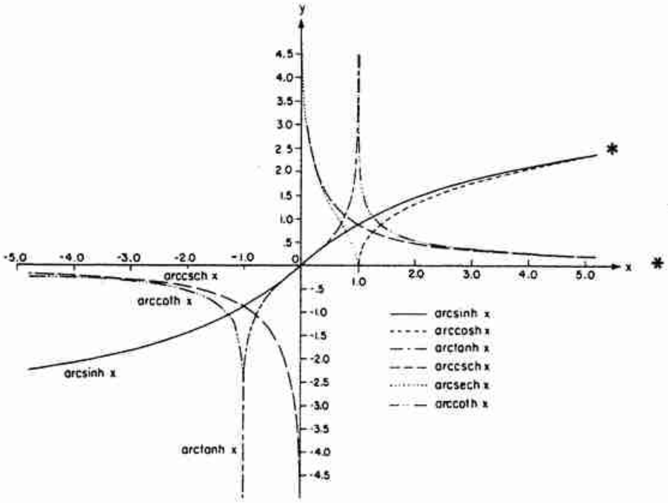

FIGURE 4.8. *Inverse hyperbolic functions.*

Addition and Subtraction of Two Inverse Hyperbolic Functions

4.6.26

$$\operatorname{Arcsinh} z_1 \pm \operatorname{Arcsinh} z_2 = \operatorname{Arcsinh} [z_1(1+z_2^2)^{\frac{1}{2}} \pm z_2(1+z_1^2)^{\frac{1}{2}}]$$

4.6.27

$$\operatorname{Arccosh} z_1 \pm \operatorname{Arccosh} z_2 = \operatorname{Arccosh} \{z_1 z_2 \pm [(z_1^2-1)(z_2^2-1)]^{\frac{1}{2}}\}$$

4.6.28

$$\operatorname{Arctanh} z_1 \pm \operatorname{Arctanh} z_2 = \operatorname{Arctanh} \left(\frac{z_1 \pm z_2}{1 \pm z_1 z_2}\right)$$

4.6.29

$$\operatorname{Arcsinh} z_1 \pm \operatorname{Arccosh} z_2 = \operatorname{Arcsinh} \{z_1 z_2 \pm [(1+z_1^2)(z_2^2-1)]^{\frac{1}{2}}\}$$
$$= \operatorname{Arccosh} [z_2(1+z_1^2)^{\frac{1}{2}} \pm z_1(z_2^2-1)^{\frac{1}{2}}]$$

4.6.30

$$\operatorname{Arctanh} z_1 \pm \operatorname{Arccoth} z_2 = \operatorname{Arctanh} \left(\frac{z_1 z_2 \pm 1}{z_2 \pm z_1}\right)$$
$$= \operatorname{Arccoth} \left(\frac{z_2 \pm z_1}{z_1 z_2 \pm 1}\right)$$

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*See page II.

Series Expansions

4.6.31

$$\operatorname{arcsinh} z = z - \frac{1}{2\cdot 3}z^3 + \frac{1\cdot 3}{2\cdot 4\cdot 5}z^5 - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}z^7 + \ldots \qquad (|z|<1)$$

$$= \ln 2z + \frac{1}{2\cdot 2z^2} - \frac{1\cdot 3}{2\cdot 4\cdot 4z^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6z^6} - \ldots \qquad (|z|>1)$$

4.6.32

$$\operatorname{arccosh} z = \ln 2z - \frac{1}{2\cdot 2z^2} - \frac{1\cdot 3}{2\cdot 4\cdot 4z^4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6z^6} - \ldots \qquad (|z|>1)$$

4.6.33 $\operatorname{arctanh} z = z + \frac{z^3}{3} + \frac{z^5}{5} + \frac{z^7}{7} + \ldots \qquad (|z|<1)$

4.6.34 $\operatorname{arccoth} z = \frac{1}{z} + \frac{1}{3z^3} + \frac{1}{5z^5} + \frac{1}{7z^7} + \ldots \qquad (|z|>1)$

Continued Fractions

4.6.35 $\operatorname{arctanh} z = \frac{z}{1-}\,\frac{z^2}{3-}\,\frac{4z^2}{5-}\,\frac{9z^2}{7-}\ldots$

(z in the cut plane of Figure 4.7.)

4.6.36

$$\frac{\operatorname{arcsinh} z}{\sqrt{1+z^2}} = \frac{z}{1+}\,\frac{1\cdot 2z^2}{3+}\,\frac{1\cdot 2z^2}{5+}\,\frac{3\cdot 4z^2}{7+}\,\frac{3\cdot 4z^2}{9+}\ldots$$

Differentiation Formulas

4.6.37 $\frac{d}{dz}\operatorname{arcsinh} z = (1+z^2)^{-\frac{1}{2}}$

4.6.38 $\frac{d}{dz}\operatorname{arccosh} z = (z^2-1)^{-\frac{1}{2}}$

4.6.39 $\frac{d}{dz}\operatorname{arctanh} z = (1-z^2)^{-1}$

4.6.40 $\frac{d}{dz}\operatorname{arccsch} z = \mp\frac{1}{z(1+z^2)^{\frac{1}{2}}}$

(according as $\mathscr{R}z \gtreqless 0$)

4.6.41 $\frac{d}{dz}\operatorname{arcsech} z = \mp\frac{1}{z(1-z^2)^{\frac{1}{2}}}$

4.6.42 $\frac{d}{dz}\operatorname{arccoth} z = (1-z^2)^{-1}$

Integration Formulas

4.6.43 $\int \operatorname{arcsinh} z\,dz = z\operatorname{arcsinh} z - (1+z^2)^{\frac{1}{2}}$

4.6.44 $\int \operatorname{arccosh} z\,dz = z\operatorname{arccosh} z - (z^2-1)^{\frac{1}{2}}$

4.6.45 $\int \operatorname{arctanh} z\,dz = z\operatorname{arctanh} z + \frac{1}{2}\ln(1-z^2)$

4.6.46 $\int \operatorname{arccsch} z\,dz = z\operatorname{arccsch} z \pm \operatorname{arcsinh} z$ *

(according as $\mathscr{R}z \gtreqless 0$)

4.6.47 $\int \operatorname{arcsech} z\,dz = z\operatorname{arcsech} z \pm \arcsin z$ *

4.6.48 $\int \operatorname{arccoth} z\,dz = z\operatorname{arccoth} z + \frac{1}{2}\ln(z^2-1)$

4.6.49

$$\int z\operatorname{arcsinh} z\,dz = \frac{2z^2+1}{4}\operatorname{arcsinh} z - \frac{z}{4}(z^2+1)^{\frac{1}{2}}$$

4.6.50

$$\int z^n\operatorname{arcsinh} z\,dz = \frac{z^{n+1}}{n+1}\operatorname{arcsinh} z - \frac{1}{n+1}\int\frac{z^{n+1}}{(1+z^2)^{\frac{1}{2}}}dz \qquad (n\neq -1)$$

4.6.51

$$\int z\operatorname{arccosh} z\,dz = \frac{2z^2-1}{4}\operatorname{arccosh} z - \frac{z}{4}(z^2-1)^{\frac{1}{2}}$$

4.6.52

$$\int z^n\operatorname{arccosh} z\,dz = \frac{z^{n+1}}{n+1}\operatorname{arccosh} z - \frac{1}{n+1}\int\frac{z^{n+1}}{(z^2-1)^{\frac{1}{2}}}dz \qquad (n\neq -1)$$

4.6.53

$$\int z\operatorname{arctanh} z\,dz = \frac{z^2-1}{2}\operatorname{arctanh} z + \frac{z}{2}$$

4.6.54

$$\int z^n\operatorname{arctanh} z\,dz = \frac{z^{n+1}}{n+1}\operatorname{arctanh} z - \frac{1}{n+1}\int\frac{z^{n+1}}{1-z^2}dz \qquad (n\neq -1)$$

4.6.55

$$\int z\operatorname{arccsch} z\,dz = \frac{z^2}{2}\operatorname{arccsh} z \pm \frac{1}{2}(1+z^2)^{\frac{1}{2}} \qquad *$$

(according as $\mathscr{R}z \gtreqless 0$)

4.6.56

$$\int z^n\operatorname{arccsch} z\,dz = \frac{z^{n+1}}{n+1}\operatorname{arccsch} z \pm \frac{1}{n+1}\int\frac{z^n}{(z^2+1)^{\frac{1}{2}}}dz \qquad *$$

$(n\neq -1)$

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*See page II.

* **4.6.57**
$$\int z\,\operatorname{arcsech} z\,dz=\frac{z^2}{2}\operatorname{arcsech} z\mp\frac{1}{2}(1-z^2)^{\frac{1}{2}}$$
(according as $\mathscr{R}z\gtrless 0$)

4.6.58
$$\int z^n\operatorname{arcsech} z\,dz=\frac{z^{n+1}}{n+1}\operatorname{arcsech} z\pm\frac{1}{n+1}\int\frac{z^n}{(1-z^2)^{\frac{1}{2}}}\,dz$$
$(n\neq-1)$

4.6.59
$$\int z\,\operatorname{arccoth} z\,dz=\frac{z^2-1}{2}\operatorname{arccoth} z+\frac{z}{2}$$

4.6.60
$$\int z^n\operatorname{arccoth} z\,dz=\frac{z^{n+1}}{n+1}\operatorname{arccoth} z+\frac{1}{n+1}\int\frac{z^{n+1}}{z^2-1}\,dz$$
$(n\neq-1)$

Numerical Methods

4.7. Use and Extension of the Tables

NOTE: In the examples given it is assumed that the arguments are exact.

Example 1. Computation of Common Logarithms.

To compute common logarithms, the number must be expressed in the form $x\cdot 10^q$, $(1\leq x<10, -\infty\leq q\leq\infty)$. The common logarithm of $x\cdot 10^q$ consists of an integral part which is called the characteristic and a decimal part which is called the mantissa. **Table 4.1** gives the common logarithm of x.

x	$x\cdot 10^q$	$\log_{10} x\cdot 10^q$
.009836	$9.836\cdot 10^{-3}$	$\bar{3}.99281\ 85=(-2.00718\ 15)$
.09836	$9.836\cdot 10^{-2}$	$\bar{2}.99281\ 85=(-1.00718\ 15)$
.9836	$9.836\cdot 10^{-1}$	$\bar{1}.99281\ 85=(-0.00718\ 15)$
9.836	$9.836\cdot 10^{0}$	0.99281 85
98.36	$9.836\cdot 10^{1}$	1.99281 85
983.6	$9.836\cdot 10^{2}$	2.99281 85

Interpolation in **Table 4.1** between 983 and 984 gives .99281 85 as the mantissa of 9836.

Note that $\bar{3}.99281\ 85=-3+.99281\ 85$. When q is negative the common logarithm can be expressed in the alternative forms

$$\log_{10}(.009836)=\bar{3}.99281\ 85=7.99281\ 85\text{–}10$$
$$=-2.00718\ 15.$$

The last form is convenient for conversion from common logarithms to natural logarithms.

The inverse of $\log_{10} x$ is called the antilogarithm of x, and is written antilog x or $\log^{-1} x$. The logarithm of the reciprocal of a number is called the cologarithm, written colog.

Example 2.

Compute $x^{-3/4}$ for $x=9.19826$ to 10D using the Table of Common Logarithms.

From **Table 4.1,** four-point Lagrangian interpolation gives $\log_{10}(9.19826)=.96370\ 56812$. Then, $-\frac{3}{4}\log_{10}(x)=-.72277\ 92609=9.27722\ 07391-10$. Linear inverse interpolation in **Table 4.1** yields antilog $(\bar{1}.27722)=.18933$. For 10 place accuracy subtabulation with 4-point Lagrangian interpolants produces the table

N	$\log_{10} N$	Δ	Δ^2
.18933	.27721 94350	2 29379	
.18934	.27724 23729		-13
.18935	.27726 53095	2 29366	

By linear inverse interpolation

$$x^{-3/4}=.18933\ 05685.$$

Example 3.

Convert $\log_{10} x$ to $\ln x$ for $x=.009836$.

Using **4.1.23** and **Table 4.1,** $\ln(.009836)=\ln 10\log_{10}(.009836)=2.30258\ 5093\ (-2.00718\ 15)=-4.62170\ 62$.

Example 4.

Compute $\ln x$ for $x=.00278$ to 6D.

Using **4.1.7, 4.1.11** and **Table 4.2,** $\ln(.00278)=\ln(.278\cdot 10^{-2})=\ln(.278)-2\ln 10=-5.885304$.

Linear interpolation between $x=.002$ and $x=.003$ would give $\ln(.00278)=-5.8\mathit{98}$. To obtain 5 decimal place accuracy with linear interpolation it is necessary that $x>.175$.

Example 5.

Compute $\ln x$ for $x=1131.718$ to 8D.

Using **4.1.7, 4.1.11** and **Table 4.2**

$$\ln 1131.718=\ln\left(\frac{1131.718}{1131}1131\right)$$
$$=\ln\frac{1131.718}{1131}+\ln 1.131+\ln 10^3$$
$$=\ln(1.00063\ 4836)+\ln 1.131+3\ln 10.$$

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*See page II.

Example 25.

Compute arcsec 2.8 to 5D.
Using **4.3.45** and **Table 4.14**

$$\operatorname{arcsec} z=\arcsin \frac{(z^2-1)^{\frac{1}{2}}}{z}$$

$$\operatorname{arcsec} 2.8=\arcsin \frac{[(2.8)^2-1]^{\frac{1}{2}}}{2.8}$$

$$=\arcsin .93404\ 97735$$

$$=1.20559$$

or using **4.3.45** and **Table 4.14**

$$\operatorname{arcsec} z=\arctan\ (z^2-1)^{\frac{1}{2}}$$

$$\operatorname{arcsec} 2.8=\arctan 2.61533\ 9366$$

$$=\frac{\pi}{2}-\arctan .38235\ 95564,$$

from **4.4.3** and **4.4.8**

$$=1.570796-.365207$$

$$=1.20559.$$

Example 26.

Compute arctanh x for $x=.96035$ to 6D.
From **4.6.22** and **Table 4.2**

$$\operatorname{arctanh} .96035=\tfrac{1}{2}\ln\frac{1+.96035}{1-.96035}=\tfrac{1}{2}\ln\frac{1.96035}{.03965}$$

$$=\tfrac{1}{2}\ln 49.44136\ 191$$

$$=\tfrac{1}{2}(3.90078\ 7359)=1.950394.$$

Example 27.

Compute arccosh x for $x=1.5368$ to 6D.
Using **Table 4.17**

$$\frac{\operatorname{arccosh} x}{(x^2-1)^{\frac{1}{2}}}=\frac{\operatorname{arccosh} 1.5368}{[(1.5368)^2-1]^{\frac{1}{2}}}=.852346$$

$$\operatorname{arccosh} 1.5368=(.852346)(1.361754)^{\frac{1}{2}}$$

$$=(.852346)(1.166942)$$

$$=.994638.$$

Example 28.

Compute arccosh x for $x=31.2$ to 5D.
Using **Tables 4.2** and **4.17** with $1/x=1/31.2$ $=.03205\ 128205$

$$\operatorname{arccosh} 31.2-\ln 31.2=.692886$$

$$\operatorname{arccosh} 31.2=.692886+3.440418=4.13330.$$

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*See page II.

5. Exponential Integral and Related Functions

WALTER GAUTSCHI [1] AND WILLIAM F. CAHILL [2]

Contents

The authors acknowledge the assistance of David S. Liepman in the preparation and checking of the tables, Robert L. Durrah for the computation of **Table 5.2**, and Alfred E. Beam for the computation of **Table 5.6**.

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[2] National Bureau of Standards. (Presently NASA.)

5. Exponential Integral and Related Functions

Mathematical Properties

5.1. Exponential Integral

Definitions

5.1.1 $$E_1(z)=\int_z^\infty \frac{e^{-t}}{t}\,dt \qquad (|\arg z|<\pi)$$

5.1.2 $$\mathrm{Ei}(x)=-\fint_{-x}^\infty \frac{e^{-t}}{t}\,dt=\fint_{-\infty}^{x} \frac{e^{t}}{t}\,dt \qquad (x>0)$$

5.1.3 $$\mathrm{li}(x)=\fint_0^x \frac{dt}{\ln t}=\mathrm{Ei}(\ln x) \qquad (x>1)$$

5.1.4

$$E_n(z)=\int_1^\infty \frac{e^{-zt}}{t^n}\,dt \qquad (n=0,1,2,\ldots;\ \mathscr{R}z>0)$$

5.1.5

$$\alpha_n(z)=\int_1^\infty t^n e^{-zt}dt \qquad (n=0,1,2,\ldots;\ \mathscr{R}z>0)$$

5.1.6 $$\beta_n(z)=\int_{-1}^1 t^n e^{-zt}dt \qquad (n=0,1,2,\ldots)$$

In **5.1.1** it is assumed that the path of integration excludes the origin and does not cross the negative real axis.

Analytic continuation of the functions in **5.1.1**, **5.1.2**, and **5.1.4** for $n>0$ yields multi-valued functions with branch points at $z=0$ and $z=\infty$.[3] They are single-valued functions in the z-plane cut along the negative real axis.[4] The function $\mathrm{li}(z)$, the logarithmic integral, has an additional branch point at $z=1$.

Interrelations

5.1.7

$$E_1(-x\pm i0)=-\mathrm{Ei}(x)\mp i\pi,$$
$$-\mathrm{Ei}(x)=\tfrac{1}{2}[E_1(-x+i0)+E_1(-x-i0)] \qquad (x>0)$$

Explicit Expressions for $\alpha_n(z)$ and $\beta_n(z)$

5.1.8 $$\alpha_n(z)=n!z^{-n-1}e^{-z}\left(1+z+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}\right)$$

5.1.9

$$\beta_n(z)=n!z^{-n-1}\left\{e^z\left[1-z+\frac{z^2}{2!}-\ldots+(-1)^n\frac{z^n}{n!}\right]-e^{-z}\left(1+z+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}\right)\right\}$$

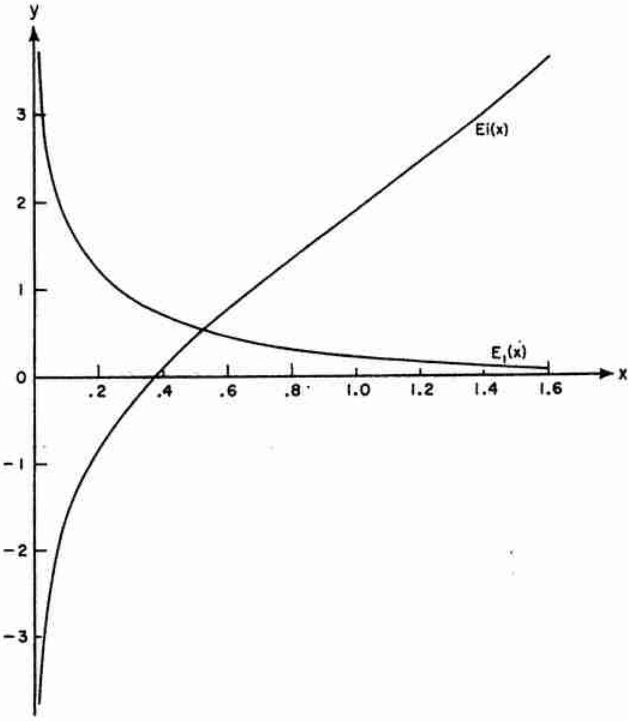

FIGURE 5.1. $y=\mathrm{Ei}(x)$ *and* $y=E_1(x)$.

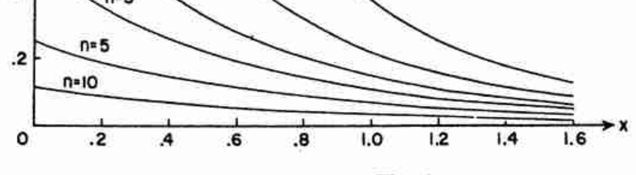

FIGURE 5.2. $y=E_n(x)$
$n=0, 1, 2, 3, 5, 10$

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[3] Some authors [5.14], [5.16] use the entire function $\int_0^z (1-e^{-t})dt/t$ as the basic function and denote it by $\mathrm{Ein}(z)$. We have $\mathrm{Ein}(z)=E_1(z)+\ln z+\gamma$.

[4] Various authors define the integral $\int_{-\infty}^{z}(e^t/t)dt$ in the z-plane cut along the positive real axis and denote it also by $\mathrm{Ei}(z)$. For $z=x>0$ additional notations such as $\overline{\mathrm{Ei}}(x)$ (e.g., in [5.10], [5.25]), $E^*(x)$ (in [5.2]), $\mathrm{Ei}^*(x)$ (in [5.6]) are then used to designate the principal value of the integral. Correspondingly, $E_1(x)$ is often denoted by $-\mathrm{Ei}(-x)$.

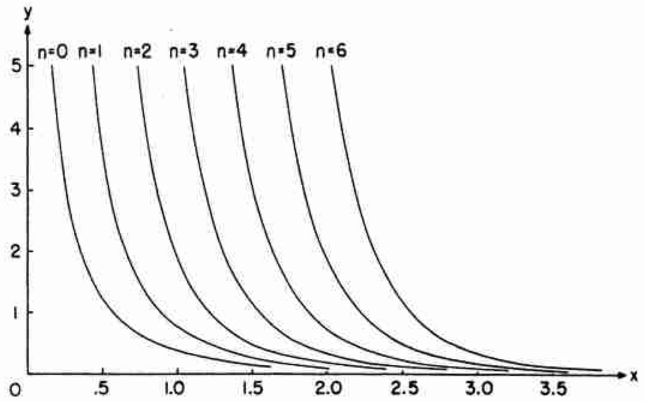

FIGURE 5.3. $y=\alpha_n(x)$
$n=0(1)6$

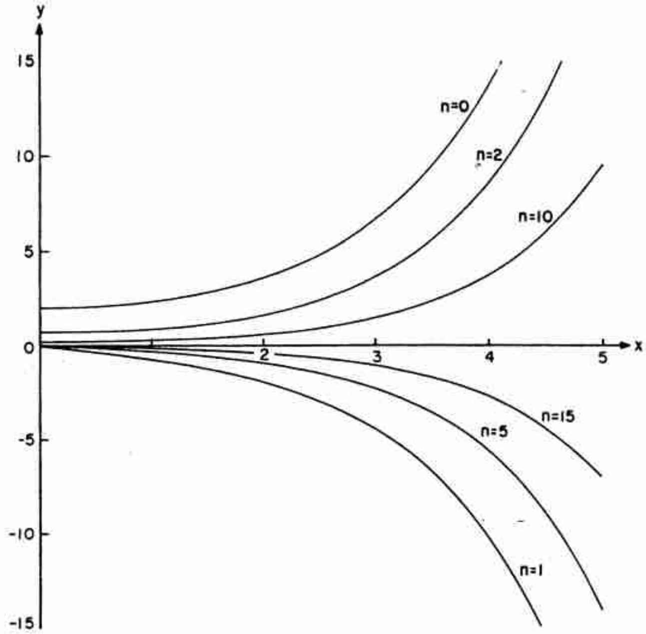

FIGURE 5.4. $y=\beta_n(x)$
$n=0, 1, 2, 5, 10, 15$

Series Expansions

5.1.10 $$\mathrm{Ei}(x)=\gamma+\ln x+\sum_{n=1}^{\infty}\frac{x^n}{nn!} \qquad (x>0)$$

5.1.11
$$E_1(z)=-\gamma-\ln z-\sum_{n=1}^{\infty}\frac{(-1)^n z^n}{nn!} \qquad (|\arg z|<\pi)$$

5.1.12
$$E_n(z)=\frac{(-z)^{n-1}}{(n-1)!}[-\ln z+\psi(n)]-\sum_{\substack{m=0\\ m\neq n-1}}^{\infty}{}' \frac{(-z)^m}{(m-n+1)m!}$$
$$(|\arg z|<\pi)$$

$$\psi(1)=-\gamma,\ \psi(n)=-\gamma+\sum_{m=1}^{n-1}\frac{1}{m} \qquad (n>1)$$

$\gamma=.57721\ 56649\ \ldots$ is Euler's constant.

Symmetry Relation

5.1.13 $$E_n(\bar{z})=\overline{E_n(z)}$$

Recurrence Relations

5.1.14
$$E_{n+1}(z)=\frac{1}{n}[e^{-z}-zE_n(z)] \quad (n=1,2,3,\ldots)$$

5.1.15 $$z\alpha_n(z)=e^{-z}+n\alpha_{n-1}(z) \qquad (n=1,2,3,\ldots)$$

5.1.16
$$z\beta_n(z)=(-1)^n e^z-e^{-z}+n\beta_{n-1}(z) \qquad (n=1,2,3,\ldots)$$

Inequalities [5.8], [5.4]

5.1.17
$$\frac{n-1}{n}E_n(x)<E_{n+1}(x)<E_n(x) \quad (x>0; n=1,2,3,\ldots)$$

5.1.18
$$E_n^2(x)<E_{n-1}(x)E_{n+1}(x) \qquad (x>0; n=1,2,3,\ldots)$$

5.1.19
$$\frac{1}{x+n}<e^xE_n(x)\leq\frac{1}{x+n-1} \qquad (x>0; n=1,2,3,\ldots)$$

5.1.20
$$\tfrac{1}{2}\ln\left(1+\frac{2}{x}\right)<e^xE_1(x)<\ln\left(1+\frac{1}{x}\right) \qquad (x>0)$$

5.1.21
$$\frac{d}{dx}\left[\frac{E_n(x)}{E_{n-1}(x)}\right]>0 \qquad (x>0; n=1,2,3,\ldots)$$

Continued Fraction

5.1.22
$$E_n(z)=e^{-z}\left(\frac{1}{z+}\,\frac{n}{1+}\,\frac{1}{z+}\,\frac{n+1}{1+}\,\frac{2}{z+}\cdots\right) \quad (|\arg z|<\pi)$$

Special Values

5.1.23 $$E_n(0)=\frac{1}{n-1} \qquad (n>1)$$

5.1.24 $$E_0(z)=\frac{e^{-z}}{z}$$

5.1.25 $$\alpha_0(z)=\frac{e^{-z}}{z},\ \beta_0(z)=\frac{2}{z}\sinh z$$

Derivatives

5.1.26 $$\frac{dE_n(z)}{dz}=-E_{n-1}(z) \qquad (n=1,2,3,\ldots)$$

5.1.27
$$\frac{d^n}{dz^n}[e^zE_1(z)]=\frac{d^{n-1}}{dz^{n-1}}[e^zE_1(z)]+\frac{(-1)^n(n-1)!}{z^n} \quad (n=1,2,3,\ldots)$$

Definite and Indefinite Integrals

(For more extensive tables of integrals see [5.3], [5.6], [5.11], [5.12], [5.13]. For integrals involving $E_n(x)$ see [5.9].)

5.1.28 $$\int_0^\infty \frac{e^{-at}}{b+t}\,dt=e^{ab}E_1(ab)$$

5.1.29
$$\int_0^\infty \frac{e^{iat}}{b+t}\,dt=e^{-iab}E_1(-iab) \qquad (a>0,\, b>0)$$

5.1.30
$$\int_0^\infty \frac{t-ib}{t^2+b^2}\,e^{iat}dt=e^{ab}E_1(ab) \qquad (a>0,\, b>0)$$

5.1.31
$$\int_0^\infty \frac{t+ib}{t^2+b^2}\,e^{iat}dt=e^{-ab}(-\mathrm{Ei}(ab)+i\pi) \qquad (a>0,\, b>0)$$

5.1.32 $$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt=\ln\frac{b}{a}$$

5.1.33 $$\int_0^\infty E_1^2(t)dt=2\ln 2$$

5.1.34
$$\int_0^\infty e^{-at}E_n(t)dt=\frac{(-1)^{n-1}}{a^n}[\ln(1+a)+\sum_{k=1}^{n-1}\frac{(-1)^ka^k}{k}] \qquad (a>-1)$$

5.1.35
$$\int_0^1 \frac{e^{at}\sin bt}{t}\,dt=\pi-\arctan\frac{b}{a}+\mathscr{I}E_1(-a+ib) \qquad (a>0, b>0)$$

5.1.36
$$\int_0^1 \frac{e^{-at}\sin bt}{t}\,dt=\arctan\frac{b}{a}+\mathscr{I}E_1(a+ib) \qquad (a>0, b \text{ real})$$

5.1.37
$$\int_0^1 \frac{e^{at}(1-\cos bt)}{t}\,dt=\frac{1}{2}\ln\left(1+\frac{b^2}{a^2}\right)+\mathrm{Ei}(a)+\mathscr{R}E_1(-a+ib) \quad (a>0, b \text{ real})$$

5.1.38
$$\int_0^1 \frac{e^{-at}(1-\cos bt)}{t}\,dt=\frac{1}{2}\ln\left(1+\frac{b^2}{a^2}\right)-E_1(a)+\mathscr{R}E_1(a+ib) \qquad (a>0, b \text{ real})$$

5.1.39 $$\int_0^z \frac{1-e^{-t}}{t}\,dt=E_1(z)+\ln z+\gamma$$

5.1.40 $$\int_0^x \frac{e^t-1}{t}\,dt=\mathrm{Ei}\,(x)-\ln x-\gamma \qquad (x>0)$$

5.1.41
$$\int \frac{e^{ix}}{a^2+x^2}\,dx=\frac{i}{2a}[e^{-a}E_1(-a-ix)-e^aE_1(a-ix)]+\text{const.}$$

5.1.42
$$\int \frac{xe^{ix}}{a^2+x^2}\,dx=-\frac{1}{2}[e^{-a}E_1(-a-ix)+e^aE_1(a-ix)]+\text{const.}$$

5.1.43
$$\int \frac{e^x}{a^2+x^2}\,dx=-\frac{1}{a}\,\mathscr{I}(e^{ia}E_1(-x+ia))+\text{const.} \quad (a>0)$$

5.1.44
$$\int \frac{xe^x}{a^2+x^2}\,dx=-\mathscr{R}(e^{ia}E_1(-x+ia))+\text{const.} \qquad (a>0)$$

Relation to Incomplete Gamma Function (see 6.5)

5.1.45 $$E_n(z)=z^{n-1}\Gamma(1-n,\, z)$$

5.1.46 $$\alpha_n(z)=z^{-n-1}\Gamma(n+1,\, z)$$

5.1.47 $$\beta_n(z)=z^{-n-1}[\Gamma(n+1,\, -z)-\Gamma(n+1,\, z)]$$

Relation to Spherical Bessel Functions (see 10.2)

5.1.48 $$\alpha_0(z)=\sqrt{\frac{2}{\pi z}}\,K_{\frac{1}{2}}(z),\ \beta_0(z)=\sqrt{\frac{2\pi}{z}}\,I_{\frac{1}{2}}(z)$$

5.1.49 $$\alpha_1(z)=\sqrt{\frac{2}{\pi z}}\,K_{3/2}(z),\ \beta_1(z)=-\sqrt{\frac{2\pi}{z}}\,I_{3/2}(z)$$

Number-Theoretic Significance of li (x)

(Assuming Riemann's hypothesis that all non-real zeros of $\zeta(z)$ have a real part of $\frac{1}{2}$)

5.1.50 $\quad \text{li}\,(x)-\pi(x)=O(\sqrt{x}\ln x) \qquad (x\to\infty)$

$\pi(x)$ is the number of primes less than or equal to x.

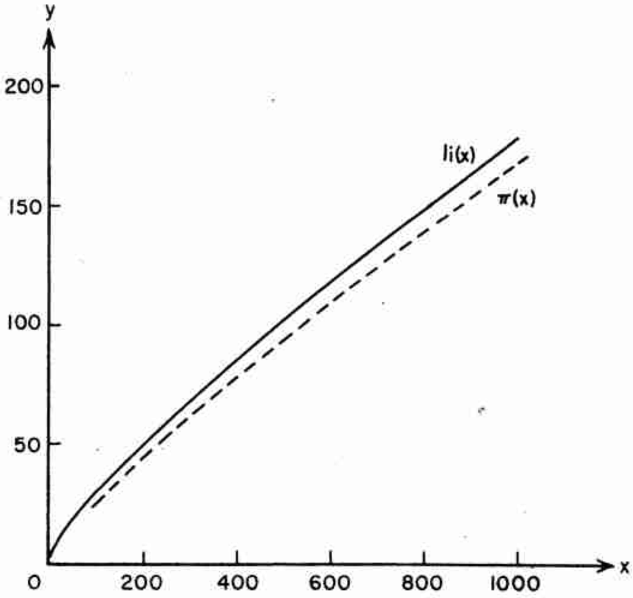

FIGURE 5.5. $y=\text{li}(x)$ *and* $y=\pi(x)$

Asymptotic Expansion

5.1.51

$$E_n(z)\sim\frac{e^{-z}}{z}\left\{1-\frac{n}{z}+\frac{n(n+1)}{z^2}-\frac{n(n+1)(n+2)}{z^3}+\ldots\right\} \qquad (|\arg z|<\tfrac{3}{2}\pi)$$

Representation of $E_n(x)$ for Large n

5.1.52

$$E_n(x)=\frac{e^{-x}}{x+n}\left\{1+\frac{n}{(x+n)^2}+\frac{n(n-2x)}{(x+n)^4}+\frac{n(6x^2-8nx+n^2)}{(x+n)^6}+R(n,x)\right\}$$

$$-.36n^{-4}\leq R(n,x)\leq\left(1+\frac{1}{x+n-1}\right)n^{-4} \quad (x>0)$$

Polynomial and Rational Approximations[5]

5.1.53 $\quad 0\leq x\leq 1$

$$E_1(x)+\ln x=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\epsilon(x)$$

$$|\epsilon(x)|<2\times10^{-7}$$

$a_0=-.57721\ 566 \qquad a_3=\ \ .05519\ 968$

$a_1=\ \ .99999\ 193 \qquad a_4=-.00976\ 004$

$a_2=-.24991\ 055 \qquad a_5=\ \ .00107\ 857$

5.1.54 $\quad 1\leq x<\infty$

$$xe^xE_1(x)=\frac{x^2+a_1x+a_2}{x^2+b_1x+b_2}+\epsilon(x)$$

$$|\epsilon(x)|<5\times10^{-5}$$

$a_1=2.334733 \qquad b_1=3.330657$

$a_2=\ .250621 \qquad b_2=1.681534$

5.1.55 $\quad 10\leq x<\infty$

$$xe^xE_1(x)=\frac{x^2+a_1x+a_2}{x^2+b_1x+b_2}+\epsilon(x)$$

$$|\epsilon(x)|<10^{-7}$$

$a_1=4.03640 \qquad b_1=5.03637$

$a_2=1.15198 \qquad b_2=4.19160$

5.1.56 $\quad 1\leq x<\infty$

$$xe^xE_1(x)=\frac{x^4+a_1x^3+a_2x^2+a_3x+a_4}{x^4+b_1x^3+b_2x^2+b_3x+b_4}+\epsilon(x)$$

$$|\epsilon(x)|<2\times10^{-8}$$

$a_1=\ 8.57332\ 87401 \qquad b_1=\ 9.57332\ 23454$

$a_2=18.05901\ 69730 \qquad b_2=25.63295\ 61486$

$a_3=\ 8.63476\ 08925 \qquad b_3=21.09965\ 30827$

$a_4=\ \ .26777\ 37343 \qquad b_4=\ 3.95849\ 69228$

5.2. Sine and Cosine Integrals

Definitions

5.2.1 $\quad \text{Si}(z)=\int_0^z\frac{\sin t}{t}\,dt$

5.2.2[6]

$$\text{Ci}(z)=\gamma+\ln z+\int_0^z\frac{\cos t-1}{t}\,dt \qquad (|\arg z|<\pi)$$

5.2.3[7] $\quad \text{Shi}(z)=\int_0^z\frac{\sinh t}{t}\,dt$

5.2.4[7]

$$\text{Chi}(z)=\gamma+\ln z+\int_0^z\frac{\cosh t-1}{t}\,dt \qquad (|\arg z|<\pi)$$

---

[5] The approximation **5.1.53** is from E. E. Allen, Note 169, MTAC 8, 240 (1954); approximations **5.1.54** and **5.1.56** are from C. Hastings, Jr., Approximations for digital computers, Princeton Univ. Press, Princeton, N.J., 1955; approximation **5.1.55** is from C. Hastings, Jr., Note 143, MTAC 7, 68 (1953) (with permission).

[6] Some authors [5.14], [5.16] use the entire function $\int_0^z(1-\cos t)dt/t$ as the basic function and denote it by $\text{Cin}(z)$. We have

$$\text{Cin}(z)=-\text{Ci}(z)+\ln z+\gamma.$$

[7] The notations $\text{Sih}(z)=\int_0^z\sinh t\,dt/t$, $\text{Cinh}(z)=\int_0^z(\cosh t-1)dt/t$ have also been proposed [5.14.]

5.2.5 $$\mathrm{si}(z)=\mathrm{Si}(z)-\frac{\pi}{2}$$

Auxiliary Functions

5.2.6 $$f(z)=\mathrm{Ci}(z)\sin z-\mathrm{si}(z)\cos z$$

5.2.7 $$g(z)=-\mathrm{Ci}(z)\cos z-\mathrm{si}(z)\sin z$$

Sine and Cosine Integrals in Terms of Auxiliary Functions

5.2.8 $$\mathrm{Si}(z)=\frac{\pi}{2}-f(z)\cos z-g(z)\sin z$$

5.2.9 $$\mathrm{Ci}(z)=f(z)\sin z-g(z)\cos z$$

Integral Representations

5.2.10 $$\mathrm{si}(z)=-\int_0^{\frac{\pi}{2}} e^{-z\cos t}\cos(z\sin t)dt$$

5.2.11 $$\mathrm{Ci}(z)+E_1(z)=\int_0^{\frac{\pi}{2}} e^{-z\cos t}\sin(z\sin t)dt$$

5.2.12 $$f(z)=\int_0^\infty \frac{\sin t}{t+z}\,dt=\int_0^\infty \frac{e^{-zt}}{t^2+1}\,dt \qquad (\mathscr{R}z>0)$$

5.2.13 $$g(z)=\int_0^\infty \frac{\cos t}{t+z}\,dt=\int_0^\infty \frac{te^{-zt}}{t^2+1}\,dt \qquad (\mathscr{R}z>0)$$

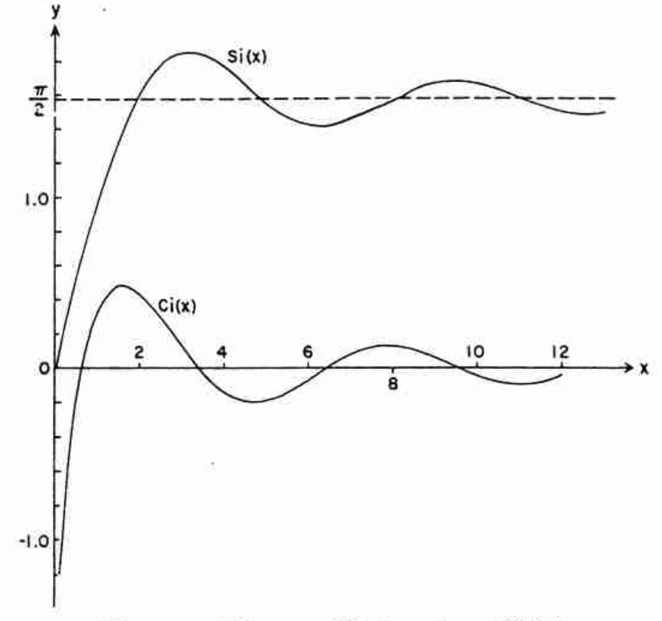

FIGURE 5.6. $y=\mathrm{Si}(x)$ *and* $y=\mathrm{Ci}(x)$

Series Expansions

5.2.14 $$\mathrm{Si}(z)=\sum_{n=0}^\infty \frac{(-1)^n z^{2n+1}}{(2n+1)(2n+1)!}$$

5.2.15 $$\mathrm{Si}(z)=\pi\sum_{n=0}^\infty J^2_{n+\frac{1}{2}}\left(\frac{z}{2}\right)$$

5.2.16 $$\mathrm{Ci}(z)=\gamma+\ln z+\sum_{n=1}^\infty \frac{(-1)^n z^{2n}}{2n(2n)!}$$

5.2.17 $$\mathrm{Shi}(z)=\sum_{n=0}^\infty \frac{z^{2n+1}}{(2n+1)(2n+1)!}$$

5.2.18 $$\mathrm{Chi}(z)=\gamma+\ln z+\sum_{n=1}^\infty \frac{z^{2n}}{2n(2n)!}$$

Symmetry Relations

5.2.19 $$\mathrm{Si}(-z)=-\mathrm{Si}(z),\ \mathrm{Si}(\bar{z})=\overline{\mathrm{Si}(z)}$$

5.2.20
$$\mathrm{Ci}(-z)=\mathrm{Ci}(z)-i\pi \qquad (0<\arg z<\pi)$$
$$\mathrm{Ci}(\bar{z})=\overline{\mathrm{Ci}(z)}$$

Relation to Exponential Integral

5.2.21
$$\mathrm{Si}(z)=\frac{1}{2i}[E_1(iz)-E_1(-iz)]+\frac{\pi}{2} \qquad \left(|\arg z|<\frac{\pi}{2}\right)$$

5.2.22 $$\mathrm{Si}(ix)=\frac{i}{2}[\mathrm{Ei}(x)+E_1(x)] \qquad (x>0)$$

5.2.23
$$\mathrm{Ci}(z)=-\frac{1}{2}[E_1(iz)+E_1(-iz)] \qquad \left(|\arg z|<\frac{\pi}{2}\right)$$

5.2.24 $$\mathrm{Ci}(ix)=\frac{1}{2}[\mathrm{Ei}(x)-E_1(x)]+i\frac{\pi}{2} \qquad (x>0)$$

Value at Infinity

5.2.25 $$\lim_{x\to\infty}\mathrm{Si}(x)=\frac{\pi}{2}$$

Integrals

(For more extensive tables of integrals see [5.3], [5.6], [5.11], [5.12], [5.13].)

5.2.26 $$\int_z^\infty \frac{\sin t}{t}\,dt=-\mathrm{si}(z) \qquad (|\arg z|<\pi)$$

5.2.27 $$\int_z^\infty \frac{\cos t}{t}\,dt=-\mathrm{Ci}(z) \qquad (|\arg z|<\pi)$$

5.2.28 $$\int_0^\infty e^{-at}\mathrm{Ci}(t)dt=-\frac{1}{2a}\ln(1+a^2) \qquad (\mathscr{R}a>0)^*$$

5.2.29 $$\int_0^\infty e^{-at}\,\mathrm{si}(t)dt=-\frac{1}{a}\arctan a \qquad (\mathscr{R}a>0)$$

5.2.30 $$\int_0^\infty \cos t\,\mathrm{Ci}(t)dt=\int_0^\infty \sin t\,\mathrm{si}(t)dt=-\frac{\pi}{4}$$

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*See page II.

5.2.31 $$\int_0^\infty \mathrm{Ci}^2(t)\,dt=\int_0^\infty \mathrm{si}^2(t)\,dt=\frac{\pi}{2}$$

5.2.32* $$\int_0^\infty \mathrm{Ci}(t)\,\mathrm{si}(t)\,dt=\ln 2$$

5.2.33

$$\int_0^1 \frac{(1-e^{-at})\cos bt}{t}\,dt=\frac{1}{2}\ln\left(1+\frac{a^2}{b^2}\right)+\mathrm{Ci}(b)$$
$$+\mathscr{R}E_1(a+ib)\quad (a \text{ real}, b>0)$$

Asymptotic Expansions

5.2.34

$$f(z)\sim\frac{1}{z}\left(1-\frac{2!}{z^2}+\frac{4!}{z^4}-\frac{6!}{z^6}+\ldots\right)\qquad (|\arg z|<\pi)$$

5.2.35

$$g(z)\sim\frac{1}{z^2}\left(1-\frac{3!}{z^2}+\frac{5!}{z^4}-\frac{7!}{z^6}+\ldots\right)\qquad (|\arg z|<\pi)$$

Rational Approximations [8]

5.2.36 $$1\leq x<\infty$$

$$f(x)=\frac{1}{x}\left(\frac{x^4+a_1x^2+a_2}{x^4+b_1x^2+b_2}\right)+\epsilon(x)$$

$$|\epsilon(x)|<2\times 10^{-4}$$

$a_1=7.241163 \qquad b_1=9.068580$

$a_2=2.463936 \qquad b_2=7.157433$

5.2.37 $$1\leq x<\infty$$

$$g(x)=\frac{1}{x^2}\left(\frac{x^4+a_1x^2+a_2}{x^4+b_1x^2+b_2}\right)+\epsilon(x)$$

$$|\epsilon(x)|<10^{-4}$$

$a_1=7.547478 \qquad b_1=12.723684$ *

$a_2=1.564072 \qquad b_2=15.723606$ *

5.2.38 $$1\leq x<\infty$$

$$f(x)=\frac{1}{x}\left(\frac{x^8+a_1x^6+a_2x^4+a_3x^2+a_4}{x^8+b_1x^6+b_2x^4+b_3x^2+b_4}\right)+\epsilon(x)$$

$$|\epsilon(x)|<5\times 10^{-7}$$

$a_1=\ 38.027264 \qquad b_1=\ 40.021433$

$a_2=265.187033 \qquad b_2=322.624911$

$a_3=335.677320 \qquad b_3=570.236280$

$a_4=\ 38.102495 \qquad b_4=157.105423$

5.2.39 $$1\leq x<\infty$$

$$g(x)=\frac{1}{x^2}\left(\frac{x^8+a_1x^6+a_2x^4+a_3x^2+a_4}{x^8+b_1x^6+b_2x^4+b_3x^2+b_4}\right)+\epsilon(x)$$

$$|\epsilon(x)|<3\times 10^{-7}$$

$a_1=\ 42.242855 \qquad b_1=\ 48.196927$

$a_2=302.757865 \qquad b_2=\ 482.485984$

$a_3=352.018498 \qquad b_3=1114.978885$

$a_4=\ 21.821899 \qquad b_4=\ 449.690326$

Numerical Methods

5.3. Use and Extension of the Tables

Example 1. Compute Ci (.25) to 5D.

From **Tables 5.1** and **4.2** we have

$$\frac{\mathrm{Ci}(.25)-\ln(.25)-\gamma}{(.25)^2}=-.249350,$$

$$\mathrm{Ci}(.25)=(.25)^2(-.249350)+(-1.38629)$$
$$+.577216=-.82466.$$

Example 2. Compute Ei (8) to 5S.

From **Table 5.1** we have $xe^{-x}\mathrm{Ei}(x)=1.18185$ for $x=8$. From **Table 4.4,** $e^8=2.98096\times 10^3$. Thus Ei $(8)=440.38$.

Example 3. Compute Si (20) to 5D.

Since $1/20=.05$ from **Table 5.2** we find $f(20)=.049757$, $g(20)=.002464$. From **Table 4.8,** sin $20=.912945$, cos $20=.408082$. Using **5.2.8**

$$\mathrm{Si}(20)=\frac{\pi}{2}-f(20)\cos 20-g(20)\sin 20$$
$$=1.570796-.022555=1.54824.$$

Example 4. Compute $E_n(x)$, $n=1(1)N$, to 5S for $x=1.275$, $N=10$.

If x is less than about five, the recurrence relation **5.1.14** can be used in increasing order of n without serious loss of accuracy.

By quadratic interpolation in **Table 5.1** we get $E_1(1.275)=.1408099$, and from **Table 4.4,** $e^{-1.275}=.2794310$. The recurrence formula **5.1.14** then yields

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*See page II.

[8] From C. Hastings, Jr., Approximations for digital computers, Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

n	$E_n(1.275)$		$E_n(1.275)$
1	.1408099	6	.0430168
2	.0998984	7	.0374307
3	.0760303	8	.0331009
4	.0608307	9	.0296534
5	.0504679	10	.0268469

Interpolating directly in **Table 5.4** for $n=10$ we get $E_{10}(1.275)=.0268470$ as a check.

Example 5. Compute $E_n(x)$, $n=1(1)N$, to 5S for $x=10$, $N=10$.

If, as in this example, x is appreciably larger than five and $N\leq x$, then the recurrence relation **5.1.14** may be safely used in decreasing order of n([5.5]). From **Table 5.5** for $x^{-1}=.1$ we get $(x+10)e^xE_{10}(x)=1.02436$ so that $E_{10}(10)=2.32529\times10^{-6}$. Using this as the initial value we obtain column (2).

n	$10^5E_n(10)$ (1)	$10^5E_n(10)$ (2)
1	.41570	.41570
2	$.3830\underline{0}$	.38302
3	$.355\underline{00}$	.35488
4	$.330\underline{00}$	.33041
5	$.31\underline{000}$	.30898
6	$.28\underline{800}$	.29005
7	$.2\underline{7667}$	.27325
8	$.2\underline{5333}$	.25822
9	$.2\underline{5084}$	.24472
10	$.2\underline{2573}$	.23253

From **Table 5.2** we get $xe^xE_1(x)=.915633$ so that $E_1(10)=4.15697\times10^{-6}$ as a check. Forward recurrence starting with $E_1(10)=4.1570\times10^{-6}$ yields the values in column (1). The underlined figures are in error.

Example 6. Compute $E_n(x)$, $n=1(1)N$, to 5S for $x=12.3$, $N=20$.

If N is appreciably larger than x, and x appreciably larger than five, then the recurrence relation **5.1.14** should be used in the backward direction to generate $E_n(x)$ for $n<n_0$, and in the forward direction to generate $E_n(x)$ for $n>n_0$, where $n_0=\langle x\rangle$.

From **5.1.52,** with $n_0=12$, $x=12.3$, we have

$$E_{n_0}(x)=\frac{e^{-12.3}}{24.3}(1+.02032-.00043-.00001)=1.91038\times10^{-7}.$$

Using the recurrence relation **5.1.14**, as indicated, we get

n	$10^6E_n(12.3)$	$10^6E_n(12.3)$	n
12	.191038	.191038	12
11	.199213	.183498	13
10	.208098	.176516	14
9	.217793	.170042	15
8	.228406	.164015	16
7	.240073	.158397	17
6	.252951	.153144	18
5	.267234	.148226	19
4	.283155	.143608	20
3	.300998		
2	.321117		
1	.343953		

From **Tables 5.2** and **5.5** we find $E_1(12.3)=.343953\times10^{-6}$, $E_{20}(12.3)=.143609\times10^{-6}$ as a check.

Example 7. Compute $\alpha_n(2)$ to 6S for $n=1(1)5$.

The recurrence formula **5.1.15** can be used for all $x>0$ in increasing order of n without loss of accuracy. From **5.1.25** we have $\alpha_0(2)=\frac{1}{2}e^{-2}=.0676676$, so we get

n	$\alpha_n(2)$
0	.0676676
1	.101501
2	.169169
3	.321421
4	.710510
5	1.84394

Independent calculation with **5.1.8** yields the same result for $\alpha_5(2)$.

The functions $\alpha_0(x)$ and $\alpha_1(x)$ can be obtained from **Table 10.8** using **5.1.48, 5.1.49.**

Example 8. Compute $\beta_n(x)$, $n=0(1)N$ to 6S for $x=1$, $N=5$.

Use the recurrence relation **5.1.16** in increasing order of n if

$$x>.368N+.184\ln N+.821$$

and in decreasing order of n otherwise [5.5].

From **5.1.9** with $n=5$ we get $\beta_5(1)=-.324297$ correctly rounded to 6D. Using the recurrence formula **5.1.16** in decreasing order of n and carrying 9D we get the values in column (2).

n	$\beta_n(1)$ (1)	$\beta_n(1)$ (2)
0	2.35040 2	$2.35040\ 238\underline{9}$
1	$-.73575\ 9\underline{269}$	$-.73575\ 888\underline{0}$
2	$.87888\ \underline{3849}$	$.87888\ 462\underline{9}$
3	$-.44950\ \underline{9722}$	$-.44950\ 73\underline{83}$
4	$.5523\underline{6\ 3499}$	$.55237\ 2\underline{854}$
5	$-.324\underline{34\ 3774}$	$-.32429\ 7$

Using forward recurrence instead, starting with

$\beta_0(1)=2\sinh 1=2.350402$ and again carrying 9D, we obtain column (1). The underlined figures are in error. The above shows that three significant figures are lost in forward recurrence, whereas about three significant figures are gained in backward recurrence!

An alternative procedure is to start with an arbitrary value for n sufficiently large (see also [5.1]). To illustrate, starting with the value zero at $n=11$ we get

n	$\beta_n(1)$	n	$\beta_n(1)$
11	$0.$	5	$-.324297$
10	$.28\underline{0560}$	4	$.552373$
9	$-.20\underline{6984}$	3	$-.449507$
8	$.319\underline{908}$	2	$.878885$
7	$-.2538\underline{12}$	1	$-.735759$
6	$.40462\underline{1}$	0	2.350402

The functions $\beta_0(x)$ and $\beta_1(x)$ can be obtained from **Table 10.8** using **5.1.48, 5.1.49.**

Example 9. Compute $E_1(z)$ for $z=3.2578+6.8943i$.

From **Table 5.6** we have for $z_0=x_0+iy_0=3+7i$

$$z_0e^{z_0}E_1(z_0)=.934958+.095598i,$$

$$e^{z_0}E_1(z_0)=.059898-.107895i.$$

From Taylor's formula with $f(z)=e^zE_1(z)$ we have

$$f(z)=f(z_0+\Delta z)=f(z_0)+\frac{f'(z_0)}{1!}\Delta z+\frac{f''(z_0)}{2!}(\Delta z)^2+\cdots$$

with $\Delta z=z-z_0=.2578-.1057i$. Thus with **5.1.27** we get

k	$f^{(k)}(z_0)/k!$		$(\Delta z)^kf^{(k)}(z_0)/k!$	
0	$.059898$	$-.107895i$	$.059898$	$-.107895i$
1	$.008174$	$+.012795i$	$.003460$	$+.002435i$
2	$-.001859$	$+.000155i$	$-.000094$	$+.000110i$
3	$.000088$	$-.000212i$	$-.000003$	$-.000004i$

$$\begin{aligned} f(z)&=.063261 &&-.105354i\\ e^{-z}&=.031510 &&-.022075i\\ E_1(z)&=-.000332 &&-.004716i \end{aligned}$$

Repeating the calculation with $z_0=3+6i$ and $\Delta z=.2578+.8943i$ we get the same result.

An alternative procedure is to perform bivariate interpolation in the real and imaginary parts of $ze^zE_1(z)$.

Example 10. Compute $E_1(z)$ for $z=-4.2+12.7i$.

Using the formula at the bottom of **Table 5.6**

$$e^zE_1(z)\approx\frac{.711093}{-3.784225+12.7i}+\frac{.278518}{-1.90572+12.7i}+\frac{.010389}{2.0900+12.7i}$$

$$=-.0184106-.0736698i$$

$$E_1(z)\approx-1.87133-4.70540i.$$

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[5.31] J. Miller and R. P. Hurst, Simplified calculation of the exponential integral, Math. Tables Aids Comp. **12,** 187–193 (1958). $e^{-x}\mathrm{Ei}(x)$, $\mathrm{Ei}(x)$, $e^xE_1(x)$, $E_1(x)$, $x=.2(.05)5(.1)10(.2)20(.5)50(1)80$; 16S.

[5.32] National Bureau of Standards, Tables of sine, cosine and exponential integrals, vol. I (U.S. Government Printing Office, Washington, D.C., 1940). $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$, $\mathrm{Ei}(x)$, $E_1(x)$, $x=0(.0001)2$, $x=0(.1)10$; 9D.

[5.33] National Bureau of Standards, Tables of sine, cosine and exponential integrals, vol. II (U.S. Government Printing Office, Washington, D.C., 1940). $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$, $\mathrm{Ei}(x)$, $E_1(x)$, $x=0(.001)10$, 9–10 D or S; $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$, $x=10(.1)40$, 10D; $\mathrm{Ei}(x)$, $E_1(x)$, $x=10(.1)15$, 7–11S.

[5.34] National Bureau of Standards, Table of sine and cosine integrals for arguments from 10 to 100, Applied Math. Series 32 (U.S. Government Printing Office, Washington, D.C., 1954). $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$, $x=10(.01)100$, 10D.

[5.35] National Bureau of Standards, Tables of functions and of zeros of functions, Collected short tables of the Computation Laboratory, Applied Math. Series 37 (U.S. Government Printing Office, Washington, D.C., 1954). $E_n(x)$, $n=0(1)20$, $x=0(.01)2(.1)10$, 4–9S; $E_2(x)-x\ln x$, $x=0(.01).5$, 7S; $E_3(x)+\frac{1}{2}x^2\ln x$, $x=0(.01).1$, 7S.

[5.36] National Bureau of Standards, Tables of the exponential integral for complex arguments, Applied Math. Series 51 (U.S. Government Printing Office, Washington, D.C., 1958). $E_1(z)+\ln z$, 6D, $x=0(.02)1$, $y=0(.02)1$, $x=-1(.1)0$, $y=0(.1)1$; $E_1(z)$, 6D, $x=0(.02)4$, $y=0(.02)3(.05)10$, $x=0(1)20$, $y=0(1)20$, $x=-3.1(.1)0$, $y=0(.1)3.1$, $x=-4.5(.5)0$, $y=0(.1)4(.5)10$, $x=-10(.5)-4.5$, $y=0(.5)10$, $x=-20(1)0$, $y=0(1)20$; $e^zE_1(z)$, 6D, $x=4(.1)10$, $y=0(.5)10$.

[5.37] S. Oberländer, Tabellen von Exponentialfunktionen und-integralen zur Anwendung auf Gebieten der Thermodynamik, Halbleitertheorie und Gaskinetik (Akademie-Verlag, Berlin, Germany, 1959). $\frac{\Delta E}{kT}$, $\frac{kT}{\Delta E}$, $\exp\left(\frac{-\Delta E}{kT}\right)$, $\frac{kT}{\Delta E}\exp\left(\frac{-\Delta E}{kT}\right)$, $E_1\left(\frac{\Delta E}{kT}\right)$, $\frac{k}{\Delta E}\int_0^T\exp\left(\frac{-\Delta E}{kT}\right)dT$, $\frac{\Delta E}{kT}\exp\left(\frac{\Delta E}{kT}\right)\times E_1\left(\frac{\Delta E}{kT}\right)$, $1-\frac{\Delta E}{kT}\exp\left(\frac{\Delta E}{kT}\right)E_1\left(\frac{\Delta E}{kT}\right)$; $\Delta E=.2(.2)2$, $T=25(25)1000$, $T=150(10)390$, 3–4S; x^{-1}, $\exp(-x^{-1})$, $x\exp(-x^{-1})$, $E_1(x^{-1})$, $\int_0^x\exp(-t^{-1})dt$, $x^{-1}\exp(x^{-1})E_1(x^{-1})$, $1-x^{-1}\exp(x^{-1})E_1(x^{-1})$; $x=.01(.0001).1$, 5–6S.

[5.38] V. I. Pagurova, Tables of the exponential integral $E_\nu(x)=\int_1^\infty e^{-xu}u^{-\nu}du$. Translated from the Russian by D. G. Fry (Pergamon Press, New York, N.Y.; Oxford, London, England; Paris, France, 1961). $E_n(x)$, $n=0(1)20$, $x=0(.01)2(.1)10$, 4–9S; $E_2(x)-x\ln x$, $x=0(.01)5$, 7S; $E_3(x)+\frac{1}{2}x^2\ln x$, $x=0(.01).1$, 7S; $e^xE_n(x)$, $n=2(1)10$, $x=10(.1)20$, 7D; $e^xE_\nu(x)$, $\nu=0(.1)1$, $x=.01(.01)7(.05)12(.1)20$, 7 S or D.

[5.39] Tablitsy integral'nogo sinusa i kosinusa (Izdat. Akad. Nauk SSSR., Moscow, U.S.S.R., 1954). $\mathrm{Si}(x)$, $\mathrm{Ci}(x)$, $x=0(.0001)2(.001)10(.005)100$, 7D; $\mathrm{Ci}(x)-\ln x$, $x=0(.0001).01$, 7D.

[5.40] Tablitsy integral'noĭ pokazatel'noĭ funktsii (Izdat. Akad. Nauk SSSR., Moscow, U.S.S.R., 1954). $\mathrm{Ei}(x)$, $E_1(x)$, $x=0(.0001)1.3(.001)3(.0005)10(.1)15$, 7D.

[5.41] D. K. Trubey, A table of three exponential integrals, Oak Ridge National Laboratory Report 2750, Oak Ridge, Tenn. (June 1959). $E_1(x)$, $E_2(x)$, $E_3(x)$, $x=0(.0005).1(.001)2(.01)10(.1)20$, 6S.

6. Gamma Function and Related Functions

PHILIP J. DAVIS[1]

Contents

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[1] National Bureau of Standards.

6. Gamma Function and Related Functions

Mathematical Properties

6.1. Gamma (Factorial) Function

Euler's Integral

6.1.1 $$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt \qquad (\mathscr{R}z>0)$$

$$=k^z\int_0^\infty t^{z-1}e^{-kt}\,dt \qquad (\mathscr{R}z>0,\ \mathscr{R}k>0)$$

Euler's Formula

6.1.2
$$\Gamma(z)=\lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)\ldots(z+n)} \qquad (z\neq 0,-1,-2,\ldots)$$

Euler's Infinite Product

6.1.3 $$\frac{1}{\Gamma(z)}=ze^{\gamma z}\prod_{n=1}^{\infty}\left[\left(1+\frac{z}{n}\right)e^{-z/n}\right] \qquad (|z|<\infty)$$

$$\gamma=\lim_{m\to\infty}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{m}-\ln m\right]$$
$$=.57721\ 56649\ldots$$

γ is known as Euler's constant and is given to 25 decimal places in chapter 1. $\Gamma(z)$ is single valued and analytic over the entire complex plane, save for the points $z=-n(n=0, 1, 2, \ldots)$ where it possesses simple poles with residue $(-1)^n/n!$. Its reciprocal $1/\Gamma(z)$ is an entire function possessing simple zeros at the points $z=-n(n=0, 1, 2, \ldots)$.

Hankel's Contour Integral

6.1.4 $$\frac{1}{\Gamma(z)}=\frac{i}{2\pi}\int_C(-t)^{-z}e^{-t}\,dt \qquad (|z|<\infty)$$

The path of integration C starts at $+\infty$ on the real axis, circles the origin in the counterclockwise direction and returns to the starting point.

Factorial and Π Notations

6.1.5 $$\Pi(z)=z!=\Gamma(z+1)$$

Integer Values

6.1.6 $$\Gamma(n+1)=1\cdot 2\cdot 3\ldots(n-1)n=n!$$

6.1.7
$$\lim_{z\to n}\frac{1}{\Gamma(-z)}=0=\frac{1}{(-n-1)!} \qquad (n=0, 1, 2, \ldots)$$

Fractional Values

6.1.8
$$\Gamma(\tfrac{1}{2})=2\int_0^\infty e^{-t^2}\,dt=\pi^{\frac{1}{2}}=1.77245\ 38509\ldots=(-\tfrac{1}{2})!$$

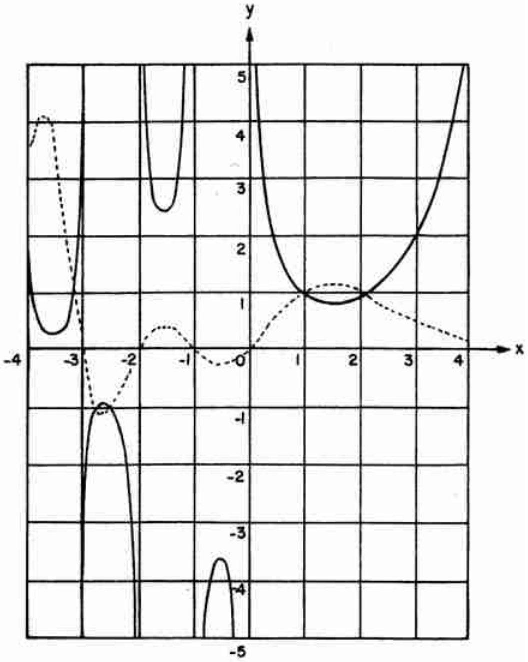

FIGURE 6.1. *Gamma function.* *

———, $y=\Gamma(x)$, – – – –, $y=1/\Gamma(x)$

6.1.9 $$\Gamma(3/2)=\tfrac{1}{2}\pi^{\frac{1}{2}}=.88622\ 69254\ldots=(\tfrac{1}{2})!$$

6.1.10 $$\Gamma(n+\tfrac{1}{4})=\frac{1\cdot 5\cdot 9\cdot 13\ldots(4n-3)}{4^n}\Gamma(\tfrac{1}{4})$$

$$\Gamma(\tfrac{1}{4})=3.62560\ 99082\ldots$$

6.1.11 $$\Gamma(n+\tfrac{1}{3})=\frac{1\cdot 4\cdot 7\cdot 10\ldots(3n-2)}{3^n}\Gamma(\tfrac{1}{3})$$

$$\Gamma(\tfrac{1}{3})=2.67893\ 85347\ldots$$

6.1.12 $$\Gamma(n+\tfrac{1}{2})=\frac{1\cdot 3\cdot 5\cdot 7\ldots(2n-1)}{2^n}\Gamma(\tfrac{1}{2})$$

6.1.13 $$\Gamma(n+\tfrac{2}{3})=\frac{2\cdot 5\cdot 8\cdot 11\ldots(3n-1)}{3^n}\Gamma(\tfrac{2}{3})$$

$$\Gamma(\tfrac{2}{3})=1.35411\ 79394\ldots$$

6.1.14 $$\Gamma(n+\tfrac{3}{4})=\frac{3\cdot 7\cdot 11\cdot 15\ldots(4n-1)}{4^n}\Gamma(\tfrac{3}{4})$$

$$\Gamma(\tfrac{3}{4})=1.22541\ 67024\ \ldots$$

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*See page II.

Recurrence Formulas

6.1.15 $$\Gamma(z+1)=z\Gamma(z)=z!=z(z-1)!$$

6.1.16

$$\begin{aligned}\Gamma(n+z)&=(n-1+z)(n-2+z)\ldots(1+z)\Gamma(1+z)\\&=(n-1+z)!\\&=(n-1+z)(n-2+z)\ldots(1+z)z!\end{aligned}$$

Reflection Formula

6.1.17 $$\Gamma(z)\Gamma(1-z)=-z\Gamma(-z)\Gamma(z)=\pi\csc\pi z$$
$$=\int_0^\infty\frac{t^{z-1}}{1+t}dt\qquad(0<\mathscr{R}z<1)$$

Duplication Formula

6.1.18 $$\Gamma(2z)=(2\pi)^{-\frac{1}{2}}2^{2z-\frac{1}{2}}\Gamma(z)\Gamma(z+\tfrac{1}{2})$$

Triplication Formula

6.1.19 $$\Gamma(3z)=(2\pi)^{-1}3^{3z-\frac{1}{2}}\Gamma(z)\Gamma(z+\tfrac{1}{3})\Gamma(z+\tfrac{2}{3})$$

Gauss' Multiplication Formula

6.1.20 $$\Gamma(nz)=(2\pi)^{\frac{1}{2}(1-n)}n^{nz-\frac{1}{2}}\prod_{k=0}^{n-1}\Gamma\left(z+\frac{k}{n}\right)$$

Binomial Coefficient

6.1.21 $$\binom{z}{w}=\frac{z!}{w!(z-w)!}=\frac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}$$

Pochhammer's Symbol

6.1.22

$$(z)_0=1,$$

$$(z)_n=z(z+1)(z+2)\ldots(z+n-1)=\frac{\Gamma(z+n)}{\Gamma(z)}$$

Gamma Function in the Complex Plane

6.1.23 $$\Gamma(\overline{z})=\overline{\Gamma(z)};\ \ln\Gamma(\overline{z})=\overline{\ln\Gamma(z)}$$

6.1.24 $$\arg\Gamma(z+1)=\arg\Gamma(z)+\arctan\frac{y}{x}$$

6.1.25 $$\left|\frac{\Gamma(x+iy)}{\Gamma(x)}\right|^2=\prod_{n=0}^{\infty}\left[1+\frac{y^2}{(x+n)^2}\right]^{-1}$$

6.1.26 $$|\Gamma(x+iy)|\leq|\Gamma(x)|$$

6.1.27

$$\arg\Gamma(x+iy)=y\psi(x)+\sum_{n=0}^{\infty}\left(\frac{y}{x+n}-\arctan\frac{y}{x+n}\right)$$
$$(x+iy\neq0,-1,-2,\ldots)$$

where $$\psi(z)=\Gamma'(z)/\Gamma(z)$$

6.1.28 $$\Gamma(1+iy)=iy\,\Gamma(iy)$$

6.1.29 $$\Gamma(iy)\Gamma(-iy)=|\Gamma(iy)|^2=\frac{\pi}{y\sinh\pi y}$$

6.1.30 $$\Gamma(\tfrac{1}{2}+iy)\Gamma(\tfrac{1}{2}-iy)=|\Gamma(\tfrac{1}{2}+iy)|^2=\frac{\pi}{\cosh\pi y}$$

6.1.31 $$\Gamma(1+iy)\Gamma(1-iy)=|\Gamma(1+iy)|^2=\frac{\pi y}{\sinh\pi y}$$

6.1.32 $$\Gamma(\tfrac{1}{4}+iy)\Gamma(\tfrac{3}{4}-iy)=\frac{\pi\sqrt{2}}{\cosh\pi y+i\sinh\pi y}$$

Power Series

6.1.33

$$\ln\Gamma(1+z)=-\ln(1+z)+z(1-\gamma)$$
$$+\sum_{n=2}^{\infty}(-1)^n[\zeta(n)-1]z^n/n\quad(|z|<2)$$

$\zeta(n)$ is the Riemann Zeta Function (see chapter **23**).

Series Expansion[2] for $1/\Gamma(z)$

6.1.34 $$\frac{1}{\Gamma(z)}=\sum_{k=1}^{\infty}c_kz^k\qquad(|z|<\infty)$$

k	c_k
1	1. 00000 00000 000000
2	0. 57721 56649 015329
3	—0. 65587 80715 202538
4	—0. 04200 26350 340952
5	0. 16653 86113 822915
6	—0. 04219 77345 555443
7	—0. 00962 19715 278770
8	0. 00721 89432 466630
9	—0. 00116 51675 918591
10	—0. 00021 52416 741149
11	0. 00012 80502 823882
12	—0. 00002 01348 547807
13	−0. 00000 12504 934821
14	0. 00000 11330 272320
15	−0. 00000 02056 338417
16	0. 00000 00061 160950
17	0. 00000 00050 020075
18	−0. 00000 00011 812746
19	0. 00000 00001 043427
20	0. 00000 00000 077823
21	−0. 00000 00000 036968
22	0. 00000 00000 005100
23	−0. 00000 00000 000206
24	−0. 00000 00000 000054
25	0. 00000 00000 000014
26	0. 00000 00000 000001

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[2] The coefficients c_k are from H. T. Davis, Tables of higher mathematical functions, 2 vols., Principia Press, Bloomington, Ind., 1933, 1935 (with permission); with corrections due to H. E. Salzer.

Polynomial Approximations[3]

6.1.35 $\qquad 0 \leq x \leq 1$

$$\Gamma(x+1)=x!=1+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\epsilon(x)$$

$$|\epsilon(x)| \leq 5 \times 10^{-5}$$

$$\begin{array}{ll} a_1=-.57486\ 46 & a_4=\ \ .42455\ 49 \\ a_2=\ \ .95123\ 63 & a_5=-.10106\ 78 \\ a_3=-.69985\ 88 & \end{array}$$

6.1.36 $\qquad 0 \leq x \leq 1$

$$\Gamma(x+1)=x!=1+b_1x+b_2x^2+\ .\ .\ .\ +b_8x^8+\epsilon(x)$$

$$|\epsilon(x)| \leq 3 \times 10^{-7}$$

$$\begin{array}{ll} b_1=-.57719\ 1652 & b_5=-.75670\ 4078 \\ b_2=\ \ .98820\ 5891 & b_6=\ \ .48219\ 9394 \\ b_3=-.89705\ 6937 & b_7=-.19352\ 7818 \\ b_4=\ \ .91820\ 6857 & b_8=\ \ .03586\ 8343 \end{array}$$

Stirling's Formula

6.1.37

$$\Gamma(z) \sim e^{-z}z^{z-\frac{1}{2}}(2\pi)^{\frac{1}{2}}\left[1+\frac{1}{12z}+\frac{1}{288z^2}-\frac{139}{51840z^3}\right.$$
$$\left.-\frac{571}{2488320z^4}+\cdots\right] \qquad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

6.1.38

$$x!=\sqrt{2\pi}\, x^{x+\frac{1}{2}}\exp\left(-x+\frac{\theta}{12x}\right) \qquad (x>0,\ 0<\theta<1)$$

Asymptotic Formulas

6.1.39

$$\Gamma(az+b) \sim \sqrt{2\pi}\, e^{-az}(az)^{az+b-\frac{1}{2}} \qquad (|\arg z|<\pi,\ a>0)$$

6.1.40

$$\ln \Gamma(z) \sim (z-\tfrac{1}{2}) \ln z-z+\tfrac{1}{2} \ln (2\pi)$$
$$+\sum_{m=1}^{\infty} \frac{B_{2m}}{2m(2m-1)z^{2m-1}} \qquad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

For B_n see chapter **23**

6.1.41

$$\ln \Gamma(z) \sim (z-\tfrac{1}{2}) \ln z-z+\tfrac{1}{2} \ln (2\pi)+\frac{1}{12z}-\frac{1}{360z^3}$$
$$+\frac{1}{1260z^5}-\frac{1}{1680z^7}+\cdots \qquad (z \rightarrow \infty \text{ in } |\arg z| < \pi)$$

Error Term for Asymptotic Expansion

6.1.42

If

$$R_n(z)= \ln \Gamma\ (z)-(z-\tfrac{1}{2}) \ln z+z-\tfrac{1}{2} \ln (2\pi)$$
$$-\sum_{m=1}^{n} \frac{B_{2m}}{2m(2m-1)z^{2m-1}}$$

then

$$|R_n(z)| \leq \frac{|B_{2n+2}|K(z)}{(2n+1)(2n+2)|z|^{2n+1}}$$

where

$$K(z)=\underset{u \geq 0}{\text{upper bound}}|z^2/(u^2+z^2)|$$

For z real and positive, R_n is less in absolute value than the first term neglected and has the same sign.

6.1.43

$$\mathscr{R} \ln \Gamma(iy)=\mathscr{R} \ln \Gamma(-iy)$$
$$=\tfrac{1}{2} \ln\left(\frac{\pi}{y \sinh \pi y}\right)$$
$$\sim\tfrac{1}{2} \ln (2\pi)-\tfrac{1}{2}\pi y-\tfrac{1}{2}\ln y, \qquad (y \rightarrow +\infty)$$

6.1.44

$$\mathscr{I} \ln \Gamma(iy)=\arg \Gamma(iy)=-\arg \Gamma(-iy)$$
$$=-\mathscr{I} \ln \Gamma(-iy)$$
$$\sim y \ln y-y-\tfrac{1}{4}\pi-\sum_{n=1}^{\infty} \frac{(-1)^{n-1} B_{2n}}{(2n-1)(2n)y^{2n-1}}$$
$$(y \rightarrow +\infty)$$

6.1.45 $\qquad \displaystyle\lim_{|y| \rightarrow \infty}(2\pi)^{-\frac{1}{2}}|\Gamma(x+iy)|e^{\frac{1}{2}\pi|y|}|y|^{\frac{1}{2}-x}=1$

6.1.46 $\qquad \displaystyle\lim_{n \rightarrow \infty} n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)}=1$

6.1.47

$$z^{b-a} \frac{\Gamma(z+a)}{\Gamma(z+b)} \sim 1+\frac{(a-b)(a+b-1)}{2z}$$
$$+\frac{1}{12}\binom{a-b}{2}\left(3(a+b-1)^2-a+b-1\right)\frac{1}{z^2}+\ .\ .\ .$$

as $z \rightarrow \infty$ along any curve joining $z=0$ and $z=\infty$, providing $z \neq -a, -a-1, \ .\ .\ .\ ;\ z \neq -b, -b-1,$ $.\ .\ .\ .$

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[3] From C. Hastings, Jr., Approximations for digital computers, Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

Continued Fraction

6.1.48

$$\ln \Gamma(z)+z-(z-\tfrac{1}{2}) \ln z-\tfrac{1}{2}\ln (2\pi)$$

$$=\frac{a_0}{z+}\,\frac{a_1}{z+}\,\frac{a_2}{z+}\,\frac{a_3}{z+}\,\frac{a_4}{z+}\,\frac{a_5}{z+}\ldots \qquad (\mathscr{R} z>0)$$

$$a_0=\frac{1}{12},\ a_1=\frac{1}{30},\ a_2=\frac{53}{210},\ a_3=\frac{195}{371},$$

$$a_4=\frac{22999}{22737},\ a_5=\frac{29944523}{19733142},\ a_6=\frac{109535241009}{48264275462}$$

Wallis' Formula[4]

6.1.49

$$\frac{2}{\pi}\int_0^{\pi/2}\begin{pmatrix}\sin\\ \cos\end{pmatrix}^{2n} x\,dx=\frac{1\cdot3\cdot5\ldots(2n-1)}{2\cdot4\cdot6\ldots(2n)}$$

$$=\frac{(2n)!}{2^{2n}(n!)^2}=\frac{1}{2^{2n}}\binom{2n}{n}=\frac{\Gamma(n+\frac{1}{2})}{\pi^{\frac{1}{2}}\Gamma(n+1)}$$

$$\sim\frac{1}{\pi^{\frac{1}{2}}n^{\frac{1}{2}}}\left[1-\frac{1}{8n}+\frac{1}{128n^2}-\cdots\right]$$

$$(n\to\infty)$$

Some Definite Integrals

6.1.50

$$\ln \Gamma(z)=\int_0^\infty\left[(z-1)\,e^{-t}-\frac{e^{-t}-e^{-zt}}{1-e^{-t}}\right]\frac{dt}{t} \quad (\mathscr{R} z>0)$$

$$=(z-\tfrac{1}{2})\ln z-z+\tfrac{1}{2}\ln 2\pi$$

$$+2\int_0^\infty\frac{\arctan (t/z)}{e^{2\pi t}-1}\,dt \qquad (\mathscr{R} z>0)$$

6.2. Beta Function

6.2.1

$$B(z,w)=\int_0^1 t^{z-1}\,(1-t)^{w-1}\,dt=\int_0^\infty\frac{t^{z-1}}{(1+t)^{z+w}}\,dt$$

$$=2\int_0^{\pi/2}(\sin t)^{2z-1}\,(\cos t)^{2w-1}\,dt$$

$$(\mathscr{R} z>0,\ \mathscr{R} w>0)$$

6.2.2 $$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}=B(w,z)$$

6.3. Psi (Digamma) Function[5]

6.3.1 $$\psi(z)=d[\ln\Gamma(z)]/dz=\Gamma'(z)/\Gamma(z)$$

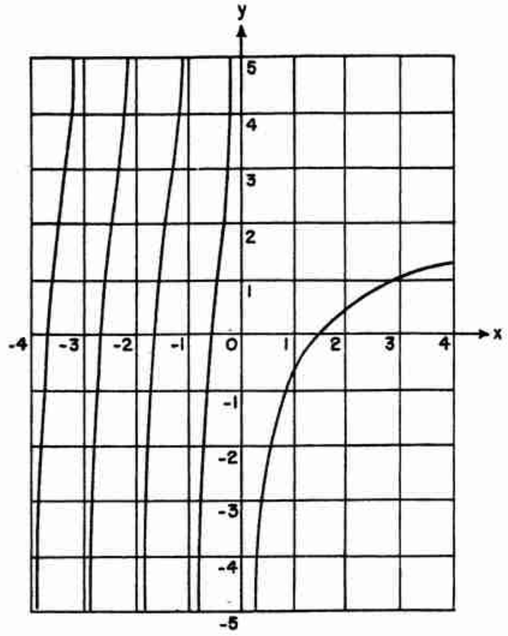

FIGURE 6.2. *Psi function.*

$y=\psi(x)=d\ln\Gamma(x)/dx$

Integer Values

6.3.2 $$\psi(1)=-\gamma,\ \psi(n)=-\gamma+\sum_{k=1}^{n-1}k^{-1} \qquad (n\geq2)$$

Fractional Values

6.3.3

$$\psi(\tfrac{1}{2})=-\gamma-2\ln 2=-1.96351\ 00260\ 21423\ \ldots$$

6.3.4

$$\psi(n+\tfrac{1}{2})=-\gamma-2\ln 2+2\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}\right)$$

$$(n\geq1)$$

Recurrence Formulas

6.3.5 $$\psi(z+1)=\psi(z)+\frac{1}{z}$$

6.3.6

$$\psi(n+z)=\frac{1}{(n-1)+z}+\frac{1}{(n-2)+z}+\cdots$$

$$+\frac{1}{2+z}+\frac{1}{1+z}+\psi(1+z)$$

---

[4] Some authors employ the special double factorial notation as follows:

$(2n)\,!\,!=2\cdot4\cdot6\ \ldots\ (2n)=2^n n\,!$

$(2n-1)\,!\,!=1\cdot3\cdot5\ \ldots\ (2n-1)=\pi^{-\frac{1}{2}}\,2^n\,\Gamma(n+\frac{1}{2})$

[5] Some authors write $\psi(z)=\frac{d}{dz}\ln\Gamma(z+1)$ and similarly for the polygamma functions.

Reflection Formula

6.3.7 $$\psi(1-z)=\psi(z)+\pi \cot \pi z$$

Duplication Formula

6.3.8 $$\psi(2z)=\tfrac{1}{2}\psi(z)+\tfrac{1}{2}\psi(z+\tfrac{1}{2})+\ln 2$$

Psi Function in the Complex Plane

6.3.9 $$\psi(\bar{z})=\overline{\psi(z)}$$

6.3.10

$$\mathscr{R}\psi(iy)=\mathscr{R}\psi(-iy)=\mathscr{R}\psi(1+iy)=\mathscr{R}\psi(1-iy)$$

6.3.11 $$\mathscr{I}\psi(iy)=\tfrac{1}{2}y^{-1}+\tfrac{1}{2}\pi \coth \pi y$$

6.3.12 $$\mathscr{I}\psi(\tfrac{1}{2}+iy)=\tfrac{1}{2}\pi \tanh \pi y$$

6.3.13 $$\mathscr{I}\psi(1+iy)=-\frac{1}{2y}+\tfrac{1}{2}\pi \coth \pi y$$

$$=y\sum_{n=1}^{\infty}(n^2+y^2)^{-1}$$

Series Expansions

6.3.14 $$\psi(1+z)=-\gamma+\sum_{n=2}^{\infty}(-1)^n\zeta(n)z^{n-1} \qquad (|z|<1)$$

6.3.15

$$\psi(1+z)=\tfrac{1}{2}z^{-1}-\tfrac{1}{2}\pi \cot \pi z-(1-z^2)^{-1}+1-\gamma$$

$$-\sum_{n=1}^{\infty}[\zeta(2n+1)-1]z^{2n} \qquad (|z|<2)$$

6.3.16

$$\psi(1+z)=-\gamma+\sum_{n=1}^{\infty}\frac{z}{n(n+z)} \quad (z\neq-1,-2,-3, \ldots)$$

6.3.17

$$\mathscr{R}\psi(1+iy)=1-\gamma-\frac{1}{1+y^2}$$

$$+\sum_{n=1}^{\infty}(-1)^{n+1}[\zeta(2n+1)-1]y^{2n}$$

$$(|y|<2)$$

$$=-\gamma+y^2\sum_{n=1}^{\infty}n^{-1}(n^2+y^2)^{-1}$$

$$(-\infty<y<\infty)$$

Asymptotic Formulas

6.3.18

$$\psi(z)\sim\ln z-\frac{1}{2z}-\sum_{n=1}^{\infty}\frac{B_{2n}}{2nz^{2n}}$$

$$=\ln z-\frac{1}{2z}-\frac{1}{12z^2}+\frac{1}{120z^4}-\frac{1}{252z^6}+\ldots$$

$$(z\to\infty \text{ in } |\arg z|<\pi)$$

6.3.19

$$\mathscr{R}\psi(1+iy)\sim\ln y+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}B_{2n}}{2ny^{2n}}$$

$$=\ln y+\frac{1}{12y^2}+\frac{1}{120y^4}+\frac{1}{252y^6}+\ldots$$

$$(y\to\infty)$$

Extrema[6] of $\Gamma(x)$ — Zeros of $\psi(x)$

$$\Gamma'(x_n)=\psi(x_n)=0$$

n	x_n	$\Gamma(x_n)$
0	+1. 462	+0. 886
1	−0. 504	−3. 545
2	−1. 573	+2. 302
3	−2. 611	−0. 888
4	−3. 635	+0. 245
5	−4. 653	−0. 053
6	−5. 667	+0. 009
7	−6. 678	−0. 001

$$x_0=1.46163 \quad 21449 \quad 68362$$

$$\Gamma(x_0)=\ .88560 \quad 31944 \quad 10889$$

6.3.20 $$x_n=-n+(\ln n)^{-1}+o[(\ln n)^{-2}]$$

Definite Integrals

6.3.21

$$\psi(z)=\int_0^{\infty}\left[\frac{e^{-t}}{t}-\frac{e^{-zt}}{1-e^{-t}}\right]dt \qquad (\mathscr{R}z>0)$$

$$=\int_0^{\infty}\left[e^{-t}-\frac{1}{(1+t)^z}\right]\frac{dt}{t}$$

$$=\ln z-\frac{1}{2z}-2\int_0^{\infty}\frac{tdt}{(t^2+z^2)(e^{2\pi t}-1)}$$

$$\left(|\arg z|<\frac{\pi}{2}\right)$$

6.3.22

$$\psi(z)+\gamma=\int_0^{\infty}\frac{e^{-t}-e^{-zt}}{1-e^{-t}}\,dt=\int_0^1\frac{1-t^{z-1}}{1-t}\,dt$$

$$\gamma=\int_0^{\infty}\left(\frac{1}{e^t-1}-\frac{1}{te^t}\right)dt$$

$$=\int_0^{\infty}\left(\frac{1}{1+t}-e^{-t}\right)\frac{dt}{t}$$

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[6] From W. Sibagaki, Theory and applications of the gamma function, Iwanami Syoten, Tokyo, Japan, 1952 (with permission).

6.4. Polygamma Functions[7]

6.4.1

$$\psi^{(n)}(z)=\frac{d^n}{dz^n}\psi(z)=\frac{d^{n+1}}{dz^{n+1}}\ln\Gamma(z) \qquad (n=1,2,3,\ldots)$$

* $$=(-1)^{n+1}\int_0^\infty \frac{t^n e^{-zt}}{1-e^{-t}}dt \qquad (\mathscr{R}z>0)$$

$\psi^{(n)}(z),(n=0,1,\ldots)$, is a single valued analytic function over the entire complex plane save at the points $z=-m(m=0,1,2,\ldots)$ where it possesses poles of order $(n+1)$.

Integer Values

6.4.2

$$\psi^{(n)}(1)=(-1)^{n+1}n!\zeta(n+1) \qquad (n=1,2,3,\ldots)$$

6.4.3

$$\psi^{(m)}(n+1)=(-1)^m m!\left[-\zeta(m+1)+1+\frac{1}{2^{m+1}}+\ldots+\frac{1}{n^{m+1}}\right]$$

Fractional Values

6.4.4

$$\psi^{(n)}(\tfrac{1}{2})=(-1)^{n+1}n!(2^{n+1}-1)\zeta(n+1) \qquad (n=1,2,\ldots)$$

6.4.5 $$\psi'(n+\tfrac{1}{2})=\tfrac{1}{2}\pi^2-4\sum_{k=1}^{n}(2k-1)^{-2}$$

Recurrence Formula

6.4.6 $$\psi^{(n)}(z+1)=\psi^{(n)}(z)+(-1)^n n! z^{-n-1}$$

Reflection Formula

6.4.7

$$\psi^{(n)}(1-z)+(-1)^{n+1}\psi^{(n)}(z)=(-1)^n\pi\frac{d^n}{dz^n}\cot\pi z$$

Multiplication Formula

6.4.8

* $$\psi^{(n)}(mz)=\delta\ln m+\frac{1}{m^{n+1}}\sum_{k=0}^{m-1}\psi^{(n)}\left(z+\frac{k}{m}\right)$$

$$\delta=1,\quad n=0$$

$$\delta=0,\quad n>0$$

Series Expansions

6.4.9

$$\psi^{(n)}(1+z)=(-1)^{n+1}\left[n!\zeta(n+1)-\frac{(n+1)!}{1!}\zeta(n+2)z+\frac{(n+2)!}{2!}\zeta(n+3)z^2-\ldots\right] \qquad (|z|<1)$$

6.4.10

$$\psi^{(n)}(z)=(-1)^{n+1}n!\sum_{k=0}^{\infty}(z+k)^{-n-1} \qquad (z\neq 0,-1,-2,\ldots)$$

Asymptotic Formulas

6.4.11

$$\psi^{(n)}(z)\sim(-1)^{n-1}\left[\frac{(n-1)!}{z^n}+\frac{n!}{2z^{n+1}}+\sum_{k=1}^{\infty}B_{2k}\frac{(2k+n-1)!}{(2k)!z^{2k+n}}\right] \qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

6.4.12

$$\psi'(z)\sim\frac{1}{z}+\frac{1}{2z^2}+\frac{1}{6z^3}-\frac{1}{30z^5}+\frac{1}{42z^7}-\frac{1}{30z^9}+\ldots \qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

6.4.13

$$\psi''(z)\sim-\frac{1}{z^2}-\frac{1}{z^3}-\frac{1}{2z^4}+\frac{1}{6z^6}-\frac{1}{6z^8}+\frac{3}{10z^{10}}-\frac{5}{6z^{12}}+\ldots \qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

6.4.14

$$\psi^{(3)}(z)\sim\frac{2}{z^3}+\frac{3}{z^4}+\frac{2}{z^5}-\frac{1}{z^7}+\frac{4}{3z^9}-\frac{3}{z^{11}}+\frac{10}{z^{13}}-\ldots \qquad (z\to\infty \text{ in } |\arg z|<\pi)$$

6.5. Incomplete Gamma Function

(see also **26.4**)

6.5.1

$$P(a,x)=\frac{1}{\Gamma(a)}\int_0^x e^{-t}t^{a-1}\,dt \qquad (\mathscr{R}a>0)$$

6.5.2

$$\gamma(a,x)=P(a,x)\Gamma(a)=\int_0^x e^{-t}t^{a-1}\,dt \qquad (\mathscr{R}a>0)$$

6.5.3

$$\Gamma(a,x)=\Gamma(a)-\gamma(a,x)=\int_x^\infty e^{-t}t^{a-1}\,dt$$

6.5.4

$$\gamma^*(a,x)=x^{-a}P(a,x)=\frac{x^{-a}}{\Gamma(a)}\gamma(a,x)$$

γ^* is a single valued analytic function of a and x possessing no finite singularities.

---

[7] ψ' is known as the trigamma function. ψ'', $\psi^{(3)}$, $\psi^{(4)}$ are the tetra-, penta-, and hexagamma functions respectively. Some authors write $\psi(z)=d[\ln\Gamma(z+1)]/dz$, and similarly for the polygamma functions.

*See page II.

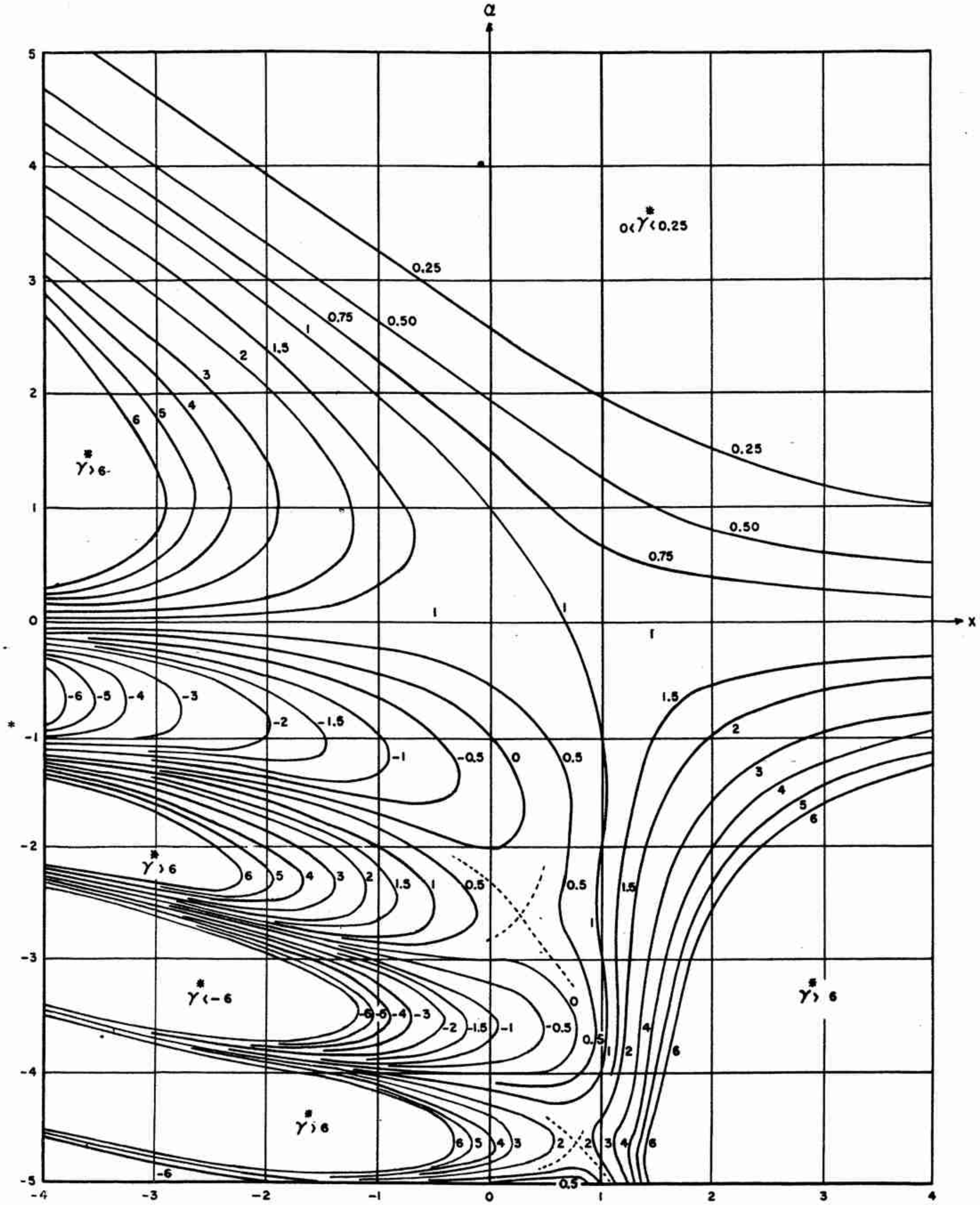

FIGURE 6.3. *Incomplete gamma function.*

$$\gamma^*(a,x)=\frac{x^{-a}}{\Gamma(a)}\int_0^x e^{-t}t^{a-1}dt$$

From F. G. Tricomi, Sulla funzione gamma incompleta, Annali di Matematica, IV, 33, 1950 (with permission).

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*See page II.

6.5.5

Probability Integral of the χ^2-Distribution

$$P(\chi^2|\nu)=\frac{1}{2^{\frac{1}{2}\nu}\Gamma\left(\frac{\nu}{2}\right)}\int_0^{\chi^2} t^{\frac{1}{2}\nu-1}e^{-\frac{t}{2}}dt$$

6.5.6

(Pearson's Form of the Incomplete Gamma Function)

$$I(u,p)=\frac{1}{\Gamma(p+1)}\int_0^{u\sqrt{p+1}} e^{-t}t^p\,dt$$
$$=P(p+1,\,u\sqrt{p+1})$$

6.5.7 $$C(x,a)=\int_x^\infty t^{a-1}\cos t\,dt \qquad (\mathscr{R}a<1)$$

6.5.8 $$S(x,a)=\int_x^\infty t^{a-1}\sin t\,dt \qquad (\mathscr{R}a<1)$$

6.5.9

$$E_n(x)=\int_1^\infty e^{-xt}t^{-n}\,dt=x^{n-1}\Gamma(1-n,x)$$

6.5.10

$$\alpha_n(x)=\int_1^\infty e^{-xt}t^n\,dt=x^{-n-1}\Gamma(1+n,x)$$

6.5.11 $$e_n(x)=\sum_{j=0}^{n}\frac{x^j}{j!}$$

Incomplete Gamma Function as a Confluent Hypergeometric Function (see chapter 13)

6.5.12 $$\gamma(a,x)=a^{-1}x^ae^{-x}M(1,\,1+a,x)$$
$$=a^{-1}x^a\,M(a,\,1+a,-x)$$

Special Values

6.5.13

$$P(n,x)=1-\left(1+x+\frac{x^2}{2!}+\ldots+\frac{x^{n-1}}{(n-1)!}\right)e^{-x}$$
$$=1-e_{n-1}(x)\,e^{-x}$$

For relation to the Poisson distribution, see **26.4**.

6.5.14 $$\gamma^*(-n,\,x)=x^n$$

6.5.15 $$\Gamma(0,\,x)=\int_x^\infty e^{-t}t^{-1}\,dt=E_1(x)$$

6.5.16 $$\gamma(\tfrac{1}{2},\,x^2)=2\int_0^x e^{-t^2}\,dt=\sqrt{\pi}\,\mathrm{erf}\,x$$

6.5.17 $$\Gamma(\tfrac{1}{2},x^2)=2\int_x^\infty e^{-t^2}dt=\sqrt{\pi}\,\mathrm{erfc}\,x$$

6.5.18 $$\tfrac{1}{2}\sqrt{\pi}\,x\,\gamma^*(\tfrac{1}{2},-x^2)=\int_0^x e^{t^2}dt$$

6.5.19 $$\Gamma(-n,x)=\frac{(-1)^n}{n!}\left[E_1(x)-e^{-x}\sum_{j=0}^{n-1}\frac{(-1)^jj!}{x^{j+1}}\right]$$

6.5.20 $$\Gamma(a,ix)=e^{\frac{1}{2}\pi ia}\,[C(x,a)-iS(x,a)]$$

Recurrence Formulas

6.5.21 $$P(a+1,\,x)=P(a,\,x)-\frac{x^ae^{-x}}{\Gamma(a+1)}$$

6.5.22 $$\gamma(a+1,x)=a\gamma(a,x)-x^ae^{-x}$$

6.5.23 $$\gamma^*(a-1,x)=x\gamma^*(a,x)+\frac{e^{-x}}{\Gamma(a)}$$

Derivatives and Differential Equations

6.5.24

$$\left(\frac{\partial\gamma^*}{\partial a}\right)_{a=0}=-\int_x^\infty\frac{e^{-t}dt}{t}-\ln x=-E_1(x)-\ln x$$

6.5.25 $$\frac{\partial\gamma(a,x)}{\partial x}=-\frac{\partial\Gamma(a,x)}{\partial x}=x^{a-1}e^{-x}$$

6.5.26

$$\frac{\partial^n}{\partial x^n}[x^{-a}\Gamma(a,x)]=(-1)^nx^{-a-n}\Gamma(a+n,x)$$
$$(n=0,1,2,\ldots)$$

6.5.27

$$\frac{\partial^n}{\partial x^n}[e^xx^a\gamma^*(a,x)]=e^xx^{a-n}\gamma^*(a-n,\,x)$$
$$(n=0,1,2,\ldots)$$

6.5.28 $$x\frac{\partial^2\gamma^*}{\partial x^2}+(a+1+x)\frac{\partial\gamma^*}{\partial x}+a\gamma^*=0$$

Series Developments

6.5.29

$$\gamma^*(a,z)=e^{-z}\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(a+n+1)}=\frac{1}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{(-z)^n}{(a+n)n!}$$
$$(|z|<\infty)$$

6.5.30

$$\gamma(a,\, x+y)-\gamma(a,\, x)$$
$$=e^{-x}x^{a-1}\sum_{n=0}^{\infty}\frac{(a-1)(a-2)\,.\,.\,.\,(a-n)}{x^n}[1-e^{-y}e_n(y)]$$
$$(|y|<|x|)$$

Continued Fraction

6.5.31

$$\Gamma(a,x)=e^{-x}x^a\left(\frac{1}{x+}\,\frac{1-a}{1+}\,\frac{1}{x+}\,\frac{2-a}{1+}\,\frac{2}{x+}\cdots\right)$$
$$(x>0, |a|<\infty)$$

Asymptotic Expansions

6.5.32

$$\Gamma(a,\, z)\sim z^{a-1}e^{-z}\left[1+\frac{a-1}{z}+\frac{(a-1)(a-2)}{z^2}+\cdots\right]$$
$$\left(z\to\infty \text{ in } |\arg z|<\frac{3\pi}{2}\right)$$

Suppose $R_n(a,z)=u_{n+1}(a,z)+\ .\ .\ .$ is the remainder after n terms in this series. Then if a,z are real, we have for $n>a-2$

$$|R_n(a,z)|\leq|u_{n+1}(a,z)|$$

and sign $R_n(a,z)=$sign $u_{n+1}(a,z)$.

6.5.33 $$\gamma(a,z)\sim\sum_{n=0}^{\infty}\frac{(-1)^n z^{a+n}}{(a+n)n!}\qquad (a\to+\infty)$$

6.5.34 $$\lim_{n\to\infty}\frac{e_n(\alpha n)}{e^{\alpha n}}=\begin{cases}0 \text{ for } \alpha>1\\ \frac{1}{2} \text{ for } \alpha=1\\ 1 \text{ for } 0\leq\alpha<1\end{cases}$$

6.5.35

$$\Gamma(z+1,z)\sim e^{-z}z^z\left(\sqrt{\frac{\pi}{2}}\,z^{\frac{1}{2}}+\frac{2}{3}+\frac{\sqrt{2\pi}}{24}\,\frac{1}{z^{\frac{1}{2}}}+\cdots\right)$$
$$(z\to\infty \text{ in } |\arg z|<\tfrac{1}{2}\pi)$$

Definite Integrals

6.5.36

$$\int_0^{\infty}e^{-at}\,\Gamma(b,ct)\,dt=\frac{\Gamma(b)}{a}\left[1-\frac{c^b}{(a+c)^b}\right]$$ *
$$(\mathscr{R}(a+c)>0, \mathscr{R}b>-1)$$

6.5.37

$$\int_0^{\infty}t^{a-1}\Gamma(b,t)\,dt=\frac{\Gamma(a+b)}{a}$$
$$(\mathscr{R}(a+b)>0,\quad \mathscr{R}a>0)$$

6.6. Incomplete Beta Function

6.6.1 $$B_x(a,b)=\int_0^x t^{a-1}(1-t)^{b-1}dt$$

6.6.2 $$I_x(a,b)=B_x(a,b)/B(a,b)$$

For statistical applications, see **26.5**.

Symmetry

6.6.3 $$I_x(a,b)=1-I_{1-x}(b,a)$$

Relation to Binomial Expansion

6.6.4 $$I_p(a,n-a+1)=\sum_{j=a}^{n}\binom{n}{j}p^j(1-p)^{n-j}$$

For binomial distribution, see **26.1**.

Recurrence Formulas

6.6.5 $$I_x(a,b)=xI_x(a-1,b)+(1-x)I_x(a,b-1)$$

6.6.6 $$(a+b-ax)I_x(a,b)$$ *
$$=a(1-x)I_x(a+1,b-1)+bI_x(a,b+1)$$

6.6.7 $$(a+b)I_x(a,b)=aI_x(a+1,b)+bI_x(a,b+1)$$

Relation to Hypergeometric Function

6.6.8 $$B_x(a,b)=a^{-1}x^aF(a,1-b;\ a+1;\ x)$$

Numerical Methods

6.7. Use and Extension of the Tables

Example 1. Compute Γ(6.38) to 8S. Using the recurrence relation **6.1.16** and **Table 6.1** we have,

$$\Gamma(6.38)=[(5.38)(4.38)(3.38)(2.38)(1.38)]\Gamma(1.38)$$
$$=232.43671.$$

Example 2. Compute ln Γ(56.38), using **Table 6.4** and linear interpolation in f_2. We have

$$\ln\Gamma(56.38)=(56.38-\tfrac{1}{2})\ln(56.38)-(56.38)$$
$$+f_2(56.38)$$

The error of linear interpolation in the table of the function f_2 is smaller than 10^{-7} in this region. Hence, $f_2(56.38)=.92041\ 67$ and $\ln\Gamma(56.38)=169.85497\ 42$.

Direct interpolation in **Table 6.4** of $\log_{10}\Gamma(n)$ eliminates the necessity of employing logarithms. However, the error of linear interpolation is .002 so that $\log_{10}\Gamma(n)$ is obtained with a relative error of 10^{-5}.

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*See page II.

Example 3. Compute $\psi(6.38)$ to 8S. Using the recurrence relation **6.3.6** and **Table 6.1**.

$$\begin{aligned}\psi(6.38)&=\frac{1}{5.38}+\frac{1}{4.38}+\frac{1}{3.38}+\frac{1}{2.38}+\frac{1}{1.38}+\psi(1.38)\\&=1.77275\;59.\end{aligned}$$

Example 4. Compute $\psi(56.38)$. Using **Table 6.3** we have $\psi(56.38)=\ln\,56.38-f_3(56.38)$.

The error of linear interpolation in the table of the function f_3 is smaller than 8×10^{-7} in this region. Hence, $f_3(56.38)=.00889\;53$ and $\psi(56.38)=4.023219$.

Example 5. Compute $\ln\,\Gamma(1-i)$. From the reflection principle **6.1.23** and **Table 6.7**, $\ln\,\Gamma(1-i)=\overline{\ln\,\Gamma(1+i)}=-.6509+.3016i$.

Example 6. Compute $\ln\,\Gamma(\frac{1}{2}+\frac{1}{2}i)$. Taking the logarithm of the recurrence relation **6.1.15** we have,

$$\begin{aligned}\ln\,\Gamma(\tfrac{1}{2}+\tfrac{1}{2}i)&=\ln\,\Gamma(\tfrac{3}{2}+\tfrac{1}{2}i)-\ln\,(\tfrac{1}{2}+\tfrac{1}{2}i)\\&=-.23419+.03467i\\&\quad-(\tfrac{1}{2}\ln\tfrac{1}{2}+i\arctan 1)\\&=.11239-.75073i\end{aligned}$$

The logarithms of complex numbers are found from **4.1.2**.

Example 7. Compute $\ln\,\Gamma(3+7i)$ using the duplication formula **6.1.18**. Taking the logarithm of **6.1.18**, we have

$$\begin{aligned}-\tfrac{1}{2}\ln 2\pi&=-\;\;.91894\\(\tfrac{5}{2}+7i)\ln 2&=\;\;1.73287+\;\;4.85203i\\\ln\,\Gamma(\tfrac{3}{2}+\tfrac{7}{2}i)&=-3.31598+\;\;2.32553i\\\ln\,\Gamma(2+\tfrac{7}{2}i)&=\underline{-2.66047+\;\;2.93869i}\\\ln\,\Gamma(3+7i)&=-5.16252+10.11625i\end{aligned}$$

Example 8. Compute $\ln\,\Gamma(3+7i)$ to 5D using the asymptotic formula **6.1.41**. We have

$$\ln\,(3+7i)=2.03022\;15+1.16590\;45i.$$

Then,

$$\begin{aligned}(2.5+7i)\ln\,(3+7i)&=-3.0857779+17.1263119i\\-(3+7i)&=-3.0000000-\;\;7.0000000i\\\tfrac{1}{2}\ln\,(2\pi)&=\quad\;.9189385\\[12(3+7i)]^{-1}&=\quad\;.0043103-\;\;\;.0100575i\\-[360(3+7i)^3]^{-1}&=\quad\;\underline{.0000059-\;\;\;.0000022i}\\\ln\,\Gamma(3+7i)&=-5.16252\quad+10.11625i\end{aligned}$$

6.8. Summation of Rational Series by Means of Polygamma Functions

An infinite series whose general term is a rational function of the index may always be reduced to a finite series of psi and polygamma functions. The method will be illustrated by writing the explicit formula when the denominator contains a triple root.

Let the general term of an infinite series have the form

$$u_n=\frac{p(n)}{d_1(n)d_2(n)d_3(n)}$$

where

$$d_1(n)=(n+\alpha_1)(n+\alpha_2)\;\;.\;.\;.\;\;(n+\alpha_m)$$

$$d_2(n)=(n+\beta_1)^2(n+\beta_2)^2\;\;.\;.\;.\;\;(n+\beta_r)^2$$

$$d_3(n)=(n+\gamma_1)^3(n+\gamma_2)^3\;\;.\;.\;.\;\;(n+\gamma_s)^3$$

where $p(n)$ is a polynomial of degree $m+2r+3s-2$ at most and where the constants α_i, β_i, and γ_i are distinct. Expand u_n in partial fractions as follows

$$\begin{aligned}u_n=\sum_{k=1}^{m}\frac{a_k}{(n+\alpha_k)}&+\sum_{k=1}^{r}\frac{b_{1k}}{(n+\beta_k)}+\frac{b_{2k}}{(n+\beta_k)^2}\\&+\sum_{k=1}^{s}\frac{c_{1k}}{(n+\gamma_k)}+\frac{c_{2k}}{(n+\gamma_k)^2}+\frac{c_{3k}}{(n+\gamma_k)^3}\end{aligned}$$

$$\sum_{k=1}^{m}a_k+\sum_{k=1}^{r}b_{1k}+\sum_{k=1}^{s}c_{1k}=0.$$

Then, we may express $\sum_{n=1}^{\infty}u_n$ in terms of the constants appearing in this partial fraction expansion as follows

$$\begin{aligned}\sum_{n=1}^{\infty}u_n=&-\sum_{j=1}^{m}a_j\psi(1+\alpha_j)\\&-\sum_{j=1}^{r}b_{1j}\psi(1+\beta_j)+\sum_{j=1}^{r}b_{2j}\psi'(1+\beta_j)\\&-\sum_{j=1}^{s}c_{1j}\psi(1+\gamma_j)+\sum_{j=1}^{s}c_{2j}\psi'(1+\gamma_j)\\&-\sum_{j=1}^{s}\frac{c_{3j}}{2!}\psi''(1+\gamma_j).\end{aligned}$$

Higher order repetitions in the denominator are handled similarly. If the denominator contains only simple or double roots, omit the corresponding lines.

Example 9. Find

$$s=\sum_{n=1}^{\infty}\frac{1}{(n+1)(2n+1)(4n+1)}.$$

Since

$$\frac{1}{(n+1)(2n+1)(4n+1)}=\frac{\frac{1}{3}}{n+1}-\frac{1}{n+\frac{1}{2}}+\frac{\frac{2}{3}}{n+\frac{1}{4}},$$

we have

$$\alpha_1=1,\ \alpha_2=\tfrac{1}{2},\ \alpha_3=\tfrac{1}{4},\ a_1=\tfrac{1}{3},\ a_2=-1,\ a_3=\tfrac{2}{3}.$$

Thus,

$$s=-\tfrac{1}{3}\psi(2)+\psi(1\tfrac{1}{2})-\tfrac{2}{3}\psi(1\tfrac{1}{4})=.047198.$$

Example 10.

Find $s=\sum_{n=1}^{\infty}\frac{1}{n^2(8n+1)^2}$.

Since $\frac{1}{n^2(8n+1)^2}=-\frac{16}{n}+\frac{16}{n+\frac{1}{8}}+\frac{1}{n^2}+\frac{1}{(n+\frac{1}{8})^2}$,

we have,

$\beta_1=0,\ \beta_2=\tfrac{1}{8},\ b_{11}=-16,\ b_{12}=16,\ b_{21}=1,\ b_{22}=1.$

Therefore

$$s=16\psi(1)-16\psi(1\tfrac{1}{8})+\psi'(1)+\psi'(1\tfrac{1}{8})=.013499.$$

Example 11.

Evaluate $s=\sum_{n=1}^{\infty}\frac{1}{(n^2+1)(n^2+4)}$ (see also **6.3.13**).

We have, $\frac{1}{(n^2+1)(n^2+4)}=\frac{i}{6}\left(\frac{1}{n+i}-\frac{1}{n-i}\right)$

$$-\frac{i}{12}\left(\frac{1}{n+2i}-\frac{1}{n-2i}\right).$$

Hence, $a_1=\frac{i}{6},\ a_2=\frac{-i}{6},\ a_3=\frac{-i}{12},\ a_4=\frac{i}{12},$

$$\alpha_1=i,\ \alpha_2=-i,\ \alpha_3=2i,\ \alpha_4=-2i,$$

and therefore

$$s=\frac{-i}{6}\left[\psi(1+i)-\psi(1-i)\right]+\frac{i}{12}\left[\psi(1+2i)-\psi(1-2i)\right].$$

By **6.3.9**, this reduces to

$$s=\frac{1}{3}\mathscr{I}\psi(1+i)-\frac{1}{6}\mathscr{I}\psi(1+2i).$$

From **Table 6.8**, $s=.13876$.

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Contains an extensive bibliography.

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Real and imaginary parts of $\ln\Gamma(\frac{1}{4}k+\frac{1}{2}ia)$, $k=0(1)3$, $a=0(.1)5(.2)20$, 8D; $(|\Gamma(\frac{3}{4}+\frac{1}{2}ia)/\Gamma(\frac{1}{4}+\frac{1}{2}ia)|)^{-1/2}$ $a=0(.02)1(.1)5(.2)20$, 8D.

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For references to tabular material on the incomplete gamma and incomplete beta functions, see the references in chapter **26**.

7. Error Function and Fresnel Integrals

WALTER GAUTSCHI[1]

Contents

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[1] Guest worker, National Bureau of Standards, from The American University. (Presently Purdue University.)

7. Error Function and Fresnel Integrals

Mathematical Properties

7.1. Error Function

Definitions

7.1.1 $$\operatorname{erf} z=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$$

7.1.2 $$\operatorname{erfc} z=\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}dt=1-\operatorname{erf} z$$

7.1.3 $$w(z)=e^{-z^2}\left(1+\frac{2i}{\sqrt{\pi}}\int_0^z e^{t^2}dt\right)=e^{-z^2}\operatorname{erfc}(-iz)$$

In **7.1.2** the path of integration is subject to the restriction $\arg t\rightarrow\alpha$ with $|\alpha|<\frac{\pi}{4}$ as $t\rightarrow\infty$ along the path. ($\alpha=\frac{\pi}{4}$ is permissible if $\mathscr{R}t^2$ remains bounded to the left.)

Integral Representation

7.1.4

$$w(z)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}dt}{z-t}=\frac{2iz}{\pi}\int_0^\infty\frac{e^{-t^2}dt}{z^2-t^2}\qquad(\mathscr{I}z>0)$$

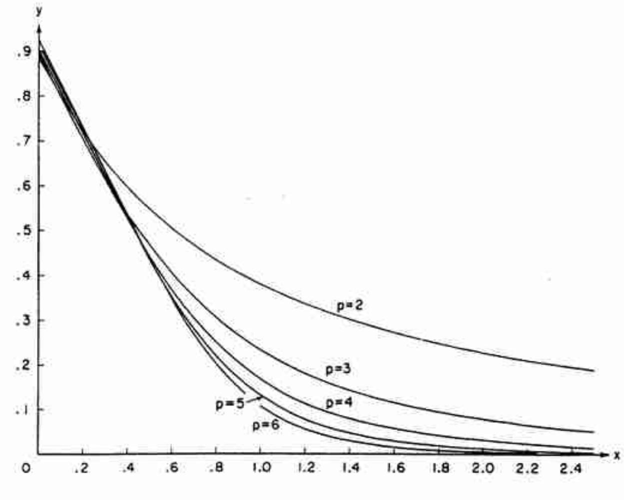

FIGURE 7.1. $y=e^{x^p}\int_x^\infty e^{-t^p}dt$.

$p=2(1)6$

Series Expansions

7.1.5 $$\operatorname{erf} z=\frac{2}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{n!(2n+1)}$$

7.1.6 $$=\frac{2}{\sqrt{\pi}}e^{-z^2}\sum_{n=0}^{\infty}\frac{2^n}{1\cdot 3\ldots(2n+1)}z^{2n+1}$$

7.1.7 $$=\sqrt{2}\sum_{n=0}^{\infty}(-1)^n[I_{2n+1/2}(z^2)-I_{2n+3/2}(z$$

7.1.8 $$w(z)=\sum_{n=0}^{\infty}\frac{(iz)^n}{\Gamma\left(\frac{n}{2}+1\right)}$$

For $I_{n-\frac{1}{2}}(x)$, see chapter **10.**

Symmetry Relations

7.1.9 $$\operatorname{erf}(-z)=-\operatorname{erf} z$$

7.1.10 $$\operatorname{erf}\bar{z}=\overline{\operatorname{erf} z}$$

7.1.11 $$w(-z)=2e^{-z^2}-w(z)$$

7.1.12 $$w(\bar{z})=\overline{w(-z)}$$

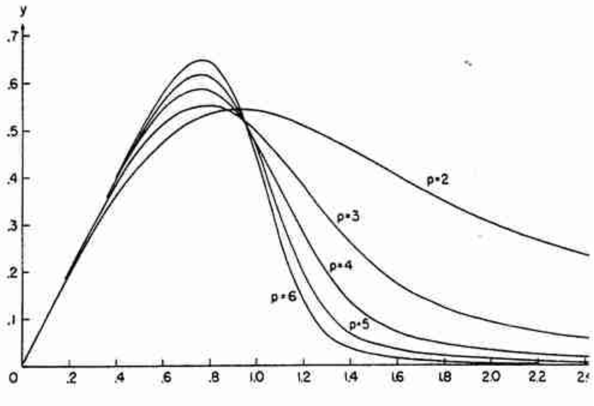

FIGURE 7.2. $y=e^{-x^p}\int_0^x e^{t^p}dt$.

$p=2(1)6$

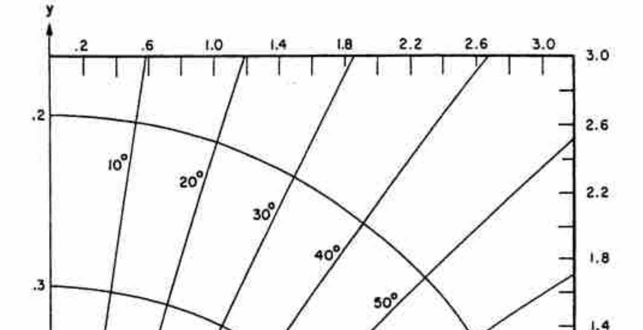

FIGURE 7.3. *Altitude Chart of $w(z)$.*

Inequalities [7.11], [7.17]

7.1.13

$$\frac{1}{x+\sqrt{x^2+2}}<e^{x^2}\int_x^\infty e^{-t^2}dt\leq\frac{1}{x+\sqrt{x^2+\frac{4}{\pi}}}\qquad(x\geq 0)$$

(For other inequalities see [7.2].)

Continued Fractions

7.1.14

$$2e^{z^2}\int_z^\infty e^{-t^2}dt=\frac{1}{z+}\,\frac{1/2}{z+}\,\frac{1}{z+}\,\frac{3/2}{z+}\,\frac{2}{z+}\cdots\quad(\mathscr{R}z>0)$$

7.1.15

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty\frac{e^{-t^2}dt}{z-t}=\frac{1}{z-}\,\frac{1/2}{z-}\,\frac{1}{z-}\,\frac{3/2}{z-}\,\frac{2}{z-}\cdots$$

$$=\frac{1}{\sqrt{\pi}}\lim_{n\to\infty}\sum_{k=1}^n\frac{H_k^{(n)}}{z-x_k^{(n)}}\qquad(\mathscr{I}z\neq 0)$$

$x_k^{(n)}$ and $H_k^{(n)}$ are the zeros and weight factors of the Hermite polynomials. For numerical values see chapter **25**.

Value at Infinity

7.1.16 $\operatorname{erf} z\to 1\left(z\to\infty\text{ in }|\arg z|<\frac{\pi}{4}\right)$

Maximum and Inflection Points for Dawson's Integral [7.31]

$$F(x)=e^{-x^2}\int_0^x e^{t^2}dt$$

7.1.17 $F(.92413\ 88730\ldots)=.54104\ 42246\ldots$

7.1.18 $F(1.50197\ 52682\ldots)=.42768\ 66160\ldots$

Derivatives

7.1.19

$$\frac{d^{n+1}}{dz^{n+1}}\operatorname{erf} z=(-1)^n\frac{2}{\sqrt{\pi}}H_n(z)e^{-z^2}\qquad(n=0,1,2,\ldots)$$

7.1.20

$$w^{(n+2)}(z)+2zw^{(n+1)}(z)+2(n+1)w^{(n)}(z)=0$$

$$(n=0,1,2,\ldots)$$

$$w^{(0)}(z)=w(z),\qquad w'(z)=-2zw(z)+\frac{2i}{\sqrt{\pi}}$$

(For the Hermite polynomials $H_n(z)$ see chapter **22**.)

Relation to Confluent Hypergeometric Function (see chapter 13)

7.1.21

$$\operatorname{erf} z=\frac{2z}{\sqrt{\pi}}M\left(\frac{1}{2},\frac{3}{2},-z^2\right)=\frac{2z}{\sqrt{\pi}}e^{-z^2}M\left(1,\frac{3}{2},z^2\right)$$

The Normal Distribution Function With Mean *m* and Standard Deviation σ (see chapter 26)

7.1.22 $$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^x e^{-\frac{(t-m)^2}{2\sigma^2}}dt=\frac{1}{2}\left(1+\operatorname{erf}\left(\frac{x-m}{\sigma\sqrt{2}}\right)\right)$$

Asymptotic Expansion

7.1.23

$$\sqrt{\pi}ze^{z^2}\operatorname{erfc} z\sim 1+\sum_{m=1}^\infty(-1)^m\frac{1\cdot 3\ldots(2m-1)}{(2z^2)^m}$$

$$\left(z\to\infty,\ |\arg z|<\frac{3\pi}{4}\right)$$

If $R_n(z)$ is the remainder after n terms then

7.1.24

$$R_n(z)=(-1)^n\frac{1\cdot 3\ldots(2n-1)}{(2z^2)^n}\theta,$$

$$\theta=\int_0^\infty e^{-t}\left(1+\frac{t}{z^2}\right)^{-n-\frac{1}{2}}dt \qquad \left(|\arg z|<\frac{\pi}{2}\right)$$

$$|\theta|<1 \qquad \left(|\arg z|<\frac{\pi}{4}\right)$$

For x real, $R_n(x)$ is less in absolute value than the first neglected term and of the same sign.

Rational Approximations[2] $(0\leq x<\infty)$

7.1.25

$$\operatorname{erf} x=1-(a_1t+a_2t^2+a_3t^3)\,e^{-x^2}+\epsilon(x), \qquad t=\frac{1}{1+px}$$

$$|\epsilon(x)|\leq 2.5\times 10^{-5}$$

$p=.47047$ $\quad a_1=.34802\ 42$ $\quad a_2=-.09587\ 98$ $\quad a_3=.74785\ 56$

7.1.26

$$\operatorname{erf} x=1-(a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5)\,e^{-x^2}+\epsilon(x),$$

$$t=\frac{1}{1+px}$$

$$|\epsilon(x)|\leq 1.5\times 10^{-7}$$

$p=.32759\ 11$ $\quad a_1=.25482\ 9592$

$a_2=-.28449\ 6736$ $\quad a_3=1.42141\ 3741$

$a_4=-1.45315\ 2027$ $\quad a_5=1.06140\ 5429$

7.1.27

$$\operatorname{erf} x=1-\frac{1}{[1+a_1x+a_2x^2+a_3x^3+a_4x^4]^4}+\epsilon(x)$$

$$|\epsilon(x)|\leq 5\times 10^{-4}$$

$a_1=.278393$ $\quad a_2=.230389$

$a_3=.000972$ $\quad a_4=.078108$

7.1.28

$$\operatorname{erf} x=1-\frac{1}{[1+a_1x+a_2x^2+\cdots+a_6x^6]^{16}}+\epsilon(x)$$

$$|\epsilon(x)|\leq 3\times 10^{-7}$$

$a_1=.07052\ 30784$ $\quad a_2=.04228\ 20123$

$a_3=.00927\ 05272$ $\quad a_4=.00015\ 20143$

$a_5=.00027\ 65672$ $\quad a_6=.00004\ 30638$

Infinite Series Approximation for Complex Error Function [7.19]

7.1.29

$$\operatorname{erf}(x+iy)=\operatorname{erf} x+\frac{e^{-x^2}}{2\pi x}[(1-\cos 2xy)+i\sin 2xy]$$

$$+\frac{2}{\pi}e^{-x^2}\sum_{n=1}^{\infty}\frac{e^{-\frac{1}{4}n^2}}{n^2+4x^2}[f_n(x,y)+ig_n(x,y)]+\epsilon(x,y)$$

where

$$f_n(x,y)=2x-2x\cosh ny\cos 2xy+n\sinh ny\sin 2xy$$

$$g_n(x,y)=2x\cosh ny\sin 2xy+n\sinh ny\cos 2xy$$

$$|\epsilon(x,y)|\approx 10^{-16}|\operatorname{erf}(x+iy)|$$

7.2. Repeated Integrals of the Error Function

Definition

7.2.1

$$\mathrm{i}^n\operatorname{erfc} z=\int_z^\infty \mathrm{i}^{n-1}\operatorname{erfc} t\,dt \qquad (n=0,1,2,\ldots)$$

$$\mathrm{i}^{-1}\operatorname{erfc} z=\frac{2}{\sqrt{\pi}}e^{-z^2},\ \mathrm{i}^0\operatorname{erfc} z=\operatorname{erfc} z$$

Differential Equation

7.2.2 $$\frac{d^2y}{dz^2}+2z\frac{dy}{dz}-2ny=0$$

$$y=A\mathrm{i}^n\operatorname{erfc} z+B\mathrm{i}^n\operatorname{erfc}(-z)$$

(A and B are constants.)

Expression as a Single Integral

7.2.3 $$\mathrm{i}^n\operatorname{erfc} z=\frac{2}{\sqrt{\pi}}\int_z^\infty\frac{(t-z)^n}{n!}e^{-t^2}dt$$

Power Series[3]

7.2.4 $$\mathrm{i}^n\operatorname{erfc} z=\sum_{k=0}^{\infty}\frac{(-1)^kz^k}{2^{n-k}k!\,\Gamma\left(1+\frac{n-k}{2}\right)}$$

Recurrence Relations

7.2.5

$$\mathrm{i}^n\operatorname{erfc} z=-\frac{z}{n}\mathrm{i}^{n-1}\operatorname{erfc} z+\frac{1}{2n}\mathrm{i}^{n-2}\operatorname{erfc} z$$

$$(n=1,2,3,\ldots)$$

7.2.6

$$2(n+1)(n+2)\mathrm{i}^{n+2}\operatorname{erfc} z$$

$$=(2n+1+2z^2)\mathrm{i}^n\operatorname{erfc} z-\frac{1}{2}\mathrm{i}^{n-2}\operatorname{erfc} z$$

$$(n=1,2,3,\ldots)$$

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[2] Approximations **7.1.25–7.1.28** are from C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N. J., 1955 (with permission).

[3] The terms in this series corresponding to $k=n+2$, $n+4$, $n+6$, . . . are understood to be zero.

Value at Zero

7.2.7

$$\mathrm{i}^n \operatorname{erfc} 0=\frac{1}{2^n\Gamma\left(\frac{n}{2}+1\right)} \qquad (n=-1,0,1,2,\ldots)$$

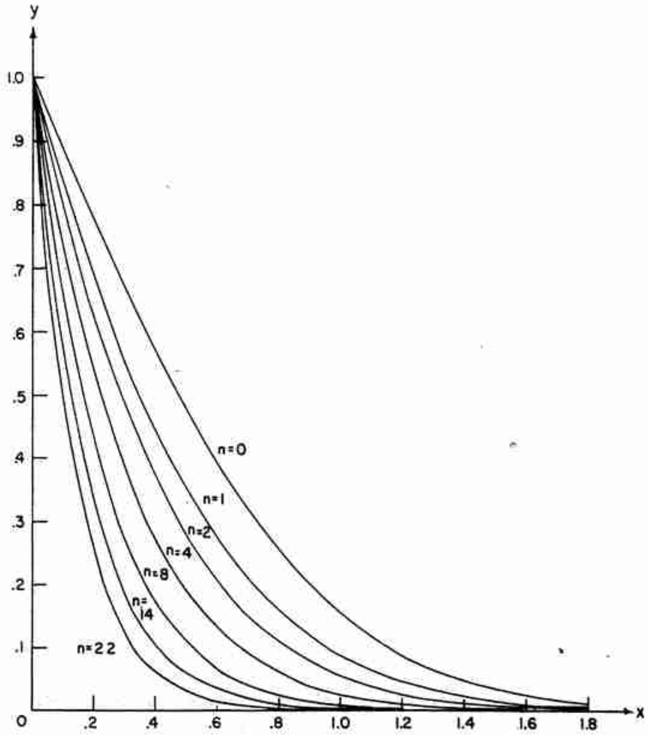

Figure 7.4. *Repeated Integrals of the Error Function.*

$$y=2^n\Gamma\left(\frac{n}{2}+1\right) \mathrm{i}^n \operatorname{erfc} x$$

$n=0, 1, 2, 4, 8, 14, 22$

Derivatives

7.2.8 $$\frac{d}{dz}\, \mathrm{i}^n \operatorname{erfc} z=-\mathrm{i}^{n-1} \operatorname{erfc} z \qquad (n=0,1,2,\ldots)$$

7.2.9

$$\frac{d^n}{dz^n}\,(e^{z^2} \operatorname{erfc} z)=(-1)^n2^nn!\,e^{z^2}\mathrm{i}^n \operatorname{erfc} z \qquad (n=0,1,2,\ldots)$$

Relation to $Hh_n(z)$ (see 19.14)

7.2.10 $$\mathrm{i}^n \operatorname{erfc} z=\frac{1}{(2^{n-1}\pi)^{\frac{1}{2}}}\, Hh_n(\sqrt{2}z)$$

Relation to Hermite Polynomials (see chapter 22)

7.2.11 $$(-1)^n\mathrm{i}^n \operatorname{erfc} z+\mathrm{i}^n \operatorname{erfc} (-z)=\frac{i^{-n}}{2^{n-1}n!}H_n(iz)$$

Relation to the Confluent Hypergeometric Function (see chapter 13)

7.2.12

$$\mathrm{i}^n \operatorname{erfc} z=e^{-z^2}\left[\frac{1}{2^n\Gamma\left(\frac{n}{2}+1\right)}M\left(\frac{n+1}{2},\frac{1}{2},z^2\right)-\frac{z}{2^{n-1}\Gamma\left(\frac{n+1}{2}\right)}M\left(\frac{n}{2}+1,\frac{3}{2},z^2\right)\right]$$

Relation to Parabolic Cylinder Functions (see chapter 19)

7.2.13 $$\mathrm{i}^n \operatorname{erfc} z=\frac{e^{-\frac{1}{2}z^2}}{(2^{n-1}\pi)^{\frac{1}{2}}}\, D_{-n-1}(z\sqrt{2})$$

Asymptotic Expansion

7.2.14

$$\mathrm{i}^n \operatorname{erfc} z\sim\frac{2}{\sqrt{\pi}}\,\frac{e^{-z^2}}{(2z)^{n+1}}\sum_{m=0}^{\infty}\frac{(-1)^m(2m+n)!}{n!m!(2z)^{2m}} \qquad \left(z\rightarrow\infty,\ |\arg z|<\frac{3\pi}{4}\right)$$

7.3. Fresnel Integrals

Definition

7.3.1 $$C(z)=\int_0^z \cos\left(\frac{\pi}{2}\,t^2\right)dt$$

7.3.2 $$S(z)=\int_0^z \sin\left(\frac{\pi}{2}\,t^2\right)dt$$

The following functions are also in use:

7.3.3

$$C_1(x)=\sqrt{\frac{2}{\pi}}\int_0^x \cos t^2dt,\ C_2(x)=\frac{1}{\sqrt{2\pi}}\int_0^x\frac{\cos t}{\sqrt{t}}\,dt$$

7.3.4

$$S_1(x)=\sqrt{\frac{2}{\pi}}\int_0^x \sin t^2dt,\ S_2(x)=\frac{1}{\sqrt{2\pi}}\int_0^x\frac{\sin t}{\sqrt{t}}\,dt$$

Auxiliary Functions

7.3.5

$$f(z)=\left[\frac{1}{2}-S(z)\right]\cos\left(\frac{\pi}{2}\,z^2\right)-\left[\frac{1}{2}-C(z)\right]\sin\left(\frac{\pi}{2}\,z^2\right)$$

7.3.6

$$g(z)=\left[\frac{1}{2}-C(z)\right]\cos\left(\frac{\pi}{2}\,z^2\right)+\left[\frac{1}{2}-S(z)\right]\sin\left(\frac{\pi}{2}\,z^2\right)$$

Interrelations

7.3.7 $$C(x)=C_1\left(x\sqrt{\frac{\pi}{2}}\right)=C_2\left(\frac{\pi}{2}\,x^2\right)$$

7.3.8 $$S(x)=S_1\left(x\sqrt{\frac{\pi}{2}}\right)=S_2\left(\frac{\pi}{2}x^2\right)$$

7.3.9 $$C(z)=\frac{1}{2}+f(z)\sin\left(\frac{\pi}{2}z^2\right)-g(z)\cos\left(\frac{\pi}{2}z^2\right)$$

7.3.10 $$S(z)=\frac{1}{2}-f(z)\cos\left(\frac{\pi}{2}z^2\right)-g(z)\sin\left(\frac{\pi}{2}z^2\right)$$

Series Expansions

7.3.11 $$C(z)=\sum_{n=0}^{\infty}\frac{(-1)^n(\pi/2)^{2n}}{(2n)!(4n+1)}z^{4n+1}$$

7.3.12

$$C(z)=\cos\left(\frac{\pi}{2}z^2\right)\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{1\cdot3\ldots(4n+1)}z^{4n+1}$$
$$+\sin\left(\frac{\pi}{2}z^2\right)\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{1\cdot3\ldots(4n+3)}z^{4n+3}$$

7.3.13 $$S(z)=\sum_{n=0}^{\infty}\frac{(-1)^n(\pi/2)^{2n+1}}{(2n+1)!(4n+3)}z^{4n+3}$$

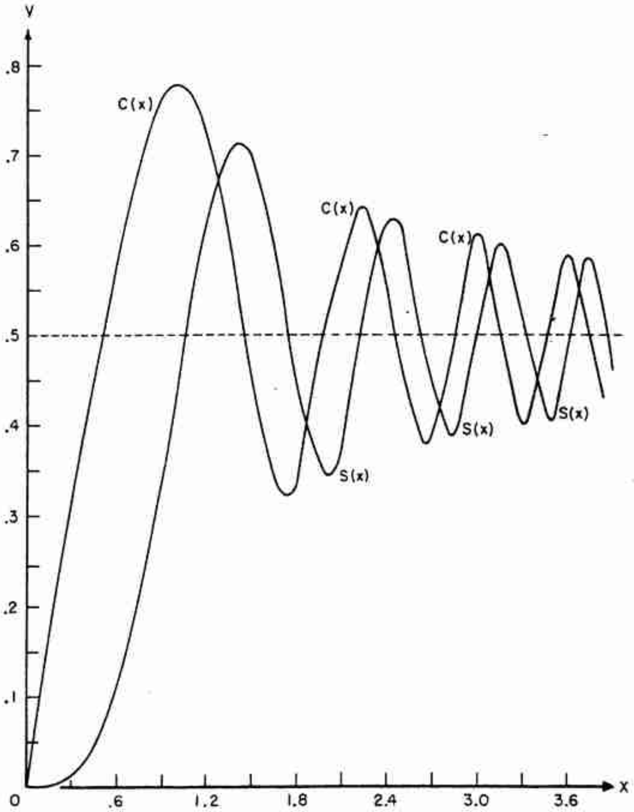

FIGURE 7.5. *Fresnel Integrals.*
$y=C(x)$, $y=S(x)$

7.3.14

$$S(z)=-\cos\left(\frac{\pi}{2}z^2\right)\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n+1}}{1\cdot3\ldots(4n+3)}z^{4n+3}$$
$$+\sin\left(\frac{\pi}{2}z^2\right)\sum_{n=0}^{\infty}\frac{(-1)^n\pi^{2n}}{1\cdot3\ldots(4n+1)}z^{4n+1}$$

7.3.15 $$C_2(z)=J_{1/2}(z)+J_{5/2}(z)+J_{9/2}(z)+\ldots$$

7.3.16 $$S_2(z)=J_{3/2}(z)+J_{7/2}(z)+J_{11/2}(z)+\ldots$$

For Bessel functions $J_{n+1/2}(z)$ see chapter **10.**

Symmetry Relations

7.3.17 $$C(-z)=-C(z),\quad S(-z)=-S(z)$$

7.3.18 $$C(iz)=iC(z),\quad S(iz)=-iS(z)$$

7.3.19 $$C(\bar{z})=\overline{C(z)},\quad S(\bar{z})=\overline{S(z)}$$

Value at Infinity

7.3.20 $$C(x)\to\frac{1}{2},\qquad S(x)\to\frac{1}{2}\qquad(x\to\infty)$$

Derivatives

7.3.21 $$\frac{df(x)}{dx}=-\pi xg(x),\qquad\frac{dg(x)}{dx}=\pi xf(x)-1$$

Relation to Error Function (see 7.1.1, 7.1.3)

7.3.22

$$C(z)+iS(z)=\frac{1+i}{2}\operatorname{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)z\right]$$
$$=\frac{1+i}{2}\left\{1-e^{i\frac{\pi}{2}z^2}w\left[\frac{\sqrt{\pi}}{2}(1+i)z\right]\right\}$$

7.3.23 $$g(x)=\mathscr{R}\left\{\frac{1+i}{2}w\left[\frac{\sqrt{\pi}}{2}(1+i)x\right]\right\}$$

7.3.24 $$f(x)=\mathscr{I}\left\{\frac{1+i}{2}w\left[\frac{\sqrt{\pi}}{2}(1+i)x\right]\right\}$$

Relation to Confluent Hypergeometric Function (see chapter 13)

7.3.25

$$C(z)+iS(z)=zM\left(\frac{1}{2},\frac{3}{2},i\frac{\pi}{2}z^2\right)$$
$$=ze^{i\frac{\pi}{2}z^2}M\left(1,\frac{3}{2},-i\frac{\pi}{2}z^2\right)$$

Relation to Spherical Bessel Functions (see chapter 10)

7.3.26 $$C_2(z)=\frac{1}{2}\int_0^z J_{-\frac{1}{2}}(t)dt,\ S_2(z)=\frac{1}{2}\int_0^z J_{\frac{1}{2}}(t)dt$$

Asymptotic Expansions

7.3.27

$$\pi z f(z) \sim 1+\sum_{m=1}^{\infty}(-1)^m \frac{1\cdot 3 \ldots (4m-1)}{(\pi z^2)^{2m}} \qquad \left(z\to\infty,\ |\arg z|<\frac{\pi}{2}\right)$$

7.3.28

$$\pi z g(z) \sim \sum_{m=0}^{\infty}(-1)^m \frac{1\cdot 3 \ldots (4m+1)}{(\pi z^2)^{2m+1}} \qquad \left(z\to\infty,\ |\arg z|<\frac{\pi}{2}\right)$$

If $R_n^{(f)}(z)$, $R_n^{(g)}(z)$ are the remainders after n terms in **7.3.27**, **7.3.28**, respectively, then

7.3.29

$$R_n^{(f)}(z)=(-1)^n\frac{1\cdot 3\ldots(4n-1)}{(\pi z^2)^{2n}}\theta^{(f)},$$

$$\theta^{(f)}=\frac{1}{\Gamma(2n+\frac{1}{2})}\int_0^\infty \frac{e^{-t}t^{2n-\frac{1}{2}}}{1+\left(\frac{2t}{\pi z^2}\right)^2}dt \left(|\arg z|<\frac{\pi}{4}\right)$$

7.3.30

$$R_n^{(g)}(z)=(-1)^n\frac{1\cdot 3\ldots(4n+1)}{(\pi z^2)^{2n}}\theta^{(g)},$$

$$\theta^{(g)}=\frac{1}{\Gamma(2n+\frac{3}{2})}\int_0^\infty \frac{e^{-t}t^{2n+\frac{1}{2}}}{1+\left(\frac{2t}{\pi z^2}\right)^2}dt \left(|\arg z|<\frac{\pi}{4}\right)$$

7.3.31 $|\theta^{(f)}|<1,\ |\theta^{(g)}|<1 \qquad \left(|\arg z|\leq\frac{\pi}{8}\right)$

For x real, $R_n^{(f)}(x)$ and $R_n^{(g)}(x)$ are less in absolute value than the first neglected term and of the same sign.

Rational Approximations[4] $(0\leq x\leq\infty)$

7.3.32

$$f(x)=\frac{1+.926x}{2+1.792x+3.104x^2}+\epsilon(x) \qquad |\epsilon(x)|\leq 2\times 10^{-3}$$

7.3.33

$$g(x)=\frac{1}{2+4.142x+3.492x^2+6.670x^3}+\epsilon(x) \qquad |\epsilon(x)|\leq 2\times 10^{-3}$$

(For more accurate approximations see [7.1].)

7.4. Definite and Indefinite Integrals

For a more extensive list of integrals see [7.5], [7 8], [7.15].

7.4.1 $$\int_0^\infty e^{-t^2}dt=\frac{\sqrt{\pi}}{2}$$

7.4.2

$$\int_0^\infty e^{-(at^2+2bt+c)}dt=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{\frac{b^2-ac}{a}}\operatorname{erfc}\frac{b}{\sqrt{a}} \qquad (\mathscr{R}a>0)$$

7.4.3

$$\int_0^\infty e^{-at^2-\frac{b}{t^2}}dt=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-2\sqrt{ab}} \qquad (\mathscr{R}a>0,\ \mathscr{R}b>0)$$

7.4.4

$$\int_0^\infty t^{2n}e^{-at^2}dt=\frac{1\cdot 3\ldots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}$$

$$=\frac{\Gamma(n+\frac{1}{2})}{2a^{n+\frac{1}{2}}} \qquad (\mathscr{R}a>0; n=0, 1, 2, \ldots)$$

7.4.5

$$\int_0^\infty t^{2n+1}e^{-at^2}dt=\frac{n!}{2a^{n+1}} \qquad (\mathscr{R}a>0; n=0, 1, 2, \ldots)$$

7.4.6

$$\int_0^\infty e^{-at^2}\cos(2xt)dt=\frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{-\frac{x^2}{a}} \qquad (\mathscr{R}a>0)$$

7.4.7

$$\int_0^\infty e^{-at^2}\sin(2xt)dt=\frac{1}{\sqrt{a}}\,e^{-x^2/a}\int_0^{x/\sqrt{a}}e^{t^2}dt \qquad (\mathscr{R}a>0)$$

7.4.8

$$\int_0^\infty \frac{e^{-at}dt}{\sqrt{t+z^2}}=\sqrt{\frac{\pi}{a}}\,e^{az^2}\operatorname{erfc}\sqrt{a}z \qquad (\mathscr{R}a>0,\ \mathscr{R}z>0)$$

7.4.9

$$\int_0^\infty \frac{e^{-at}dt}{\sqrt{t}(t+z)}=\frac{\pi}{\sqrt{z}}\,e^{az}\operatorname{erfc}\sqrt{az} \qquad (\mathscr{R}a>0,\ z\neq 0, |\arg z|<\pi)$$

7.4.10

$$\int_0^\infty \frac{e^{-at^2}dt}{t+x}=e^{-ax^2}\left[\sqrt{\pi}\int_0^{\sqrt{a}x}e^{t^2}dt-\frac{1}{2}\operatorname{Ei}(ax^2)\right] \quad * \qquad (a>0, x>0)$$

7.4.11

$$\int_0^\infty \frac{e^{-at^2}dt}{t^2+x^2}=\frac{\pi}{2x}\,e^{ax^2}\operatorname{erfc}\sqrt{a}x \qquad (a>0, x>0)$$

7.4.12 $$\int_0^1 \frac{e^{-at^2}dt}{t^2+1}=\frac{\pi}{4}\,e^a[1-(\operatorname{erf}\sqrt{\bar{a}})^2] \qquad (a>0)$$

7.4.13

$$\int_{-\infty}^\infty \frac{ye^{-t^2}dt}{(x-t)^2+y^2}=\pi\,\mathscr{R}w(x+iy) \qquad (x \text{ real}, y>0)$$

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[4] Approximations **7.3.32**, **7.3.33** are based on those given in C. Hastings, Jr., Approximations for calculating Fresnel integrals, Approximation Newsletter, April 1956, Note 10. [See also MTAC **10**, 173, 1956.]

*See page II.

7.4.14

$$\int_{-\infty}^{\infty} \frac{(x-t)e^{-t^2}dt}{(x-t)^2+y^2}=\pi\mathscr{I}w(x+iy) \qquad (x \text{ real}, y>0)$$

7.4.15

$$\int_0^{\infty} \frac{[t^2-(x^2-y^2)]e^{-t^2}dt}{t^4-2(x^2-y^2)t^2+(x^2+y^2)^2}=\frac{\pi}{2}\mathscr{R}\frac{w(x+iy)}{y-ix}$$

$$(x \text{ real}, y>0)$$

7.4.16

$$\int_0^{\infty} \frac{2xye^{-t^2}dt}{t^4-2(x^2-y^2)t^2+(x^2+y^2)^2}=\frac{\pi}{2}\mathscr{I}\frac{w(x+iy)}{y-ix}$$

$$(x \text{ real}, y>0)$$

7.4.17

$$\int_0^{\infty} e^{-at} \operatorname{erf} bt\, dt=\frac{1}{a}e^{\frac{a^2}{4b^2}} \operatorname{erfc}\frac{a}{2b}$$

$$\left(\mathscr{R}a>0, |\arg b|<\frac{\pi}{4}\right)$$

7.4.18

$$\int_0^{\infty} \sin(2at) \operatorname{erfc} bt\, dt=\frac{1}{2a}[1-e^{-(a/b)^2}] (a>0, \mathscr{R}b>0)$$

7.4.19

$$\int_0^{\infty} e^{-at} \operatorname{erf}\sqrt{bt}\, dt=\frac{1}{a}\sqrt{\frac{b}{a+b}} \qquad (\mathscr{R}(a+b)>0)$$

7.4.20

$$\int_0^{\infty} e^{-at} \operatorname{erfc}\sqrt{\frac{b}{t}}\, dt=\frac{1}{a}e^{-2\sqrt{ab}} \qquad (\mathscr{R}a>0, \mathscr{R}b>0)$$

7.4.21

$$\int_0^{\infty} e^{(a-b)t} \operatorname{erfc}\left(\sqrt{at}+\sqrt{\frac{c}{t}}\right)dt=\frac{e^{-2(\sqrt{ac}+\sqrt{bc})}}{\sqrt{b}(\sqrt{a}+\sqrt{b})}$$

$$(\mathscr{R}b>0, \mathscr{R}c>0)$$

7.4.22

$$\int_0^{\infty} e^{-at} \cos(t^2)dt=\sqrt{\frac{\pi}{2}}\left\{\left[\frac{1}{2}-S\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right)\right]\cos\left(\frac{a^2}{4}\right)\right.$$

$$\left.-\left[\frac{1}{2}-C\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right)\right]\sin\left(\frac{a^2}{4}\right)\right\} \qquad (\mathscr{R}a>0)$$

7.4.23

$$\int_0^{\infty} e^{-at} \sin(t^2)dt=\sqrt{\frac{\pi}{2}}\left\{\left[\frac{1}{2}-C\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right)\right]\cos\left(\frac{a^2}{4}\right)\right.$$

$$\left.+\left[\frac{1}{2}-S\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right)\right]\sin\left(\frac{a^2}{4}\right)\right\} \qquad (\mathscr{R}a>0)$$

7.4.24

$$\int_0^{\infty} e^{-at}\frac{\sin(t^2)}{t}dt=\frac{\pi}{2}\left[\frac{1}{2}-C\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right)\right]^2$$

$$+\frac{\pi}{2}\left[\frac{1}{2}-S\left(\frac{a}{2}\sqrt{\frac{2}{\pi}}\right)\right]^2 \qquad (\mathscr{R}a>0)$$

7.4.25

$$\int_0^{\infty}\frac{e^{-at}\sqrt{t}}{t^2+b^2}dt=\pi\sqrt{\frac{2}{b}}\left\{\left[\frac{1}{2}-C\left(\sqrt{\frac{2ab}{\pi}}\right)\right]\cos(ab)\right.$$

$$\left.+\left[\frac{1}{2}-S\left(\sqrt{\frac{2ab}{\pi}}\right)\right]\sin(ab)\right\} \qquad (\mathscr{R}a>0, \mathscr{R}b>0)$$

7.4.26

$$\int_0^{\infty}\frac{e^{-at}dt}{\sqrt{t}(t^2+b^2)}=\frac{\pi}{b}\sqrt{\frac{2}{b}}\left\{\left[\frac{1}{2}-S\left(\sqrt{\frac{2ab}{\pi}}\right)\right]\cos(ab)\right.$$

$$\left.-\left[\frac{1}{2}-C\left(\sqrt{\frac{2ab}{\pi}}\right)\right]\sin(ab)\right\} \qquad (\mathscr{R}a>0, \mathscr{R}b>0)$$

7.4.27

$$\int_0^{\infty} e^{-at}C(t)dt=\frac{1}{a}\left\{\left[\frac{1}{2}-S\left(\frac{a}{\pi}\right)\right]\cos\left(\frac{a^2}{2\pi}\right)\right.$$

$$\left.-\left[\frac{1}{2}-C\left(\frac{a}{\pi}\right)\right]\sin\left(\frac{a^2}{2\pi}\right)\right\} \qquad (\mathscr{R}a>0)$$

7.4.28

$$\int_0^{\infty} e^{-at}S(t)dt=\frac{1}{a}\left\{\left[\frac{1}{2}-C\left(\frac{a}{\pi}\right)\right]\cos\left(\frac{a^2}{2\pi}\right)\right.$$

$$\left.+\left[\frac{1}{2}-S\left(\frac{a}{\pi}\right)\right]\sin\left(\frac{a^2}{2\pi}\right)\right\} \qquad (\mathscr{R}a>0)$$

7.4.29

$$\int_0^{\infty} e^{-at}C\left(\sqrt{\frac{2t}{\pi}}\right)dt=\frac{1}{2a(\sqrt{a^2+1}-a)^{\frac{1}{2}}\sqrt{a^2+1}}$$

$$(\mathscr{R}a>0)$$

7.4.30

$$\int_0^{\infty} e^{-at}S\left(\sqrt{\frac{2t}{\pi}}\right)dt=\frac{1}{2a(\sqrt{a^2+1}+a)^{\frac{1}{2}}\sqrt{a^2+1}}$$

$$(\mathscr{R}a>0)$$

7.4.31 $\int_0^{\infty}\left\{\left[\frac{1}{2}-C(t)\right]^2+\left[\frac{1}{2}-S(t)\right]^2\right\}dt=\frac{1}{\pi}$

7.4.32

$$\int e^{-(ax^2+2bx+c)}dx=\frac{1}{2}\sqrt{\frac{\pi}{a}}e^{\frac{b^2-ac}{a}}\operatorname{erf}\left(\sqrt{a}x+\frac{b}{\sqrt{a}}\right)+\text{const.}$$

$$(a\neq 0)$$

7.4.33

$$\int e^{-a^2x^2-\frac{b^2}{x^2}}dx=\frac{\sqrt{\pi}}{4a}\left[e^{2ab}\operatorname{erf}\left(ax+\frac{b}{x}\right)\right.$$
$$\left.+e^{-2ab}\operatorname{erf}\left(ax-\frac{b}{x}\right)\right]+\text{const.}\qquad (a\neq 0)$$

7.4.34

$$\int e^{-a^2x^2+\frac{b^2}{x^2}}dx=-\frac{\sqrt{\pi}}{4a}\,e^{-a^2x^2+\frac{b^2}{x^2}}\left[w\left(\frac{b}{x}+iax\right)\right.$$
$$\left.+w\left(-\frac{b}{x}+iax\right)\right]+\text{const.}\qquad (a\neq 0)$$

7.4.35 $$\int \operatorname{erf} x\,dx=x\operatorname{erf} x+\frac{1}{\sqrt{\pi}}\,e^{-x^2}+\text{const.}$$

7.4.36

$$\int e^{ax}\operatorname{erf} bx\,dx=\frac{1}{a}\left[e^{ax}\operatorname{erf} bx-e^{\frac{a^2}{4b^2}}\operatorname{erf}\left(bx-\frac{a}{2b}\right)\right]$$
$$+\text{const.}\qquad (a\neq 0)$$

7.4.37

$$\int e^{ax}\operatorname{erf}\sqrt{\frac{b}{x}}\,dx=\frac{1}{a}\left\{e^{ax}\operatorname{erf}\sqrt{\frac{b}{x}}\right.$$
$$\left.+\frac{1}{2}\,e^{ax-\frac{b}{x}}\left[w\left(\sqrt{ax}+i\sqrt{\frac{b}{x}}\right)+w\left(-\sqrt{ax}+i\sqrt{\frac{b}{x}}\right)\right]\right\}$$
$$+\text{const.}\qquad (a\neq 0)$$

7.4.38

$$\int \cos\,(ax^2+2bx+c)dx$$
$$=\sqrt{\frac{\pi}{2a}}\left\{\cos\left(\frac{b^2-ac}{a}\right)C\left[\sqrt{\frac{2}{a\pi}}\,(ax+b)\right]\right.$$
$$\left.+\sin\left(\frac{b^2-ac}{a}\right)S\left[\sqrt{\frac{2}{a\pi}}\,(ax+b)\right]\right\}+\text{const.}$$

7.4.39

$$\int \sin\,(ax^2+2bx+c)dx$$
$$=\sqrt{\frac{\pi}{2a}}\left\{\cos\left(\frac{b^2-ac}{a}\right)S\left[\sqrt{\frac{2}{a\pi}}\,(ax+b)\right]\right.$$
$$\left.-\sin\left(\frac{b^2-ac}{a}\right)C\left[\sqrt{\frac{2}{a\pi}}\,(ax+b)\right]\right\}+\text{const.}$$

7.4.40 $$\int C(x)dx=xC(x)-\frac{1}{\pi}\sin\left(\frac{\pi}{2}\,x^2\right)+\text{const.}$$

7.4.41 $$\int S(x)dx=xS(x)+\frac{1}{\pi}\cos\left(\frac{\pi}{2}\,x^2\right)+\text{const.}$$

Numerical Methods

7.5. Use and Extension of the Tables

Example 1. Compute erf .745 and $e^{-(.745)^2}$ using Taylor's series.

With the aid of Taylor's theorem and **7.1.19** it can be shown that

$$\operatorname{erf}\,(x_0+ph)=\operatorname{erf} x_0$$
$$+\frac{2}{\sqrt{\pi}}\,e^{-x_0^2}ph\left[1-phx_0+\frac{1}{3}\,p^2h^2(2x_0^2-1)\right]+\epsilon$$

$$e^{-(x_0+ph)^2}=e^{-x_0^2}\left[1-2phx_0+p^2h^2\,(2x_0^2-1)\right.$$
$$\left.-\frac{2}{3}\,p^3h^3x_0(2x_0^2-3)\right]+\eta$$

where $|\epsilon|<1.2\times 10^{-10}$, $|\eta|<3.2\times 10^{-10}$ if $h=10^{-2}$, $|p|\leq\frac{1}{2}$. With $x_0=.74$, $p=.5$ and using **Table 7.1**

$$\operatorname{erf}\,.745=.70467\;80779+(.5)(.00652\;58247)\times$$
$$[1-(.005)(.74)+(.00000\;83333)(.0952)]$$
$$=.70792\;8920$$

$$e^{-(.745)^2}=\frac{\sqrt{\pi}}{2}\,(.65258\;24665)\;[1-.0074$$
$$+(.000025)(.0952)+(.00000\;\;00833)(.74)(1.9048)]$$
$$=.57405\;7910.$$

As a check the computation was repeated with $x_0=.75$, $p=-.5$.

Example 2. Compute erfc x to 5S for $x=4.8$.

We have $1/x^2=.0434028$. With **Table 7.2** and linear interpolation in **Table 7.3,** we obtain

$$\operatorname{erfc} 4.8=\frac{1}{4.8}\,(1.11253)(10^{-10})(.552669)\,\frac{\sqrt{\pi}}{2}$$
$$=(1.1352)10^{-11}.$$

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and

$$D_p(x)=(p!)^{-1}\int_0^\infty \epsilon^p(\epsilon^2+x^2)^{-2}e^{-\epsilon}d\epsilon$$

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$z=x$; $x=0(.001)2(.01)10$; 5D;
$z=\rho e^{i\theta}$; $\theta=2.5°(2.5°)30°(1.25°)35°(.625°)40°$;
$\rho=\rho_\theta(.001)\rho_\theta'(.01)\rho_\theta''(.0002)5$, $0\le\rho_\theta\le\rho_\theta'\le\rho_\theta''\le5$, 5D;
$z=iy$; $y=0(.001)3(.0002)5$, 5S.

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$\theta=45°(.3125°)48.75°(.625°)55°(1.25°)65°(2.5°)90°$,
$\rho=\rho_\theta(.001)\rho_\theta'(.01)\rho_\theta''$, $0\le\rho_\theta<\rho_\theta'\le\rho_\theta''\le5$, 5D;
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COMPLEX ZEROS OF THE ERROR FUNCTION

Table 7.10

$$\operatorname{erf} z_n=0 \qquad z_n=x_n+iy_n$$

n	x_n	y_n	n	x_n	y_n
1	1.45061 616	1.88094 300	6	4.15899 840	4.43557 144
2	2.24465 928	2.61657 514	7	4.51631 940	4.78044 764
3	2.83974 105	3.17562 810	8	4.84797 031	5.10158 804
4	3.33546 074	3.64617 438	9	5.15876 791	5.40333 264
5	3.76900 557	4.06069 723	10	5.45219 220	5.68883 744

$$\operatorname{erf} z_n=\operatorname{erf}(-z_n)=\operatorname{erf}\bar{z}_n=\operatorname{erf}(-\bar{z}_n)=0$$

$$\begin{matrix}x_n\\ y_n\end{matrix}\approx\frac{1}{2}\sqrt{\pi\left(4n-\frac{1}{2}\right)}\mp\frac{\ln\left(\pi\sqrt{2n-\frac{1}{4}}\right)}{2\sqrt{\pi\left(4n-\frac{1}{2}\right)}} \qquad (n>0)$$

From H. E. Salzer, Complex zeros of the error function, J. Franklin Inst. **260,** 209–211, 1955 (with permission).

COMPLEX ZEROS OF FRESNEL INTEGRALS

Table 7.11

$$C(z_n)=0 \qquad z_n=x_n+iy_n$$

$$S(z_n^*)=0 \qquad z_n^*=x_n^*+iy_n^*$$

n	x_n	y_n	x_n^*	y_n^*
0	0.0000	0.0000	0.0000	0.0000
1	1.7437	0.3057	2.0093	0.2886
2	2.6515	0.2529	2.8335	0.2443
3	3.3208	0.2239	3.4675	0.2185
4	3.8759	0.2047	4.0026	0.2008
5	4.3611	0.1909	4.4742	0.1877

$$\substack{C\\ S}(z_n)=\substack{C\\ S}(-z_n)=\substack{C\\ S}(\bar{z}_n)=\substack{C\\ S}(-\bar{z}_n)=\substack{C\\ S}(iz_n)=\substack{C\\ S}(-iz_n)=\substack{C\\ S}(-i\bar{z}_n)=\substack{C\\ S}(i\bar{z}_n)=0$$

$$x_n\approx\sqrt{4n-1}-\frac{\ln(\pi\sqrt{4n-1})}{\pi^2(4n-1)^{3/2}} \qquad y_n\approx\frac{\ln(\pi\sqrt{4n-1})}{\pi\sqrt{4n-1}}$$

$$(n>0)$$

$$x_n^*\approx2\sqrt{n}-\frac{\ln(2\pi\sqrt{n})}{8\pi^2n^{3/2}} \qquad y_n^*\approx\frac{\ln(2\pi\sqrt{n})}{2\pi\sqrt{n}}$$

MAXIMA AND MINIMA OF FRESNEL INTEGRALS

Table 7.12

$$M_n=C(\sqrt{4n+1}) \qquad m_n=C(\sqrt{4n+3}) \qquad M_n^*=S(\sqrt{4n+2}) \qquad m_n^*=S(\sqrt{4n+4})$$

n	M_n	m_n	M_n^*	m_n^*
0	0.779893	0.321056	0.713972	0.343415
1	0.640807	0.380389	0.628940	0.387969
2	0.605721	0.404260	0.600361	0.408301
3	0.588128	0.417922	0.584942	0.420516
4	0.577121	0.427036	0.574957	0.428877
5	0.569413	0.433666	0.567822	0.435059

$$M_n\sim\frac{1}{2}+\frac{\pi^2(4n+1)^2-3}{\pi^3(4n+1)^{5/2}} \qquad m_n\sim\frac{1}{2}-\frac{\pi^2(4n+3)^2-3}{\pi^3(4n+3)^{5/2}} \qquad (n\to\infty)$$

$$M_n^*\sim\frac{1}{2}+\frac{\pi^2(4n+2)^2-3}{\pi^3(4n+2)^{5/2}} \qquad m_n^*\sim\frac{1}{2}-\frac{16\pi^2(n+1)^2-3}{32\pi^3(n+1)^{5/2}}$$

From G. N. Watson, A treatise on the theory of Bessel functions, 2d ed. Cambridge Univ. Press, Cambridge, England, 1958 (with permission).

8. Legendre Functions

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Contents

The author acknowledges the assistance of Ruth E. Capuano, Elizabeth F. Godefroy, David S. Liepman, and Bertha H. Walter in the preparation and checking of the tables and examples.

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[1] National Bureau of Standards.

8. Legendre Functions

Mathematical Properties

Notation

The conventions used are $z=x+iy$, x, y real, and in particular, x always means a real number in the interval $-1 \leq x \leq +1$ with $\cos\theta = x$ where θ is likewise a real number; n and m are positive integers or zero; ν and μ are unrestricted except where otherwise indicated.

Other notations are:

$$P^n(x) \text{ for } \frac{n!P_n(x)}{(2n-1)!!}$$

$$P_{nm}(x) \text{ for } (-1)^m P_n^m(x)$$

$$T_n^m(x) \text{ for } (-1)^m P_n^m(x)$$

$$\overline{P_n^m}(x) \text{ for } (-1)^m \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}}\, P_n^m(x)$$

$$\mathfrak{P}_\nu^\mu(z) \text{ for } P_\nu^\mu(z),\ \mathfrak{Q}_\nu^\mu(z) \text{ for } Q_\nu^\mu(z) \qquad (\mathscr{R}z>1)$$

$$\mathfrak{Q}_\nu^\mu(z) \text{ for } e^{\mu\pi i} Q_\nu^\mu(z)$$

$$Q_\nu^\mu(z) \text{ for } \frac{\sin(\nu+u)\pi}{\sin\nu\pi}\, Q_\nu^\mu(z)$$

Various other definitions of the functions occur as well as mixing of definitions.

8.1. Differential Equation

8.1.1

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right]w = 0$$

Solutions

(Degree ν and order μ with singularities at $z=\pm 1$, ∞ as ordinary branch points—μ, ν arbitrary complex constants.)

$P_\nu^\mu(z)$, $Q_\nu^\mu(z)$—Associated Legendre Functions (Spherical Harmonics) of the First and Second Kinds [2]

$$|\arg(z\pm 1)| < \pi, \qquad |\arg z| < \pi$$

$$(z^2-1)^{\frac{1}{2}\mu} = (z-1)^{\frac{1}{2}\mu}(z+1)^{\frac{1}{2}\mu}$$

(For $P_\nu^\mu(z)$, $\mu=0$, Legendre polynomials, see chapter **22.**)

8.1.2

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left[\frac{z+1}{z-1}\right]^{\frac{1}{2}\mu} F\left(-\nu, \nu+1; 1-\mu; \frac{1-z}{2}\right) \qquad (|1-z|<2)$$

(For $F(a, b; c; z)$ see chapter **15.**)

8.1.3

$$Q_\nu^\mu(z) = e^{i\mu\pi} 2^{-\nu-1}\pi^{\frac{1}{2}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} z^{-\nu-\mu-1}(z^2-1)^{\frac{1}{2}\mu} F\left(1+\frac{\nu}{2}+\frac{\mu}{2}, \frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2}; \nu+\frac{3}{2}; \frac{1}{z^2}\right) \qquad (|z|>1)$$

Alternate Forms

(Additional forms may be obtained by means of the transformation formulas of the hypergeometric function, see [8.1].)

8.1.4

$$P_\nu^\mu(z) = 2^\mu \pi^{\frac{1}{2}} (z^2-1)^{-\frac{1}{2}\mu} \left\{ \frac{F\left(-\frac{\nu}{2}-\frac{\mu}{2}, \frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}; \frac{1}{2}; z^2\right)}{\Gamma\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}\right)\Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}\right)} - 2z \frac{F\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}, 1+\frac{\nu}{2}-\frac{\mu}{2}; \frac{3}{2}; z^2\right)}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}\right)\Gamma\left(-\frac{\nu}{2}-\frac{\mu}{2}\right)} \right\} \qquad (|z^2|<1)$$

8.1.5

$$\begin{aligned} P_\nu^\mu(z) = {} & \frac{2^{-\nu-1}\pi^{-\frac{1}{2}}\Gamma(-\frac{1}{2}-\nu) z^{-\nu+\mu-1}}{(z^2-1)^{\mu/2}\Gamma(-\nu-\mu)} F\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}, 1+\frac{\nu}{2}-\frac{\mu}{2}; \nu+\frac{3}{2}; z^{-2}\right) \\ & + \frac{2^\nu \Gamma(\frac{1}{2}+\nu) z^{\nu+\mu}}{(z^2-1)^{\mu/2}\Gamma(1+\nu-\mu)} F\left(-\frac{\nu}{2}-\frac{\mu}{2}, \frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2}; \frac{1}{2}-\nu; z^{-2}\right) \qquad (|z^{-2}|<1) \end{aligned}$$

8.1.6

$$\begin{aligned} e^{-i\mu\pi} Q_\nu^\mu(z) = {} & \frac{\Gamma(1+\nu+\mu)\Gamma(-\mu)(z-1)^{\frac{1}{2}\mu}(z+1)^{-\frac{1}{2}\mu}}{2\Gamma(1+\nu-\mu)} F\left(-\nu, 1+\nu; 1+\mu; \frac{1-z}{2}\right) \\ & + \frac{1}{2}\Gamma(\mu)(z+1)^{\frac{1}{2}\mu}(z-1)^{-\frac{1}{2}\mu} F\left(-\nu, 1+\nu; 1-\mu; \frac{1-z}{2}\right) \qquad (|1-z|<2) \end{aligned}$$ *

---

[2] The functions $Y_n^m(\theta, \varphi) = \left.\begin{matrix}\cos m\varphi \\ \sin m\varphi\end{matrix}\right\} P_n^m(\cos\theta)$ called surface harmonics of the first kind, tesseral for $m<n$ and sectoral for $m=n$. With $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$, they are everywhere one valued and continuous functions on the surface of the unit sphere $x^2+y^2+z^2=1$ where $x=\sin\theta\cos\varphi$, $y=\sin\theta\sin\varphi$ and $z=\cos\theta$.

*See page II.

8.1.7 $$e^{-i\mu\pi}Q_\nu^\mu(z)=\pi^{\frac{1}{2}}2^\mu(z^2-1)^{-\frac{1}{2}\mu}\left\{\frac{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}+\frac{\mu}{2}\right)}{2\Gamma\left(1+\frac{\nu}{2}-\frac{\mu}{2}\right)}e^{\pm i\frac{1}{2}\pi(\mu-\nu-1)}F\left(-\frac{\nu}{2}-\frac{\mu}{2},\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2};\frac{1}{2};z^2\right)\right.$$
$$\left.+\frac{z\Gamma\left(1+\frac{\nu}{2}+\frac{\mu}{2}\right)e^{\pm i\frac{1}{2}\pi(\mu-\nu)}}{\Gamma\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}\right)}F\left(\frac{1}{2}-\frac{\nu}{2}-\frac{\mu}{2},1+\frac{\nu}{2}-\frac{\mu}{2};\frac{3}{2};z^2\right)\right\}\qquad(|z^2|<1)$$

(Upper and lower signs according as $\mathscr{I}z\gtrless 0$.)

Wronskian

8.1.8
$$W\{P_\nu^\mu(z),Q_\nu^\mu(z)\}=\frac{e^{i\mu\pi}2^{2\mu}\Gamma\left(\frac{\nu+\mu+2}{2}\right)\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{(1-z^2)\Gamma\left(\frac{\nu-\mu+2}{2}\right)\Gamma\left(\frac{\nu-\mu+1}{2}\right)}$$

8.1.9 $$W\{P_n(z),Q_n(z)\}=-(z^2-1)^{-1}$$

8.2. Relations Between Legendre Functions

Negative Degree

8.2.1 $$P_{-\nu-1}^\mu(z)=P_\nu^\mu(z)$$

8.2.2
$$Q_{-\nu-1}^\mu(z)=\{-\pi e^{i\mu\pi}\cos\nu\pi P_\nu^\mu(z)$$
$$+Q_\nu^\mu(z)\sin[\pi(\nu+\mu)]\}/\sin[\pi(\nu-\mu)]$$

Negative Argument ($\mathscr{I}z\gtrless 0$)

8.2.3
$$P_\nu^\mu(-z)=e^{\mp i\nu\pi}P_\nu^\mu(z)-\frac{2}{\pi}e^{-i\mu\pi}\sin[\pi(\nu+\mu)]Q_\nu^\mu(z)$$

8.2.4 $$Q_\nu^\mu(-z)=-e^{\pm i\nu\pi}Q_\nu^\mu(z)$$

Negative Order

8.2.5
$$P_\nu^{-\mu}(z)=\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\left[P_\nu^\mu(z)-\frac{2}{\pi}e^{-i\mu\pi}\sin(\mu\pi)Q_\nu^\mu(z)\right]$$

8.2.6 $$Q_\nu^{-\mu}(z)=e^{-2i\mu\pi}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}Q_\nu^\mu(z)$$

Degree $\mu+\frac{1}{2}$ and Order $\nu+\frac{1}{2}$ *

$$\mathscr{R}z>0$$

8.2.7 $$P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}\left(\frac{z}{(z^2-1)^{1/2}}\right)=\frac{(z^2-1)^{1/4}e^{-i\mu\pi}Q_\nu^\mu(z)}{(\frac{1}{2}\pi)^{1/2}\Gamma(\nu+\mu+1)}$$ *

8.2.8
$$Q_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}\left(\frac{z}{(z^2-1)^{1/2}}\right)$$ *
$$=-i(\tfrac{1}{2}\pi)^{1/2}\Gamma(-\nu-\mu)(z^2-1)^{1/4}e^{-i\nu\pi}P_\nu^\mu(z)$$

8.3. Values on the Cut

$$(-1<x<1)$$

8.3.1
$$P_\nu^\mu(x)=\tfrac{1}{2}[e^{\frac{1}{2}i\mu\pi}P_\nu^\mu(x+i0)+e^{-\frac{1}{2}i\mu\pi}P_\nu^\mu(x-i0)]$$

8.3.2
$$P_\nu^\mu(x)=e^{\pm\frac{1}{2}i\mu\pi}P_\nu^\mu(x\pm i0)$$ *

8.3.3
$$=i\pi^{-1}e^{-i\mu\pi}[e^{-\frac{1}{2}i\mu\pi}Q_\nu^\mu(x+i0)$$ *
$$-e^{\frac{1}{2}i\mu\pi}Q_\nu^\mu(x-i0)]$$ *

8.3.4
$$Q_\nu^\mu(x)=\tfrac{1}{2}e^{-i\mu\pi}[e^{-\frac{1}{2}i\mu\pi}Q_\nu^\mu(x+i0)+e^{\frac{1}{2}i\mu\pi}Q_\nu^\mu(x-i0)]$$

(Formulas for $P_\nu^\mu(x)$ and $Q_\nu^\mu(x)$ are obtained with the replacement of $z-1$ by $(1-x)e^{\pm i\pi}$, (z^2-1) by $(1-x^2)e^{\pm i\pi}$, $z+1$ by $x+1$ for $z=x\pm i0$.)

8.4. Explicit Expressions

$$(x=\cos\theta)$$

8.4.1 $$P_0(z)=1\qquad P_0(x)=1$$

8.4.2
$$Q_0(z)=\frac{1}{2}\ln\left(\frac{z+1}{z-1}\right)\qquad Q_0(x)=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$
$$=xF(\tfrac{1}{2},1;\tfrac{3}{2};x^2)$$

8.4.3 $$P_1(z)=z\qquad P_1(x)=x=\cos\theta$$

8.4.4
$$Q_1(z)=\frac{z}{2}\ln\left(\frac{z+1}{z-1}\right)-1\qquad Q_1(x)=\frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)-1$$

8.4.5
$$P_2(z)=\tfrac{1}{2}(3z^2-1)\qquad P_2(x)=\tfrac{1}{2}(3x^2-1)$$
$$=\tfrac{1}{4}(3\cos 2\theta+1)$$

8.4.6
$$Q_2(z)=\frac{1}{2}P_2(z)\ln\left(\frac{z+1}{z-1}\right)\qquad Q_2(x)=$$
$$-\frac{3z}{2}\qquad\qquad\left(\frac{3x^2-1}{4}\right)\ln\left(\frac{1+x}{1-x}\right)-\frac{3x}{2}.$$

8.5. Recurrence Relations

(Both P_ν^μ and Q_ν^μ satisfy the same recurrence relations.)

Varying Order

8.5.1
$$P_\nu^{\mu+1}(z)=(z^2-1)^{-\frac{1}{2}}\{(\nu-\mu)zP_\nu^\mu(z)-(\nu+\mu)P_{\nu-1}^\mu(z)\}$$

---

*See page II.

8.5.2

$$(z^2-1)\,\frac{dP_\nu^\mu(z)}{dz}=(\nu+\mu)(\nu-\mu+1)(z^2-1)^{\frac{1}{2}}P_\nu^{\mu-1}(z)-\mu zP_\nu^\mu(z)$$

Varying Degree

8.5.3

$$(\nu-\mu+1)P_{\nu+1}^\mu(z)=(2\nu+1)zP_\nu^\mu(z)-(\nu+\mu)P_{\nu-1}^\mu(z)$$

8.5.4 $$(z^2-1)\,\frac{dP_\nu^\mu(z)}{dz}=\nu zP_\nu^\mu(z)-(\nu+\mu)P_{\nu-1}^\mu(z)$$

Varying Order and Degree

8.5.5 $$P_{\nu+1}^\mu(z)=P_{\nu-1}^\mu(z)+(2\nu+1)(z^2-1)^{\frac{1}{2}}P_\nu^{\mu-1}(z)$$

8.6. Special Values

$x=0$

8.6.1

$$P_\nu^\mu(0)=2^\mu\pi^{-\frac{1}{2}}\cos\left[\tfrac{1}{2}\pi(\nu+\mu)\right]\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+\tfrac{1}{2})/\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}\mu+1)$$

8.6.2

$$Q_\nu^\mu(0)=-2^{\mu-1}\pi^{\frac{1}{2}}\sin\left[\tfrac{1}{2}\pi(\nu+\mu)\right]\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+\tfrac{1}{2})/\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}\mu+1)$$

8.6.3

$$\left[\frac{dP_\nu^\mu(x)}{dx}\right]_{x=0}=2^{\mu+1}\pi^{-\frac{1}{2}}\sin\left[\tfrac{1}{2}\pi(\nu+\mu)\right]\,\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+1)/\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}\mu+\tfrac{1}{2})$$

8.6.4

$$\left[\frac{dQ_\nu^\mu(x)}{dx}\right]_{x=0}=2^\mu\pi^{\frac{1}{2}}\cos\left[\tfrac{1}{2}\pi(\nu+\mu)\right]\,\Gamma(\tfrac{1}{2}\nu+\tfrac{1}{2}\mu+1)/\Gamma(\tfrac{1}{2}\nu-\tfrac{1}{2}\mu+\tfrac{1}{2})$$

8.6.5

$$W\{P_\nu^\mu(x),Q_\nu^\mu(x)\}_{x=0}=\frac{2^{2\mu}\Gamma(\frac{1}{2}\nu+\frac{1}{2}\mu+1)\Gamma(\frac{1}{2}\nu+\frac{1}{2}\mu+\frac{1}{2})}{\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu+1)\Gamma(\frac{1}{2}\nu-\frac{1}{2}\mu+\frac{1}{2})}$$

$\mu=m=1, 2, 3, \ldots$

8.6.6

$$P_\nu^m(z)=(z^2-1)^{\frac{1}{2}m}\,\frac{d^mP_\nu(z)}{dz^m},$$

$$P_\nu^m(x)=(-1)^m(1-x^2)^{\frac{1}{2}m}\,\frac{d^mP_\nu(x)}{dx^m}$$

8.6.7

$$Q_\nu^m(z)=(z^2-1)^{\frac{1}{2}m}\,\frac{d^mQ_\nu(z)}{dz^m},$$

$$Q_\nu^m(x)=(-1)^m(1-x^2)^{\frac{1}{2}m}\,\frac{d^mQ_\nu(x)}{dx^m}$$

$\mu=\pm\frac{1}{2}$

8.6.8

$$P_\nu^{\frac{1}{2}}(z)=(z^2-1)^{-1/4}(2\pi)^{-1/2}\{[z+(z^2-1)^{1/2}]^{\nu+\frac{1}{2}}+[z+(z^2-1)^{1/2}]^{-\nu-\frac{1}{2}}\}$$

8.6.9

$$P_\nu^{-\frac{1}{2}}(z)=\left(\frac{2}{\pi}\right)^{1/2}\frac{(z^2-1)^{-1/4}}{2\nu+1}\,\{[z+(z^2-1)^{1/2}]^{\nu+\frac{1}{2}}-[z+(z^2-1)^{1/2}]^{-\nu-\frac{1}{2}}\}$$

8.6.10

$$Q_\nu^{\frac{1}{2}}(z)=i(\tfrac{1}{2}\pi)^{1/2}(z^2-1)^{-1/4}[z+(z^2-1)^{1/2}]^{-\nu-\frac{1}{2}}$$

8.6.11

$$Q_\nu^{-\frac{1}{2}}(z)=-i(2\pi)^{1/2}\,\frac{(z^2-1)^{-1/4}}{2\nu+1}[z+(z^2-1)^{1/2}]^{-\nu-\frac{1}{2}}\quad *$$

8.6.12

$$P_\nu^{\frac{1}{2}}(\cos\theta)=(\tfrac{1}{2}\pi)^{-\frac{1}{2}}\,(\sin\theta)^{-\frac{1}{2}}\cos\left[(\nu+\tfrac{1}{2})\theta\right]$$

8.6.13

$$Q_\nu^{\frac{1}{2}}(\cos\theta)=-(\tfrac{1}{2}\pi)^{\frac{1}{2}}\,(\sin\theta)^{-\frac{1}{2}}\sin\left[(\nu+\tfrac{1}{2})\theta\right]$$

8.6.14

$$P_\nu^{-\frac{1}{2}}(\cos\theta)=(\tfrac{1}{2}\pi)^{-\frac{1}{2}}(\nu+\tfrac{1}{2})^{-1}(\sin\theta)^{-\frac{1}{2}}\sin\left[(\nu+\tfrac{1}{2})\theta\right]$$

8.6.15

$$Q_\nu^{-\frac{1}{2}}(\cos\theta)=(2\pi)^{\frac{1}{2}}(2\nu+1)^{-1}(\sin\theta)^{-\frac{1}{2}}\cos\left[(\nu+\tfrac{1}{2})\theta\right]^{*}$$

$\mu=-\nu$

8.6.16 $$P_\nu^{-\nu}(z)=\frac{2^{-\nu}(z^2-1)^{\frac{1}{2}\nu}}{\Gamma(\nu+1)}$$

8.6.17 $$P_\nu^{-\nu}(\cos\theta)=\frac{2^{-\nu}(\sin\theta)^\nu}{\Gamma(\nu+1)}$$

$\mu=0,\ \nu=n$

(Rodrigues' Formula)

8.6.18 $$P_n(z)=\frac{1}{2^nn!}\,\frac{d^n(z^2-1)^n}{dz^n}$$

8.6.19 $$Q_n(x)=\frac{1}{2}\,P_n(x)\,\ln\frac{1+x}{1-x}-W_{n-1}(x)$$

where

$$W_{n-1}(x)=\frac{2n-1}{1\cdot n}\,P_{n-1}(x)+\frac{2n-5}{3(n-1)}\,P_{n-3}(x)+\frac{2n-9}{5(n-2)}\,P_{n-5}(x)+\ldots$$

$$=\sum_{m=1}^{n}\frac{1}{m}\,P_{m-1}(x)P_{n-m}(x)$$

$W_{-1}(x)=0$

---

*See page II.

$$\nu=0,1$$

8.6.20 $$\left[\frac{\partial P_\nu(\cos\theta)}{\partial\nu}\right]_{\nu=0}=2\ln(\cos\tfrac{1}{2}\theta)$$

8.6.21 $$\left[\frac{\partial P_\nu^{-1}(\cos\theta)}{\partial\nu}\right]_{\nu=0}=-\tan\tfrac{1}{2}\theta-2\cot\tfrac{1}{2}\theta\ln(\cos\tfrac{1}{2}\theta)$$

8.6.22 $$\left[\frac{\partial P_\nu^{-1}(\cos\theta)}{\partial\nu}\right]_{\nu=1}=-\tfrac{1}{2}\tan\tfrac{1}{2}\theta\sin^2\tfrac{1}{2}\theta+\sin\theta\ln(\cos\tfrac{1}{2}\theta)$$

8.7. Trigonometric Expansions $(0<\theta<\pi)$

8.7.1 $$P_\nu^\mu(\cos\theta)=\pi^{-1/2}2^{\mu+1}(\sin\theta)^\mu\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}\sum_{k=0}^{\infty}\frac{(\mu+\frac{1}{2})_k(\nu+\mu+1)_k}{k!\,(\nu+\frac{3}{2})_k}\sin[(\nu+\mu+2k+1)\theta]$$

8.7.2 $$Q_\nu^\mu(\cos\theta)=\pi^{1/2}2^\mu(\sin\theta)^\mu\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}\sum_{k=0}^{\infty}\frac{(\mu+\frac{1}{2})_k(\nu+\mu+1)_k}{k!(\nu+\frac{3}{2})_k}\cos[(\nu+\mu+2k+1)\theta]$$

8.7.3 $$P_n(\cos\theta)=\frac{2^{2n+2}(n!)^2}{\pi(2n+1)!}\left[\sin(n+1)\theta+\frac{n+1}{2n+3}\sin(n+3)\theta+\frac{1\cdot3}{2!}\frac{(n+1)(n+2)}{(2n+3)(2n+5)}\sin(n+5)\theta+\dots\right]$$

8.7.4 $$Q_n(\cos\theta)=\frac{2^{2n+1}(n!)^2}{(2n+1)!}\left[\cos(n+1)\theta+\frac{n+1}{2n+3}\cos(n+3)\theta+\frac{1\cdot3}{2!}\frac{(n+1)(n+2)}{(2n+3)(2n+5)}\cos(n+5)\theta+\dots\right]$$

8.8. Integral Representations

(z not on the real axis between -1 and $-\infty$) *

8.8.1 $$P_\nu^\mu(z)=\frac{2^{-\nu}(z^2-1)^{-\frac{1}{2}\mu}}{\Gamma(-\nu-\mu)\Gamma(\nu+1)}\int_0^\infty(z+\cosh t)^{\mu-\nu-1}(\sinh t)^{2\nu+1}dt \qquad (\mathscr{R}(-\mu)>\mathscr{R}\nu>-1)$$

8.8.2 $$Q_\nu^\mu(z)=\frac{e^{i\mu\pi}\sqrt{\pi}2^{-\mu}}{\Gamma(\mu+\frac{1}{2})}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}(z^2-1)^{\frac{1}{2}\mu}\int_0^\infty[z+(z^2-1)^{\frac{1}{2}}\cosh t]^{-\nu-\mu-1}(\sinh t)^{2\mu}dt \qquad (\mathscr{R}(\nu\pm\mu+1)>0)$$ *

8.8.3 $$Q_n(z)=\frac{1}{2}\int_{-1}^{1}(z-t)^{-1}P_n(t)\,dt=(-1)^{n+1}Q_n(-z)$$

(For other integral representations see [8.2].)

8.9. Summation Formulas

8.9.1 $$(\xi-z)\sum_{m=0}^{n}(2m+1)P_m(z)P_m(\xi)=(n+1)[P_{n+1}(\xi)P_n(z)-P_n(\xi)P_{n+1}(z)]$$

8.9.2 $$(\xi-z)\sum_{m=0}^{n}(2m+1)P_m(z)Q_m(\xi)=1-(n+1)[P_{n+1}(z)Q_n(\xi)-P_n(z)Q_{n+1}(\xi)]$$

8.10. Asymptotic Expansions

For fixed z and ν and $\mathscr{R}\mu\to\infty$, **8.10.1–8.10.3** are asymptotic expansions if z is not on the real axis between $-\infty$ and -1 and $+\infty$ and $+1$. (Upper or lower signs according as $\mathscr{I}z\gtreqless0$.)

8.10.1 $$P_\nu^\mu(z)=\frac{\Gamma(\nu+\mu+1)\Gamma(\mu-\nu)}{\pi\Gamma(\mu+1)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu}\sin\mu\pi\left[F(-\nu,\nu+1;1+\mu;\tfrac{1}{2}+\tfrac{1}{2}z)\right.$$
$$\left.-\frac{\sin\nu\pi}{\sin\mu\pi}e^{\mp i\mu\pi}\left(\frac{z-1}{z+1}\right)^\mu F(-\nu,\nu+1;1+\mu;\tfrac{1}{2}-\tfrac{1}{2}z)\right]$$

8.10.2 $$Q_\nu^\mu(z)=\tfrac{1}{2}e^{i\mu\pi}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu}\Gamma(\mu-\nu)\left[F(-\nu,\nu+1;1+\mu;\tfrac{1}{2}+\tfrac{1}{2}z)\right.$$
$$\left.-e^{\mp i\nu\pi}\left(\frac{z-1}{z+1}\right)^\mu F(-\nu,\nu+1;1+\mu;\tfrac{1}{2}-\tfrac{1}{2}z)\right]$$

---

*See page II.

8.10.3 $$Q_\nu^{-\mu}(z)=\frac{e^{-i\mu\pi}\csc[\pi(\nu-\mu)]}{2\pi\Gamma(1+\mu)}\Bigg[e^{\mp i\nu\pi}\left(\frac{z+1}{z-1}\right)^{-\frac{1}{2}\mu}F(-\nu,\nu+1;1+\mu;\tfrac{1}{2}-\tfrac{1}{2}z)$$
$$-\left(\frac{z-1}{z+1}\right)^{-\frac{1}{2}\mu}F(-\nu,\nu+1;1+\mu;\tfrac{1}{2}+\tfrac{1}{2}z)\Bigg]$$

With μ replaced by $-\mu$, **8.1.2** is an asymptotic expansion for $P_\nu^{-\mu}(z)$ for fixed z and ν and $\mathscr{R}\,\mu\rightarrow\infty$ if z is not on the real axis between $-\infty$ and -1.

For fixed z and μ and $\mathscr{R}\nu\rightarrow\infty$, **8.10.4** and **8.10.6** are asymptotic expansions if z is not on the real axis between $-\infty$ and -1 and $+\infty$ and $+1$; **8.10.5** if z is not on the real axis between $-\infty$ and $+1$.

8.10.4 $$P_\nu^\mu(z)=(2\pi)^{-\frac{1}{2}}(z^2-1)^{-1/4}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}\Bigg\{[z+(z^2-1)^{\frac{1}{2}}]^{\nu+\frac{1}{2}}F(\tfrac{1}{2}+\mu,\tfrac{1}{2}-\mu;\tfrac{3}{2}+\nu;\frac{z+(z^2-1)^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}})$$
$$+ie^{-i\mu\pi}[z-(z^2-1)^{\frac{1}{2}}]^{\nu+\frac{1}{2}}F(\tfrac{1}{2}+\mu,\tfrac{1}{2}-\mu;\tfrac{3}{2}+\nu;\frac{-z+(z^2-1)^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}})\Bigg\}$$

8.10.5 $$Q_\nu^\mu(z)=e^{i\mu\pi}(\tfrac{1}{2}\pi)^{\frac{1}{2}}(z^2-1)^{-1/4}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}[z-(z^2-1)^{\frac{1}{2}}]^{\nu+\frac{1}{2}}F(\tfrac{1}{2}+\mu,\tfrac{1}{2}-\mu;\tfrac{3}{2}+\nu;\frac{-z+(z^2-1)^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}})$$

8.10.6 $$Q_{-\nu}^\mu(z)=\frac{e^{i\mu\pi}(\frac{1}{2}\pi)^{\frac{1}{2}}(z^2-1)^{-1/4}}{\sin[\pi(\mu-\nu)]}\frac{\Gamma(\mu+\nu)}{\Gamma(\frac{1}{2}-\mu)}\Bigg\{\cos\nu\pi[z+(z^2-1)^{\frac{1}{2}}]^{\nu-\frac{1}{2}}F(\tfrac{1}{2}+\mu,\tfrac{1}{2}-\mu;\tfrac{1}{2}+\nu;\frac{z+(z^2-1)^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}})$$
$$+ie^{i\nu\pi}\cos\mu\pi[z-(z^2-1)^{\frac{1}{2}}]^{\nu-\frac{1}{2}}F(\tfrac{1}{2}+\mu,\tfrac{1}{2}-\mu;\tfrac{1}{2}+\nu;\frac{-z+(z^2-1)^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}})\Bigg\}$$

The related asymptotic expansion for $P_{-\nu}^\mu(z)$ may be derived from **8.10.4** together with **8.2.1.**

8.10.7 $$P_\nu^\mu(\cos\theta)=\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}(\tfrac{1}{2}\pi\sin\theta)^{-\frac{1}{2}}\cos[(\nu+\tfrac{1}{2})\theta-\frac{\pi}{4}+\frac{\mu\pi}{2}]+O(\nu^{-1})$$

8.10.8 $$Q_\nu^\mu(\cos\theta)=\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})}\left(\frac{\pi}{2\sin\theta}\right)^{\frac{1}{2}}\cos[(\nu+\tfrac{1}{2})\theta+\frac{\pi}{4}+\frac{\mu\pi}{2}]+O(\nu^{-1})\qquad(\epsilon<\theta<\pi-\epsilon,\ \epsilon>0)$$

For other asymptotic expansions, see [8.7] and [8.9].

8.11. Toroidal Functions (or Ring Functions)

(Only special properties are given; other properties and representations follow from the earlier sections.)

8.11.1 $$P_{\nu-\frac{1}{2}}^\mu(\cosh\eta)=[\Gamma(1-\mu)]^{-1}2^{2\mu}(1-e^{-2\eta})^{-\mu}e^{-(\nu+\frac{1}{2})\eta}F(\tfrac{1}{2}-\mu,\tfrac{1}{2}+\nu-\mu;1-2\mu;1-e^{-2\eta})$$

8.11.2 $$P_{n-\frac{1}{2}}^m(\cosh\eta)=\frac{\Gamma(n+m+\frac{1}{2})(\sinh\eta)^m}{\Gamma(n-m+\frac{1}{2})2^m\sqrt{\pi}\Gamma(m+\frac{1}{2})}\int_0^\pi\frac{(\sin\varphi)^{2m}d\varphi}{(\cosh\eta+\cos\varphi\sinh\eta)^{n+m+\frac{1}{2}}}$$

8.11.3 $$Q_{\nu-\frac{1}{2}}^\mu(\cosh\eta)=[\Gamma(1+\nu)]^{-1}\sqrt{\pi}e^{i\mu\pi}\Gamma(\tfrac{1}{2}+\nu+\mu)(1-e^{-2\eta})^\mu e^{-(\nu+\frac{1}{2})\eta}F(\tfrac{1}{2}+\mu,\tfrac{1}{2}+\nu+\mu;1+\nu;e^{-2\eta})\qquad *$$

8.11.4 $$Q_{n-\frac{1}{2}}^m(\cosh\eta)=\frac{(-1)^m\Gamma(n+\frac{1}{2})}{\Gamma(n-m+\frac{1}{2})}\int_0^\infty\frac{\cosh mt\,dt}{(\cosh\eta+\cosh t\sinh\eta)^{n+\frac{1}{2}}}\qquad *\qquad(n>m)$$

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*See page II.

8.12. Conical Functions

$$\left(P^{\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta),\ Q^{\mu}_{-\frac{1}{2}+i\lambda}(\cos\theta)\right)$$

(Only special properties are given as other properties and representations follow from earlier sections with $\nu=-\frac{1}{2}+i\lambda$ (λ, a real parameter) and $z=\cos\theta$.)

8.12.1

$$P_{-\frac{1}{2}+i\lambda}(\cos\theta)=1+\frac{4\lambda^2+1^2}{2^2}\sin^2\frac{\theta}{2}+\frac{(4\lambda^2+1^2)(4\lambda^2+3^2)}{2^2 4^2}\sin^4\frac{\theta}{2}+\ldots \qquad (0\leq\theta<\pi)$$

8.12.2 $$P_{-\frac{1}{2}+i\lambda}(\cos\theta)=P_{-\frac{1}{2}-i\lambda}(\cos\theta)$$

8.12.3 $$P_{-\frac{1}{2}+i\lambda}(\cos\theta)=\frac{2}{\pi}\int_0^{\theta}\frac{\cosh\lambda t\,dt}{\sqrt{2(\cos t-\cos\theta)}}.$$

8.12.4

$$Q_{-\frac{1}{2}\mp i\lambda}(\cos\theta)=\pm i\sinh\lambda\pi\int_0^{\infty}\frac{\cos\lambda t\,dt}{\sqrt{2(\cosh t+\cos\theta)}}+\int_0^{\infty}\frac{\cosh\lambda t\,dt}{\sqrt{2(\cosh t-\cos\theta)}}$$

8.12.5

$$P_{-\frac{1}{2}+i\lambda}(-\cos\theta)=\frac{\cosh\lambda\pi}{\pi}\left[Q_{-\frac{1}{2}+i\lambda}(\cos\theta)+Q_{-\frac{1}{2}-i\lambda}(\cos\theta)\right]$$

* 8.13. Relation to Elliptic Integrals (see chapter 17) ($\mathscr{R}\eta>0$)

8.13.1 $$P_{-\frac{1}{2}}(z)=\frac{2}{\pi}\sqrt{\frac{2}{z+1}}\,K\left(\sqrt{\frac{z-1}{z+1}}\right)$$

8.13.2 $$P_{-\frac{1}{2}}(\cosh\eta)=\left[\frac{\pi}{2}\cosh\frac{\eta}{2}\right]^{-1}K\left(\tanh\frac{\eta}{2}\right)$$

8.13.3 $$Q_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{z+1}}\,K\left(\sqrt{\frac{2}{z+1}}\right)$$

8.13.4 $$Q_{-\frac{1}{2}}(\cosh\eta)=2e^{-\eta/2}K(e^{-\eta})$$

8.13.5

$$P_{\frac{1}{2}}(z)=\frac{2}{\pi}\left(z+\sqrt{z^2-1}\right)^{\frac{1}{2}}E\left(\sqrt{\frac{2(z^2-1)^{1/2}}{z+(z^2-1)^{1/2}}}\right)$$

8.13.6 $$P_{\frac{1}{2}}(\cosh\eta)=\frac{2}{\pi}e^{\eta/2}E\left(\sqrt{1-e^{-2\eta}}\right)$$

8.13.7

$$Q_{\frac{1}{2}}(z)=z\sqrt{\frac{2}{z+1}}\,K\left(\sqrt{\frac{2}{z+1}}\right)-[2(z+1)]^{\frac{1}{2}}E\left(\sqrt{\frac{2}{z+1}}\right) \qquad *$$

$$(-1<x<1)$$

8.13.8 $$P_{-\frac{1}{2}}(x)=\frac{2}{\pi}K\left(\sqrt{\frac{1-x}{2}}\right)$$

8.13.9 $$P_{-\frac{1}{2}}(\cos\theta)=\frac{2}{\pi}K\left(\sin\frac{\theta}{2}\right)$$

8.13.10 $$Q_{-\frac{1}{2}}(x)=K\left(\sqrt{\frac{1+x}{2}}\right) \quad *$$

8.13.11 $$P_{\frac{1}{2}}(x)=\frac{2}{\pi}\left[2E\left(\sqrt{\frac{1-x}{2}}\right)-K\left(\sqrt{\frac{1-x}{2}}\right)\right]$$

8.13.12 $$Q_{\frac{1}{2}}(x)=K\left(\sqrt{\frac{1+x}{2}}\right)-2E\left(\sqrt{\frac{1+x}{2}}\right) \quad *$$

8.14. Integrals

8.14.1 $$\int_1^{\infty}P_\nu(x)Q_\rho(x)dx=[(\rho-\nu)(\rho+\nu+1)]^{-1} \qquad (\mathscr{R}\rho>\mathscr{R}\nu>0)$$

8.14.2 $$\int_1^{\infty}Q_\nu(x)Q_\rho(x)dx=[(\rho-\nu)(\rho+\nu+1)]^{-1}[\psi(\rho+1)-\psi(\nu+1)] \qquad (\mathscr{R}(\rho+\nu)>-1,\ \rho+\nu+1\neq0;\ \nu,\ \rho\neq-1,\ -2,\ -3,\ \ldots)$$

8.14.3 $$\int_1^{\infty}[Q_\nu(x)]^2dx=(2\nu+1)^{-1}\psi'(\nu+1) \qquad (\mathscr{R}\nu>-\tfrac{1}{2})$$

8.14.4 $$\int_{-1}^{1}P_\nu(x)P_\rho(x)dx=\frac{2}{\pi^2}[(\rho-\nu)(\rho+\nu+1)]^{-1}\{2\sin\pi\nu\sin\pi\rho[\psi(\nu+1)-\psi(\rho+1)]+\pi\sin(\pi\rho-\pi\nu)\} \qquad (\rho+\nu+1\neq0)$$

8.14.5 $$\int_{-1}^{1}[P_\nu(x)]^2dx=\frac{\pi^2-2(\sin\pi\nu)^2\psi'(\nu+1)}{\pi^2(\nu+\frac{1}{2})} \qquad *$$

8.14.6 $$\int_{-1}^{1}Q_\nu(x)Q_\rho(x)dx=[(\rho-\nu)(\rho+\nu+1)]^{-1}\{[\psi(\nu+1)-\psi(\rho+1)][1+\cos\rho\pi\cos\nu\pi]-\tfrac{1}{2}\pi\sin(\nu\pi-\rho\pi)\} \quad *$$

$$(\rho+\nu+1\neq0;\ \nu,\ \rho\neq-1,\ -2,\ -3,\ \ldots)$$

8.14.7 $$\int_{-1}^{1}[Q_\nu(x)]^2dx=(2\nu+1)^{-1}\{\tfrac{1}{2}\pi^2-\psi'(\nu+1)[1+(\cos\nu\pi)^2]\} \qquad (\nu\neq-1,\ -2,\ -3,\ \ldots)$$

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*See page II.

8.14.8 $$\int_{-1}^{1} P_\nu(x)Q_\rho(x)dx=[(\nu-\rho)(\rho+\nu+1)]^{-1}\left\{1-\cos(\rho\pi-\nu\pi)-\frac{2}{\pi}\sin\pi\nu\cos\pi\nu[\psi(\nu+1)-\psi(\rho+1)]\right\}$$

$(\mathscr{R}\nu>0,\ \mathscr{R}\rho>0,\ \rho\neq\nu)$

8.14.9 $$\int_{-1}^{1} P_\nu(x)Q_\nu(x)dx=-\frac{1}{\pi}(2\nu+1)^{-1}\sin 2\nu\pi\psi'(\nu+1) \qquad (\mathscr{R}\nu>0)$$

(m, n, l positive integers)

8.14.10

$$\int_{-1}^{1} Q_n^m(x)P_l^m(x)dx=(-1)^m\frac{1-(-1)^{l+n}(n+m)!}{(l-n)(l+n+1)(n-m)!}$$

8.14.11 $$\int_{-1}^{1} P_n^m(x)P_l^m(x)dx=0 \qquad (l\neq n)$$

8.14.12 $$\int_{-1}^{1} P_n^m(x)P_n^l(x)(1-x^2)^{-1}dx=0 \qquad (l\neq m)$$

8.14.13 $$\int_{-1}^{1}[P_n^m(x)]^2dx=(n+\tfrac{1}{2})^{-1}(n+m)!/(n-m)!$$

8.14.14

$$\int_{-1}^{1}(1-x^2)^{-1}[P_n^m(x)]^2dx=(n+m)!/m(n-m)!$$

8.14.15

$$\int_0^1 P_\nu(x)x^\rho dx=\frac{\pi^{\frac{1}{2}}2^{-\rho-1}\Gamma(1+\rho)}{\Gamma(1+\frac{1}{2}\rho-\frac{1}{2}\nu)\Gamma(\frac{1}{2}\rho+\frac{1}{2}\nu+\frac{3}{2})}$$

$(\mathscr{R}\rho>-1)$

8.14.16

$$\int_0^\pi(\sin t)^{\alpha-1}P_\nu^{-\mu}(\cos t)dt=\frac{2^{-\mu}\pi\Gamma(\frac{1}{2}\alpha+\frac{1}{2}\mu)\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\mu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\alpha+\frac{1}{2}\nu)\Gamma(\frac{1}{2}\alpha-\frac{1}{2}\nu)\Gamma(\frac{1}{2}\mu+\frac{1}{2}\nu+1)\Gamma(\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2})} \qquad (\mathscr{R}(\alpha\pm\mu)>0)$$

8.14.17

$$P_\nu^{-m}(z)=(z^2-1)^{-\frac{1}{2}m}\int_1^z\cdots\int_1^z P_\nu(z)(dz)^m$$

8.14.18

$$Q_\nu^{-m}(z)=(-1)^m(z^2-1)^{-\frac{1}{2}m}\int_z^\infty\cdots\int_z^\infty Q_\nu(z)(dz)^m$$

For other integrals, see [8.2], [8.4] and chapter 22.

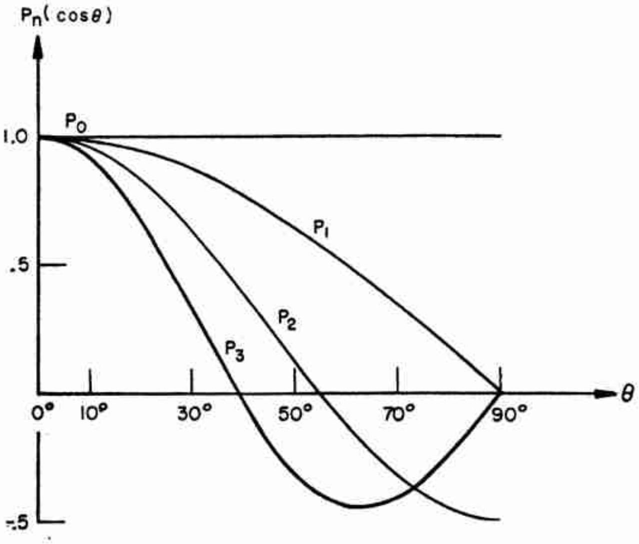

FIGURE 8.1. $P_n(\cos\theta)$. $n=0(1)3$.

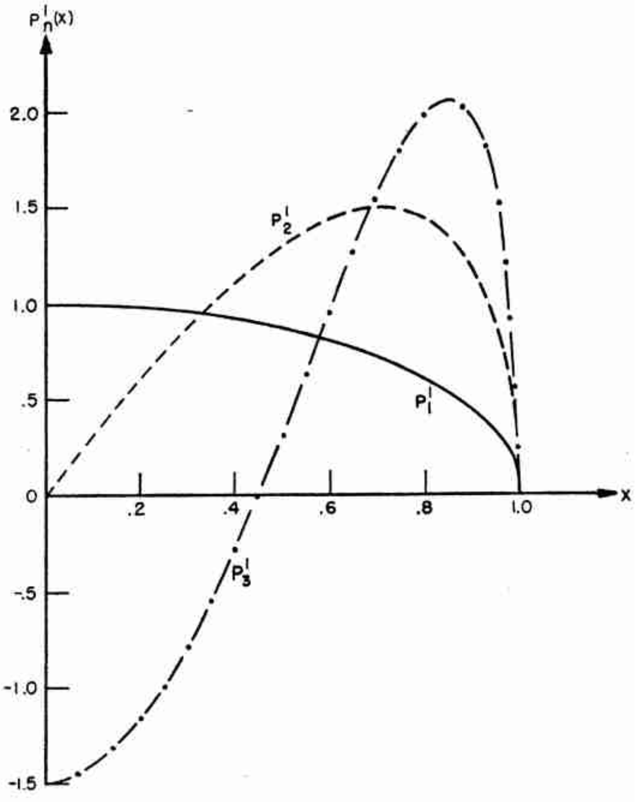

FIGURE 8.2. $P_n^1(x)$. $n=1(1)3, x\leq 1$.

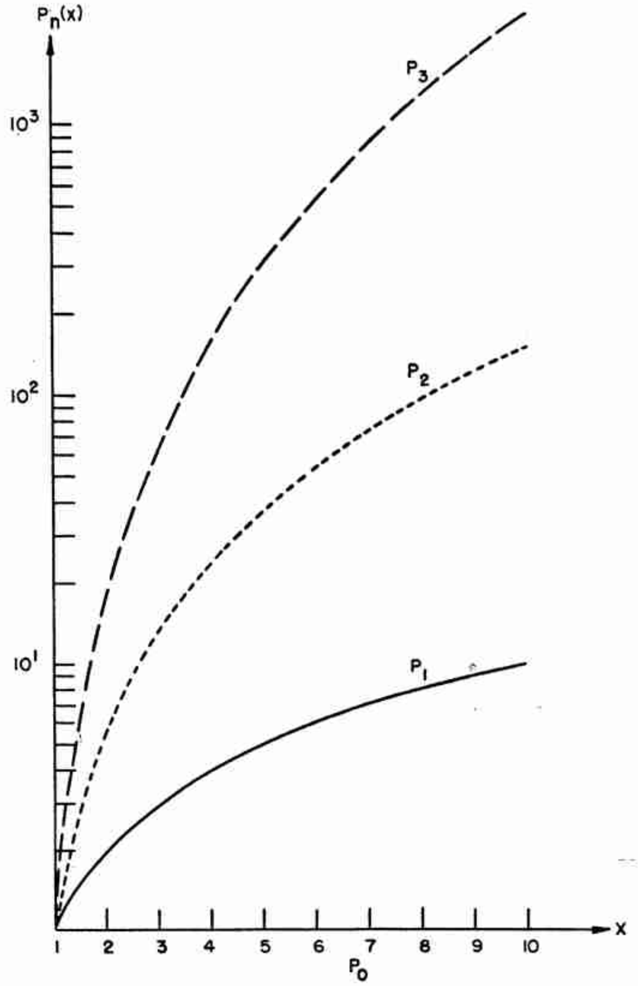

FIGURE 8.3. $P_n(x)$. $n=0(1)3, x\geq 1$.

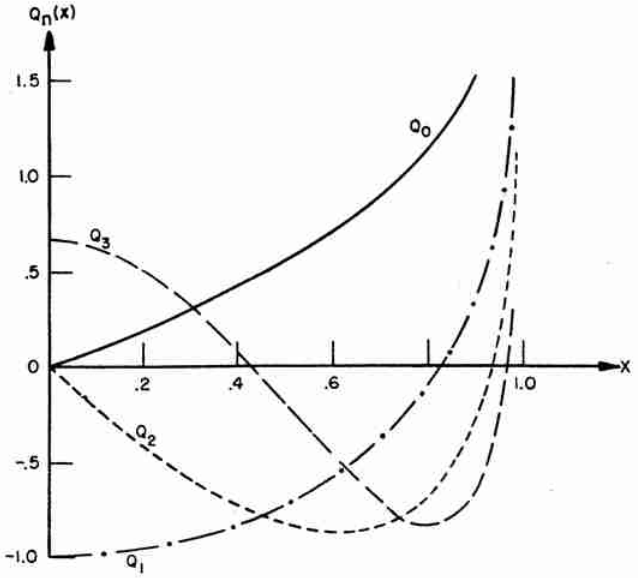

FIGURE 8.4. $Q_n(x)$. $n=0(1)3, x<1$.

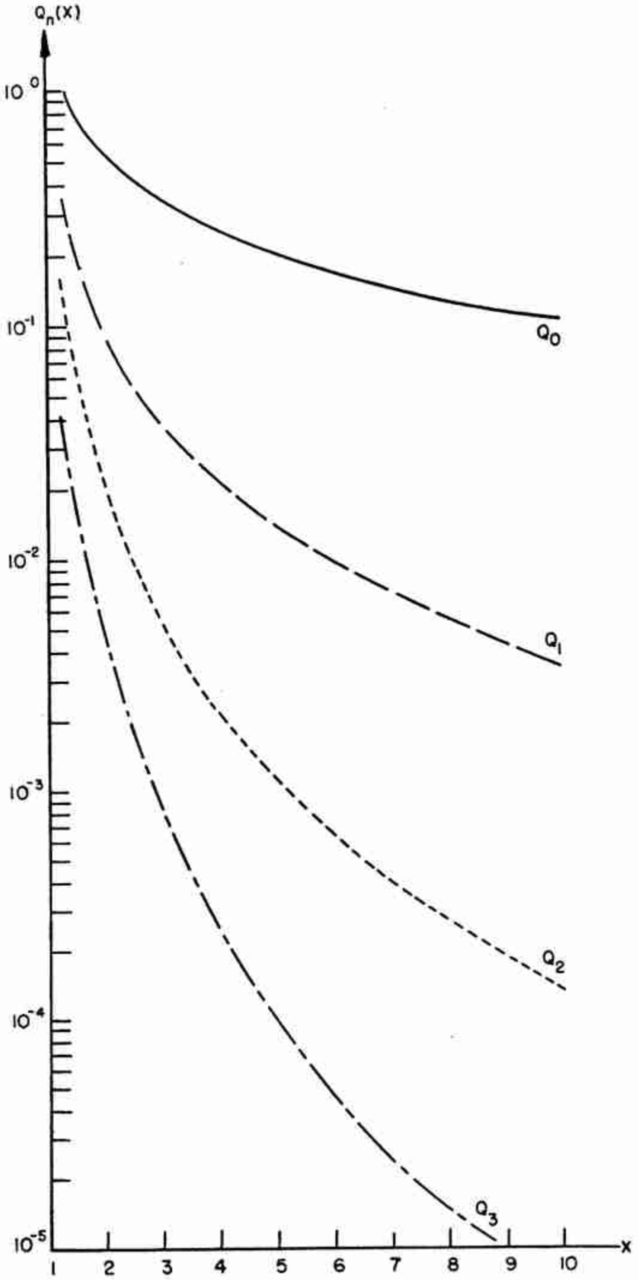

FIGURE 8.5. $Q_n(x)$. $n=0(1)3, x>1$.

Numerical Methods

8.15. Use and Extension of the Tables

Computation of $P_n(x)$

For all values of x there is very little loss of significant figures (except at zeros) in using the recurrence relation **8.5.3** for increasing values of n.

Example 1. Compute $P_n(x)$ for $x=.31415\ 92654$ and $x=2.6$ for $n=2(1)8$.

n	$P_n(.31415\ 92654)$	$P_n(2.6)$
0	1	1
1	.31415 92654	2.6
2	−.35195 59340	9.64
3	−.39372 32064	40.04
4	.04750 63122	174.952
5	.34184 27517	786.74336
6	.15729 86975	3604.350016
7	−.20123 39354	16729.51005
8	−.25617 29328	78402.55522

Computing $P_8(x)$ using **Table 22.9** carrying ten significant figures, $P_8(.31415\ 92654)=-.25617\ 2933$ and $P_8(2.6)=78402.55526$.

Computation of $Q_n(x)$

For $x<1$, use of **8.5.3** for increasing values of n leads to very little loss of significant figures. However, for $x>1$, the recurrence relation **8.5.3** should be used only for decreasing values of n, after having first obtained Q_n using the formulas in terms of hypergeometric functions.

Example 2. Compute $Q_n(x)$ for $x=.31415\ 92654$ and $n=0(1)4$.

With the aid of **8.4.2** and **8.4.4** we obtain

n	$Q_n(.31415\ 92654)$
0	.32515 34813
1	−.89785 00212
2	−.58567 85953
3	.29190 60854
4	.59974 26989

Using the results of **Example 1** together with **8.6.19**. we find $Q_4(x)=\frac{1}{2}P_4(x)\ln\left(\frac{1+x}{1-x}\right)-W_3(x)$ where $W_3=\frac{7}{4}P_3+\frac{1}{3}P_1$, giving $Q_4(.31415\ 92654)=.59974\ 26989$.

Example 3. Compute $Q_5(x)$ for $x=2.6$.

Ten terms in the series for $F\left(\frac{\nu+2}{2},\frac{\nu+1}{2};\nu+\frac{3}{2};\frac{1}{z^2}\right)$ of **8.1.3** are necessary to obtain nine significant figures giving $Q_5(2.6)=4.8182\ 4468\times10^{-5}$. Using **8.5.3** with increasing values of n carrying ten significant figures we obtain

n	$Q_n(2.6)$
0	.40546 51081
1	.05420 928
2	.00868 364
3	.00148 95
4	.00026 49
5	.00004 81

where Q_0 and Q_1 are obtained using **8.4.2** and **8.4.4.**

Computation of $P_{\pm\frac{1}{2}}(x)$, $Q_{\pm\frac{1}{2}}(x)$

For all values of x, $P_{\pm\frac{1}{2}}(x)$ and $Q_{\pm\frac{1}{2}}(x)$ are most easily computed by means of **8.13.**

Example 4. Compute $Q_{-\frac{1}{2}}(x)$ for $x=2.6$.

Using **8.13.3** and interpolating in **Table 17.1** for $K(.5)$, we find

$$Q_{-\frac{1}{2}}(2.6)=\sqrt{\frac{2}{x+1}}\,K\left(\sqrt{\frac{2}{x+1}}\right)$$

$$=(.74535\ 59925)(1.90424\ 1417)$$

$$=1.41933\ 7751.$$

On the other hand, at least nine terms in the expansion of $F\left(\frac{\nu+2}{2},\frac{\nu+1}{2};\nu+\frac{3}{2};\frac{1}{z^2}\right)$ of **8.1.3** are necessary to obtain comparable accuracy.

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9. Bessel Functions of Integer Order

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Contents

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[1] National Bureau of Standards, on leave from the National Physical Laboratory.

9. Bessel Functions of Integer Order

Mathematical Properties

Notation

The tables in this chapter are for Bessel functions of integer order; the text treats general orders. The conventions used are:

$z=x+iy;\ x,\ y$ real.

n is a positive integer or zero.

$\nu,\ \mu$ are unrestricted except where otherwise indicated; ν is supposed real in the sections devoted to Kelvin functions **9.9, 9.10,** and **9.11.**

The notation used for the Bessel functions is that of Watson [9.15] and the British Association and Royal Society Mathematical Tables. The function $Y_\nu(z)$ is often denoted $N_\nu(z)$ by physicists and European workers.

Other notations are those of:

Aldis, Airey:

$$G_n(z) \text{ for } -\tfrac{1}{2}\pi Y_n(z), K_n(z) \text{ for } (-)^n K_n(z).$$

Clifford:

$$C_n(x) \text{ for } x^{-\frac{1}{2}n} J_n(2\sqrt{x}).$$

Gray, Mathews and MacRobert [9.9]:

$$Y_n(z) \text{ for } \tfrac{1}{2}\pi Y_n(z)+(\ln 2-\gamma)J_n(z),$$

$$\overline{Y}_\nu(z) \text{ for } \pi e^{\nu\pi i} \sec(\nu\pi) Y_\nu(z),$$

$$G_\nu(z) \text{ for } \tfrac{1}{2}\pi i H_\nu^{(1)}(z).$$

Jahnke, Emde and Lösch [9.32]:

$$\Lambda_\nu(z) \text{ for } \Gamma(\nu+1)(\tfrac{1}{2}z)^{-\nu} J_\nu(z).$$

Jeffreys:

$$Hs_\nu(z) \text{ for } H_\nu^{(1)}(z),\ Hi_\nu(z) \text{ for } H_\nu^{(2)}(z),$$

$$Kh_\nu(z) \text{ for } (2/\pi) K_\nu(z).$$

Heine:

$$K_n(z) \text{ for} -\tfrac{1}{2}\pi Y_n(z).$$

Neumann:

$$Y^n(z) \text{ for } \tfrac{1}{2}\pi Y_n(z)+(\ln 2-\gamma)J_n(z).$$

Whittaker and Watson [9.18]:

$$K_\nu(z) \text{ for } \cos(\nu\pi) K_\nu(z).$$

Bessel Functions J and Y

9.1. Definitions and Elementary Properties

Differential Equation

9.1.1
$$z^2 \frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-\nu^2)w=0$$

Solutions are the Bessel functions of the first kind $J_{\pm\nu}(z)$, of the second kind $Y_\nu(z)$ (also called Weber's function) and of the third kind $H_\nu^{(1)}(z)$, $H_\nu^{(2)}(z)$ (also called the Hankel functions). Each is a regular (holomorphic) function of z throughout the z-plane cut along the negative real axis, and for fixed $z(\neq 0)$ each is an entire (integral) function of ν. When $\nu=\pm n$, $J_\nu(z)$ has no branch point and is an entire (integral) function of z.

Important features of the various solutions are as follows: $J_\nu(z)(\mathscr{R}\nu\geq 0)$ is bounded as $z\rightarrow 0$ in any bounded range of arg z. $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent except when ν is an integer. $J_\nu(z)$ and $Y_\nu(z)$ are linearly independent for all values of ν.

$H_\nu^{(1)}(z)$ tends to zero as $|z|\rightarrow\infty$ in the sector $0<\arg z<\pi$; $H_\nu^{(2)}(z)$ tends to zero as $|z|\rightarrow\infty$ in the sector $-\pi<\arg z<0$. For all values of ν, $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$ are linearly independent.

Relations Between Solutions

9.1.2
$$Y_\nu(z)=\frac{J_\nu(z)\cos(\nu\pi)-J_{-\nu}(z)}{\sin(\nu\pi)}$$

The right of this equation is replaced by its limiting value if ν is an integer or zero.

9.1.3

$$H_\nu^{(1)}(z)=J_\nu(z)+iY_\nu(z)$$
$$=i\csc(\nu\pi)\{e^{-\nu\pi i}J_\nu(z)-J_{-\nu}(z)\}$$

9.1.4

$$H_\nu^{(2)}(z)=J_\nu(z)-iY_\nu(z)$$
$$=i\csc(\nu\pi)\{J_{-\nu}(z)-e^{\nu\pi i}J_\nu(z)\}$$

9.1.5 $\quad J_{-n}(z)=(-)^n J_n(z) \qquad Y_{-n}(z)=(-)^n Y_n(z)$

9.1.6 $\quad H_{-\nu}^{(1)}(z)=e^{\nu\pi i}H_\nu^{(1)}(z) \qquad H_{-\nu}^{(2)}(z)=e^{-\nu\pi i}H_\nu^{(2)}(z)$

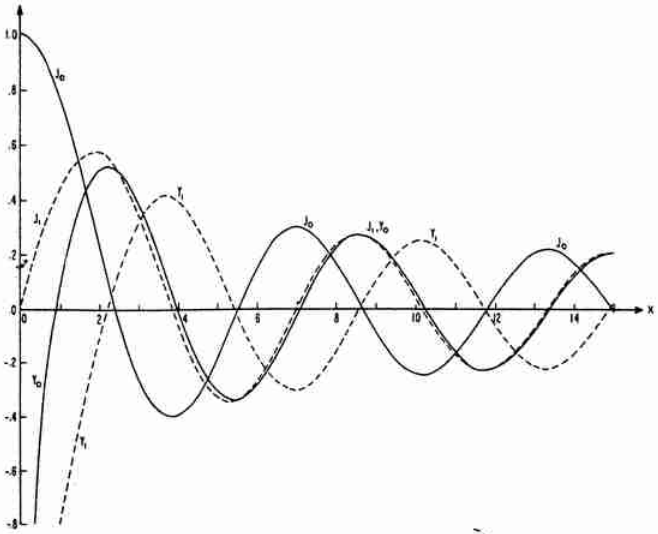


FIGURE 9.1. $J_0(x)$, $Y_0(x)$, $J_1(x)$, $Y_1(x)$.

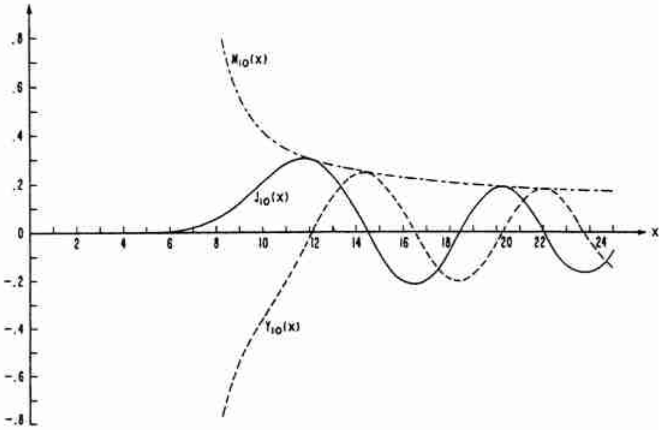


FIGURE 9.2. $J_{10}(x)$, $Y_{10}(x)$, and $M_{10}(x)=\sqrt{J_{10}^2(x)+Y_{10}^2(x)}$.

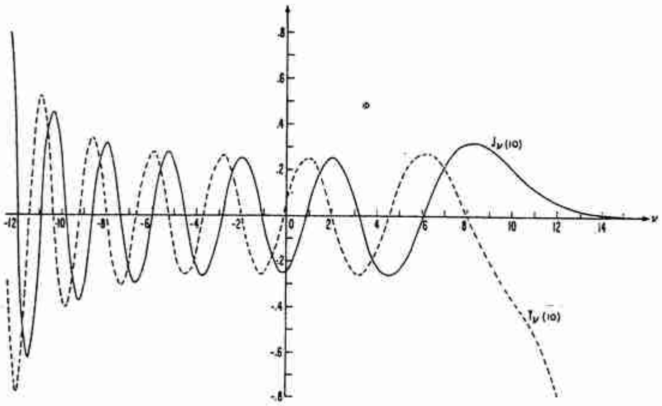


FIGURE 9.3. $J_\nu(10)$ and $Y_\nu(10)$.

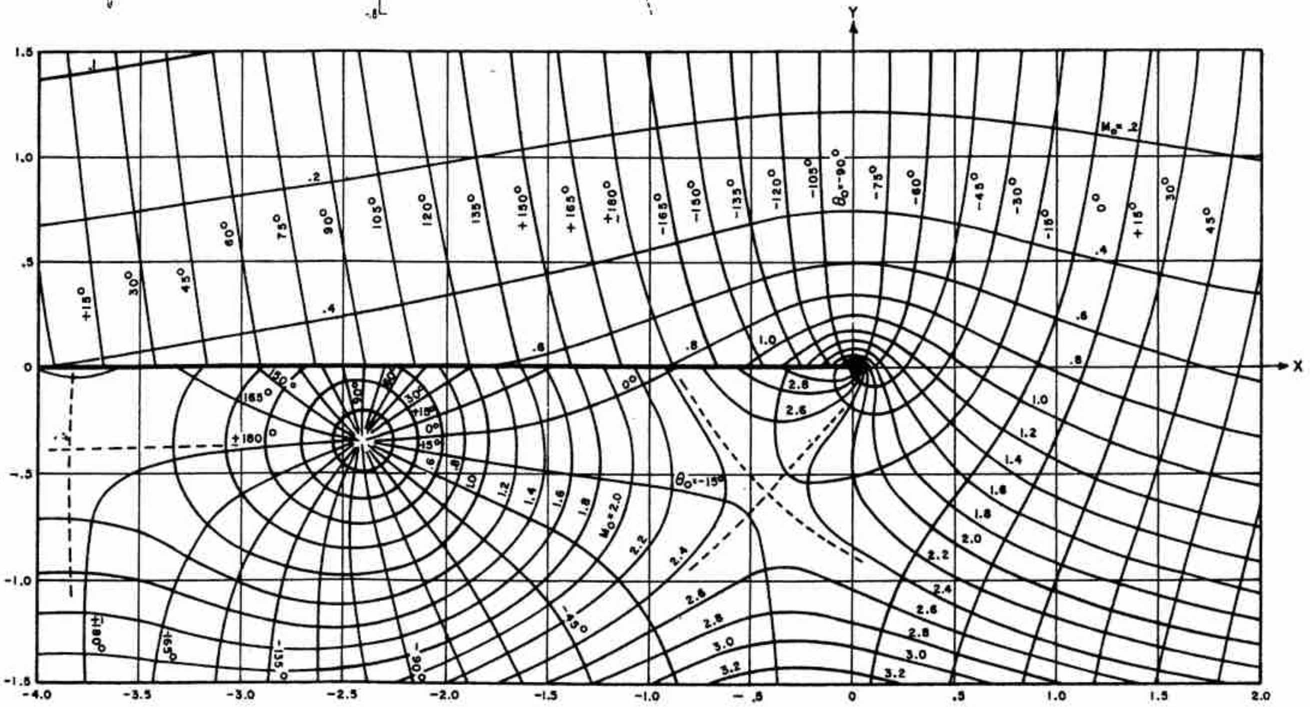


FIGURE 9.4. *Contour lines of the modulus and phase of the Hankel Function* $H_0^{(1)}(x+iy)=M_0e^{i\theta_0}$. From E. Jahnke, F. Emde, and F. Lösch, Tables of higher functions, McGraw-Hill Book Co., Inc., New York, N.Y., 1960 (with permission).

Limiting Forms for Small Arguments

When ν is fixed and $z \to 0$

9.1.7

$$J_\nu(z) \sim (\tfrac{1}{2}z)^\nu / \Gamma(\nu+1) \qquad (\nu \neq -1, -2, -3, \ldots)$$

9.1.8 $$Y_0(z) \sim -iH_0^{(1)}(z) \sim iH_0^{(2)}(z) \sim (2/\pi) \ln z$$

9.1.9

$$Y_\nu(z) \sim -iH_\nu^{(1)}(z) \sim iH_\nu^{(2)}(z) \sim -(1/\pi)\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu} \qquad (\mathscr{R}\nu > 0)$$

Ascending Series

9.1.10 $$J_\nu(z) = (\tfrac{1}{2}z)^\nu \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)^k}{k!\,\Gamma(\nu+k+1)}$$

9.1.11

$$\begin{aligned} Y_n(z) = &-\frac{(\frac{1}{2}z)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} (\tfrac{1}{4}z^2)^k \\ &+\frac{2}{\pi} \ln (\tfrac{1}{2}z) J_n(z) \\ &-\frac{(\frac{1}{2}z)^n}{\pi} \sum_{k=0}^{\infty} \{\psi(k+1) + \psi(n+k+1)\} \frac{(-\frac{1}{4}z^2)^k}{k!\,(n+k)!} \end{aligned}$$

where $\psi(n)$ is given by **6.3.2.**

9.1.12 $$J_0(z) = 1 - \frac{\frac{1}{4}z^2}{(1!)^2} + \frac{(\frac{1}{4}z^2)^2}{(2!)^2} - \frac{(\frac{1}{4}z^2)^3}{(3!)^2} + \ldots$$

9.1.13

$$\begin{aligned} Y_0(z) = \frac{2}{\pi} \{\ln (\tfrac{1}{2}z) + \gamma\} J_0(z) + \frac{2}{\pi} \{ \frac{\frac{1}{4}z^2}{(1!)^2} \\ -(1+\tfrac{1}{2}) \frac{(\frac{1}{4}z^2)^2}{(2!)^2} + (1+\tfrac{1}{2}+\tfrac{1}{3}) \frac{(\frac{1}{4}z^2)^3}{(3!)^2} - \ldots \} \end{aligned}$$

9.1.14

$$J_\nu(z) J_\mu(z) =$$

$$(\tfrac{1}{2}z)^{\nu+\mu} \sum_{k=0}^{\infty} \frac{(-)^k \Gamma(\nu+\mu+2k+1)\,(\frac{1}{4}z^2)^k}{\Gamma(\nu+k+1)\Gamma(\mu+k+1)\Gamma(\nu+\mu+k+1)\; k!}$$

Wronskians

9.1.15

$$\begin{aligned} W\{J_\nu(z), J_{-\nu}(z)\} &= J_{\nu+1}(z) J_{-\nu}(z) + J_\nu(z) J_{-(\nu+1)}(z) \\ &= -2 \sin (\nu\pi)/(\pi z) \end{aligned}$$

9.1.16

$$\begin{aligned} W\{J_\nu(z), Y_\nu(z)\} &= J_{\nu+1}(z) Y_\nu(z) - J_\nu(z) Y_{\nu+1}(z) \\ &= 2/(\pi z) \end{aligned}$$

9.1.17

$$\begin{aligned} W\{H_\nu^{(1)}(z), H_\nu^{(2)}(z)\} &= H_{\nu+1}^{(1)}(z) H_\nu^{(2)}(z) - H_\nu^{(1)}(z) H_{\nu+1}^{(2)}(z) \\ &= -4i/(\pi z) \end{aligned}$$

Integral Representations

9.1.18

$$J_0(z) = \frac{1}{\pi} \int_0^\pi \cos (z \sin \theta) d\theta = \frac{1}{\pi} \int_0^\pi \cos (z \cos \theta) d\theta$$

9.1.19

$$Y_0(z) = \frac{4}{\pi^2} \int_0^{\frac{1}{2}\pi} \cos (z \cos \theta) \{\gamma + \ln (2z \sin^2 \theta)\} \, d\theta$$

9.1.20

$$\begin{aligned} J_\nu(z) &= \frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}} \Gamma(\nu+\frac{1}{2})} \int_0^\pi \cos(z \cos \theta) \sin^{2\nu} \theta d\theta \\ &= \frac{2(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}} \Gamma(\nu+\frac{1}{2})} \int_0^1 (1-t^2)^{\nu-\frac{1}{2}} \cos (zt) dt \quad (\mathscr{R}\nu > -\tfrac{1}{2}) \end{aligned}$$

9.1.21

$$\begin{aligned} J_n(z) &= \frac{1}{\pi} \int_0^\pi \cos (z \sin \theta - n\theta) \, d\theta \\ &= \frac{i^{-n}}{\pi} \int_0^\pi e^{iz \cos \theta} \cos (n\theta) d\theta \end{aligned}$$

9.1.22

$$\begin{aligned} J_\nu(z) = &\frac{1}{\pi} \int_0^\pi \cos(z \sin \theta - \nu\theta) d\theta \\ &- \frac{\sin(\nu\pi)}{\pi} \int_0^\infty e^{-z \sinh t - \nu t} dt \quad (|\arg z| < \tfrac{1}{2}\pi) \end{aligned}$$

$$\begin{aligned} Y_\nu(z) = &\frac{1}{\pi} \int_0^\pi \sin(z \sin \theta - \nu\theta) d\theta \\ &- \frac{1}{\pi} \int_0^\infty \{e^{\nu t} + e^{-\nu t} \cos (\nu\pi)\} e^{-z \sinh t} dt \quad (|\arg z| < \tfrac{1}{2}\pi) \end{aligned}$$

9.1.23

$$J_0(x) = \frac{2}{\pi} \int_0^\infty \sin(x \cosh t) dt \quad (x > 0)$$

$$Y_0(x) = -\frac{2}{\pi} \int_0^\infty \cos(x \cosh t) dt \quad (x > 0)$$

9.1.24

$$J_\nu(x) = \frac{2(\frac{1}{2}x)^{-\nu}}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-\nu)} \int_1^\infty \frac{\sin(xt)\, dt}{(t^2-1)^{\nu+\frac{1}{2}}} \quad (|\mathscr{R}\nu| < \tfrac{1}{2}, x > 0)$$

$$Y_\nu(x) = -\frac{2(\frac{1}{2}x)^{-\nu}}{\pi^{\frac{1}{2}} \Gamma(\frac{1}{2}-\nu)} \int_1^\infty \frac{\cos(xt) dt}{(t^2-1)^{\nu+\frac{1}{2}}} \quad (|\mathscr{R}\nu| < \tfrac{1}{2}, x > 0)$$

9.1.25

$$H_\nu^{(1)}(z) = \frac{1}{\pi i} \int_{-\infty}^{\infty+\pi i} e^{z \sinh t - \nu t} dt \quad (|\arg z| < \tfrac{1}{2}\pi)$$

$$H_\nu^{(2)}(z) = -\frac{1}{\pi i} \int_{-\infty}^{\infty-\pi i} e^{z \sinh t - \nu t} dt \quad (|\arg z| < \tfrac{1}{2}\pi)$$

9.1.26

$$J_\nu(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma(-t)(\frac{1}{2}x)^{\nu+2t}}{\Gamma(\nu+t+1)} dt \quad (\mathscr{R}\nu > 0, x > 0)$$

In the last integral the path of integration must lie to the left of the points $t = 0, 1, 2, \ldots$.

Recurrence Relations

9.1.27

$$\mathscr{C}_{\nu-1}(z)+\mathscr{C}_{\nu+1}(z)=\frac{2\nu}{z}\mathscr{C}_\nu(z)$$

$$\mathscr{C}_{\nu-1}(z)-\mathscr{C}_{\nu+1}(z)=2\mathscr{C}'_\nu(z)$$

$$\mathscr{C}'_\nu(z)=\mathscr{C}_{\nu-1}(z)-\frac{\nu}{z}\mathscr{C}_\nu(z)$$

$$\mathscr{C}'_\nu(z)=-\mathscr{C}_{\nu+1}(z)+\frac{\nu}{z}\mathscr{C}_\nu(z)$$

$\mathscr{C}$ denotes $J, Y, H^{(1)}, H^{(2)}$ or any linear combination of these functions, the coefficients in which are independent of z and ν.

9.1.28 $J'_0(z)=-J_1(z)\qquad Y'_0(z)=-Y_1(z)$

If $f_\nu(z)=z^p\mathscr{C}_\nu(\lambda z^q)$ where p, q, λ are independent of ν, then

9.1.29

$$f_{\nu-1}(z)+f_{\nu+1}(z)=(2\nu/\lambda)z^{-q}f_\nu(z)$$

$$(p+\nu q)f_{\nu-1}(z)+(p-\nu q)f_{\nu+1}(z)=(2\nu/\lambda)z^{1-q}f'_\nu(z)$$

$$zf'_\nu(z)=\lambda qz^qf_{\nu-1}(z)+(p-\nu q)f_\nu(z)$$

$$zf'_\nu(z)=-\lambda qz^qf_{\nu+1}(z)+(p+\nu q)f_\nu(z)$$

Formulas for Derivatives

9.1.30

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\{z^\nu\mathscr{C}_\nu(z)\}=z^{\nu-k}\mathscr{C}_{\nu-k}(z)$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\{z^{-\nu}\mathscr{C}_\nu(z)\}=(-)^kz^{-\nu-k}\mathscr{C}_{\nu+k}(z)\qquad(k=0,1,2,\ldots)$$

9.1.31

$$\mathscr{C}^{(k)}_\nu(z)=\frac{1}{2^k}\{\mathscr{C}_{\nu-k}(z)-\binom{k}{1}\mathscr{C}_{\nu-k+2}(z)+\binom{k}{2}\mathscr{C}_{\nu-k+4}(z)-\ldots+(-)^k\mathscr{C}_{\nu+k}(z)\}\qquad(k=0,1,2,\ldots)$$

Recurrence Relations for Cross-Products

If

9.1.32

$$p_\nu=J_\nu(a)Y_\nu(b)-J_\nu(b)Y_\nu(a)$$

$$q_\nu=J_\nu(a)Y'_\nu(b)-J'_\nu(b)Y_\nu(a)$$

$$r_\nu=J'_\nu(a)Y_\nu(b)-J_\nu(b)Y'_\nu(a)$$

$$s_\nu=J'_\nu(a)Y'_\nu(b)-J'_\nu(b)Y'_\nu(a)$$

then

9.1.33

$$p_{\nu+1}-p_{\nu-1}=-\frac{2\nu}{a}q_\nu-\frac{2\nu}{b}r_\nu$$

$$q_{\nu+1}+r_\nu=\frac{\nu}{a}p_\nu-\frac{\nu+1}{b}p_{\nu+1}$$

$$r_{\nu+1}+q_\nu=\frac{\nu}{b}p_\nu-\frac{\nu+1}{a}p_{\nu+1}$$

$$s_\nu=\frac{1}{2}p_{\nu+1}+\frac{1}{2}p_{\nu-1}-\frac{\nu^2}{ab}p_\nu$$

and

9.1.34 $$p_\nu s_\nu-q_\nu r_\nu=\frac{4}{\pi^2ab}$$

Analytic Continuation

In **9.1.35** to **9.1.38**, m is an integer.

9.1.35 $$J_\nu(ze^{m\pi i})=e^{m\nu\pi i}J_\nu(z)$$

9.1.36

$$Y_\nu(ze^{m\pi i})=e^{-m\nu\pi i}Y_\nu(z)+2i\sin(m\nu\pi)\cot(\nu\pi)J_\nu(z)$$

9.1.37

$$\sin(\nu\pi)H^{(1)}_\nu(ze^{m\pi i})=-\sin\{(m-1)\nu\pi\}H^{(1)}_\nu(z)-e^{-\nu\pi i}\sin(m\nu\pi)H^{(2)}_\nu(z)$$

9.1.38

$$\sin(\nu\pi)H^{(2)}_\nu(ze^{m\pi i})=\sin\{(m+1)\nu\pi\}H^{(2)}_\nu(z)+e^{\nu\pi i}\sin(m\nu\pi)H^{(1)}_\nu(z)$$

9.1.39

$$H^{(1)}_\nu(ze^{\pi i})=-e^{-\nu\pi i}H^{(2)}_\nu(z)$$

$$H^{(2)}_\nu(ze^{-\pi i})=-e^{\nu\pi i}H^{(1)}_\nu(z)$$

9.1.40

$$J_\nu(\bar{z})=\overline{J_\nu(z)}\qquad Y_\nu(\bar{z})=\overline{Y_\nu(z)}$$

$$H^{(1)}_\nu(\bar{z})=\overline{H^{(2)}_\nu(z)}\qquad H^{(2)}_\nu(\bar{z})=\overline{H^{(1)}_\nu(z)}\qquad(\nu\text{ real})$$

Generating Function and Associated Series

9.1.41 $$e^{\frac{1}{2}z(t-1/t)}=\sum_{k=-\infty}^{\infty}t^kJ_k(z)\qquad(t\neq0)$$

9.1.42 $$\cos(z\sin\theta)=J_0(z)+2\sum_{k=1}^{\infty}J_{2k}(z)\cos(2k\theta)$$

9.1.43 $$\sin(z\sin\theta)=2\sum_{k=0}^{\infty}J_{2k+1}(z)\sin\{(2k+1)\theta\}$$

9.1.44

$$\cos(z\cos\theta)=J_0(z)+2\sum_{k=1}^{\infty}(-)^kJ_{2k}(z)\cos(2k\theta)$$

9.1.45

$$\sin(z\cos\theta)=2\sum_{k=0}^{\infty}(-)^kJ_{2k+1}(z)\cos\{(2k+1)\theta\}$$

9.1.46 $$1=J_0(z)+2J_2(z)+2J_4(z)+2J_6(z)+\ldots$$

9.1.47

$$\cos z=J_0(z)-2J_2(z)+2J_4(z)-2J_6(z)+\ldots$$

9.1.48 $$\sin z=2J_1(z)-2J_3(z)+2J_5(z)-\ldots$$

Other Differential Equations

9.1.49 $w''+\left(\lambda^2-\frac{\nu^2-\frac{1}{4}}{z^2}\right)w=0, \qquad w=z^{\frac{1}{2}}\mathscr{C}_\nu(\lambda z)$

9.1.50 $w''+\left(\frac{\lambda^2}{4z}-\frac{\nu^2-1}{4z^2}\right)w=0, \qquad w=z^{\frac{1}{2}}\mathscr{C}_\nu(\lambda z^{\frac{1}{2}})$

9.1.51 $w''+\lambda^2z^{p-2}w=0, \qquad w=z^{\frac{1}{2}}\mathscr{C}_{1/p}(2\lambda z^{\frac{1}{2}p}/p)$

9.1.52

$$w''-\frac{2\nu-1}{z}w'+\lambda^2w=0, \qquad w=z^\nu\mathscr{C}_\nu(\lambda z)$$

9.1.53

$$z^2w''+(1-2p)zw'+(\lambda^2q^2z^{2q}+p^2-\nu^2q^2)w=0,$$
$$w=z^p\mathscr{C}_\nu(\lambda z^q)$$

9.1.54

$$w''+(\lambda^2e^{2z}-\nu^2)w=0, \qquad w=\mathscr{C}_\nu(\lambda e^z)$$

9.1.55

$$z^2(z^2-\nu^2)w''+z(z^2-3\nu^2)w'$$
$$+\{(z^2-\nu^2)^2-(z^2+\nu^2)\}w=0, \qquad w=\mathscr{C}'_\nu(z)$$

9.1.56

$$w^{(2n)}=(-)^n\lambda^{2n}z^{-n}w, \qquad w=z^{\frac{1}{2}n}\mathscr{C}_n(2\lambda\alpha z^{\frac{1}{2}})$$

where α is any of the $2n$ roots of unity.

Differential Equations for Products

In the following $\vartheta\equiv z\frac{d}{dz}$ and $\mathscr{C}_\nu(z)$, $\mathscr{D}_\mu(z)$ are any cylinder functions of orders ν, μ respectively.

9.1.57

$$\{\vartheta^4-2(\nu^2+\mu^2)\vartheta^2+(\nu^2-\mu^2)^2\}w$$
$$+4z^2(\vartheta+1)(\vartheta+2)w=0, \qquad w=\mathscr{C}_\nu(z)\mathscr{D}_\mu(z)$$

9.1.58

$$\vartheta(\vartheta^2-4\nu^2)w+4z^2(\vartheta+1)w=0, \qquad w=\mathscr{C}_\nu(z)\mathscr{D}_\nu(z)$$

9.1.59

$$z^3w'''+z(4z^2+1-4\nu^2)w'+(4\nu^2-1)w=0,$$
$$w=z\mathscr{C}_\nu(z)\mathscr{D}_\nu(z)$$

Upper Bounds

9.1.60 $|J_\nu(x)|\leq 1 \; (\nu\geq 0), \; |J_\nu(x)|\leq 1/\sqrt{2} \qquad (\nu\geq 1)$

9.1.61 $0<J_\nu(\nu)<\frac{2^{\frac{1}{3}}}{3^{\frac{2}{3}}\Gamma(\frac{2}{3})\nu^{\frac{1}{3}}} \qquad (\nu>0)$

9.1.62 $|J_\nu(z)|\leq\frac{|\frac{1}{2}z|^\nu e^{|\mathscr{I}z|}}{\Gamma(\nu+1)} \qquad (\nu\geq-\frac{1}{2})$ *

9.1.63 $|J_n(nz)|\leq\left|\frac{z^n\exp\{n\sqrt{(1-z^2)}\}}{\{1+\sqrt{(1-z^2)}\}^n}\right|$

Derivatives With Respect to Order

9.1.64

$$\frac{\partial}{\partial\nu}J_\nu(z)=J_\nu(z)\ln(\tfrac{1}{2}z)$$
$$-(\tfrac{1}{2}z)^\nu\sum_{k=0}^{\infty}(-)^k\frac{\psi(\nu+k+1)}{\Gamma(\nu+k+1)}\frac{(\frac{1}{4}z^2)^k}{k!}$$

9.1.65

$$\frac{\partial}{\partial\nu}Y_\nu(z)=\cot(\nu\pi)\left\{\frac{\partial}{\partial\nu}J_\nu(z)-\pi Y_\nu(z)\right\}$$
$$-\csc(\nu\pi)\frac{\partial}{\partial\nu}J_{-\nu}(z)-\pi J_\nu(z)$$
$$(\nu\neq 0, \pm 1, \pm 2, \ldots)$$

9.1.66

$$\left[\frac{\partial}{\partial\nu}J_\nu(z)\right]_{\nu=n}=\frac{\pi}{2}Y_n(z)+\frac{n!(\frac{1}{2}z)^{-n}}{2}\sum_{k=0}^{n-1}\frac{(\frac{1}{2}z)^kJ_k(z)}{(n-k)k!}$$

9.1.67

$$\left[\frac{\partial}{\partial\nu}Y_\nu(z)\right]_{\nu=n}=-\frac{\pi}{2}J_n(z)+\frac{n!(\frac{1}{2}z)^{-n}}{2}\sum_{k=0}^{n-1}\frac{(\frac{1}{2}z)^kY_k(z)}{(n-k)k!}$$

9.1.68

$$\left[\frac{\partial}{\partial\nu}J_\nu(z)\right]_{\nu=0}=\frac{\pi}{2}Y_0(z), \left[\frac{\partial}{\partial\nu}Y_\nu(z)\right]_{\nu=0}=-\frac{\pi}{2}J_0(z)$$

Expressions in Terms of Hypergeometric Functions

9.1.69

$$J_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\,{}_0F_1(\nu+1; -\tfrac{1}{4}z^2)$$
$$=\frac{(\frac{1}{2}z)^\nu e^{-iz}}{\Gamma(\nu+1)}M(\nu+\tfrac{1}{2}, 2\nu+1, 2iz)$$

9.1.70

$$J_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\lim F\left(\lambda, \mu; \nu+1; -\frac{z^2}{4\lambda\mu}\right)$$

as λ, $\mu\to\infty$ through real or complex values; z, ν being fixed.

(${}_0F_1$ is the generalized hypergeometric function. For $M(a, b, z)$ and $F(a, b; c; z)$ see chapters **13** and **15**.)

Connection With Legendre Functions

If μ and x are fixed and $\nu\to\infty$ through real positive values

9.1.71

$$\lim\left\{\nu^\mu P_\nu^{-\mu}\left(\cos\frac{x}{\nu}\right)\right\}=J_\mu(x) \qquad (x>0)$$

---

*See page II.

9.1.72

$$\lim \{\nu^{\mu} Q_{\nu}^{-\mu}\left(\cos\frac{x}{\nu}\right)\} = -\tfrac{1}{2}\pi Y_{\mu}(x) \qquad (x>0)$$

For $P_{\nu}^{-\mu}$ and $Q_{\nu}^{-\mu}$, see chapter **8**.

Continued Fractions

9.1.73

$$\frac{J_{\nu}(z)}{J_{\nu-1}(z)} = \frac{1}{2\nu z^{-1}-}\;\frac{1}{2(\nu+1)z^{-1}-}\;\frac{1}{2(\nu+2)z^{-1}-}\cdots$$
$$= \frac{\frac{1}{2}z/\nu}{1-}\;\frac{\frac{1}{4}z^2/\{\nu(\nu+1)\}}{1-}\;\frac{\frac{1}{4}z^2/\{(\nu+1)(\nu+2)\}}{1-}\cdots$$

Multiplication Theorem

9.1.74

$$\mathscr{C}_{\nu}(\lambda z) = \lambda^{\pm\nu} \sum_{k=0}^{\infty} \frac{(\mp)^k(\lambda^2-1)^k(\frac{1}{2}z)^k}{k!}\,\mathscr{C}_{\nu\pm k}(z) \qquad (|\lambda^2-1|<1)$$

If $\mathscr{C}=J$ and the upper signs are taken, the restriction on λ is unnecessary.

This theorem will furnish expansions of $\mathscr{C}_{\nu}(re^{i\theta})$ in terms of $\mathscr{C}_{\nu\pm k}(r)$.

Addition Theorems

Neumann's

9.1.75 $$\mathscr{C}_{\nu}(u\pm v) = \sum_{k=-\infty}^{\infty} \mathscr{C}_{\nu\mp k}(u) J_k(v) \qquad (|v|<|u|)$$

The restriction $|v|<|u|$ is unnecessary when $\mathscr{C}=J$ and ν is an integer or zero. Special cases are

9.1.76 $$1 = J_0^2(z) + 2\sum_{k=1}^{\infty} J_k^2(z)$$

9.1.77

$$0 = \sum_{k=0}^{2n} (-)^k J_k(z) J_{2n-k}(z) + 2\sum_{k=1}^{\infty} J_k(z) J_{2n+k}(z) \quad (n\geq 1)$$

9.1.78

$$J_n(2z) = \sum_{k=0}^{n} J_k(z) J_{n-k}(z) + 2\sum_{k=1}^{\infty} (-)^k J_k(z) J_{n+k}(z)$$

Graf's

9.1.79

$$\mathscr{C}_{\nu}(w) \begin{matrix}\cos\\ \sin\end{matrix} \nu\chi = \sum_{k=-\infty}^{\infty} \mathscr{C}_{\nu+k}(u) J_k(v) \begin{matrix}\cos\\ \sin\end{matrix} k\alpha (|ve^{\pm i\alpha}|<|u|)$$

Gegenbauer's

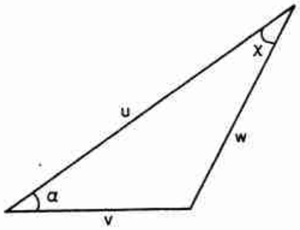
9.1.80

$$\frac{\mathscr{C}_{\nu}(w)}{w^{\nu}} = 2^{\nu}\Gamma(\nu) \sum_{k=0}^{\infty} (\nu+k) \frac{\mathscr{C}_{\nu+k}(u)}{u^{\nu}} \frac{J_{\nu+k}(v)}{v^{\nu}} C_k^{(\nu)}(\cos\alpha)$$
$$(\nu\neq 0,-1,\ldots, |ve^{\pm i\alpha}|<|u|)$$

In **9.1.79** and **9.1.80**,

$$w = \sqrt{(u^2+v^2-2uv\cos\alpha)},$$
$$u-v\cos\alpha = w\cos\chi,\; v\sin\alpha = w\sin\chi$$

the branches being chosen so that $w\to u$ and $\chi\to 0$ as $v\to 0$. $C_k^{(\nu)}(\cos\alpha)$ is Gegenbauer's polynomial (see chapter **22**).



Gegenbauer's addition theorem.

If u, v are real and positive and $0\leq\alpha\leq\pi$, then w, χ are real and non-negative, and the geometrical relationship of the variables is shown in the diagram.

The restrictions $|ve^{\pm i\alpha}|<|u|$ are unnecessary in **9.1.79** when $\mathscr{C}=J$ and ν is an integer or zero, and in **9.1.80** when $\mathscr{C}=J$.

Degenerate Form $(u=\infty)$:

9.1.81

$$e^{iv\cos\alpha} = \Gamma(\nu)(\tfrac{1}{2}v)^{-\nu} \sum_{k=0}^{\infty} (\nu+k) i^k J_{\nu+k}(v) C_k^{(\nu)}(\cos\alpha) \qquad (\nu\neq 0,-1,\ldots)$$

Neumann's Expansion of an Arbitrary Function in a Series of Bessel Functions

9.1.82 $$f(z) = a_0 J_0(z) + 2\sum_{k=1}^{\infty} a_k J_k(z) \qquad (|z|<c)$$

where c is the distance of the nearest singularity of $f(z)$ from $z=0$,

9.1.83 $$a_k = \frac{1}{2\pi i}\int_{|z|=c'} f(t) O_k(t)\,dt \qquad (0<c'<c)$$

and $O_k(t)$ is Neumann's polynomial. The latter is defined by the generating function

9.1.84

$$\frac{1}{t-z} = J_0(z) O_0(t) + 2\sum_{k=1}^{\infty} J_k(z) O_k(t) \qquad (|z|<|t|)$$

$O_n(t)$ is a polynomial of degree $n+1$ in $1/t$; $O_0(t)=1/t$,

9.1.85

$$O_n(t) = \frac{1}{4}\sum_{k=0}^{\leq\frac{1}{2}n} \frac{n(n-k-1)!}{k!}\left(\frac{2}{t}\right)^{n-2k+1} \qquad (n=1,2,\ldots)$$

The more general form of expansion

9.1.86 $$f(z) = a_0 J_{\nu}(z) + 2\sum_{k=1}^{\infty} a_k J_{\nu+k}(z)$$

also called a Neumann expansion, is investigated in [9.7] and [9.15] together with further generalizations. Examples of Neumann expansions are **9.1.41** to **9.1.48** and the Addition Theorems. Other examples are

9.1.87

$$(\tfrac{1}{2}z)^\nu=\sum_{k=0}^{\infty}\frac{(\nu+2k)\Gamma(\nu+k)}{k!}J_{\nu+2k}(z)\qquad(\nu\neq 0,-1,-2,\ldots)$$

9.1.88

$$Y_n(z)=-\frac{n!(\tfrac{1}{2}z)^{-n}}{\pi}\sum_{k=0}^{n-1}\frac{(\tfrac{1}{2}z)^kJ_k(z)}{(n-k)k!}+\frac{2}{\pi}\{\ln(\tfrac{1}{2}z)-\psi(n+1)\}J_n(z)-\frac{2}{\pi}\sum_{k=1}^{\infty}(-)^k\frac{(n+2k)J_{n+2k}(z)}{k(n+k)}$$

where $\psi(n)$ is given by **6.3.2.**

9.1.89

$$Y_0(z)=\frac{2}{\pi}\{\ln(\tfrac{1}{2}z)+\gamma\}J_0(z)-\frac{4}{\pi}\sum_{k=1}^{\infty}(-)^k\frac{J_{2k}(z)}{k}$$

9.2. Asymptotic Expansions for Large Arguments

Principal Asymptotic Forms

When ν is fixed and $|z|\to\infty$

9.2.1

$$J_\nu(z)=\sqrt{2/(\pi z)}\{\cos(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)+e^{|\mathscr{I}z|}O(|z|^{-1})\}\qquad(|\arg z|<\pi)$$

9.2.2

$$Y_\nu(z)=\sqrt{2/(\pi z)}\{\sin(z-\tfrac{1}{2}\nu\pi-\tfrac{1}{4}\pi)+e^{|\mathscr{I}z|}O(|z|^{-1})\}\qquad(|\arg z|<\pi)$$

9.2.3

$$H_\nu^{(1)}(z)\sim\sqrt{2/(\pi z)}e^{i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}\qquad(-\pi<\arg z<2\pi)$$

9.2.4

$$H_\nu^{(2)}(z)\sim\sqrt{2/(\pi z)}e^{-i(z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi)}\qquad(-2\pi<\arg z<\pi)$$

Hankel's Asymptotic Expansions

When ν is fixed and $|z|\to\infty$

9.2.5

$$J_\nu(z)=\sqrt{2/(\pi z)}\{P(\nu,z)\cos\chi-Q(\nu,z)\sin\chi\}\qquad(|\arg z|<\pi)$$

9.2.6

$$Y_\nu(z)=\sqrt{2/(\pi z)}\{P(\nu,z)\sin\chi+Q(\nu,z)\cos\chi\}\qquad(|\arg z|<\pi)$$

9.2.7

$$H_\nu^{(1)}(z)=\sqrt{2/(\pi z)}\{P(\nu,z)+iQ(\nu,z)\}e^{i\chi}\qquad(-\pi<\arg z<2\pi)$$

9.2.8

$$H_\nu^{(2)}(z)=\sqrt{2/(\pi z)}\{P(\nu,z)-iQ(\nu,z)\}e^{-i\chi}\qquad(-2\pi<\arg z<\pi)$$

where $\chi=z-(\tfrac{1}{2}\nu+\tfrac{1}{4})\pi$ and, with $4\nu^2$ denoted by μ,

9.2.9

$$P(\nu,z)\sim\sum_{k=0}^{\infty}(-)^k\frac{(\nu,2k)}{(2z)^{2k}}=1-\frac{(\mu-1)(\mu-9)}{2!(8z)^2}+\frac{(\mu-1)(\mu-9)(\mu-25)(\mu-49)}{4!(8z)^4}-\ldots$$

9.2.10

$$Q(\nu,z)\sim\sum_{k=0}^{\infty}(-)^k\frac{(\nu,2k+1)}{(2z)^{2k+1}}=\frac{\mu-1}{8z}-\frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3}+\ldots$$

If ν is real and non-negative and z is positive, the remainder after k terms in the expansion of $P(\nu,z)$ does not exceed the $(k+1)$th term in absolute value and is of the same sign, provided that $k>\tfrac{1}{2}\nu-\tfrac{1}{4}$. The same is true of $Q(\nu,z)$ provided that $k>\tfrac{1}{2}\nu-\tfrac{3}{4}$.

Asymptotic Expansions of Derivatives

With the conditions and notation of the preceding subsection

9.2.11

$$J_\nu'(z)=\sqrt{2/(\pi z)}\{-R(\nu,z)\sin\chi-S(\nu,z)\cos\chi\}\qquad(|\arg z|<\pi)$$

9.2.12

$$Y_\nu'(z)=\sqrt{2/(\pi z)}\{R(\nu,z)\cos\chi-S(\nu,z)\sin\chi\}\qquad(|\arg z|<\pi)$$

9.2.13

$$H_\nu^{(1)\prime}(z)=\sqrt{2/(\pi z)}\{iR(\nu,z)-S(\nu,z)\}e^{i\chi}\qquad(-\pi<\arg z<2\pi)$$

9.2.14

$$H_\nu^{(2)\prime}(z)=\sqrt{2/(\pi z)}\{-iR(\nu,z)-S(\nu,z)\}e^{-i\chi}\qquad(-2\pi<\arg z<\pi)$$

9.2.15

$$R(\nu, z)\sim\sum_{k=0}^{\infty} (-)^k \frac{4\nu^2+16k^2-1}{4\nu^2-(4k-1)^2} \frac{(\nu, 2k)}{(2z)^{2k}}$$

$$=1-\frac{(\mu-1)(\mu+15)}{2!(8z)^2}+\ldots$$

9.2.16

$$S(\nu, z)\sim\sum_{k=0}^{\infty} (-)^k \frac{4\nu^2+4(2k+1)^2-1}{4\nu^2-(4k+1)^2} \frac{(\nu, 2k+1)}{(2z)^{2k+1}}$$

$$=\frac{\mu+3}{8z}-\frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8z)^3}+\ldots$$

Modulus and Phase

For real ν and positive x

9.2.17

$$M_\nu=|H_\nu^{(1)}(x)|=\sqrt{\{J_\nu^2(x)+Y_\nu^2(x)\}}$$
$$\theta_\nu=\arg H_\nu^{(1)}(x)=\arctan\{Y_\nu(x)/J_\nu(x)\}$$

9.2.18

$$N_\nu=|H_\nu^{(1)\prime}(x)|=\sqrt{\{J_\nu'^2(x)+Y_\nu'^2(x)\}}$$
$$\varphi_\nu=\arg H_\nu^{(1)\prime}(x)=\arctan\{Y_\nu'(x)/J_\nu'(x)\}$$

9.2.19 $J_\nu(x)=M_\nu \cos\theta_\nu, \qquad Y_\nu(x)=M_\nu\sin\theta_\nu,$

9.2.20 $J_\nu'(x)=N_\nu\cos\varphi_\nu, \qquad Y_\nu'(x)=N_\nu\sin\varphi_\nu.$

In the following relations, primes denote differentiations with respect to x.

9.2.21 $M_\nu^2\theta_\nu'=2/(\pi x) \qquad N_\nu^2\varphi_\nu'=2(x^2-\nu^2)/(\pi x^3)$

9.2.22 $N_\nu^2=M_\nu'^2+M_\nu^2\theta_\nu'^2=M_\nu'^2+4/(\pi x M_\nu)^2$

9.2.23 $(x^2-\nu^2)M_\nu M_\nu'+x^2N_\nu N_\nu'+xN_\nu^2=0$

9.2.24

$$\tan(\varphi_\nu-\theta_\nu)=M_\nu\theta_\nu'/M_\nu'=2/(\pi x M_\nu M_\nu')$$
$$M_\nu N_\nu\sin(\varphi_\nu-\theta_\nu)=2/(\pi x)$$

9.2.25 $x^2M_\nu''+xM_\nu'+(x^2-\nu^2)M_\nu-4/(\pi^2M_\nu^3)=0$

9.2.26

$$x^3w'''+x(4x^2+1-4\nu^2)w'+(4\nu^2-1)w=0, \quad w=xM_\nu^2$$

9.2.27 $$\theta_\nu'^2+\frac{1}{2}\frac{\theta_\nu'''}{\theta_\nu'}-\frac{3}{4}\left(\frac{\theta_\nu''}{\theta_\nu'}\right)^2=1-\frac{\nu^2-\frac{1}{4}}{x^2}$$

Asymptotic Expansions of Modulus and Phase

When ν is fixed, x is large and positive, and $\mu=4\nu^2$

9.2.28

$$M_\nu^2\sim\frac{2}{\pi x}\left\{1+\frac{1}{2}\frac{\mu-1}{(2x)^2}+\frac{1\cdot3}{2\cdot4}\frac{(\mu-1)(\mu-9)}{(2x)^4}\right.$$
$$\left.+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\frac{(\mu-1)(\mu-9)(\mu-25)}{(2x)^6}+\ldots\right\}$$

9.2.29

$$\theta_\nu\sim x-(\tfrac{1}{2}\nu+\tfrac{1}{4})\pi+\frac{\mu-1}{2(4x)}$$
$$+\frac{(\mu-1)(\mu-25)}{6(4x)^3}+\frac{(\mu-1)(\mu^2-114\mu+1073)}{5(4x)^5}$$
$$+\frac{(\mu-1)(5\mu^3-1535\mu^2+54703\mu-375733)}{14(4x)^7}+\ldots$$

9.2.30

$$N_\nu^2\sim\frac{2}{\pi x}\left\{1-\frac{1}{2}\frac{\mu-3}{(2x)^2}-\frac{1\cdot1}{2\cdot4}\frac{(\mu-1)(\mu-45)}{(2x)^4}-\ldots\right\}$$

The general term in the last expansion is given by

$$-\frac{1\cdot1\cdot3\ldots(2k-3)}{2\cdot4\cdot6\ldots(2k)}$$
$$\times\frac{(\mu-1)(\mu-9)\ldots\{\mu-(2k-3)^2\}\{\mu-(2k+1)(2k-1)^2\}}{(2x)^{2k}}$$

9.2.31

$$\phi_\nu\sim x-(\tfrac{1}{2}\nu-\tfrac{1}{4})\pi+\frac{\mu+3}{2(4x)}+\frac{\mu^2+46\mu-63}{6(4x)^3}$$
$$+\frac{\mu^3+185\mu^2-2053\mu+1899}{5(4x)^5}+\ldots$$

If $\nu\geq0$, the remainder after k terms in **9.2.28** does not exceed the $(k+1)$th term in absolute value and is of the same sign, provided that $k>\nu-\frac{1}{2}$.

9.3. Asymptotic Expansions for Large Orders

Principal Asymptotic Forms

In the following equations it is supposed that $\nu\to\infty$ through real positive values, the other variables being fixed.

9.3.1

$$J_\nu(z)\sim\frac{1}{\sqrt{2\pi\nu}}\left(\frac{ez}{2\nu}\right)^\nu$$

$$Y_\nu(z)\sim-\sqrt{\frac{2}{\pi\nu}}\left(\frac{ez}{2\nu}\right)^{-\nu}$$

9.3.2

$$J_\nu(\nu\operatorname{sech}\alpha)\sim\frac{e^{\nu(\tanh\alpha-\alpha)}}{\sqrt{2\pi\nu\tanh\alpha}} \qquad (\alpha>0)$$

$$Y_\nu(\nu\operatorname{sech}\alpha)\sim-\frac{e^{\nu(\alpha-\tanh\alpha)}}{\sqrt{\frac{1}{2}\pi\nu\tanh\alpha}} \qquad (\alpha>0)$$

---

*See page II.

9.3.3

$$J_\nu(\nu \sec \beta)=\sqrt{2/(\pi\nu \tan \beta)}\,\{\cos(\nu\tan\beta-\nu\beta-\tfrac{1}{4}\pi)+O(\nu^{-1})\}\qquad (0<\beta<\tfrac{1}{2}\pi)$$

$$Y_\nu(\nu \sec \beta)=\sqrt{2/(\pi\nu \tan \beta)}\,\{\sin(\nu\tan\beta-\nu\beta-\tfrac{1}{4}\pi)+O(\nu^{-1})\}\qquad (0<\beta<\tfrac{1}{2}\pi)$$

9.3.4

$$J_\nu(\nu+z\nu^{1/3})=2^{1/3}\nu^{-1/3}\,\mathrm{Ai}(-2^{1/3}z)+O(\nu^{-1})$$

$$Y_\nu(\nu+z\nu^{1/3})=-2^{1/3}\nu^{-1/3}\,\mathrm{Bi}(-2^{1/3}z)+O(\nu^{-1})$$

9.3.5

$$J_\nu(\nu)\sim\frac{2^{1/3}}{3^{2/3}\Gamma(\frac{2}{3})}\,\frac{1}{\nu^{1/3}}$$

$$Y_\nu(\nu)\sim-\frac{2^{1/3}}{3^{1/6}\Gamma(\frac{2}{3})}\,\frac{1}{\nu^{1/3}}$$

9.3.6

$$J_\nu(\nu z)=\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\left\{\frac{\mathrm{Ai}\,(\nu^{2/3}\zeta)}{\nu^{1/3}}+\frac{\exp\,(-\frac{2}{3}\nu\zeta^{3/2})}{1+\nu^{1/6}|\zeta|^{1/4}}\,O\left(\frac{1}{\nu^{4/3}}\right)\right\}\qquad (|\arg z|<\pi)$$

$$Y_\nu(\nu z)=-\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\left\{\frac{\mathrm{Bi}\,(\nu^{2/3}\zeta)}{\nu^{1/3}}+\frac{\exp\,|\mathscr{R}(\frac{2}{3}\nu\zeta^{3/2})|}{1+\nu^{1/6}|\zeta|^{1/4}}\,O\left(\frac{1}{\nu^{4/3}}\right)\right\}\qquad (|\arg z|<\pi)$$

In the last two equations ζ is given by **9.3.38** and **9.3.39** below.

Debye's Asymptotic Expansions

(i) If α is fixed and positive and ν is large and positive

9.3.7

$$J_\nu(\nu\,\mathrm{sech}\,\alpha)\sim\frac{e^{\nu(\tanh\alpha-\alpha)}}{\sqrt{2\pi\nu\tanh\alpha}}\left\{1+\sum_{k=1}^{\infty}\frac{u_k\,(\coth\alpha)}{\nu^k}\right\}$$

9.3.8

$$Y_\nu(\nu\,\mathrm{sech}\,\alpha)\sim-\frac{e^{\nu(\alpha-\tanh\alpha)}}{\sqrt{\frac{1}{2}\pi\nu\tanh\alpha}}\left\{1+\sum_{k=1}^{\infty}(-)^k\,\frac{u_k\,(\coth\alpha)}{\nu^k}\right\}$$

where

9.3.9

$u_0(t)=1$

$u_1(t)=(3t-5t^3)/24$

$u_2(t)=(81t^2-462t^4+385t^6)/1152$

$u_3(t)=(30375t^3-3\ 69603t^5+7\ 65765t^7-4\ 25425t^9)/4\ 14720$

$u_4(t)=(44\ 65125t^4-941\ 21676t^6+3499\ 22430t^8-4461\ 85740t^{10}+1859\ 10725t^{12})/398\ 13120$

For $u_5(t)$ and $u_6(t)$ see [9.4] or [9.21].

9.3.10

$$u_{k+1}(t)=\tfrac{1}{2}t^2(1-t^2)u_k'(t)+\frac{1}{8}\int_0^t(1-5t^2)u_k(t)dt\qquad (k=0,\,1,\,\ldots)$$

Also

9.3.11

$$J_\nu'(\nu\,\mathrm{sech}\,\alpha)\sim\sqrt{\frac{\sinh 2\alpha}{4\pi\nu}}\,e^{\nu(\tanh\alpha-\alpha)}\left\{1+\sum_{k=1}^{\infty}\frac{v_k\,(\coth\alpha)}{\nu^k}\right\}$$

9.3.12

$$Y_\nu'(\nu\,\mathrm{sech}\,\alpha)\sim\sqrt{\frac{\sinh 2\alpha}{\pi\nu}}\,e^{\nu(\alpha-\tanh\alpha)}\left\{1+\sum_{k=1}^{\infty}(-)^k\frac{v_k\,(\coth\alpha)}{\nu^k}\right\}$$

where

9.3.13

$v_0(t)=1$

$v_1(t)=(-9t+7t^3)/24$

$v_2(t)=(-135t^2+594t^4-455t^6)/1152$

$v_3(t)=(-42525t^3+4\ 51737t^5-8\ 83575t^7+4\ 75475t^9)/4\ 14720$

9.3.14

$$v_k(t)=u_k(t)+t(t^2-1)\{\tfrac{1}{2}u_{k-1}(t)+tu_{k-1}'(t)\}\qquad (k=1,\,2,\,\ldots)$$

(ii) If β is fixed, $0<\beta<\frac{1}{2}\pi$ and ν is large and positive

9.3.15

$$J_\nu(\nu\sec\beta)=\sqrt{2/(\pi\nu\tan\beta)}\{L(\nu,\beta)\cos\Psi+M(\nu,\beta)\sin\Psi\}$$

9.3.16

$$Y_\nu(\nu\sec\beta)=\sqrt{2/(\pi\nu\tan\beta)}\{L(\nu,\beta)\sin\Psi-M(\nu,\beta)\cos\Psi\}$$

where $\Psi=\nu(\tan\beta-\beta)-\frac{1}{4}\pi$

9.3.17

$$L(\nu,\beta)\sim\sum_{k=0}^{\infty}\frac{u_{2k}(i\cot\beta)}{\nu^{2k}}=1-\frac{81\cot^2\beta+462\cot^4\beta+385\cot^6\beta}{1152\nu^2}+\ldots$$

9.3.18

$$M(\nu,\beta)\sim -i\sum_{k=0}^{\infty}\frac{u_{2k+1}(i\cot\beta)}{\nu^{2k+1}}$$
$$=\frac{3\cot\beta+5\cot^3\beta}{24\nu}-\ldots$$

Also

9.3.19

$$J'_\nu(\nu\sec\beta)=\sqrt{(\sin 2\beta)/(\pi\nu)}\{-N(\nu,\beta)\sin\Psi-O(\nu,\beta)\cos\Psi\}$$

9.3.20

$$Y'_\nu(\nu\sec\beta)=\sqrt{(\sin 2\beta)/(\pi\nu)}\{N(\nu,\beta)\cos\Psi-O(\nu,\beta)\sin\Psi\}$$

where

9.3.21

$$N(\nu,\beta)\sim\sum_{k=0}^{\infty}\frac{v_{2k}(i\cot\beta)}{\nu^{2k}}$$
$$=1+\frac{135\cot^2\beta+594\cot^4\beta+455\cot^6\beta}{1152\nu^2}-\ldots$$

9.3.22

$$O(\nu,\beta)\sim i\sum_{k=0}^{\infty}\frac{v_{2k+1}(i\cot\beta)}{\nu^{2k+1}}=\frac{9\cot\beta+7\cot^3\beta}{24\nu}-\ldots$$

Asymptotic Expansions in the Transition Regions

When z is fixed, $|\nu|$ is large and $|\arg\nu|<\frac{1}{2}\pi$

9.3.23

$$J_\nu(\nu+z\nu^{1/3})\sim\frac{2^{1/3}}{\nu^{1/3}}\operatorname{Ai}(-2^{1/3}z)\{1+\sum_{k=1}^{\infty}\frac{f_k(z)}{\nu^{2k/3}}\}+\frac{2^{2/3}}{\nu}\operatorname{Ai}'(-2^{1/3}z)\sum_{k=0}^{\infty}\frac{g_k(z)}{\nu^{2k/3}}$$

9.3.24

$$Y_\nu(\nu+z\nu^{1/3})\sim-\frac{2^{1/3}}{\nu^{1/3}}\operatorname{Bi}(-2^{1/3}z)\{1+\sum_{k=1}^{\infty}\frac{f_k(z)}{\nu^{2k/3}}\}-\frac{2^{2/3}}{\nu}\operatorname{Bi}'(-2^{1/3}z)\sum_{k=0}^{\infty}\frac{g_k(z)}{\nu^{2k/3}}$$

where

9.3.25

$$f_1(z)=-\frac{1}{5}z$$

$$f_2(z)=-\frac{9}{100}z^5+\frac{3}{35}z^2$$

$$f_3(z)=\frac{957}{7000}z^6-\frac{173}{3150}z^3-\frac{1}{225}$$

$$f_4(z)=\frac{27}{20000}z^{10}-\frac{23573}{147000}z^7+\frac{5903}{138600}z^4+\frac{947}{346500}z$$

9.3.26

$$g_0(z)=\frac{3}{10}z^2$$

$$g_1(z)=-\frac{17}{70}z^3+\frac{1}{70}$$

$$g_2(z)=-\frac{9}{1000}z^7+\frac{611}{3150}z^4-\frac{37}{3150}z$$

$$g_3(z)=\frac{549}{28000}z^8-\frac{110767}{693000}z^5+\frac{79}{12375}z^2$$

The corresponding expansions for $H_\nu^{(1)}(\nu+z\nu^{1/3})$ and $H_\nu^{(2)}(\nu+z\nu^{1/3})$ are obtained by use of **9.1.3** and **9.1.4**; they are valid for $-\frac{1}{2}\pi<\arg\nu<\frac{3}{2}\pi$ and $-\frac{3}{2}\pi<\arg\nu<\frac{1}{2}\pi$, respectively.

9.3.27

$$J'_\nu(\nu+z\nu^{1/3})\sim-\frac{2^{2/3}}{\nu^{2/3}}\operatorname{Ai}'(-2^{1/3}z)\{1+\sum_{k=1}^{\infty}\frac{h_k(z)}{\nu^{2k/3}}\}+\frac{2^{1/3}}{\nu^{4/3}}\operatorname{Ai}(-2^{1/3}z)\sum_{k=0}^{\infty}\frac{l_k(z)}{\nu^{2k/3}}$$

9.3.28

$$Y'_\nu(\nu+z\nu^{1/3})\sim\frac{2^{2/3}}{\nu^{2/3}}\operatorname{Bi}'(-2^{1/3}z)\{1+\sum_{k=1}^{\infty}\frac{h_k(z)}{\nu^{2k/3}}\}-\frac{2^{1/3}}{\nu^{4/3}}\operatorname{Bi}(-2^{1/3}z)\sum_{k=0}^{\infty}\frac{l_k(z)}{\nu^{2k/3}}$$

where

9.3.29

$$h_1(z)=-\frac{4}{5}z$$

$$h_2(z)=-\frac{9}{100}z^5+\frac{57}{70}z^2$$

$$h_3(z)=\frac{699}{3500}z^6-\frac{2617}{3150}z^3+\frac{23}{3150}$$

$$h_4(z)=\frac{27}{20000}z^{10}-\frac{46631}{147000}z^7+\frac{3889}{4620}z^4-\frac{1159}{115500}z$$

9.3.30

$$l_0(z)=\frac{3}{5}z^3-\frac{1}{5}$$

$$l_1(z)=-\frac{131}{140}z^4+\frac{1}{5}z$$

$$l_2(z)=-\frac{9}{500}z^8+\frac{5437}{4500}z^5-\frac{593}{3150}z^2$$

$$l_3(z)=\frac{369}{7000}z^9-\frac{999443}{693000}z^6+\frac{31727}{173250}z^3+\frac{947}{346500}$$

9.3.31 $$J_\nu(\nu)\sim\frac{a}{\nu^{1/3}}\{1+\sum_{k=1}^{\infty}\frac{\alpha_k}{\nu^{2k}}\}-\frac{b}{\nu^{5/3}}\sum_{k=0}^{\infty}\frac{\beta_k}{\nu^{2k}}$$

9.3.32 $$Y_\nu(\nu)\sim-\frac{3^{1/2}a}{\nu^{1/3}}\{1+\sum_{k=1}^{\infty}\frac{\alpha_k}{\nu^{2k}}\}-\frac{3^{1/2}b}{\nu^{5/3}}\sum_{k=0}^{\infty}\frac{\beta_k}{\nu^{2k}}$$

9.3.33 $$J'_\nu(\nu)\sim\frac{b}{\nu^{2/3}}\{1+\sum_{k=1}^{\infty}\frac{\gamma_k}{\nu^{2k}}\}-\frac{a}{\nu^{4/3}}\sum_{k=0}^{\infty}\frac{\delta_k}{\nu^{2k}}$$

9.3.34 $$Y'_\nu(\nu)\sim\frac{3^{1/2}b}{\nu^{2/3}}\{1+\sum_{k=1}^{\infty}\frac{\gamma_k}{\nu^{2k}}\}+\frac{3^{1/2}a}{\nu^{4/3}}\sum_{k=0}^{\infty}\frac{\delta_k}{\nu^{2k}}$$

where

$$a=\frac{2^{1/3}}{3^{2/3}\Gamma(\frac{2}{3})}=.44730\ 73184,\qquad 3^{\frac{1}{2}}a=.77475\ 90021$$

$$b=\frac{2^{2/3}}{3^{1/3}\Gamma(\frac{1}{3})}=.41085\ 01939,\qquad 3^{\frac{1}{2}}b=.71161\ 34101$$

$$\alpha_0=1,\qquad \alpha_1=-\frac{1}{225}=-.00\dot{4},$$

$$\alpha_2=.00069\ 3735\ \ldots,\qquad \alpha_3=-.00035\ 38\ \ldots$$

$$\beta_0=\frac{1}{70}=.01428\ 57143\ \ldots,$$

$$\beta_1=-\frac{1213}{10\ 23750}=-.00118\ 48596\ldots,$$

$$\beta_2=.00043\ 78\ \ldots,\qquad \beta_3=-.00038\ \ldots$$

$$\gamma_0=1,\qquad \gamma_1=\frac{23}{3150}=.00730\ 15873\ \ldots,$$

$$\gamma_2=-.00093\ 7300\ \ldots,\qquad \gamma_3=.00044\ 40\ \ldots$$

$$\delta_0=\frac{1}{5},\quad \delta_1=-\frac{947}{3\ 46500}=-.00273\ 30447\ \ldots,$$

$$\delta_2=.00060\ 47\ \ldots,\qquad \delta_3=-.00038\ \ldots$$

Uniform Asymptotic Expansions

These are more powerful than the previous expansions of this section, save for **9.3.31** and **9.3.32**, but their coefficients are more complicated. They reduce to **9.3.31** and **9.3.32** when the argument equals the order.

9.3.35

$$J_\nu(\nu z)\sim\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\{\frac{\mathrm{Ai}\,(\nu^{2/3}\zeta)}{\nu^{1/3}}\sum_{k=0}^{\infty}\frac{a_k(\zeta)}{\nu^{2k}}+\frac{\mathrm{Ai}'(\nu^{2/3}\zeta)}{\nu^{5/3}}\sum_{k=0}^{\infty}\frac{b_k(\zeta)}{\nu^{2k}}\}$$

9.3.36

$$Y_\nu(\nu z)\sim-\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\{\frac{\mathrm{Bi}\,(\nu^{2/3}\zeta)}{\nu^{1/3}}\sum_{k=0}^{\infty}\frac{a_k(\zeta)}{\nu^{2k}}+\frac{\mathrm{Bi}'(\nu^{2/3}\zeta)}{\nu^{5/3}}\sum_{k=0}^{\infty}\frac{b_k(\zeta)}{\nu^{2k}}\}$$

9.3.37

$$H^{(1)}_\nu(\nu z)\sim 2e^{-\pi i/3}\left(\frac{4\zeta}{1-z^2}\right)^{1/4}\{\frac{\mathrm{Ai}\,(e^{2\pi i/3}\nu^{2/3}\zeta)}{\nu^{1/3}}\sum_{k=0}^{\infty}\frac{a_k(\zeta)}{\nu^{2k}}+\frac{e^{2\pi i/3}\mathrm{Ai}'(e^{2\pi i/3}\nu^{2/3}\zeta)}{\nu^{5/3}}\sum_{k=0}^{\infty}\frac{b_k(\zeta)}{\nu^{2k}}\}$$

When $\nu\rightarrow+\infty$, these expansions hold uniformly with respect to z in the sector $|\arg z|\leq\pi-\epsilon$, where ϵ is an arbitrary positive number. The corresponding expansion for $H^{(2)}_\nu(\nu z)$ is obtained by changing the sign of i in **9.3.37.**

Here

9.3.38

$$\frac{2}{3}\zeta^{3/2}=\int_z^1\frac{\sqrt{1-t^2}}{t}\,dt=\ln\frac{1+\sqrt{1-z^2}}{z}-\sqrt{1-z^2}$$

equivalently,

9.3.39

$$\frac{2}{3}(-\zeta)^{3/2}=\int_1^z\frac{\sqrt{t^2-1}}{t}\,dt=\sqrt{z^2-1}-\mathrm{arccos}\left(\frac{1}{z}\right)$$

the branches being chosen so that ζ is real when z is positive. The coefficients are given by

9.3.40

$$a_k(\zeta)=\sum_{s=0}^{2k}\mu_s\zeta^{-3s/2}u_{2k-s}\{(1-z^2)^{-\frac{1}{2}}\}$$

$$b_k(\zeta)=-\zeta^{-\frac{1}{2}}\sum_{s=0}^{2k+1}\lambda_s\zeta^{-3s/2}u_{2k-s+1}\{(1-z^2)^{-\frac{1}{2}}\}$$

where u_k is given by **9.3.9** and **9.3.10,** $\lambda_0=\mu_0=1$ and

9.3.41

$$\lambda_s=\frac{(2s+1)\,(2s+3)\,\ldots\,(6s-1)}{s!\,(144)^s},\quad \mu_s=-\frac{6s+1}{6s-1}\lambda_s$$

Thus $a_0(\zeta)=1$,

9.3.42

$$b_0(\zeta)=-\frac{5}{48\zeta^2}+\frac{1}{\zeta^{\frac{1}{2}}}\{\frac{5}{24(1-z^2)^{3/2}}-\frac{1}{8(1-z^2)^{\frac{1}{2}}}\}$$

$$=-\frac{5}{48\zeta^2}+\frac{1}{(-\zeta)^{\frac{1}{2}}}\{\frac{5}{24(z^2-1)^{3/2}}+\frac{1}{8(z^2-1)^{\frac{1}{2}}}\}$$

Tables of the early coefficients are given below. For more extensive tables of the coefficients and for bounds on the remainder terms in **9.3.35** and **9.3.36** see [9.38].

Uniform Expansions of the Derivatives

With the conditions of the preceding subsection

9.3.43

$$J_\nu'(\nu z)\sim-\frac{2}{z}\left(\frac{1-z^2}{4\zeta}\right)^{\frac{1}{4}}\left\{\frac{\mathrm{Ai}\,(\nu^{2/3}\zeta)}{\nu^{4/3}}\sum_{k=0}^{\infty}\frac{c_k(\zeta)}{\nu^{2k}}+\frac{\mathrm{Ai}'\,(\nu^{2/3}\zeta)}{\nu^{2/3}}\sum_{k=0}^{\infty}\frac{d_k(\zeta)}{\nu^{2k}}\right\}$$

9.3.44

$$Y_\nu'(\nu z)\sim\frac{2}{z}\left(\frac{1-z^2}{4\zeta}\right)^{\frac{1}{4}}\left\{\frac{\mathrm{Bi}\,(\nu^{2/3}\zeta)}{\nu^{4/3}}\sum_{k=0}^{\infty}\frac{c_k(\zeta)}{\nu^{2k}}+\frac{\mathrm{Bi}'\,(\nu^{2/3}\zeta)}{\nu^{2/3}}\sum_{k=0}^{\infty}\frac{d_k(\zeta)}{\nu^{2k}}\right\}$$

9.3.45

$$H_\nu^{(1)\prime}(\nu z)\sim\frac{4e^{2\pi i/3}}{z}\left(\frac{1-z^2}{4\zeta}\right)^{\frac{1}{4}}\left\{\frac{\mathrm{Ai}\,(e^{2\pi i/3}\nu^{2/3}\zeta)}{\nu^{4/3}}\sum_{k=0}^{\infty}\frac{c_k(\zeta)}{\nu^{2k}}+\frac{e^{2\pi i/3}\,\mathrm{Ai}'\,(e^{2\pi i/3}\nu^{2/3}\zeta)}{\nu^{2/3}}\sum_{k=0}^{\infty}\frac{d_k(\zeta)}{\nu^{2k}}\right\}$$

where

9.3.46

$$c_k(\zeta)=-\zeta^{\frac{1}{2}}\sum_{s=0}^{2k+1}\mu_s\zeta^{-3s/2}v_{2k-s+1}\{(1-z^2)^{-\frac{1}{2}}\}$$

$$d_k(\zeta)=\sum_{s=0}^{2k}\lambda_s\zeta^{-3s/2}v_{2k-s}\{(1-z^2)^{-\frac{1}{2}}\}$$

and v_k is given by **9.3.13** and **9.3.14.** For bounds on the remainder terms in **9.3.43** and **9.3.44** see [9.38].

ζ	$b_0(\zeta)$	$a_1(\zeta)$	$c_0(\zeta)$	$d_1(\zeta)$
0	0. 0180	−0. 004	0. 1587	0. 007
1	. 0278	−. 004	. 1785	. 009
2	. 0351	−. 001	. 1862	. 007
3	. 0366	+. 002	. 1927	. 005
4	. 0352	. 003	. 2031	. 004
5	. 0331	. 004	. 2155	. 003
6	. 0311	. 004	. 2284	. 003
7	. 0294	. 004	. 2413	. 003
8	. 0278	. 004	. 2539	. 003
9	. 0265	. 004	. 2662	. 003
10	. 0253	. 004	. 2781	. 003

$-\zeta$	$b_0(\zeta)$	$a_1(\zeta)$	$c_0(\zeta)$	$d_1(\zeta)$
0	0. 0180	−0. 004	0. 1587	0. 007
1	. 0109	−. 003	. 1323	. 004
2	. 0067	−. 002	. 1087	. 002
3	. 0044	−. 001	. 0903	. 001
4	. 0031	−. 001	. 0764	. 001
5	. 0022	−. 000	. 0658	. 000
6	. 0017	−. 000	. 0576	. 000
7	. 0013	−. 000	. 0511	. 000
8	. 0011	−. 000	. 0459	. 000
9	. 0009	−. 000	. 0415	. 000
10	. 0007	−. 000	. 0379	. 000

For $\zeta>10$ use

$$b_0(\zeta)\sim\frac{1}{12}\zeta^{-\frac{1}{2}}-.104\zeta^{-2},\qquad a_1(\zeta)=.003,$$

$$c_0(\zeta)\sim\frac{1}{12}\zeta^{\frac{1}{2}}+.146\zeta^{-1},\qquad d_1(\zeta)=.003.$$

For $\zeta<-10$ use

$$b_0(\zeta)\sim\frac{1}{12}\zeta^{-2},\qquad a_1(\zeta)=.000,$$

$$c_0(\zeta)\sim-\frac{5}{12}\zeta^{-1}-1.33(-\zeta)^{-5/2},\qquad d_1(\zeta)=.000.$$

Maximum values of higher coefficients:

$$|b_1(\zeta)|=.003,\qquad |a_2(\zeta)|=.0008,\qquad |d_2(\zeta)|=.001$$

$$|c_1(\zeta)|=.008\ (\zeta<10),\quad c_1(\zeta)\sim-.003\zeta^{\frac{1}{2}}\text{ as }\zeta\to+\infty.$$

9.4. Polynomial Approximations [2]

9.4.1 $\qquad -3\leq x\leq 3$

$$J_0(x)=1-2.24999\,97(x/3)^2+1.26562\,08(x/3)^4-.31638\,66(x/3)^6+.04444\,79(x/3)^8-.00394\,44(x/3)^{10}+.00021\,00(x/3)^{12}+\epsilon$$

$$|\epsilon|<5\times10^{-8}$$

9.4.2 $\qquad 0<x\leq 3$

$$Y_0(x)=(2/\pi)\ln(\tfrac{1}{2}x)J_0(x)+.36746\,691+.60559\,366(x/3)^2-.74350\,384(x/3)^4+.25300\,117(x/3)^6-.04261\,214(x/3)^8+.00427\,916(x/3)^{10}-.00024\,846(x/3)^{12}+\epsilon$$

$$|\epsilon|<1.4\times10^{-8}$$

9.4.3 $\qquad 3\leq x<\infty$

$$J_0(x)=x^{-\frac{1}{2}}f_0\cos\theta_0\qquad Y_0(x)=x^{-\frac{1}{2}}f_0\sin\theta_0$$

$$f_0=.79788\,456-.00000\,077(3/x)-.00552\,740(3/x)^2-.00009\,512(3/x)^3+.00137\,237(3/x)^4-.00072\,805(3/x)^5+.00014\,476(3/x)^6+\epsilon$$

$$|\epsilon|<1.6\times10^{-8}$$

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[2] Equations **9.4.1** to **9.4.6** and **9.8.1** to **9.8.8** are taken from E. E. Allen, Analytical approximations, Math. Tables Aids Comp. **8**, 240–241 (1954), and Polynomial approximations to some modified Bessel functions, Math. Tables Aids Comp. **10**, 162–164 (1956) (with permission). They were checked at the National Physical Laboratory by systematic tabulation; new bounds for the errors, ϵ, given here were obtained as a result.

$$\theta_0 = x - .78539\,816 - .04166\,397(3/x) - .00003\,954(3/x)^2 + .00262\,573(3/x)^3 - .00054\,125(3/x)^4 - .00029\,333(3/x)^5 + .00013\,558(3/x)^6 + \epsilon$$

$$|\epsilon| < 7 \times 10^{-8}$$

9.4.4 $$-3 \leq x \leq 3$$

$$x^{-1}\,J_1(x) = \tfrac{1}{2} - .56249\,985(x/3)^2 + .21093\,573(x/3)^4 - .03954\,289(x/3)^6 + .00443\,319(x/3)^8 - .00031\,761(x/3)^{10} + .00001\,109(x/3)^{12} + \epsilon$$

$$|\epsilon| < 1.3 \times 10^{-8}$$

9.4.5 $$0 < x \leq 3$$

$$xY_1(x) = (2/\pi)x \ln(\tfrac{1}{2}x)J_1(x) - .63661\,98 + .22120\,91(x/3)^2 + 2.16827\,09(x/3)^4 - 1.31648\,27(x/3)^6 + .31239\,51(x/3)^8 - .04009\,76(x/3)^{10} + .00278\,73(x/3)^{12} + \epsilon$$

$$|\epsilon| < 1.1 \times 10^{-7}$$

9.4.6 $$3 \leq x < \infty$$

$$J_1(x) = x^{-\frac{1}{2}} f_1 \cos \theta_1, \qquad Y_1(x) = x^{-\frac{1}{2}} f_1 \sin \theta_1$$

$$f_1 = .79788\,456 + .00000\,156(3/x) + .01659\,667(3/x)^2 + .00017\,105(3/x)^3 - .00249\,511(3/x)^4 + .00113\,653(3/x)^5 - .00020\,033(3/x)^6 + \epsilon$$

$$|\epsilon| < 4 \times 10^{-8}$$

$$\theta_1 = x - 2.35619\,449 + .12499\,612(3/x) + .00005\,650(3/x)^2 - .00637\,879(3/x)^3 + .00074\,348(3/x)^4 + .00079\,824(3/x)^5 - .00029\,166(3/x)^6 + \epsilon$$

$$|\epsilon| < 9 \times 10^{-8}$$

For expansions of $J_0(x)$, $Y_0(x)$, $J_1(x)$, and $Y_1(x)$ in series of Chebyshev polynomials for the ranges $0 \leq x \leq 8$ and $0 \leq 8/x \leq 1$, see [9.37].

9.5. Zeros

Real Zeros

When ν is real, the functions $J_\nu(z)$, $J'_\nu(z)$, $Y_\nu(z)$ and $Y'_\nu(z)$ each have an infinite number of real zeros, all of which are simple with the possible exception of $z=0$. For *non-negative* ν the sth positive zeros of these functions are denoted by $j_{\nu,s}$, $j'_{\nu,s}$, $y_{\nu,s}$ and $y'_{\nu,s}$ respectively, except that $z=0$ is counted as the first zero of $J'_0(z)$. Since $J'_0(z) = -J_1(z)$, it follows that

9.5.1 $$j'_{0,1} = 0, \qquad j'_{0,s} = j_{1,s-1} \qquad (s=2,\ 3,\ \ldots)$$

The zeros interlace according to the inequalities

9.5.2

$$j_{\nu,1} < j_{\nu+1,1} < j_{\nu,2} < j_{\nu+1,2} < j_{\nu,3} < \ldots$$

$$y_{\nu,1} < y_{\nu+1,1} < y_{\nu,2} < y_{\nu+1,2} < y_{\nu,3} < \ldots$$

$$\nu \leq j'_{\nu,1} < y_{\nu,1} < y'_{\nu,1} < j_{\nu,1} < j'_{\nu,2} < y_{\nu,2} < y'_{\nu,2} < j_{\nu,2} < j'_{\nu,3} < \ldots$$

The positive zeros of any two real distinct cylinder functions of the same order are interlaced, as are the positive zeros of any real cylinder function $\mathscr{C}_\nu(z)$, defined as in **9.1.27,** and the contiguous function $\mathscr{C}_{\nu+1}(z)$.

If ρ_ν is a zero of the cylinder function

9.5.3 $$\mathscr{C}_\nu(z) = J_\nu(z) \cos(\pi t) + Y_\nu(z) \sin(\pi t)$$

where t is a parameter, then

9.5.4 $$\mathscr{C}'_\nu(\rho_\nu) = \mathscr{C}_{\nu-1}(\rho_\nu) = -\mathscr{C}_{\nu+1}(\rho_\nu)$$

If σ_ν is a zero of $\mathscr{C}'_\nu(z)$ then

9.5.5 $$\mathscr{C}_\nu(\sigma_\nu) = \frac{\sigma_\nu}{\nu} \mathscr{C}_{\nu-1}(\sigma_\nu) = \frac{\sigma_\nu}{\nu} \mathscr{C}_{\nu+1}(\sigma_\nu)$$

The parameter t may be regarded as a continuous variable and ρ_ν, σ_ν as functions $\rho_\nu(t)$, $\sigma_\nu(t)$ of t. If these functions are fixed by

9.5.6 $$\rho_\nu(0) = 0, \qquad \sigma_\nu(0) = j'_{\nu,1}$$

then

9.5.7

$$j_{\nu,s} = \rho_\nu(s), \qquad y_{\nu,s} = \rho_\nu(s - \tfrac{1}{2}) \qquad (s=1, 2, \ldots)$$

9.5.8

$$j'_{\nu,s} = \sigma_\nu(s-1), \qquad y'_{\nu,s} = \sigma_\nu(s - \tfrac{1}{2}) \qquad (s=1, 2, \ldots)$$

9.5.9 $$\mathscr{C}'_\nu(\rho_\nu) = \left(\frac{\rho_\nu}{2}\frac{d\rho_\nu}{dt}\right)^{-\frac{1}{2}}, \mathscr{C}_\nu(\sigma_\nu) = \left(\frac{\sigma_\nu^2 - \nu^2}{2\sigma_\nu}\frac{d\sigma_\nu}{dt}\right)^{-\frac{1}{2}}$$

Infinite Products

9.5.10 $$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)} \prod_{s=1}^{\infty} \left(1 - \frac{z^2}{j_{\nu,s}^2}\right)$$

9.5.11 $$J'_\nu(z) = \frac{(\frac{1}{2}z)^{\nu-1}}{2\Gamma(\nu)} \prod_{s=1}^{\infty} \left(1 - \frac{z^2}{j'^2_{\nu,s}}\right) \qquad (\nu > 0)$$

McMahon's Expansions for Large Zeros

When ν is fixed, $s>>\nu$ and $\mu=4\nu^2$

9.5.12

$$j_{\nu,s},y_{\nu,s}\sim\beta-\frac{\mu-1}{8\beta}-\frac{4(\mu-1)(7\mu-31)}{3(8\beta)^3}-\frac{32(\mu-1)(83\mu^2-982\mu+3779)}{15(8\beta)^5}-\frac{64(\mu-1)(6949\mu^3-1\ 53855\mu^2+15\ 85743\mu-62\ 77237)}{105(8\beta)^7}-\ldots$$

where $\beta=(s+\frac{1}{2}\nu-\frac{1}{4})\pi$ for $j_{\nu,s}$, $\beta=(s+\frac{1}{2}\nu-\frac{3}{4})\pi$ for $y_{\nu,s}$. With $\beta=(t+\frac{1}{2}\nu-\frac{1}{4})\pi$, the right of **9.5.12** is the asymptotic expansion of $\rho_\nu(t)$ for large t.

9.5.13

$$j'_{\nu,s},y'_{\nu,s}\sim\beta'-\frac{\mu+3}{8\beta'}-\frac{4(7\mu^2+82\mu-9)}{3(8\beta')^3}-\frac{32(83\mu^3+2075\mu^2-3039\mu+3537)}{15(8\beta')^5}-\frac{64(6949\mu^4+2\ 96492\mu^3-12\ 48002\mu^2+74\ 14380\mu-58\ 53627)}{105(8\beta')^7}-\ldots$$

where $\beta'=(s+\frac{1}{2}\nu-\frac{3}{4})\pi$ for $j'_{\nu,s}$, $\beta'=(s+\frac{1}{2}\nu-\frac{1}{4})\pi$ for $y'_{\nu,s}$, $\beta'=(t+\frac{1}{2}\nu+\frac{1}{4})\pi$ for $\sigma_\nu(t)$. For higher terms in **9.5.12** and **9.5.13** see [9.4] or [9.40].

Asymptotic Expansions of Zeros and Associated Values for Large Orders

9.5.14

$$j_{\nu,1}\sim\nu+1.85575\ 71\nu^{1/3}+1.03315\ 0\nu^{-1/3}-.00397\nu^{-1}-.0908\nu^{-5/3}+.043\nu^{-7/3}+\ldots$$

9.5.15

$$y_{\nu,1}\sim\nu+.93157\ 68\nu^{1/3}+.26035\ 1\nu^{-1/3}+.01198\nu^{-1}-.0060\nu^{-5/3}-.001\nu^{-7/3}+\ldots$$

9.5.16

$$j'_{\nu,1}\sim\nu+.80861\ 65\nu^{1/3}+.07249\ 0\nu^{-1/3}-.05097\nu^{-1}+.0094\nu^{-5/3}+\ldots$$

9.5.17

$$y'_{\nu,1}\sim\nu+1.82109\ 80\nu^{1/3}+.94000\ 7\nu^{-1/3}-.05808\nu^{-1}-.0540\nu^{-5/3}+\ldots$$

9.5.18

$$J'_\nu(j_{\nu,1})\sim-1.11310\ 28\nu^{-2/3}/(1+1.48460\ 6\nu^{-2/3}+.43294\nu^{-4/3}-.1943\nu^{-2}+.019\nu^{-8/3}+\ldots)$$

9.5.19

$$Y'_\nu(y_{\nu,1})\sim.95554\ 86\nu^{-2/3}/(1+.74526\ 1\nu^{-2/3}+.10910\nu^{-4/3}-.0185\nu^{-2}-.003\nu^{-8/3}+\ldots)$$

9.5.20

$$J_\nu(j'_{\nu,1})\sim.67488\ 51\nu^{-1/3}(1-.16172\ 3\nu^{-2/3}+.02918\nu^{-4/3}-.0068\nu^{-2}+\ldots)$$

9.5.21

$$Y_\nu(y'_{\nu,1})\sim.57319\ 40\nu^{-1/3}(1-.36422\ 0\nu^{-2/3}+.09077\nu^{-4/3}+.0237\nu^{-2}+\ldots)$$

Corresponding expansions for $s=2$, 3 are given in [9.40]. These expansions become progressively weaker as s increases; those which follow do not suffer from this defect.

Uniform Asymptotic Expansions of Zeros and Associated Values for Large Orders

9.5.22 $j_{\nu,s}\sim\nu z(\zeta)+\sum_{k=1}^{\infty}\frac{f_k(\zeta)}{\nu^{2k-1}}$ with $\zeta=\nu^{-2/3}a_s$

9.5.23

$$J'_\nu(j_{\nu,s})\sim-\frac{2}{\nu^{2/3}}\frac{\mathrm{Ai}'(a_s)}{z(\zeta)h(\zeta)}\{1+\sum_{k=1}^{\infty}\frac{F_k(\zeta)}{\nu^{2k}}\}\quad\text{with } \zeta=\nu^{-2/3}a_s$$

9.5.24 $j'_{\nu,s}\sim\nu z(\zeta)+\sum_{k=1}^{\infty}\frac{g_k(\zeta)}{\nu^{2k-1}}$ with $\zeta=\nu^{-2/3}a'_s$

9.5.25

$$J_\nu(j'_{\nu,s})\sim\mathrm{Ai}(a'_s)\frac{h(\zeta)}{\nu^{1/3}}\{1+\sum_{k=1}^{\infty}\frac{G_k(\zeta)}{\nu^{2k}}\}\text{ with } \zeta=\nu^{-2/3}a'_s$$

where a_s, a'_s are the sth negative zeros of $\mathrm{Ai}(z)$, $\mathrm{Ai}'(z)$ (see **10.4**), $z=z(\zeta)$ is the inverse function defined implicitly by **9.3.39**, and

9.5.26

$$h(\zeta)=\{4\zeta/(1-z^2)\}^{1/4}$$

$$f_1(\zeta)=\tfrac{1}{2}z(\zeta)\{h(\zeta)\}^2b_0(\zeta)$$

$$g_1(\zeta)=\tfrac{1}{2}\zeta^{-1}z(\zeta)\{h(\zeta)\}^2c_0(\zeta)$$

where $b_0(\zeta)$, $c_0(\zeta)$ appear in **9.3.42** and **9.3.46.** Tables of the leading coefficients follow. More extensive tables are given in [9.40].

The expansions of $y_{\nu,s}$, $Y'_\nu(y_{\nu,s})$, $y'_{\nu,s}$ and $Y_\nu(y'_{\nu,s})$ corresponding to **9.5.22** to **9.5.25** are obtained by changing the symbols j, J, Ai, Ai$'$, a_s and a'_s to y, Y, $-$Bi, $-$Bi$'$, b_s and b'_s respectively.

$-\zeta$	$z(\zeta)$	$h(\zeta)$	$f_1(\zeta)$	$F_1(\zeta)$	$(-\zeta)g_1(\zeta)$	$(-\zeta)^3g_2(\zeta)$	$(-\zeta)^2G_1(\zeta)$
0.0	1.000000	1.25992	0.0143	−0.007	−0.1260	−0.010	0.000
0.2	1.166284	1.22076	.0142	−.005	−.1335	−.010	.002
0.4	1.347557	1.18337	.0139	−.004	−.1399	−.009	.004
0.6	1.543615	1.14780	.0135	−.003	−.1453	−.009	.005
0.8	1.754187	1.11409	.0131	−.003	−.1498	−.008	.006
1.0	1.978963	1.08220	0.0126	−0.002	−0.1533	−0.008	0.006

$-\zeta$	$z(\zeta)$	$h(\zeta)$	$f_1(\zeta)$	$F_1(\zeta)$	$g_1(\zeta)$	$g_2(\zeta)$	$G_1(\zeta)$
1.0	1.978963	1.08220	0.0126	−0.002	−0.1533	−0.008	0.006
1.2	2.217607	1.05208	.0121	−.002	−.1301	−.004	.004
1.4	2.469770	1.02367	.0115	−.001	−.1130	−.002	.003
1.6	2.735103	0.99687	.0110	−.001	−.0998	−.001	.002
1.8	3.013256	.97159	.0105	−.001	−.0893	−.001	.002
2.0	3.303889	0.94775	0.0100	−0.001	−0.0807	−0.001	0.001
2.2	3.606673	.92524	.0095	−0.001	−.0734		.001
2.4	3.921292	.90397	.0091		−.0673		.001
2.6	4.247441	.88387	.0086		−.0619		.001
2.8	4.584833	.86484	.0082		−.0573		0.001
3.0	4.933192	0.84681	0.0078		−0.0533		
3.2	5.292257	.82972	.0075		−.0497		
3.4	5.661780	.81348	.0071		−.0464		
3.6	6.041525	.79806	.0068		−.0436		
3.8	6.431269	.78338	.0065		−.0410		
4.0	6.830800	0.76939	0.0062		−0.0386		
4.2	7.239917	.75605	.0060		−.0365		
4.4	7.658427	.74332	.0057		−.0345		
4.6	8.086150	.73115	.0055		−.0328		
4.8	8.522912	.71951	.0052		−.0311		
5.0	8.968548	0.70836	0.0050		−0.0296		
5.2	9.422900	.69768	.0048		−.0282		
5.4	9.885820	.68742	.0047		−.0270		
5.6	10.357162	.67758	.0045		−.0258		
5.8	10.836791	.66811	.0043		−.0246		
6.0	11.324575	0.65901	0.0042		−0.0236		
6.2	11.820388	.65024	.0040		−.0227		
6.4	12.324111	.64180	.0039		−.0218		
6.6	12.835627	.63366	.0037		−.0209		
6.8	13.354826	.62580	.0036		−.0201		
7.0	13.881601	0.61821	0.0035		−0.0194		

$(-\zeta)^{-\frac{3}{2}}$	$z(\zeta)-\frac{2}{3}(-\zeta)^{\frac{3}{2}}$	$(-\zeta)^{\frac{1}{2}}h(\zeta)$	$f_1(\zeta)$	$g_1(\zeta)$
0.40	1.528915	1.62026	0.0040	−0.0224
.35	1.541532	1.65351	.0029	−.0158
.30	1.551741	1.68067	.0020	−.0104
.25	1.559490	1.70146	.0012	−.0062
.20	1.564907	1.71607	.0006	−.0033
0.15	1.568285	1.72523	0.0003	−0.0014
.10	1.570048	1.73002	.0001	−.0004
.05	1.570703	1.73180	.0000	−.0001
.00	1.570796	1.73205	.0000	−.0000

Maximum Values of Higher Coefficients

$|f_2(\zeta)|=.001,\ |F_2(\zeta)|=.0004 \qquad (0\leq-\zeta<\infty)$

$|g_3(\zeta)|=.001,\ |G_2(\zeta)|=.0007 \qquad (1\leq-\zeta<\infty)$

$|(-\zeta)^5g_3(\zeta)|=.002,\ |(-\zeta)^4G_2(\zeta)|=.0007$

$(0\leq-\zeta\leq1)$

Complex Zeros of $J_\nu(z)$

When $\nu\geq-1$ the zeros of $J_\nu(z)$ are all real. If $\nu<-1$ and ν is not an integer the number of complex zeros of $J_\nu(z)$ is twice the integer part of $(-\nu)$; if the integer part of $(-\nu)$ is odd two of these zeros lie on the imaginary axis.

If $\nu\geq0$, all zeros of $J_\nu'(z)$ are real.

Complex Zeros of $Y_\nu(z)$

When ν is real the pattern of the complex zeros of $Y_\nu(z)$ and $Y_\nu'(z)$ depends on the non-integer part of ν. Attention is confined here to the case $\nu=n$, a positive integer or zero.

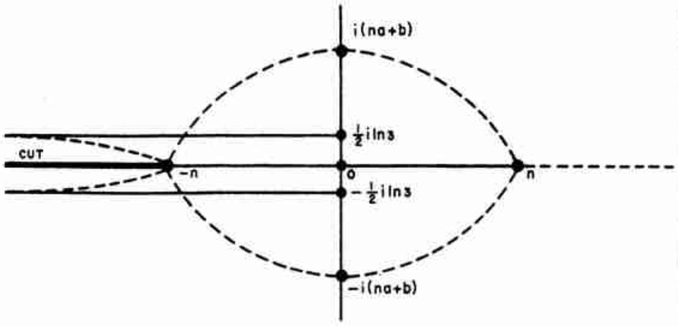

FIGURE 9.5. *Zeros of $Y_n(z)$ and $Y_n'(z)$ . . . $|\arg z|\leq\pi$.*

Figure 9.5 shows the approximate distribution of the complex zeros of $Y_n(z)$ in the region $|\arg z|\leq\pi$. The figure is symmetrical about the real axis. The two curves on the left extend to infinity, having the asymptotes

$$\mathscr{I}z=\pm\tfrac{1}{2}\ln 3=\pm.54931\ldots$$

There are an infinite number of zeros near each of these curves.

The two curves extending from $z=-n$ to $z=n$ and bounding an eye-shaped domain intersect the imaginary axis at the points $\pm i(na+b)$, where

$$a=\sqrt{t_0^2-1}=.66274\ldots$$

$$b=\tfrac{1}{2}\sqrt{1-t_0^{-2}}\ln 2=.19146\ldots$$

and $t_0=1.19968\ldots$ is the positive root of $\coth t=t$. There are n zeros near each of these curves. Asymptotic expansions of these zeros for large n are given by the right of **9.5.22** with $\nu=n$ and $\zeta=n^{-2/3}\beta_s$ or $n^{-2/3}\overline{\beta}_s$, where β_s, $\overline{\beta}_s$ are the complex zeros of $\mathrm{Bi}(z)$ (see **10.4**).

Figure 9.5 is also applicable to the zeros of $Y_n'(z)$. There are again an infinite number near the infinite curves, and n near each of the finite curves. Asymptotic expansions of the latter for large n are given by the right of **9.5.24** with $\nu=n$ and $\zeta=n^{-2/3}\beta_s'$ or $n^{-2/3}\overline{\beta}_s'$; where β_s' and $\overline{\beta}_s'$ are the complex zeros of $\mathrm{Bi}'(z)$.

Numerical values of the three smallest complex zeros of $Y_0(z)$, $Y_1(z)$ and $Y_1'(z)$ in the region $0<\arg z<\pi$ are given below.

For further details see [9.36] and [9.13]. The latter reference includes tables to facilitate computation.

Complex Zeros of the Hankel Functions

The approximate distribution of the zeros of $H_n^{(1)}(z)$ and its derivative in the region $|\arg z|\leq\pi$ is indicated in a similar manner on **Figure 9.6.**

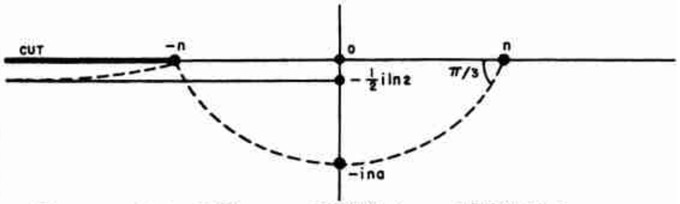

FIGURE 9.6. *Zeros of $H_n^{(1)}(z)$ and $H_n^{(1)\prime}(z)$ . . . $|\arg z|\leq\pi$.*

The asymptote of the solitary infinite curve is given by

$$\mathscr{I}z=-\tfrac{1}{2}\ln 2=-.34657\ldots$$

Zeros of $Y_0(z)$ and Values of $Y_1(z)$ at the Zeros[3]

Zero		Y_1	
Real	*Imag.*	*Real*	*Imag.*
−2.40301 6632	+.53988 2313	+.10074 7689	−.88196 7710
−5.51987 6702	+.54718 0011	−.02924 6418	+.58716 9503
−8.65367 2403	+.54841 2067	+.01490 8063	−.46945 8752

Zeros of $Y_1(z)$ and Values of $Y_0(z)$ at the Zeros

Zero		Y_0	
Real	*Imag.*	*Real*	*Imag.*
−0.50274 3273	+.78624 3714	−.45952 7684	+1.31710 1937
−3.83353 5193	+.56235 6538	+.04830 1909	−0.69251 2884
−7.01590 3683	+.55339 3046	−.02012 6949	+0.51864 2833

Zeros of $Y_1'(z)$ and Values of $Y_1(z)$ at the Zeros

Zero		Y_1	
Real	*Imag.*	*Real*	*Imag.*
+0.57678 5129	+.90398 4792	−.76349 7088	+.58924 4865
−1.94047 7342	+.72118 5919	+.16206 4006	−.95202 7886
−5.33347 8617	+.56721 9637	−.03179 4008	+.59685 3673

---

[3] From National Bureau of Standards, Tables of the Bessel functions $Y_0(z)$ and $Y_1(z)$ for complex arguments, Columbia Univ. Press, New York, N.Y., 1950 (with permission).

There are n zeros of each function near the finite curve extending from $z=-n$ to $z=n$; the asymptotic expansions of these zeros for large n are given by the right side of **9.5.22** or **9.5.24** with $\nu=n$ and $\zeta=e^{-2\pi i/3}n^{-2/3}a_s$ or $\zeta=e^{-2\pi i/3}n^{-2/3}a'_s$.

Zeros of Cross-Products

If ν is real and λ is positive, the zeros of the function

9.5.27 $$J_\nu(z)Y_\nu(\lambda z)-J_\nu(\lambda z)Y_\nu(z)$$

are real and simple. If $\lambda>1$, the asymptotic expansion of the sth zero is

9.5.28 $$\beta+\frac{p}{\beta}+\frac{q-p^2}{\beta^3}+\frac{r-4pq+2p^3}{\beta^5}+\ldots$$

where with $4\nu^2$ denoted by μ,

9.5.29
$$\beta=s\pi/(\lambda-1)$$
$$p=\frac{\mu-1}{8\lambda},\qquad q=\frac{(\mu-1)(\mu-25)(\lambda^3-1)}{6(4\lambda)^3(\lambda-1)}$$
$$r=\frac{(\mu-1)(\mu^2-114\mu+1073)(\lambda^5-1)}{5(4\lambda)^5(\lambda-1)}$$

The asymptotic expansion of the large positive zeros (not necessarily the sth) of the function

9.5.30 $$J'_\nu(z)Y'_\nu(\lambda z)-J'_\nu(\lambda z)Y'_\nu(z)\qquad(\lambda>1)$$

is given by **9.5.28** with the same value of β, but instead of **9.5.29** we have

9.5.31
$$p=\frac{\mu+3}{8\lambda},\qquad q=\frac{(\mu^2+46\mu-63)(\lambda^3-1)}{6(4\lambda)^3(\lambda-1)}$$
$$r=\frac{(\mu^3+185\mu^2-2053\mu+1899)(\lambda^5-1)}{5(4\lambda)^5(\lambda-1)}$$

The asymptotic expansion of the large positive zeros of the function

9.5.32 $$J'_\nu(z)Y_\nu(\lambda z)-Y'_\nu(z)J_\nu(\lambda z)$$

is given by **9.5.28** with

9.5.33
$$\beta=(s-\tfrac{1}{2})\pi/(\lambda-1)$$
$$p=\frac{(\mu+3)\lambda-(\mu-1)}{8\lambda(\lambda-1)}$$
$$q=\frac{(\mu^2+46\mu-63)\lambda^3-(\mu-1)(\mu-25)}{6(4\lambda)^3(\lambda-1)}$$
$$5(4\lambda)^5(\lambda-1)r=(\mu^3+185\mu^2-2053\mu+1899)\lambda^5-(\mu-1)(\mu^2-114\mu+1073)$$

Modified Bessel Functions I and K

9.6. Definitions and Properties

Differential Equation

9.6.1 $$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}-(z^2+\nu^2)w=0$$

Solutions are $I_{\pm\nu}(z)$ and $K_\nu(z)$. Each is a regular function of z throughout the z-plane cut along the negative real axis, and for fixed $z(\neq 0)$ each is an entire function of ν. When $\nu=\pm n$, $I_\nu(z)$ is an entire function of z.

$I_\nu(z)$ $(\mathscr{R}\nu\geq 0)$ is bounded as $z\rightarrow 0$ in any bounded range of arg z. $I_\nu(z)$ and $I_{-\nu}(z)$ are linearly independent except when ν is an integer. $K_\nu(z)$ tends to zero as $|z|\rightarrow\infty$ in the sector $|\arg z|<\frac{1}{2}\pi$, and for all values of ν, $I_\nu(z)$ and $K_\nu(z)$ are linearly independent. $I_\nu(z)$, $K_\nu(z)$ are real and positive when $\nu>-1$ and $z>0$.

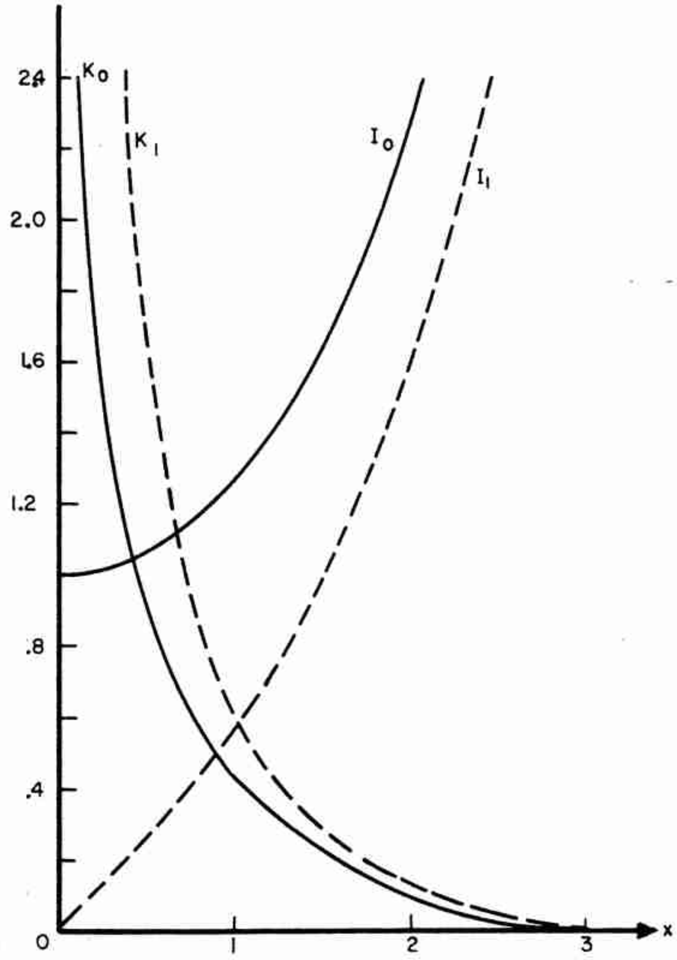

FIGURE 9.7. $I_0(x)$, $K_0(x)$, $I_1(x)$ *and* $K_1(x)$.

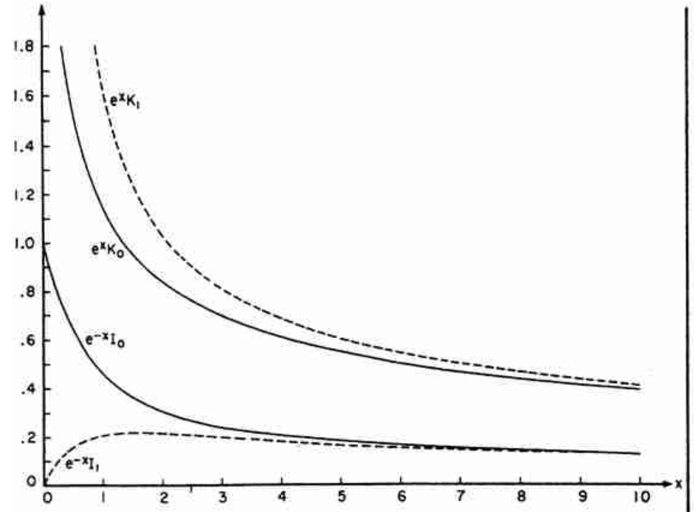

FIGURE 9.8. $e^{-x}I_0(x), e^{-x}I_1(x), e^xK_0(x)$ *and* $e^xK_1(x)$.

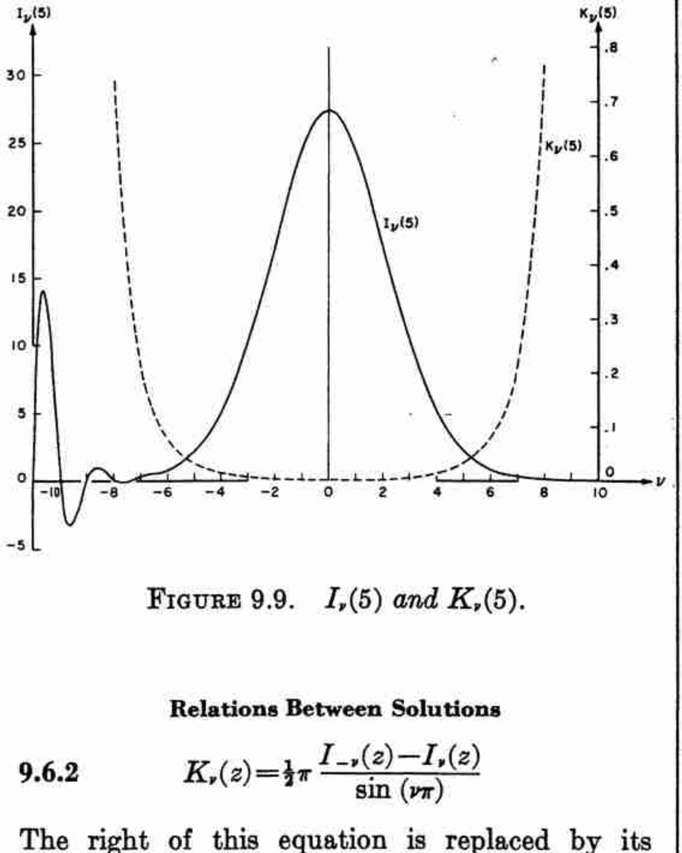

FIGURE 9.9. $I_\nu(5)$ *and* $K_\nu(5)$.

Relations Between Solutions

9.6.2
$$K_\nu(z)=\frac{1}{2}\pi\,\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\nu\pi)}$$

The right of this equation is replaced by its limiting value if ν is an integer or zero.

9.6.3
$$I_\nu(z)=e^{-\frac{1}{2}\nu\pi i}J_\nu(ze^{\frac{1}{2}\pi i})\qquad(-\pi<\arg z\le\tfrac{1}{2}\pi)$$
$$I_\nu(z)=e^{3\nu\pi i/2}J_\nu(ze^{-3\pi i/2})\qquad(\tfrac{1}{2}\pi<\arg z\le\pi)$$

9.6.4
$$K_\nu(z)=\tfrac{1}{2}\pi ie^{\frac{1}{2}\nu\pi i}H_\nu^{(1)}(ze^{\frac{1}{2}\pi i})\qquad(-\pi<\arg z\le\tfrac{1}{2}\pi)$$
$$K_\nu(z)=-\tfrac{1}{2}\pi ie^{-\frac{1}{2}\nu\pi i}H_\nu^{(2)}(ze^{-\frac{1}{2}\pi i})\,(-\tfrac{1}{2}\pi<\arg z\le\pi)$$

9.6.5
$$Y_\nu(ze^{\frac{1}{2}\pi i})=e^{\frac{1}{2}(\nu+1)\pi i}I_\nu(z)-(2/\pi)e^{-\frac{1}{2}\nu\pi i}K_\nu(z)$$
$$(-\pi<\arg z\le\tfrac{1}{2}\pi)$$

9.6.6 $I_{-n}(z)=I_n(z),\ K_{-\nu}(z)=K_\nu(z)$

Most of the properties of modified Bessel functions can be deduced immediately from those of ordinary Bessel functions by application of these relations.

Limiting Forms for Small Arguments

When ν is fixed and $z\rightarrow0$

9.6.7
$$I_\nu(z)\sim(\tfrac{1}{2}z)^\nu/\Gamma(\nu+1)\qquad(\nu\neq-1,-2,\ldots)$$

9.6.8 $K_0(z)\sim-\ln z$

9.6.9 $K_\nu(z)\sim\tfrac{1}{2}\Gamma(\nu)(\tfrac{1}{2}z)^{-\nu}\qquad(\mathscr{R}\nu>0)$

Ascending Series

9.6.10
$$I_\nu(z)=(\tfrac{1}{2}z)^\nu\sum_{k=0}^{\infty}\frac{(\frac{1}{4}z^2)^k}{k!\Gamma(\nu+k+1)}$$

9.6.11
$$K_n(z)=\tfrac{1}{2}(\tfrac{1}{2}z)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(-\tfrac{1}{4}z^2)^k$$
$$+(-)^{n+1}\ln(\tfrac{1}{2}z)I_n(z)$$
$$+(-)^n\tfrac{1}{2}(\tfrac{1}{2}z)^n\sum_{k=0}^{\infty}\{\psi(k+1)+\psi(n+k+1)\}\frac{(\frac{1}{4}z^2)^k}{k!(n+k)!}$$

where $\psi(n)$ is given by **6.3.2.**

9.6.12
$$I_0(z)=1+\frac{\frac{1}{4}z^2}{(1!)^2}+\frac{(\frac{1}{4}z^2)^2}{(2!)^2}+\frac{(\frac{1}{4}z^2)^3}{(3!)^2}+\ldots$$

9.6.13
$$K_0(z)=-\{\ln(\tfrac{1}{2}z)+\gamma\}I_0(z)+\frac{\frac{1}{4}z^2}{(1!)^2}$$
$$+(1+\tfrac{1}{2})\frac{(\frac{1}{4}z^2)^2}{(2!)^2}+(1+\tfrac{1}{2}+\tfrac{1}{3})\frac{(\frac{1}{4}z^2)^3}{(3!)^2}+\ldots$$

Wronskians

9.6.14
$$W\{I_\nu(z),I_{-\nu}(z)\}=I_\nu(z)I_{-(\nu+1)}(z)-I_{\nu+1}(z)I_{-\nu}(z)$$
$$=-2\sin(\nu\pi)/(\pi z)$$

9.6.15
$$W\{K_\nu(z),I_\nu(z)\}=I_\nu(z)K_{\nu+1}(z)+I_{\nu+1}(z)K_\nu(z)=1/z$$

Integral Representations

9.6.16

$$I_0(z)=\frac{1}{\pi}\int_0^\pi e^{\pm z\cos\theta}\,d\theta=\frac{1}{\pi}\int_0^\pi \cosh\,(z\cos\theta)d\theta$$

9.6.17 $$K_0(z)=-\frac{1}{\pi}\int_0^\pi e^{\pm z\cos\theta}\,\{\gamma+\ln\,(2z\sin^2\theta)\}\,d\theta$$

9.6.18

$$I_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int_0^\pi e^{\pm z\cos\theta}\sin^{2\nu}\theta\,d\theta$$

$$=\frac{(\frac{1}{2}z)^\nu}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\int_{-1}^1 (1-t^2)^{\nu-\frac{1}{2}}e^{\pm zt}dt \qquad (\mathscr{R}\nu>-\tfrac{1}{2})$$

9.6.19 $$I_n(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos\,(n\theta)d\theta$$

9.6.20

$$I_\nu(z)=\frac{1}{\pi}\int_0^\pi e^{z\cos\theta}\cos\,(\nu\theta)d\theta$$

$$-\frac{\sin\,(\nu\pi)}{\pi}\int_0^\infty e^{-z\cosh t-\nu t}dt \qquad (|\arg z|<\tfrac{1}{2}\pi)$$

9.6.21

$$K_0(x)=\int_0^\infty \cos\,(x\sinh t)dt=\int_0^\infty \frac{\cos\,(xt)}{\sqrt{t^2+1}}dt$$

$$(x>0)$$

9.6.22

$$K_\nu(x)=\sec\,(\tfrac{1}{2}\nu\pi)\int_0^\infty \cos\,(x\sinh t)\cosh\,(\nu t)dt$$

$$=\csc\,(\tfrac{1}{2}\nu\pi)\int_0^\infty \sin\,(x\sinh t)\sinh\,(\nu t)dt$$

$$(|\mathscr{R}\nu|<1,\ x>0)$$

9.6.23

$$K_\nu(z)=\frac{\pi^{\frac{1}{2}}(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty e^{-z\cosh t}\sinh^{2\nu}t\,dt$$

$$=\frac{\pi^{\frac{1}{2}}(\frac{1}{2}z)^\nu}{\Gamma(\nu+\frac{1}{2})}\int_1^\infty e^{-zt}(t^2-1)^{\nu-\frac{1}{2}}\,dt$$

$$(\mathscr{R}\nu>-\tfrac{1}{2},\ |\arg z|<\tfrac{1}{2}\pi)$$

9.6.24 $$K_\nu(z)=\int_0^\infty e^{-z\cosh t}\cosh\,(\nu t)\,dt \quad (|\arg z|<\tfrac{1}{2}\pi)$$

9.6.25

$$K_\nu(xz)=\frac{\Gamma(\nu+\frac{1}{2})(2z)^\nu}{\pi^{\frac{1}{2}}x^\nu}\int_0^\infty \frac{\cos\,(xt)dt}{(t^2+z^2)^{\nu+\frac{1}{2}}}$$

$$(\mathscr{R}\nu>-\tfrac{1}{2},\ x>0,\ |\arg z|<\tfrac{1}{2}\pi)^*$$

Recurrence Relations

9.6.26

$$\mathscr{Z}_{\nu-1}(z)-\mathscr{Z}_{\nu+1}(z)=\frac{2\nu}{z}\mathscr{Z}_\nu(z)$$

$$\mathscr{Z}'_\nu(z)=\mathscr{Z}_{\nu-1}(z)-\frac{\nu}{z}\mathscr{Z}_\nu(z)$$

$$\mathscr{Z}_{\nu-1}(z)+\mathscr{Z}_{\nu+1}(z)=2\mathscr{Z}'_\nu(z)$$

$$\mathscr{Z}'_\nu(z)=\mathscr{Z}_{\nu+1}(z)+\frac{\nu}{z}\mathscr{Z}_\nu(z)$$

$\mathscr{Z}_\nu$ denotes I_ν, $e^{\nu\pi i}K_\nu$ or any linear combination of these functions, the coefficients in which are independent of z and ν.

9.6.27 $$I'_0(z)=I_1(z), \qquad K'_0(z)=-K_1(z)$$

Formulas for Derivatives

9.6.28

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\{z^\nu\mathscr{Z}_\nu(z)\}=z^{\nu-k}\mathscr{Z}_{\nu-k}(z)$$

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k\{z^{-\nu}\mathscr{Z}_\nu(z)\}=z^{-\nu-k}\mathscr{Z}_{\nu+k}(z) \qquad (k=0,1,2,\ldots)$$

9.6.29

$$\mathscr{Z}_\nu^{(k)}(z)=\frac{1}{2^k}\{\mathscr{Z}_{\nu-k}(z)+\binom{k}{1}\mathscr{Z}_{\nu-k+2}(z)$$

$$+\binom{k}{2}\mathscr{Z}_{\nu-k+4}(z)+\ldots+\mathscr{Z}_{\nu+k}(z)\}$$

$$(k=0,1,2,\ldots)$$

Analytic Continuation

9.6.30 $$I_\nu(ze^{m\pi i})=e^{m\nu\pi i}I_\nu(z) \qquad (m\text{ an integer})$$

9.6.31

$$K_\nu(ze^{m\pi i})=e^{-m\nu\pi i}K_\nu(z)-\pi i\sin\,(m\nu\pi)\csc\,(\nu\pi)I_\nu(z)$$

$$(m\text{ an integer})$$

9.6.32 $$I_\nu(\bar{z})=\overline{I_\nu(z)}, \qquad K_\nu(\bar{z})=\overline{K_\nu(z)} \qquad (\nu\text{ real})$$

Generating Function and Associated Series

9.6.33 $$e^{\frac{1}{2}z(t+1/t)}=\sum_{k=-\infty}^{\infty}t^kI_k(z) \qquad (t\neq 0)$$

9.6.34 $$e^{z\cos\theta}=I_0(z)+2\sum_{k=1}^{\infty}I_k(z)\cos(k\theta)$$

9.6.35

$$e^{z\sin\theta}=I_0(z)+2\sum_{k=0}^{\infty}(-)^kI_{2k+1}(z)\sin\{(2k+1)\theta\}$$

$$+2\sum_{k=1}^{\infty}(-)^kI_{2k}(z)\cos(2k\theta)$$

9.6.36 $$1=I_0(z)-2I_2(z)+2I_4(z)-2I_6(z)+\ldots$$

9.6.37 $$e^z=I_0(z)+2I_1(z)+2I_2(z)+2I_3(z)+\ldots$$

9.6.38 $$e^{-z}=I_0(z)-2I_1(z)+2I_2(z)-2I_3(z)+\ldots$$

9.6.39

$$\cosh z=I_0(z)+2I_2(z)+2I_4(z)+2I_6(z)+\ldots$$

9.6.40 $$\sinh z=2I_1(z)+2I_3(z)+2I_5(z)+\ldots$$

[illegible]

Other Differential Equations

The quantity λ^2 in equations **9.1.49** to **9.1.54** and **9.1.56** can be replaced by $-\lambda^2$ if at the same time the symbol $\mathscr{C}$ in the given solutions is replaced by $\mathscr{Z}$.

9.6.41

$$z^2w''+z(1\pm 2z)w'+(\pm z-\nu^2)w=0, \qquad w=e^{\mp z}\mathscr{Z}_\nu(z)$$

Differential equations for products may be obtained from **9.1.57** to **9.1.59** by replacing z by iz.

Derivatives With Respect to Order

9.6.42

$$\frac{\partial}{\partial\nu}I_\nu(z)=I_\nu(z)\ln(\tfrac{1}{2}z)-(\tfrac{1}{2}z)^\nu\sum_{k=0}^{\infty}\frac{\psi(\nu+k+1)}{\Gamma(\nu+k+1)}\frac{(\frac{1}{4}z^2)^k}{k!}$$

9.6.43

$$\frac{\partial}{\partial\nu}K_\nu(z)=\tfrac{1}{2}\pi\csc(\nu\pi)\left\{\frac{\partial}{\partial\nu}I_{-\nu}(z)-\frac{\partial}{\partial\nu}I_\nu(z)\right\}$$
$$-\pi\cot(\nu\pi)K_\nu(z) \qquad (\nu\neq 0,\pm 1,\pm 2,\ldots)$$

9.6.44

$$(-)^n\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=n}=$$
$$-K_n(z)+\frac{n!(\frac{1}{2}z)^{-n}}{2}\sum_{k=0}^{n-1}(-)^k\frac{(\frac{1}{2}z)^kI_k(z)}{(n-k)k!}$$

9.6.45

$$\left[\frac{\partial}{\partial\nu}K_\nu(z)\right]_{\nu=n}=\frac{n!(\frac{1}{2}z)^{-n}}{2}\sum_{k=0}^{n-1}\frac{(\frac{1}{2}z)^kK_k(z)}{(n-k)k!}$$

9.6.46

$$\left[\frac{\partial}{\partial\nu}I_\nu(z)\right]_{\nu=0}=-K_0(z), \qquad \left[\frac{\partial}{\partial\nu}K_\nu(z)\right]_{\nu=0}=0$$

Expressions in Terms of Hypergeometric Functions

9.6.47

$$I_\nu(z)=\frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\,{}_0F_1\,(\nu+1;\tfrac{1}{4}z^2)$$
$$=\frac{(\frac{1}{2}z)^\nu e^{-z}}{\Gamma(\nu+1)}M\,(\nu+\tfrac{1}{2},2\nu+1,2z)=\frac{z^{-\frac{1}{2}}M_{0,\nu}(2z)}{2^{2\nu+\frac{1}{2}}\Gamma(\nu+1)}$$

9.6.48 $$K_\nu(z)=\left(\frac{\pi}{2z}\right)^{\frac{1}{2}}W_{0,\nu}(2z)$$

(${}_0F_1$ is the generalized hypergeometric function. For $M(a,b,z)$, $M_{0,\nu}(z)$ and $W_{0,\nu}(z)$ see chapter **13.**)

Connection With Legendre Functions

If μ and z are fixed, $\mathscr{R}z>0$, and $\nu\rightarrow\infty$ through real positive values

9.6.49 $$\lim\left\{\nu^\mu P_\nu^{-\mu}\left(\cosh\frac{z}{\nu}\right)\right\}=I_\mu(z)$$

9.6.50 $$\lim\left\{\nu^{-\mu}e^{-\mu\pi i}Q_\nu^\mu\left(\cosh\frac{z}{\nu}\right)\right\}=K_\mu(z)$$

For the definition of $P_\nu^{-\mu}$ and Q_ν^μ, see chapter **8.**

Multiplication Theorems

9.6.51

$$\mathscr{Z}_\nu(\lambda z)=\lambda^{\pm\nu}\sum_{k=0}^{\infty}\frac{(\lambda^2-1)^k(\frac{1}{2}z)^k}{k!}\mathscr{Z}_{\nu\pm k}(z) \qquad (|\lambda^2-1|<1)$$

If $\mathscr{Z}=I$ and the upper signs are taken, the restriction on λ is unnecessary.

9.6.52

$$I_\nu(z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}J_{\nu+k}(z), \qquad J_\nu(z)=\sum_{k=0}^{\infty}(-)^k\frac{z^k}{k!}I_{\nu+k}(z)$$

Neumann Series for $K_n(z)$

9.6.53

$$K_n(z)=(-)^{n-1}\{\ln(\tfrac{1}{2}z)-\psi(n+1)\}I_n(z)$$
$$+\frac{n!(\frac{1}{2}z)^{-n}}{2}\sum_{k=0}^{n-1}(-)^k\frac{(\frac{1}{2}z)^kI_k(z)}{(n-k)k!}$$
$$+(-)^n\sum_{k=1}^{\infty}\frac{(n+2k)I_{n+2k}(z)}{k(n+k)}$$

9.6.54 $$K_0(z)=-\{\ln(\tfrac{1}{2}z)+\gamma\}I_0(z)+2\sum_{k=1}^{\infty}\frac{I_{2k}(z)}{k}$$

Zeros

Properties of the zeros of $I_\nu(z)$ and $K_\nu(z)$ may be deduced from those of $J_\nu(z)$ and $H_\nu^{(1)}(z)$ respectively, by application of the transformations **9.6.3** and **9.6.4.**

For example, if ν is real the zeros of $I_\nu(z)$ are all complex unless $-2k<\nu<-(2k-1)$ for some positive integer k, in which event $I_\nu(z)$ has two real zeros.

The approximate distribution of the zeros of $K_n(z)$ in the region $-\frac{3}{2}\pi\leq\arg z\leq\frac{1}{2}\pi$ is obtained on rotating **Figure 9.6** through an angle $-\frac{1}{2}\pi$ so that the cut lies along the positive imaginary axis. The zeros in the region $-\frac{1}{2}\pi\leq\arg z\leq\frac{3}{2}\pi$ are their conjugates. $K_n(z)$ has no zeros in the region $|\arg z|\leq\frac{1}{2}\pi$; this result remains true when n is replaced by any real number ν.

9.7. Asymptotic Expansions

Asymptotic Expansions for Large Arguments

When ν is fixed, $|z|$ is large and $\mu=4\nu^2$

9.7.1

$$I_\nu(z)\sim\frac{e^z}{\sqrt{2\pi z}}\left\{1-\frac{\mu-1}{8z}+\frac{(\mu-1)(\mu-9)}{2!(8z)^2}\right.$$
$$\left.-\frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3}+\ldots\right\} \qquad (|\arg z|<\tfrac{1}{2}\pi)$$

9.7.2

$$K_\nu(z)\sim\sqrt{\frac{\pi}{2z}}\,e^{-z}\{1+\frac{\mu-1}{8z}+\frac{(\mu-1)(\mu-9)}{2!(8z)^2}+\frac{(\mu-1)(\mu-9)(\mu-25)}{3!(8z)^3}+\ldots\}\quad(|\arg z|<\tfrac{3}{2}\pi)$$

9.7.3

$$I'_\nu(z)\sim\frac{e^z}{\sqrt{2\pi z}}\{1-\frac{\mu+3}{8z}+\frac{(\mu-1)(\mu+15)}{2!(8z)^2}-\frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8z)^3}+\ldots\}\quad(|\arg z|<\tfrac{1}{2}\pi)$$

9.7.4

$$K'_\nu(z)\sim-\sqrt{\frac{\pi}{2z}}e^{-z}\{1+\frac{\mu+3}{8z}+\frac{(\mu-1)(\mu+15)}{2!(8z)^2}+\frac{(\mu-1)(\mu-9)(\mu+35)}{3!(8z)^3}+\ldots\}\quad(|\arg z|<\tfrac{3}{2}\pi)$$

The general terms in the last two expansions can be written down by inspection of **9.2.15** and **9.2.16.**

If ν is real and non-negative and z is positive the remainder after k terms in the expansion **9.7.2** does not exceed the $(k+1)$th term in absolute value and is of the same sign, provided that $k\geq\nu-\frac{1}{2}$.

9.7.5

$$I_\nu(z)K_\nu(z)\sim\frac{1}{2z}\{1-\frac{1}{2}\frac{\mu-1}{(2z)^2}+\frac{1\cdot3}{2\cdot4}\frac{(\mu-1)(\mu-9)}{(2z)^4}-\ldots\}\quad(|\arg z|<\tfrac{1}{2}\pi)$$

9.7.6

$$I'_\nu(z)K'_\nu(z)\sim-\frac{1}{2z}\{1+\frac{1}{2}\frac{\mu-3}{(2z)^2}-\frac{1\cdot1}{2\cdot4}\frac{(\mu-1)(\mu-45)}{(2z)^4}+\ldots\}\quad(|\arg z|<\tfrac{1}{2}\pi)$$

The general terms can be written down by inspection of **9.2.28** and **9.2.30.**

Uniform Asymptotic Expansions for Large Orders

9.7.7 $$I_\nu(\nu z)\sim\frac{1}{\sqrt{2\pi\nu}}\frac{e^{\nu\eta}}{(1+z^2)^{1/4}}\{1+\sum_{k=1}^{\infty}\frac{u_k(t)}{\nu^k}\}$$

9.7.8

$$K_\nu(\nu z)\sim\sqrt{\frac{\pi}{2\nu}}\frac{e^{-\nu\eta}}{(1+z^2)^{1/4}}\{1+\sum_{k=1}^{\infty}(-)^k\frac{u_k(t)}{\nu^k}\}$$

9.7.9 $$I'_\nu(\nu z)\sim\frac{1}{\sqrt{2\pi\nu}}\frac{(1+z^2)^{1/4}}{z}e^{\nu\eta}\{1+\sum_{k=1}^{\infty}\frac{v_k(t)}{\nu^k}\}$$

9.7.10

$$K'_\nu(\nu z)\sim-\sqrt{\frac{\pi}{2\nu}}\frac{(1+z^2)^{1/4}}{z}e^{-\nu\eta}\{1+\sum_{k=1}^{\infty}(-)^k\frac{v_k(t)}{\nu^k}\}$$

When $\nu\rightarrow+\infty$, these expansions hold uniformly with respect to z in the sector $|\arg z|\leq\frac{1}{2}\pi-\epsilon$, where ϵ is an arbitrary positive number. Here

9.7.11 $$t=1/\sqrt{1+z^2},\qquad\eta=\sqrt{1+z^2}+\ln\frac{z}{1+\sqrt{1+z^2}}$$

and $u_k(t)$, $v_k(t)$ are given by **9.3.9, 9.3.10, 9.3.13** and **9.3.14**. See [9.38] for tables of η, $u_k(t)$, $v_k(t)$, and also for bounds on the remainder terms in **9.7.7** to **9.7.10**.

9.8. Polynomial Approximations [4]

In equations **9.8.1** to **9.8.4**, $t=x/3.75$.

9.8.1 $-3.75\leq x\leq3.75$

$$I_0(x)=1+3.51562\,29t^2+3.08994\,24t^4+1.20674\,92t^6+.26597\,32t^8+.03607\,68t^{10}+.00458\,13t^{12}+\epsilon$$

$$|\epsilon|<1.6\times10^{-7}$$

9.8.2 $3.75\leq x<\infty$

$$x^{\frac{1}{2}}e^{-x}I_0(x)=.39894\,228+.01328\,592t^{-1}+.00225\,319t^{-2}-.00157\,565t^{-3}+.00916\,281t^{-4}-.02057\,706t^{-5}+.02635\,537t^{-6}-.01647\,633t^{-7}+.00392\,377t^{-8}+\epsilon$$

$$|\epsilon|<1.9\times10^{-7}$$

9.8.3 $-3.75\leq x\leq3.75$

$$x^{-1}I_1(x)=\tfrac{1}{2}+.87890\,594t^2+.51498\,869t^4+.15084\,934t^6+.02658\,733t^8+.00301\,532t^{10}+.00032\,411t^{12}+\epsilon$$

$$|\epsilon|<8\times10^{-9}$$

9.8.4 $3.75\leq x<\infty$

$$x^{\frac{1}{2}}e^{-x}I_1(x)=.39894\,228-.03988\,024t^{-1}-.00362\,018t^{-2}+.00163\,801t^{-3}-.01031\,555t^{-4}+.02282\,967t^{-5}-.02895\,312t^{-6}+.01787\,654t^{-7}-.00420\,059t^{-8}+\epsilon$$

$$|\epsilon|<2.2\times10^{-7}$$

---

[4] See footnote 2, section **9.4**.

9.8.5 $0<x\leq 2$

$$\begin{aligned}K_0(x)=-\ln\,(x/2)I_0(x)-.57721\ 566\\ +.42278\ 420(x/2)^2+.23069\ 756(x/2)^4\\ +.03488\ 590(x/2)^6+.00262\ 698(x/2)^8\\ +.00010\ 750(x/2)^{10}+.00000\ 740(x/2)^{12}+\epsilon\end{aligned}$$

$$|\epsilon|<1\times 10^{-8}$$

9.8.6 $2\leq x<\infty$

$$\begin{aligned}x^{\frac{1}{2}}e^xK_0(x)=1.25331\ 414-.07832\ 358(2/x)\\ +.02189\ 568(2/x)^2-.01062\ 446(2/x)^3\\ +.00587\ 872(2/x)^4-.00251\ 540(2/x)^5\\ +.00053\ 208(2/x)^6+\epsilon\end{aligned}$$

$$|\epsilon|<1.9\times 10^{-7}$$

9.8.7 $0<x\leq 2$

$$\begin{aligned}xK_1(x)=x\ln\,(x/2)I_1(x)+1+.15443\ 144(x/2)^2\\ -.67278\ 579(x/2)^4-.18156\ 897(x/2)^6\\ -.01919\ 402(x/2)^8-.00110\ 404(x/2)^{10}\\ -.00004\ 686(x/2)^{12}+\epsilon\end{aligned}$$

$$|\epsilon|<8\times 10^{-9}$$

9.8.8 $2\leq x<\infty$

$$\begin{aligned}x^{\frac{1}{2}}e^xK_1(x)=1.25331\ 414+.23498\ 619(2/x)\\ -.03655\ 620(2/x)^2+.01504\ 268(2/x)^3\\ -.00780\ 353(2/x)^4+.00325\ 614(2/x)^5\\ -.00068\ 245(2/x)^6+\epsilon\end{aligned}$$

$$|\epsilon|<2.2\times 10^{-7}$$

For expansions of $I_0(x)$, $K_0(x)$, $I_1(x)$, and $K_1(x)$ in series of Chebyshev polynomials for the ranges $0\leq x\leq 8$ and $0\leq 8/x\leq 1$, see [9.37].

Kelvin Functions

9.9. Definitions and Properties

In this and the following section ν is real, x is real and non-negative, and n is again a positive integer or zero.

Definitions

9.9.1

$$\begin{aligned}\mathrm{ber}_\nu\,x+i\,\mathrm{bei}_\nu\,x=J_\nu(xe^{3\pi i/4})=e^{\nu\pi i}J_\nu(xe^{-\pi i/4})\\ =e^{\frac{1}{2}\nu\pi i}I_\nu(xe^{\pi i/4})=e^{3\nu\pi i/2}I_\nu(xe^{-3\pi i/4})\end{aligned}$$

9.9.2

$$\begin{aligned}\mathrm{ker}_\nu\,x+i\,\mathrm{kei}_\nu\,x=e^{-\frac{1}{2}\nu\pi i}K_\nu(xe^{\pi i/4})\\ =\tfrac{1}{2}\pi iH_\nu^{(1)}(xe^{3\pi i/4})=-\tfrac{1}{2}\pi ie^{-\nu\pi i}H_\nu^{(2)}(xe^{-\pi i/4})\end{aligned}$$

When $\nu=0$, suffices are usually suppressed.

Differential Equations

9.9.3

$$\begin{aligned}x^2w''+xw'-(ix^2+\nu^2)w=0,\\ w=\mathrm{ber}_\nu\,x+i\,\mathrm{bei}_\nu\,x,\quad \mathrm{ber}_{-\nu}\,x+i\,\mathrm{bei}_{-\nu}\,x,\\ \mathrm{ker}_\nu\,x+i\,\mathrm{kei}_\nu\,x,\quad \mathrm{ker}_{-\nu}\,x+i\,\mathrm{kei}_{-\nu}\,x\end{aligned}$$

9.9.4

$$\begin{aligned}x^4w^{iv}+2x^3w'''-(1+2\nu^2)(x^2w''-xw')\\ +(\nu^4-4\nu^2+x^4)w=0,\\ w=\mathrm{ber}_{\pm\nu}\,x,\ \mathrm{bei}_{\pm\nu}\,x,\ \mathrm{ker}_{\pm\nu}\,x,\ \mathrm{kei}_{\pm\nu}\,x\end{aligned}$$

Relations Between Solutions

9.9.5

$$\begin{aligned}\mathrm{ber}_{-\nu}\,x=\cos(\nu\pi)\,\mathrm{ber}_\nu\,x+\sin(\nu\pi)\,\mathrm{bei}_\nu\,x\\ +(2/\pi)\sin(\nu\pi)\,\mathrm{ker}_\nu\,x\end{aligned}$$

$$\begin{aligned}\mathrm{bei}_{-\nu}\,x=-\sin(\nu\pi)\,\mathrm{ber}_\nu\,x+\cos(\nu\pi)\,\mathrm{bei}_\nu\,x\\ +(2/\pi)\sin(\nu\pi)\,\mathrm{kei}_\nu\,x\end{aligned}$$

9.9.6

$$\mathrm{ker}_{-\nu}\,x=\cos(\nu\pi)\,\mathrm{ker}_\nu\,x-\sin(\nu\pi)\,\mathrm{kei}_\nu\,x$$

$$\mathrm{kei}_{-\nu}\,x=\sin(\nu\pi)\,\mathrm{ker}_\nu\,x+\cos(\nu\pi)\,\mathrm{kei}_\nu\,x$$

9.9.7 $\mathrm{ber}_{-n}\,x=(-)^n\,\mathrm{ber}_n\,x,\quad \mathrm{bei}_{-n}\,x=(-)^n\,\mathrm{bei}_n\,x$

9.9.8 $\mathrm{ker}_{-n}\,x=(-)^n\,\mathrm{ker}_n\,x,\quad \mathrm{kei}_{-n}\,x=(-)^n\,\mathrm{kei}_n\,x$

Ascending Series

9.9.9

$$\mathrm{ber}_\nu\,x=(\tfrac{1}{2}x)^\nu\sum_{k=0}^{\infty}\frac{\cos\{(\frac{3}{4}\nu+\frac{1}{2}k)\pi\}}{k!\Gamma(\nu+k+1)}(\tfrac{1}{4}x^2)^k$$

$$\mathrm{bei}_\nu\,x=(\tfrac{1}{2}x)^\nu\sum_{k=0}^{\infty}\frac{\sin\{(\frac{3}{4}\nu+\frac{1}{2}k)\pi\}}{k!\Gamma(\nu+k+1)}(\tfrac{1}{4}x^2)^k$$

9.9.10

$$\mathrm{ber}\,x=1-\frac{(\frac{1}{4}x^2)^2}{(2!)^2}+\frac{(\frac{1}{4}x^2)^4}{(4!)^2}-\cdots$$

$$\mathrm{bei}\,x=\tfrac{1}{4}x^2-\frac{(\frac{1}{4}x^2)^3}{(3!)^2}+\frac{(\frac{1}{4}x^2)^5}{(5!)^2}-\cdots$$

9.9.11

$$\begin{aligned}\mathrm{ker}_n\,x=\tfrac{1}{2}(\tfrac{1}{2}x)^{-n}\sum_{k=0}^{n-1}\cos\{(\tfrac{3}{4}n+\tfrac{1}{2}k)\pi\}\\ \times\frac{(n-k-1)!}{k!}(\tfrac{1}{4}x^2)^k-\ln\,(\tfrac{1}{2}x)\,\mathrm{ber}_n\,x+\tfrac{1}{4}\pi\,\mathrm{bei}_n\,x\\ +\tfrac{1}{2}(\tfrac{1}{2}x)^n\sum_{k=0}^{\infty}\cos\{(\tfrac{3}{4}n+\tfrac{1}{2}k)\pi\}\\ \times\frac{\{\psi(k+1)+\psi(n+k+1)\}}{k!(n+k)!}(\tfrac{1}{4}x^2)^k\end{aligned}$$

$$\operatorname{kei}_n x=-\tfrac{1}{2}(\tfrac{1}{2}x)^{-n}\sum_{k=0}^{n-1}\sin\{(\tfrac{3}{4}n+\tfrac{1}{2}k)\pi\}$$
$$\times\frac{(n-k-1)!}{k!}(\tfrac{1}{4}x^2)^k-\ln(\tfrac{1}{2}x)\operatorname{bei}_n x-\tfrac{1}{4}\pi\operatorname{ber}_n x$$
$$+\tfrac{1}{2}(\tfrac{1}{2}x)^n\sum_{k=0}^{\infty}\sin\{(\tfrac{3}{4}n+\tfrac{1}{2}k)\pi\}$$
$$\times\frac{\{\psi(k+1)+\psi(n+k+1)\}}{k!(n+k)!}(\tfrac{1}{4}x^2)^k$$

where $\psi(n)$ is given by **6.3.2.**

9.9.12

$$\ker x=-\ln(\tfrac{1}{2}x)\operatorname{ber} x+\tfrac{1}{4}\pi\operatorname{bei} x$$
$$+\sum_{k=0}^{\infty}(-)^k\frac{\psi(2k+1)}{\{(2k)!\}^2}(\tfrac{1}{4}x^2)^{2k}$$

$$\operatorname{kei} x=-\ln(\tfrac{1}{2}x)\operatorname{bei} x-\tfrac{1}{4}\pi\operatorname{ber} x$$
$$+\sum_{k=0}^{\infty}(-)^k\frac{\psi(2k+2)}{\{(2k+1)!\}^2}(\tfrac{1}{4}x^2)^{2k+1}$$

Functions of Negative Argument

In general Kelvin functions have a branch point at $x=0$ and individual functions with arguments $xe^{\pm\pi i}$ are complex. The branch point is absent however in the case of ber_ν and bei_ν when ν is an integer, and

9.9.13

$$\operatorname{ber}_n(-x)=(-)^n\operatorname{ber}_n x,\qquad \operatorname{bei}_n(-x)=(-)^n\operatorname{bei}_n x$$

Recurrence Relations

9.9.14

$$f_{\nu+1}+f_{\nu-1}=-\frac{\nu\sqrt{2}}{x}(f_\nu-g_\nu)$$
$$f'_\nu=\frac{1}{2\sqrt{2}}(f_{\nu+1}+g_{\nu+1}-f_{\nu-1}-g_{\nu-1})$$
$$f'_\nu-\frac{\nu}{x}f_\nu=\frac{1}{\sqrt{2}}(f_{\nu+1}+g_{\nu+1})$$
$$f'_\nu+\frac{\nu}{x}f_\nu=-\frac{1}{\sqrt{2}}(f_{\nu-1}+g_{\nu-1})$$

where

9.9.15

$$\left.\begin{aligned}f_\nu&=\operatorname{ber}_\nu x\\ g_\nu&=\operatorname{bei}_\nu x\end{aligned}\right\}\qquad\left.\begin{aligned}f_\nu&=\operatorname{bei}_\nu x\\ g_\nu&=-\operatorname{ber}_\nu x\end{aligned}\right\}$$
$$\left.\begin{aligned}f_\nu&=\ker_\nu x\\ g_\nu&=\operatorname{kei}_\nu x\end{aligned}\right\}\qquad\left.\begin{aligned}f_\nu&=\operatorname{kei}_\nu x\\ g_\nu&=-\ker_\nu x\end{aligned}\right\}$$

9.9.16

$$\sqrt{2}\operatorname{ber}' x=\operatorname{ber}_1 x+\operatorname{bei}_1 x$$
$$\sqrt{2}\operatorname{bei}' x=-\operatorname{ber}_1 x+\operatorname{bei}_1 x$$

9.9.17

$$\sqrt{2}\ker' x=\ker_1 x+\operatorname{kei}_1 x$$
$$\sqrt{2}\operatorname{kei}' x=-\ker_1 x+\operatorname{kei}_1 x$$

Recurrence Relations for Cross-Products

If

9.9.18

$$p_\nu=\operatorname{ber}_\nu^2 x+\operatorname{bei}_\nu^2 x$$
$$q_\nu=\operatorname{ber}_\nu x\operatorname{bei}'_\nu x-\operatorname{ber}'_\nu x\operatorname{bei}_\nu x$$
$$r_\nu=\operatorname{ber}_\nu x\operatorname{ber}'_\nu x+\operatorname{bei}_\nu x\operatorname{bei}'_\nu x$$
$$s_\nu=\operatorname{ber}'^2_\nu x+\operatorname{bei}'^2_\nu x$$

then

9.9.19

$$p_{\nu+1}=p_{\nu-1}-\frac{4\nu}{x}r_\nu$$
$$q_{\nu+1}=-\frac{\nu}{x}p_\nu+r_\nu=-q_{\nu-1}+2r_\nu$$
$$r_{\nu+1}=-\frac{(\nu+1)}{x}p_{\nu+1}+q_\nu$$
$$s_\nu=\frac{1}{2}p_{\nu+1}+\frac{1}{2}p_{\nu-1}-\frac{\nu^2}{x^2}p_\nu$$

and

9.9.20 $$p_\nu s_\nu=r_\nu^2+q_\nu^2$$

The same relations hold with ber, bei replaced throughout by ker, kei, respectively.

Indefinite Integrals

In the following f_ν, g_ν are any one of the pairs given by equations **9.9.15** and f^*_ν, g^*_ν are either the same pair or any other pair.

9.9.21

$$\int x^{1+\nu}f_\nu dx=-\frac{x^{1+\nu}}{\sqrt{2}}(f_{\nu+1}-g_{\nu+1})=-x^{1+\nu}\left(\frac{\nu}{x}g_\nu-g'_\nu\right)$$

9.9.22

$$\int x^{1-\nu}f_\nu dx=\frac{x^{1-\nu}}{\sqrt{2}}(f_{\nu-1}-g_{\nu-1})=x^{1-\nu}\left(\frac{\nu}{x}g_\nu+g'_\nu\right)$$

9.9.23

$$\int x(f_\nu g^*_\nu-g_\nu f^*_\nu)dx=\frac{x}{2\sqrt{2}}\{f^*_\nu(f_{\nu+1}+g_{\nu+1})$$
$$-g^*_\nu(f_{\nu+1}-g_{\nu+1})-f_\nu(f^*_{\nu+1}+g^*_{\nu+1})+g_\nu(f^*_{\nu+1}-g^*_{\nu+1})\}$$
$$=\frac{1}{2}x(f'_\nu f^*_\nu-f_\nu f^{*\prime}_\nu+g'_\nu g^*_\nu-g_\nu g^{*\prime}_\nu)$$

9.9.24

$$\int x(f_\nu g_\nu^* + g_\nu f_\nu^*)dx = \frac{1}{4}x^2(2f_\nu g_\nu^* - f_{\nu-1}g_{\nu+1}^*$$
$$-f_{\nu+1}g_{\nu-1}^* + 2g_\nu f_\nu^* - g_{\nu-1}f_{\nu+1}^* - g_{\nu+1}f_{\nu-1}^*)$$

9.9.25

$$\int x(f_\nu^2+g_\nu^2)dx = x(f_\nu g_\nu' - f_\nu' g_\nu)$$
$$= -(x/\sqrt{2})(f_\nu f_{\nu+1} + g_\nu g_{\nu+1} - f_\nu g_{\nu+1} + f_{\nu+1}g_\nu)$$

9.9.26

$$\int x f_\nu g_\nu dx = \frac{1}{4}x^2(2f_\nu g_\nu - f_{\nu-1}g_{\nu+1} - f_{\nu+1}g_{\nu-1})$$

9.9.27

$$\int x(f_\nu^2 - g_\nu^2)dx = \frac{1}{2}x^2(f_\nu^2 - f_{\nu-1}f_{\nu+1} - g_\nu^2 + g_{\nu-1}g_{\nu+1})$$

Ascending Series for Cross-Products

9.9.28

$$\mathrm{ber}_\nu^2\, x + \mathrm{bei}_\nu^2\, x =$$
$$(\tfrac{1}{2}x)^{2\nu} \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu+k+1)\Gamma(\nu+2k+1)} \frac{(\frac{1}{4}x^2)^{2k}}{k!}$$

9.9.29

$$\mathrm{ber}_\nu\, x\, \mathrm{bei}_\nu'\, x - \mathrm{ber}_\nu'\, x\, \mathrm{bei}_\nu\, x$$
$$= (\tfrac{1}{2}x)^{2\nu+1} \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu+k+1)\Gamma(\nu+2k+2)} \frac{(\frac{1}{4}x^2)^{2k}}{k!}$$

9.9.30

$$\mathrm{ber}_\nu\, x\, \mathrm{ber}_\nu'\, x + \mathrm{bei}_\nu\, x\, \mathrm{bei}_\nu'\, x$$
$$= \tfrac{1}{2}(\tfrac{1}{2}x)^{2\nu-1} \sum_{k=0}^{\infty} \frac{1}{\Gamma(\nu+k+1)\Gamma(\nu+2k)} \frac{(\frac{1}{4}x^2)^{2k}}{k!}$$

9.9.31

$$\mathrm{ber}_\nu'^2\, x + \mathrm{bei}_\nu'^2\, x$$
$$= (\tfrac{1}{2}x)^{2\nu-2} \sum_{k=0}^{\infty} \frac{(2k^2+2\nu k+\frac{1}{4}\nu^2)}{\Gamma(\nu+k+1)\Gamma(\nu+2k+1)} \frac{(\frac{1}{4}x^2)^{2k}}{k!}$$

Expansions in Series of Bessel Functions

9.9.32

$$\mathrm{ber}_\nu\, x + i\, \mathrm{bei}_\nu\, x = \sum_{k=0}^{\infty} \frac{e^{(3\nu+k)\pi i/4} x^k J_{\nu+k}(x)}{2^{\frac{1}{2}k}\, k!}$$
$$= \sum_{k=0}^{\infty} \frac{e^{(3\nu+3k)\pi i/4} x^k I_{\nu+k}(x)}{2^{\frac{1}{2}k}\, k!}$$

9.9.33

$$\mathrm{ber}_n(x\sqrt{2}) = \sum_{k=-\infty}^{\infty} (-)^{n+k} J_{n+2k}(x) I_{2k}(x)$$

$$\mathrm{bei}_n(x\sqrt{2}) = \sum_{k=-\infty}^{\infty} (-)^{n+k} J_{n+2k+1}(x) I_{2k+1}(x)$$

Zeros of Functions of Order Zero [5]

	ber x	bei x	ker x	kei x
1st zero	2. 84892	5. 02622	1. 71854	3. 91467
2nd zero	7. 23883	9. 45541	6. 12728	8. 34422
3rd zero	11. 67396	13. 89349	10. 56294	12. 78256
4th zero	16. 11356	18. 33398	15. 00269	17. 22314
5th zero	20. 55463	22. 77544	19. 44381	21. 66464

	ber′ x	bei′ x	ker′ x	kei′ x
1st zero	6. 03871	3. 77320	2. 66584	4. 93181
2nd zero	10. 51364	8. 28099	7. 17212	9. 40405
3rd zero	14. 96844	12. 74215	11. 63218	13. 85827
4th zero	19. 41758	17. 19343	16. 08312	18. 30717
5th zero	23. 86430	21. 64114	20. 53068	22. 75379

9.10. Asymptotic Expansions

Asymptotic Expansions for Large Arguments

When ν is fixed and x is large

9.10.1

$$\mathrm{ber}_\nu\, x = \frac{e^{x/\sqrt{2}}}{\sqrt{2\pi x}}\{f_\nu(x)\cos\alpha + g_\nu(x)\sin\alpha\}$$
$$-\frac{1}{\pi}\{\sin(2\nu\pi)\,\mathrm{ker}_\nu\, x + \cos(2\nu\pi)\,\mathrm{kei}_\nu\, x\}$$

9.10.2

$$\mathrm{bei}_\nu\, x = \frac{e^{x/\sqrt{2}}}{\sqrt{2\pi x}}\{f_\nu(x)\sin\alpha - g_\nu(x)\cos\alpha\}$$
$$+\frac{1}{\pi}\{\cos(2\nu\pi)\,\mathrm{ker}_\nu\, x - \sin(2\nu\pi)\,\mathrm{kei}_\nu\, x\}$$

9.10.3

$$\mathrm{ker}_\nu\, x = \sqrt{\pi/(2x)}\, e^{-x/\sqrt{2}}\{f_\nu(-x)\cos\beta - g_\nu(-x)\sin\beta\}$$

9.10.4

$$\mathrm{kei}_\nu\, x = \sqrt{\pi/(2x)}\, e^{-x/\sqrt{2}}\{-f_\nu(-x)\sin\beta - g_\nu(-x)\cos\beta\}$$

where

9.10.5

$$\alpha = (x/\sqrt{2}) + (\tfrac{1}{2}\nu - \tfrac{1}{8})\pi, \qquad \beta = (x/\sqrt{2}) + (\tfrac{1}{2}\nu + \tfrac{1}{8})\pi = \alpha + \tfrac{1}{4}\pi$$

and, with $4\nu^2$ denoted by μ,

9.10.6

$$f_\nu(\pm x)$$
$$\sim 1 + \sum_{k=1}^{\infty} (\mp)^k \frac{(\mu-1)(\mu-9)\ldots\{\mu-(2k-1)^2\}}{k!(8x)^k} \cos\left(\frac{k\pi}{4}\right)$$

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[5] From British Association for the Advancement of Science, Annual Report (J. R. Airey), 254 (1927) with permission. This reference also gives 5-decimal values of the next five zeros of each function.

9.10.7

$$g_\nu(\pm x)$$

$$\sim\sum_{k=1}^{\infty}(\mp)^k\frac{(\mu-1)(\mu-9)\ldots\{\mu-(2k-1)^2\}}{k!(8x)^k}\sin\left(\frac{k\pi}{4}\right)$$

The terms[6] in $\mathrm{ker}_\nu x$ and $\mathrm{kei}_\nu x$ in equations **9.10.1** and **9.10.2** are asymptotically negligible compared with the other terms, but their inclusion in numerical calculations yields improved accuracy.

The corresponding series for $\mathrm{ber}'_\nu x$, $\mathrm{b\,ei}'_\nu x$, $\mathrm{ker}'_\nu x$ and $\mathrm{kei}'_\nu x$ can be derived from **9.2.11** and **9.2.13** with $z=xe^{3\pi i/4}$; the extra terms in the expansions of $\mathrm{ber}'_\nu x$ and $\mathrm{bei}'_\nu x$ are respectively

$$-(1/\pi)\{\sin(2\nu\pi)\mathrm{ker}'_\nu x+\cos(2\nu\pi)\mathrm{kei}'_\nu x\}$$

and

$$(1/\pi)\{\cos(2\nu\pi)\mathrm{ker}'_\nu x-\sin(2\nu\pi)\mathrm{kei}'_\nu x\}.$$

Modulus and Phase

9.10.8

$$M_\nu=\sqrt{(\mathrm{ber}^2_\nu x+\mathrm{bei}^2_\nu x)},\qquad \theta_\nu=\arctan\,(\mathrm{bei}_\nu x/\mathrm{ber}_\nu x)$$

9.10.9 $\mathrm{ber}_\nu x=M_\nu\cos\theta_\nu,\qquad \mathrm{bei}_\nu x=M_\nu\sin\theta_\nu$

9.10.10 $M_{-n}=M_n,\qquad \theta_{-n}=\theta_n-n\pi$

9.10.11

$$\begin{aligned}\mathrm{ber}'_\nu x&=\tfrac{1}{2}M_{\nu+1}\cos(\theta_{\nu+1}-\tfrac{1}{4}\pi)-\tfrac{1}{2}M_{\nu-1}\cos(\theta_{\nu-1}-\tfrac{1}{4}\pi)\\&=(\nu/x)M_\nu\cos\theta_\nu+M_{\nu+1}\cos(\theta_{\nu+1}-\tfrac{1}{4}\pi)\\&=-(\nu/x)M_\nu\cos\theta_\nu-M_{\nu-1}\cos(\theta_{\nu-1}-\tfrac{1}{4}\pi)\end{aligned}$$

9.10.12

$$\begin{aligned}\mathrm{bei}'_\nu x&=\tfrac{1}{2}M_{\nu+1}\sin(\theta_{\nu+1}-\tfrac{1}{4}\pi)-\tfrac{1}{2}M_{\nu-1}\sin(\theta_{\nu-1}-\tfrac{1}{4}\pi)\\&=(\nu/x)M_\nu\sin\theta_\nu+M_{\nu+1}\sin(\theta_{\nu+1}-\tfrac{1}{4}\pi)\\&=-(\nu/x)M_\nu\sin\theta_\nu-M_{\nu-1}\sin(\theta_{\nu-1}-\tfrac{1}{4}\pi)\end{aligned}$$

9.10.13

$$\mathrm{ber}'\,x=M_1\cos(\theta_1-\tfrac{1}{4}\pi),\qquad \mathrm{bei}'\,x=M_1\sin(\theta_1-\tfrac{1}{4}\pi)$$

9.10.14

$$\begin{aligned}M'_\nu&=(\nu/x)M_\nu+M_{\nu+1}\cos(\theta_{\nu+1}-\theta_\nu-\tfrac{1}{4}\pi)\\&=-(\nu/x)M_\nu-M_{\nu-1}\cos(\theta_{\nu-1}-\theta_\nu-\tfrac{1}{4}\pi)\end{aligned}$$

9.10.15

$$\begin{aligned}\theta'_\nu&=(M_{\nu+1}/M_\nu)\sin(\theta_{\nu+1}-\theta_\nu-\tfrac{1}{4}\pi)\\&=-(M_{\nu-1}/M_\nu)\sin(\theta_{\nu-1}-\theta_\nu-\tfrac{1}{4}\pi)\end{aligned}$$

9.10.16

$$\begin{aligned}M'_0&=M_1\cos(\theta_1-\theta_0-\tfrac{1}{4}\pi)\\ \theta'_0&=(M_1/M_0)\sin(\theta_1-\theta_0-\tfrac{1}{4}\pi)\end{aligned}$$

9.10.17

$$d(xM^2_\nu\theta'_\nu)/dx=xM^2_\nu,\qquad x^2M''_\nu+xM'_\nu-\nu^2M_\nu=x^2M_\nu\theta'^2_\nu$$

9.10.18

$$N_\nu=\sqrt{(\mathrm{ker}^2_\nu x+\mathrm{kei}^2_\nu x)},\qquad \phi_\nu=\arctan\,(\mathrm{kei}_\nu x/\mathrm{ker}_\nu x)$$

9.10.19 $\mathrm{ker}_\nu x=N_\nu\cos\phi_\nu,\qquad \mathrm{kei}_\nu x=N_\nu\sin\phi_\nu$

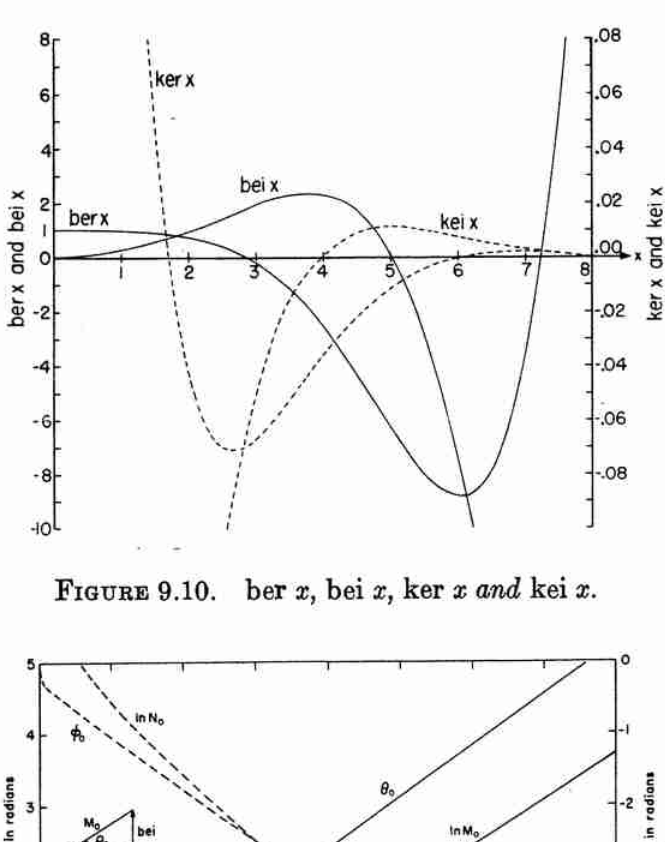

FIGURE 9.10. ber x, bei x, ker x *and* kei x.

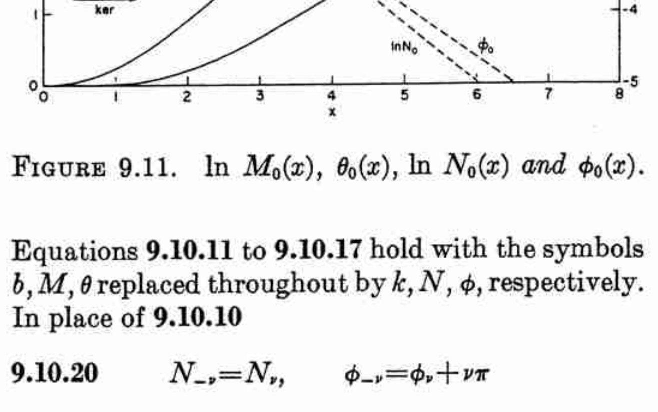

FIGURE 9.11. ln $M_0(x)$, $\theta_0(x)$, ln $N_0(x)$ *and* $\phi_0(x)$.

Equations **9.10.11** to **9.10.17** hold with the symbols b, M, θ replaced throughout by k, N, ϕ, respectively. In place of **9.10.10**

9.10.20 $N_{-\nu}=N_\nu,\qquad \phi_{-\nu}=\phi_\nu+\nu\pi$

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[6] The coefficients of these terms given in [9.17] are incorrect. The present results are due to Mr. G. F. Miller.

Asymptotic Expansions of Modulus and Phase

When ν is fixed, x is large and $\mu=4\nu^2$

9.10.21

$$M_\nu=\frac{e^{x/\sqrt{2}}}{\sqrt{2\pi x}}\{1-\frac{\mu-1}{8\sqrt{2}}\frac{1}{x}+\frac{(\mu-1)^2}{256}\frac{1}{x^2}-\frac{(\mu-1)(\mu^2+14\mu-399)}{6144\sqrt{2}}\frac{1}{x^3}+O\left(\frac{1}{x^4}\right)\}$$

9.10.22

$$\ln M_\nu=\frac{x}{\sqrt{2}}-\tfrac{1}{2}\ln(2\pi x)-\frac{\mu-1}{8\sqrt{2}}\frac{1}{x}-\frac{(\mu-1)(\mu-25)}{384\sqrt{2}}\frac{1}{x^3}-\frac{(\mu-1)(\mu-13)}{128}\frac{1}{x^4}+O\left(\frac{1}{x^5}\right)$$

9.10.23

$$\theta_\nu=\frac{x}{\sqrt{2}}+\left(\frac{1}{2}\nu-\frac{1}{8}\right)\pi+\frac{\mu-1}{8\sqrt{2}}\frac{1}{x}+\frac{\mu-1}{16}\frac{1}{x^2}-\frac{(\mu-1)(\mu-25)}{384\sqrt{2}}\frac{1}{x^3}+O\left(\frac{1}{x^5}\right)$$

9.10.24

$$N_\nu=\sqrt{\frac{\pi}{2x}}\,e^{-x/\sqrt{2}}\{1+\frac{\mu-1}{8\sqrt{2}}\frac{1}{x}+\frac{(\mu-1)^2}{256}\frac{1}{x^2}+\frac{(\mu-1)(\mu^2+14\mu-399)}{6144\sqrt{2}}\frac{1}{x^3}+O\left(\frac{1}{x^4}\right)\}$$

9.10.25

$$\ln N_\nu=-\frac{x}{\sqrt{2}}+\tfrac{1}{2}\ln\left(\frac{\pi}{2x}\right)+\frac{\mu-1}{8\sqrt{2}}\frac{1}{x}+\frac{(\mu-1)(\mu-25)}{384\sqrt{2}}\frac{1}{x^3}-\frac{(\mu-1)(\mu-13)}{128}\frac{1}{x^4}+O\left(\frac{1}{x^5}\right)$$

9.10.26

$$\phi_\nu=-\frac{x}{\sqrt{2}}-\left(\frac{1}{2}\nu+\frac{1}{8}\right)\pi-\frac{\mu-1}{8\sqrt{2}}\frac{1}{x}+\frac{\mu-1}{16}\frac{1}{x^2}+\frac{(\mu-1)(\mu-25)}{384\sqrt{2}}\frac{1}{x^3}+O\left(\frac{1}{x^5}\right)$$

Asymptotic Expansions of Cross-Products

If x is large

9.10.27

$$\mathrm{ber}^2\,x+\mathrm{bei}^2\,x\sim\frac{e^{x\sqrt{2}}}{2\pi x}\left(1+\frac{1}{4\sqrt{2}}\frac{1}{x}+\frac{1}{64}\frac{1}{x^2}-\frac{33}{256\sqrt{2}}\frac{1}{x^3}-\frac{1797}{8192}\frac{1}{x^4}+\ldots\right)$$

9.10.28

$$\mathrm{ber}\,x\,\mathrm{bei}'\,x-\mathrm{ber}'\,x\,\mathrm{bei}\,x\sim\frac{e^{x\sqrt{2}}}{2\pi x}\left(\frac{1}{\sqrt{2}}+\frac{1}{8}\frac{1}{x}+\frac{9}{64\sqrt{2}}\frac{1}{x^2}+\frac{39}{512}\frac{1}{x^3}+\frac{75}{8192\sqrt{2}}\frac{1}{x^4}+\ldots\right)$$

9.10.29

$$\mathrm{ber}\,x\,\mathrm{ber}'\,x+\mathrm{bei}\,x\,\mathrm{bei}'\,x\sim\frac{e^{x\sqrt{2}}}{2\pi x}\left(\frac{1}{\sqrt{2}}-\frac{3}{8}\frac{1}{x}-\frac{15}{64\sqrt{2}}\frac{1}{x^2}-\frac{45}{512}\frac{1}{x^3}+\frac{315}{8192\sqrt{2}}\frac{1}{x^4}+\ldots\right)$$

9.10.30

$$\mathrm{ber}'^2\,x+\mathrm{bei}'^2\,x\sim\frac{e^{x\sqrt{2}}}{2\pi x}\left(1-\frac{3}{4\sqrt{2}}\frac{1}{x}+\frac{9}{64}\frac{1}{x^2}+\frac{75}{256\sqrt{2}}\frac{1}{x^3}+\frac{2475}{8192}\frac{1}{x^4}+\ldots\right)$$

9.10.31

$$\mathrm{ker}^2\,x+\mathrm{kei}^2\,x\sim\frac{\pi}{2x}\,e^{-x\sqrt{2}}\left(1-\frac{1}{4\sqrt{2}}\frac{1}{x}+\frac{1}{64}\frac{1}{x^2}+\frac{33}{256\sqrt{2}}\frac{1}{x^3}-\frac{1797}{8192}\frac{1}{x^4}+\ldots\right)$$

9.10.32

$$\mathrm{ker}\,x\,\mathrm{kei}'\,x-\mathrm{ker}'\,x\,\mathrm{kei}\,x\sim-\frac{\pi}{2x}\,e^{-x\sqrt{2}}\left(\frac{1}{\sqrt{2}}-\frac{1}{8}\frac{1}{x}+\frac{9}{64\sqrt{2}}\frac{1}{x^2}-\frac{39}{512}\frac{1}{x^3}+\frac{75}{8192\sqrt{2}}\frac{1}{x^4}+\ldots\right)$$

9.10.33

$$\mathrm{ker}\,x\,\mathrm{ker}'\,x+\mathrm{kei}\,x\,\mathrm{kei}'\,x\sim-\frac{\pi}{2x}\,e^{-x\sqrt{2}}\left(\frac{1}{\sqrt{2}}+\frac{3}{8}\frac{1}{x}-\frac{15}{64\sqrt{2}}\frac{1}{x^2}+\frac{45}{512}\frac{1}{x^3}+\frac{315}{8192\sqrt{2}}\frac{1}{x^4}+\ldots\right)$$

9.10.34

$$\mathrm{ker}'^2\,x+\mathrm{kei}'^2\,x\sim\frac{\pi}{2x}\,e^{-x\sqrt{2}}\left(1+\frac{3}{4\sqrt{2}}\frac{1}{x}+\frac{9}{64}\frac{1}{x^2}-\frac{75}{256\sqrt{2}}\frac{1}{x^3}+\frac{2475}{8192}\frac{1}{x^4}+\cdots\right)$$

Asymptotic Expansions of Large Zeros

Let

9.10.35

$$f(\delta)=\frac{\mu-1}{16\delta}+\frac{\mu-1}{32\delta^2}+\frac{(\mu-1)(5\mu+19)}{1536\delta^3}+\frac{3(\mu-1)^2}{512\delta^4}+\cdots$$

where $\mu=4\nu^2$. Then if s is a large positive integer

9.10.36

Zeros of $\mathrm{ber}_\nu\,x\sim\sqrt{2}\{\delta-f(\delta)\}$, $\quad\delta=(s-\tfrac{1}{2}\nu-\tfrac{3}{8})\pi$

Zeros of $\mathrm{bei}_\nu\,x\sim\sqrt{2}\{\delta-f(\delta)\}$, $\quad\delta=(s-\tfrac{1}{2}\nu+\tfrac{1}{8})\pi$

Zeros of $\mathrm{ker}_\nu\,x\sim\sqrt{2}\{\delta+f(-\delta)\}$, $\quad\delta=(s-\tfrac{1}{2}\nu-\tfrac{5}{8})\pi$

Zeros of $\mathrm{kei}_\nu\,x\sim\sqrt{2}\{\delta+f(-\delta)\}$, $\quad\delta=(s-\tfrac{1}{2}\nu-\tfrac{1}{8})\pi$

For $\nu=0$ these expressions give the sth zero of each function; for other values of ν the zeros represented may not be the sth.

Uniform Asymptotic Expansions for Large Orders

When ν is large and positive

9.10.37

$$\operatorname{ber}_\nu(\nu x)+i\operatorname{bei}_\nu(\nu x)\sim \frac{e^{\nu\xi}}{\sqrt{2\pi\nu\xi}}\left(\frac{xe^{3\pi i/4}}{1+\xi}\right)^{\nu}\left\{1+\sum_{k=1}^{\infty}\frac{u_k(\xi^{-1})}{\nu^k}\right\}$$

9.10.38

$$\operatorname{ker}_\nu(\nu x)+i\operatorname{kei}_\nu(\nu x)\sim\sqrt{\frac{\pi}{2\nu\xi}}\,e^{-\nu\xi}\left(\frac{xe^{3\pi i/4}}{1+\xi}\right)^{-\nu}\left\{1+\sum_{k=1}^{\infty}(-)^k\frac{u_k(\xi^{-1})}{\nu^k}\right\}$$

9.10.39

$$\operatorname{ber}'_\nu(\nu x)+i\operatorname{bei}'_\nu(\nu x)\sim\sqrt{\frac{\xi}{2\pi\nu}}\,\frac{e^{\nu\xi}}{x}\left(\frac{xe^{3\pi i/4}}{1+\xi}\right)^{\nu}\left\{1+\sum_{k=1}^{\infty}\frac{v_k(\xi^{-1})}{\nu^k}\right\}$$

9.10.40

$$\operatorname{ker}'_\nu(\nu x)+i\operatorname{kei}'_\nu(\nu x)\sim-\sqrt{\frac{\pi\xi}{2\nu}}\,\frac{e^{-\nu\xi}}{x}\left(\frac{xe^{3\pi i/4}}{1+\xi}\right)^{-\nu}\left\{1+\sum_{k=1}^{\infty}(-)^k\frac{v_k(\xi^{-1})}{\nu^k}\right\}$$

where

9.10.41 $$\xi=\sqrt{1+ix^2}$$

and $u_k(t)$, $v_k(t)$ are given by **9.3.9** and **9.3.13.** All fractional powers take their principal values.

9.11. Polynomial Approximations

9.11.1 $$-8\leq x\leq 8$$

$$\begin{aligned}\operatorname{ber} x=1&-64(x/8)^4+113.77777\;774(x/8)^8\\&-32.36345\;652(x/8)^{12}+2.64191\;397(x/8)^{16}\\&-.08349\;609(x/8)^{20}+.00122\;552(x/8)^{24}\\&-.00000\;901(x/8)^{28}+\epsilon\end{aligned}$$

$$|\epsilon|<1\times10^{-9}$$

9.11.2 $$-8\leq x\leq 8$$

$$\begin{aligned}\operatorname{bei} x=16(x/8)^2&-113.77777\;774(x/8)^6\\&+72.81777\;742(x/8)^{10}-10.56765\;779(x/8)^{14}\\&+.52185\;615(x/8)^{18}-.01103\;667(x/8)^{22}\\&+.00011\;346(x/8)^{26}+\epsilon\end{aligned}$$

$$|\epsilon|<6\times10^{-9}$$

9.11.3 $$0<x\leq 8$$

$$\begin{aligned}\operatorname{ker} x=-\ln(\tfrac{1}{2}x)\operatorname{ber} x&+\tfrac{1}{4}\pi\operatorname{bei} x-.57721\;566\\&-59.05819\;744(x/8)^4+171.36272\;133(x/8)^8\\&-60.60977\;451(x/8)^{12}+5.65539\;121(x/8)^{16}\\&-.19636\;347(x/8)^{20}+.00309\;699(x/8)^{24}\\&-.00002\;458(x/8)^{28}+\epsilon\end{aligned}$$

$$|\epsilon|<1\times10^{-8}$$

9.11.4 $$0<x\leq 8$$

$$\begin{aligned}\operatorname{kei} x=-\ln(\tfrac{1}{2}x)\operatorname{bei} x&-\tfrac{1}{4}\pi\operatorname{ber} x+6.76454\;936(x/8)^2\\&-142.91827\;687(x/8)^6+124.23569\;650(x/8)^{10}\\&-21.30060\;904(x/8)^{14}+1.17509\;064(x/8)^{18}\\&-.02695\;875(x/8)^{22}+.00029\;532(x/8)^{26}+\epsilon\end{aligned}$$

$$|\epsilon|<3\times10^{-9}$$

9.11.5 $$-8\leq x\leq 8$$

$$\begin{aligned}\operatorname{ber}' x=x[&-4(x/8)^2+14.22222\;222(x/8)^6\\&-6.06814\;810(x/8)^{10}+.66047\;849(x/8)^{14}\\&-.02609\;253(x/8)^{18}+.00045\;957(x/8)^{22}\\&-.00000\;394(x/8)^{26}]+\epsilon\end{aligned}$$

$$|\epsilon|<2.1\times10^{-8}$$

9.11.6 $$-8\leq x\leq 8$$

$$\begin{aligned}\operatorname{bei}' x=x[&\tfrac{1}{2}-10.66666\;666(x/8)^4\\&+11.37777\;772(x/8)^8-2.31167\;514(x/8)^{12}\\&+.14677\;204(x/8)^{16}-.00379\;386(x/8)^{20}\\&+.00004\;609(x/8)^{24}]+\epsilon\end{aligned}$$

$$|\epsilon|<7\times10^{-8}$$

9.11.7 $$0<x\leq 8$$

$$\begin{aligned}\operatorname{ker}' x=-\ln(\tfrac{1}{2}x)\operatorname{ber}' x&-x^{-1}\operatorname{ber} x+\tfrac{1}{4}\pi\operatorname{bei}' x\\&+x[-3.69113\;734(x/8)^2+21.42034\;017(x/8)^6\\&-11.36433\;272(x/8)^{10}+1.41384\;780(x/8)^{14}\\&-.06136\;358(x/8)^{18}+.00116\;137(x/8)^{22}\\&-.00001\;075(x/8)^{26}]+\epsilon\end{aligned}$$

$$|\epsilon|<8\times10^{-8}$$

9.11.8 $0<x\leq 8$

$$\text{kei}'\,x=-\ln\left(\tfrac{1}{2}x\right)\text{bei}'\,x-x^{-1}\,\text{bei}\,x-\tfrac{1}{4}\pi\,\text{ber}'\,x$$
$$+x[.21139\,217-13.39858\,846(x/8)^4$$
$$+19.41182\,758(x/8)^8-4.65950\,823(x/8)^{12}$$
$$+.33049\,424(x/8)^{16}-.00926\,707(x/8)^{20}$$
$$+.00011\,997(x/8)^{24}]+\epsilon$$

$$|\epsilon|<7\times 10^{-8}$$

9.11.9 $8\leq x<\infty$

$$\text{ker}\,x+i\,\text{kei}\,x=f(x)(1+\epsilon_1)$$

$$f(x)=\sqrt{\frac{\pi}{2x}}\exp\left[-\frac{1+i}{\sqrt{2}}\,x+\theta(-x)\right]$$

$$|\epsilon_1|<1\times 10^{-7}$$

9.11.10 $8\leq x<\infty$

$$\text{ber}\,x+i\,\text{bei}\,x-\frac{i}{\pi}\,(\text{ker}\,x+i\,\text{kei}\,x)=g(x)(1+\epsilon_2)$$

$$g(x)=\frac{1}{\sqrt{2\pi x}}\exp\left[\frac{1+i}{\sqrt{2}}\,x+\theta(x)\right]$$

$$|\epsilon_2|<3\times 10^{-7}$$

where

9.11.11

$$\theta(x)=(.00000\,00-.39269\,91i)$$
$$+(.01104\,86-.01104\,85i)(8/x)$$
$$+(.00000\,00-.00097\,65i)(8/x)^2$$
$$+(-.00009\,06-.00009\,01i)(8/x)^3$$
$$+(-.00002\,52+.00000\,00i)(8/x)^4$$
$$+(-.00000\,34+.00000\,51i)(8/x)^5$$
$$+(.00000\,06+.00000\,19i)(8/x)^6$$

9.11.12 $8\leq x<\infty$

$$\text{ker}'\,x+i\,\text{kei}'\,x=-f(x)\phi(-x)(1+\epsilon_3)$$

$$|\epsilon_3|<2\times 10^{-7}$$

9.11.13 $8\leq x<\infty$

$$\text{ber}'\,x+i\,\text{bei}'\,x-\frac{i}{\pi}\,(\text{ker}'\,x+i\,\text{kei}'\,x)=g(x)\phi(x)(1+\epsilon_4)$$

$$|\epsilon_4|<3\times 10^{-7}$$

where

9.11.14

$$\phi(x)=(.70710\,68+.70710\,68i)$$
$$+(-.06250\,01-.00000\,01i)(8/x)$$
$$+(-.00138\,13+.00138\,11i)(8/x)^2$$
$$+(.00000\,05+.00024\,52i)(8/x)^3$$
$$+(.00003\,46+.00003\,38i)(8/x)^4$$
$$+(.00001\,17-.00000\,24i)(8/x)^5$$
$$+(.00000\,16-.00000\,32i)(8/x)^6$$

Numerical Methods

9.12. Use and Extension of the Tables

Example 1. To evaluate $J_n(1.55)$, $n=0, 1, 2, \ldots$, each to 5 decimals.

The recurrence relation

$$J_{n-1}(x)+J_{n+1}(x)=(2n/x)J_n(x)$$

can be used to compute $J_0(x)$, $J_1(x)$, $J_2(x)$, . . ., successively provided that $n<x$, otherwise severe accumulation of rounding errors will occur. Since, however, $J_n(x)$ is a decreasing function of n when $n>x$, recurrence can always be carried out in the direction of decreasing n.

Inspection of **Table 9.2** shows that $J_n(1.55)$ vanishes to 5 decimals when $n>7$. Taking arbitrary values zero for J_9 and unity for J_8, we compute by recurrence the entries in the second column of the following table, rounding off to the nearest integer at each step.

n	*Trial values*	$J_n(1.55)$
9	0	.00000
8	1	.00000
7	10	.00003
6	89	.00028
5	679	.00211
4	4292	.01331
3	21473	.06661
2	78829	.24453
1	181957	.56442
0	155954	.48376

We normalize the results by use of the equation **9.1.46**, namely

$$J_0(x)+2J_2(x)+2J_4(x)+\ .\ .\ .\ =1$$

This yields the normalization factor

$$1/322376=.00000\ 31019\ 7$$

and multiplying the trial values by this factor we obtain the required results, given in the third column. As a check we may verify the value of $J_0(1.55)$ by interpolation in **Table 9.1.**

Remarks. (i) In this example it was possible to estimate immediately the value of $n=N$, say, at which to begin the recurrence. This may not always be the case and an arbitrary value of N may have to be taken. The number of correct significant figures in the final values is the same as the number of digits in the respective trial values. If the chosen N is too small the trial values will have too few digits and insufficient accuracy is obtained in the results. The calculation must then be repeated taking a higher value. On the other hand if N were too large unnecessary effort would be expended. This could be offset to some extent by discarding significant figures in the trial values which are in excess of the number of decimals required in J_n.

(ii) If we had required, say, $J_0(1.55)$, $J_1(1.55)$, . . ., $J_{10}(1.55)$, each to 5 significant figures, we would have found the values of $J_{10}(1.55)$ and $J_{11}(1.55)$ to 5 significant figures by interpolation in **Table 9.3** and then computed by recurrence J_9, J_8, . . ., J_0, no normalization being required. Alternatively, we could begin the recurrence at a higher value of N and retain only 5 significant figures in the trial values for $n \leq 10$.

(iii) Exactly similar methods can be used to compute the modified Bessel function $I_n(x)$ by means of the relations **9.6.26** and **9.6.36.** If x is large, however, considerable cancellation will take place in using the latter equation, and it is preferable to normalize by means of **9.6.37.**

Example 2. To evaluate $Y_n(1.55)$, $n=0, 1, 2, \ldots, 10$, each to 5 significant figures.

The recurrence relation

$$Y_{n-1}(x)+Y_{n+1}(x)=(2n/x)Y_n(x)$$

can be used to compute $Y_n(x)$ in the direction of increasing n both for $n<x$ and $n>x$, because in the latter event $Y_n(x)$ is a numerically increasing function of n.

We therefore compute $Y_0(1.55)$ and $Y_1(1.55)$ by interpolation in **Table 9.1**, generate $Y_2(1.55)$, $Y_3(1.55)$, . . ., $Y_{10}(1.55)$ by recurrence and check $Y_{10}(1.55)$ by interpolation in **Table 9.3.**

n	$Y_n(1.55)$	n	$Y_n(1.55)$
0	+0. 40225	6	-1.9917×10^2
1	−0. 37970	7	-1.5100×10^3
2	−0. 89218	8	-1.3440×10^4
3	−1. 9227	9	-1.3722×10^5
4	−6. 5505	10	-1.5801×10^6
5	−31. 886		

Remarks. (i) An alternative way of computing $Y_0(x)$, should $J_0(x)$, $J_2(x)$, $J_4(x)$, . . ., be available (see **Example 1**), is to use formula **9.1.89.** The other starting value for the recurrence, $Y_1(x)$, can then be found from the Wronskian relation $J_1(x)Y_0(x)-J_0(x)Y_1(x)=2/(\pi x)$. This is a convenient procedure for use with an automatic computer.

(ii) Similar methods can be used to compute the modified Bessel function $K_n(x)$ by means of the recurrence relation **9.6.26** and the relation **9.6.54**, except that if x is large severe cancellation will occur in the use of **9.6.54** and other methods for evaluating $K_0(x)$ may be preferable, for example, use of the asymptotic expansion **9.7.2** or the polynomial approximation **9.8.6.**

Example 3. To evaluate $J_0(.36)$ and $Y_0(.36)$ each to 5 decimals, using the multiplication theorem.

From **9.1.74** we have

$$\mathscr{C}_0(\lambda z)=\sum_{k=0}^{\infty}a_k\mathscr{C}_k(z), \text{ where } a_k=\frac{(-)^k(\lambda^2-1)^k(\frac{1}{2}z)^k}{k!}.$$

We take $z=.4$. Then $\lambda=.9$, $(\lambda^2-1)(\frac{1}{2}z)=-.038$, and extracting the necessary values of $J_k(.4)$ and $Y_k(.4)$ from **Tables 9.1** and **9.2,** we compute the required results as follows:

k	a_k	$a_kJ_k(.4)$	$a_kY_k(.4)$
0	$+1.0$	+.96040	−.60602
1	$+0.038$	+.00745	−.06767
2	$+0.7220\times10^{-3}$	+.00001	−.00599
3	$+0.914\times10^{-5}$		−.00074
4	$+0.87\times10^{-7}$		−.00011
5	$+0.7\times10^{-9}$		−.00002
		$J_0(.36)=+.96786$	$Y_0(.36)=-.68055$

Remark. This procedure is equivalent to interpolating by means of the Taylor series

$$\mathscr{C}_0(z+h)=\sum_{k=0}^{\infty}\frac{h^k}{k!}\mathscr{C}_0^{(k)}(z)$$

at $z=.4$, and expressing the derivatives $\mathscr{C}_0^{(k)}(z)$ in terms of $\mathscr{C}_k(z)$ by means of the recurrence relations and differential equation for the Bessel functions.

Example 4. To evaluate $J_\nu(x)$, $J'_\nu(x)$, $Y_\nu(x)$ and $Y'_\nu(x)$ for $\nu=50$, $x=75$, each to 6 decimals.

We use the asymptotic expansions **9.3.35, 9.3.36, 9.3.43,** and **9.3.44.** Here $z=x/\nu=3/2$. From **9.3.39** we find

$$\frac{2}{3}(-\zeta)^{3/2}=\frac{1}{2}\sqrt{5}-\arccos\frac{2}{3}=+.2769653.$$

Hence

$$\zeta=-.5567724 \text{ and } \left(\frac{4\zeta}{1-z^2}\right)^{1/4}=+1.155332.$$

Next,

$$\nu^{1/3}=3.684031, \qquad \nu^{2/3}\zeta=-7.556562.$$

Interpolating in **Table 10.11**, we find that

$$\mathrm{Ai}(\nu^{2/3}\zeta)=+.299953, \qquad \mathrm{Ai}'(\nu^{2/3}\zeta)=+.451441,$$

$$\mathrm{Bi}(\nu^{2/3}\zeta)=-.160565, \qquad \mathrm{Bi}'(\nu^{2/3}\zeta)=+.819542.$$

As a check on the interpolation, we may verify that $\mathrm{Ai}\,\mathrm{Bi}'-\mathrm{Ai}'\mathrm{Bi}=1/\pi$.

Interpolating in the table following **9.3.46** we obtain

$$b_0(\zeta)=+.0136, \qquad c_0(\zeta)=+.1442.$$

The contributions of the terms involving $a_1(\zeta)$ and $d_1(\zeta)$ are negligible, and substituting in the asymptotic expansions we find that

$$J_{50}(75)=+1.155332(50^{-1/3}\times.299953 \\ +50^{-5/3}\times.451441\times.0136)=+.094077,$$

$$J'_{50}(75)=-(4/3)(1.155332)^{-1}(50^{-4/3}\times.299953 \\ \times.1442+50^{-2/3}\times.451441)=-.038658,$$

$$Y_{50}(75)=-1.155332(-50^{-1/3}\times.160565 \\ +50^{-5/3}\times.819542\times.0136)=+.050335,$$

$$Y'_{50}(75)=+(4/3)(1.155332)^{-1}(-50^{-4/3}\times.160565 \\ \times.1442+50^{-2/3}\times.819542)=+.069543.$$

As a check we may verify that

$$JY'-J'Y=2/(75\pi).$$

Remarks. This example may also be computed using the Debye expansions **9.3.15, 9.3.16, 9.3.19,** and **9.3.20.** Four terms of each of these series are required, compared with two in the computations above. The closer the argument-order ratio is to unity, the less effective the Debye expansions become. In the neighborhood of unity the expansions **9.3.23, 9.3.24, 9.3.27,** and **9.3.28** will furnish results of moderate accuracy; for high-accuracy work the uniform expansions should again be used.

Example 5. To evaluate the 5th positive zero of $J_{10}(x)$ and the corresponding value of $J'_{10}(x)$, each to 5 decimals.

We use the asymptotic expansions **9.5.22** and **9.5.23** setting $\nu=10$, $s=5$. From **Table 10.11** we find

$$a_5=-7.944134, \qquad \mathrm{Ai}'(a_5)=+.947336.$$

Hence

$$\zeta=10^{-2/3}a_5=.21544347a_5=-1.7115118.$$

Interpolating in the table following **9.5.26** we obtain

$$z(\zeta)=+2.888631, \qquad h(\zeta)=+.98259,$$
$$f_1(\zeta)=+.0107, \qquad F_1(\zeta)=-.001.$$

The bounds given at the foot of the table show that the contributions of higher terms to the asymptotic series are negligible. Hence

$$j_{10,5}=28.88631+.00107+\ .\ .\ .\ =28.88738,$$

$$J'_{10}(j_{10,5})=-\frac{2}{10^{2/3}}\,\frac{.947336}{2.888631\times.98259} \\ \times(1-.00001+\ .\ .\ .)=-.14381.$$

Example 6. To evaluate the first root of $J_0(x)Y_0(\lambda x)-Y_0(x)J_0(\lambda x)=0$ for $\lambda=\frac{3}{2}$ to 4 significant figures.

Let $\alpha_\lambda^{(1)}$ denote the root. Direct interpolation in **Table 9.7** is impracticable owing to the divergence of the differences. Inspection of **9.5.28** suggests that a smoother function is $(\lambda-1)\alpha_\lambda^{(1)}$. Using **Table 9.7** we compute the following values

$1/\lambda$	$(\lambda-1)\alpha_\lambda^{(1)}$	δ	δ^2
0. 4	3. 110		
		+21	
0. 6	3. 131		−12
		+9	
0. 8	3. 140		−7
		+2	
1. 0	3. 142(π)		

Interpolating for $1/\lambda=.667$, we obtain $(\lambda-1)\alpha_\lambda^{(1)}=3.134$ and thence the required root $\alpha_{1.5}^{(1)}=6.268$.

Example 7. To evaluate $\mathrm{ber}_n\,1.55$, $\mathrm{bei}_n\,1.55$, $n=0, 1, 2, \ldots$, each to 5 decimals.

We use the recurrence relation

$$J_{n-1}(xe^{3\pi i/4})+J_{n+1}(xe^{3\pi i/4}) \\ =-\frac{n\sqrt{2}}{x}(1+i)J_n(xe^{3\pi i/4}),$$

taking arbitrary values zero for $J_9(xe^{3\pi i/4})$ and $1+0i$ for $J_8(xe^{3\pi i/4})$ (see **Example 1**).

n	Real trial values	Imag. trial values	$\text{ber}_n x$	$\text{bei}_n x$
9	0	0	.00000	.00000
8	+1	0	.00000	.00000
7	−7	−7	−.00002	−.00003
6	−1	+89	−.00003	+.00030
5	+500	−475	+.00181	−.00148
4	−4447	−203	−.01494	−.00180
3	+14989	+17446	+.04614	+.06258
2	+11172	−88578	+.05994	−.29580
1	−197012	+123804	−.69531	+.36781
0	+281539	+155373	+.91004	+.59461
Σ	+106734	+207449	+.30763	+.72619

The values of $\text{ber}_n x$ and $\text{bei}_n x$ are computed by multiplication of the trial values by the normalizing factor

$$1/(294989-22011i)=(.337119+.025155i)\times 10^{-5},$$

obtained from the relation

$$J_0(xe^{3\pi i/4})+2J_2(xe^{3\pi i/4})+2J_4(xe^{3\pi i/4})+\ldots=1.$$

Adequate checks are furnished by interpolating in **Table 9.12** for ber 1.55 and bei 1.55, and the use of a simple sum check on the normalization.

Should $\text{ker}_n x$ and $\text{kei}_n x$ be required they can be computed by forward recurrence using formulas **9.9.14**, taking the required starting values for $n=0$ and 1 from **Table 9.12** (see **Example 2**). If an independent check on the recurrence is required the asymptotic expansion **9.10.38** can be used.

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10. Bessel Functions of Fractional Order

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Contents

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10. Bessel Functions of Fractional Order

Mathematical Properties

10.1. Spherical Bessel Functions

Definitions

Differential Equation

10.1.1

$$z^2w''+2zw'+[z^2-n(n+1)]w=0 \qquad (n=0,\pm 1,\ \pm 2,\ \ldots)$$

Particular solutions are the *Spherical Bessel functions of the first kind*

$$j_n(z)=\sqrt{\tfrac{1}{2}\pi/z}J_{n+\frac{1}{2}}(z),$$

the *Spherical Bessel functions of the second kind*

$$y_n(z)=\sqrt{\tfrac{1}{2}\pi/z}Y_{n+\frac{1}{2}}(z),$$

and the *Spherical Bessel functions of the third kind*

$$h_n^{(1)}(z)=j_n(z)+iy_n(z)=\sqrt{\tfrac{1}{2}\pi/z}H_{n+\frac{1}{2}}^{(1)}(z),$$

$$h_n^{(2)}(z)=j_n(z)-iy_n(z)=\sqrt{\tfrac{1}{2}\pi/z}H_{n+\frac{1}{2}}^{(2)}(z).$$

The pairs $j_n(z)$, $y_n(z)$ and $h_n^{(1)}(z)$, $h_n^{(2)}(z)$ are linearly independent solutions for every n. For general properties see the remarks after **9.1.1.**

Ascending Series (See 9.1.2, 9.1.10)

10.1.2

$$j_n(z)=\frac{z^n}{1\cdot 3\cdot 5\ \ldots\ (2n+1)}\left\{1-\frac{\frac{1}{2}z^2}{1!(2n+3)}+\frac{(\frac{1}{2}z^2)^2}{2!(2n+3)(2n+5)}-\ldots\right\}$$

10.1.3

$$y_n(z)=-\frac{1\cdot 3\cdot 5\ \ldots\ (2n-1)}{z^{n+1}}\left\{1-\frac{\frac{1}{2}z^2}{1!(1-2n)}+\frac{(\frac{1}{2}z^2)^2}{2!(1-2n)(3-2n)}-\ldots\right\}$$

$$(n=0,\ 1,\ 2,\ \ldots)$$

Limiting Values as $z\to 0$

10.1.4 $$z^{-n}j_n(z)\to\frac{1}{1\cdot 3\cdot 5\ \ldots\ (2n+1)}$$

10.1.5

$$z^{n+1}y_n(z)\to -1\cdot 3\cdot 5\ \ldots\ (2n-1) \qquad (n=0,\ 1,\ 2,\ \ldots)$$

Wronskians

10.1.6 $$W\{j_n(z),\ y_n(z)\}=z^{-2}$$

10.1.7

$$W\{h_n^{(1)}(z),\ h_n^{(2)}(z)\}=-2iz^{-2} \qquad (n=0,\ 1,\ 2,\ \ldots)$$

Representations by Elementary Functions

10.1.8

$$j_n(z)=z^{-1}[P(n+\tfrac{1}{2},\ z)\sin(z-\tfrac{1}{2}n\pi)+Q(n+\tfrac{1}{2},\ z)\cos(z-\tfrac{1}{2}n\pi)]$$

10.1.9

$$y_n(z)=(-1)^{n+1}z^{-1}[P(n+\tfrac{1}{2},\ z)\cos(z+\tfrac{1}{2}n\pi)-Q(n+\tfrac{1}{2},\ z)\sin(z+\tfrac{1}{2}n\pi)]$$

$$P(n+\tfrac{1}{2},\ z)=1-\frac{(n+2)!}{2!\,\Gamma(n-1)}(2z)^{-2}+\frac{(n+4)!}{4!\,\Gamma(n-3)}(2z)^{-4}-\ldots$$

$$=\sum_{0}^{[\frac{1}{2}n]}(-1)^k(n+\tfrac{1}{2},2k)(2z)^{-2k}$$

$$Q(n+\tfrac{1}{2},\ z)=\frac{(n+1)!}{1!\,\Gamma(n)}(2z)^{-1}-\frac{(n+3)!}{3!\,\Gamma(n-2)}(2z)^{-3}+\frac{(n+5)!}{5!\,\Gamma(n-4)}(2z)^{-5}-\ldots$$

$$=\sum_{0}^{[\frac{1}{2}(n-1)]}(-1)^k(n+\tfrac{1}{2},2k+1)(2z)^{-2k-1}$$

$$(n=0,1,2,\ \ldots)$$

$$(n+\tfrac{1}{2},\ k)=\frac{(n+k)!}{k!\,\Gamma(n-k+1)}$$

n \ k	1	2	3	4	5
1	2				
2	6	12			
3	12	60	120		
4	20	180	840	1680	
5	30	420	3360	15120	30240

10.1.10

$$j_n(z)=f_n(z)\sin z+(-1)^{n+1}f_{-n-1}(z)\cos z$$

$$f_0(z)=z^{-1},\qquad f_1(z)=z^{-2}$$

$$f_{n-1}(z)+f_{n+1}(z)=(2n+1)z^{-1}f_n(z)$$

$$(n=0,\ \pm1,\ \pm2,\ \ldots)$$

The Functions $j_n(z)$, $y_n(z)$ for $n=0, 1, 2$

10.1.11 $$j_0(z)=\frac{\sin z}{z}$$

$$j_1(z)=\frac{\sin z}{z^2}-\frac{\cos z}{z}$$

$$j_2(z)=\left(\frac{3}{z^3}-\frac{1}{z}\right)\sin z-\frac{3}{z^2}\cos z$$ *

10.1.12

$$y_0(z)=-j_{-1}(z)=-\frac{\cos z}{z}$$

$$y_1(z)=j_{-2}(z)=-\frac{\cos z}{z^2}-\frac{\sin z}{z}$$

$$y_2(z)=-j_{-3}(z)=\left(-\frac{3}{z^3}+\frac{1}{z}\right)\cos z-\frac{3}{z^2}\sin z$$ *

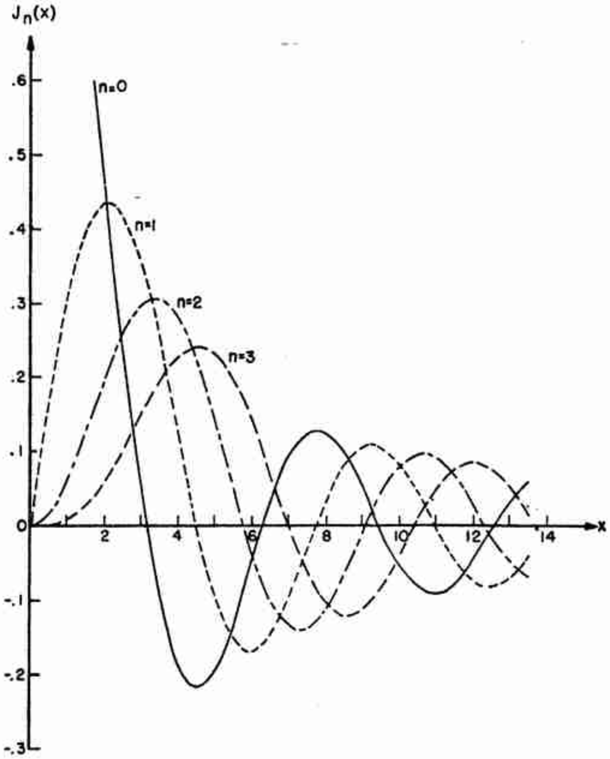

FIGURE 10.1. $j_n(x)$. $n=0(1)3$.

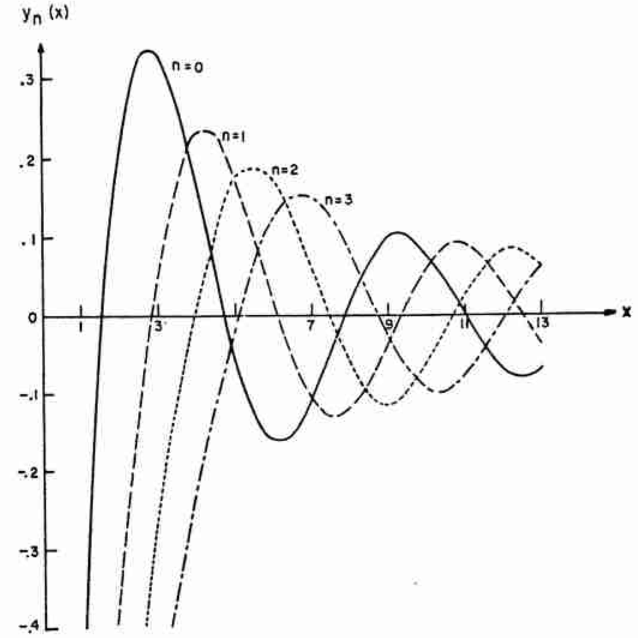

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FIGURE 10.2. $y_n(x)$. $n=0(1)3$.

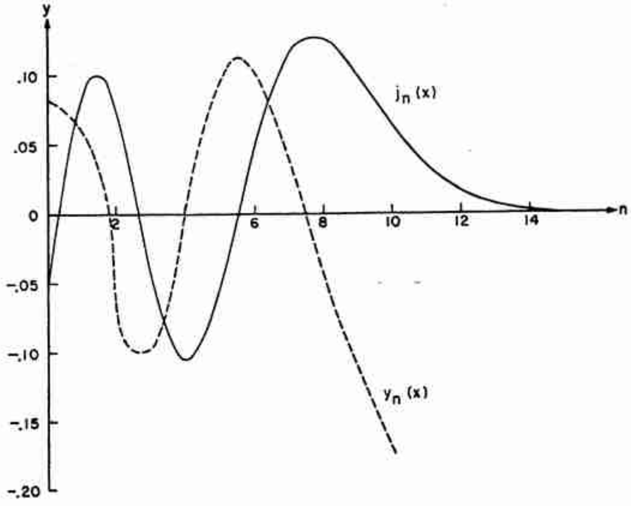

FIGURE 10.3. $j_n(x)$, $y_n(x)$. $x=10$.

Poisson's Integral and Gegenbauer's Generalization

10.1.13 $$j_n(z)=\frac{z^n}{2^{n+1}n!}\int_0^\pi \cos(z\cos\theta)\sin^{2n+1}\theta\,d\theta$$

(See **9.1.20.**)

10.1.14

$$=\frac{1}{2}(-i)^n\int_0^\pi e^{iz\cos\theta}P_n(\cos\theta)\sin\theta\,d\theta$$

$$(n=0,\ 1,\ 2,\ \ldots)$$

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*See page II.

Spherical Bessel Functions of the Second and Third Kind

10.1.15

$$y_n(z)=(-1)^{n+1}j_{-n-1}(z) \qquad (n=0, \pm 1, \pm 2, \ldots)$$

10.1.16

$$h_n^{(1)}(z)=i^{-n-1}z^{-1}e^{iz}\sum_0^n (n+\tfrac{1}{2}, k)(-2iz)^{-k}$$

10.1.17

$$h_n^{(2)}(z)=i^{n+1}z^{-1}e^{-iz}\sum_0^n (n+\tfrac{1}{2}, k)\ (2iz)^{-k} \qquad *$$

10.1.18

$$h_{-n-1}^{(1)}(z)=i(-1)^n h_n^{(1)}(z)$$
$$h_{-n-1}^{(2)}(z)=-i(-1)^n h_n^{(2)}(z) \qquad (n=0, 1, 2, \ldots)$$

Elementary Properties

Recurrence Relations

$f_n(z): j_n(z), y_n(z), h_n^{(1)}(z), h_n^{(2)}(z)$

$(n=0, \pm 1, \pm 2, \ldots)$

10.1.19 $$f_{n-1}(z)+f_{n+1}(z)=(2n+1)z^{-1}f_n(z)$$

10.1.20 $$nf_{n-1}(z)-(n+1)f_{n+1}(z)=(2n+1)\frac{d}{dz}f_n(z)$$

10.1.21 $$\frac{n+1}{z}f_n(z)+\frac{d}{dz}f_n(z)=f_{n-1}(z)$$

(See **10.1.23.**)

10.1.22 $$\frac{n}{z}f_n(z)-\frac{d}{dz}f_n(z)=f_{n+1}(z)$$

(See **10.1.24.**)

Differentiation Formulas

$f_n(z): j_n(z), y_n(z), h_n^{(1)}(z), h_n^{(2)}(z)$

$(n=0, \pm 1, \pm 2, \ldots)$

10.1.23 $$\left(\frac{1}{z}\frac{d}{dz}\right)^m [z^{n+1}f_n(z)]=z^{n-m+1}f_{n-m}(z)$$

10.1.24 $$\left(\frac{1}{z}\frac{d}{dz}\right)^m [z^{-n}f_n(z)]=(-1)^m z^{-n-m}f_{n+m}(z)$$

$(m=1, 2, 3, \ldots)$

Rayleigh's Formulas

10.1.25

$$j_n(z)=z^n\left(-\frac{1}{z}\frac{d}{dz}\right)^n\frac{\sin z}{z}$$

10.1.26

$$y_n(z)=-z^n\left(-\frac{1}{z}\frac{d}{dz}\right)^n\frac{\cos z}{z} \qquad (n=0, 1, 2, \ldots)$$

Modulus and Phase

$$j_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,M_{n+\frac{1}{2}}(z)\cos\theta_{n+\frac{1}{2}}(z),$$
$$y_n(z)=\sqrt{\tfrac{1}{2}\pi/z}\,M_{n+\frac{1}{2}}(z)\sin\theta_{n+\frac{1}{2}}(z)$$

(See **9.2.17.**)

10.1.27

$$(\tfrac{1}{2}\pi/z)\,M_{n+\frac{1}{2}}^2(z)=\frac{1}{z^2}\sum_0^n\frac{(2n-k)!(2n-2k)!}{k![(n-k)!]^2}(2z)^{2k-2n}$$

(See **9.2.28.**)

10.1.28 $$(\tfrac{1}{2}\pi/z)M_{1/2}^2(z)=j_0^2(z)+y_0^2(z)=z^{-2}$$

10.1.29

$$(\tfrac{1}{2}\pi/z)M_{3/2}^2(z)=j_1^2(z)+y_1^2(z)=z^{-2}+z^{-4}$$

10.1.30

$$(\tfrac{1}{2}\pi/z)M_{5/2}^2(z)=j_2^2(z)+y_2^2(z)=z^{-2}+3z^{-4}+9z^{-6}$$

Cross Products

10.1.31 $$j_n(z)y_{n-1}(z)-j_{n-1}(z)y_n(z)=z^{-2}$$

10.1.32

$$j_{n+1}(z)y_{n-1}(z)-j_{n-1}(z)y_{n+1}(z)=(2n+1)z^{-3}$$

10.1.33

$$j_0(z)j_n(z)+y_0(z)y_n(z)$$
$$=z^{-2}\sum_0^{[\frac{1}{2}n]}(-1)^k 2^{n-2k}\left(k+\tfrac{1}{2}\right)_{n-2k}\binom{n-k}{k}z^{2k-n}$$

$(n=0, 1, 2, \ldots)$

Analytic Continuation

10.1.34 $$j_n(ze^{m\pi i})=e^{mn\pi i}j_n(z)$$

10.1.35 $$y_n(ze^{m\pi i})=(-1)^m e^{mn\pi i}y_n(z)$$

10.1.36 $$h_n^{(1)}(ze^{(2m+1)\pi i})=(-1)^n h_n^{(2)}(z)$$

10.1.37 $$h_n^{(2)}(ze^{(2m+1)\pi i})=(-1)^n h_n^{(1)}(z)$$

10.1.38 $$h_n^{(l)}(ze^{2m\pi i})=h_n^{(l)}(z)$$

$(l=1, 2; m, n=0, 1, 2, \ldots)$

Generating Functions

10.1.39

$$\frac{1}{z}\sin\sqrt{z^2+2zt}=\sum_0^\infty\frac{(-t)^n}{n!}y_{n-1}(z) \qquad (2|t|<|z|)$$

10.1.40 $$\frac{1}{z}\cos\sqrt{z^2-2zt}=\sum_0^\infty\frac{t^n}{n!}j_{n-1}(z)$$

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*See page II

Derivatives With Respect to Order

10.1.41

$$\left[\frac{\partial}{\partial\nu}j_\nu(x)\right]_{\nu=0}=(\tfrac{1}{2}\pi/x)\{\mathrm{Ci}(2x)\sin x-\mathrm{Si}(2x)\cos x\}$$

10.1.42

$$\left[\frac{\partial}{\partial\nu}j_\nu(x)\right]_{\nu=-1}=(\tfrac{1}{2}\pi/x)\{\mathrm{Ci}(2x)\cos x+\mathrm{Si}(2x)\sin x\}$$

10.1.43

$$\left[\frac{\partial}{\partial\nu}y_\nu(x)\right]_{\nu=0}=(\tfrac{1}{2}\pi/x)\{\mathrm{Ci}(2x)\cos x+[\mathrm{Si}(2x)-\pi]\sin x\}$$

10.1.44

$$\left[\frac{\partial}{\partial\nu}y_\nu(x)\right]_{\nu=-1}=$$

$$(\tfrac{1}{2}\pi/x)\{\mathrm{Ci}(2x)\sin x-[\mathrm{Si}(2x)-\pi]\cos x\}$$

Addition Theorems and Degenerate Forms

r, ρ, θ, λ arbitrary complex; $R=\sqrt{(r^2+\rho^2-2r\rho\cos\theta)}$

10.1.45 $$\frac{\sin\lambda R}{\lambda R}=\sum_0^\infty(2n+1)j_n(\lambda r)j_n(\lambda\rho)P_n(\cos\theta)$$

***10.1.46** $$-\frac{\cos\lambda R}{\lambda R}=\sum_0^\infty(2n+1)j_n(\lambda r)y_n(\lambda\rho)P_n(\cos\theta)$$

$$|re^{\pm i\theta}|<|\rho|$$

10.1.47 $$e^{iz\cos\theta}=\sum_0^\infty(2n+1)e^{\frac{1}{2}n\pi i}j_n(z)P_n(\cos\theta)$$

10.1.48

$$J_0(z\sin\theta)=\sum_0^\infty(4n+1)\frac{(2n)!}{2^{2n}(n!)^2}j_{2n}(z)P_{2n}(\cos\theta)$$

Duplication Formula

10.1.49

$$j_n(2z)=$$

* $$-n!z^{n+1}\sum_0^n\frac{2n-2k+1}{k!(2n-k+1)!}j_{n-k}(z)y_{n-k}(z)$$

Some Infinite Series Involving $j_n^2(z)$

10.1.50 $$\sum_0^\infty(2n+1)j_n^2(z)=1$$

10.1.51 $$\sum_0^\infty(-1)^n(2n+1)j_n^2(z)=\frac{\sin 2z}{2z}$$

10.1.52 $$\sum_0^\infty j_n^2(z)=\frac{\mathrm{Si}(2z)}{2z}$$

Fresnel Integrals

10.1.53

$$C(\sqrt{2x/\pi})=\frac{1}{2}\int_0^x J_{-\frac{1}{2}}(t)dt$$

$$=\sqrt{2}[\cos\tfrac{1}{2}x\sum_0^\infty(-1)^nJ_{2n+\frac{1}{2}}(\tfrac{1}{2}x)$$

$$+\sin\tfrac{1}{2}x\sum_0^\infty(-1)^nJ_{2n+3/2}(\tfrac{1}{2}x)]$$

10.1.54

$$S(\sqrt{2x/\pi})=\frac{1}{2}\int_0^x J_{\frac{1}{2}}(t)dt$$

$$=\sqrt{2}[\sin\tfrac{1}{2}x\sum_0^\infty(-1)^nJ_{2n+\frac{1}{2}}(\tfrac{1}{2}x)$$

$$-\cos\tfrac{1}{2}x\sum_0^\infty(-1)^nJ_{2n+3/2}(\tfrac{1}{2}x)].$$

(See also **11.1.1, 11.1.2.**)

Zeros and Their Asymptotic Expansions

The zeros of $j_n(x)$ and $y_n(x)$ are the same as the zeros of $J_{n+\frac{1}{2}}(x)$ and $Y_{n+\frac{1}{2}}(x)$ and the formulas for $j_{\nu,s}$ and $y_{\nu,s}$ given in **9.5** are applicable with $\nu=n+\frac{1}{2}$. There are, however, no simple relations connecting the zeros of the derivatives. Accordingly, we now give formulas for $a'_{n,s}$, $b'_{n,s}$, the s-th positive zero of $j'_n(z)$, $y'_n(z)$, respectively; $z=0$ is counted as the first zero of $j'_0(z)$.

(Tables of $a'_{n,s}$, $b'_{n,s}$, $j_n(a'_{n,s})$, $y_n(b'_{n,s})$ are given in [10.31].)

Elementary Relations

$$f_n(z)=j_n(z)\cos\pi t+y_n(z)\sin\pi t$$

(t a real parameter, $0\leq t\leq 1$)

If τ_n is a zero of $f'_n(z)$ then

10.1.55 $f_n(\tau_n)=[\tau_n/(n+1)]f_{n-1}(\tau_n)$

(See **10.1.21.**)

10.1.56 $=(\tau_n/n)f_{n+1}(\tau_n)$

(See **10.1.22.**)

10.1.57 $$=\left\{\frac{1}{\pi}[\tau_n^2-n(n+1)]\frac{d\tau_n}{d\tau}\right\}^{-\frac{1}{2}}$$

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*See page II.

McMahon's Expansions for n Fixed and s Large

10.1.58

$$a'_{n,s},\ b'_{n,s}\sim\beta-(\mu+7)(8\beta)^{-1}$$
$$-\frac{4}{3}(7\mu^2+154\mu+95)(8\beta)^{-3}$$
$$-\frac{32}{15}(85\mu^3+3535\mu^2+3561\mu+6133)(8\beta)^{-5}$$
$$-\frac{64}{105}(6949\mu^4+474908\mu^3+330638\mu^2$$
$$+9046780\mu-5075147)(8\beta)^{-7}-\ldots$$

$\beta=\pi(s+\frac{1}{2}n-\frac{1}{2})$ for $a'_{n,s}$, $\beta=\pi(s+\frac{1}{2}n)$ for $b'_{n,s}$;

$$\mu=(2n+1)^2$$

Asymptotic Expansions of Zeros and Associated Values for n Large

10.1.59

$$a'_{n,1}\sim(n+\tfrac{1}{2})+.8086165(n+\tfrac{1}{2})^{1/3}-.236680(n+\tfrac{1}{2})^{-1/3}$$
$$-.20736(n+\tfrac{1}{2})^{-1}+.0233(n+\tfrac{1}{2})^{-5/3}+\ldots$$

10.1.60

$$b'_{n,1}\sim(n+\tfrac{1}{2})+1.8210980(n+\tfrac{1}{2})^{1/3}$$
$$+.802728(n+\tfrac{1}{2})^{-1/3}-.11740(n+\tfrac{1}{2})^{-1}$$
$$+.0249(n+\tfrac{1}{2})^{-5/3}+\ldots$$

10.1.61

$$j_n(a'_{n,1})\sim.8458430(n+\tfrac{1}{2})^{-5/6}\{1-.566032(n+\tfrac{1}{2})^{-2/3}$$
$$+.38081(n+\tfrac{1}{2})^{-4/3}-.2203(n+\tfrac{1}{2})^{-2}+\ldots\}$$

10.1.62

$$y_n(b'_{n,1})\sim.7183921(n+\tfrac{1}{2})^{-5/6}\{1-1.274769(n+\tfrac{1}{2})^{-2/3}$$
$$+1.23038(n+\tfrac{1}{2})^{-4/3}-1.0070(n+\tfrac{1}{2})^{-2}+\ldots\}$$

See [10.31] for corresponding expansions for $s=2, 3$.

Uniform Asymptotic Expansions of Zeros and Associated Values for n Large

10.1.63

$$a'_{n,s}\sim(n+\tfrac{1}{2})\{z[(n+\tfrac{1}{2})^{-2/3}a'_s]$$
$$+\sum_{k=1}^{\infty}h_k[(n+\tfrac{1}{2})^{-2/3}a'_s](n+\tfrac{1}{2})^{-2k}\}$$

10.1.64

$$b'_{n,s}\sim(n+\tfrac{1}{2})\{z[(n+\tfrac{1}{2})^{-2/3}b'_s]$$
$$+\sum_{k=1}^{\infty}h_k[(n+\tfrac{1}{2})^{-2/3}b'_s](n+\tfrac{1}{2})^{-2k}\}$$

10.1.65

$$j_n(a'_{n,s})\sim\sqrt{\tfrac{1}{2}\pi}\,\mathrm{Ai}(a'_s)(n+\tfrac{1}{2})^{-5/6}$$
$$h[(n+\tfrac{1}{2})^{-2/3}a'_s](z[(n+\tfrac{1}{2})^{-2/3}a'_s])^{-1/2}$$
$$\{1+\sum_{k=1}^{\infty}H_k[(n+\tfrac{1}{2})^{-2/3}a'_s](n+\tfrac{1}{2})^{-2k}\}$$

10.1.66

$$y_n(b'_{n,s})\sim-\sqrt{\tfrac{1}{2}\pi}\,\mathrm{Bi}(b'_s)(n+\tfrac{1}{2})^{-5/6}$$
$$h[(n+\tfrac{1}{2})^{-2/3}b'_s](z[(n+\tfrac{1}{2})^{-2/3}b'_s])^{-1/2}$$
$$\{1+\sum_{k=1}^{\infty}H_k[(n+\tfrac{1}{2})^{-2/3}b'_s](n+\tfrac{1}{2})^{-2k}\}$$

$h(\xi)$, $z(\xi)$ are defined as in **9.5.26, 9.3.38, 9.3.39.** a'_s, b'_s s-th (negative) real zero of $\mathrm{Ai}'(z)$, $\mathrm{Bi}'(z)$ (see **10.4.95, 10.4.99.**)

Complex Zeros of $h_n^{(1)}(z)$, $h_n^{(1)\prime}(z)$

$h_n^{(1)}(z)$ and $h_n^{(1)}(ze^{2m\pi i})$, m any integer, have the same zeros.

$h_n^{(1)}(z)$ has n zeros, symmetrically distributed with respect to the imaginary axis and lying approximately on the finite arc joining $z=-n$ and $z=n$ shown in **Figure 9.6.** If n is odd, one zero lies on the imaginary axis.

$h_n^{(1)\prime}(z)$ has $n+1$ zeros lying approximately on the same curve. If n is even, one zero lies on the imaginary axis.

10.2. Modified Spherical Bessel Functions

Definitions

Differential Equation

10.2.1

$$z^2w''+2zw'-[z^2+n(n+1)]w=0 \qquad (n=0, \pm 1, \pm 2, \ldots)$$

Particular solutions are the *Modified Spherical Bessel functions of the first kind,*

10.2.2

$$\begin{aligned}\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z)&=e^{-n\pi i/2}j_n(ze^{\pi i/2}) \qquad (-\pi<\arg z\leq\tfrac{1}{2}\pi)\\&=e^{3n\pi i/2}j_n(ze^{-3\pi i/2}) \qquad (\tfrac{1}{2}\pi<\arg z\leq\pi)\end{aligned}$$

of the second kind,

10.2.3

$$\begin{aligned}\sqrt{\tfrac{1}{2}\pi/z}I_{-n-\frac{1}{2}}(z)&=e^{3(n+1)\pi i/2}y_n(ze^{\pi i/2}) \qquad (-\pi<\arg z\leq\tfrac{1}{2}\pi)\\&=e^{-(n+1)\pi i/2}y_n(ze^{-3\pi i/2}) \qquad (\tfrac{1}{2}\pi<\arg z\leq\pi)\end{aligned}$$

of the third kind,

10.2.4

$$\sqrt{\tfrac{1}{2}\pi/z}K_{n+\frac{1}{2}}(z)=\tfrac{1}{2}\pi(-1)^{n+1}\sqrt{\tfrac{1}{2}\pi/z}[I_{n+\frac{1}{2}}(z)-I_{-n-\frac{1}{2}}(z)]$$

The pairs

$$\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z),\ \sqrt{\tfrac{1}{2}\pi/z}I_{-n-\frac{1}{2}}(z)$$

and

$$\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z),\ \sqrt{\tfrac{1}{2}\pi/z}K_{n+\frac{1}{2}}(z)$$

are linearly independent solutions for every n.

Most properties of the Modified Spherical Bessel functions can be derived from those of the Spherical Bessel functions by use of the above relations.

Ascending Series

10.2.5

$$\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z)=\frac{z^n}{1\cdot 3\cdot 5\ldots(2n+1)}\left\{1+\frac{\frac{1}{2}z^2}{1!(2n+3)}+\frac{(\frac{1}{2}z^2)^2}{2!(2n+3)(2n+5)}+\ldots\right\}$$

10.2.6

$$\sqrt{\tfrac{1}{2}\pi/z}I_{-n-\frac{1}{2}}(z)=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{(-1)^nz^{n+1}}\left\{1+\frac{\frac{1}{2}z^2}{1!(1-2n)}+\frac{(\frac{1}{2}z^2)^2}{2!(1-2n)(3-2n)}+\ldots\right\} \qquad (n=0, 1, 2, \ldots)$$

Wronskians

10.2.7

$$W\{\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z),\ \sqrt{\tfrac{1}{2}\pi/z}I_{-n-\frac{1}{2}}(z)\}=(-1)^{n+1}z^{-2}$$

10.2.8

$$W\{\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z),\ \sqrt{\tfrac{1}{2}\pi/z}K_{n+\frac{1}{2}}(z)\}=-\tfrac{1}{2}\pi z^{-2}$$

Representations by Elementary Functions

10.2.9

$$\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z)=(2z)^{-1}[R(n+\tfrac{1}{2},-z)e^z-(-1)^nR(n+\tfrac{1}{2},z)e^{-z}]$$

10.2.10

$$\sqrt{\tfrac{1}{2}\pi/z}I_{-n-\frac{1}{2}}(z)=(2z)^{-1}[R(n+\tfrac{1}{2},-z)e^z+(-1)^nR(n+\tfrac{1}{2},z)e^{-z}]$$

10.2.11

$$\begin{aligned}R(n+\tfrac{1}{2}, z)&=1+\frac{(n+1)!}{1!\Gamma(n)}(2z)^{-1}+\frac{(n+2)!}{2!\Gamma(n-1)}(2z)^{-2}+\ldots\\&=\sum_0^n(n+\tfrac{1}{2}, k)(2z)^{-k} \qquad (n=0, 1, 2, \ldots)\end{aligned}$$

(See **10.1.9.**)

10.2.12

$$\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z)=g_n(z)\sinh z+g_{-n-1}(z)\cosh z$$

$$g_0(z)=z^{-1},\ g_1(z)=-z^{-2}$$

$$g_{n-1}(z)-g_{n+1}(z)=(2n+1)z^{-1}g_n(z) \qquad (n=0, \pm 1, \pm 2, \ldots)$$

The Functions $\sqrt{\frac{1}{2}\pi/z}I_{\pm(n+\frac{1}{2})}(z)$, $n=0, 1, 2$

10.2.13

$$\sqrt{\tfrac{1}{2}\pi/z}I_{1/2}(z)=\frac{\sinh z}{z}$$

$$\sqrt{\tfrac{1}{2}\pi/z}I_{3/2}(z)=-\frac{\sinh z}{z^2}+\frac{\cosh z}{z}$$

$$\sqrt{\tfrac{1}{2}\pi/z}I_{5/2}(z)=\left(\frac{3}{z^3}+\frac{1}{z}\right)\sinh z-\frac{3}{z^2}\cosh z$$

10.2.14

$$\sqrt{\tfrac{1}{2}\pi/z}I_{-1/2}(z)=\frac{\cosh z}{z}$$

$$\sqrt{\tfrac{1}{2}\pi/z}I_{-3/2}(z)=\frac{\sinh z}{z}-\frac{\cosh z}{z^2}$$

$$\sqrt{\tfrac{1}{2}\pi/z}I_{-5/2}(z)=-\frac{3}{z^2}\sinh z+\left(\frac{3}{z^3}+\frac{1}{z}\right)\cosh z$$

Modified Spherical Bessel Functions of the Third Kind

10.2.15

$$\sqrt{\tfrac{1}{2}\pi/z}K_{n+\frac{1}{2}}(z)=\tfrac{1}{2}\pi i e^{(n+1)\pi i/2}h_n^{(1)}(ze^{\frac{1}{2}\pi i})$$

$$(-\pi<\arg z\leq\tfrac{1}{2}\pi)$$

$$=-\tfrac{1}{2}\pi i e^{-(n+1)\pi i/2}h_n^{(2)}(ze^{-\frac{1}{2}\pi i})$$

$$(\tfrac{1}{2}\pi<\arg z\leq\pi)$$

$$=(\tfrac{1}{2}\pi/z)e^{-z}\sum_0^n (n+\tfrac{1}{2}, k)(2z)^{-k}$$

10.2.16

$$K_{n+\frac{1}{2}}(z)=K_{-n-\frac{1}{2}}(z) \qquad (n=0, 1, 2, \ldots)$$

The Functions $\sqrt{\frac{1}{2}\pi/z}K_{n+\frac{1}{2}}(z), n=0, 1, 2$

10.2.17 $\sqrt{\tfrac{1}{2}\pi/z}K_{1/2}(z)=(\tfrac{1}{2}\pi/z)e^{-z}$

$\sqrt{\tfrac{1}{2}\pi/z}K_{3/2}(z)=(\tfrac{1}{2}\pi/z)e^{-z}(1+z^{-1})$

$\sqrt{\tfrac{1}{2}\pi/z}K_{5/2}(z)=(\tfrac{1}{2}\pi/z)e^{-z}(1+3z^{-1}+3z^{-2})$

Elementary Properties

Recurrence Relations

$f_n(z)$: $\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z)$, $(-1)^{n+1}\sqrt{\tfrac{1}{2}\pi/z}K_{n+\frac{1}{2}}(z)$

$(n=0, \pm 1, \pm 2, \ldots)$

10.2.18 $f_{n-1}(z)-f_{n+1}(z)=(2n+1)z^{-1}f_n(z)$

10.2.19 $nf_{n-1}(z)+(n+1)f_{n+1}(z)=(2n+1)\dfrac{d}{dz}f_n(z)$

10.2.20 $\dfrac{n+1}{z}f_n(z)+\dfrac{d}{dz}f_n(z)=f_{n-1}(z)$

(See **10.2.22.**)

10.2.21 $-\dfrac{n}{z}f_n(z)+\dfrac{d}{dz}f_n(z)=f_{n+1}(z)$

(See **10.2.23.**)

Differentiation Formulas

$f_n(z)$: $\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z)$, $(-1)^{n+1}\sqrt{\tfrac{1}{2}\pi/z}K_{n+\frac{1}{2}}(z)$

$(n=0, \pm 1, \pm 2, \ldots)$

10.2.22 $\left(\dfrac{1}{z}\dfrac{d}{dz}\right)^m[z^{n+1}f_n(z)]=z^{n-m+1}f_{n-m}(z)$

10.2.23 $\left(\dfrac{1}{z}\dfrac{d}{dz}\right)^m[z^{-n}f_n(z)]=z^{-n-m}f_{n+m}(z)$

$(m=1, 2, 3, \ldots)$

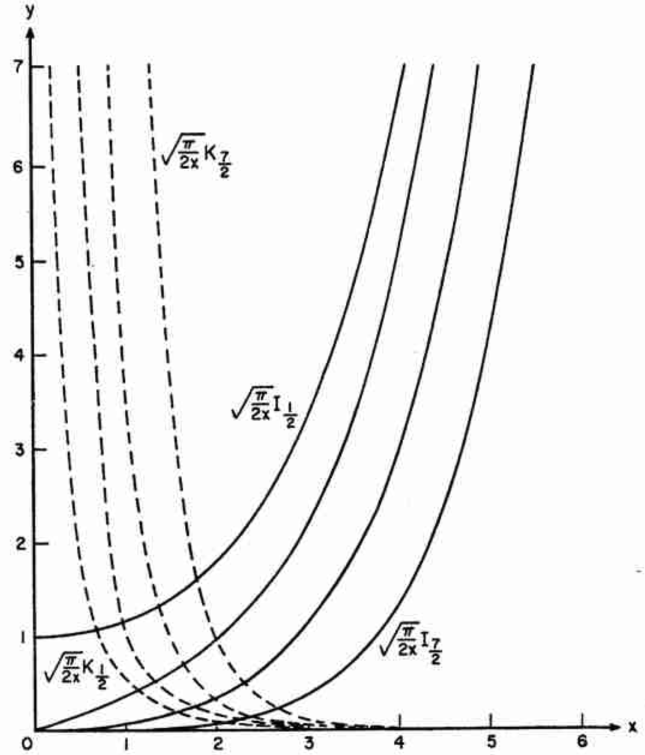

FIGURE 10.4. $\sqrt{\frac{\pi}{2x}}I_{n+\frac{1}{2}}(x)$, $\sqrt{\frac{\pi}{2x}}K_{n+\frac{1}{2}}(x)$. $n=0(1)3$.

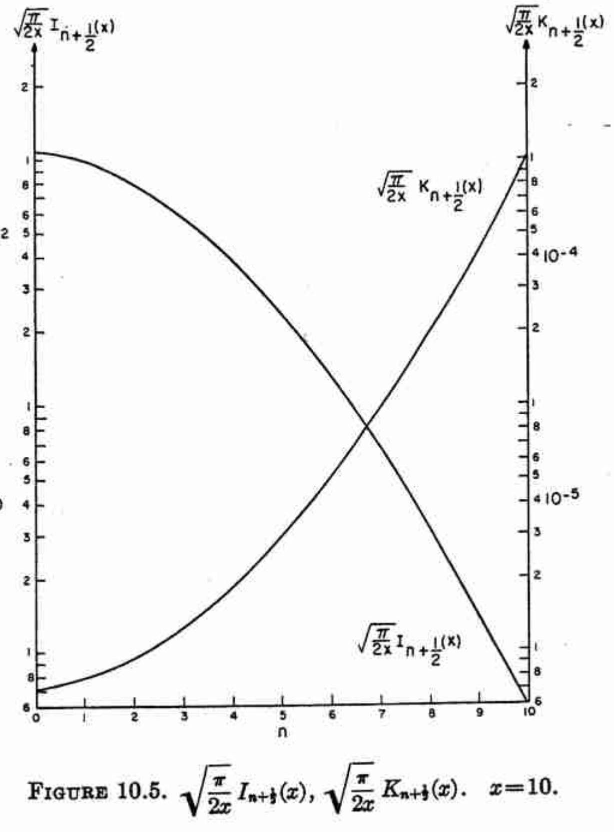

FIGURE 10.5. $\sqrt{\frac{\pi}{2x}}I_{n+\frac{1}{2}}(x)$, $\sqrt{\frac{\pi}{2x}}K_{n+\frac{1}{2}}(x)$. $x=10$.

Formulas of Rayleigh's Type

10.2.24 $\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z)=z^n\left(\frac{1}{z}\frac{d}{dz}\right)^n\frac{\sinh z}{z}$

10.2.25

$$\sqrt{\tfrac{1}{2}\pi/z}I_{-n-\frac{1}{2}}(z)=z^n\left(\frac{1}{z}\frac{d}{dz}\right)^n\frac{\cosh z}{z}$$

$$(n=0, 1, 2, \ldots)$$

Formulas for $I^2_{n+\frac{1}{2}}(z)-I^2_{-n-\frac{1}{2}}(z)$

10.2.26

$$(\tfrac{1}{2}\pi/z)[I^2_{n+\frac{1}{2}}(z)-I^2_{-n-\frac{1}{2}}(z)]$$

$$=\frac{1}{z^2}\sum_0^n(-1)^{k+1}\frac{(2n-k)!(2n-2k)!}{k![(n-k)!]^2}(2z)^{2k-2n}$$

$$(n=0, 1, 2, \ldots)$$

10.2.27 $(\tfrac{1}{2}\pi/z)[I^2_{1/2}(z)-I^2_{-1/2}(z)]=-z^{-2}$

10.2.28 $(\tfrac{1}{2}\pi/z)[I^2_{3/2}(z)-I^2_{-3/2}(z)]=z^{-2}-z^{-4}$

10.2.29

$$(\tfrac{1}{2}\pi/z)[I^2_{5/2}(z)-I^2_{-5/2}(z)]=-z^{-2}+3z^{-4}-9z^{-6}$$

Generating Functions

10.2.30

$$\frac{1}{z}\sinh\sqrt{z^2-2izt}=\sum_0^\infty\frac{(-it)^n}{n!}[\sqrt{\tfrac{1}{2}\pi/z}I_{-n+\frac{1}{2}}(z)]$$

$$(2|t|<|z|)$$

10.2.31

$$\frac{1}{z}\cosh\sqrt{z^2+2izt}=\sum_0^\infty\frac{(it)^n}{n!}[\sqrt{\tfrac{1}{2}\pi/z}I_{n-\frac{1}{2}}(z)]$$

Derivatives With Respect to Order

10.2.32

$$[\frac{\partial}{\partial\nu}I_\nu(x)]_{\nu=\frac{1}{2}}=-\frac{1}{2\pi x}[\mathrm{Ei}(2x)e^{-x}-E_1(-2x)e^x]$$

10.2.33

$$\left[\frac{\partial}{\partial\nu}I_\nu(x)\right]_{\nu=-\frac{1}{2}}=\frac{1}{2\pi x}[\mathrm{Ei}(2x)e^{-x}+E_1(-2x)e^x]$$

10.2.34 $\left[\frac{\partial}{\partial\nu}K_\nu(x)\right]_{\nu=\pm\frac{1}{2}}=\mp\sqrt{\pi/2x}\mathrm{Ei}(-2x)e^x$

For $E_1(x)$ and $\mathrm{Ei}(x)$, see **5.1.1, 5.1.2.**

Addition Theorems and Degenerate Forms

r, ρ, θ, λ arbitrary complex; $R=\sqrt{r^2+\rho^2-2r\rho\cos\theta}$

10.2.35

$$\frac{e^{-\lambda R}}{\lambda R}=\frac{2}{\pi}\sum_0^\infty(2n+1)[\sqrt{\tfrac{1}{2}\pi/\lambda r}I_{n+\frac{1}{2}}(\lambda r)]$$

$$[\sqrt{\tfrac{1}{2}\pi/\lambda\rho}K_{n+\frac{1}{2}}(\lambda\rho)]P_n(\cos\theta)$$

10.2.36

$$e^{z\cos\theta}=\sum_0^\infty(2n+1)[\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z)]P_n(\cos\theta)$$

10.2.37

$$e^{-z\cos\theta}=\sum_0^\infty(-1)^n(2n+1)[\sqrt{\tfrac{1}{2}\pi/z}I_{n+\frac{1}{2}}(z)]P_n(\cos\theta)$$

Duplication Formula

10.2.38

$$K_{n+\frac{1}{2}}(2z)=$$

$$n!\pi^{-\frac{1}{2}}z^{n+\frac{1}{2}}\sum_0^n\frac{(-1)^k(2n-2k+1)}{k!(2n-k+1)!}K^2_{n-k+\frac{1}{2}}(z)$$

10.3. Riccati-Bessel Functions

Differential Equation

10.3.1

$$z^2w''+[z^2-n(n+1)]w=0$$

$$(n=0, \pm1, \pm2, \ldots)$$

Pairs of linearly independent solutions are

$$zj_n(z),\ zy_n(z)$$

$$zh_n^{(1)}(z),\ zh_n^{(2)}(z)$$

All properties of these functions follow directly from those of the Spherical Bessel functions.

The Functions $zj_n(z)$, $zy_n(z)$, $n=0, 1, 2$

10.3.2

$zj_0(z)=\sin z,\qquad zj_1(z)=z^{-1}\sin z-\cos z$

$zj_2(z)=(3z^{-2}-1)\sin z-3z^{-1}\cos z$ *

10.3.3

$zy_0(z)=-\cos z,\qquad zy_1(z)=-\sin z-z^{-1}\cos z$

$zy_2(z)=-3z^{-1}\sin z-(3z^{-2}-1)\cos z$ *

Wronskians

10.3.4 $W\{zj_n(z), zy_n(z)\}=1$

10.3.5 $W\{zh_n^{(1)}(z), zh_n^{(2)}(z)\}=-2i$

$$(n=0, 1, 2, \ldots)$$

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*See page II.

10.4. Airy Functions

Definitions and Elementary Properties

Differential Equation

10.4.1 $$w''-zw=0$$

Pairs of linearly independent solutions are

$$\mathrm{Ai}\,(z),\ \mathrm{Bi}\,(z),$$
$$\mathrm{Ai}\,(z),\ \mathrm{Ai}\,(ze^{2\pi i/3}),$$
$$\mathrm{Ai}\,(z),\ \mathrm{Ai}\,(ze^{-2\pi i/3}).$$

Ascending Series

10.4.2 $\mathrm{Ai}\,(z)=c_1f(z)-c_2g(z)$

10.4.3 $\mathrm{Bi}\,(z)=\sqrt{3}\,[c_1f(z)+c_2g(z)]$

$$f(z)=1+\frac{1}{3!}z^3+\frac{1\cdot 4}{6!}z^6+\frac{1\cdot 4\cdot 7}{9!}z^9+\ldots$$
$$=\sum_0^\infty 3^k\left(\frac{1}{3}\right)_k\frac{z^{3k}}{(3k)!}$$
$$g(z)=z+\frac{2}{4!}z^4+\frac{2\cdot 5}{7!}z^7+\frac{2\cdot 5\cdot 8}{10!}z^{10}+\ldots$$
$$=\sum_0^\infty 3^k\left(\frac{2}{3}\right)_k\frac{z^{3k+1}}{(3k+1)!}$$
$$\left(\alpha+\frac{1}{3}\right)_0=1$$
$$3^k\left(\alpha+\frac{1}{3}\right)_k=(3\alpha+1)(3\alpha+4)\ldots(3\alpha+3k-2)$$
$$(\alpha \text{ arbitrary};\ k=1,2,3,\ldots)$$

(See **6.1.22.**)

10.4.4

$$c_1=\mathrm{Ai}\,(0)=\mathrm{Bi}\,(0)/\sqrt{3}=3^{-2/3}/\Gamma(2/3)$$
$$=.35502\ 80538\ 87817$$

10.4.5

$$c_2=-\mathrm{Ai}'\,(0)=\mathrm{Bi}'\,(0)/\sqrt{3}=3^{-1/3}/\Gamma(1/3)$$
$$=.25881\ 94037\ 92807$$

Relations Between Solutions

10.4.6 $\mathrm{Bi}\,(z)=e^{\pi i/6}\,\mathrm{Ai}\,(ze^{2\pi i/3})+e^{-\pi i/6}\,\mathrm{Ai}\,(ze^{-2\pi i/3})$

10.4.7

$$\mathrm{Ai}\,(z)+e^{2\pi i/3}\,\mathrm{Ai}\,(ze^{2\pi i/3})+e^{-2\pi i/3}\,\mathrm{Ai}\,(ze^{-2\pi i/3})=0$$

10.4.8

$$\mathrm{Bi}\,(z)+e^{2\pi i/3}\,\mathrm{Bi}\,(ze^{2\pi i/3})+e^{-2\pi i/3}\,\mathrm{Bi}\,(ze^{-2\pi i/3})=0$$

10.4.9 $\mathrm{Ai}\,(ze^{\pm 2\pi i/3})=\frac{1}{2}e^{\pm\pi i/3}[\mathrm{Ai}\,(z)\mp i\,\mathrm{Bi}\,(z)]$

Wronskians

10.4.10 $W\{\mathrm{Ai}\,(z),\mathrm{Bi}\,(z)\}=\pi^{-1}$

10.4.11 $W\{\mathrm{Ai}\,(z),\mathrm{Ai}\,(ze^{2\pi i/3})\}=\frac{1}{2}\pi^{-1}e^{-\pi i/6}$

10.4.12 $W\{\mathrm{Ai}\,(z),\mathrm{Ai}\,(ze^{-2\pi i/3})\}=\frac{1}{2}\pi^{-1}e^{\pi i/6}$

10.4.13 $W\{\mathrm{Ai}\,(ze^{2\pi i/3}),\mathrm{Ai}\,(ze^{-2\pi i/3})\}=\frac{1}{2}i\pi^{-1}$

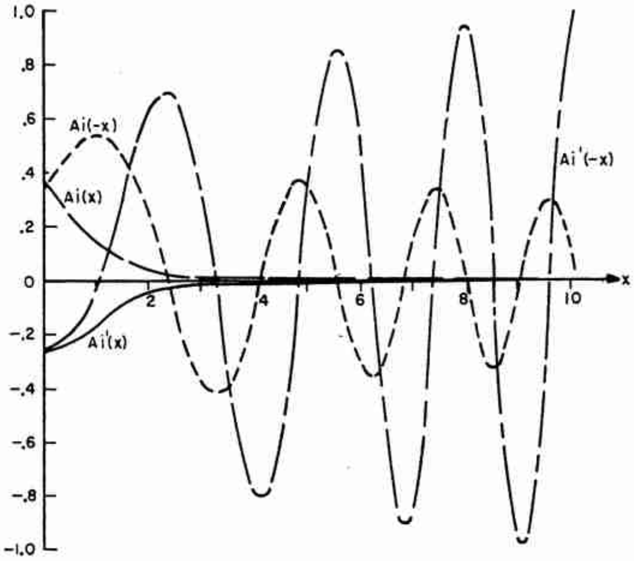

FIGURE 10.6. Ai $(\pm x)$, Ai′ $(\pm x)$.

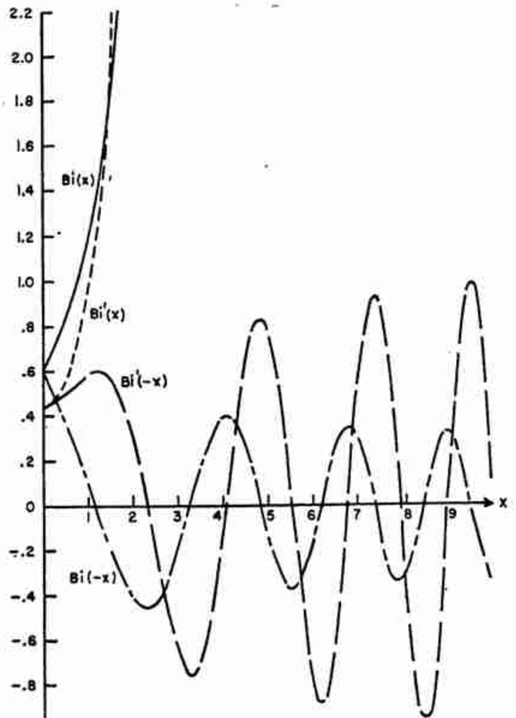

FIGURE 10.7. Bi $(\pm x)$, Bi′ $(\pm x)$.

Representations in Terms of Bessel Functions

$$\zeta=\tfrac{2}{3}z^{3/2}$$

10.4.14

$$\mathrm{Ai}\,(z)=\tfrac{1}{3}\sqrt{z}[I_{-1/3}(\zeta)-I_{1/3}(\zeta)]=\pi^{-1}\sqrt{z/3}K_{1/3}(\zeta)$$

10.4.15

$$\begin{aligned}\mathrm{Ai}\,(-z)&=\tfrac{1}{3}\sqrt{z}[J_{1/3}(\zeta)+J_{-1/3}(\zeta)]\\&=\tfrac{1}{2}\sqrt{z/3}[e^{\pi i/6}H^{(1)}_{1/3}(\zeta)+e^{-\pi i/6}H^{(2)}_{1/3}(\zeta)]\end{aligned}$$

10.4.16

* $$-\mathrm{Ai}'(z)=\tfrac{1}{3}z[I_{-2/3}(\zeta)-I_{2/3}(\zeta)]=\pi^{-1}(z/\sqrt{3})K_{2/3}(\zeta)$$

10.4.17

$$\begin{aligned}\mathrm{Ai}'\,(-z)&=-\tfrac{1}{3}z[J_{-2/3}(\zeta)-J_{2/3}(\zeta)]\\&=\tfrac{1}{2}(z/\sqrt{3})[e^{-\pi i/6}H^{(1)}_{2/3}(\zeta)+e^{\pi i/6}H^{(2)}_{2/3}(\zeta)]\end{aligned}$$

10.4.18 $$\mathrm{Bi}\,(z)=\sqrt{z/3}[I_{-1/3}(\zeta)+I_{1/3}(\zeta)]$$

10.4.19

$$\begin{aligned}\mathrm{Bi}\,(-z)&=\sqrt{z/3}[J_{-1/3}(\zeta)-J_{1/3}(\zeta)]\\&=\tfrac{1}{2}i\sqrt{z/3}[e^{\pi i/6}H^{(1)}_{1/3}(\zeta)-e^{-\pi i/6}H^{(2)}_{1/3}(\zeta)]\end{aligned}$$

10.4.20 $$\mathrm{Bi}'\,(z)=(z/\sqrt{3})[I_{-2/3}(\zeta)+I_{2/3}(\zeta)]$$

10.4.21

$$\begin{aligned}\mathrm{Bi}'\,(-z)&=(z/\sqrt{3})[J_{-2/3}(\zeta)+J_{2/3}(\zeta)]\\&=\tfrac{1}{2}i(z/\sqrt{3})[e^{-\pi i/6}H^{(1)}_{2/3}(\zeta)-e^{\pi i/6}H^{(2)}_{2/3}(\zeta)]\end{aligned}$$

Representations of Bessel Functions in Terms of Airy Functions

$$z=\left(\frac{3}{2}\zeta\right)^{2/3}$$

10.4.22 $$J_{\pm 1/3}(\zeta)=\tfrac{1}{2}\sqrt{3/z}[\sqrt{3}\,\mathrm{Ai}\,(-z)\mp\mathrm{Bi}\,(-z)]$$

***10.4.23** $$H^{(1)}_{\pm 1/3}(\zeta)=e^{\mp\pi i/6}\sqrt{3/z}[\mathrm{Ai}\,(-z)-i\,\mathrm{Bi}\,(-z)]$$

10.4.24 $$H^{(2)}_{\pm 1/3}(\zeta)=e^{\pm\pi i/6}\sqrt{3/z}[\mathrm{Ai}\,(-z)+i\,\mathrm{Bi}\,(-z)]$$

10.4.25 $$I_{\pm 1/3}(\zeta)=\tfrac{1}{2}\sqrt{3/z}[\mp\sqrt{3}\,\mathrm{Ai}\,(z)+\mathrm{Bi}\,(z)]$$

10.4.26 $$K_{\pm 1/3}(\zeta)=\pi\sqrt{3/z}\,\mathrm{Ai}\,(z)$$

10.4.27 $$J_{\pm 2/3}(\zeta)=(\sqrt{3}/2z)[\pm\sqrt{3}\,\mathrm{Ai}'\,(-z)+\mathrm{Bi}'\,(-z)]$$

10.4.28

$$\begin{aligned}H^{(1)}_{2/3}(\zeta)&=e^{-2\pi i/3}H^{(1)}_{-2/3}(\zeta)\\&=e^{\pi i/6}(\sqrt{3}/z)[\mathrm{Ai}'\,(-z)-i\,\mathrm{Bi}'\,(-z)]\end{aligned}$$

10.4.29

$$\begin{aligned}H^{(2)}_{2/3}(\zeta)&=e^{2\pi i/3}H^{(2)}_{-2/3}(\zeta)\\&=e^{-\pi i/6}(\sqrt{3}/z)[\mathrm{Ai}'\,(-z)+i\,\mathrm{Bi}'\,(-z)]\end{aligned}$$

10.4.30 $$I_{\pm 2/3}(\zeta)=(\sqrt{3}/2z)[\pm\sqrt{3}\,\mathrm{Ai}'\,(z)+\mathrm{Bi}'\,(z)]$$

10.4.31 $$K_{\pm 2/3}(\zeta)=-\pi(\sqrt{3}/z)\,\mathrm{Ai}'\,(z)$$

Integral Representations

10.4.32

$$(3a)^{-1/3}\pi\,\mathrm{Ai}\,[\pm(3a)^{-1/3}x]=\int_0^\infty \cos\,(at^3\pm xt)dt$$

10.4.33

$$\begin{aligned}&(3a)^{-1/3}\pi\,\mathrm{Bi}\,[\pm(3a)^{-1/3}x]\\&\qquad=\int_0^\infty[\exp\,(-at^3\pm xt)+\sin\,(at^3\pm xt)]dt\end{aligned}$$

The Integrals $\int_0^z \mathrm{Ai}\,(\pm t)dt$, $\int_0^z \mathrm{Bi}\,(\pm t)dt$

$$\zeta=\tfrac{2}{3}z^{3/2}$$

10.4.34 $$\int_0^z \mathrm{Ai}\,(t)dt=\frac{1}{3}\int_0^\zeta[I_{-1/3}(t)-I_{1/3}(t)]dt$$

10.4.35 $$\int_0^z \mathrm{Ai}\,(-t)dt=\frac{1}{3}\int_0^\zeta[J_{-1/3}(t)+J_{1/3}(t)]dt$$

10.4.36 $$\int_0^z \mathrm{Bi}\,(t)dt=\frac{1}{\sqrt{3}}\int_0^\zeta[I_{-1/3}(t)+I_{1/3}(t)]dt$$

10.4.37 $$\int_0^z \mathrm{Bi}\,(-t)dt=\frac{1}{\sqrt{3}}\int_0^\zeta[J_{-1/3}(t)-J_{1/3}(t)]dt$$

Ascending Series for $\int_0^z \mathrm{Ai}\,(\pm t)dt$, $\int_0^z \mathrm{Bi}\,(\pm t)dt$

10.4.38 $$\int_0^z \mathrm{Ai}\,(t)dt=c_1F(z)-c_2G(z)$$

(See **10.4.2.**)

10.4.39 $$\int_0^z \mathrm{Ai}\,(-t)dt=-c_1F(-z)+c_2G(-z)$$

10.4.40 $$\int_0^z \mathrm{Bi}\,(t)dt=\sqrt{3}[c_1F(z)+c_2G(z)]$$

(See **10.4.3.**)

10.4.41

$$\int_0^z \mathrm{Bi}\,(-t)dt=-\sqrt{3}[c_1F(-z)+c_2G(-z)]$$

$$\begin{aligned}F(z)&=z+\frac{1}{4!}z^4+\frac{1\cdot 4}{7!}z^7+\frac{1\cdot 4\cdot 7}{10!}z^{10}+\dots\\&=\sum_0^\infty 3^k\left(\frac{1}{3}\right)_k\frac{z^{3k+1}}{(3k+1)!}\end{aligned}$$

$$\begin{aligned}G(z)&=\frac{1}{2!}z^2+\frac{2}{5!}z^5+\frac{2\cdot 5}{8!}z^8+\frac{2\cdot 5\cdot 8}{11!}z^{11}+\dots\\&=\sum_0^\infty 3^k\left(\frac{2}{3}\right)_k\frac{z^{3k+2}}{(3k+2)!}\end{aligned}$$

The constants c_1, c_2 are given in **10.4.4, 10.4.5.**

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*See page II.

The Functions Gi(z), Hi(z)

10.4.42

$$\mathrm{Gi}\,(z)=\pi^{-1}\int_0^\infty \sin\left(\frac{1}{3}t^3+zt\right)dt$$

$$=\frac{1}{3}\mathrm{Bi}\,(z)+\int_0^z[\mathrm{Ai}\,(z)\,\mathrm{Bi}\,(t)-\mathrm{Ai}\,(t)\,\mathrm{Bi}\,(z)]dt$$

10.4.43

$$\mathrm{Gi}'\,(z)=\frac{1}{3}\mathrm{Bi}'\,(z)+\int_0^z[\mathrm{Ai}'(z)\,\mathrm{Bi}\,(t)-\mathrm{Ai}\,(t)\,\mathrm{Bi}'(z)]dt$$

10.4.44

$$\mathrm{Hi}(z)=\pi^{-1}\int_0^\infty \exp\left(-\frac{1}{3}t^3+zt\right)dt$$

$$=\frac{2}{3}\mathrm{Bi}\,(z)+\int_0^z[\mathrm{Ai}\,(t)\,\mathrm{Bi}\,(z)-\mathrm{Ai}(z)\,\mathrm{Bi}\,(t)]dt$$

10.4.45

$$\mathrm{Hi}'\,(z)=\frac{2}{3}\mathrm{Bi}'\,(z)+\int_0^z[\mathrm{Ai}\,(t)\,\mathrm{Bi}'(z)-\mathrm{Ai}'(z)\,\mathrm{Bi}\,(t)]dt$$

10.4.46 $$\mathrm{Gi}\,(z)+\mathrm{Hi}\,(z)=\mathrm{Bi}\,(z)$$

Representations of $\int_0^z \mathrm{Ai}(\pm t)dt$, $\int_0^z \mathrm{Bi}(\pm t)dt$ by Gi $(\pm z)$, Hi $(\pm z)$

10.4.47

$$\int_0^z \mathrm{Ai}\,(t)dt=\frac{1}{3}+\pi[\mathrm{Ai}'\,(z)\mathrm{Gi}(z)-\mathrm{Ai}\,(z)\mathrm{Gi}'(z)]$$

10.4.48

$$=-\frac{2}{3}-\pi[\mathrm{Ai}'\,(z)\,\mathrm{Hi}\,(z)-\mathrm{Ai}\,(z)\,\mathrm{Hi}'\,(z)]$$

10.4.49

$$\int_0^z \mathrm{Ai}\,(-t)dt=-\frac{1}{3}-\pi[\mathrm{Ai}'\,(-z)\,\mathrm{Gi}\,(-z)-\mathrm{Ai}\,(-z)\,\mathrm{Gi}'\,(-z)]$$

10.4.50

$$=\frac{2}{3}+\pi[\mathrm{Ai}'\,(-z)\,\mathrm{Hi}\,(-z)-\mathrm{Ai}\,(-z)\mathrm{Hi}'(-z)]$$

10.4.51

$$\int_0^z \mathrm{Bi}\,(t)dt=\pi[\mathrm{Bi}'\,(z)\,\mathrm{Gi}\,(z)-\mathrm{Bi}\,(z)\,\mathrm{Gi}'\,(z)]$$

10.4.52 $$=-\pi[\mathrm{Bi}'\,(z)\,\mathrm{Hi}\,(z)-\mathrm{Bi}\,(z)\,\mathrm{Hi}'(z)]$$

10.4.53

$$\int_0^z \mathrm{Bi}\,(-t)dt=-\pi[\mathrm{Bi}'\,(-z)\,\mathrm{Gi}\,(-z)-\mathrm{Bi}\,(-z)\,\mathrm{Gi}'\,(-z)]$$

10.4.54 $$=\pi[\mathrm{Bi}'\,(-z)\,\mathrm{Hi}\,(-z)-\mathrm{Bi}\,(-z)\,\mathrm{Hi}'\,(-z)]$$

Differential Equations for Gi (z), Hi (z)

10.4.55 $$w''-zw=-\pi^{-1}$$

$$w(0)=\frac{1}{3}\,\mathrm{Bi}\,(0)=\frac{1}{\sqrt{3}}\,\mathrm{Ai}\,(0)=.20497\ 55424\ 78$$

$$w'(0)=\frac{1}{3}\,\mathrm{Bi}'\,(0)=-\frac{1}{\sqrt{3}}\,\mathrm{Ai}'\,(0)=.14942\ 94524\ 49$$

$$w(z)=\mathrm{Gi}(z)$$

10.4.56 $$w''-zw=\pi^{-1}$$

$$w(0)=\frac{2}{3}\,\mathrm{Bi}\,(0)=\frac{2}{\sqrt{3}}\,\mathrm{Ai}\,(0)=.40995\ 10849\ 56$$

$$w'(0)=\frac{2}{3}\,\mathrm{Bi}'\,(0)=-\frac{2}{\sqrt{3}}\,\mathrm{Ai}'\,(0)=.29885\ 89048\ 98$$

$$w(z)=\mathrm{Hi}\,(z)$$

Differential Equation for Products of Airy Functions

10.4.57 $$w'''-4zw'-2w=0$$

Linearly independent solutions are $\mathrm{Ai}^2\,(z)$, $\mathrm{Ai}\,(z)\,\mathrm{Bi}\,(z)$, $\mathrm{Bi}^2\,(z)$.

Wronskian for Products of Airy Functions

10.4.58 $$W\{\mathrm{Ai}^2\,(z),\mathrm{Ai}\,(z)\,\mathrm{Bi}\,(z),\mathrm{Bi}^2\,(z)\}=2\pi^{-3}$$

Asymptotic Expansions for $|z|$ Large

$$c_0=1,\ c_k=\frac{\Gamma(3k+\frac{1}{2})}{54^k k!\,\Gamma(k+\frac{1}{2})}=\frac{(2k+1)(2k+3)\ldots(6k-1)}{216^k k!},$$

$$d_0=1,\ d_k=-\frac{6k+1}{6k-1}c_k\qquad(k=1,2,3,\ldots)$$

$$\zeta=\frac{2}{3}z^{3/2}$$

10.4.59

$$\mathrm{Ai}\,(z)\sim\frac{1}{2}\pi^{-1/2}z^{-1/4}e^{-\zeta}\sum_0^\infty(-1)^k c_k\zeta^{-k}\qquad(|\arg z|<\pi)$$

10.4.60

$$\mathrm{Ai}\,(-z)\sim\pi^{-1/2}z^{-1/4}\left[\sin\left(\zeta+\frac{\pi}{4}\right)\sum_0^\infty(-1)^k c_{2k}\zeta^{-2k}-\cos\left(\zeta+\frac{\pi}{4}\right)\sum_0^\infty(-1)^k c_{2k+1}\zeta^{-2k-1}\right]$$

$$(|\arg z|<\tfrac{2}{3}\pi)$$

10.4.61

$$\mathrm{Ai}'\,(z)\sim-\tfrac{1}{2}\pi^{-1/2}z^{1/4}e^{-\zeta}\sum_0^\infty(-1)^k d_k\zeta^{-k}$$

$$(|\arg z|<\pi)$$

10.4.62

$$\mathrm{Ai}'(-z)\sim-\pi^{-\frac{1}{2}}z^{\frac{1}{4}}\left[\cos\left(\zeta+\frac{\pi}{4}\right)\sum_{0}^{\infty}(-1)^k d_{2k}\zeta^{-2k}+\sin\left(\zeta+\frac{\pi}{4}\right)\sum_{0}^{\infty}(-1)^k d_{2k+1}\zeta^{-2k-1}\right]\qquad(|\arg z|<\tfrac{2}{3}\pi)$$

10.4.63

$$\mathrm{Bi}(z)\sim\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}e^{\zeta}\sum_{0}^{\infty}c_k\zeta^{-k}\qquad(|\arg z|<\tfrac{1}{3}\pi)$$

10.4.64

$$\mathrm{Bi}(-z)\sim\pi^{-\frac{1}{2}}z^{-\frac{1}{4}}\left[\cos\left(\zeta+\frac{\pi}{4}\right)\sum_{0}^{\infty}(-1)^k c_{2k}\zeta^{-2k}+\sin\left(\zeta+\frac{\pi}{4}\right)\sum_{0}^{\infty}(-1)^k c_{2k+1}\zeta^{-2k-1}\right]\qquad(|\arg z|<\tfrac{2}{3}\pi)$$

10.4.65

$$\mathrm{Bi}(ze^{\pm\pi i/3})\sim\sqrt{2/\pi}\,e^{\pm\pi i/6}z^{-\frac{1}{4}}\left[\sin\left(\zeta+\frac{\pi}{4}\mp\frac{i}{2}\ln 2\right)\sum_{0}^{\infty}(-1)^k c_{2k}\zeta^{-2k}-\cos\left(\zeta+\frac{\pi}{4}\mp\frac{i}{2}\ln 2\right)\sum_{0}^{\infty}(-1)^k c_{2k+1}\zeta^{-2k-1}\right]\qquad(|\arg z|<\tfrac{2}{3}\pi)$$

10.4.66

* $$\mathrm{Bi}'(z)\sim\pi^{-\frac{1}{2}}z^{\frac{1}{4}}e^{\zeta}\sum_{0}^{\infty}d_k\zeta^{-k}\quad(|\arg z|<\tfrac{1}{3}\pi)$$

10.4.67

$$\mathrm{Bi}'(-z)\sim\pi^{-\frac{1}{2}}z^{\frac{1}{4}}\left[\sin\left(\zeta+\frac{\pi}{4}\right)\sum_{0}^{\infty}(-1)^k d_{2k}\zeta^{-2k}-\cos\left(\zeta+\frac{\pi}{4}\right)\sum_{0}^{\infty}(-1)^k d_{2k+1}\zeta^{-2k-1}\right]\qquad(|\arg z|<\tfrac{2}{3}\pi)$$

10.4.68

$$\mathrm{Bi}'(ze^{\pm\pi i/3})\sim\sqrt{2/\pi}\,e^{\mp\pi i/6}z^{\frac{1}{4}}\left[\cos\left(\zeta+\frac{\pi}{4}\mp\frac{i}{2}\ln 2\right)\sum_{0}^{\infty}(-1)^k d_{2k}\zeta^{-2k}+\sin\left(\zeta+\frac{\pi}{4}\mp\frac{i}{2}\ln 2\right)\sum_{0}^{\infty}(-1)^k d_{2k+1}\zeta^{-2k-1}\right]\qquad(|\arg z|<\tfrac{2}{3}\pi)$$

Modulus and Phase

10.4.69

$$\mathrm{Ai}(-x)=M(x)\cos\theta(x),\ \mathrm{Bi}(-x)=M(x)\sin\theta(x)$$

$$M(x)=\sqrt{[\mathrm{Ai}^2(-x)+\mathrm{Bi}^2(-x)]},\qquad \theta(x)=\arctan\,[\mathrm{Bi}(-x)/\mathrm{Ai}(-x)]$$

10.4.70

$$\mathrm{Ai}'(-x)=N(x)\cos\phi(x),\ \mathrm{Bi}'(-x)=N(x)\sin\phi(x)$$

$$N(x)=\sqrt{[\mathrm{Ai}'^2(-x)+\mathrm{Bi}'^2(-x)]},\qquad \phi(x)=\arctan\,[\mathrm{Bi}'(-x)/\mathrm{Ai}'(-x)]$$

Differential Equations for Modulus and Phase

Primes denote differentiation with respect to x

10.4.71 $$M^2\theta'=-\pi^{-1},\ N^2\phi'=-\pi^{-1}x$$

10.4.72 $$N^2=M'^2+M^2\theta'^2=M'^2+\pi^{-2}M^{-2}$$ *

10.4.73 $$NN'=-xMM'$$

10.4.74

$$\tan(\phi-\theta)=M\theta'/M'=-(\pi MM')^{-1},$$
$$MN\sin(\phi-\theta)=\pi^{-1}$$

10.4.75 $$M''+xM-\pi^{-2}M^{-3}=0$$

10.4.76 $$(M^2)'''+4x(M^2)'-2M^2=0$$

10.4.77 $$\theta'^2+\tfrac{1}{2}(\theta'''/\theta')-\tfrac{3}{4}(\theta''/\theta')^2=x$$

Asymptotic Expansions of Modulus and Phase for Large x

10.4.78 $$M^2(x)\sim\frac{1}{\pi}x^{-1/2}\sum_{0}^{\infty}\frac{(-1)^k}{12^k k!}2^{3k}\left(\frac{1}{2}\right)_{3k}(2x)^{-3k}$$

10.4.79

$$\theta(x)\sim\frac{1}{4}\pi-\frac{2}{3}x^{3/2}\left[1-\frac{5}{4}(2x)^{-3}+\frac{1105}{96}(2x)^{-6}-\frac{82825}{128}(2x)^{-9}+\frac{12820\,31525}{14336}(2x)^{-12}-\ldots\right]$$

10.4.80

$$N^2(x)\sim\frac{1}{\pi}x^{\frac{1}{2}}\sum_{0}^{\infty}\frac{(-1)^{k+1}}{12^k k!}\frac{6k+1}{6k-1}2^{3k}\left(\frac{1}{2}\right)_{3k}(2x)^{-3k}$$

10.4.81

$$\phi(x)\sim\frac{3}{4}\pi-\frac{2}{3}x^{3/2}\left[1+\frac{7}{4}(2x)^{-3}-\frac{1463}{96}(2x)^{-6}+\frac{495271}{640}(2x)^{-9}-\frac{2065\ 30429}{2048}(2x)^{-12}+\ldots\right]$$

Asymptotic Forms of $\int_0^x \mathrm{Ai}(\pm t)\,dt$, $\int_0^x \mathrm{Bi}(\pm t)\,dt$ for Large x

10.4.82 $$\int_0^x\mathrm{Ai}(t)dt\sim\frac{1}{3}-\frac{1}{2}\pi^{-1/2}x^{-3/4}\exp\left(-\frac{2}{3}x^{3/2}\right)$$

10.4.83

$$\int_0^x\mathrm{Ai}(-t)dt\sim\frac{2}{3}-\pi^{-1/2}x^{-3/4}\cos\left(\frac{2}{3}x^{3/2}+\frac{\pi}{4}\right)$$

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*See page II.

10.4.84 $$\int_0^x \mathrm{Bi}\,(t)dt \sim \pi^{-1/2}x^{-3/4}\exp\left(\frac{2}{3}x^{3/2}\right)$$

10.4.85 $$\int_0^x \mathrm{Bi}\,(-t)dt \sim \pi^{-1/2}x^{-3/4}\sin\left(\frac{2}{3}x^{3/2}+\frac{\pi}{4}\right)$$

Asymptotic Forms of Gi $(\pm x)$, Gi′ $(\pm x)$, Hi $(\pm x)$, Hi′ $(\pm x)$ for Large x

10.4.86 $$\mathrm{Gi}\,(x)\sim\pi^{-1}x^{-1}$$

10.4.87 $$\mathrm{Gi}\,(-x)\sim\pi^{-1/2}x^{-1/4}\cos\left(\frac{2}{3}x^{3/2}+\frac{\pi}{4}\right)$$

10.4.88 $$\mathrm{Gi}'\,(x)\sim\frac{7}{96}\pi^{-1}x^{-2}$$

10.4.89 $$\mathrm{Gi}'\,(-x)\sim\pi^{-1/2}x^{1/4}\sin\left(\frac{2}{3}x^{3/2}+\frac{\pi}{4}\right)$$

10.4.90 $$\mathrm{Hi}\,(x)\sim\pi^{-1/2}x^{-1/4}\exp\,(\tfrac{2}{3}x^{3/2})$$

10.4.91 $$\mathrm{Hi}\,(-x)\sim\pi^{-1}x^{-1}$$

10.4.92 $$\mathrm{Hi}'\,(x)\sim\pi^{-1/2}x^{1/4}\exp\,(\tfrac{2}{3}x^{3/2})$$

10.4.93 $$\mathrm{Hi}'\,(-x)\sim-\frac{3}{2}\pi^{-1}x^{-2}$$

Zeros and Their Asymptotic Expansions

Ai (z), Ai′ (z) have zeros on the negative real axis only. Bi (z), Bi′ (z) have zeros on the negative real axis and in the sector $\frac{1}{3}\pi<|\arg z|<\frac{1}{2}\pi$. a_s, a'_s; b_s, b'_s s-th (real) negative zero of Ai (z), Ai′ (z); Bi (z), Bi′ (z), respectively. β_s, β'_s; $\overline{\beta}_s$, $\overline{\beta}'_s$ s-th complex zero of Bi (z), Bi′ (z) in the sectors $\frac{1}{3}\pi<\arg z<\frac{1}{2}\pi$, $-\frac{1}{2}\pi<\arg z<-\frac{1}{3}\pi$, respectively.

10.4.94 $$a_s=-f[3\pi(4s-1)/8]$$

10.4.95 $$a'_s=-g[3\pi(4s-3)/8]$$

10.4.96 $$\mathrm{Ai}'\,(a_s)=(-1)^{s-1}f_1[3\pi(4s-1)/8]$$

10.4.97 $$\mathrm{Ai}\,(a'_s)=(-1)^{s-1}g_1[3\pi(4s-3)/8]$$

10.4.98 $$b_s=-f[3\pi(4s-3)/8]$$

10.4.99 $$b'_s=-g[3\pi(4s-1)/8]$$

10.4.100 $$\mathrm{Bi}'\,(b_s)=(-1)^{s-1}f_1[3\pi(4s-3)/8]$$

10.4.101 $$\mathrm{Bi}\,(b'_s)=(-1)^{s}g_1[3\pi(4s-1)/8]$$

10.4.102 $$\beta_s=e^{\pi i/3}f\left[\frac{3\pi}{8}(4s-1)+\frac{3i}{4}\ln 2\right]$$

10.4.103 $$\beta'_s=e^{\pi i/3}g\left[\frac{3\pi}{8}(4s-3)+\frac{3i}{4}\ln 2\right]$$

10.4.104
$$\mathrm{Bi}'\,(\beta_s)=(-1)^s\sqrt{2}e^{-\pi i/6}f_1\left[\frac{3\pi}{8}(4s-1)+\frac{3i}{4}\ln 2\right]$$

10.4.105
$$\mathrm{Bi}\,(\beta'_s)=(-1)^{s-1}\sqrt{2}e^{\pi i/6}g_1\left[\frac{3\pi}{8}(4s-3)+\frac{3i}{4}\ln 2\right]$$

$|z|$ sufficiently large

$$f(z)\sim z^{2/3}\left(1+\frac{5}{48}z^{-2}-\frac{5}{36}z^{-4}+\frac{77125}{82944}z^{-6}-\frac{1080\,56875}{69\,67296}z^{-8}+\frac{16\,23755\,96875}{3344\,30208}z^{-10}-\ldots\right)$$

$$g(z)\sim z^{2/3}\left(1-\frac{7}{48}z^{-2}+\frac{35}{288}z^{-4}-\frac{181223}{207360}z^{-6}+\frac{186\,83371}{12\,44160}z^{-8}-\frac{9\,11458\,84361}{1911\,02976}z^{-10}+\ldots\right)$$

$$f_1(z)\sim\pi^{-1/2}z^{1/6}\left(1+\frac{5}{48}z^{-2}-\frac{1525}{4608}z^{-4}+\frac{23\,97875}{6\,63552}z^{-6}-\ldots\right)$$

$$g_1(z)\sim\pi^{-1/2}z^{-1/6}\left(1-\frac{7}{96}z^{-2}+\frac{1673}{6144}z^{-4}-\frac{843\,94709}{265\,42080}z^{-6}+\ldots\right)$$ *

Formal and Asymptotic Solutions of Ordinary Differential Equations of Second Order With Turning Points

An equation

10.4.106 $$W''+a(z,\lambda)W'+b(z,\lambda)W=0$$

in which λ is a real or complex parameter and, for fixed λ, $a(z,\lambda)$ is analytic in z and $b(z,\lambda)$ is continuous in z in some region of the z-plane, may be reduced by the transformation

10.4.107 $$W(z)=w(z)\exp\left(-\frac{1}{2}\int^z a(t,\lambda)dt\right)$$

to the equation

10.4.108
$$w''+\varphi(z,\lambda)w=0$$

$$\varphi(z,\lambda)=b(z,\lambda)-\frac{1}{4}a^2(z,\lambda)-\frac{1}{2}\frac{d}{dz}a(z,\lambda).$$

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*See page II.

If $\varphi(z, \lambda)$ can be written in the form

10.4.109 $$\varphi(z, \lambda)=\lambda^2 p(z)+q(z, \lambda)$$

where $q(z, \lambda)$ is bounded in a region R of the z-plane, then the zeros of $p(z)$ in R are said to be turning points of the equation **10.4.108.**

The Special Case $w''+[\lambda^2 z+q(z, \lambda)]w=0$

Let $\lambda=|\lambda|e^{i\omega}$ vary over a sectorial domain S: $|\lambda|\geq\lambda_0(>0)$, $\omega_1\leq\omega\leq\omega_2$, and suppose that $q(z, \lambda)$ is continuous in z for $|z|<r$ and λ in S, and $q(z, \lambda) \sim\sum_0^\infty q_n(z)\lambda^{-n}$ as $\lambda\rightarrow\infty$ in S.

Formal Series Solution

10.4.110

$$w(z)=u(z)\sum_0^\infty \varphi_n(z)\lambda^{-n}+\lambda^{-1}u'(z)\sum_0^\infty \psi_n(z)\lambda^{-n}$$

$$u''+\lambda^2 zu=0$$

$$\varphi_0(z)=c_0, \qquad \psi_0(z)=z^{-\frac{1}{2}}c_1, \qquad c_0,\ c_1 \text{ constants}$$

$$\varphi_{n+1}(z)=-\frac{1}{2}\psi_n'(z)-\frac{1}{2}\int_0^z\sum_0^n q_{n-k}(t)\psi_k(t)dt$$

$$\psi_n(z)=\frac{1}{2}z^{-\frac{1}{2}}\int_0^z t^{-\frac{1}{2}}\left[\varphi_n''(t)+\sum_0^n q_{n-k}(t)\varphi_k(t)\right]dt$$

$$(n=0, 1, 2, \ldots)$$

Uniform Asymptotic Expansions of Solutions

For z real, i.e. for the equation

10.4.111 $$y''+[\lambda^2 x+q(x, \lambda)]y=0$$

where x varies in a bounded interval $a\leq x\leq b$ that includes the origin and where, for each fixed λ in S, $q(x, \lambda)$ is continuous in x for $a\leq x\leq b$, the following asymptotic representations hold.

(i) If λ is real and positive, there are solutions $y_0(x)$, $y_1(x)$ such that, uniformly in x on $a\leq x\leq 0$,

10.4.112

$$y_0(x)=\mathrm{Ai}\,(-\lambda^{2/3}x)[1+O(\lambda^{-1})] \qquad (\lambda\rightarrow\infty)$$
$$y_1(x)=\mathrm{Bi}\,(-\lambda^{2/3}x)[1+O(\lambda^{-1})]$$

and, uniformly in x on $0\leq x\leq b$

10.4.113

$$y_0(x)=\mathrm{Ai}\,(-\lambda^{2/3}x)[1+O(\lambda^{-1})]+\mathrm{Bi}\,(-\lambda^{2/3}x)O(\lambda^{-1}),$$
$$y_1(x)=\mathrm{Bi}\,(-\lambda^{2/3}x)[1+O(\lambda^{-1})]+\mathrm{Ai}\,(-\lambda^{2/3}x)O(\lambda^{-1})$$
$$(\lambda\rightarrow\infty)$$

(ii) If $\mathscr{R}\lambda\geq 0$, $\mathscr{I}\lambda\neq 0$, there are solutions $y_0(x)$, $y_1(x)$ such that, uniformly in x on $a\leq x\leq b$,

10.4.114

$$y_0(x)=\mathrm{Ai}\,(-\lambda^{2/3}x)[1+O(\lambda^{-1})]$$
$$y_1(x)=\mathrm{Bi}\,(-\lambda^{2/3}x)[1+O(\lambda^{-1})] \qquad (|\lambda|\rightarrow\infty)$$

For further representations and details, we refer to [10.4].

When z is complex (bounded or unbounded), conditions under which the formal series **10.4.110** yields a uniform asymptotic expansion of a solution are given in [10.12] if $q(z, \lambda)$ is independent of λ and $|\lambda|\rightarrow\infty$ with fixed ω, and in [10.14] if λ lies in any region of the complex plane. Further references are [10.2; 10.9; 10.10].

The General Case $w''+[\lambda^2 p(z)+q(z, \lambda)]w=0$

Let $\lambda=|\lambda|e^{i\omega}$ where $|\lambda|\geq\lambda_0(>0)$ and $-\pi\leq\omega\leq\pi$; suppose that $p(z)$ is analytic in a region R and has a zero $z=z_0$ in R, and that, for fixed λ, $q(z, \lambda)$ is analytic in z for z in R. The transformation $\xi=\xi(z)$, $v=[p(z)/\xi]^{1/4}w(z)$, where ξ is defined as the (unique) solution of the equation

10.4.115 $$\xi\left(\frac{d\xi}{dz}\right)^2=p(z),$$

yields the special case

10.4.116 $$\frac{d^2v}{d\xi^2}+[\lambda^2\xi+f(\xi, \lambda)]v=0, \qquad *$$

$$f(\xi, \lambda)=\left(\frac{d\xi}{dz}\right)^{-2}q(z, \lambda)-\left(\frac{d\xi}{dz}\right)^{-\frac{1}{2}}\frac{d^2}{d\xi^2}\left[\left(\frac{d\xi}{dz}\right)^{\frac{1}{2}}\right].$$

Example:

Consider the equation

10.4.117 $$y''+[\lambda^2-(\lambda^2-\tfrac{1}{4})\,x^{-2}]y=0$$

for which the points $x=0$, ∞ are singular points and $x=1$ is a turning point. It has the functions $x^{\frac{1}{2}}J_\lambda(\lambda x)$, $x^{\frac{1}{2}}Y_\lambda(\lambda x)$ as particular solutions (see **9.1.49**).

The equation **10.4.115** becomes

$$\xi\left(\frac{d\xi}{dx}\right)^2=\frac{x^2-1}{x^2}$$

whence

$$\tfrac{2}{3}(-\xi)^{3/2}=-\sqrt{1-x^2}+\ln x^{-1}(1+\sqrt{1-x^2}) \qquad (0<x\leq 1)$$

$$\tfrac{2}{3}\xi^{3/2}=\sqrt{x^2-1}-\arccos x^{-1} \qquad (1\leq x<\infty).$$

Thus

10.4.118 $$v(\xi)=\left(\frac{x^2-1}{x^2\xi}\right)^{1/4}y(x)$$

---

*See page II.

satisfies the equation

10.4.119 $$\frac{d^2v}{d\xi^2}+\left[\lambda^2\xi-\frac{5}{16\xi^2}+\frac{\xi^2}{4}\frac{x^2(x^2+4)}{(x^2-1)^3}\right]v=0$$

which is of the form **10.4.111** with x replaced by ξ and $q(\xi, \lambda)$ independent of λ.

Suppose $\mathscr{R}\lambda \geq 0$, $\mathscr{I}\lambda \neq 0$. By the first equation of **10.4.114** there is a solution $v_0(\xi)$ of **10.4.119**, i.e., a solution $y_0(x)$ of **10.4.117** for which the representation

10.4.120
$$v_0(\xi)=\left(\frac{x^2-1}{x^2\xi}\right)^{1/4}y_0(x)=\text{Ai}(-\lambda^{2/3}\xi)[1+O(\lambda^{-1})]$$

holds uniformly in x on $0<x<\infty$ as $|\lambda|\to\infty$.

To identify $y_0(x)$ in terms of $x^{\frac{1}{2}}J_\lambda(\lambda x)$, $x^{\frac{1}{2}}Y_\lambda(\lambda x)$, restrict x to $0<x\leq b<1$ so that by **10.4.118** ξ is negative, and replace the Airy function by its asymptotic representation **10.4.59.** This yields

10.4.121
$$\begin{aligned}y_0(x)\\&=\left(\frac{x^2-1}{x^2\xi}\right)^{-1/4}\frac{1}{2}\pi^{-1/2}\lambda^{-1/6}(-\xi)^{1/4}\exp\left(\frac{2}{3}\lambda(-\xi)^{3/2}\right)\\&\qquad[1+O(\lambda^{-1})]\\&=\frac{1}{2}\pi^{-1/2}\lambda^{-1/6}\left(\frac{1-x^2}{x^2}\right)^{-1/4}\exp\left(\frac{2}{3}\lambda(-\xi)^{3/2}\right)\\&\qquad[1+O(\lambda^{-1})]\end{aligned}$$

Let now λ be fixed and $x\to0$ in **10.4.121.** There results

10.4.122 $$y_0(x)\sim\tfrac{1}{2}\pi^{-1/2}\lambda^{-1/6}x^{1/2}\left(\tfrac{1}{2}x\right)^\lambda e^\lambda.$$

On the other hand, $y_0(x)$ is a solution of **10.4.117** and therefore it can be written in the form

10.4.123 $$y_0(x)=x^{1/2}[c_1J_\lambda(\lambda x)+c_2Y_\lambda(\lambda x)]$$

where, from **9.1.7** for λ fixed and $x\to0$

$$J_\lambda(\lambda x)\sim\frac{(\frac{1}{2}\lambda x)^\lambda}{\Gamma(\lambda+1)},$$

$$Y_\lambda(\lambda x)\sim\frac{(\frac{1}{2}\lambda x)^\lambda}{\Gamma(\lambda+1)}\cot\lambda\pi-\frac{(\frac{1}{2}\lambda x)^{-\lambda}}{\Gamma(1-\lambda)}\csc\lambda\pi.$$

Thus, letting $x\to0$ in **10.4.123** and comparing the resulting relation with **10.4.122** one finds that $c_2=0$ and

10.4.124 $$y_0(x)=\tfrac{1}{2}\pi^{-1/2}\lambda^{-\lambda-1/6}e^\lambda\Gamma(\lambda+1)x^{1/2}J_\lambda(\lambda x).$$

It follows from **10.4.120** that uniformly in x on $0<x<\infty$

10.4.125

$$\begin{aligned}&J_\lambda(\lambda x)\\&=\frac{2\pi^{1/2}}{\Gamma(\lambda+1)}\lambda^{\lambda+1/6}e^{-\lambda}\left(\frac{x^2-1}{\xi}\right)^{-1/4}\text{Ai}(-\lambda^{2/3}\xi)[1+O(\lambda^{-1})]\\&\qquad(|\lambda|\to\infty)\end{aligned}$$

Numerical Methods

10.5. Use and Extension of the Tables

Spherical Bessel Functions

To compute $j_n(x)$, $y_n(x)$, $n=0, 1, 2$, for values of x outside the range of **Table 10.1,** use formulas **10.1.11, 10.1.12** and obtain values for the circular functions from **Tables 4.6–4.8.**

Example 1. Compute $j_1(x)$ for $x=11.425$.

From **10.1.11,** $j_1(x)=\frac{\sin x}{x^2}-\frac{\cos x}{x}$. Hence, using **Tables 4.6** and **4.8,**

$$\begin{aligned}j_1(11.425)&=-\frac{.90920\,500}{(11.425)^2}-\frac{.41634\,873}{11.425}\\&=-.00696\,54535-.03644\,1902\\&=-.04340\,7356.\end{aligned}$$

To compute $j_n(x)$, $11\leq n\leq20$, for a value of x within the range of **Table 10.3,** obtain from **Table 10.3,** directly or possibly by linear interpolation, $j_{21}(x)$, $j_{20}(x)$ and use these as starting values in the recurrence relation **10.1.19** for decreasing n.

An alternative procedure which often yields better accuracy and which also applies to computations of $j_n(x)$ when both n and x are outside the range of **Table 10.1** is the following device essentially due to J. C. P. Miller [9.20].

At some value N larger than the desired value n, assume tentatively $F_{N+1}=0$, $F_N=1$ and use recurrence relation **10.1.19** for decreasing N to obtain the sequence $F_{N-1}, \ldots, F_0$. If N was chosen large enough, each term of this sequence up to F_n is proportional, to a certain number of significant figures, to the corresponding term in the sequence $j_{N-1}(x), \ldots, j_0(x)$ of true values. The factor of proportionality, p, may be obtained by comparing, say, F_0 with the true value $j_0(x)$ computed separately. The terms in the sequence $pF_0, \ldots pF_n$ are then accurate to the number of significant figures present in the tentative values. If the accuracy obtained is not sufficient, the process may be repeated by starting from a larger value N.

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$$I=\int_{x_1}^{x_2} f(x)e^{ip(x)}dx$$

and the tabulation of the function

$$\mathrm{Gi}\ (z)=(1/\pi)\int_0^\infty \sin\ (uz+1/3u^3)du,$$

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11. Integrals of Bessel Functions

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Contents

The author acknowledges the assistance of Geraldine Coombs, Betty Kahn, Marilyn Kemp, Betty Ruhlman, and Anna Lee Samuels for checking formulas and developing numerical examples, only a portion of which could be accommodated here.

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[1] Midwest Research Institute. (Prepared under contract with the National Bureau of Standards.)

11. Integrals of Bessel Functions

Mathematical Properties

11.1. Simple Integrals of Bessel Functions

$$\underline{\int_0^z t^\mu J_\nu(t)dt}$$

11.1.1

$$\int_0^z t^\mu J_\nu(t)dt=\frac{z^\mu\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)}$$
$$\times\sum_{k=0}^\infty\frac{(\nu+2k+1)\Gamma\left(\frac{\nu-\mu+1}{2}+k\right)}{\Gamma\left(\frac{\nu+\mu+3}{2}+k\right)}J_{\nu+2k+1}(z)$$
$$(\mathscr{R}(\mu+\nu+1)>0)$$

11.1.2

$$\int_0^z J_\nu(t)dt=2\sum_{k=0}^\infty J_{\nu+2k+1}(z)\,(\mathscr{R}\nu>-1)$$

11.1.3 $$\int_0^z J_{2n}(t)dt=\int_0^z J_0(t)dt-2\sum_{k=0}^{n-1}J_{2k+1}(z)$$

11.1.4 $$\int_0^z J_{2n+1}(t)dt=1-J_0(z)-2\sum_{k=1}^{n}J_{2k}(z)$$

Recurrence Relations

11.1.5

$$\int_0^z J_{n+1}(t)dt=\int_0^z J_{n-1}(t)dt-2J_n(z)\qquad(n>0)$$

11.1.6 $$\int_0^z J_1(t)dt=1-J_0(z)$$

$$\underline{\int J_0(t)dt,\int Y_0(t)dt,\int I_0(t)dt,\int K_0(t)dt}$$

11.1.7

$$\int_0^z \mathscr{C}_0(t)dt=x\mathscr{C}_0(x)+\frac{1}{2}\pi x\{\mathbf{H}_0(x)\mathscr{C}_1(x)-\mathbf{H}_1(x)\mathscr{C}_0(x)\}$$
$$\mathscr{C}_\nu(x)=AJ_\nu(x)+BY_\nu(x),\nu=0,1$$

A and B are constants.

11.1.8

$$\int_0^z Z_0(t)dt=xZ_0(x)+\frac{1}{2}\pi x\{-\mathbf{L}_0(x)Z_1(x)+\mathbf{L}_1(x)Z_0(x)\}$$
$$Z_\nu(x)=AI_\nu(x)+Be^{i\nu\pi}K_\nu(x),\nu=0,1$$

A and B are constants.

$\mathbf{H}_\nu(x)$ and $\mathbf{L}_\nu(x)$ are Struve functions (see chapter **12**).

11.1.9

$$\int_0^z K_0(t)dt=-\left(\gamma+\ln\frac{x}{2}\right)x\sum_{k=0}^\infty\frac{(x/2)^{2k}}{(k!)^2(2k+1)}$$
$$+x\sum_{k=0}^\infty\frac{(x/2)^{2k}}{(k!)^2(2k+1)^2}$$
$$+x\sum_{k=1}^\infty\frac{(x/2)^{2k}}{(k!)^2(2k+1)}\left(1+\frac{1}{2}+\ldots+\frac{1}{k}\right)$$

γ (Euler's constant)$=.57721\ 56649\ \ldots$

In this and all other integrals of **11.1**, x is real and positive although all the results remain valid for extended portions of the complex plane unless stated to the contrary.

11.1.10

$$\int_0^{-iz}K_0(t)dt=\frac{\pi}{2}\int_0^z J_0(t)dt+i\frac{\pi}{2}\int_0^z Y_0(t)dt$$

Asymptotic Expansions

11.1.11

$$\int_x^\infty[J_0(t)+iY_0(t)]dt\sim\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}e^{i(x-\pi/4)}$$
$$\times\left[\sum_{k=0}^\infty(-)^k a_{2k+1}x^{-2k-1}+i\sum_{k=0}^\infty(-)^k a_{2k}x^{-2k}\right]$$

11.1.12 $$a_k=\frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}\sum_{s=0}^k\frac{\Gamma(s+\frac{1}{2})}{2^s s!\Gamma(\frac{1}{2})}$$

11.1.13

$$2(k+1)a_{k+1}=3\left(k+\frac{1}{2}\right)\left(k+\frac{5}{6}\right)a_k$$
$$-\left(k+\frac{1}{2}\right)^2\left(k-\frac{1}{2}\right)a_{k-1}$$

11.1.14 $x^{\frac{1}{2}}e^{-x}\int_0^x I_0(t)dt \sim (2\pi)^{-\frac{1}{2}}\sum_{k=0}^{\infty} a_k x^{-k}$

where the a_k are defined as in **11.1.12.**

11.1.15 $x^{\frac{1}{2}}e^{x}\int_x^{\infty} K_0(t)dt \sim \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\sum_{k=0}^{\infty} (-)^k a_k x^{-k}$

where the a_k are defined as in **11.1.12.**

Polynomial Approximations[2]

11.1.16 $8 \leq x \leq \infty$

$$\int_x^{\infty} [J_0(t)+iY_0(t)]dt$$

$$=x^{-\frac{1}{2}}e^{i(x-\pi/4)}\left[\sum_{k=0}^{7} (-)^k a_k (x/8)^{-2k-1} + i\sum_{k=0}^{7} (-)^k b_k (x/8)^{-2k}+\epsilon(x)\right]$$

$$|\epsilon(x)| \leq 2\times 10^{-9}$$

k	a_k	b_k
0	.06233 47304	.79788 45600
1	.00404 03539	.01256 42405
2	.00100 89872	.00178 70944
3	.00053 66169	.00067 40148
4	.00039 92825	.00041 00676
5	.00027 55037	.00025 43955
6	.00012 70039	.00011 07299
7	.00002 68482	.00002 26238

11.1.17 $8 \leq x \leq \infty$

$$x^{\frac{1}{2}}e^{-x}\int_0^x I_0(t)\,dt=\sum_{k=0}^{6} d_k (x/8)^{-k}+\epsilon(x)$$

$$|\epsilon(x)| \leq 2\times 10^{-6}$$

k	d_k
0	.39894 23
1	.03117 34
2	.00591 91
3	.00559 56
4	−.01148 58
5	.01774 40
6	−.00739 95

11.1.18 $7 \leq x \leq \infty$

$$x^{\frac{1}{2}}e^{x}\int_x^{\infty} K_0(t)dt=\sum_{k=0}^{6} (-)^k e_k (x/7)^{-k}+\epsilon(x)$$

$$|\epsilon(x)| \leq 2\times 10^{-7}$$

k	e_k
0	1.25331 414
1	0.11190 289
2	.02576 646
3	.00933 994
4	.00417 454
5	.00163 271
6	.00033 934

$\int \frac{J_0(t)dt}{t}, \int \frac{Y_0(t)dt}{t}, \int \frac{K_0(t)dt}{t}$

11.1.19

$$\int_0^x \frac{1-J_0(t)}{t}\,dt$$

$$=2x^{-1}\sum_{k=0}^{\infty} (2k+3)[\psi(k+2)-\psi(1)]\,J_{2k+3}(x)$$

$$=1-2x^{-1}J_1(x)$$

$$+2x^{-1}\sum_{k=0}^{\infty} (2k+5)[\psi(k+3)-\psi(1)-1]J_{2k+5}(x)$$

For $\psi(z)$, see **6.3.**

11.1.20

$$\int_x^{\infty} \frac{J_0(t)dt}{t}+\gamma+\ln\frac{x}{2}=\int_0^x \frac{[1-J_0(t)]dt}{t}$$

$$=-\sum_{k=1}^{\infty} \frac{(-)^k\left(\frac{x}{2}\right)^{2k}}{2k(k!)^2}$$

11.1.21

$$\int_x^{\infty} \frac{Y_0(t)dt}{t}=-\frac{1}{\pi}\left(\ln\frac{x}{2}\right)^2-\frac{2\gamma}{\pi}\left(\ln\frac{x}{2}\right)+\frac{1}{\pi}\left(\frac{\pi^2}{6}-\gamma^2\right)$$

$$+\frac{2}{\pi}\sum_{k=1}^{\infty} \frac{(-)^k\left(\frac{x}{2}\right)^{2k}}{2k(k!)^2}\left\{\psi(k+1)+\frac{1}{2k}-\ln\frac{x}{2}\right\}$$

11.1.22

$$\int_x^{\infty} \frac{K_0(t)dt}{t}=\frac{1}{2}\left(\ln\frac{x}{2}\right)^2+\gamma\ln\frac{x}{2}+\frac{\pi^2}{24}+\frac{\gamma^2}{2}$$

$$-\sum_{k=1}^{\infty} \frac{\left(\frac{x}{2}\right)^{2k}}{2k(k!)^2}\left\{\psi(k+1)+\frac{1}{2k}-\ln\frac{x}{2}\right\}$$

11.1.23

$$\int_{-ix}^{-i\infty} \frac{K_0(t)dt}{t}=\frac{i\pi}{2}\int_x^{\infty} \frac{J_0(t)dt}{t}-\frac{\pi}{2}\int_x^{\infty} \frac{Y_0(t)dt}{t}$$

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[2] Approximation **11.1.16** is from A. J. M. Hitchcock. Polynomial approximations to Bessel functions of order zero and one and to related functions, Math. Tables Aids Comp. **11**, 86–88 (1957) (with permission).

Asymptotic Expansions

11.1.24 $$\int_x^\infty \frac{\mathscr{C}_0(t)dt}{t}=\frac{2g_1(x)\mathscr{C}_0(x)}{x^2}-\frac{g_0(x)\mathscr{C}_1(x)}{x}$$

where

$$g_0(x)\sim\sum_{k=0}^\infty(-)^k\left(\frac{x}{2}\right)^{-2k}(k!)^2,$$

$$g_1(x)\sim\sum_{k=0}^\infty(-)^k\left(\frac{x}{2}\right)^{-2k}k!(k+1)!$$

11.1.25 $$g_0(x)=2x^2\int_x^\infty\frac{g_1(t)dt}{t^3}$$

11.1.26 $$x^{3/2}e^x\int_x^\infty\frac{K_0(t)dt}{t}\sim\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\sum_{k=0}^\infty(-)^kc_kx^{-k}$$

where

11.1.27 $$c_0=1,c_1=\frac{13}{8}$$

$$2(k+1)c_{k+1}=\left[3(k+1)^2+\frac{1}{4}\right]c_k-\left(k+\frac{1}{2}\right)^3c_{k-1}$$

11.1.28 $$x^{3/2}e^{-x}\int_0^x\frac{[I_0(t)-1]dt}{t}\sim(2\pi)^{-\frac{1}{2}}\sum_{k=0}^\infty c_kx^{-k}$$

where c_k is defined as in **11.1.27.**

Polynomial Approximations

11.1.29 $5\leq x\leq\infty$

$$\int_x^\infty\frac{\mathscr{C}_0(t)dt}{t}=\frac{2g_1(x)\mathscr{C}_0(x)}{x^2}-\frac{g_0(x)\mathscr{C}_1(x)}{x}$$

where

$$g_0(x)=\sum_{k=0}^9(-)^ka_k(x/5)^{-2k}+\epsilon(x),$$

$$g_1(x)=\sum_{k=0}^9(-)^kb_k(x/5)^{-2k}+\epsilon(x)$$

$$|\epsilon(x)|\leq2\times10^{-7}$$

k	a_k	b_k
0	1. 0	1. 0
1	0. 15999 2815	0. 31998 5629
2	. 10161 9385	. 30485 8155
3	. 13081 1585	. 52324 6341
4	. 20740 4022	1. 03702 0112
5	. 28330 0508	1. 69980 3050
6	. 27902 9488	1. 95320 6413
7	. 17891 5710	1. 43132 5684
8	. 06622 8328	0. 59605 4956
9	. 01070 2234	. 10702 2336

11.1.30 $4\leq x\leq\infty$

$$x^{\frac{3}{2}}e^x\int_x^\infty\frac{K_0(t)dt}{t}=\sum_{k=0}^6(-)^kd_k\left(\frac{x}{4}\right)^{-k}+\epsilon(x)$$

$$|\epsilon(x)|\leq6\times10^{-6}$$

k	d_k
0	1. 25331 41
1	0. 50913 39
2	. 32191 84
3	. 26214 46
4	. 20601 26
5	. 11103 96
6	. 02724 00

11.1.31 $5\leq x\leq\infty$

$$x^{\frac{3}{2}}e^{-x}\int_0^x\frac{[I_0(t)-1]dt}{t}=\sum_{k=0}^{10}f_k\left(\frac{x}{5}\right)^{-k}+\epsilon(x)$$

$$|\epsilon(x)|\leq1.1\times10^{-5}$$

k	f_k
0	0. 39893 14
1	. 13320 55
2	−. 04938 43
3	1. 47800 44
4	−8. 65560 13
5	28. 12214 78
6	−48. 05241 15
7	40. 39473 40
8	−11. 90943 95
9	−3. 51950 09
10	2. 19454 64

11.2. Repeated Integrals of $J_n(z)$ and $K_0(z)$

Repeated Integrals of $J_n(z)$

Let

11.2.1

$$f_{0,n}(z)=J_n(z),$$

$$f_{1,n}(z)=\int_0^zJ_n(t)dt,\ \ldots,\ f_{r,n}(z)=\int_0^zf_{r-1,n}(t)dt$$

11.2.2 $$f_{-r,n}(z)=\frac{d^r}{dz^r}J_n(z)$$

Then

11.2.3

$$f_{r,n}(z)=\frac{1}{\Gamma(r)}\int_0^z(z-t)^{r-1}J_n(t)dt\quad(\mathscr{R}r>0)$$

11.2.4 $$f_{r,n}(z)=\frac{2^r}{\Gamma(r)}\sum_{k=0}^\infty\frac{\Gamma(k+r)}{k!}J_{n+r+2k}(z)$$

Recurrence Relations

11.2.5

$$r(r-1)f_{r+1,\,n}(z)=2(r-1)zf_{r,\,n}(z)$$
$$-[(1-r)^2-n^2+z^2]f_{r-1,\,n}(z)$$
$$+(2r-3)zf_{r-2,\,n}(z)-z^2f_{r-3,\,n}(z)$$

11.2.6

$$rf_{r+1,\,0}(z)=zf_{r,\,0}(z)-(r-1)f_{r-1,\,0}(z)+zf_{r-2,\,0}(z)$$

11.2.7 $$f_{r+1,\,n+1}(z)=f_{r+1,\,n-1}(z)-2f_{r,\,n}(z)$$

Repeated Integrals of $K_0(z)$

Let

11.2.8

$$\mathrm{Ki}_0(z)=K_0(z),$$
$$\mathrm{Ki}_1(z)=\int_z^\infty K_0(t)dt,\ \ldots,\ \mathrm{Ki}_r(z)=\int_z^\infty \mathrm{Ki}_{r-1}(t)dt$$

11.2.9 $$\mathrm{Ki}_{-r}(z)=(-)^r\,\frac{d^r}{dz^r}\,K_0(z)$$

Then

11.2.10

$$\mathrm{Ki}_r(z)=\int_0^\infty \frac{e^{-z\cosh t}\,dt}{\cosh^r t}$$
$$(\mathscr{R}z\geq 0,\ \mathscr{R}r>0,\ \mathscr{R}z>0,\ r=0)$$

11.2.11

$$\mathrm{Ki}_r(z)=\frac{1}{\Gamma(r)}\int_z^\infty (t-z)^{r-1}K_0(t)dt$$
$$(\mathscr{R}z\geq 0,\ \mathscr{R}r>0)$$

11.2.12 $$\mathrm{Ki}_{2r}(0)=\frac{\Gamma(r)\Gamma(\frac{1}{2})}{\Gamma(r+\frac{1}{2})}\qquad(\mathscr{R}r>0)$$

11.2.13 $$\mathrm{Ki}_{2r+1}(0)=\frac{\frac{\pi}{2}\,\Gamma(r+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(r+1)}\qquad\left(\mathscr{R}r>-\frac{1}{2}\right)$$

11.2.14

$$r\mathrm{Ki}_{r+1}(z)=-z\mathrm{Ki}_r(z)+(r-1)\mathrm{Ki}_{r-1}(z)+z\mathrm{Ki}_{r-2}(z)$$

11.3. Reduction Formulas for Indefinite Integrals

Let

11.3.1 $$g_{\mu,\nu}(z)=\int^z e^{-pt}t^\mu Z_\nu(t)dt$$

where $Z_\nu(z)$ represents any of the Bessel functions of the first three kinds or the modified Bessel functions. The parameters a and b appearing in the reduction formulae are associated with the particular type of Bessel function as delineated in the following table.

11.3.2

$Z_\nu(z)$	a	b
$J_\nu(z),\ Y_\nu(z),\ H_\nu^{(1)}(z),\ H_\nu^{(2)}(z)$	1	1
$I_\nu(z)$	-1	1
$K_\nu(z)$	1	-1

11.3.3

$$pg_{\mu,\,\nu}(z)=-e^{-pz}z^\mu Z_\nu(z)$$
$$+(\mu+\nu)g_{\mu-1,\,\nu}(z)-ag_{\mu,\nu+1}(z)$$

11.3.4

$$pg_{\mu,\,\nu+1}(z)=-e^{-pz}z^\mu Z_{\nu+1}(z)$$
$$+(\mu-\nu-1)g_{\mu-1,\nu+1}(z)+bg_{\mu,\nu}(z)$$

11.3.5

$$(p^2+ab)g_{\mu,\nu}(z)=ae^{-pz}z^\mu Z_{\nu+1}(z)$$
$$+(\mu-\nu-1)e^{-pz}z^{\mu-1}Z_\nu(z)-pe^{-pz}z^\mu Z_\nu(z)$$
$$+p(2\mu-1)g_{\mu-1,\nu}(z)+[\nu^2-(\mu-1)^2]g_{\mu-2,\nu}(z)$$

11.3.6

$$a(\nu-\mu)g_{\mu,\nu+1}(z)=-2\nu e^{-pz}z^\mu Z_\nu(z)-2\nu pg_{\mu,\nu}(z)$$
$$+b(\mu+\nu)g_{\mu,\nu-1}(z)$$

Case 1: $p^2+ab=0,\ \nu=\pm(\mu-1)$

11.3.7 $$g_{\nu,\,\nu}(z)=\frac{e^{-pz}z^{\nu+1}}{2\nu+1}\left\{Z_\nu(z)-\frac{a}{p}\,Z_{\nu+1}(z)\right\}$$

11.3.8 $$g_{-\nu,\,\nu}(z)=-\frac{e^{-pz}z^{-\nu+1}}{2\nu-1}\left\{Z_\nu(z)+\frac{b}{p}\,Z_{\nu-1}(z)\right\}$$

11.3.9

$$\int_0^z e^{it}t^\nu J_\nu(t)dt=\frac{e^{iz}z^{\nu+1}}{2\nu+1}\,[J_\nu(z)-iJ_{\nu+1}(z)]$$
$$(\mathscr{R}\nu>-\tfrac{1}{2})$$

11.3.10

$$\int_0^z e^{it}t^{-\nu}J_\nu(t)dt=-\frac{e^{iz}z^{-\nu+1}}{2\nu-1}\,[J_\nu(z)+iJ_{\nu-1}(z)]$$
$$+\frac{i}{2^{\nu-1}(2\nu-1)\Gamma(\nu)}\qquad(\nu\neq\tfrac{1}{2})$$

11.3.11

$$\int_0^z e^{it}t^\nu Y_\nu(t)dt=\frac{e^{iz}z^{\nu+1}}{2\nu+1}\,[Y_\nu(z)-iY_{\nu+1}(z)]$$
$$-\frac{i2^{\nu+1}\Gamma(\nu+1)}{\pi(2\nu+1)}\qquad(\mathscr{R}\nu>-\tfrac{1}{2})$$

11.3.12

$$\int_0^z e^{\pm t}t^\nu I_\nu(t)dt=\frac{e^{\pm z}z^{\nu+1}}{2\nu+1}\,[I_\nu(z)\mp I_{\nu+1}(z)]$$
$$(\mathscr{R}\nu>-\tfrac{1}{2})$$

11.3.13

$$\int_0^z e^{-t} I_n(t)dt = ze^{-z}[I_0(z)+I_1(z)] + n[e^{-z}I_0(z)-1] + 2e^{-z}\sum_{k=1}^{n-1}(n-k)I_k(z)$$

11.3.14

$$\int_0^z e^{\pm t} t^{-\nu} I_\nu(t)dt = -\frac{e^{\pm z}z^{-\nu+1}}{2\nu-1}[I_\nu(z)\mp I_{\nu-1}(z)] \mp \frac{1}{2^{\nu-1}(2\nu-1)\Gamma(\nu)} \qquad (\nu\neq\tfrac{1}{2})$$

11.3.15

$$\int_0^z e^{\pm t} t^{\nu} K_\nu(t)dt = \frac{e^{\pm z}z^{\nu+1}}{2\nu+1}[K_\nu(z)\pm K_{\nu+1}(z)] \mp \frac{2^\nu\Gamma(\nu+1)}{2\nu+1} \qquad (\mathscr{R}\nu>-\tfrac{1}{2})$$

King's integral (see [11.5])

11.3.16 $\int_0^z e^t K_0(t)dt = ze^z[K_0(z)+K_1(z)]-1$

11.3.17

$$\int_z^\infty e^t t^{-\nu} K_\nu(t)dt = \frac{e^z z^{-\nu+1}}{2\nu-1}[K_\nu(z)+K_{\nu-1}(z)] \qquad (\mathscr{R}\nu>\tfrac{1}{2})$$

Case 2: $p=0, \mu=\pm\nu$

11.3.18 $bg_{\nu,\nu-1}(z)=z^\nu Z_\nu(z)$

11.3.19 $ag_{-\nu,\nu+1}(z)=-z^{-\nu}Z_\nu(z)$

11.3.20 $\int_0^z t^\nu J_{\nu-1}(t)dt = z^\nu J_\nu(z) \qquad (\mathscr{R}\nu>0)$

11.3.21 $\int_0^z t^{-\nu} J_{\nu+1}(t)dt = \frac{1}{2^\nu\Gamma(\nu+1)} - z^{-\nu}J_\nu(z)$

11.3.22

$$2n\int_0^z \frac{J_{2n}(t)dt}{t} = 1-\frac{2}{z}\sum_{k=1}^{n}(2k-1)J_{2k-1}(z) = \frac{2}{z}\sum_{k=n+1}^{\infty}(2k-1)J_{2k-1}(z) \qquad (n>0)$$

11.3.23

$$(2n+1)\int_0^z \frac{J_{2n+1}(t)dt}{t} = \int_0^z J_0(t)dt - J_1(z) - \frac{4}{z}\sum_{k=1}^{n}kJ_{2k}(z)$$

11.3.24

$$\int_0^z t^\nu Y_{\nu-1}(t)dt = z^\nu Y_\nu(z) + \frac{2^\nu\Gamma(\nu)}{\pi} \qquad (\mathscr{R}\nu>0)$$

11.3.25 $\int_0^z t^\nu I_{\nu-1}(t)dt = z^\nu I_\nu(z) \qquad (\mathscr{R}\nu>0)$

11.3.26 $\int_0^z t^{-\nu} I_{\nu+1}(t)dt = z^{-\nu}I_\nu(z) - \frac{1}{2^\nu\Gamma(\nu+1)}$

11.3.27

$$\int_0^z t^\nu K_{\nu-1}(t)dt = -z^\nu K_\nu(z) + 2^{\nu-1}\Gamma(\nu) \qquad (\mathscr{R}\nu>0)$$

11.3.28 $\int_z^\infty t^{-\nu} K_{\nu+1}(t)dt = z^{-\nu}K_\nu(z)$

Indefinite Integrals of Products of Bessel Functions

Let $\mathscr{C}_\mu(z)$ and $\mathscr{D}_\nu(z)$ denote any two cylinder functions of orders μ and ν respectively.

11.3.29

$$\int^z \left\{(k^2-l^2)t - \frac{(\mu^2-\nu^2)}{t}\right\}\mathscr{C}_\mu(kt)\mathscr{D}_\nu(lt)dt = z\{k\mathscr{C}_{\mu+1}(kz)\mathscr{D}_\nu(lz) - l\mathscr{C}_\mu(kz)\mathscr{D}_{\nu+1}(lz)\} \quad * \quad -(\mu-\nu)\mathscr{C}_\mu(kz)\mathscr{D}_\nu(lz)$$

11.3.30

$$\int^z t^{-\mu-\nu-1}\mathscr{C}_{\mu+1}(t)\mathscr{D}_{\nu+1}(t)dt = -\frac{z^{-\mu-\nu}}{2(\mu+\nu+1)}\{\mathscr{C}_\mu(z)\mathscr{D}_\nu(z)+\mathscr{C}_{\mu+1}(z)\mathscr{D}_{\nu+1}(z)\}$$

11.3.31

$$\int^z t^{\mu+\nu+1}\mathscr{C}_\mu(t)\mathscr{D}_\nu(t)dt = \frac{z^{\mu+\nu+2}}{2(\mu+\nu+1)}\{\mathscr{C}_\mu(z)\mathscr{D}_\nu(z)+\mathscr{C}_{\mu+1}(z)\mathscr{D}_{\nu+1}(z)\}$$

11.3.32

$$\int_0^z tJ_{\nu-1}^2(t)dt = 2\sum_{k=0}^{\infty}(\nu+2k)J_{\nu+2k}^2(z) \qquad (\mathscr{R}\nu>0)$$

11.3.33

$$\int_0^z t[J_{\nu-1}^2(t)-J_{\nu+1}^2(t)]dt = 2\nu J_\nu^2(z) \qquad (\mathscr{R}\nu>0)$$

11.3.34 $\int_0^z tJ_0^2(t)dt = \frac{z^2}{2}[J_0^2(z)+J_1^2(z)]$

11.3.35

$$\int_0^z J_n(t)J_{n+1}(t)dt = \frac{1}{2}[1-J_0^2(z)] - \sum_{k=1}^{n}J_k^2(z) = \sum_{k=n+1}^{\infty}J_k^2(z)$$

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*See page II.

11.3.36

$$(\mu+\nu)\int^{z} t^{-1}\mathscr{C}_\mu(t)\mathscr{D}_\nu(t)dt$$

$$-(\mu+\nu+2n)\int^{z} t^{-1}\mathscr{C}_{\mu+n}(t)\mathscr{D}_{\nu+n}(t)dt$$

$$=\mathscr{C}_\mu(z)\mathscr{D}_\nu(z)+\mathscr{C}_{\mu+n}(z)\mathscr{D}_{\nu+n}(z)+2\sum_{k=1}^{n-1}\mathscr{C}_{\mu+k}(z)\mathscr{D}_{\nu+k}(z)$$

Convolution Type Integrals

11.3.37

$$\int_0^z J_\mu(t)J_\nu(z-t)dt=2\sum_{k=0}^{\infty}(-)^k J_{\mu+\nu+2k+1}(z)$$

$$(\mathscr{R}\mu>-1, \mathscr{R}\nu>-1)$$

11.3.38

$$\int_0^z J_\nu(t)J_{1-\nu}(z-t)dt=J_0(z)-\cos z \quad (-1<\mathscr{R}\nu<2)$$

11.3.39

$$\int_0^z J_\nu(t)J_{-\nu}(z-t)\,dt=\sin z \quad (|\mathscr{R}\nu|<1)$$

11.3.40

$$\int_0^z t^{-1}J_\mu(t)J_\nu(z-t)dt=\frac{J_{\mu+\nu}(z)}{\mu}$$

$$(\mathscr{R}\mu>0, \mathscr{R}\nu>-1)$$

11.3.41

$$\int_0^z \frac{J_\mu(t)J_\nu(z-t)dt}{t(z-t)}=\frac{(\mu+\nu)J_{\mu+\nu}(z)}{\mu\nu z}$$

$$(\mathscr{R}\mu>0, \mathscr{R}\nu>0)$$

11.4. Definite Integrals

Orthogonality Properties of Bessel Functions

Let $\mathscr{C}_\nu(z)$ be a cylinder function of order ν. In particular, let

11.4.1 $$\mathscr{C}_\nu(z)=AJ_\nu(z)+BY_\nu(z)$$

where A and B are real constants. Then

11.4.2

$$\int_a^b t\mathscr{C}_\nu(\lambda_m t)\mathscr{C}_\nu(\lambda_n t)dt=0\ (m\neq n)$$

$$=\left[\frac{1}{2}t^2\left\{\left(1-\frac{\nu^2}{\lambda_n^2t^2}\right)\mathscr{C}_\nu^2(\lambda_n t)+\mathscr{C}_\nu'^2(\lambda_n t)\right\}\right]_a^b$$

$$(m=n)(0<a<b)$$

provided the following two conditions hold:

1. λ_n is a real zero of

11.4.3 $$h_1\lambda\mathscr{C}_{\nu+1}(\lambda b)-h_2\mathscr{C}_\nu(\lambda b)=0$$

2. There must exist numbers k_1 and k_2 (both not zero) so that for all n

11.4.4 $$k_1\lambda_n\mathscr{C}_{\nu+1}(\lambda_n a)-k_2\mathscr{C}_\nu(\lambda_n a)=0$$

In connection with these formulae, see **11.3.29.** If $a=0$, the above is valid provided $B=0$. This case is covered by the following result.

11.4.5

$$\int_0^1 tJ_\nu(\alpha_m t)J_\nu(\alpha_n t)dt=0 \qquad (m\neq n, \nu>-1)$$

$$=\tfrac{1}{2}[J_\nu'(\alpha_n)]^2$$

$$(m=n, b=0, \nu>-1)$$

$$=\frac{1}{2\alpha_n^2}\left[\frac{a^2}{b^2}+\alpha_n^2-\nu^2\right][J_\nu(\alpha_n)]^2$$

$$(m=n,\ b\neq 0,\ \nu\geq -1)$$

$\alpha_1, \alpha_2, \ldots$ are the positive zeros of $aJ_\nu(x)+bxJ_\nu'(x)=0$, where a and b are real constants.

11.4.6

$$\int_0^\infty t^{-1}J_{\nu+2n+1}(t)J_{\nu+2m+1}(t)dt=0 \qquad (m\neq n)$$

$$=\frac{1}{2(2n+\nu+1)}$$

$$(m=n)(\nu+n+m>-1)$$

Definite Integrals Over a Finite Range

11.4.7 $$\int_0^{\frac{\pi}{2}} J_{2n}(2z\sin t)dt=\frac{\pi}{2}J_n^2(z)$$

11.4.8 $$\int_0^\pi J_0(2z\sin t)\cos 2nt dt=\pi J_n^2(z)$$

11.4.9 $$\int_0^{\frac{\pi}{2}} Y_0(2z\sin t)\cos 2nt dt=\frac{\pi}{2}J_n(z)Y_n(z)$$

11.4.10

$$\int_0^{\frac{\pi}{2}} J_\mu(z\sin t)\sin^{\mu+1}t\cos^{2\nu+1}t\,dt$$

$$=\frac{2^\nu\Gamma(\nu+1)}{z^{\nu+1}}J_{\mu+\nu+1}(z) \qquad (\mathscr{R}\mu>-1, \mathscr{R}\nu>-1)$$

11.4.11

$$\int_0^{\frac{\pi}{2}} J_\mu(z\sin^2 t)J_\nu(z\cos^2 t)\csc 2t dt$$

$$=\frac{(\mu+\nu)}{4\mu\nu}J_{\mu+\nu}(z) \qquad (\mathscr{R}\mu>0, \mathscr{R}\nu>0)$$

Infinite Integrals

Integrals of the Form $\int_0^\infty e^{-pt} t^\mu Z_\nu(t)dt$

11.4.12

$$\int_0^\infty e^{it} t^{\mu-1} J_\nu(t)dt = \frac{e^{\frac{1}{2}i\pi(\mu+\nu)}\Gamma(\mu+\nu)\Gamma(\frac{1}{2}-\mu)}{\Gamma(\frac{1}{2})2^\mu\Gamma(\nu-\mu+1)}$$

$$\left(\mathscr{R}\mu<\frac{1}{2}, \mathscr{R}(\mu+\nu)>0\right)$$

11.4.13

$$\int_0^\infty e^{-t} t^{\mu-1} I_\nu(t)dt = \frac{\Gamma(\mu+\nu)\Gamma(\frac{1}{2}-\mu)}{\Gamma(\frac{1}{2})2^\mu\Gamma(\nu-\mu+1)}$$

$$\left(\mathscr{R}\mu<\frac{1}{2}, \mathscr{R}(\mu+\nu)>0\right)$$

11.4.14

$$\int_0^\infty \cos bt\, K_0(t)\,dt = \frac{\frac{1}{2}\pi}{(1+b^2)^{\frac{1}{2}}} \qquad (|\mathscr{I}b|<1)$$

11.4.15

$$\int_0^\infty \sin bt\, K_0(t)dt = \frac{\text{arc sinh } b}{(1+b^2)^{\frac{1}{2}}} \qquad (|\mathscr{I}b|<1)$$

11.4.16 $$\int_0^\infty t^\mu J_\nu(t)dt = \frac{2^\mu\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)}$$

$$\left(\mathscr{R}(\mu+\nu)>-1, \mathscr{R}\mu<\frac{1}{2}\right)$$

11.4.17 $$\int_0^\infty J_\nu(t)dt = 1 \qquad (\mathscr{R}\nu>-1)$$

11.4.18

$$\int_0^\infty \frac{[1-J_0(t)]dt}{t^\mu} = \frac{\Gamma\left(\frac{\mu-1}{2}\right)\Gamma\left(\frac{3-\mu}{2}\right)}{2^\mu\left\{\Gamma\left(\frac{\mu+1}{2}\right)\right\}^2} \; (1<\mathscr{R}\mu<3)$$

11.4.19

$$\int_0^\infty t^\mu Y_\nu(t)dt = \frac{2^\mu}{\pi}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)$$

$$\times \sin\frac{\pi}{2}(\mu-\nu)\left(\mathscr{R}(\mu\pm\nu)>-1, \mathscr{R}\mu<\frac{1}{2}\right)$$

11.4.20 $$\int_0^\infty Y_\nu(t)dt = -\tan\frac{\nu\pi}{2} \qquad (|\mathscr{R}\nu|<1)$$

11.4.21 $$\int_0^\infty Y_0(t)dt = 0$$

11.4.22

$$\int_0^\infty t^\mu K_\nu(t)dt = 2^{\mu-1}\Gamma\left(\frac{\mu+\nu+1}{2}\right)\Gamma\left(\frac{\mu-\nu+1}{2}\right)$$

$$(\mathscr{R}(\mu\pm\nu)>-1)$$

11.4.23 $$\int_0^\infty K_0(t)dt = \frac{\pi}{2}$$

11.4.24 $$\int_{-\infty}^\infty e^{-i\omega t}J_n(t)dt = \frac{2(-i)^n T_n(\omega)}{(1-\omega^2)^{\frac{1}{2}}} \; (\omega^2<1)$$

$$=0(\omega^2>1)$$

where $T_n(\omega)$ is the Chebyshev polynomial of the first kind (see chapter **22**).

11.4.25

$$\int_{-\infty}^\infty t^{-1}e^{-i\omega t}J_n(t)dt$$

$$=\frac{2i}{n}(-i)^n(1-\omega^2)^{\frac{1}{2}}U_{n-1}(\omega)\,(\omega^2<1)$$

$$=0(\omega^2>1)$$

where $U_n(\omega)$ is the Chebyshev polynomial of the second kind (see chapter **22**).

11.4.26

$$\int_{-\infty}^\infty t^{-\frac{1}{2}}e^{-i\omega t}J_{n+\frac{1}{2}}(t)dt = (-i)^n(2\pi)^{\frac{1}{2}}P_n(\omega)\,(\omega^2<1)$$

$$=0(\omega^2>1)$$

where $P_n(\omega)$ is the Legendre polynomial (see chapter **22**).

11.4.27

$$\int_0^\infty e^{-t}t^{\frac{a}{2}-1}J_a[2(zt)^{\frac{1}{2}}]dt = \frac{\gamma(a,z)}{z^{a/2}} \quad (\mathscr{R}a>0, \mathscr{R}z>0)$$

where $\gamma(a, z)$ is the incomplete gamma function (see chapter **6**).

Integrals of the Form $\int_0^\infty e^{-a^2t^2}t^\mu Z_\nu(bt)\,dt$

11.4.28

$$\int_0^\infty e^{-a^2t^2}t^{\mu-1}J_\nu(bt)dt$$

$$=\frac{\Gamma\left(\frac{1}{2}\nu+\frac{1}{2}\mu\right)\left(\frac{1}{2}\frac{b}{a}\right)^\nu}{2a^\mu\Gamma(\nu+1)}M\left(\frac{1}{2}\nu+\frac{1}{2}\mu, \nu+1, -\frac{b^2}{4a^2}\right)$$

$$(\mathscr{R}(\mu+\nu)>0, \mathscr{R}a^2>0)$$

where the notation $M(a, b, z)$ stands for the confluent hypergeometric function (see chapter **13**).

11.4.29

$$\int_0^\infty e^{-a^2t^2}t^{\nu+1}J_\nu(bt)dt$$

$$=\frac{b^\nu}{(2a^2)^{\nu+1}}e^{-\frac{b^2}{4a^2}} \qquad (\mathscr{R}\nu>-1, \mathscr{R}a^2>0)$$

11.4.30

$$\int_0^\infty e^{-a^2t^2}Y_{2\nu}(bt)dt=-\frac{\pi^{\frac{1}{2}}}{2a}e^{-\frac{b^2}{8a^2}}\left[I_\nu\left(\frac{b^2}{8a^2}\right)\tan\nu\pi\right.$$

$$\left.+\frac{1}{\pi}K_\nu\left(\frac{b^2}{8a^2}\right)\sec\nu\pi\right]\qquad\left(|\Re\nu|<\frac{1}{2},\Re a^2>0\right)$$

11.4.31

$$\int_0^\infty e^{-a^2t^2}I_\nu(bt)dt=\frac{\pi^{\frac{1}{2}}}{2a}e^{\frac{b^2}{8a^2}}I_{\frac{1}{2}\nu}\left(\frac{b^2}{8a^2}\right)$$

$$(\Re\nu>-1,\Re a^2>0)$$

11.4.32

$$\int_0^\infty e^{-a^2t^2}K_0(bt)dt=\frac{\pi^{\frac{1}{2}}}{4a}e^{\frac{b^2}{8a^2}}K_0\left(\frac{b^2}{8a^2}\right)\qquad(\Re a^2>0)$$

Weber-Schafheitlin Type Integrals

11.4.33

$$\int_0^\infty\frac{J_\mu(at)J_\nu(bt)dt}{t^\lambda}=\frac{b^\nu\Gamma\left(\frac{\mu+\nu-\lambda+1}{2}\right)}{2^\lambda a^{\nu-\lambda+1}\Gamma(\nu+1)\Gamma\left(\frac{\mu-\nu+\lambda+1}{2}\right)}$$

$$\times{}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2},\frac{\nu-\mu-\lambda+1}{2};\nu+1;\frac{b^2}{a^2}\right)$$

$$(\Re(\mu+\nu-\lambda+1)>0,\Re\lambda>-1,0<b<a)$$

11.4.34

$$\int_0^\infty\frac{J_\mu(at)J_\nu(bt)dt}{t^\lambda}=\frac{a^\mu\Gamma\left(\frac{\mu+\nu-\lambda+1}{2}\right)}{2^\lambda b^{\mu-\lambda+1}\Gamma(\mu+1)\Gamma\left(\frac{\nu-\mu+\lambda+1}{2}\right)}$$

$$\times{}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2},\frac{\mu-\nu-\lambda+1}{2};\mu+1;\frac{a^2}{b^2}\right)$$

$$(\Re(\mu+\nu-\lambda+1)>0,\Re\lambda>-1,0<a<b)$$

For ${}_2F_1$, see chapter **15.**

Special Cases of the Discontinuous Weber-Schafheitlin Integral

11.4.35

$$\int_0^\infty\frac{J_\mu(at)\sin bt\,dt}{t}=\frac{1}{\mu}\sin\left[\mu\arcsin\frac{b}{a}\right]\qquad(0\leq b\leq a)$$

$$=\frac{a^\mu\sin\frac{\pi\mu}{2}}{\mu[b+(b^2-a^2)^{\frac{1}{2}}]^\mu}\qquad(b\geq a>0)$$

$$(\Re\mu>-1)$$

11.4.36

$$\int_0^\infty\frac{J_\mu(at)\cos bt\,dt}{t}=\frac{1}{\mu}\cos\left[\mu\arcsin\frac{b}{a}\right]\qquad(0\leq b\leq a)$$

$$=\frac{a^\mu\cos\frac{\pi\mu}{2}}{\mu[b+(b^2-a^2)^{\frac{1}{2}}]^\mu}\qquad(b\geq a>0)$$

$$(\Re\mu>0)$$

11.4.37

$$\int_0^\infty J_\mu(at)\cos bt\,dt=\frac{\cos\left[\mu\arcsin\frac{b}{a}\right]}{(a^2-b^2)^{\frac{1}{2}}}\qquad(0\leq b<a)$$

$$=\frac{-a^\mu\sin\frac{\pi\mu}{2}}{(b^2-a^2)^{\frac{1}{2}}[b+(b^2-a^2)^{\frac{1}{2}}]^\mu}$$

$$(b>a>0)\qquad(\Re\mu>-1)$$

11.4.38

$$\int_0^\infty J_\mu(at)\sin bt\,dt=\frac{\sin\left[\mu\arcsin\frac{b}{a}\right]}{(a^2-b^2)^{\frac{1}{2}}}\qquad(0\leq b<a)$$

$$=\frac{a^\mu\cos\frac{\pi\mu}{2}}{(b^2-a^2)^{\frac{1}{2}}[b+(b^2-a^2)^{\frac{1}{2}}]^\mu}$$

$$(b>a>0)\qquad(\Re\mu>-2)$$

11.4.39 $$\int_0^\infty e^{ibt}J_0(at)dt=\frac{1}{(a^2-b^2)^{\frac{1}{2}}}\qquad(0\leq b<a)$$

$$=\frac{i}{(b^2-a^2)^{\frac{1}{2}}}\qquad(0<a<b)$$

11.4.40

$$\int_0^\infty e^{ibt}Y_0(at)dt=\frac{2i}{\pi(a^2-b^2)^{\frac{1}{2}}}\arcsin\frac{b}{a}\qquad(0\leq b<a)$$

$$=\frac{-1}{(b^2-a^2)^{\frac{1}{2}}}+\frac{2i}{\pi(b^2-a^2)^{\frac{1}{2}}}$$

$$\times\ln\left\{\frac{b-(b^2-a^2)^{\frac{1}{2}}}{a}\right\}\quad(0<a<b)$$

11.4.41

$$\int_0^\infty t^{\mu-\nu+1}J_\mu(at)J_\nu(bt)dt=0\qquad(0<b<a)$$

$$=\frac{2^{\mu-\nu+1}a^\mu(b^2-a^2)^{\nu-\mu-1}}{b^\nu\Gamma(\nu-\mu)}$$

$$(b>a>0)$$

$$(\Re\nu>\Re\mu>-1)$$

11.4.42 $$\int_0^\infty J_\mu(at)J_{\mu-1}(bt)dt=\frac{b^{\mu-1}}{a^\mu}\qquad(0<b<a)$$

$$=\frac{1}{2b}\qquad(0<b=a)$$

$$=0\qquad(b>a>0)$$

$$(\Re\mu>0)$$

11.4.43 $$\int_0^\infty\frac{J_0(at)}{t}\{1-J_0(bt)\}dt=0\qquad(0<b\leq a)$$

$$=\ln\frac{b}{a}\quad(b\geq a>0)$$

Hankel-Nicholson Type Integrals

11.4.44

$$\int_0^\infty \frac{t^{\nu+1}J_\nu(at)dt}{(t^2+z^2)^{\mu+1}}=\frac{a^\mu z^{\nu-\mu}}{2^\mu\Gamma(\mu+1)}K_{\nu-\mu}(az)$$

$$\left(a>0,\ \mathscr{R}z>0, -1<\mathscr{R}\nu<2\mathscr{R}\mu+\frac{3}{2}\right)$$

11.4.45

$$\int_0^\infty \frac{J_\nu(at)dt}{t^\nu(t^2+z^2)}=\frac{\pi}{2z^{\nu+1}}[I_\nu(az)-\mathbf{L}_\nu(az)]$$

$$\left(a>0,\ \mathscr{R}z>0,\ \mathscr{R}\nu>-\frac{5}{2}\right)$$

11.4.46

$$\int_0^\infty \frac{Y_0(at)dt}{t^2+z^2}=-\frac{K_0(az)}{z} \qquad (a>0,\ \mathscr{R}z>0)$$

11.4.47

$$\int_0^\infty \frac{K_\nu(at)dt}{t^\nu(t^2+z^2)}=\frac{\pi^2}{4z^{\nu+1}\cos\nu\pi}[\mathbf{H}_\nu(az)-Y_\nu(az)]$$

$$(\mathscr{R}a>0,\ \mathscr{R}z>0,\ \mathscr{R}\nu<\tfrac{1}{2})$$

11.4.48

$$\int_0^\infty \frac{J_\nu(at)dt}{(t^2+z^2)^{\frac{1}{2}}}=I_{\frac{1}{2}\nu}(\tfrac{1}{2}az)K_{\frac{1}{2}\nu}(\tfrac{1}{2}az)$$

$$(a>0,\ \mathscr{R}z>0,\ \mathscr{R}\nu>-1)$$

11.4.49

$$\int_0^\infty \frac{J_\nu(at)dt}{t^\nu(t^2+z^2)^{\nu+\frac{1}{2}}}=\frac{\left(\frac{2a}{z^2}\right)^\nu\Gamma(\nu+1)}{\Gamma(2\nu+1)}I_\nu(\tfrac{1}{2}az)K_\nu(\tfrac{1}{2}az)$$

$$(a>0,\ \mathscr{R}z>0,\ \mathscr{R}\nu>-\tfrac{1}{2})$$

Numerical Methods

11.5. Use and Extension of the Tables

$$\int_0^x J_0(t)dt, \int_0^x Y_0(t)dt, \int_0^x I_0(t)dt, \int_x^\infty K_0(t)dt$$

For moderate values of x, use **11.1.2** and **11.1.7–11.1.10** as appropriate. For x sufficiently large, use the asymptotic expansions or the polynomial approximations **11.1.11–11.1.18.**

Example 1. Compute $\int_0^{3.05} J_0(t)dt$ to 5D. Using **11.1.2** and interpolating in **Tables 9.1** and **9.2,** we have

$$\begin{aligned}\int_0^{3.05} J_0(t)dt&=2[.32019\ 09+.31783\ 69+.04611\ 52\\&\quad+.00283\ 19+.00009\ 72+.00000\ 21]\\&=1.37415\end{aligned}$$

Example 2. Compute $\int_0^{3.05} J_0(t)dt$ to 5D by interpolation of **Table 11.1** using Taylor's formula. We have

$$\begin{aligned}\int_0^{x+h} J_0(t)dt&=\int_0^x J_0(t)dt+hJ_0(x)-\frac{h^2}{2}J_1(x)\\&\quad+\frac{h^3}{12}[J_2(x)-J_0(x)]+\frac{h^4}{96}[3J_1(x)-J_3(x)]+\dots\end{aligned}$$

Then with $x=3.0$ and $h=.05$,

$$\begin{aligned}\int_0^{3.05} J_0(t)dt&=1.387567+(.05)(-.260052)\\&\quad-(.00125)(.339059)\\&\quad+(.000010)(.746143)=1.37415\end{aligned}$$

This value is readily checked using $x=3.1$ and $h=-.05$. Now $|J_0(x)|\leq 1$ for all x and $|J_n(x)|<2^{-\frac{1}{2}}$, $n\geq 1$ for all x. In **Table 11.1,** we can always choose $|h|\leq .05$. Thus if all terms of $O(h^4)$ and higher are neglected, then a bound for the absolute error is $2^{\frac{1}{2}}h^4/48<.2\cdot 10^{-6}$ for all x if $|h|\leq .05$. Similarly, the absolute error for quadratic interpolation does not exceed

$$h^3(2^{\frac{1}{2}}+2)/24<.2\cdot 10^{-4}.$$

Example 3. Interpolation of $\int_0^x J_0(t)dt$ using Simpson's rule. We have

$$\int_0^{x+h} J_0(t)dt=\int_0^x J_0(t)dt+\int_x^{x+h} J_0(t)dt$$

$$\int_x^{x+h} J_0(t)dt=\frac{h}{6}\left[J_0(x)+4J_0\left(x+\frac{h}{2}\right)+J_0(x+h)\right]+R$$

$$R=-\frac{h^5}{2880}J_0^{(4)}(\xi), \qquad x<\xi<x+h$$

Now

$$J_0^{(4)}(x)=\frac{1}{8}[J_4(x)-4J_2(x)+3J_0(x)]$$

$$|J_0^{(4)}(x)|<\frac{6+5\sqrt{2}}{16}<.82$$

and with $|h|\leq .05$, it follows that

$$|R|<.9\cdot 10^{-10}$$

Thus if $x=3.0$ and $h=.05$

$$\begin{aligned}\int_0^{3.05} J_0(t)dt&=1.38756\ 72520+\frac{(.05)}{6}[-.26005\ 19549\\&\quad+4(-.26841\ 13883)-.27653\ 49599]\\&=1.37414\ 86481\end{aligned}$$

which is correct to 10D. The above procedure gives high accuracy though it may be necessary to interpolate twice in $J_0(x)$ to compute $J_0\left(x+\frac{h}{2}\right)$ and $J_0(x+h)$. A similar technique based on the trapezoidal rule is less accurate, but at most only one interpolation of $J_0(x)$ is required.

Example 4. Compute $\int_0^3 J_0(t)dt$ and $\int_0^3 Y_0(t)dt$ to 5D using the representation in terms of Struve functions and the tables in chapters **9** and **12.**

For $x=3$, from **Tables 9.1** and **12.1**

$$\begin{array}{ll} J_0=-.260052 & J_1=\ .339059 \\ Y_0=\ .376850 & Y_1=\ .324674 \\ \mathbf{H}_0=\ .574306 & \mathbf{H}_1=1.020110 \end{array}$$

Using **11.1.7**, we have

$$\int_0^3 J_0(t)dt=3(-.260052)+\frac{3\pi}{2}[(.574306)(.339059) \\ -(1.020110)(-.260052)] \\ =1.38757$$

Similarly,

$$\int_0^3 Y_0(t)dt=.19766$$

Using **11.1.8** and **Tables 9.8** and **12.1,** one can compute $\int_0^x I_0(t)dt$ and $\int_0^x K_0(t)dt$.

$\int_x^\infty \frac{J_0(t)dt}{t}, \int_x^\infty \frac{Y_0(t)dt}{t}, \int_0^x \frac{[I_0(t)-1]dt}{t}, \int_x^\infty \frac{K_0(t)dt}{t}$

For moderate values of x, use **11.1.19–11.1.23.** For x sufficiently large, use the asymptotic expansions or the polynomial approximations **11.1.24–11.1.31.**

Repeated Integrals of $J_0(x)$

For moderate values of x and r, use **11.2.4.** If $r=1$, see **Example 1.** For moderate values of x, use the recurrence formula **11.2.5.** If x is large and $x\gg r$, see the discussion below.

Example 5. Compute $f_{r,0}(x)=f_r(x)$ to 5D for $x=2$ and $r=0(1)5$ using **11.2.6.** We have

$$rf_{r+1}(x)=xf_r(x)-(r-1)f_{r-1}(x)+xf_{r-2}(x)$$

$$f_{-1}(x)=-J_1(x), f_0(x)=J_0(x), f_1(x)=\int_0^x J_0(t)\,dt$$

and the terms on this last line are tabulated. Thus for $x=2$,

$$f_{-1}=-.57672\ 48, f_0=.22389\ 08, f_1=1.42577\ 03$$

The recurrence formula gives

$$f_2=2(f_1+f_{-1})=1.69809\ 10$$

Similarly,

$$f_3=1.20909\ 66, f_4=.62451\ 73, f_5=.25448\ 17$$

When $x>>r$, it is convenient to use the auxiliary function

$$g_r(x)=(r-1)!x^{-r+1}f_r(x)$$

This satisfies the recurrence relation

$$x^2g_{r+1}(x)=x^2g_r-(r-1)^2g_{r-1}(x) \\ +(r-1)(r-2)g_{r-2}(x),\ r\geq 3$$

$$g_1(x)=\int_0^x J_0(t)dt,\ g_2(x)=g_1(x)-J_1(x) \\ g_3(x)=[x^2g_2(x)-g_1(x)+xJ_0(x)]/x^2$$

Example 6. Compute $g_r(x)$ to 5D for $x=10$ and $r=0(1)6$. We have for $x=10$,

$$J_0=-.24593\ 58,\ J_1=.04347\ 27,\ g_1=1.06701\ 13$$

Thus

$$g_2=1.02353\ 86,\ g_3=.98827\ 49$$

and the forward recurrence formula gives

$$g_4=.96867\ 36,\ g_5=.94114\ 12,\ g_6=.90474\ 64$$

For tables of $2^{-r}f_r(x)$, see [11.16].

Repeated Integrals of $K_0(x)$

For moderate values of x, use the recurrence formula **11.2.14** for all r.

Example 7. Compute $\mathrm{Ki}_r(x)$ to 5D for $x=2$ and $r=0(1)5$. We have

$$r\mathrm{Ki}_{r+1}(x)=-x\mathrm{Ki}_r(x)+(r-1)\mathrm{Ki}_{r-1}(x)+x\mathrm{Ki}_{r-2}(x)$$

$$\mathrm{Ki}_{-1}(x)=K_1(x),\ \mathrm{Ki}_0(x)=K_0(x),\ \mathrm{Ki}_1(x)=\int_x^\infty K_0(t)dt$$

and the functions on this last line are tabulated Thus for $x=2$,

$$K_0=.11389\ 39,\ K_1=.13986\ 59,\ \mathrm{Ki}_1=.09712\ 06$$

and

$$\mathrm{Ki}_2=-2\mathrm{Ki}_1+2K_1=.08549\ 06$$

Similarly,

$$\mathrm{Ki}_3=.07696\ 36,\ \mathrm{Ki}_4=.07043\ 17,\ \mathrm{Ki}_5=.06525\ 22$$

If x/r is not large the formula can still be used provided that the starting values are sufficiently accurate to offset the growth of rounding error.

For tables of $\mathrm{Ki}_r(x)$, see [11.11].

$$f_m(x)=x^{-m}\int_0^x t^m K_0(t)dt$$

Now

$$f_0(x)=\int_0^x K_0(t)dt, f_1(x)=[1-xK_1(x)]/x$$

the latter following from **11.3.27** with $\nu=1$. In **11.3.5,** put $a=1$, $b=-1$, $p=0$ and $\nu=0$. Let $\mu=m$. Then

$$f_m(x)=[(m-1)^2 f_{m-2}(x)-x^2K_1(x)-x(m-1)K_0(x)]/x^2 \qquad (m>1)$$

Using tabular values of f_0 and f_1, one can compute in succession $f_2, f_3, \ldots$ provided that m/x is not large.

Example 8. Compute $f_m(x)$ to 5D for $x=5$ and $m=0(1)6$. We have, retaining two additional decimals

$$K_0= .00369\ 11 \qquad K_1=.00404\ 46$$

$$f_0=1.56738\ 74 \qquad f_1=.19595\ 54$$

Thus

$$f_2=.05791\ 27, f_4=.01458\ 93, f_6=.00685\ 36$$

Similarly starting with f_1, we can compute f_3 and f_5.

If $m>x$, employ the recurrence formula in backward form and write

$$f_{m-2}(x)=[x^2f_m(x)+x^2K_1(x)+x(m-1)K_0(x)]/(m-1)^2$$

In the latter expression, replace f_m by g_m. Fix x. Take $r>m$ and assume $g_r=0$. Compute g_{r-2}, g_{r-4}, etc. Then

$$\lim_{r\to\infty} g_{r-2k}(x)=f_m(x),\ m=r-2k$$

Apart from round-off error, the value of r needed to achieve a stated accuracy for given x and m can be determined a priori. Let

$$\epsilon_r=|g_r-f_r|$$

Then

$$\epsilon_{r-2k}=\frac{x^{2k}\epsilon_r}{(r-1)^2(r-3)^2\ldots(r-2k+1)^2}$$

$$\epsilon_r\leq[x^2K_1(x)+x(r-1)K_0(x)]/(r-1)^2$$

since for x fixed, $f_r(x)$ is positive and decreases as r increases.

Example 9. Compute $f_m(x)$ to 5D for $x=3$ and $m=0(2)10$. We have

$$K_0=.03473\ 95 \qquad K_1=.04015\ 64$$

If $r=16$,

$$\epsilon_{16}<.86\cdot10^{-2} \qquad \epsilon_{10}<1.4\cdot10^{-6}$$

Taking $g_{16}=0$, we compute the following values of $g_{14}, g_{12}, \ldots, g_0$ by recurrence. Also recorded are the required values of f_m to 5D.

m	g_m	f_m
14	.00855 42	
12	.01061 09	
10	.01325 05	.01325
8	.01751 39	.01751
6	.02548 09	.02548
4	.04447 31	.04447
2	.11936 90	.11937
0	1.53994 71	1.53995

For tables of $f_m(x)$, see [11.21].

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12. Struve Functions and Related Functions

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Contents

The author acknowledges the assistance of Bertha H. Walter in the preparation and checking of the tables.

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12. Struve Functions and Related Functions

Mathematical Properties

12.1. Struve Function $\mathbf{H}_\nu(z)$

Differential Equation and General Solution

12.1.1

$$z^2\frac{d^2w}{dz^2}+z\frac{dw}{dz}+(z^2-\nu^2)w=\frac{4(\frac{1}{2}z)^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}$$

The general solution is

12.1.2 $w=aJ_\nu(z)+bY_\nu(z)+\mathbf{H}_\nu(z)$ (a,b, constants)

where $z^{-\nu}\mathbf{H}_\nu(z)$ is an entire function of z.

Power Series Expansion

12.1.3

$$\mathbf{H}_\nu(z)=(\tfrac{1}{2}z)^{\nu+1}\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{1}{2}z)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\nu+\frac{3}{2})}$$

12.1.4 $$\mathbf{H}_0(z)=\frac{2}{\pi}\left[z-\frac{z^3}{1^2\cdot 3^2}+\frac{z^5}{1^2\cdot 3^2\cdot 5^2}-\cdots\right]$$

12.1.5

$$\mathbf{H}_1(z)=\frac{2}{\pi}\left[\frac{z^2}{1^2\cdot 3}-\frac{z^4}{1^2\cdot 3^2\cdot 5}+\frac{z^6}{1^2\cdot 3^2\cdot 5^2\cdot 7}-\cdots\right]$$

Integral Representations

If $\mathscr{R}\nu>-\frac{1}{2}$,

12.1.6

$$\mathbf{H}_\nu(z)=\frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^1(1-t^2)^{\nu-\frac{1}{2}}\sin(z\,t)\,dt$$

12.1.7 $$=\frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{\pi}{2}}\sin(z\cos\theta)\sin^{2\nu}\theta\,d\theta$$

12.1.8 $$=Y_\nu(z)+\frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{1}{2})}\int_0^\infty e^{-zt}(1+t^2)^{\nu-\frac{1}{2}}dt \qquad \left(|\arg z|<\frac{\pi}{2}\right)$$

Recurrence Relations

12.1.9 $$\mathbf{H}_{\nu-1}+\mathbf{H}_{\nu+1}=\frac{2\nu}{z}\mathbf{H}_\nu+\frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}$$

12.1.10 $$\mathbf{H}_{\nu-1}-\mathbf{H}_{\nu+1}=2\mathbf{H}'_\nu-\frac{(\frac{1}{2}z)^\nu}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}$$

12.1.11 $$\mathbf{H}'_0=(2/\pi)-\mathbf{H}_1$$

12.1.12 $$\frac{d}{dz}(z^\nu\mathbf{H}_\nu)=z^\nu\mathbf{H}_{\nu-1}$$

12.1.13 $$\frac{d}{dz}(z^{-\nu}\mathbf{H}_\nu)=\frac{1}{\sqrt{\pi}\,2^\nu\Gamma(\nu+\frac{3}{2})}-z^{-\nu}\mathbf{H}_{\nu+1}$$

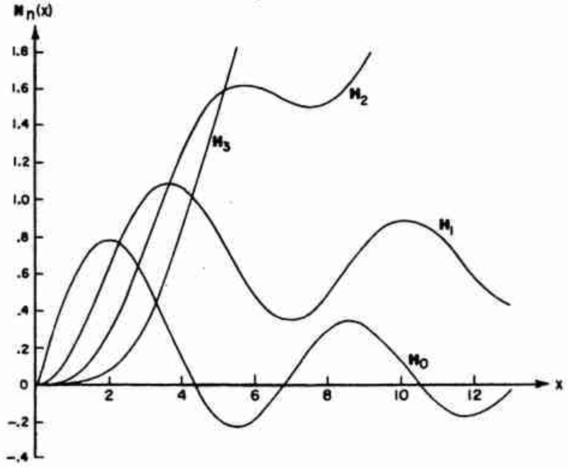

FIGURE 12.1. *Struve functions.*

$\mathbf{H}_n(x)$, $n=0(1)3$

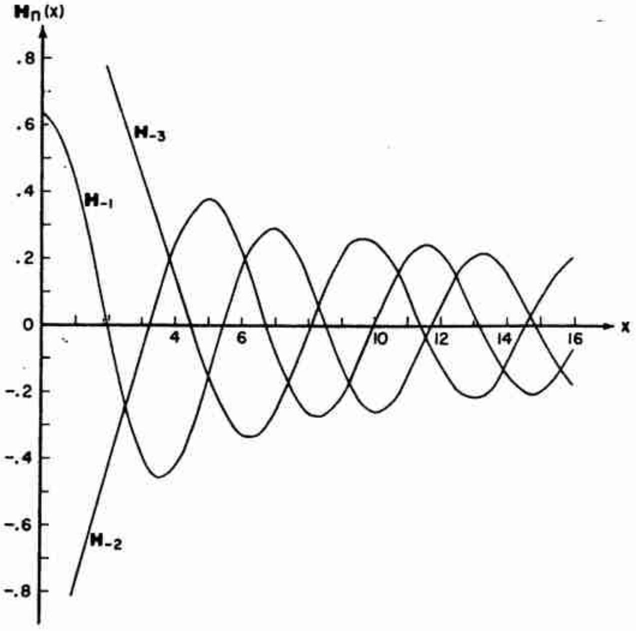

FIGURE 12.2. *Struve functions.*

$\mathbf{H}_n(x)$, $-n=1(1)3$

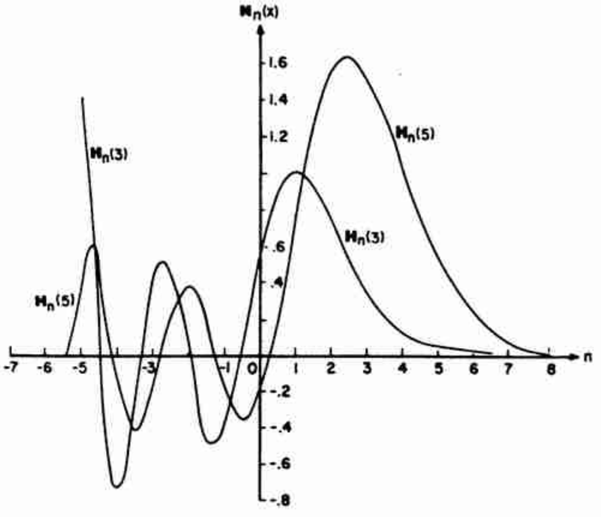

FIGURE 12.3. *Struve functions.*

$\mathbf{H}_n(x),\ x=3,\ 5$

Special Properties

12.1.14 $$\mathbf{H}_\nu(x)\geq 0 \qquad (x>0 \text{ and } \nu\geq \tfrac{1}{2})$$

12.1.15
$$\mathbf{H}_{-(n+\frac{1}{2})}(z)=(-1)^n J_{n+\frac{1}{2}}(z) \quad (n \text{ an integer}\geq 0)$$

12.1.16 $$\mathbf{H}_{\frac{1}{2}}(z)=\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}(1-\cos z)$$

12.1.17
$$\mathbf{H}_{\frac{3}{2}}(z)=\left(\frac{z}{2\pi}\right)^{\frac{1}{2}}\left(1+\frac{2}{z^2}\right)-\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}\left(\sin z+\frac{\cos z}{z}\right)$$

12.1.18 $$\mathbf{H}_\nu(ze^{m\pi i})=e^{m(\nu+1)\pi i}\,\mathbf{H}_\nu(z) \ (m \text{ an integer})$$

12.1.19 $$\mathbf{H}_0(z)=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{J_{2k+1}(z)}{2k+1}$$

12.1.20 $$\mathbf{H}_1(z)=\frac{2}{\pi}-\frac{2}{\pi}J_0(z)+\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{J_{2k}(z)}{4k^2-1}$$

12.1.21 $$\mathbf{H}_\nu(z)=\frac{2(z/2)^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}\,{}_1F_2\left(1;\frac{3}{2}+\nu,\frac{3}{2};-\frac{z^2}{4}\right)$$

Integrals (See chapter 11)

12.1.22 $$\int_0^\infty t^{-1}\,\mathbf{H}_0(t)dt=\frac{\pi}{2}$$

12.1.23
$$\int_0^z \mathbf{H}_0(t)dt=\frac{2}{\pi}\left[\frac{z^2}{2}-\frac{z^4}{1^2{\cdot}3^2{\cdot}4}+\frac{z^6}{1^2{\cdot}3^2{\cdot}5^2{\cdot}6}-\cdots\right]$$

12.1.24 $$\int_0^z t^{-\nu}\,\mathbf{H}_{\nu+1}(t)dt=\frac{z}{2^\nu\sqrt{\pi}\,\Gamma(\nu+\frac{3}{2})}-z^{-\nu}\,\mathbf{H}_\nu(z)$$

Struve's Integral

12.1.25
$$\frac{4}{\pi}\int_z^\infty t^{-2}\,\mathbf{H}_1(t)dt=\frac{2}{\pi z}\mathbf{H}_1(z)+\frac{2}{\pi}\int_z^\infty t^{-1}\,\mathbf{H}_0(t)dt$$

12.1.26
$$\frac{2}{\pi}\int_z^\infty t^{-1}\,\mathbf{H}_0(t)dt=1-\frac{4}{\pi^2}\left[z-\frac{z^3}{1^2{\cdot}3^2{\cdot}3}+\frac{z^5}{1^2{\cdot}3^2{\cdot}5^2{\cdot}5}-\cdots\right]$$

12.1.27
$$\int_0^\infty t^{\mu-\nu-1}\mathbf{H}_\nu(t)dt=\frac{\Gamma(\frac{1}{2}\mu)2^{\mu-\nu-1}\tan(\frac{1}{2}\pi\mu)}{\Gamma(\nu-\frac{1}{2}\mu+1)}$$
$$(|\mathscr{R}\mu|<1,\ \mathscr{R}\nu>\mathscr{R}\mu-\tfrac{3}{2})$$

If $f_\nu(z)=\int_0^z \mathbf{H}_\nu(t)t^\nu dt$

12.1.28
$$f_{\nu+1}=(2\nu+1)f_\nu(z)-z^{\nu+1}\mathbf{H}_\nu(z)+\frac{z^{2\nu+2}}{(\nu+1)2^{\nu+1}\Gamma(\frac{1}{2})\Gamma(\nu+\frac{3}{2})} \ (\mathscr{R}\nu>-\tfrac{1}{2})$$

Asymptotic Expansions for Large $|z|$

12.1.29
$$\mathbf{H}_\nu(z)-Y_\nu(z)=\frac{1}{\pi}\sum_{k=0}^{m-1}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2}-k)\left(\frac{z}{2}\right)^{2k-\nu+1}}+R_m$$
$$(|\arg z|<\pi)$$

where $R_m=O(|z|^{\nu-2m-1})$. If ν is real, z positive * and $m+\frac{1}{2}-\nu\geq 0$, the remainder after m terms is of the same sign and numerically less than the first term neglected.

12.1.30
$$\mathbf{H}_0(z)-Y_0(z)\sim\frac{2}{\pi}\left[\frac{1}{z}-\frac{1}{z^3}+\frac{1^2\cdot 3^2}{z^5}-\frac{1^2\cdot 3^2\cdot 5^2}{z^7}+\ldots\right]$$
$$(|\arg z|<\pi)$$

12.1.31
$$\mathbf{H}_1(z)-Y_1(z)\sim\frac{2}{\pi}\left[1+\frac{1}{z^2}-\frac{1^2\cdot 3}{z^4}+\frac{1^2\cdot 3^2\cdot 5}{z^6}-\ldots\right]$$
$$(|\arg z|<\pi)$$

12.1.32
$$\int_0^z[\mathbf{H}_0(t)-Y_0(t)]dt-\frac{2}{\pi}[\ln(2z)+\gamma]\sim\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}(2k)!(2k-1)!}{(k!)^2(2z)^{2k}} \quad (|\arg z|<\pi)$$

where $\gamma=.57721\ 56649\ \ldots$ is Euler's constant.

12.1.33
$$\int_z^\infty t^{-1}[\mathbf{H}_0(t)-Y_0(t)]dt\sim\frac{2}{\pi z}\sum_{k=0}^{\infty}\frac{(-1)^k[(2k)!]^2}{(k!)^2(2k+1)(2z)^{2k}}$$
$$(|\arg z|<\pi)$$

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*See page II.

Asymptotic Expansions for Large Orders

12.1.34

$$\mathbf{H}_\nu(z)-Y_\nu(z)\sim\frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\sum_{k=0}^{\infty}\frac{k!b_k}{z^{k+1}}$$

$$(|\arg z|<\tfrac{1}{2}\pi, |\nu|<|z|)$$

$b_0=1$, $b_1=2\nu/z$, $b_2=6(\nu/z)^2-\frac{1}{2}$, $b_3=20(\nu/z)^3-4(\nu/z)$

12.1.35

$$\mathbf{H}_\nu(z)+iJ_\nu(z)\sim\frac{2(\frac{1}{2}z)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\sum_{k=0}^{\infty}\frac{k!b_k}{z^{k+1}}\qquad(|\nu|>|z|)$$

12.2. Modified Struve Function $\mathbf{L}_\nu(z)$

Power Series Expansion

12.2.1 $$\mathbf{L}_\nu(z)=-ie^{-\frac{i\nu\pi}{2}}\mathbf{H}_\nu(iz)$$

$$=(\tfrac{1}{2}z)^{\nu+1}\sum_{k=0}^{\infty}\frac{(z/2)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\nu+\frac{3}{2})}$$

Integral Representations

12.2.2 $$\mathbf{L}_\nu(z)=\frac{2(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^{\frac{\pi}{2}}\sinh(z\cos\theta)\sin^{2\nu}\theta d\theta$$

$$(\mathscr{R}\nu>-\tfrac{1}{2})$$

12.2.3

$$I_{-\nu}(x)-\mathbf{L}_\nu(x)=\frac{2(x/2)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}\int_0^\infty\sin(tx)(1+t^2)^{\nu-\frac{1}{2}}dt$$

$$(\mathscr{R}\nu<\tfrac{1}{2}, x>0)$$

Recurrence Relations

12.2.4 $$\mathbf{L}_{\nu-1}-\mathbf{L}_{\nu+1}=\frac{2\nu}{z}\mathbf{L}_\nu+\frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})}$$

12.2.5 $$\mathbf{L}_{\nu-1}+\mathbf{L}_{\nu+1}=2\mathbf{L}_\nu'-\frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu+\frac{3}{2})}$$

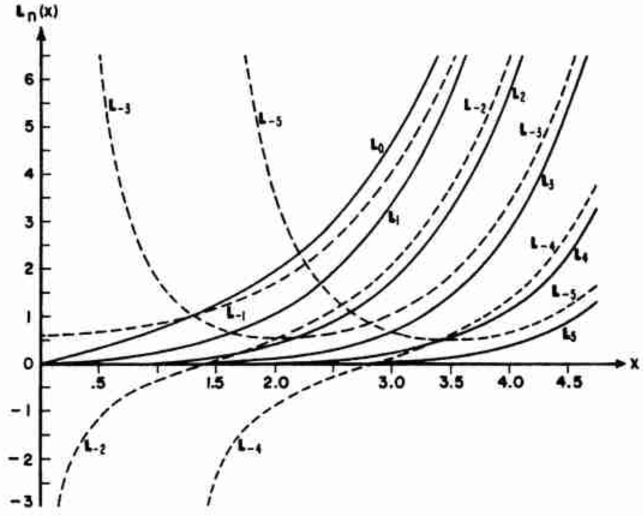

FIGURE 12.4. *Modified Struve functions.*

$\mathbf{L}_n(x)$, $\pm n=0(1)5$

Asymptotic Expansion for Large $|z|$

12.2.6 $$\mathbf{L}_\nu(z)-I_{-\nu}(z)$$

$$\sim\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\Gamma(k+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2}-k)\left(\frac{z}{2}\right)^{2k-\nu+1}}\ (|\arg z|<\tfrac{1}{2}\pi)$$

Integrals

12.2.7

$$\int_0^z\mathbf{L}_0(t)dt=\frac{2}{\pi}\left[\frac{z^2}{2}+\frac{z^4}{1^2\cdot3^2\cdot4}+\frac{z^6}{1^2\cdot3^2\cdot5^2\cdot6}+\cdots\right]$$

12.2.8 $$\int_0^z[I_0(t)-\mathbf{L}_0(t)]dt-\frac{2}{\pi}[\ln(2z)+\gamma]$$

$$\sim-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(2k)!\,(2k-1)!}{(k!)^2(2z)^{2k}}\ (|\arg z|<\tfrac{1}{2}\pi)$$

12.2.9 $$\int_0^z\mathbf{L}_1(t)dt=\mathbf{L}_0(z)-\frac{2}{\pi}z$$

Relation to Modified Spherical Bessel Function

12.2.10 $\mathbf{L}_{-(n+\frac{1}{2})}(z)=I_{(n+\frac{1}{2})}(z)$ (n an integer ≥0)

12.3. Anger and Weber Functions

Anger's Function

12.3.1 $$\mathbf{J}_\nu(z)=\frac{1}{\pi}\int_0^\pi\cos(\nu\theta-z\sin\theta)\,d\theta$$

12.3.2 $\mathbf{J}_n(z)=J_n(z)$ (n an integer)

Weber's Function

12.3.3 $$\mathbf{E}_\nu(z)=\frac{1}{\pi}\int_0^\pi\sin(\nu\theta-z\sin\theta)\,d\theta$$

Relations Between Anger's and Weber's Function

12.3.4 $\sin(\nu\pi)\,\mathbf{J}_\nu(z)=\cos(\nu\pi)\,\mathbf{E}_\nu(z)-\mathbf{E}_{-\nu}(z)$

12.3.5 $\sin(\nu\pi)\,\mathbf{E}_\nu(z)=\mathbf{J}_{-\nu}(z)-\cos(\nu\pi)\,\mathbf{J}_\nu(z)$

Relations Between Weber's Function and Struve's Function

If n is a positive integer or zero,

12.3.6 $$\mathbf{E}_n(z)=\frac{1}{\pi}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{\Gamma(k+\frac{1}{2})(\frac{1}{2}z)^{n-2k-1}}{\Gamma(n+\frac{1}{2}-k)}-\mathbf{H}_n(z)\quad *$$

12.3.7

$$\mathbf{E}_{-n}(z)=\frac{(-1)^{n+1}}{\pi}\sum_{k=0}^{\left[\frac{n-1}{2}\right]}\frac{\Gamma(n-k-\frac{1}{2})(\frac{1}{2}z)^{-n+2k+1}}{\Gamma(k+\frac{3}{2})}-\mathbf{H}_{-n}(z)\quad *$$

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*See page II.

12.3.8 $$\mathbf{E}_0(z)=-\mathbf{H}_0(z)$$

12.3.9 $$\mathbf{E}_1(z)=\frac{2}{\pi}-\mathbf{H}_1(z)$$

12.3.10 $$\mathbf{E}_2(z)=\frac{2z}{3\pi}-\mathbf{H}_2(z)$$

Numerical Methods

12.4. Use and Extension of the Tables

Example 1. Compute $\mathbf{L}_0(2)$ to 6D. From **Table 12.1** $I_0(2)-\mathbf{L}_0(2)=.342152$; from **Table 9.11** we have $I_0(2)=2.279585$ so that $\mathbf{L}_0(2)=1.937433$.

Example 2. Compute $\mathbf{H}_0(10)$ to 6D. From **Table 12.2** for $x^{-1}=.1$, $\mathbf{H}_0(10)-Y_0(10)=.063072$; from **Table 9.1** we have $Y_0(10)=.055671$. Thus, $\mathbf{H}_0(10)=.118743$.

Example 3. Compute $\int_0^x \mathbf{H}_0(t)dt$ for $x=6$ to 5D. Using **Tables 12.2, 11.1** and **4.2,** we have

$$\int_0^6 \mathbf{H}_0(t)dt=\int_0^6 Y_0(t)dt+\frac{2}{\pi}\ln 6+f_1(6)$$

$$=-.125951+(.636620)(1.791759)$$
$$+.816764$$

$$=1.83148$$

Example 4. Compute $\mathbf{H}_n(x)$ for $x=4$, $-n=0(1)8$ to 6S. From **Table 12.1** we have $\mathbf{H}_0(4)=.1350146$, $\mathbf{H}_1(4)=1.0697267$. Using **12.1.9** we find

$\mathbf{H}_{-1}(4)=-.433107$
$\mathbf{H}_{-2}(4)=\ \ .240694$
$\mathbf{H}_{-3}(4)=\ \ .152624$
$\mathbf{H}_{-4}(4)=-.439789$
$\mathbf{H}_{-5}(4)=\ \ .689652$
$\mathbf{H}_{-6}(4)=-1.21906$
$\mathbf{H}_{-7}(4)=\ \ 2.82066$
$\mathbf{H}_{-8}(4)=-8.24933$

Example 5. Compute $\mathbf{H}_n(x)$ for $x=4$, $n=0(1)10$ to 7S. Starting with the values of $\mathbf{H}_0(4)$ and $\mathbf{H}_1(4)$ and using **12.1.9** with forward recurrence, we get

$\mathbf{H}_0(4)=\ .13501\ 46$
$\mathbf{H}_1(4)=1.06972\ 67$
$\mathbf{H}_2(4)=1.24867\ 51$
$\mathbf{H}_3(4)=\ .85800\ 95$
$\mathbf{H}_4(4)=\ .42637\ 41$
$\mathbf{H}_5(4)=\ .16719\ 87$
$\mathbf{H}_6(4)=.05433\ 5\mathit{4}$
$\mathbf{H}_7(4)=.01510\ 3\mathit{7}$
$\mathbf{H}_8(4)=.00367\ \mathit{33}$
$\mathbf{H}_9(4)=.0008\mathit{0}\ \mathit{02}$
$\mathbf{H}_{10}(4)=.0001\mathit{8}\ \mathit{25}$

We note that for $n>6$ there is a rapid loss of significant figures. On the other hand using **12.1.3** for $x=4$ we find $\mathbf{H}_9(4)=.0007935729$, $\mathbf{H}_{10}(4)=.00015447630$ and backward recurrence with **12.1.9** gives

$\mathbf{H}_8(4)=.00367\ 1495$
$\mathbf{H}_7(4)=.01510\ 315$
$\mathbf{H}_6(4)=.05433\ 519$
$\mathbf{H}_5(4)=.16719\ 87$
$\mathbf{H}_4(4)=.42637\ 43$
$\mathbf{H}_3(4)=\ .85800\ 94$
$\mathbf{H}_2(4)=1.24867\ 6$
$\mathbf{H}_1(4)=1.06972\ 7$
$\mathbf{H}_0(4)=\ .13501\ 4$

Example 6. Compute $\mathbf{L}_n(.5)$ for $n=0(1)5$ to 8S. From **12.2.1** we find $\mathbf{L}_5(.5)=9.6307462\times10^{-7}$, $\mathbf{L}_4(.5)=2.1212342\times10^{-5}$. Then, with **12.2.4** we get

$\mathbf{L}_3(.5)=3.82465\ 03\times10^{-4}$
$\mathbf{L}_2(.5)=5.36867\ 34\times10^{-3}$
$\mathbf{L}_1(.5)=.05394\ 2181$
$\mathbf{L}_0(.5)=.32724\ 068$

Example 7. Compute $\mathbf{L}_n(.5)$ for $-n=0(1)5$ to 6S. From **Tables 12.1** and **9.8** we find $\mathbf{L}_0(.5)=.327240$, $\mathbf{L}_1(.5)=.053942$. Then employing **12.2.4** with backward recurrence we get

$\mathbf{L}_{-1}(.5)=\ \ .690562$
$\mathbf{L}_{-2}(.5)=-1.16177$
$\mathbf{L}_{-3}(.5)=\ \ 7.43824$
$\mathbf{L}_{-4}(.5)=-75.1418$
$\mathbf{L}_{-5}(.5)=1056.92$

Example 8. Compute $\mathbf{L}_n(x)$ for $x=6$ and $-n=0(1)6$ to 8S. From **Tables 12.2** and **9.8** we find $\mathbf{L}_0(6)=67.124454$, $\mathbf{L}_1(6)=60.725011$. Using **12.2.4** we get

$\mathbf{L}_{-1}(6)=61.361631$
$\mathbf{L}_{-2}(6)=46.776680$
$\mathbf{L}_{-3}(6)=30.159494$
$\mathbf{L}_{-4}(6)=16.626028$
$\mathbf{L}_{-5}(6)=\ 7.984089$
$\mathbf{L}_{-6}(6)=\ 3.32780$

We note that there is no essential loss of accuracy until $n=-6$. However, if further values were necessary the recurrence procedure becomes unstable. To avoid the instability use the methods described in **Examples 5** and **6.**

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Texts

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Tables

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Table 12.2 STRUVE FUNCTIONS FOR LARGE ARGUMENTS

x^{-1}	$\mathbf{H}_0(x)-Y_0(x)$	$\mathbf{H}_1(x)-Y_1(x)$	$f_1(x)$	$I_0(x)-\mathbf{L}_0(x)$	$I_1(x)-\mathbf{L}_1(x)$	$f_2(x)$	$f_3(x)$	$<x>$
0.20	0.123301	0.659949	0.819924	0.133955	0.607426	0.793280	0.125868	5
0.19	0.117449	0.657819	0.818935	0.126683	0.610467	0.794902	0.119694	5
0.18	0.111556	0.655774	0.817981	0.119468	0.613348	0.796448	0.113505	6
0.17	0.105625	0.653818	0.817062	0.112319	0.616060	0.797910	0.107299	6
0.16	0.099655	0.651952	0.816182	0.105242	0.618598	0.799279	0.101079	6
0.15	0.093647	0.650180	0.815341	0.098241	0.620955	0.800551	0.094843	7
0.14	0.087602	0.648504	0.814541	0.091318	0.623129	0.801721	0.088593	7
0.13	0.081521	0.646927	0.813785	0.084474	0.625119	0.802787	0.082328	8
0.12	0.075404	0.645452	0.813074	0.077706	0.626927	0.803750	0.076051	8
0.11	0.069254	0.644081	0.812411	0.071010	0.628558	0.804611	0.069761	9
0.10	0.063072	0.642817	0.811796	0.064379	0.630018	0.805374	0.063460	10
0.09	0.056860	0.641663	0.811232	0.057805	0.631315	0.806047	0.057147	11
0.08	0.050620	0.640622	0.810722	0.051279	0.632457	0.806634	0.050824	13
0.07	0.044354	0.639696	0.810266	0.044793	0.633450	0.807140	0.044492	14
0.06	0.038064	0.638888	0.809866	0.038340	0.634302	0.807572	0.038152	17
0.05	0.031753	0.638200	0.809525	0.031912	0.635016	0.807933	0.031805	20
0.04	0.025425	0.637634	0.809244	0.025506	0.635596	0.808225	0.025451	25
0.03	0.019082	0.637191	0.809023	0.019116	0.636045	0.808450	0.019093	33
0.02	0.012727	0.636874	0.808865	0.012738	0.636365	0.808611	0.012731	50
0.01	0.006366	0.636683	0.808770	0.006367	0.636556	0.808706	0.006366	100
0.00	0.000000	0.636620	0.808738	0.000000	0.636620	0.808738	0.000000	∞
	$\left[{(-6)5 \atop 3}\right]$	$\left[{(-5)2 \atop 3}\right]$	$\left[{(-6)8 \atop 3}\right]$	$\left[{(-5)1 \atop 3}\right]$	$\left[{(-5)2 \atop 3}\right]$	$\left[{(-5)1 \atop 3}\right]$	$\left[{(-6)2 \atop 3}\right]$	

$$\int_0^x [\mathbf{H}_0(t)-Y_0(t)]dt=\frac{2}{\pi}\ln x+f_1(x)$$

$$\int_0^x [\mathbf{L}_0(t)-I_0(t)]dt=\frac{2}{\pi}\ln x+f_2(x)$$

$$\int_x^\infty \left[\frac{\mathbf{H}_0(t)-Y_0(t)}{t}\right]dt=f_3(x)$$

$<x>$ = nearest integer to x.

Starting with $\mathbf{H}_0(x)$ and $\mathbf{H}_1(x)$, recurrence formula **12.1.9** may be used to generate $\mathbf{H}_n(x)$ for $n<0$. As long as $n<x/2$ (approx.), $\mathbf{H}_n(x)$ may be generated by forward recurrence. When $n>x/2$, forward recurrence is unstable. To avoid the instability, choose $n>>x$, compute $\mathbf{H}_k(x)$ and $\mathbf{H}_{k+1}(x)$ with **12.1.3**, and then use backward recurrence with **12.1.9**.

If $n>0$, $\mathbf{L}_n(x)$ must be generated by backward recurrence. If $n<0$, $\mathbf{L}_n(x)$ may be generated by backward recurrence as long as $\mathbf{L}_n(x)$ increases. If $n<0$ and $\mathbf{L}_n(x)$ is decreasing, forward recurrence should be used.

See Examples 4–8.

13. Confluent Hypergeometric Functions

LUCY JOAN SLATER [1]

Contents

The tables were calculated by the author on the electronic calculator EDSACI in the Mathematical Laboratory of Cambridge University, by kind permission of its director, Dr. M. V. Wilkes. The table of $M(a, b, x)$ was recomputed by Alfred E. Beam for uniformity to eight significant figures.

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[1] University Mathematical Laboratory, Cambridge. (Prepared under contract with the National Bureau of Standards.)

13. Confluent Hypergeometric Functions

Mathematical Properties

13.1. Definitions of Kummer and Whittaker Functions

Kummer's Equation

13.1.1 $$z\frac{d^2w}{dz^2}+(b-z)\frac{dw}{dz}-aw=0$$

It has a regular singularity at $z=0$ and an irregular singularity at ∞.

Independent solutions are

Kummer's Function

13.1.2

$$M(a,b,z)=1+\frac{az}{b}+\frac{(a)_2z^2}{(b)_2 2!}+\ldots+\frac{(a)_nz^n}{(b)_n n!}+\ldots$$

where

$$(a)_n=a(a+1)(a+2)\ldots(a+n-1),\ (a)_0=1,$$

and

13.1.3

$$U(a,b,z)=\frac{\pi}{\sin \pi b}\left\{\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)}-z^{1-b}\frac{M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right\}$$

Parameters *(m, n positive integers)*		$M(a, b, z)$
$b\neq-n$	$a\neq-m$	a convergent series for all values of a, b and z
$b\neq-n$	$a=-m$	a polynomial of degree m in z
$b=-n$	$a\neq-m$	
$b=-n$	$a=-m$, $m>n$	a simple pole at $b=-n$
$b=-n$	$a=-m$, $m\leq n$	undefined

$U(a, b, z)$ is defined even when $b\rightarrow\pm n$

As $|z|\rightarrow\infty$,

13.1.4

$$M(a,b,z)=\frac{\Gamma(b)}{\Gamma(a)}e^zz^{a-b}[1+O(|z|^{-1})] \qquad (\mathscr{R}z>0)$$

and

13.1.5

$$M(a,b,z)=\frac{\Gamma(b)}{\Gamma(b-a)}(-z)^{-a}[1+O(|z|^{-1})] \quad (\mathscr{R}z<0)$$

$U(a, b, z)$ is a many-valued function. Its principal branch is given by $-\pi<\arg z\leq\pi$.

Logarithmic Solution

13.1.6

$$U(a,n+1,z)=\frac{(-1)^{n+1}}{n!\Gamma(a-n)}\left[M(a,n+1,z)\ln z+\sum_{r=0}^{\infty}\frac{(a)_rz^r}{(n+1)_rr!}\{\psi(a+r)-\psi(1+r)-\psi(1+n+r)\}\right]+\frac{(n-1)!}{\Gamma(a)}z^{-n}M(a-n,1-n,z)_n$$

for $n=0, 1, 2, \ldots$, where the last function is the sum to n terms. It is to be interpreted as zero when $n=0$, and $\psi(a)=\Gamma'(a)/\Gamma(a)$.

13.1.7 $$U(a,1-n,z)=z^nU(a+n,1+n,z)$$

As $\mathscr{R}z\rightarrow\infty$

13.1.8 $$U(a,b,z)=z^{-a}[1+O(|z|^{-1})]$$

Analytic Continuation

13.1.9

$$U(a,b,ze^{\pm\pi i})=\frac{\pi}{\sin \pi b}e^{-z}\left\{\frac{M(b-a,b,z)}{\Gamma(1+a-b)\Gamma(b)}-\frac{e^{\pm\pi i(1-b)}z^{1-b}M(1-a,2-b,z)}{\Gamma(a)\Gamma(2-b)}\right\}$$

where either upper or lower signs are to be taken throughout.

13.1.10

$$U(a,b,ze^{2\pi in})=[1-e^{-2\pi ibn}]\frac{\Gamma(1-b)}{\Gamma(1+a-b)}M(a,b,z)+e^{-2\pi ibn}U(a,b,z)$$

Alternative Notations

${}_1F_1(a;\,b;\,z)$ or $\Phi(a;\,b;\,z)$ for $M(a, b, z)$

$z^{-a}{}_2F_0(a, 1+a-b;\, ;-1/z)$ or $\Psi(a;\,b;\,z)$ for $U(a, b, z)$

Complete Solution

13.1.11 $$y=AM(a,b,z)+BU(a,b,z)$$

where A and B are arbitrary constants, $b\neq-n$.

Eight Solutions

13.1.12 $$y_1=M(a,b,z)$$

13.1.13 $$y_2=z^{1-b}M(1+a-b,2-b,z)$$

13.1.14 $$y_3=e^zM(b-a,b,-z)$$

13.1.15 $y_4=z^{1-b}e^zM(1-a,\ 2-b,\ -z)$

13.1.16 $y_5=U(a,\ b,\ z)$

13.1.17 $y_6=z^{1-b}U(1+a-b,\ 2-b,\ z)$

13.1.18 $y_7=e^zU(b-a,\ b,\ -z)$

13.1.19 $y_8=z^{1-b}e^zU(1-a,\ 2-b,\ -z)$

Wronskians

If $W\{m,\ n\}=y_my'_n-y_ny'_m$ and

$$\epsilon=sgn\ (\mathscr{I}z)=1 \text{ if } \mathscr{I}z>0,\qquad =-1 \text{ if } \mathscr{I}z\leq 0$$

13.1.20

$$W\{1,\ 2\}=W\{3,\ 4\}=W\{1,\ 4\}=-W\{2,\ 3\}=(1-b)z^{-b}e^z$$

13.1.21

$$W\{1,\ 3\}=W\{2,\ 4\}=W\{5,\ 6\}=W\{7,\ 8\}=0$$

13.1.22 $W\{1,\ 5\}=-\Gamma(b)z^{-b}e^z/\Gamma(a)$

13.1.23 $W\{1,\ 7\}=\Gamma(b)e^{\epsilon\pi ib}z^{-b}e^z/\Gamma(b-a)$

13.1.24 $W\{2,\ 5\}=-\Gamma(2-b)z^{-b}e^z/\Gamma(1+a-b)$

13.1.25 $W\{2,\ 7\}=-\Gamma(2-b)z^{-b}e^z/\Gamma(1-a)$

13.1.26 $W\{5,\ 7\}=e^{\epsilon\pi i(b-a)}z^{-b}e^z$

Kummer Transformations

13.1.27 $M(a,\ b,\ z)=e^zM(b-a,\ b,\ -z)$

13.1.28

$$z^{1-b}M(1+a-b,\ 2-b,\ z)=z^{1-b}e^zM(1-a,\ 2-b,\ -z)$$

13.1.29 $U(a,\ b,\ z)=z^{1-b}U(1+a-b,\ 2-b,\ z)$

13.1.30

$$e^zU(b-a,\ b,\ -z)=e^{\epsilon\pi i(1-b)}e^zz^{1-b}U(1-a,\ 2-b,\ -z)$$

Whittaker's Equation

13.1.31 $$\frac{d^2w}{dz^2}+\left[-\frac{1}{4}+\frac{\kappa}{z}+\frac{(\frac{1}{4}-\mu^2)}{z^2}\right]w=0$$

Solutions:

Whittaker's Functions

13.1.32 $M_{\kappa,\mu}(z)=e^{-\frac{1}{2}z}z^{\frac{1}{2}+\mu}M(\tfrac{1}{2}+\mu-\kappa,\ 1+2\mu,\ z)$

13.1.33

$$W_{\kappa,\mu}(z)=e^{-\frac{1}{2}z}z^{\frac{1}{2}+\mu}U(\tfrac{1}{2}+\mu-\kappa,\ 1+2\mu,\ z)$$

$$(-\pi<\arg z\leq\pi,\ \kappa=\tfrac{1}{2}b-a,\ \mu=\tfrac{1}{2}b-\tfrac{1}{2})$$

13.1.34

$$W_{\kappa,\mu}(z)=\frac{\Gamma(-2\mu)}{\Gamma(\frac{1}{2}-\mu-\kappa)}M_{\kappa,\mu}(z)+\frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2}+\mu-\kappa)}M_{\kappa,-\mu}(z)$$

General Confluent Equation

13.1.35

$$w''+\left[\frac{2A}{Z}+2f'+\frac{bh'}{h}-h'-\frac{h''}{h'}\right]w'$$

$$+\left[\left(\frac{bh'}{h}-h'-\frac{h''}{h'}\right)\left(\frac{A}{Z}+f'\right)+\frac{A(A-1)}{Z^2}\right.$$

$$\left.+\frac{2Af'}{Z}+f''+f'^2-\frac{ah'^2}{h}\right]w=0$$

Solutions:

13.1.36 $Z^{-A}e^{-f(Z)}M(a,\ b,\ h(Z))$

13.1.37 $Z^{-A}e^{-f(Z)}U(a,\ b,\ h(Z))$

13.2. Integral Representations

$$\mathscr{R}b>\mathscr{R}a>0$$

13.2.1

$$\frac{\Gamma(b-a)\Gamma(a)}{\Gamma(b)}M(a,b,z)$$

$$=\int_0^1 e^{zt}t^{a-1}(1-t)^{b-a-1}dt$$

13.2.2

$$=2^{1-b}e^{\frac{1}{2}z}\int_{-1}^{+1} e^{-\frac{1}{2}zt}(1+t)^{b-a-1}(1-t)^{a-1}dt$$

13.2.3

$$=2^{1-b}e^{\frac{1}{2}z}\int_0^\pi e^{-\frac{1}{2}z\cos\theta}\sin^{b-1}\theta\cot^{b-2a}(\tfrac{1}{2}\theta)\,d\theta$$

13.2.4

$$=e^{-Az}\int_A^B e^{zt}(t-A)^{a-1}(B-t)^{b-a-1}dt \qquad (A=B-1)$$

$$\mathscr{R}a>0,\ \mathscr{R}z>0$$

13.2.5

$$\Gamma(a)U(a,b,z)=\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}dt$$

13.2.6

$$=e^z\int_1^\infty e^{-zt}(t-1)^{a-1}t^{b-a-1}dt$$

13.2.7

$$=2^{1-b}e^{\frac{1}{2}z}\int_0^\infty e^{-\frac{1}{2}z\cosh\theta}\sinh^{b-1}\theta\coth^{b-2a}(\tfrac{1}{2}\theta)d\theta$$ *

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*See page II.

13.2.8 $\Gamma(a)U(a, b, z)$

$$= e^{Az}\int_A^\infty e^{-zt}(t-A)^{a-1}(t+B)^{b-a-1}dt \qquad (A=1-B)$$

Similar integrals for $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ can be deduced with the help of **13.1.32** and **13.1.33.**

Barnes-type Contour Integrals

13.2.9

$$\frac{\Gamma(a)}{\Gamma(b)}M(a, b, z)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(-s)\Gamma(a+s)}{\Gamma(b+s)}(-z)^s ds$$

for $|\arg(-z)|<\frac{1}{2}\pi$, $a, b\neq 0, -1, -2, \ldots$. The contour must separate the poles of $\Gamma(-s)$ from those of $\Gamma(a+s)$; c is finite.

13.2.10

$$\Gamma(a)\Gamma(1+a-b)z^a U(a, b, z)$$
$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\Gamma(-s)\Gamma(a+s)\Gamma(1+a-b+s)z^{-s}ds$$

for $|\arg z|<\frac{3\pi}{2}$, $a\neq 0, -1, -2, \ldots$, $b-a\neq 1, 2, 3, \ldots$. The contour must separate the poles of $\Gamma(-s)$ from those of $\Gamma(a+s)$ and $\Gamma(1+a-b+s)$.

13.3. Connections With Bessel Functions

(see chapters **9** and **10**)

Bessel Functions as Limiting Cases

If b and z are fixed,

13.3.1 $$\lim_{a\to\infty}\{M(a, b, z/a)/\Gamma(b)\}=z^{\frac{1}{2}-\frac{1}{2}b}I_{b-1}(2\sqrt{z})$$

13.3.2 $$\lim_{a\to\infty}\{M(a, b, -z/a)/\Gamma(b)\}=z^{\frac{1}{2}-\frac{1}{2}b}J_{b-1}(2\sqrt{z})$$

13.3.3

$$\lim_{a\to\infty}\{\Gamma(1+a-b)\,U(a, b, z/a)\}=2z^{\frac{1}{2}-\frac{1}{2}b}K_{b-1}(2\sqrt{z})$$

13.3.4

$$\lim_{a\to\infty}\{\Gamma(1+a-b)U(a, b, -z/a)\}$$
$$=-\pi i e^{\pi i b}z^{\frac{1}{2}-\frac{1}{2}b}H_{b-1}^{(1)}(2\sqrt{z}) \qquad (\mathscr{I}z>0)$$

13.3.5 $$=\pi i e^{-\pi i b}z^{\frac{1}{2}-\frac{1}{2}b}H_{b-1}^{(2)}(2\sqrt{z}) \qquad (\mathscr{I}z<0)$$

Expansions in Series

13.3.6

$$M(a, b, z)=e^{\frac{1}{2}z}\Gamma\,(b-a-\tfrac{1}{2})(\tfrac{1}{4}z)^{a-b+\frac{1}{2}}$$

* $$\cdot\sum_{n=0}^{\infty}\frac{(2b-2a-1)_n(b-2a)_n(b-a-\frac{1}{2}+n)}{n!(b)_n}$$
$$(-1)^n\,I_{b-a-\frac{1}{2}+n}(\tfrac{1}{2}z)\ (b\neq 0,-1,-2, \ldots)$$

13.3.7

$$\frac{M(a, b, z)}{\Gamma(b)}=e^{\frac{1}{2}z}(\tfrac{1}{2}bz-az)^{\frac{1}{2}-\frac{1}{2}b}$$
$$\cdot\sum_{n=0}^{\infty}A_n(\tfrac{1}{2}z)^{\frac{1}{2}n}(b-2a)^{-\frac{1}{2}n}J_{b-1+n}(\sqrt{(2zb-4za)})$$

where

$$A_0=1, A_1=0, A_2=\tfrac{1}{2}b,$$
$$(n+1)A_{n+1}=(n+b-1)A_{n-1}+(2a-b)A_{n-2}, \qquad (a \text{ real})$$

13.3.8

$$\frac{M(a, b, z)}{\Gamma(b)}$$
$$=e^{hz}\sum_{n=0}^{\infty}C_n z^n(-az)^{\frac{1}{2}(1-b-n)}J_{b-1+n}(2\sqrt{(-az)})$$

where

$$C_0=1,\ C_1=-bh,\ C_2=-\tfrac{1}{2}(2h-1)a+\tfrac{1}{2}b(b+1)h^2,$$
$$(n+1)C_{n+1}=[(1-2h)n-bh]C_n$$
$$+[(1-2h)a-h(h-1)(b+n-1)]C_{n-1}$$
$$-h(h-1)aC_{n-2} \qquad (h \text{ real})$$

13.3.9 $$M(a, b, z)=\sum_{n=0}^{\infty}C_n(a, b)I_n(z)$$

where

$$C_0=1,\ C_1(a, b)=2a/b,$$
$$C_{n+1}(a, b)=2aC_n(a+1, b+1)/b-C_{n-1}(a, b)$$

13.4. Recurrence Relations and Differential Properties

13.4.1

$$(b-a)M(a-1, b, z)+(2a-b+z)M(a, b, z)$$
$$-aM(a+1, b, z)=0$$

13.4.2

$$b(b-1)M(a, b-1, z)+b(1-b-z)M(a, b, z)$$
$$+z(b-a)M(a, b+1, z)=0$$

13.4.3

$$(1+a-b)M(a, b, z)-aM(a+1, b, z)$$
$$+(b-1)M(a, b-1, z)=0$$

13.4.4

$$bM(a, b, z)-bM(a-1, b, z)-zM(a, b+1, z)=0$$

13.4.5

$$b(a+z)M(a, b, z)+z(a-b)M(a, b+1, z)$$
$$-abM(a+1, b, z)=0$$

13.4.6

$$(a-1+z)M(a,\ b,\ z)+(b-a)M(a-1,\ b,\ z)\\ +(1-b)M(a,\ b-1,\ z)=0$$

13.4.7

$$b(1-b+z)M(a,\ b,\ z)+b(b-1)M(a-1,\ b-1,\ z)\\ -azM(a+1,\ b+1,\ z)=0$$

13.4.8 $$M'(a,\ b,\ z)=\frac{a}{b}M(a+1,\ b+1,\ z)$$

13.4.9 $$\frac{d^n}{dz^n}\{M(a,\ b,\ z)\}=\frac{(a)_n}{(b)_n}M(a+n,\ b+n,\ z)$$

13.4.10 $$aM(a+1,\ b,\ z)=aM(a,\ b,\ z)+zM'(a,\ b,\ z)$$

13.4.11

$$(b-a)M(a-1,\ b,\ z)=(b-a-z)M(a,\ b,\ z)\\ +zM'(a,\ b,\ z)$$

13.4.12

$$(b-a)M(a,\ b+1,\ z)=bM(a,\ b,\ z)-bM'(a,\ b,\ z)$$

13.4.13

$$(b-1)M(a,\ b-1,\ z)=(b-1)M(a,\ b,\ z)\\ +zM'(a,\ b,\ z)$$

13.4.14

$$(b-1)M(a-1,\ b-1,\ z)=(b-1-z)M(a,\ b,\ z)\\ +zM'(a,\ b,\ z)$$

13.4.15

$$U(a-1,\ b,\ z)+(b-2a-z)U(a,\ b,\ z)\\ +a(1+a-b)U(a+1,\ b,\ z)=0$$

13.4.16

$$(b-a-1)U(a,\ b-1,\ z)+(1-b-z)U(a,\ b,\ z)\\ +zU(a,\ b+1,\ z)=0$$

13.4.17

$$U(a,\ b,\ z)-aU(a+1,\ b,\ z)-U(a,\ b-1,\ z)=0$$

13.4.18

$$(b-a)U(a,\ b,\ z)+U(a-1,\ b,\ z)\\ -zU(a,\ b+1,\ z)=0$$

13.4.19

$$(a+z)U(a,\ b,\ z)-zU(a,\ b+1,\ z)\\ +a(b-a-1)U(a+1,\ b,\ z)=0$$

13.4.20

$$(a+z-1)U(a,\ b,\ z)-U(a-1,\ b,\ z)\\ +(1+a-b)U(a,\ b-1,\ z)=0$$

13.4.21 $$U'(a,\ b,\ z)=-aU(a+1,\ b+1,\ z)$$

13.4.22

$$\frac{d^n}{dz^n}\{U(a,\ b,\ z)\}=(-1)^n(a)_nU(a+n,\ b+n,\ z)$$

13.4.23

$$a(1+a-b)U(a+1,\ b,\ z)=aU(a,\ b,\ z)\\ +zU'(a,\ b,\ z)$$

13.4.24

$$(1+a-b)U(a,\ b-1,\ z)=(1-b)U(a,\ b,\ z)\\ -zU'(a,\ b,\ z)$$

13.4.25 $$U(a,\ b+1,\ z)=U(a,\ b,\ z)-U'(a,\ b,\ z)$$

13.4.26

$$U(a-1,\ b,\ z)=(a-b+z)U(a,\ b,\ z)-zU'(a,\ b,\ z)$$

13.4.27

$$U(a-1,\ b-1,\ z)=(1-b+z)U(a,\ b,\ z)\\ -zU'(a,\ b,\ z)$$

13.4.28 $$2\mu M_{\kappa-\frac{1}{2},\mu-\frac{1}{2}}(z)-z^{\frac{1}{2}}M_{\kappa,\mu}(z)=2\mu M_{\kappa+\frac{1}{2},\mu-\frac{1}{2}}(z)$$

13.4.29

$$(1+2\mu+2\kappa)M_{\kappa+1,\mu}(z)-(1+2\mu-2\kappa)M_{\kappa-1,\mu}(z)\\ =2(2\kappa-z)M_{\kappa,\mu}(z)$$

13.4.30

$$W_{\kappa+\frac{1}{2},\mu}(z)-z^{\frac{1}{2}}W_{\kappa,\mu+\frac{1}{2}}(z)+(\kappa+\mu)W_{\kappa-\frac{1}{2},\mu}(z)=0$$

13.4.31

$$(2\kappa-z)W_{\kappa,\mu}(z)+W_{\kappa+1,\mu}(z)\\ =(\mu-\kappa+\tfrac{1}{2})(\mu+\kappa-\tfrac{1}{2})W_{\kappa-1,\mu}(z)$$

13.4.32

$$zM'_{\kappa,\mu}(z)=(\tfrac{1}{2}z-\kappa)M_{\kappa,\mu}(z)+(\tfrac{1}{2}+\mu+\kappa)M_{\kappa+1,\mu}(z)$$

13.4.33 $$zW'_{\kappa,\mu}(z)=(\tfrac{1}{2}z-\kappa)W_{\kappa,\mu}(z)-W_{\kappa+1,\mu}(z)$$

13.5. Asymptotic Expansions and Limiting Forms

For $|z|$ large, (a, b fixed)

13.5.1

$$\frac{M(a,b,z)}{\Gamma(b)}=\frac{e^{\pm i\pi a}z^{-a}}{\Gamma(b-a)}\left\{\sum_{n=0}^{R-1}\frac{(a)_n(1+a-b)_n}{n!}(-z)^{-n}+O(|z|^{-R})\right\}+\frac{e^z z^{a-b}}{\Gamma(a)}\left\{\sum_{n=0}^{S-1}\frac{(b-a)_n(1-a)_n}{n!}z^{-n}+O(|z|^{-S})\right\}$$

the upper sign being taken if $-\frac{1}{2}\pi<\arg z<\frac{3}{2}\pi$, the lower sign if $-\frac{3}{2}\pi<\arg z\leq-\frac{1}{2}\pi$.

13.5.2

$$U(a,b,z)=z^{-a}\left\{\sum_{n=0}^{R-1}\frac{(a)_n(1+a-b)_n}{n!}(-z)^{-n}+O(|z|^{-R})\right\}\quad(-\tfrac{3}{2}\pi<\arg z<\tfrac{3}{2}\pi)$$

Converging Factors for the Remainders

13.5.3

$$O(|z|^{-R})=\frac{(a)_R(1+a-b)_R}{R!}(-z)^{-R}\left[\tfrac{1}{2}+\frac{(\frac{1}{8}+\frac{1}{4}b-\frac{1}{2}a+\frac{1}{4}z-\frac{1}{4}R)}{z}+O(|z|^{-2})\right]$$

and

13.5.4

$$O(|z|^{-S})=\frac{(b-a)_S(1-a)_S}{S!}z^{-S}\left[\tfrac{2}{3}-b+2a+z-S+O(|z|^{-1})\right]$$

where the R'th and S'th terms are the smallest in the expansions **13.5.1** and **13.5.2**.

For small z (a, b fixed)

13.5.5 As $|z|\to 0$, $M(a,b,0)=1$, $b\neq-n$

13.5.6 $$U(a,b,z)=\frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}+O(|z|^{\mathscr{R}b-2})\quad(\mathscr{R}b\geq 2,\ b\neq 2)$$

13.5.7 $$=\frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}+O(|\ln z|)\quad(b=2)$$

13.5.8 $$=\frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}+O(1)\quad(1<\mathscr{R}b<2)$$

***13.5.9** $$=-\frac{1}{\Gamma(a)}[\ln z+\psi(a)+2\gamma]+O(|z\ln z|)\quad(b=1)$$

13.5.10 $$U(a,b,z)=\frac{\Gamma(1-b)}{\Gamma(1+a-b)}+O(|z|^{1-\mathscr{R}b})\quad(0<\mathscr{R}b<1)$$

13.5.11 $$=\frac{1}{\Gamma(1+a)}+O(|z\ln z|)\quad(b=0)$$

13.5.12 $$=\frac{\Gamma(1-b)}{\Gamma(1+a-b)}+O(|z|)\quad(\mathscr{R}b\leq 0,\ b\neq 0)$$

For large a (b, z fixed)

13.5.13

$$M(a,b,z)=\Gamma(b)e^{\frac{1}{2}z}(\tfrac{1}{2}bz-az)^{\frac{1}{2}-\frac{1}{2}b}J_{b-1}(\sqrt{(2bz-4az)})\left[1+O(|\tfrac{1}{2}b-a|^{-\sigma})\right]$$

where

$|z|=\left|\frac{1}{2}b-a\right|^{\rho}$ and $\sigma=\min(1-\rho,\ \frac{1}{2}-\frac{3}{2}\rho)$, $0\leq\rho<\frac{1}{3}$.

13.5.14

$$M(a,b,x)=\Gamma(b)e^{\frac{1}{2}x}(\tfrac{1}{2}bx-ax)^{\frac{1}{4}-\frac{1}{2}b}\pi^{-\frac{1}{2}}\cos(\sqrt{(2bx-4ax)}-\tfrac{1}{2}b\pi+\tfrac{1}{4}\pi)\left[1+O(|\tfrac{1}{2}b-a|^{-\frac{1}{2}})\right]$$

as $a\to-\infty$ for b bounded, x real.

13.5.15

$$U(a,b,z)=\Gamma(\tfrac{1}{2}b-a+\tfrac{1}{2})e^{\frac{1}{2}z}z^{\frac{1}{2}-\frac{1}{2}b}[\cos(a\pi)J_{b-1}(\sqrt{(2bz-4az)})-\sin(a\pi)Y_{b-1}(\sqrt{(2bz-4az)})]\left[1+O(|\tfrac{1}{2}b-a|^{-\sigma})\right]$$

where σ is defined in **13.5.13.**

13.5.16

$$U(a,b,x)=\Gamma(\tfrac{1}{2}b-a+\tfrac{1}{4})\pi^{-\frac{1}{2}}e^{\frac{1}{2}x}x^{\frac{1}{4}-\frac{1}{2}b}\cos(\sqrt{(2bx-4ax)}-\tfrac{1}{2}b\pi+a\pi+\tfrac{1}{4}\pi)\left[1+O(|\tfrac{1}{2}b-a|^{-\frac{1}{2}})\right]$$

as $a\to-\infty$ for b bounded, x real.

For large real a, b, x

If $\cosh^2\theta=x/(2b-4a)$ so that $x>2b-a>1$,

13.5.17

$$M(a,b,x)=\Gamma(b)\sin(a\pi)\exp[(b-2a)(\tfrac{1}{2}\sinh 2\theta-\theta+\cosh^2\theta)][(b-2a)\cosh\theta]^{1-b}[\pi(\tfrac{1}{2}b-a)\sinh 2\theta]^{-\frac{1}{2}}\left[1+O(|\tfrac{1}{2}b-a|^{-1})\right]$$

13.5.18

$$U(a,b,x)=\exp[(b-2a)(\tfrac{1}{2}\sinh 2\theta-\theta+\cosh^2\theta)][(b-2a)\cosh\theta]^{1-b}[(\tfrac{1}{2}b-a)\sinh 2\theta]^{-\frac{1}{2}}\left[1+O(|\tfrac{1}{2}b-a|^{-1})\right]$$

If $x=(2b-4a)[1+t/(b-2a)^{\frac{1}{3}}]$, so that

$$x\sim 2b-4a$$

13.5.19

$$M(a, b, x)=e^{\frac{1}{2}x}(b-2a)^{\frac{1}{3}-b}\Gamma(b)[\mathrm{Ai}(t)\cos(a\pi)+\mathrm{Bi}(t)\sin(a\pi)+O(|\tfrac{1}{2}b-a|^{-\frac{2}{3}})]$$

13.5.20

$$U(a, b, x)=e^{\frac{1}{2}x+a-\frac{1}{2}b}\Gamma(\tfrac{1}{3})\pi^{-\frac{1}{2}}x6^{-\frac{1}{6}}\{1-t\Gamma(\tfrac{5}{6})(bx-2ax)^{-\frac{1}{3}}3^{\frac{1}{3}}\pi^{-\frac{1}{2}}+O(|\tfrac{1}{2}b-a|^{-\frac{2}{3}})\}$$

If $\cos^2\theta=x/(2b-4a)$ so that $2b-4a>x>0$,

13.5.21

$$M(a, b, x)=\Gamma(b)\exp\{(b-2a)\cos^2\theta\}[(b-2a)\cos\theta]^{1-b}[\pi(\tfrac{1}{2}b-a)\sin 2\theta]^{-\frac{1}{2}}[\sin(a\pi)+\sin\{(\tfrac{1}{2}b-a)(2\theta-\sin 2\theta)+\tfrac{1}{4}\pi\}+O(|\tfrac{1}{2}b-a|^{-1})]$$

13.5.22

$$U(a, b, x)=\exp[(b-2a)\cos^2\theta][(b-2a)\cos\theta]^{1-b}[(\tfrac{1}{2}b-a)\sin 2\theta]^{-\frac{1}{2}}\{\sin[(\tfrac{1}{2}b-a)(2\theta-\sin 2\theta)+\tfrac{1}{4}\pi]+O(|\tfrac{1}{2}b-a|^{-1})\}$$

13.6. Special Cases

	$M(a, b, z)$			Relation	Function
	a	b	z		
13.6.1	$\nu+\frac{1}{2}$	$2\nu+1$	$2iz$	$\Gamma(1+\nu)e^{iz}(\frac{1}{2}z)^{-\nu}J_\nu(z)$	Bessel
13.6.2	$-\nu+\frac{1}{2}$	$-2\nu+1$	$2iz$	$\Gamma(1-\nu)e^{iz}(\frac{1}{2}z)^{\nu}[\cos(\nu\pi)J_\nu(z)-\sin(\nu\pi)Y_\nu(z)]$	Bessel
13.6.3	$\nu+\frac{1}{2}$	$2\nu+1$	$2z$	$\Gamma(1+\nu)e^{z}(\frac{1}{2}z)^{-\nu}I_\nu(z)$	Modified Bessel
13.6.4	$n+1$	$2n+2$	$2iz$	$\Gamma(\frac{3}{2}+n)e^{iz}(\frac{1}{2}z)^{-n-\frac{1}{2}}J_{n+\frac{1}{2}}(z)$	Spherical Bessel
13.6.5	$-n$	$-2n$	$2iz$	$\Gamma(\frac{1}{2}-n)e^{iz}(\frac{1}{2}z)^{n+\frac{1}{2}}J_{-n-\frac{1}{2}}(z)$	Spherical Bessel
13.6.6	$n+1$	$2n+2$	$2z$	$\Gamma(\frac{3}{2}+n)e^{z}(\frac{1}{2}z)^{-n-\frac{1}{2}}I_{n+\frac{1}{2}}(z)$ *	Spherical Bessel
13.6.7	$n+\frac{1}{2}$	$2n+1$	$-2\sqrt{ix}$	$\Gamma(1+n)e^{-ix\pi}(\frac{1}{2}ix\pi)^{-n}(\mathrm{ber}_n\,x+i\,\mathrm{bei}_n\,x)$	Kelvin
13.6.8	$L+1-i\eta$	$2L+2$	$2ix$	$e^{ix}F_L(\eta, x)x^{-L-1}/C_L(\eta)$	Coulomb Wave
13.6.9	$-n$	$\alpha+1$	x	$\frac{n!}{(\alpha+1)_n}L_n^{(\alpha)}(x)$	Laguerre
13.6.10	a	$a+1$	$-x$	$ax^{-a}\gamma(a, x)$	Incomplete Gamma
13.6.11	$-n$	$1+\nu-n$	x	$\frac{(n!)^{\frac{1}{2}}x^{\frac{1}{2}n}}{(1+\nu-n)_n}\rho_n(\nu, x)$	Poisson-Charlier
13.6.12	a	a	z	e^z	Exponential
13.6.13	1	2	$-2iz$	$\frac{e^{-iz}}{z}\sin z$	Trigonometric
13.6.14	1	2	$2z$	$\frac{e^z}{z}\sinh z$	Hyperbolic
13.6.15	$-\frac{1}{2}\nu$	$\frac{1}{2}$	$\frac{1}{2}z^2$	$2^{-\frac{1}{2}}\exp(\frac{1}{4}z^2)E_\nu^{(0)}(z)$	Weber or Parabolic Cylinder
13.6.16	$\frac{1}{2}-\frac{1}{2}\nu$	$\frac{3}{2}$	$\frac{1}{2}z^2$	$\frac{\exp(\frac{1}{4}z^2)}{2z}E_\nu^{(1)}(z)$	
13.6.17	$-n$	$\frac{1}{2}$	$\frac{1}{2}x^2$	$\frac{n!}{(2n)!}(-\frac{1}{2})^{-n}He_{2n}(x)$	Hermite
13.6.18	$-n$	$\frac{3}{2}$	$\frac{1}{2}x^2$	$\frac{n!}{(2n+1)!}(-\frac{1}{2})^{-n}\frac{1}{x}He_{2n+1}(x)$ *	Hermite
13.6.19	$\frac{1}{2}$	$\frac{3}{2}$	$-x^2$	$\frac{\pi^{\frac{1}{2}}}{2x}\mathrm{erf}\,x$	Error Integral
13.6.20	$\frac{1}{2}m+\frac{1}{2}$	$1+n$	r^2	$\frac{n!r^{-2n+m-1}}{\Gamma(\frac{1}{2}m+\frac{1}{2})}e^{r^2}T(m, n, r)$ *	Toronto

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*See page II.

13.6. Special Cases—Continued

	$U(a, b, z)$			Relation	Function
	a	b	z		
13.6.21	$\nu+\frac{1}{2}$	$2\nu+1$	$2z$	$\pi^{-\frac{1}{2}}e^{z}(2z)^{-\nu}K_{\nu}(z)$	Modified Bessel
13.6.22	$\nu+\frac{1}{2}$	$2\nu+1$	$-2iz$	$\frac{1}{2}\pi^{\frac{1}{2}}e^{i[\pi(\nu+\frac{1}{2})-z]}(2z)^{-\nu}H_{\nu}^{(1)}(z)^{*}$	Hankel
13.6.23	$\nu+\frac{1}{2}$	$2\nu+1$	$2iz$	$\frac{1}{2}\pi^{\frac{1}{2}}e^{-i[\pi(\nu+\frac{1}{2})-z]}(2z)^{-\nu}H_{\nu}^{(2)}(z)^{*}$	Hankel
13.6.24	$n+1$	$2n+2$	$2z$	$\pi^{-\frac{1}{2}}e^{z}(2z)^{-n-\frac{1}{2}}K_{n+\frac{1}{2}}(z)$	Spherical Bessel
13.6.25	$\frac{5}{6}$	$\frac{5}{3}$	$\frac{4}{3}z^{3/2}$	$\pi^{\frac{1}{2}}z^{-1}\exp(\frac{2}{3}z^{3/2})2^{-2/3}3^{5/6}\,\mathrm{Ai}(z)$	Airy
13.6.26	$n+\frac{1}{2}$	$2n+1$	$\sqrt{ix}$	$i^{n}\pi^{-\frac{1}{2}}e^{\sqrt{ix}}(2\sqrt{ix})^{-n}[\mathrm{ker}_{n}\,x+i\,\mathrm{kei}_{n}\,x]$	Kelvin
13.6.27	$-n$	$\alpha+1$	x	$(-1)^{n}n!L_{n}^{(\alpha)}(x)$	Laguerre
13.6.28	$1-a$	$1-a$	x	$e^{x}\Gamma(a, x)$	Incomplete Gamma
13.6.29	1	1	$-x$	$-e^{-x}\,\mathrm{Ei}\,(x)$	Exponential Integral
13.6.30	1	1	x	$e^{x}E_{1}(x)$	Exponential Integral
13.6.31	1	1	$-\ln x$	$-\frac{1}{x}\,\mathrm{li}\,(x)$	Logarithmic Integral
13.6.32	$\frac{1}{2}m-n$	$1+m$	x	$\Gamma(1+n-\frac{1}{2}m)e^{x-\pi i(\frac{1}{2}m-n)}\omega_{n,m}(x)$	Cunningham
13.6.33	$-\frac{1}{2}\nu$	0	$2x$	$\Gamma(1+\frac{1}{2}\nu)e^{x}k_{\nu}(x)$ for $x>0$	Bateman
13.6.34	1	1	ix	$e^{ix}[-\frac{1}{2}\pi i+i\,\mathrm{Si}\,(x)-\mathrm{Ci}\,(x)]$	Sine and Cosine Integral
13.6.35	1	1	$-ix$	$e^{-ix}[\frac{1}{2}\pi i-i\,\mathrm{Si}\,(x)-\mathrm{Ci}\,(x)]$	Sine and Cosine Integral
13.6.36	$-\frac{1}{2}\nu$	$\frac{1}{2}$	$\frac{1}{2}z^{2}$	$2^{-\frac{1}{2}\nu}e^{z^{2}/4}D_{\nu}(z)$	Weber or
13.6.37	$\frac{1}{2}-\frac{1}{2}\nu$	$\frac{3}{2}$	$\frac{1}{2}z^{2}$	$2^{\frac{1}{2}-\frac{1}{2}\nu}e^{z^{2}/4}D_{\nu}(z)/z$ *	Parabolic Cylinder
13.6.38	$\frac{1}{2}-\frac{1}{2}n$	$\frac{3}{2}$	x^{2}	$2^{-n}H_{n}(x)/x$ *	Hermite
13.6.39	$\frac{1}{2}$	$\frac{1}{2}$	x^{2}	$\sqrt{\pi}\exp(x^{2})\,\mathrm{erfc}\,x$	Error Integral

13.7. Zeros and Turning Values

If $j_{b-1,r}$ is the r'th positive zero of $J_{b-1}(x)$, then a first approximation X_0 to the r'th positive zero of $M(a, b, x)$ is

13.7.1 $$X_0=j_{b-1,r}^2\{1/(2b-4a)+O(1/(\tfrac{1}{2}b-a)^2)\}$$

13.7.2 $$X_0\approx\frac{\pi^2(r+\frac{1}{2}b-\frac{3}{4})^2}{2b-4a}$$

A closer approximation is given by

13.7.3 $$X_1=X_0-M(a, b, X_0)/M'(a, b, X_0)$$

For the derivative,

13.7.4

$$M'(a, b, X_1)=M'(a, b, X_0)\{1+(b-X_0)\frac{M(a, b, X_0)}{M'(a, b, X_0)}\}$$

If X_0' is the first approximation to a turning value of $M(a, b, x)$, that is, to a zero of $M'(a, b, x)$ then a better approximation is

13.7.5 $$X_1'=X_0'-\frac{X_0'M'(a, b, X_0')}{aM(a, b, X_0')}$$

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*See page II.

The self-adjoint equation **13.1.1** can also be written

13.7.6 $$\frac{d}{dz}[z^b e^{-z} \frac{dw}{dz}]=az^{b-1}e^{-z}w$$

The Sonine-Polya Theorem

The maxima and minima of $|w|$ form an increasing or decreasing sequence according as

$$-ax^{2b-1}e^{-2x}$$

is an increasing or decreasing function of x, that is, they form an increasing sequence for $M(a, b, x)$ if $a>0$, $x<b-\frac{1}{2}$ or if $a<0$, $x>b-\frac{1}{2}$, and a decreasing sequence if $a>0$ and $x>b-\frac{1}{2}$ or if $a<0$ and $x<b-\frac{1}{2}$.

The turning values of $|w|$ lie near the curves

13.7.7

$$w=\pm\Gamma(b)\pi^{-1/2}e^{x/2}(\tfrac{1}{2}bx-ax)^{\frac{1}{4}-\frac{1}{2}b}\{1-x/(2b-4a)\}^{-1/4}$$

Numerical Methods

13.8. Use and Extension of the Tables

Calculation of $M(a, b, x)$

Kummer's Transformation

Example 1. Compute $M(.3, .2, -.1)$ to 7S. Using **13.1.27** and **Tables 4.4** and **13.1** we have $a=.3$, $b=.2$ so that

$$M(.3, .2, -.1)=e^{-.1}M(-.1, .2, .1)$$
$$=.85784\ 90.$$

Thus **13.1.27** can be used to extend **Table 13.1** to negative values of x. Kummer's transformation should also be used when a and b are large and nearly equal, for x large or small.

Example 2. Compute $M(17, 16, 1)$ to 7S. Here $a=17$, $b=16$, and

$$M(17, 16, 1)=e^1M(-1, 16, -1)$$
$$=2.71828\ 18\times1.06250\ 00$$
$$=2.88817\ 44.$$

Recurrence Relations

Example 3. Compute $M(-1.3, 1.2, .1)$ to 7S. Using **13.4.1** and **Table 13.1** we have $a=-.3$, $b=.2$ so that

$$M(-1.3, .2, .1)=2[.7\,M(-.3, .2, .1)-.3\,M(.7, .2, .1)]$$
$$=.35821\ 23.$$

By **13.4.5** when $a=-1.3$ and $b=.2$,

$$M(-1.3, 1.2, .1)=[.26\ M(-.3, .2, .1)$$
$$-.24\ M(-1.3, .2, .1)]/.15$$
$$=.89241\ 08.$$

Similarly when $a=-.3$ and $b=.2$

$$M(-.3, 1.2, .1)=.97459\ 52.$$

Check, by **13.4.6,**

$$M(-1.3, 1.2, .1)=[.2\ M(-.3, .2, .1)$$
$$+1.2\ M(-.3, 1.2, .1)]/1.5$$
$$=.89241\ 08.$$

In this way **13.4.1–13.4.7** can be used together with **13.1.27** to extend **Table 13.1** to the range

$$-10\leq a\leq10,\ -10\leq b\leq10,\ -10\leq x\leq10.$$

This extension of ten units in any direction is possible with the loss of about 1S. All the recurrence relations are stable except i) if $a<0$, $b<0$ and $|a|>|b|$, $x>0$, or ii) $b<a$, $b<0$, $|b-a|>|b|$, $x<0$, when the oscillations may become large, especially if $|x|$ also is large.

Neither interpolation nor the use of recurrence relations should be attempted in the strips $b=-n\pm.1$ where the function is very large numerically. In particular $M(a, b, x)$ cannot be evaluated in the neighborhood of the points $a=-m$, $b=-n$, $m\leq n$, as near these points small changes in a, b or x can produce very large changes in the numerical value of $M(a, b, x)$.

Example 4. At the point $(-1, -1, x)$, $M(a, b, x)$ is undefined.

When $a=-1$, $M(-1, b, x)=1-\frac{x}{b}$ for all x.

Hence $\lim_{b\to-1} M(-1, b, x)=1+x$. But $M(b, b, x)=e^x$ for all x, when $a=b$. Hence $\lim_{b\to-1} M(b, b, x)=e^x$.

In the first case $b\to-1$ along the line $a=-1$, and in the second case $b\to-1$ along the line $a=b$.

Derivatives

Example 5. To evaluate $M'(-.7, -.6, .5)$ to 7S. By **13.4.8**, when $a=-.7$ and $b=-.6$, we have

$$M'(-.7, -.6, .5)=\frac{-.7}{-.6}M(.3, .4, .5)$$
$$=1.724128.$$

Asymptotic Formulas

For $x\geq10$, a and b small, $M(a, b, x)$ should be evaluated by **13.5.1** using converging factors **13.5.3** and **13.5.4** to improve the accuracy if necessary.

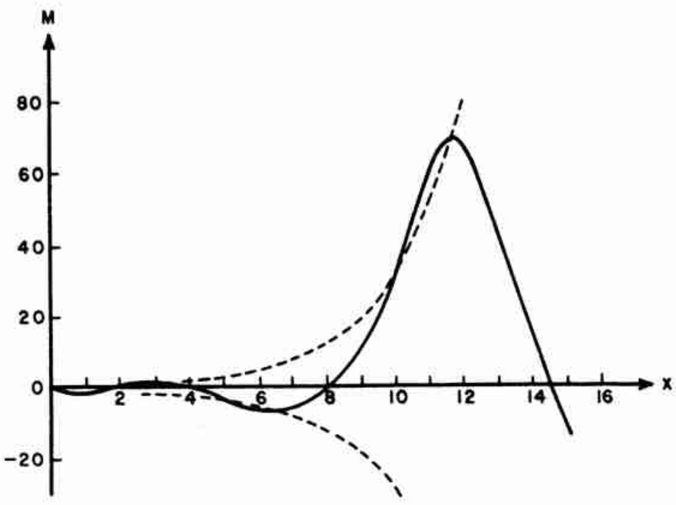

FIGURE 13.2. $M(-4.5, 1, x)$.

(From F. G. Tricomi, Funzioni, ipergeometriche confluenti, Edizioni. Cremonese, Rome, Italy, 1954, with permission.)

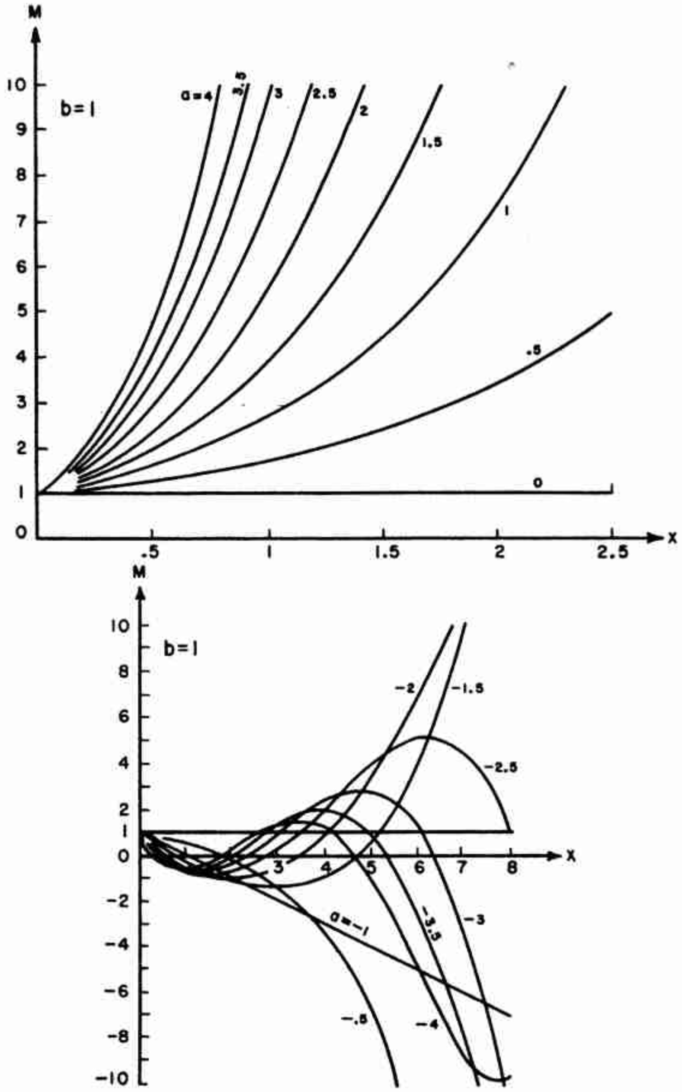

FIGURE 13.3. $M(a, 1, x)$.

(From E. Jahnke and F. Emde, Tables of functions, Dover Publications, Inc., New York, N.Y., 1945, with permission.)

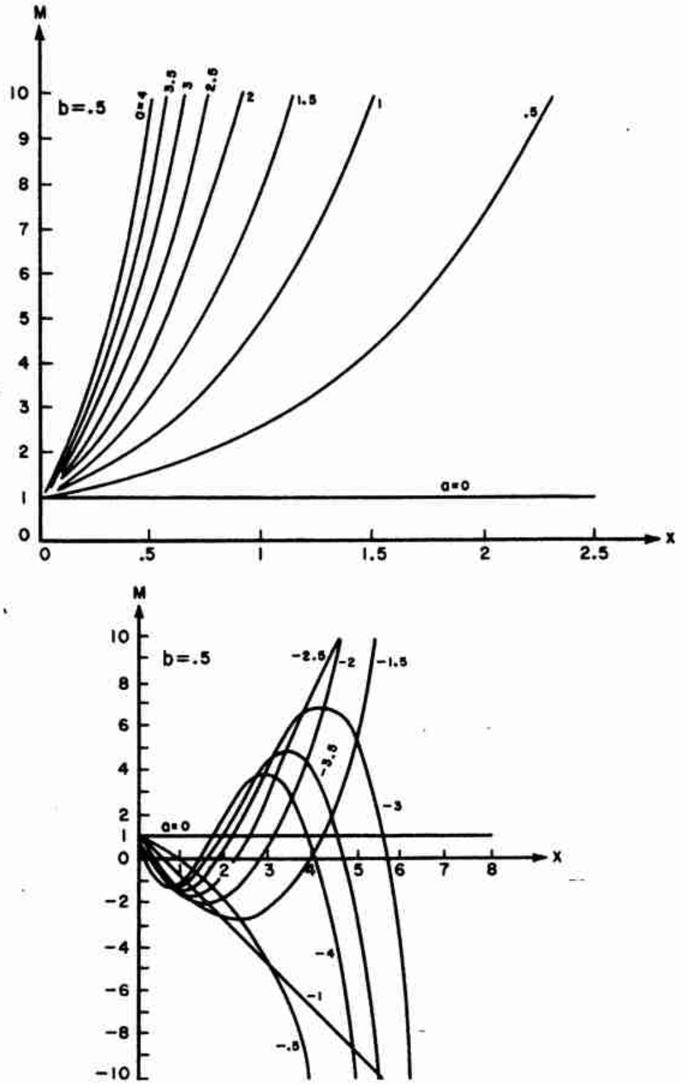

FIGURE 13.4. $M(a, .5, x)$.

(From E. Jahnke and F. Emde, Tables of functions, Dover Publications, Inc., New York, N.Y., 1945, with permission.)

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14. Coulomb Wave Functions

MILTON ABRAMOWITZ[1]

Contents

The author wishes to acknowledge the assistance of David S. Liepman in checking the formulas and tables.

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[1] National Bureau of Standards (deceased).

14. Coulomb Wave Functions

Mathematical Properties

14.1. Differential Equation, Series Expansions

Differential Equation

14.1.1

$$\frac{d^2w}{d\rho^2}+[1-\frac{2\eta}{\rho}-\frac{L(L+1)}{\rho^2}]w=0$$

$(\rho>0,\ -\infty<\eta<\infty,\ L$ a non-negative integer)

The Coulomb wave equation has a regular singularity at $\rho=0$ with indices $L+1$ and $-L$; it has an irregular singularity at $\rho=\infty$.

General Solution

14.1.2

$$w=C_1F_L(\eta,\ \rho)+C_2G_L(\eta,\ \rho) \qquad (C_1,\ C_2 \text{ constants})$$

where $F_L(\eta, \rho)$ is the regular Coulomb wave function and $G_L(\eta, \rho)$ is the irregular (logarithmic) Coulomb wave function.

Regular Coulomb Wave Function $F_L(\eta, \rho)$

14.1.3

$$F_L(\eta, \rho)=C_L(\eta)\rho^{L+1}e^{-i\rho}M(L+1-i\eta, 2L+2, 2i\rho)$$

14.1.4

$$=C_L(\eta)\rho^{L+1}\Phi_L(\eta, \rho)$$

14.1.5 $$\Phi_L(\eta, \rho)=\sum_{k=L+1}^{\infty} A_k^L(\eta)\rho^{k-L-1}$$

14.1.6

$$A_{L+1}^L=1, \quad A_{L+2}^L=\frac{\eta}{L+1},$$

$$(k+L)(k-L-1)A_k^L=2\eta A_{k-1}^L-A_{k-2}^L \qquad (k>L+2)$$

14.1.7 $$C_L(\eta)=\frac{2^Le^{-\frac{\pi\eta}{2}}|\Gamma(L+1+i\eta)|}{\Gamma(2L+2)}$$

(See chapter **6.**)

14.1.8 $$C_0^2(\eta)=2\pi\eta(e^{2\pi\eta}-1)^{-1}$$

14.1.9 $$C_L^2(\eta)=\frac{p_L(\eta)C_0^2(\eta)}{2\eta(2L+1)}$$

14.1.10 $$C_L(\eta)=\frac{(L^2+\eta^2)^{\frac{1}{2}}}{L(2L+1)}C_{L-1}(\eta)$$

14.1.11 $$\frac{p_L(\eta)}{2\eta}=\frac{(1+\eta^2)(4+\eta^2)\ldots(L^2+\eta^2)2^{2L}}{(2L+1)[(2L)!]^2}$$

14.1.12 $$F_L'=\frac{d}{d\rho}F_L(\eta, \rho)=C_L(\eta)\rho^L\Phi_L^*(\eta, \rho)$$

14.1.13 $$\Phi_L^*(\eta, \rho)=\sum_{k=L+1}^{\infty}kA_k^L(\eta)\rho^{k-L-1}$$

Irregular Coulomb Wave Function $G_L(\eta, \rho)$

14.1.14

$$G_L(\eta, \rho)=\frac{2\eta}{C_0^2(\eta)}F_L(\eta, \rho)[\ln 2\rho+\frac{q_L(\eta)}{p_L(\eta)}]+\theta_L(\eta, \rho)$$

14.1.15 $$\theta_L(\eta, \rho)=D_L(\eta)\rho^{-L}\psi_L(\eta, \rho)$$

14.1.16 $$D_L(\eta)C_L(\eta)=\frac{1}{2L+1}$$

14.1.17 $$\psi_L(\eta, \rho)=\sum_{k=-L}^{\infty}a_k^L(\eta)\rho^{k+L}$$

14.1.18

$$a_{-L}^L=1, \quad a_{L+1}^L=0,$$

$$(k-L-1)(k+L)a_k^L=2\eta a_{k-1}^L-a_{k-2}^L-(2k-1)p_L(\eta)A_k^L$$

14.1.19

$$\frac{q_L(\eta)}{p_L(\eta)}=\sum_{s=1}^{L}\frac{s}{s^2+\eta^2}-\sum_{s=1}^{2L+1}\frac{1}{s}$$
$$+\mathscr{R}\{\frac{\Gamma'(1+i\eta)}{\Gamma(1+i\eta)}\}+2\gamma+\frac{r_L(\eta)}{p_L(\eta)}$$

(See **Table 6.8.**)

14.1.20

$$r_L(\eta)=\frac{(-1)^{L+1}}{(2L)!}\mathscr{I}\{\frac{1}{2L+1}+\frac{2(i\eta-L)}{2L(1!)}$$
$$+\frac{2^2(i\eta-L)(i\eta-L+1)}{(2L-1)(2!)}+\ldots$$
$$+\frac{2^{2L}(i\eta-L)(i\eta-L+1)\ldots(i\eta+L-1)}{(2L)!}\}$$

14.1.21

$$G_L'=\frac{dG_L}{d\rho}=\frac{2\eta}{C_0^2(\eta)}\{F_L'[\ln 2\rho+\frac{q_L(\eta)}{p_L(\eta)}]+\rho^{-1}F_L(\eta, \rho)\}$$
$$+\theta_L'(\eta, \rho)$$

14.1.22 $$\theta_L'=\frac{d}{d\rho}\theta_L(\eta,\rho)=D_L(\eta)\rho^{-L-1}\psi_L^*(\eta,\rho)$$

14.1.23 $$\psi_L^*(\eta,\rho)=\sum_{k=-L}^{\infty}ka_k^L(\eta)\rho^{k+L}$$

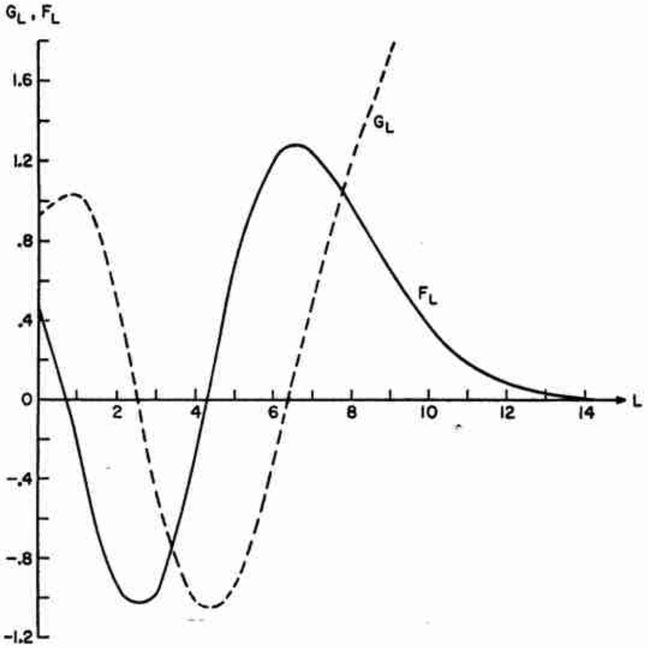

FIGURE 14.1. $F_L(\eta,\rho)$, $G_L(\eta,\rho)$.

$\eta=1$, $\rho=10$

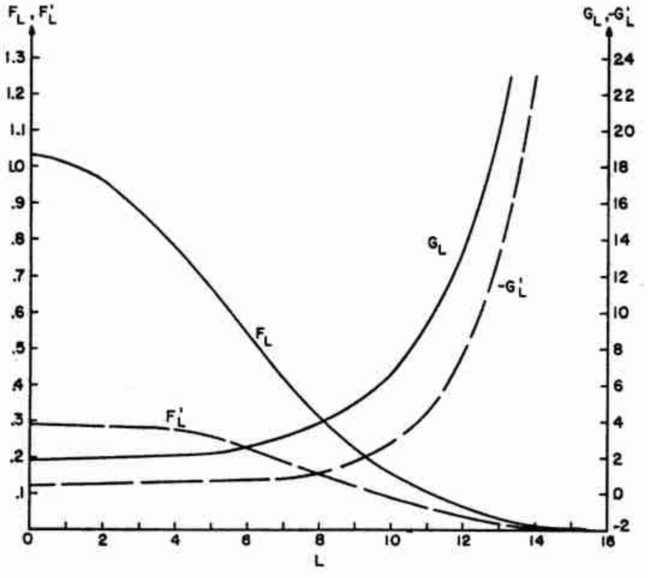

FIGURE 14.2. F_L, F_L', G_L and G_L'.

$\eta=10$, $\rho=20$

14.2. Recurrence and Wronskian Relations

Recurrence Relations

If $u_L=F_L(\eta,\rho)$ or $G_L(\eta,\rho)$,

14.2.1 $$L\frac{du_L}{d\rho}=(L^2+\eta^2)^{\frac{1}{2}}u_{L-1}-\left(\frac{L^2}{\rho}+\eta\right)u_L$$

14.2.2
$$(L+1)\frac{du_L}{d\rho}=\left[\frac{(L+1)^2}{\rho}+\eta\right]u_L-[(L+1)^2+\eta^2]^{\frac{1}{2}}u_{L+1}$$

14.2.3
$$L[(L+1)^2+\eta^2]^{\frac{1}{2}}u_{L+1}=(2L+1)\left[\eta+\frac{L(L+1)}{\rho}\right]u_L-(L+1)[L^2+\eta^2]^{\frac{1}{2}}u_{L-1}$$

Wronskian Relations

14.2.4 $$F_L'G_L-F_LG_L'=1$$

14.2.5 $$F_{L-1}G_L-F_LG_{L-1}=L(L^2+\eta^2)^{-\frac{1}{2}}$$

14.3. Integral Representations

14.3.1
$$F_L+iG_L=\frac{ie^{-i\rho}\rho^{-L}}{(2L+1)!C_L(\eta)}\int_0^{\infty}e^{-t}t^{L-i\eta}(t+2i\rho)^{L+i\eta}dt$$

14.3.2
$$F_L-iG_L=$$
$$\frac{e^{-\pi\eta}\rho^{L+1}}{(2L+1)!C_L(\eta)}\int_{-1}^{-i\infty}e^{-i\rho t}(1-t)^{L-i\eta}(1+t)^{L+i\eta}dt$$

14.3.3
$$F_L+iG_L=\frac{e^{-\pi\eta}\rho^{L+1}}{(2L+1)!C_L(\eta)}$$
$$\cdot\int_0^{\infty}\{(1-\tanh^2 t)^{L+1}\exp[-i(\rho\tanh t-2\eta t)]+i(1+t^2)^L\exp[-\rho t+2\eta\arctan t]\}dt$$

14.4. Bessel Function Expansions

Expansion in Terms of Bessel-Clifford Functions

14.4.1
$$F_L(\eta,\rho)=C_L(\eta)\frac{(2L+1)!}{(2\eta)^{2L+1}}\rho^{-L}\sum_{k=2L+1}^{\infty}b_kt^{k/2}I_k(2\sqrt{t})$$
$$(t=2\eta\rho,\ \eta>0)$$

14.4.2
$$G_L(\eta,\rho)\sim D_L(\eta)\lambda_L(\eta)\rho^{-L}\sum_{k=2L+1}^{\infty}(-1)^kb_kt^{k/2}K_k(2\sqrt{t})$$

14.4.3

$$b_{2L+1}=1,\quad b_{2L+2}=0,$$

* $$4\eta^2(k-2L)b_{k+1}+kb_{k-1}+b_{k-2}=0\qquad(k\geq 2L+2)$$

14.4.4

$$\lambda_L(\eta)\sum_{k=2L+1}^{\infty}(-1)^k(k-1)!b_k=2$$

(See chapter **9.**)

Expansion in Terms of Spherical Bessel Functions

14.4.5

$$F_L(\eta,\rho)=1\cdot 3\cdot 5\ldots(2L+1)\rho C_L(\eta)\sum_{k=L}^{\infty}b_k\sqrt{\frac{\pi}{2\rho}}J_{k+\frac{1}{2}}(\rho)$$

14.4.6

$$b_L=1,\ b_{L+1}=\frac{2L+3}{L+1}\eta$$

$$b_k=\frac{(2k+1)}{k(k+1)-L(L+1)}\left\{2\eta b_{k-1}-\frac{(k-1)(k-2)-L(L+1)}{2k-3}b_{k-2}\right\}\qquad(k>L+1)$$

14.4.7

$$F'_L(\eta,\rho)=1\cdot 3\cdot 5\ldots(2L+1)\rho C_L(\eta)\left\{\frac{(L+1)}{(2L+1)}b_L\sqrt{\frac{\pi}{2\rho}}J_{L-\frac{1}{2}}(\rho)+\frac{(L+2)}{(2L+3)}\cdot b_{L+1}\cdot\sqrt{\frac{\pi}{2\rho}}J_{L+\frac{1}{2}}(\rho)+\sum_{k=L+1}^{\infty}b'_k\sqrt{\frac{\pi}{2\rho}}J_{k+\frac{1}{2}}(\rho)\right\}$$

14.4.8 $$b'_k=\frac{(k+2)}{(2k+3)}b_{k+1}-\frac{(k-1)}{(2k-1)}b_{k-1}$$

Expansion in Terms of Airy Functions

$$x=(2\eta-\rho)/(2\eta)^{1/3}\qquad\mu=(2\eta)^{2/3},\ \eta>>0$$

$$|\rho-2\eta|<2\eta$$

14.4.9

$$\begin{matrix}F_0(\eta,\rho)\\G_0(\eta,\rho)\end{matrix}=\pi^{\frac{1}{2}}(2\eta)^{\frac{1}{6}}\left\{\begin{matrix}\mathrm{Ai}(x)\\\mathrm{Bi}(x)\end{matrix}\left[1+\frac{g_1}{\mu}+\frac{g_2}{\mu^2}+\cdots\right]+\begin{matrix}\mathrm{Ai}'(x)\\\mathrm{Bi}'(x)\end{matrix}\left[\frac{f_1}{\mu}+\frac{f_2}{\mu^2}+\cdots\right]\right\}$$

14.4.10

$$\begin{matrix}F'_0(\eta,\rho)\\G'_0(\eta,\rho)\end{matrix}=-\pi^{\frac{1}{2}}(2\eta)^{-\frac{1}{6}}\left\{\begin{matrix}\mathrm{Ai}(x)\\\mathrm{Bi}(x)\end{matrix}\left[\frac{g'_1+xf_1}{\mu}+\frac{g'_2+xf_2}{\mu^2}+\cdots\right]+\begin{matrix}\mathrm{Ai}'(x)\\\mathrm{Bi}'(x)\end{matrix}\left[1+\frac{(g_1+f'_1)}{\mu}+\frac{(g_2+f'_2)}{\mu^2}+\cdots\right]\right\}$$

$$f_1=(1/5)x^2$$

$$f_2=\frac{1}{35}(2x^3+6)$$

$$f_3=\frac{1}{63000}(84x^7+1480x^4+2320x)$$

$$g_1=-(1/5)x$$

$$g_2=\frac{1}{350}(7x^5-30x^2)$$

$$g_3=\frac{1}{63000}(1056x^6-1160x^3-2240)$$

(See chapter **10.**)

14.5. Asymptotic Expansions

Asymptotic Expansion for Large Values of ρ

14.5.1 $$F_L=g\cos\theta_L+f\sin\theta_L$$

14.5.2 $$G_L=f\cos\theta_L-g\sin\theta_L$$

14.5.3 $$F'_L=g^*\cos\theta_L+f^*\sin\theta_L$$

14.5.4 $$G'_L=f^*\cos\theta_L-g^*\sin\theta_L,\ gf^*-fg^*=1$$

14.5.5 $$\theta_L=\rho-\eta\ln 2\rho-L\frac{\pi}{2}+\sigma_L$$

14.5.6 $$\sigma_L=\arg\Gamma(L+1+i\eta)$$

(See **6.1.27, 6.1.44.**)

14.5.7 $$\sigma_{L+1}=\sigma_L+\arctan\frac{\eta}{L+1}$$

(See **Tables 4.14, 6.7.**)

14.5.8 $$f\sim\sum_{k=0}^{\infty}f_k,\ g\sim\sum_{k=0}^{\infty}g_k,\ f^*\sim\sum_{k=0}^{\infty}f^*_k,\ g^*\sim\sum_{k=0}^{\infty}g^*_k$$

where

$$f_0=1,\ g_0=0,\ f^*_0=0,\ g^*_0=1-\eta/\rho$$

$$f_{k+1}=a_kf_k-b_kg_k$$

$$g_{k+1}=a_kg_k+b_kf_k$$

$$f^*_{k+1}=a_kf^*_k-b_kg^*_k-f_{k+1}/\rho$$

$$g^*_{k+1}=a_kg^*_k+b_kf^*_k-g_{k+1}/\rho$$

$$a_k=\frac{(2k+1)\eta}{(2k+2)\rho},\quad b_k=\frac{L(L+1)-k(k+1)+\eta^2}{(2k+2)\rho}$$

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*See page II.

14.5.9

$$f+ig\sim 1+\frac{(i\eta-L)(i\eta+L+1)}{1!(2i\rho)}+\frac{(i\eta-L)(i\eta-L+1)(i\eta+L+1)(i\eta+L+2)}{2!(2i\rho)^2}$$
$$+\frac{(i\eta-L)(i\eta-L+1)(i\eta-L+2)(i\eta+L+1)(i\eta+L+2)(i\eta+L+3)}{3!(2i\rho)^3}+\ldots$$

Asymptotic Expansion for $L=0,\ \rho=2\eta>>0$

14.5.10
$$\begin{matrix}F_0(2\eta)\\ G_0(2\eta)/\sqrt{3}\end{matrix}\sim\frac{\Gamma(1/3)\beta^{1/2}}{2\sqrt{\pi}}\{1\mp\frac{2}{35}\frac{\Gamma(2/3)}{\Gamma(1/3)}\frac{1}{\beta^4}-\frac{32}{8100}\frac{1}{\beta^6}\mp\frac{92672}{7371\cdot 10^4}\frac{\Gamma(2/3)}{\Gamma(1/3)}\frac{1}{\beta^{10}}-\ldots\}$$

14.5.11
$$\begin{matrix}F_0'(2\eta)\\ G_0'(2\eta)/\sqrt{3}\end{matrix}\sim\frac{\Gamma(2/3)}{2\sqrt{\pi}\beta^{1/2}}\{\pm 1+\frac{1}{15}\frac{\Gamma(1/3)}{\Gamma(2/3)}\frac{1}{\beta^2}\pm\frac{8}{56700}\frac{1}{\beta^6}+\frac{11488}{18711\cdot 10^3}\frac{\Gamma(1/3)}{\Gamma(2/3)}\frac{1}{\beta^8}\pm\ldots\}$$

$\beta=(2\eta/3)^{1/6},\ \Gamma(1/3)=2.6789\ 38534\ldots,\ \Gamma(2/3)=1.3541\ 17939\ldots$

14.5.12
$$\begin{matrix}F_0(2\eta)\\ G_0(2\eta)\end{matrix}\sim\left\{\begin{matrix}.70633\ 26373\\ 1.22340\ 4016\end{matrix}\right\}\eta^{1/6}\{1\mp\frac{.04959\ 570165}{\eta^{4/3}}-\frac{.00888\ 88888\ 89}{\eta^2}$$
$$\mp\frac{.00245\ 51991\ 81}{\eta^{10/3}}-\frac{.00091\ 08958\ 061}{\eta^4}\mp\frac{.00025\ 34684\ 115}{\eta^{16/3}}-\ldots\}$$

14.5.13
$$\begin{matrix}F_0'(2\eta)\\ G_0'(2\eta)\end{matrix}\sim\left\{\begin{matrix}.40869\ 57323\\ -.70788\ 17734\end{matrix}\right\}\eta^{-1/6}\{1\pm\frac{.17282\ 60369}{\eta^{2/3}}+\frac{.00031\ 74603\ 174}{\eta^2}$$
$$\pm\frac{.00358\ 12148\ 50}{\eta^{8/3}}+\frac{.00031\ 17824\ 680}{\eta^4}\pm\frac{.00090\ 73966\ 427}{\eta^{14/3}}+\ldots\}$$

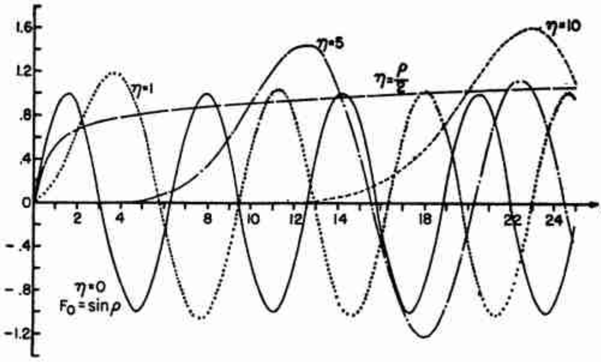

FIGURE 14.3. $F_0(\eta,\rho)$.

$\eta=0,\ 1,\ 5,\ 10,\ \rho/2$

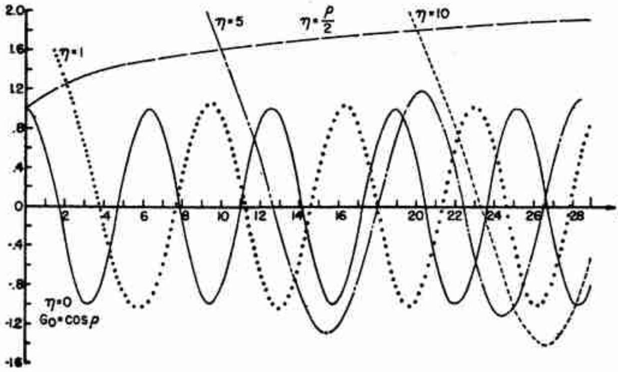

FIGURE 14.5. $G_0(\eta,\rho)$.

$\eta=0,\ 1,\ 5,\ 10,\ \rho/2$

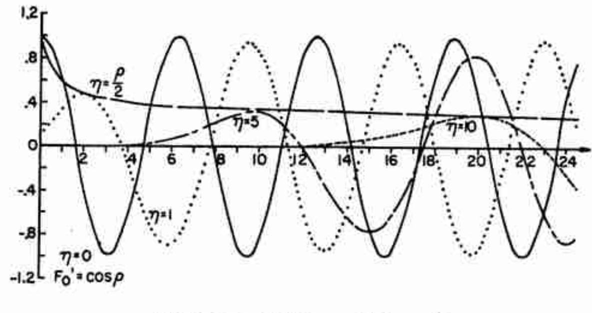

FIGURE 14.4. $F_0'(\eta,\rho)$.

$\eta=0,\ 1,\ 5,\ 10,\ \rho/2$

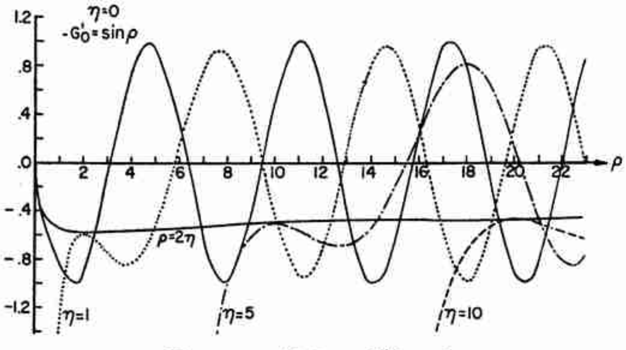

FIGURE 14.6. $G_0'(\eta,\rho)$.

$\eta=0,\ 1,\ 5,\ 10,\ \rho/2$

14.6. Special Values and Asymptotic Behavior

14.6.1 $$L>0,\ \rho=0$$
$$F_L=0,\ F'_L=0$$
$$G_L=\infty,\ G'_L=-\infty$$

14.6.2 $$L=0,\ \rho=0$$
$$F_0=0,\ F'_0=C_0(\eta)$$
$$G_0=1/C_0(\eta),\ G'_0=-\infty$$

14.6.3 $$L\rightarrow\infty$$
$$F_L\sim C_L(\eta)\rho^{L+1},\ G_L\sim D_L(\eta)\rho^{-L}$$

14.6.4 $$L=0,\ \eta=0$$
$$F_0=\sin\rho,\ F'_0=\cos\rho$$
$$G_0=\cos\rho,\ G'_0=-\sin\rho$$

14.6.5 $$\rho\rightarrow\infty$$
$$G_L+iF_L\sim\exp i[\rho-\eta\ln 2\rho-\frac{L\pi}{2}+\sigma_L]$$

14.6.6 $$L\geq 0,\ \eta=0$$
$$F_L=(\tfrac{1}{2}\pi\rho)^{\frac{1}{2}}\,J_{L+\frac{1}{2}}(\rho)$$
$$G_L=(-1)^L(\tfrac{1}{2}\pi\rho)^{\frac{1}{2}}\,J_{-(L+\frac{1}{2})}(\rho)$$

14.6.7 $$L\geq 0,\ 2\eta>>\rho$$
$$F_L\sim\frac{(2L+1)!C_L(\eta)}{(2\eta)^{L+1}}\,(2\eta\rho)^{\frac{1}{2}}I_{2L+1}[2(2\eta\rho)^{\frac{1}{2}}]$$
$$G_L\sim\frac{2(2\eta)^L}{(2L+1)!C_L(\eta)}\,(2\eta\rho)^{\frac{1}{2}}K_{2L+1}[2(2\eta\rho)^{\frac{1}{2}}]$$

14.6.8 $$L=0,\ 2\eta>>\rho$$
$$F_0\sim e^{-\pi\eta}(\pi\rho)^{\frac{1}{2}}I_1[2(2\eta\rho)^{\frac{1}{2}}]$$
$$F'_0\sim e^{-\pi\eta}(2\pi\eta)^{\frac{1}{2}}I_0[2(2\eta\rho)^{\frac{1}{2}}]$$
$$G_0\sim 2e^{\pi\eta}\left(\frac{\rho}{\pi}\right)^{\frac{1}{2}}K_1[2(2\eta\rho)^{\frac{1}{2}}]$$
$$G'_0\sim -2\left(\frac{2\eta}{\pi}\right)^{\frac{1}{2}}e^{\pi\eta}K_0[2(2\eta\rho)^{\frac{1}{2}}]$$

14.6.9 $$L=0,\ 2\eta>>\rho$$
$$F_0\sim\frac{1}{2}\,\beta e^{\alpha};\ F'_0\sim\frac{1}{2}\,\beta^{-1}e^{\alpha}$$
$$G_0\sim\beta e^{-\alpha};\ G'_0\sim-\beta^{-1}e^{-\alpha}$$
$$\alpha=2\sqrt{2\eta\rho}-\pi\eta$$
$$\beta=(\rho/2\eta)^{\frac{1}{4}}$$

14.6.10 $$L=0,\ 2\eta>>\rho$$
$$F_0\sim\frac{1}{2}\,\beta e^{\alpha};\ F'_0\sim\{\beta^{-2}+\frac{1}{8\eta}\,t^{-2}\beta^4\}F_0$$
$$G_0\sim\beta e^{-\alpha};\ G'_0\sim\{-\beta^{-2}+\frac{1}{8\eta}\,t^{-2}\beta^4\}G_0$$
$$t=\rho/2\eta$$
$$\alpha=2\eta\{[t(1-t)]^{\frac{1}{2}}+\arcsin t^{\frac{1}{2}}-\tfrac{1}{2}\pi\}$$
$$\beta=\{t/(1-t)\}^{\frac{1}{4}}$$

14.6.11 $$L=0,\ \rho>>2\eta$$
$$F_0=\alpha\sin\beta;\ F'_0=-t^2(bF_0-aG_0)$$
$$G_0=\alpha\cos\beta;\ G'_0=-t^2(aF_0+bG_0)$$
$$t=\frac{2\eta}{\rho}$$
$$\alpha=(\frac{1}{1-t})^{\frac{1}{4}}\exp[-\frac{8t^3-3t^4}{64(2\eta)^2(1-t)^3}]$$
$$\beta=\frac{\pi}{4}+2\eta\{\frac{(1-t)^{\frac{1}{2}}}{t}+\frac{1}{2}\ln[\frac{1-(1-t)^{\frac{1}{2}}}{1+(1-t)^{\frac{1}{2}}}]\}$$
$$a=t^{-2}(1-t)^{\frac{1}{2}},\ b=[8\eta(1-t)]^{-1}$$

14.6.12 $$\eta>>0,\ 2\eta\sim\rho$$
$$\begin{matrix}F_L(\eta,\rho)\\G_L(\eta,\rho)\end{matrix}\sim\sqrt{\pi}\left\{\frac{\rho_L}{1+\frac{L(L+1)}{\rho_L^2}}\right\}^{1/6}\begin{Bmatrix}\mathrm{Ai}\,(x)\\\mathrm{Bi}\,(x)\end{Bmatrix}$$
$$\rho_L=\eta+[\eta^2+L(L+1)]^{1/2}$$
$$x=(\rho_L-\rho)[\frac{1}{\rho_L}+\frac{L(L+1)}{\rho_L^3}]^{1/3}$$

14.6.13 $$\eta>>0,\ 2\eta\sim\rho$$
$$x=(2\eta-\rho)(2\eta)^{-1/3}$$
$$[G_0+iF_0]\sim\pi^{1/2}(2\eta)^{1/6}[\mathrm{Bi}(x)+i\mathrm{Ai}(x)]$$
$$[G'_0+iF'_0]\sim-\pi^{1/2}(2\eta)^{-1/6}[\mathrm{Bi}'(x)+i\mathrm{Ai}'(x)]$$

14.6.14 $$\eta>>0$$
$$\rho_L=\eta+[\eta^2+L(L+1)]^{1/2}$$
$$\begin{matrix}F_L(\rho_L)\\G_L(\rho_L)/\sqrt{3}\end{matrix}\sim\frac{\Gamma(1/3)}{2\sqrt{\pi}}\left(\frac{\rho_L}{3}\right)^{1/6}\left\{1+\frac{L(L+1)}{\rho_L^2}\right\}^{-1/6}$$
$$\begin{matrix}F'_L(\rho_L)\\G'_L(\rho_L)/\sqrt{3}\end{matrix}\sim\pm\frac{\Gamma(2/3)}{2\sqrt{\pi}}\left(\frac{\rho_L}{3}\right)^{-1/6}\left\{1+\frac{L(L+1)}{\rho_L^2}\right\}^{1/6}$$

14.6.15 $$\rho=2\eta>>0$$

$$\begin{matrix}F_0\\ G_0/\sqrt{3}\end{matrix}\sim\frac{\Gamma(1/3)}{2\sqrt{\pi}}\left(\frac{2\eta}{3}\right)^{1/6}$$

$$\begin{matrix}F_0'\\ -G_0'/\sqrt{3}\end{matrix}\sim\frac{\Gamma(2/3)}{2\sqrt{\pi}(2\eta/3)^{1/6}}$$

14.6.16 $$\eta\rightarrow\infty$$

$$\sigma_0(\eta)\sim[\frac{\pi}{4}+\eta(\ln\eta-1)]$$

$$C_0(\eta)\sim(2\pi\eta)^{1/2}e^{-\pi\eta},$$

(Equality to 8S for $\eta>3$.)

14.6.17 $$\eta\rightarrow 0$$

$$\sigma_0(\eta)\sim-\gamma\eta\qquad(\gamma=\text{Euler's constant})$$

$$C_L(\eta)\sim\frac{2^L L!}{(2L+1)!}$$

14.6.18 $$L\rightarrow\infty$$

$$C_L(\eta)\sim\frac{2^L L!}{(2L+1)!}e^{-\pi\eta/2}$$

Numerical Methods

14.7. Use and Extension of the Tables

In general the tables as presented are not simply interpolable. However, values for $L>0$ may be obtained with the help of the recurrence relations. The values of $G_L(\eta,\rho)$ may be obtained by applying the recurrence relations in increasing order of L. Forward recurrence may be used for $F_L(\eta,\rho)$ as long as the instability does not produce errors in excess of the accuracy needed. In this case the backwards recurrence scheme (see **Example 1**) should be used.

Example 1. Compute $F_L(\eta,\rho)$ and $F_L'(\eta,\rho)$ for $\eta=2$, $\rho=5$, $L=0(1)5$. Starting with $F_{10}^*=1$, $F_{11}^*=0$, where $F_L^*=cF_L$, we compute from **14.2.3** in decreasing order of L:

	(1)	(2)	(3)	(4)
L	F_L^*	F_L	F_L	F_L'
11	0.			
10	1.			
9	4. 49284			
8	17. 5225			
7	61. 3603			
6	191. 238			
5	523. 472	. 090791	. 091	. 1043
4	1238. 53	. 21481	. 215	. 2030
3	2486. 72	. 43130	. 4313	. 3205
2	4158. 46	. 72124	. 72125	. 3952
1	5727. 97	. 99346	. 99347	. 3709
0	6591. 81	1. 1433	1. 1433	. 29380

$F_0/F_0^*=1.7344\times10^{-4}=c^{-1}$.

The values in the second column are obtained from those in the first by multiplying by the normalization constant, F_0/F_0^* where F_0 is the known value obtained from **Table 14.1.**

Repetition starting with $F_{15}^*=1$ and $F_{16}^*=0$ yields the same results.

In column 3, the results have been given when **14.2.3** is used in increasing order of L.

F_L' (column 4) follows from **14.2.2.**

Example 2. Compute $G_L(\eta,\rho)$ and $G_L'(\eta,\rho)$ for $\eta=2$, $\rho=5$, $L=1(1)5$.

Using **14.2.2** and $G_0(2,5)=.79445$, $G_0'=-.67049$ from **Table 14.1** we find $G_1(2,5)=1.0815$. Then by forward recurrence using **14.2.3** we find:

L	G_L	$-G'_L$ *
1	1. 0815	. 60286
2	1. 4969	. 56619
3	2. 0487	. 79597
4	3. 0941	1. 7318
5	5. 6298	4. 5493

The values of G_L' are obtained with **14.2.1.**

Example 3. Compute $G_0(\eta,\rho)$ for $\eta=2$, $\rho=2.5$.

From **Table 14.1,** $G_0(2,2)=3.5124$, $G_0'(2,2)=-2.5554$. Successive differentiation of **14.1.1** for $L=0$ gives

$$\rho\frac{d^{k+2}w}{d\rho^{k+2}}=(2\eta-\rho)\frac{d^kw}{d\rho^k}-k\{\frac{d^{k+1}w}{d\rho^{k+1}}+\frac{d^{k-1}w}{d\rho^{k-1}}\}$$

Taylor's expansion is $w(\rho+\Delta\rho)=w(\rho)+(\Delta\rho)w'+\frac{(\Delta\rho)^2}{2!}w''+\ldots$. With $w=G_0(\eta,\rho)$ and $\Delta\rho=.5$ we get:

k	$\frac{d^kG_0}{d\rho^k}$	$\frac{(\Delta\rho)^k}{k!}\frac{d^kG_0}{d\rho^k}$
0	3. 5124	3. 5124
1	−2. 5554	−1. 2777
2	3. 5124	. 43905
3	−6. 0678	−. 12641
4	12. 136	. 03160
5	−29. 540	−. 00769
6	83. 352	. 00181
7	−268. 26	−. 00042

$$G_0(2,2.5)=2.5726$$

As a check the result is obtained with $\eta=2$, $\rho=3$, $\Delta\rho=-.5$. The derivative $G_0'(\eta,\rho)$ may be obtained using Taylor's formula with $w=G_0'(\eta,\rho)$.

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*See page II.

References

Texts

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Tables

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$$\Phi_L(\eta, \rho) \text{ and } \frac{1}{k!}\frac{d^k\Phi_k(\eta, \rho)}{d\eta^k} \text{ for } \rho=0(.2)5,$$

$$\eta=-5(1)5,\ L=0(1)5,\ 10,\ 11,\ 20,\ 21, \quad 7\text{D}.$$

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15. Hypergeometric Functions

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Contents

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15. Hypergeometric Functions

Mathematical Properties

15.1. Gauss Series, Special Elementary Cases, Special Values of the Argument

Gauss Series

The circle of convergence of the Gauss hypergeometric series

15.1.1

$$F(a,b;c;z)={}_2F_1(a,b;c;z)$$

$$=F(b,a;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

$$=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}\frac{z^n}{n!}$$

is the unit circle $|z|=1$. The behavior of this series on its circle of convergence is:

(a) Divergence when $\mathscr{R}(c-a-b)\leqq -1$.

(b) Absolute convergence when $\mathscr{R}(c-a-b)>0$.

(c) Conditional convergence when $-1<\mathscr{R}(c-a-b)\leqq 0$; the point $z=1$ is excluded. The Gauss series reduces to a polynomial of degree n in z when a or b is equal to $-n$, $(n=0, 1, 2, \ldots)$. (For these cases see also **15.4.**) The series **15.1.1** is not defined when c is equal to $-m$, $(m=0, 1, 2, \ldots)$, provided a or b is not a negative integer n with $n<m$. For $c=-m$

15.1.2

$$\lim_{c\to -m}\frac{1}{\Gamma(c)}F(a,b;c;z)=$$

$$\frac{(a)_{m+1}(b)_{m+1}}{(m+1)!}z^{m+1}F(a+m+1,b+m+1;m+2;z)$$

Special Elementary Cases of Gauss Series

(For cases involving higher functions see **15.4.**)

15.1.3 $\quad F(1,1;2;z)=-z^{-1}\ln(1-z)$ *

15.1.4 $\quad F(\tfrac{1}{2},1;\tfrac{3}{2};z^2)=\tfrac{1}{2}z^{-1}\ln\left(\frac{1+z}{1-z}\right)$

15.1.5 $\quad F(\tfrac{1}{2},1;\tfrac{3}{2};-z^2)=z^{-1}\arctan z$

15.1.6

$$F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};z^2)=(1-z^2)^{\frac{1}{2}}F(1,1;\tfrac{3}{2};z^2)=z^{-1}\arcsin z$$

15.1.7

$$F(\tfrac{1}{2},\tfrac{1}{2};\tfrac{3}{2};-z^2)=(1+z^2)^{\frac{1}{2}}F(1,1;\tfrac{3}{2};-z^2)$$
$$=z^{-1}\ln[z+(1+z^2)^{\frac{1}{2}}]$$

15.1.8 $\quad F(a,b;b;z)=(1-z)^{-a}$

15.1.9 $\quad F(a,\tfrac{1}{2}+a;\tfrac{1}{2};z^2)=\tfrac{1}{2}[(1+z)^{-2a}+(1-z)^{-2a}]$

15.1.10

$$F(a,\tfrac{1}{2}+a;\tfrac{3}{2};z^2)=$$
$$\tfrac{1}{2}z^{-1}(1-2a)^{-1}[(1+z)^{1-2a}-(1-z)^{1-2a}]$$

15.1.11

$$F(-a,a;\tfrac{1}{2};-z^2)=\tfrac{1}{2}\{[(1+z^2)^{\frac{1}{2}}+z]^{2a}+[(1+z^2)^{\frac{1}{2}}-z]^{2a}\}$$

15.1.12

$$F(a,1-a;\tfrac{1}{2};-z^2)=$$
$$\tfrac{1}{2}(1+z^2)^{-\frac{1}{2}}\{[(1+z^2)^{\frac{1}{2}}+z]^{2a-1}+[(1+z^2)^{\frac{1}{2}}-z]^{2a-1}\}$$

15.1.13

$$F(a,\tfrac{1}{2}+a;1+2a;z)=2^{2a}[1+(1-z)^{\frac{1}{2}}]^{-2a}$$
$$=(1-z)^{\frac{1}{2}}F(1+a,\tfrac{1}{2}+a;1+2a;z)$$

15.1.14

$$F(a,\tfrac{1}{2}+a;2a;z)=2^{2a-1}(1-z)^{-\frac{1}{2}}[1+(1-z)^{\frac{1}{2}}]^{1-2a}$$

15.1.15 $\quad F(a,1-a;\tfrac{3}{2};\sin^2 z)=\dfrac{\sin[(2a-1)z]}{(2a-1)\sin z}$

15.1.16 $\quad F(a,2-a;\tfrac{3}{2};\sin^2 z)=\dfrac{\sin[(2a-2)z]}{(a-1)\sin(2z)}$

15.1.17 $\quad F(-a,a;\tfrac{1}{2};\sin^2 z)=\cos(2az)$

15.1.18 $\quad F(a,1-a;\tfrac{1}{2};\sin^2 z)=\dfrac{\cos[(2a-1)z]}{\cos z}$

15.1.19 $\quad F(a,\tfrac{1}{2}+a;\tfrac{1}{2};-\tan^2 z)=\cos^{2a}z\cos(2az)$

Special Values of the Argument

15.1.20

$$F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$(c\neq 0,-1,-2,\ldots,\mathscr{R}(c-a-b)>0)$$

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*See page II.

15.1.21

$$F(a,\ b;\ a-b+1;\ -1)=2^{-a}\pi^{\frac{1}{2}}\frac{\Gamma(1+a-b)}{\Gamma(1+\frac{1}{2}a-b)\Gamma(\frac{1}{2}+\frac{1}{2}a)}$$

$$(1+a-b\neq 0,\ -1,\ -2,\ .\ .\ .)$$

15.1.22

$$F(a,\ b;\ a-b+2;-1)=2^{-a}\pi^{1/2}(b-1)^{-1}\Gamma(a-b+2)$$

$$\left[\frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{3}{2}+\frac{1}{2}a-b)}-\frac{1}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(1+\frac{1}{2}a-b)}\right]$$

$$(a-b+2\neq 0,\ -1,\ -2,\ .\ .\ .)$$

15.1.23 $F(1,\ a;\ a+1;\ -1)=\frac{1}{2}a[\psi(\frac{1}{2}+\frac{1}{2}a)-\psi(\frac{1}{2}a)]$

15.1.24

$$F\ (a,\ b;\ \tfrac{1}{2}a+\tfrac{1}{2}b+\tfrac{1}{2};\tfrac{1}{2})=\pi^{\frac{1}{2}}\frac{\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)}$$

$$(\tfrac{1}{2}a+\tfrac{1}{2}b+\tfrac{1}{2}\neq 0,\ -1,\ -2,\ .\ .\ .)$$

15.1.25

$$F(a,\ b;\ \tfrac{1}{2}a+\tfrac{1}{2}b+1;\ \tfrac{1}{2})=2\pi^{\frac{1}{2}}(a-b)^{-1}\Gamma(1+\tfrac{1}{2}a+\tfrac{1}{2}b)$$

$$\{[\Gamma(\tfrac{1}{2}a)\Gamma(\tfrac{1}{2}+\tfrac{1}{2}b)]^{-1}-[\Gamma(\tfrac{1}{2}+\tfrac{1}{2}a)\Gamma(\tfrac{1}{2}b)]^{-1}\}$$

$$(\tfrac{1}{2}(a+b)+1\neq 0,\ -1,\ -2,\ .\ .\ .)$$

15.1.26

$$F(a,\ 1-a;\ b;\ \tfrac{1}{2})=$$

$$2^{1-b}\pi^{\frac{1}{2}}\Gamma(b)\ [\Gamma(\tfrac{1}{2}a+\tfrac{1}{2}b)\ \Gamma\ (\tfrac{1}{2}+\tfrac{1}{2}b-\tfrac{1}{2}a)]^{-1}$$

$$(b\neq 0,\ -1,\ -2,\ .\ .\ .)$$

15.1.27

$$F(1,\ 1;\ a+1;\ \tfrac{1}{2})=a[\psi(\tfrac{1}{2}+\tfrac{1}{2}a)-\psi(\tfrac{1}{2}a)]$$

$$(a\neq -1,\ -2,\ -3,\ .\ .\ .)$$

15.1.28

$$F(a,\ a;\ a+1;\ \tfrac{1}{2})=2^{a-1}a[\psi(\tfrac{1}{2}+\tfrac{1}{2}a)-\psi(\tfrac{1}{2}a)]$$

$$(a\neq -1,\ -2,\ -3,\ .\ .\ .)$$

15.1.29

$$F\ (a,\ \tfrac{1}{2}+a;\ \tfrac{3}{2}-2a;-\tfrac{1}{3})=(\tfrac{8}{9})^{-2a}\frac{\Gamma(\frac{4}{3})\Gamma(\frac{3}{2}-2a)}{\Gamma(\frac{3}{2})\Gamma(\frac{4}{3}-2a)}$$

$$(\tfrac{3}{2}-2a\neq 0,\ -1,\ -2,\ .\ .\ .)$$

15.1.30

$$F\ (a,\ \tfrac{1}{2}+a;\ \tfrac{5}{6}+a;\ \tfrac{1}{9})=(\tfrac{3}{4})^{a}\pi^{\frac{1}{2}}\frac{\Gamma(\frac{5}{6}+\frac{2}{3}a)}{\Gamma(\frac{1}{2}+\frac{1}{3}a)\Gamma(\frac{5}{6}+\frac{1}{3}a)}$$

$$(\tfrac{5}{6}+\tfrac{2}{3}a\neq 0,\ -1,\ -2,\ .\ .\ .)$$

15.1.31

$$F\ (a,\ \tfrac{1}{3}a+\tfrac{1}{3};\ \tfrac{2}{3}a+\tfrac{2}{3};\ e^{i\pi/3})$$

$$=2^{\frac{2}{3}a+\frac{2}{3}}\pi^{\frac{1}{2}}3^{-\frac{1}{2}(a+1)}e^{i\pi a/6}\frac{\Gamma(\frac{1}{3}a+\frac{5}{6})}{\Gamma(\frac{1}{3}a+\frac{2}{3})\Gamma(\frac{2}{3})}$$

$$(\tfrac{1}{3}\ a\neq -\tfrac{5}{6},-\tfrac{11}{6},-\tfrac{17}{6},\ .\ .\ .)$$

15.2. Differentiation Formulas and Gauss' Relations for Contiguous Functions

Differentiation Formulas

15.2.1 $\frac{d}{dz}F(a,\ b;\ c;\ z)=\frac{ab}{c}F(a+1,\ b+1;\ c+1;\ z)$

15.2.2

$$\frac{d^n}{dz^n}F(a,\ b;\ c;\ z)=\frac{(a)_n(b)_n}{(c)_n}F(a+n,\ b+n;\ c+n;\ z)$$

15.2.3

$$\frac{d^n}{dz^n}[z^{a+n-1}F(a,\ b;\ c;\ z)]=(a)_nz^{a-1}F(a+n,\ b;\ c;\ z)$$

15.2.4

$$\frac{d^n}{dz^n}[z^{c-1}F(a,\ b;\ c;\ z)]=(c-n)_nz^{c-n-1}F(a,\ b;\ c-n;\ z)$$

15.2.5

$$\frac{d^n}{dz^n}[z^{c-a+n-1}(1-z)^{a+b-c}F(a,b;c;z)]$$

$$=(c-a)_nz^{c-a-1}(1-z)^{a+b-c-n}F(a-n,b;c;z)$$

15.2.6

$$\frac{d^n}{dz^n}[(1-z)^{a+b-c}F(a,b;c;z)]$$

$$=\frac{(c-a)_n(c-b)_n}{(c)_n}(1-z)^{a+b-c-n}F(a,b;c+n;z)$$

15.2.7

$$\frac{d^n}{dz^n}[(1-z)^{a+n-1}F(a,b;c;z)]$$

$$=\frac{(-1)^n(a)_n(c-b)_n}{(c)_n}(1-z)^{a-1}F(a+n,b;c+n;z)$$

15.2.8

$$\frac{d^n}{dz^n}[z^{c-1}(1-z)^{b-c+n}F(a,b;c;z)]$$

$$=(c-n)_nz^{c-n-1}(1-z)^{b-c}F(a-n,b;c-n;z)$$

15.2.9

$$\frac{d^n}{dz^n}[z^{c-1}(1-z)^{a+b-c}F(a,b;c;z)]$$

$$=(c-n)_nz^{c-n-1}(1-z)^{a+b-c-n}F(a-n,b-n;c-n;z)$$

Gauss' Relations for Contiguous Functions

The six functions $F(a\pm 1,b;c;z)$, $F(a,b\pm 1;c;z)$, $F(a,\ b;\ c\pm 1;\ z)$ are called contiguous to $F(a,\ b;\ c;\ z)$. Relations between $F(a,\ b;\ c;\ z)$ and any two contiguous functions have been given by Gauss. By repeated application of these relations the function $F(a+m, b+n; c+l; z)$ with integral $m, n, l (c+l\neq 0, -1, -2, \ldots)$ can be expressed as a linear combination of $F(a, b; c; z)$ and one of its contiguous functions with coefficients which are rational functions of a, b, c, z.

15.2.10

$$(c-a)F(a-1,b;c;z)+(2a-c-az+bz)F(a,b;c;z)$$
$$+a(z-1)F(a+1, b; c; z)=0$$

15.2.11

$$(c-b)F(a,b-1;c;z)+(2b-c-bz+az)F(a,b;c;z)$$
$$+b(z-1)F(a, b+1; c; z)=0$$

15.2.12

$$c(c-1)(z-1)F(a, b; c-1; z)$$
$$+c[c-1-(2c-a-b-1)z]F(a, b; c; z)$$
$$+(c-a)(c-b)zF(a, b; c+1; z)=0$$

15.2.13

$$[c-2a-(b-a)z]F(a, b; c; z)$$
$$+a(1-z)F(a+1, b; c; z)$$
$$-(c-a)F(a-1, b; c; z)=0$$

15.2.14

$$(b-a)F(a, b; c; z)+aF(a+1, b; c; z)$$
$$-bF(a, b+1; c; z)=0$$

15.2.15

$$(c-a-b)F(a, b; c; z)+a(1-z)F(a+1, b; c; z)$$
$$-(c-b)F(a, b-1; c; z)=0$$

15.2.16

$$c[a-(c-b)z]F(a, b; c; z)-ac(1-z)F(a+1, b; c; z)$$
$$+(c-a)(c-b)zF(a, b; c+1; z)=0$$

15.2.17

$$(c-a-1)F(a, b; c; z)+aF(a+1, b; c; z)$$
$$-(c-1)F(a, b; c-1; z)=0$$

15.2.18

$$(c-a-b)F(a, b; c; z)-(c-a)F(a-1, b; c; z)$$
$$+b(1-z)F(a, b+1; c; z)=0$$

15.2.19

$$(b-a)(1-z)F(a, b; c; z)-(c-a)F(a-1, b; c; z)$$
$$+(c-b)F(a, b-1; c; z)=0$$

15.2.20

$$c(1-z)F(a, b; c; z)-cF(a-1, b; c; z)$$
$$+(c-b)zF(a, b; c+1; z)=0$$

15.2.21

$$[a-1-(c-b-1)z]F(a, b; c; z)$$
$$+(c-a)F(a-1, b; c; z)$$
$$-(c-1)(1-z)F(a, b; c-1; z)=0$$

15.2.22

$$[c-2b+(b-a)z]F(a, b; c; z)$$
$$+b(1-z)F(a, b+1; c; z)$$
$$-(c-b)F(a, b-1; c; z)=0$$

15.2.23

$$c[b-(c-a)z]F(a, b; c; z)-bc(1-z)F(a, b+1; c; z)$$
$$+(c-a)(c-b)zF(a, b; c+1; z)=0$$

15.2.24

$$(c-b-1)F(a, b; c; z)+bF(a, b+1; c; z)$$
$$-(c-1)F(a, b; c-1; z)=0$$

15.2.25

$$c(1-z)F(a, b; c; z)-cF(a, b-1; c; z)$$
$$*\quad +(c-a)zF(a, b; c+1; z)=0$$

15.2.26

$$[b-1-(c-a-1)z]F(a, b; c; z)$$
$$+(c-b)F(a, b-1; c; z)$$
$$-(c-1)(1-z)F(a, b; c-1; z)=0$$

15.2.27

$$c[c-1-(2c-a-b-1)z]F(a, b; c; z)$$
$$+(c-a)(c-b)zF(a, b; c+1; z)$$
$$-c(c-1)(1-z)F(a, b; c-1; z)=0$$

15.3. Integral Representations and Transformation Formulas

Integral Representations

15.3.1

$$F(a, b; c; z)=$$
$$\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt$$
$$(\mathscr{R}c>\mathscr{R}b>0)$$

The integral represents a one valued analytic function in the z-plane cut along the real axis from 1 to ∞ and hence **15.3.1** gives the analytic continuation of **15.1.1**, $F(a, b; c; z)$. Another integral representation is in the form of a Mellin-Barnes integral

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*See page II.

15.3.2 $$F(a, b; c; z)=\frac{\Gamma(c)}{2\pi i\Gamma(a)\Gamma(b)}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)}(-z)^s ds$$
$$=\tfrac{1}{2}i\,\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(1+s)\Gamma(c+s)}\csc(\pi s)(-z)^s ds$$

Here $-\pi<\arg(-z)<\pi$ and the path of integration is chosen such that the poles of $\Gamma(a+s)$ and $\Gamma(b+s)$ i.e. the points $s=-a-n$ and $s=-b-m(n,\ m=0,\ 1,\ 2,\ \ldots)$ respectively, are at its left side and the poles of $\csc(\pi s)$ or $\Gamma(-s)$ i.e. $s=0, 1, 2,$ are at its right side. The cases in which $-a$, $-b$ or $-c$ are non-negative integers or $a-b$ equal to an integer are excluded.

Linear Transformation Formulas

From **15.3.1** and **15.3.2** a number of transformation formulas for $F(a, b; c; z)$ can be derived.

15.3.3 $$F(a, b; c; z)=(1-z)^{c-a-b}F(c-a, c-b; c; z)$$

15.3.4 $$=(1-z)^{-a}F\left(a, c-b; c; \frac{z}{z-1}\right)$$

15.3.5 $$=(1-z)^{-b}F\left(b, c-a; c; \frac{z}{z-1}\right)$$

15.3.6 $$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a, b; a+b-c+1; 1-z)$$
$$+(1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a, c-b; c-a-b+1; 1-z)$$
$$(|\arg(1-z)|<\pi)$$

15.3.7 $$=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F\left(a, 1-c+a; 1-b+a; \frac{1}{z}\right)$$
$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}F\left(b, 1-c+b; 1-a+b; \frac{1}{z}\right)\qquad(|\arg(-z)|<\pi)$$

15.3.8 $$=(1-z)^{-a}\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}F\left(a, c-b; a-b+1; \frac{1}{1-z}\right)$$
$$+(1-z)^{-b}\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}F\left(b, c-a; b-a+1; \frac{1}{1-z}\right)\qquad(|\arg(1-z)|<\pi)$$

15.3.9 $$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}z^{-a}F\left(a, a-c+1; a+b-c+1; 1-\frac{1}{z}\right)$$
$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}z^{a-c}F\left(c-a, 1-a; c-a-b+1; 1-\frac{1}{z}\right)$$
$$(|\arg z|<\pi,\ |\arg(1-z)|<\pi)$$

Each term of **15.3.6** has a pole when $c=a+b\pm m$, $(m=0, 1, 2, \ldots)$; this case is covered by

15.3.10 $$F(a, b; a+b; z)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}[2\psi(n+1)-\psi(a+n)-\psi(b+n)-\ln(1-z)](1-z)^n$$
$$(|\arg(1-z)|<\pi,\ |1-z|<1)$$

Furthermore for $m=1, 2, 3, \ldots$

15.3.11 $$F(a, b; a+b+m; z)=\frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)}\sum_{n=0}^{m-1}\frac{(a)_n(b)_n}{n!(1-m)_n}(1-z)^n$$
$$-\frac{\Gamma(a+b+m)}{\Gamma(a)\Gamma(b)}(z-1)^m\sum_{n=0}^{\infty}\frac{(a+m)_n(b+m)_n}{n!(n+m)!}(1-z)^n[\ln(1-z)-\psi(n+1)$$
$$-\psi(n+m+1)+\psi(a+n+m)+\psi(b+n+m)]\qquad(|\arg(1-z)|<\pi, |1-z|<1)$$

15.3.12
$$F(a,b;a+b-m;z)=\frac{\Gamma(m)\Gamma(a+b-m)}{\Gamma(a)\Gamma(b)}(1-z)^{-m}\sum_{n=0}^{m-1}\frac{(a-m)_n(b-m)_n}{n!(1-m)_n}(1-z)^n$$
$$-\frac{(-1)^m\Gamma(a+b-m)}{\Gamma(a-m)\Gamma(b-m)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!(n+m)!}(1-z)^n[\ln(1-z)-\psi(n+1)$$
$$-\psi(n+m+1)+\psi(a+n)+\psi(b+n)]$$
$$(|\arg(1-z)|<\pi, |1-z|<1)$$

Similarly each term of **15.3.7** has a pole when $b=a\pm m$ or $b-a=\pm m$ and the case is covered by

15.3.13
$$F(a,a;c;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}(-z)^{-a}\sum_{n=0}^{\infty}\frac{(a)_n(1-c+a)_n}{(n!)^2}z^{-n}[\ln(-z)+2\psi(n+1)-\psi(a+n)-\psi(c-a-n)]$$
$$(|\arg(-z)|<\pi, |z|>1, (c-a)\neq 0, \pm 1, \pm 2, \ldots)$$

The case $b-a=m$, $(m=1, 2, 3, \ldots)$ is covered by

15.3.14
$$F(a,a+m;c;z)=F(a+m,a;c;z)$$
$$=\frac{\Gamma(c)(-z)^{-a-m}}{\Gamma(a+m)\Gamma(c-a)}\sum_{n=0}^{\infty}\frac{(a)_{n+m}(1-c+a)_{n+m}}{n!(n+m)!}z^{-n}[\ln(-z)+\psi(1+m+n)+\psi(1+n)$$
$$-\psi(a+m+n)-\psi(c-a-m-n)]+(-z)^{-a}\frac{\Gamma(c)}{\Gamma(a+m)}\sum_{n=0}^{m-1}\frac{\Gamma(m-n)(a)_n}{n!\Gamma(c-a-n)}z^{-n}$$
$$(|\arg(-z)|<\pi, |z|>1, (c-a)\neq 0, \pm 1, \pm 2, \ldots)$$

The case $c-a=0, -1, -2, \ldots$ becomes elementary, **15.3.3**, and the case $c-a=1, 2, 3, \ldots$ can be obtained from **15.3.14**, by a limiting process (see [15.2]).

Quadratic Transformation Formulas

If, and only if the numbers $\pm(1-c)$, $\pm(a-b)$, $\pm(a+b-c)$ are such, that two of them are equal or one of them is equal to $\frac{1}{2}$, then there exists a quadratic transformation. The basic formulas are due to Kummer [15.7] and a complete list is due to Goursat [15.3]. See also [15.2].

15.3.15
$$F(a,b;2b;z)=(1-z)^{-\frac{1}{2}a}F\left(\tfrac{1}{2}a, b-\tfrac{1}{2}a; b+\tfrac{1}{2}; \frac{z^2}{4z-4}\right)$$

15.3.16
$$=(1-\tfrac{1}{2}z)^{-a}F(\tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a; b+\tfrac{1}{2}; z^2(2-z)^{-2})$$

15.3.17
$$=(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-z})^{-2a}F\left[a, a-b+\tfrac{1}{2}; b+\tfrac{1}{2}; \left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2\right]$$

15.3.18
$$=(1-z)^{-\frac{1}{2}a}F\left(a, 2b-a; b+\tfrac{1}{2}; -\frac{(1-\sqrt{1-z})^2}{4\sqrt{1-z}}\right)$$

15.3.19
$$F(a,a+\tfrac{1}{2};c;z)=(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-z})^{-2a}F\left(2a, 2a-c+1; c; \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)$$

15.3.20
$$=(1\pm\sqrt{z})^{-2a}F\left(2a, c-\tfrac{1}{2}; 2c-1; \pm\frac{2\sqrt{z}}{1\pm\sqrt{z}}\right)$$

15.3.21
$$=(1-z)^{-a}F\left(2a, 2c-2a-1; c; \frac{\sqrt{1-z}-1}{2\sqrt{1-z}}\right)$$

15.3.22
$$F(a,b;a+b+\tfrac{1}{2};z)=F(2a, 2b; a+b+\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-z})$$

15.3.23
$$=(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-z})^{-2a}F\left(2a, a-b+\tfrac{1}{2}; a+b+\tfrac{1}{2}; \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}\right)$$

15.3.24 $F(a,\,b;\,a+b-\tfrac{1}{2};\,z)=(1-z)^{-\frac{1}{2}}F(2a-1,2b-1;a+b-\tfrac{1}{2};\tfrac{1}{2}-\tfrac{1}{2}\sqrt{1-z})$

15.3.25 $=(1-z)^{-\frac{1}{2}}\,(\tfrac{1}{2}+\tfrac{1}{2}\sqrt{1-z})^{1-2a}\,F\left(2a-1,\,a-b+\tfrac{1}{2};\,a+b-\tfrac{1}{2};\,\frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}\right)$

15.3.26 $F(a,\,b;\,a-b+1;\,z)=(1+z)^{-a}F\,(\tfrac{1}{2}a,\,\tfrac{1}{2}a+\tfrac{1}{2};\,a-b+1;\,4z(1+z)^{-2})$

15.3.27 $=(1\pm\sqrt{z})^{-2a}F(a,\,a-b+\tfrac{1}{2};\,2a-2b+1;\,\pm 4\sqrt{z}(1\pm\sqrt{z})^{-2})$

15.3.28 $=(1-z)^{-a}F(\tfrac{1}{2}a,\,\tfrac{1}{2}a-b+\tfrac{1}{2};\,a-b+1;-4z(1-z)^{-2})$

15.3.29 $F\,(a,\,b;\,\tfrac{1}{2}a+\tfrac{1}{2}b+\tfrac{1}{2};\,z)=(1-2z)^{-a}F\left(\tfrac{1}{2}a,\,\tfrac{1}{2}a+\tfrac{1}{2};\,\tfrac{1}{2}a+\tfrac{1}{2}b+\tfrac{1}{2};\,\frac{4z^2-4z}{(1-2z)^2}\right)$

15.3.30 $=F(\tfrac{1}{2}a,\,\tfrac{1}{2}b;\,\tfrac{1}{2}a+\tfrac{1}{2}b+\tfrac{1}{2};\,4z-4z^2)$

15.3.31 $F(a,\,1-a;\,c;\,z)=(1-z)^{c-1}F(\tfrac{1}{2}c-\tfrac{1}{2}a,\,\tfrac{1}{2}c+\tfrac{1}{2}a-\tfrac{1}{2};\,c;\,4z-4z^2)$

15.3.32 $=(1-z)^{c-1}(1-2z)^{a-c}F\,(\tfrac{1}{2}c-\tfrac{1}{2}a,\,\tfrac{1}{2}c-\tfrac{1}{2}a+\tfrac{1}{2};\,c;\,(4z^2-4z)(1-2z)^{-2})$

Cubic transformations are listed in [15.2] and [15.3].

In the formulas above, the square roots are defined so that their value is real and positive when $0\leqq z<1$. All formulas are valid in the neighborhood of $z=0$.

15.4. Special Cases of $F(a,\,b;\,c;\,z)$

Polynomials

When a or b is equal to a negative integer, then

15.4.1 $F(-m,\,b;\,c;\,z)=\sum_{n=0}^{m}\frac{(-m)_n(b)_n}{(c)_n}\frac{z^n}{n!}$

This formula is also valid when $c=-m-l;\ m,\,l=0,\,1,\,2,\,\ldots$

15.4.2 $F(-m,\,b;-m-l;\,z)=\sum_{n=0}^{m}\frac{(-m)_n(b)_n}{(-m-l)_n}\frac{z^n}{n!}$

Some particular cases are

15.4.3 $F(-n,\,n;\,\tfrac{1}{2};\,x)=T_n(1-2x)$

15.4.4 $F(-n,\,n+1;\,1;\,x)=P_n(1-2x)$

15.4.5 $F\left(-n,\,n+2\alpha;\,\alpha+\frac{1}{2};\,x\right)=\frac{n!}{(2\alpha)_n}\,C_n^{(\alpha)}(1-2x)$

15.4.6 $F(-n,\,\alpha+1+\beta+n;\,\alpha+1;\,x)=\frac{n!}{(\alpha+1)_n}\,P_n^{(\alpha,\beta)}(1-2x)$

Here T_n, P_n, $C_n^{(\alpha)}$, $P_n^{(\alpha,\beta)}$ denote Chebyshev, Legendre's, Gegenbauer's and Jacobi's polynomials respectively (see chapter **22**).

Legendre Functions

Legendre functions are connected with those special cases of the hypergeometric function for which a quadratic transformation exists (see **15.3**).

15.4.7 $F(a,\,b;2b;z)=2^{2b-1}\Gamma(\tfrac{1}{2}+b)z^{\frac{1}{2}-b}\,(1-z)^{\frac{1}{2}(b-a-\frac{1}{2})}P^{\frac{1}{2}-b}_{a-b-\frac{1}{2}}\left[\left(1-\frac{z}{2}\right)(1-z)^{-\frac{1}{2}}\right]$

15.4.8 $=2^{2b}\pi^{-\frac{1}{2}}\,\frac{\Gamma(\frac{1}{2}+b)}{\Gamma(2b-a)}\,z^{-b}(1-z)^{\frac{1}{2}(b-a)}\,e^{i\pi(a-b)}\,Q^{b-a}_{b-1}\left(\frac{2}{z}-1\right)^*$

15.4.9 $F(a,b;2b;-z)=2^{2b}\pi^{-\frac{1}{2}}\,\frac{\Gamma(\frac{1}{2}+b)}{\Gamma(a)}\,z^{-b}(1+z)^{\frac{1}{2}(b-a)}e^{-i\pi(a-b)}\,Q^{a-b}_{b-1}\left(1+\frac{2}{z}\right)(|\arg z|<\pi,\,|\arg(1\pm z)|<\pi)^*$

15.4.10 $F(a,a+\frac{1}{2};c;z)=2^{c-1}\Gamma(c)z^{\frac{1}{2}-\frac{1}{2}c}(1-z)^{\frac{1}{2}c-a-\frac{1}{2}}P^{1-c}_{2a-c}[(1-z)^{-\frac{1}{2}}]$

($|\arg z|<\pi$, $|\arg (1-z)|<\pi$, z not between 0 and $-\infty$)

15.4.11 $F(a,a+\frac{1}{2};c;x)=2^{c-1}\Gamma(c)(-x)^{\frac{1}{2}-\frac{1}{2}c}(1-x)^{\frac{1}{2}c-a-\frac{1}{2}}P^{1-c}_{2a-c}[(1-x)^{-\frac{1}{2}}]$ $(-\infty<x<0)$

15.4.12 $F(a,b;a+b+\frac{1}{2};z)=2^{a+b-\frac{1}{2}}\Gamma(\frac{1}{2}+a+b)(-z)^{\frac{1}{2}(\frac{1}{2}-a-b)}P^{\frac{1}{2}-a-b}_{a-b-\frac{1}{2}}[(1-z)^{\frac{1}{2}}]$

($|\arg (-z)|<\pi$, z not between 0 and 1)

15.4.13 $F(a,b;a+b+\frac{1}{2};x)=2^{a+b-\frac{1}{2}}\Gamma(\frac{1}{2}+a+b)x^{\frac{1}{2}(\frac{1}{2}-a-b)}P^{\frac{1}{2}-a-b}_{a-b-\frac{1}{2}}[(1-x)^{\frac{1}{2}}]$ $(0<x<1)$

15.4.14 $F(a,b;a-b+1;z)=\Gamma(a-b+1)z^{\frac{1}{2}b-\frac{1}{2}a}(1-z)^{-b}P^{b-a}_{-b}\left(\frac{1+z}{1-z}\right)$

($|\arg (1-z)|<\pi$, z not between 0 and $-\infty$)

15.4.15 $F(a,b;a-b+1;x)=\Gamma(a-b+1)(1-x)^{-b}(-x)^{\frac{1}{2}b-\frac{1}{2}a}P^{b-a}_{-b}\left(\frac{1+x}{1-x}\right)$ $(-\infty<x<0)$

15.4.16 $F(a,1-a;c;z)=\Gamma(c)(-z)^{\frac{1}{2}-\frac{1}{2}c}(1-z)^{\frac{1}{2}c-\frac{1}{2}}P^{1-c}_{-a}(1-2z)$

($|\arg (-z)|<\pi$, $|\arg (1-z)|<\pi$, z not between 0 and 1)

15.4.17 $F(a,1-a;c;x)=\Gamma(c)x^{\frac{1}{2}-\frac{1}{2}c}(1-x)^{\frac{1}{2}c-\frac{1}{2}}P^{1-c}_{-a}(1-2x)$ $(0<x<1)$

15.4.18 $F(a,b;\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2};z)=\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)[z(z-1)]^{\frac{1}{4}(1-a-b)}P^{\frac{1}{2}(1-a-b)}_{\frac{1}{2}(a-b-1)}(1-2z)$

($|\arg z|<\pi$, $|\arg (z-1)|<\pi$, z not between 0 and 1)

15.4.19 $F(a,b;\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2};x)=\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)(x-x^2)^{\frac{1}{4}(1-a-b)}P^{\frac{1}{2}(1-a-b)}_{\frac{1}{2}(a-b-1)}(1-2x)$ $(0<x<1)$

15.4.20 $F(a,b;a+b-\frac{1}{2};z)=2^{a+b-\frac{3}{2}}\Gamma(a+b-\frac{1}{2})(-z)^{\frac{1}{2}(\frac{3}{2}-a-b)}(1-z)^{-\frac{1}{2}}P^{\frac{3}{2}-a-b}_{b-a-\frac{1}{2}}[(1-z)^{\frac{1}{2}}]$

($|\arg (-z)|<\pi$, $|\arg (1-z)|<\pi$, $\mathscr{R}[(1-z)^{\frac{1}{2}}]>0$, z not between 0 and 1)

15.4.21 $F(a,b;a+b-\frac{1}{2};x)=2^{a+b-\frac{3}{2}}\Gamma(a+b-\frac{1}{2})x^{\frac{1}{2}(\frac{3}{2}-a-b)}(1-x)^{-\frac{1}{2}}P^{\frac{3}{2}-a-b}_{b-a-\frac{1}{2}}[(1-x)^{\frac{1}{2}}]$ $(0<x<1)$

15.4.22 $F(a,b;\frac{1}{2};z)=\pi^{-\frac{1}{2}}2^{a+b-\frac{3}{2}}\Gamma(\frac{1}{2}+a)\Gamma(\frac{1}{2}+b)(z-1)^{\frac{1}{2}(\frac{1}{2}-a-b)}[P^{\frac{1}{2}-a-b}_{a-b-\frac{1}{2}}(z^{\frac{1}{2}})+P^{\frac{1}{2}-a-b}_{a-b-\frac{1}{2}}(-z^{\frac{1}{2}})]$

($|\arg z|<\pi$, $|\arg(z-1)|<\pi$, z not between 0 and 1)

15.4.23 $F(a,b;\frac{1}{2};x)=\pi^{-\frac{1}{2}}2^{a+b-\frac{3}{2}}\Gamma(\frac{1}{2}+a)\Gamma(\frac{1}{2}+b)(1-x)^{\frac{1}{2}(\frac{1}{2}-a-b)}[P^{\frac{1}{2}-a-b}_{a-b-\frac{1}{2}}(x^{\frac{1}{2}})+P^{\frac{1}{2}-a-b}_{a-b-\frac{1}{2}}(-x^{\frac{1}{2}})]$ $(0<x<1)$

15.4.24 $F(a,b;\frac{1}{2};-z)=\pi^{-\frac{1}{2}}2^{a-b-1}\Gamma(\frac{1}{2}+a)\Gamma(1-b)(z+1)^{-\frac{1}{2}a-\frac{1}{2}b}e^{\pm i\frac{\pi}{2}(b-a)}\{P^{b-a}_{a+b-1}[z^{\frac{1}{2}}(1+z)^{-\frac{1}{2}}]$

$+P^{b-a}_{a+b-1}[-z^{\frac{1}{2}}(1+z)^{-\frac{1}{2}}]\}$

($\pm$ according as $\mathscr{I}z\gtrless 0$, z not between 0 and ∞)

15.4.25 $F(a,b;\frac{1}{2};-x)=\pi^{-\frac{1}{2}}2^{a-b-1}\Gamma(\frac{1}{2}+a)\Gamma(1-b)(1+x)^{-\frac{1}{2}a-\frac{1}{2}b}\{P^{b-a}_{a+b-1}[x^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}]+P^{b-a}_{a+b-1}[-x^{1/2}(1+x)^{-\frac{1}{2}}]\}$

$(0<x<\infty)$

15.4.26 $F(a,b;\frac{3}{2};x)=-\pi^{-\frac{1}{2}}2^{a+b-\frac{5}{2}}\Gamma(a-\frac{1}{2})\Gamma(b-\frac{1}{2})x^{-\frac{1}{2}}(1-x)^{\frac{1}{2}(\frac{3}{2}-a-b)}\{P^{\frac{3}{2}-a-b}_{a-b-\frac{1}{2}}(x^{\frac{1}{2}})-P^{\frac{3}{2}-a-b}_{a-b-\frac{1}{2}}(-x^{\frac{1}{2}})\}$ $(0<x<1)$

15.5. The Hypergeometric Differential Equation

The hypergeometric differential equation

15.5.1 $$z(1-z)\frac{d^2w}{dz^2}+[c-(a+b+1)z]\frac{dw}{dz}-abw=0$$

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*See page II.

has three (regular) singular points $z=0, 1, \infty$. The pairs of exponents at these points are

15.5.2 $$\rho^{(0)}_{1,2}=0, 1-c, \qquad \rho^{(1)}_{1,2}=0, c-a-b, \qquad \rho^{(\infty)}_{1,2}=a, b$$

respectively. The general theory of differential equations of the Fuchsian type distinguishes between the following cases.

A. None of the numbers c, $c-a-b$; $a-b$ is equal to an integer. Then two linearly independent solutions of **15.5.1** in the neighborhood of the singular points 0, 1, ∞ are respectively

15.5.3 $$w_{1(0)}=F(a, b; c; z)=(1-z)^{c-a-b}F(c-a, c-b; c; z)$$

15.5.4 $$w_{2(0)}=z^{1-c}F(a-c+1, b-c+1; 2-c; z)=z^{1-c}(1-z)^{c-a-b}F(1-a, 1-b; 2-c; z)$$

15.5.5 $$w_{1(1)}=F(a, b; a+b+1-c; 1-z)=z^{1-c}F(1+b-c, 1+a-c; a+b+1-c; 1-z)$$

15.5.6 $$w_{2(1)}=(1-z)^{c-a-b}F(c-b, c-a; c-a-b+1; 1-z)=z^{1-c}(1-z)^{c-a-b}F(1-a, 1-b; c-a-b+1; 1-z)$$

15.5.7 $$w_{1(\infty)}=z^{-a}F(a, a-c+1; a-b+1; z^{-1})=z^{b-c}(z-1)^{c-a-b}F(1-b, c-b; a-b+1; z^{-1})$$

15.5.8 $$w_{2(\infty)}=z^{-b}F(b, b-c+1; b-a+1; z^{-1})=z^{a-c}(z-1)^{c-a-b}F(1-a, c-a; b-a+1; z^{-1})$$

The second set of the above expressions is obtained by applying **15.3.3** to the first set.

Another set of representations is obtained by applying **15.3.4** to **15.5.3** through **15.5.8.** This gives **15.5.9–15.5.14.**

15.5.9 $$w_{1(0)}=(1-z)^{-a}F\left(a, c-b; c; \frac{z}{z-1}\right)=(1-z)^{-b}F\left(b, c-a; c; \frac{z}{z-1}\right)$$

15.5.10 $$w_{2(0)}=z^{1-c}(1-z)^{c-a-1}F\left(a-c+1, 1-b; 2-c; \frac{z}{z-1}\right)=z^{1-c}(1-z)^{c-b-1}F\left(b-c+1, 1-a; 2-c; \frac{z}{z-1}\right)$$

15.5.11 $$w_{1(1)}=z^{-a}F(a, a-c+1; a+b-c+1; 1-z^{-1})=z^{-b}F(b, b-c+1; a+b-c+1; 1-z^{-1})$$

15.5.12

$$w_{2(1)}=z^{a-c}(1-z)^{c-a-b}F(c-a, 1-a; c-a-b+1; 1-z^{-1})=z^{b-c}(1-z)^{c-a-b}F(c-b, 1-b; c-a-b+1; 1-z^{-1})$$

15.5.13 $$w_{1(\infty)}=(z-1)^{-a}F\left(a, c-b; a-b+1; \frac{1}{1-z}\right)=(z-1)^{-b}F\left(b, c-a; b-a+1; \frac{1}{1-z}\right)$$

15.5.14

$$w_{2(\infty)}=z^{1-c}(z-1)^{c-a-1}F\left(a-c+1, 1-b; a-b+1; \frac{1}{1-z}\right)=z^{1-c}(z-1)^{c-b-1}F\left(b-c+1, 1-a; b-a+1; \frac{1}{1-z}\right)$$

15.5.3 to **15.5.14** constitute Kummer's 24 solutions of the hypergeometric equation. The analytic continuation of $w_{1,2(0)}(z)$ can then be obtained by means of **15.3.3** to **15.3.9.**

B. One of the numbers a, b, $c-a$, $c-b$ is an integer. Then one of the hypergeometric series for instance $w_{1,2(0)}$, **15.5.3, 15.5.4** terminates and the corresponding solution is of the form

15.5.15 $$w=z^{\alpha}(1-z)^{\beta}p_n(z)$$

where $p_n(z)$ is a polynomial in z of degree n. This case is referred to as the degenerate case of the hypergeometric differential equation and its solutions are listed and discussed in great detail in [15.2].

C. The number $c-a-b$ is an integer, c nonintegral. Then **15.3.10** to **15.3.12** give the analytic continuation of $w_{1,2(0)}$ into the neighborhood of $z=1$. Similarly **15.3.13** and **15.3.14** give the analytic continuation of $w_{1,2(0)}$ into the neighborhood of $z=\infty$ in case $a-b$ is an integer but not c, subject of course to the further restrictions $c-a=0, \pm 1, \pm 2 \ldots$ (For a detailed discussion of all possible cases, see [15.2]).

D. The number $c=1$. Then **15.5.3, 15.5.4** are replaced by

15.5.16 $$w_{1(0)}=F(a, b; 1; z)$$

15.5.17 $$w_{2(0)}=F(a,b;1;z)\ln z+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(n!)^2}z^n[\psi(a+n)-\psi(a)+\psi(b+n)-\psi(b)-2\psi(n+1)+2\psi(1)] \qquad (|z|<1)$$

E. The number $c=m+1$, $m=1, 2, 3, \ldots$. A fundamental system is

15.5.18 $$w_{1(0)}=F(a,b;m+1;z)$$

15.5.19 $$w_{2(0)}=F(a,b;m+1;z)\ln z+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(1+m)_n n!}z^n[\psi(a+n)-\psi(a)+\psi(b+n)-\psi(b)-\psi(m+1+n)$$
$$+\psi(m+1)-\psi(n+1)+\psi(1)]-\sum_{n=1}^{m}\frac{(n-1)!(-m)_n}{(1-a)_n(1-b)_n}z^{-n} \qquad (|z|<1 \text{ and } a, b\neq 0, 1, 2, \ldots (m-1))$$

F. The number $c=1-m$, $m=1, 2, 3, \ldots$. A fundamental system is

15.5.20 $$w_{1(0)}=z^mF(a+m, b+m; 1+m; z)$$

15.5.21
$$w_{2(0)}=z^mF(a+m, b+m; 1+m; z)\ln z+z^m\sum_{n=1}^{\infty}z^n\frac{(a+m)_n(b+m)_n}{(1+m)_n n!}[\psi(a+m+n)-\psi(a+m)+\psi(b+m+n)$$
$$-\psi(b+m)-\psi(m+1+n)+\psi(m+1)-\psi(n+1)+\psi(1)]-\sum_{n=1}^{m}\frac{(n-1)!(-m)_n}{(1-a-m)_n(1-b-m)_n}z^{m-n}$$
$$(|z|<1 \text{ and } a, b\neq 0, -1, -2, \ldots -(m-1))$$

15.6. Riemann's Differential Equation

The hypergeometric differential equation **15.5.1** with the (regular) singular points 0, 1, ∞ is a special case of Riemann's differential equation with three (regular) singular points a, b, c

15.6.1

$$\frac{d^2w}{dz^2}+\left[\frac{1-\alpha-\alpha'}{z-a}+\frac{1-\beta-\beta'}{z-b}+\frac{1-\gamma-\gamma'}{z-c}\right]\frac{dw}{dz}$$
$$+\left[\frac{\alpha\alpha'(a-b)(a-c)}{z-a}+\frac{\beta\beta'(b-c)(b-a)}{z-b}\right.$$
$$\left.+\frac{\gamma\gamma'(c-a)(c-b)}{z-c}\right]\frac{w}{(z-a)(z-b)(z-c)}=0$$

The pairs of the exponents with respect to the singular points a; b; c are α, α'; β, β'; γ, γ' respectively subject to the condition

15.6.2 $$\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$$

The complete set of solutions of **15.6.1** is denoted by the symbol

15.6.3 $$w=P\left\{\begin{matrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$

Special Cases of Riemann's *P* Function

(a) The generalized hypergeometric function

15.6.4 $$w=P\left\{\begin{matrix} 0 & \infty & 1 & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{matrix}\right\}$$

(b) The hypergeometric function $F(a, b; c; z)$

15.6.5 $$w=P\left\{\begin{matrix} 0 & \infty & 1 & \\ 0 & a & 0 & z \\ 1-c & b & c-a-b & \end{matrix}\right\}$$

(c) The Legendre functions $P^\mu_\nu(z)$, $Q^\mu_\nu(z)$

15.6.6 $$w=P\left\{\begin{matrix} 0 & \infty & 1 & \\ -\frac{1}{2}\nu & \frac{1}{2}\mu & 0 & (1-z^2)^{-1} \\ \frac{1}{2}+\frac{1}{2}\nu & -\frac{1}{2}\mu & \frac{1}{2} & \end{matrix}\right\}$$

(d) The confluent hypergeometric function

15.6.7 $$w=P\left\{\begin{matrix} 0 & \infty & c & \\ \frac{1}{2}+u & -c & c-k & z \\ \frac{1}{2}-u & 0 & k & \end{matrix}\right\}$$

provided lim $c\rightarrow\infty$.

Transformation Formulas for Riemann's *P* Function

15.6.8
$$\left(\frac{z-a}{z-b}\right)^k\left(\frac{z-c}{z-b}\right)^l P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}=P\begin{Bmatrix} a & b & c & \\ \alpha+k & \beta-k-l & \gamma+l & z \\ \alpha'+k & \beta'-k-l & \gamma'+l & \end{Bmatrix}$$

15.6.9
$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}=P\begin{Bmatrix} a_1 & b_1 & c_1 & \\ \alpha & \beta & \gamma & z_1 \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

where

15.6.10
$$z=\frac{Az_1+B}{Cz_1+D},\ a=\frac{Aa_1+B}{Ca_1+D},\ b=\frac{Ab_1+B}{Cb_1+D},\ c=\frac{Ac_1+B}{Cc_1+D}$$

and A, B, C, D are arbitrary constants such that $AD-BC\neq 0$.

Riemann's P function reduced to the hypergeometric function is

15.6.11
$$P\begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & . \end{Bmatrix}=\left(\frac{z-a}{z-b}\right)^\alpha\left(\frac{z-c}{z-b}\right)^\gamma P\begin{Bmatrix} 0 & \infty & 1 & \\ 0 & \alpha+\beta+\gamma & 0 & \frac{(z-a)(c-b)}{(z-b)(c-a)} \\ \alpha'-\alpha & \alpha+\beta'+\gamma & \gamma'-\gamma & \end{Bmatrix}$$

The P function on the right hand side is Gauss' hypergeometric function (see **15.6.5**). If it is replaced by Kummer's 24 solutions **15.5.3** to **15.5.14** the complete set of 24 solutions for Riemann's differential equation **15.6.1** is obtained. The first of these solutions is for instance by **15.5.3** and **15.6.5**

15.6.12
$$w=\left(\frac{z-a}{z-b}\right)^\alpha\left(\frac{z-c}{z-b}\right)^\gamma F\left[\alpha+\beta+\gamma,\alpha+\beta'+\gamma;1+\alpha-\alpha';\frac{(z-a)\,(c-b)}{(z-b)\,(c-a)}\right]$$

15.7. Asymptotic Expansions

The behavior of $F(a,\ b;\ c;\ z)$ for large $|z|$ is described by the transformation formulas of **15.3.**

For fixed a, b, z and large $|c|$ one has [15.8]

15.7.1
$$F(a,b;c;z)=\sum_{n=0}^{m}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}+O(|c|^{-m-1})$$

For fixed a, c, z, ($c\neq 0, -1, -2, \ldots, 0<|z|<1$) and large $|b|$ one has [15.2]

15.7.2
$$\begin{gathered}F(a,b;c;z)=e^{-i\pi a}[\Gamma(c)/\Gamma(c-a)]\,(bz)^{-a}[1+O(|bz|^{-1})]\\ +[\Gamma(c)/\Gamma(a)]\,e^{bz}(bz)^{a-c}[1+O(|bz|^{-1})]\\ \left(-\frac{3\pi}{2}<\arg(bz)<\tfrac{1}{2}\pi\right)\end{gathered}$$

15.7.3
$$\begin{gathered}F(a,b;c;z)=e^{i\pi a}[\Gamma(c)/\Gamma(c-a)]\,(bz)^{-a}[1+O(|bz|^{-1})]\\ +[\Gamma(c)/\Gamma(a)]e^{bz}(bz)^{a-c}[1+O(|bz|^{-1})]\\ (-\tfrac{1}{2}\pi<\arg(bz)<\tfrac{3}{2}\pi)\end{gathered}$$

For the case when more than one of the parameters are large consult [15.2].

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16. Jacobian Elliptic Functions and Theta Functions

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Contents

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The author wishes to acknowledge his great indebtedness to his friend, the late Professor E. H. Neville, for invaluable assistance in reading and criticizing the manuscript. Professor Neville generously supplied material from his own work and was responsible for many improvements in matter and arrangement.

The author's best thanks are also due to David S. Liepman and Ruth Zucker for the preparation and checking of the tables and graphs.

16. Jacobian Elliptic Functions and Theta Functions

Mathematical Properties

Jacobian Elliptic Functions

16.1. Introduction

A doubly periodic meromorphic function is called an *elliptic function.*

Let m, m_1 be numbers such that

$$m+m_1=1.$$

We call m the *parameter*, m_1 the *complementary parameter*.

In what follows we shall assume that the parameter m is a real number. Without loss of generality we can then suppose that $0 \leq m \leq 1$ (see **16.10, 16.11**).

We define *quarter-periods* K and iK' by

16.1.1

$$K(m)=K=\int_0^{\pi/2} \frac{d\theta}{(1-m \sin^2 \theta)^{1/2}},$$

$$iK'(m)=iK'=i\int_0^{\pi/2} \frac{d\theta}{(1-m_1 \sin^2 \theta)^{1/2}}$$

so that K and K' are real numbers. K is called the real, iK' the imaginary quarter-period.

We note that

16.1.2 $\quad K(m)=K'(m_1)=K'(1-m).$

We also note that if any *one* of the numbers m, m_1, $K(m)$, $K'(m)$, $K'(m)/K(m)$ is given, all the rest are determined. Thus K and K' can not both be chosen arbitrarily.

In the Argand diagram denote the points 0, K, $K+iK'$, iK' by s, c, d, n respectively. These points are at the vertices of a rectangle. The translations of this rectangle by λK, $\mu iK'$, where λ, μ are given all integral values positive or negative, will lead to the lattice

.s	.c	.s	.c
.n	.d	.n	.d
.s	.c	.s	.c
.n	.d	.n	.d

the pattern being repeated indefinitely on all sides.

Let p, q be any two of the letters s, c, d, n. Then p, q determine in the lattice a minimum rectangle whose sides are of length K and K' and whose vertices s, c, d, n are in counterclockwise order.

Definition

The Jacobian elliptic function pq u is defined by the following three properties.

(i) pq u has a simple zero at p and a simple pole at q.

(ii) The step from p to q is a half-period of pq u. Those of the numbers K, iK', $K+iK'$ which differ from this step are only quarter-periods.

(iii) The coefficient of the leading term in the expansion of pq u in ascending powers of u about $u=0$ is unity. With regard to (iii) the leading term is u, $1/u$, 1 according as $u=0$ is a zero, a pole, or an ordinary point.

Thus the functions with a pole or zero at the origin (i.e., the functions in which one letter is s) are odd, and the others are even.

Should we wish to call explicit attention to the value of the parameter, we write pq $(u|m)$ instead of pq u.

The Jacobian elliptic functions can also be defined with respect to certain integrals. Thus if

16.1.3 $$u=\int_0^{\varphi} \frac{d\theta}{(1-m \sin^2 \theta)^{1/2}},$$

the angle φ is called the *amplitude*

16.1.4 $\quad \varphi=\text{am } u$

and we define

16.1.5

$$\text{sn } u=\sin \varphi, \text{ cn } u=\cos \varphi,$$

$$\text{dn } u=(1-m \sin^2 \varphi)^{1/2}=\Delta(\varphi).$$

Similarly all the functions pq u can be expressed in terms of φ. This second set of definitions, although seemingly different, is mathematically equivalent to the definition previously given in terms of a lattice. For further explanation of notations, including the interpretation, of such expressions as sn $(\varphi\backslash\alpha)$, cn $(u|m)$, dn (u, k), see **17.2.**

16.2. Classification of the Twelve Jacobian Elliptic Functions

According to Poles and Half-Periods

	Pole iK'	Pole $K+iK'$	Pole K	Pole 0	
Half period iK'	sn u	cd u	dc u	ns u	Periods $2iK'$, $4K+4iK'$, $4K$
Half period $K+iK'$	cn u	sd u	nc u	ds u	Periods $4iK'$, $2K+2iK'$, $4K$
Half period K	dn u	nd u	sc u	cs u	Periods $4iK'$, $4K+4iK'$, $2K$

The three functions in a vertical column are *copolar*.

The four functions in a horizontal line are *coperiodic*. Of the periods quoted in the last line of each row only two are independent.

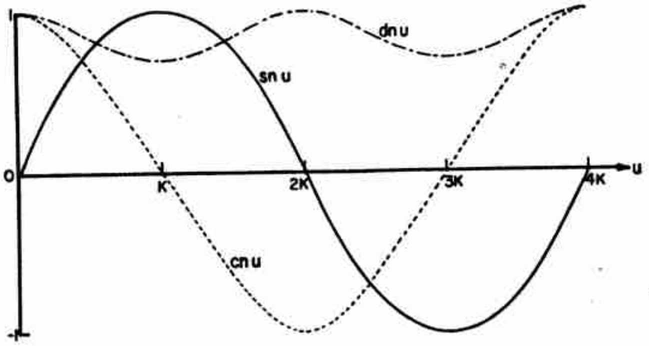

FIGURE 16.1. *Jacobian elliptic functions*

sn u, cn u, dn u

$$m=\frac{1}{2}$$

The curve for cn $(u|\frac{1}{2})$ is the boundary between those which have an inflexion and those which have not.

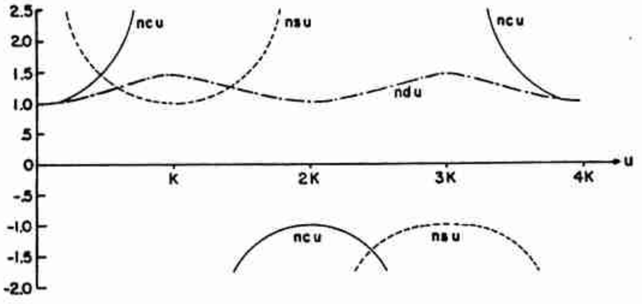

FIGURE 16.2. *Jacobian elliptic functions*

ns u, nc u, nd u

$$m=\frac{1}{2}$$

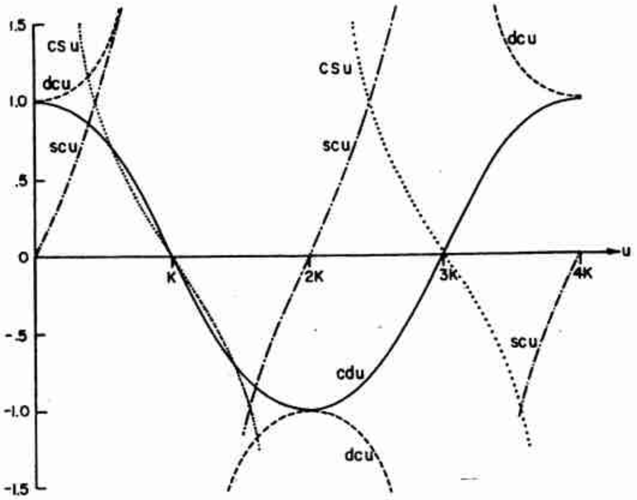

FIGURE 16.3. *Jacobian elliptic functions*

sc u, cs u, cd u, dc u

$$m=\frac{1}{2}$$

16.3. Relation of the Jacobian Functions to the Copolar Trio sn u, cn u, dn u

16.3.1 $\mathrm{cd}\,u=\dfrac{\mathrm{cn}\,u}{\mathrm{dn}\,u}$ $\qquad \mathrm{dc}\,u=\dfrac{\mathrm{dn}\,u}{\mathrm{cn}\,u}$ $\qquad \mathrm{ns}\,u=\dfrac{1}{\mathrm{sn}\,u}$

16.3.2 $\mathrm{sd}\,u=\dfrac{\mathrm{sn}\,u}{\mathrm{dn}\,u}$ $\qquad \mathrm{nc}\,u=\dfrac{1}{\mathrm{cn}\,u}$ $\qquad \mathrm{ds}\,u=\dfrac{\mathrm{dn}\,u}{\mathrm{sn}\,u}$

16.3.3 $\mathrm{nd}\,u=\dfrac{1}{\mathrm{dn}\,u}$ $\qquad \mathrm{sc}\,u=\dfrac{\mathrm{sn}\,u}{\mathrm{cn}\,u}$ $\qquad \mathrm{cs}\,u=\dfrac{\mathrm{cn}\,u}{\mathrm{sn}\,u}$

And generally if p, q, r are any three of the letters s, c, d, n,

16.3.4 $$\mathrm{pq}\,u=\frac{\mathrm{pr}\,u}{\mathrm{qr}\,u}$$

provided that when two letters are the same, e.g., pp u, the corresponding function is put equal to unity.

16.4. Calculation of the Jacobian Functions by Use of the Arithmetic-Geometric Mean (A.G.M.)

For the A.G.M. scale see **17.6.**

To calculate sn $(u|m)$, cn $(u|m)$, and dn $(u|m)$ form the A.G.M. scale starting with

16.4.1 $$a_0=1, b_0=\sqrt{m_1}, c_0=\sqrt{m},$$

terminating at the step N when c_N is negligible to the accuracy required. Find φ_N in degrees where

16.4.2 $$\varphi_N=2^N a_N u \frac{180°}{\pi}$$

and then compute successively φ_{N-1}, φ_{N-2}, . . ., φ_1, φ_0 from the recurrence relation

16.4.3 $$\sin(2\varphi_{n-1}-\varphi_n)=\frac{c_n}{a_n}\sin\varphi_n.$$

Then

16.4.4

$$\text{sn}(u|m)=\sin\varphi_0, \quad \text{cn}(u|m)=\cos\varphi_0$$

$$\text{dn}(u|m)=\frac{\cos\varphi_0}{\cos(\varphi_1-\varphi_0)}.$$

From these all the other functions can be determined.

16.5. Special Arguments

	u	sn u	cn u	dn u
16.5.1	0	0	1	1
16.5.2	$\frac{1}{2}K$	$\frac{1}{(1+m_1^{1/2})^{1/2}}$	$\frac{m_1^{1/4}}{(1+m_1^{1/2})^{1/2}}$	$m_1^{1/4}$
16.5.3	K	1	0	$m_1^{1/2}$
16.5.4	$\frac{1}{2}(iK')$	$im^{-1/4}$	$\frac{(1+m^{1/2})^{1/2}}{m^{1/4}}$	$(1+m^{1/2})^{1/2}$
* **16.5.5**	$\frac{1}{2}(K+iK')$	$2^{-1/2}m^{-1/4}[(1+m^{1/2})^{1/2}+i(1-m^{1/2})^{1/2}]$	$\left(\frac{m_1}{4m}\right)^{1/4}(1-i)$	$\left(\frac{m_1}{4}\right)^{1/4}[(1+m_1^{1/2})^{1/2}-i(1-m_1^{1/2})^{1/2}]$
16.5.6	$K+\frac{1}{2}(iK')$	$m^{-1/4}$	$-i\left(\frac{1-m^{1/2}}{m^{1/2}}\right)^{1/2}$	$(1-m^{1/2})^{1/2}$
16.5.7	iK'	∞	∞	∞
16.5.8	$\frac{1}{2}K+iK'$	$(1-m_1^{1/2})^{-1/2}$	$-i\left(\frac{m_1^{1/2}}{1-m_1^{1/2}}\right)^{1/2}$	$-im_1^{1/4}$
16.5.9	$K+iK'$	$m^{-1/2}$	$-i(m_1/m)^{1/2}$	0

16.6. Jacobian Functions when $m=0$ or 1

		$m=0$	$m=1$
16.6.1	sn $(u\|m)$	sin u	tanh u
16.6.2	cn $(u\|m)$	cos u	sech u
16.6.3	dn $(u\|m)$	1	sech u
16.6.4	cd $(u\|m)$	cos u	1
16.6.5	sd $(u\|m)$	sin u	sinh u
16.6.6	nd $(u\|m)$	1	cosh u
16.6.7	dc $(u\|m)$	sec u	1
16.6.8	nc $(u\|m)$	sec u	cosh u
16.6.9	sc $(u\|m)$	tan u	sinh u
16.6.10	ns $(u\|m)$	csc u	coth u
16.6.11	ds $(u\|m)$	csc u	csch u
16.6.12	cs $(u\|m)$	cot u	csch u
16.6.13	am $(u\|m)$	u	gd u

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*See page II

16.7. Principal Terms

When the elliptic function pq u is expanded in ascending powers of $(u-K_r)$, where K_r is one of $0, K, iK', K+iK'$, the first term of the expansion is called the principal term and has one of the forms A, $B\times(u-K_r)$, $C\div(u-K_r)$ according as K_r is an ordinary point, a zero, or a pole of pq u. The following list gives these forms, where $\times$ means that the factor $(u-K_r)$ has to be supplied and $\div$ means that the divisor $(u-K_r)$ has to be supplied.

	$K_r=$	0	K	iK'	$K+iK'$
16.7.1	$\mathrm{sn}\,u$	$1\times$	1	$m^{-1/2}\div$	$m^{-1/2}$
16.7.2	$\mathrm{cn}\,u$	1	$-m_1^{1/2}\times$	$-im^{-1/2}\div$	$-i\left(\frac{m_1}{m}\right)^{1/2}$
16.7.3	$\mathrm{dn}\,u$	1	$m_1^{1/2}$	$-i\div$	$im_1^{1/2}\times$
16.7.4	$\mathrm{cd}\,u$	1	$-1\times$	$m^{-1/2}$	$-m^{-1/2}\div$
16.7.5	$\mathrm{sd}\,u$	$1\times$	$m_1^{-1/2}$	$im^{-1/2}$	$-i\frac{1}{(mm_1)^{1/2}}\div$
16.7.6	$\mathrm{nd}\,u$	1	$m_1^{-1/2}$	$i\times$	$-im_1^{-1/2}\div$
16.7.7	$\mathrm{dc}\,u$	1	$-1\div$	$m^{1/2}$	$-m^{1/2}\times$
16.7.8	$\mathrm{nc}\,u$	1	$-m_1^{-1/2}\div$	$im^{1/2}\times$	$i\left(\frac{m}{m_1}\right)^{1/2}$
16.7.9	$\mathrm{sc}\,u$	$1\times$	$-m_1^{-1/2}\div$	i	$im_1^{-1/2}$
16.7.10	$\mathrm{ns}\,u$	$1\div$	1	$m^{1/2}\times$	$m^{1/2}$
16.7.11	$\mathrm{ds}\,u$	$1\div$	$m_1^{1/2}$	$-im^{1/2}$	$i(mm_1)^{1/2}\times$
16.7.12	$\mathrm{cs}\,u$	$1\div$	$-m_1^{1/2}\times$	$-i$	$-im_1^{1/2}$

16.8. Change of Argument

		u	$-u$	$u+K$	$u-K$	$K-u$	$u+2K$	$u-2K$	$2K-u$	$u+iK'$	$u+2iK'$	$u+K+iK'$	$u+2K$ $+2iK'$
16.8.1	sn	$\mathrm{sn}\,u$	$-\mathrm{sn}\,u$	$\mathrm{cd}\,u$	$-\mathrm{cd}\,u$	$\mathrm{cd}\,u$	$-\mathrm{sn}\,u$	$-\mathrm{sn}\,u$	$\mathrm{sn}\,u$	$m^{-1/2}\mathrm{ns}\,u$	$\mathrm{sn}\,u$	$m^{-1/2}\mathrm{dc}\,u$	$-\mathrm{sn}\,u$
16.8.2	cn	$\mathrm{cn}\,u$	$\mathrm{cn}\,u$	$-m_1^{1/2}\mathrm{sd}\,u$	$m_1^{1/2}\mathrm{sd}\,u$	$m_1^{1/2}\mathrm{sd}\,u$	$-\mathrm{cn}\,u$	$-\mathrm{cn}\,u$	$-\mathrm{cn}\,u$	$-im^{-1/2}\mathrm{ds}\,u$	$-\mathrm{cn}\,u$	$-im_1^{1/2}m^{-1/2}\mathrm{nc}\,u$	$\mathrm{cn}\,u$
16.8.3	dn	$\mathrm{dn}\,u$	$\mathrm{dn}\,u$	$m_1^{1/2}\mathrm{nd}\,u$	$m_1^{1/2}\mathrm{nd}\,u$	$m_1^{1/2}\mathrm{nd}\,u$	$\mathrm{dn}\,u$	$\mathrm{dn}\,u$	$\mathrm{dn}\,u$	$-i\mathrm{cs}\,u$	$-\mathrm{dn}\,u$	$im_1^{1/2}\mathrm{sc}\,u$	$-\mathrm{dn}\,u$
16.8.4	cd	$\mathrm{cd}\,u$	$\mathrm{cd}\,u$	$-\mathrm{sn}\,u$	$\mathrm{sn}\,u$	$\mathrm{sn}\,u$	$-\mathrm{cd}\,u$	$-\mathrm{cd}\,u$	$-\mathrm{cd}\,u$	$m^{-1/2}\mathrm{dc}\,u$	$\mathrm{cd}\,u$	$-m^{-1/2}\mathrm{ns}\,u$	$-\mathrm{cd}\,u$
16.8.5	sd	$\mathrm{sd}\,u$	$-\mathrm{sd}\,u$	$m_1^{-1/2}\mathrm{cn}\,u$	$-m_1^{-1/2}\mathrm{cn}\,u$	$m_1^{-1/2}\mathrm{cn}\,u$	$-\mathrm{sd}\,u$	$-\mathrm{sd}\,u$	$\mathrm{sd}\,u$	$im^{-1/2}\mathrm{nc}\,u$	$-\mathrm{sd}\,u$	$-im_1^{-1/2}m^{-1/2}\mathrm{ds}\,u$	$\mathrm{sd}\,u$
16.8.6	nd	$\mathrm{nd}\,u$	$\mathrm{nd}\,u$	$m_1^{-1/2}\mathrm{dn}\,u$	$m_1^{-1/2}\mathrm{dn}\,u$	$m_1^{-1/2}\mathrm{dn}\,u$	$\mathrm{nd}\,u$	$\mathrm{nd}\,u$	$\mathrm{nd}\,u$	$i\mathrm{sc}\,u$	$-\mathrm{nd}\,u$	$-im_1^{-1/2}\mathrm{cs}\,u$	$-\mathrm{nd}\,u$
16.8.7	dc	$\mathrm{dc}\,u$	$\mathrm{dc}\,u$	$-\mathrm{ns}\,u$	$\mathrm{ns}\,u$	$\mathrm{ns}\,u$	$-\mathrm{dc}\,u$	$-\mathrm{dc}\,u$	$-\mathrm{dc}\,u$	$m^{1/2}\mathrm{cd}\,u$	$\mathrm{dc}\,u$	$-m^{1/2}\mathrm{sn}\,u$	$-\mathrm{dc}\,u$
16.8.8	nc	$\mathrm{nc}\,u$	$\mathrm{nc}\,u$	$-m_1^{-1/2}\mathrm{ds}\,u$	$m_1^{-1/2}\mathrm{ds}\,u$	$m_1^{-1/2}\mathrm{ds}\,u$	$-\mathrm{nc}\,u$	$-\mathrm{nc}\,u$	$-\mathrm{nc}\,u$	$im^{1/2}\mathrm{sd}\,u$	$-\mathrm{nc}\,u$	$im_1^{-1/2}m^{1/2}\mathrm{cn}\,u$	$\mathrm{nc}\,u$
16.8.9	sc	$\mathrm{sc}\,u$	$-\mathrm{sc}\,u$	$-m_1^{-1/2}\mathrm{cs}\,u$	$-m_1^{-1/2}\mathrm{cs}\,u$	$m_1^{-1/2}\mathrm{cs}\,u$	$\mathrm{sc}\,u$	$\mathrm{sc}\,u$	$-\mathrm{sc}\,u$	$i\mathrm{nd}\,u$	$-\mathrm{sc}\,u$	$im_1^{-1/2}\mathrm{dn}\,u$	$-\mathrm{sc}\,u$
16.8.10	ns	$\mathrm{ns}\,u$	$-\mathrm{ns}\,u$	$\mathrm{dc}\,u$	$-\mathrm{dc}\,u$	$\mathrm{dc}\,u$	$-\mathrm{ns}\,u$	$-\mathrm{ns}\,u$	$\mathrm{ns}\,u$	$m^{1/2}\mathrm{sn}\,u$	$\mathrm{ns}\,u$	$m^{1/2}\mathrm{cd}\,u$	$-\mathrm{ns}\,u$
16.8.11	ds	$\mathrm{ds}\,u$	$-\mathrm{ds}\,u$	$m_1^{1/2}\mathrm{nc}\,u$	$-m_1^{1/2}\mathrm{nc}\,u$	$m_1^{1/2}\mathrm{nc}\,u$	$-\mathrm{ds}\,u$	$-\mathrm{ds}\,u$	$\mathrm{ds}\,u$	$-im^{1/2}\mathrm{cn}\,u$	$-\mathrm{ds}\,u$	$im_1^{1/2}m^{1/2}\mathrm{sd}\,u$	$\mathrm{ds}\,u$
16.8.12	cs	$\mathrm{cs}\,u$	$-\mathrm{cs}\,u$	$-m_1^{1/2}\mathrm{sc}\,u$	$-m_1^{1/2}\mathrm{sc}\,u$	$m_1^{1/2}\mathrm{sc}\,u$	$\mathrm{cs}\,u$	$\mathrm{cs}\,u$	$-\mathrm{cs}\,u$	$-i\mathrm{dn}\,u$	$-\mathrm{cs}\,u$	$-im_1^{1/2}\mathrm{nd}\,u$	$-\mathrm{cs}\,u$

16.9. Relations Between the Squares of the Functions

16.9.1 $-\mathrm{dn}^2u+m_1=-m\ \mathrm{cn}^2u=m\ \mathrm{sn}^2u-m$

16.9.2 $-m_1\mathrm{nd}^2u+m_1=-mm_1\mathrm{sd}^2u=m\ \mathrm{cd}^2u-m$

16.9.3 $m_1\mathrm{sc}^2u+m_1=m_1\mathrm{nc}^2u=\mathrm{dc}^2u-m$

16.9.4 $\mathrm{cs}^2u+m_1=\mathrm{ds}^2u=\mathrm{ns}^2u-m$

In using the above results remember that $m+m_1=1$.

If pq u, rt u are any two of the twelve functions, one entry expresses tq^2u in terms of pq^2u and another expresses qt^2u in terms of rt^2u. Since $\mathrm{tq}^2u\cdot\mathrm{qt}^2u=1$, we can obtain from the table the bilinear relation between pq^2u and rt^2u. Thus for the functions cd u, sn u we have

16.9.5 $$\mathrm{nd}^2u=\frac{1-m\ \mathrm{cd}^2u}{m_1},\ \mathrm{dn}^2u=1-m\ \mathrm{sn}^2u$$

and therefore

16.9.6 $$(1-m\ \mathrm{cd}^2u)(1-m\ \mathrm{sn}^2u)=m_1.$$

16.10. Change of Parameter

Negative Parameter

If m is a positive number, let

16.10.1 $$\mu=\frac{m}{1+m},\ \mu_1=\frac{1}{1+m},\ v=\frac{u}{\mu_1^{\frac{1}{2}}}\qquad(0<\mu<1)$$

16.10.2 $\mathrm{sn}\ (u|-m)=\mu_1^{\frac{1}{2}}\mathrm{sd}\ (v|\mu)$

16.10.3 $\mathrm{cn}\ (u|-m)=\mathrm{cd}\ (v|\mu)$

16.10.4 $\mathrm{dn}\ (u|-m)=\mathrm{nd}\ (v|\mu)$.

16.11. Reciprocal Parameter (Jacobi's Real Transformation)

16.11.1 $m>0,\ \mu=m^{-1},\ v=um^{1/2}$

16.11.2 $\mathrm{sn}\ (u|m)=\mu^{1/2}\mathrm{sn}\ (v|\mu)$

16.11.3 $\mathrm{cn}\ (u|m)=\mathrm{dn}\ (v|\mu)$

16.11.4 $\mathrm{dn}\ (u|m)=\mathrm{cn}\ (v|\mu)$

Here if $m>1$ then $m^{-1}=\mu<1$.

Thus elliptic functions whose parameter is real can be made to depend on elliptic functions whose parameter lies between 0 and 1.

16.12. Descending Landen Transformation (Gauss' Transformation)

To decrease the parameter, let

16.12.1 $$\mu=\left(\frac{1-m_1^{1/2}}{1+m_1^{1/2}}\right)^2,\ v=\frac{u}{1+\mu^{1/2}},$$

then

16.12.2 $$\mathrm{sn}\ (u|m)=\frac{(1+\mu^{1/2})\ \mathrm{sn}\ (v|\mu)}{1+\mu^{1/2}\ \mathrm{sn}^2\ (v|\mu)}$$

16.12.3 $$\mathrm{cn}\ (u|m)=\frac{\mathrm{cn}\ (v|\mu)\ \mathrm{dn}\ (v|\mu)}{1+\mu^{1/2}\ \mathrm{sn}^2\ (v|\mu)}$$

16.12.4 $$\mathrm{dn}\ (u|m)=\frac{\mathrm{dn}^2(v|\mu)-(1-\mu^{1/2})}{(1+\mu^{1/2})-\mathrm{dn}^2(v|\mu)}.$$

Note that successive applications can be made conveniently to find sn $(u|m)$ in terms of sn $(v|\mu)$ and dn $(u|m)$ in terms of dn $(v|\mu)$, but that the calculation of cn $(u|m)$ requires all three functions.

16.13. Approximation in Terms of Circular Functions

When the parameter m is so small that we may neglect m^2 and higher powers, we have the approximations

16.13.1
$$\mathrm{sn}\ (u|m)\approx\sin u-\frac{1}{4}\ m(u-\sin u\cos u)\cos u$$

16.13.2
$$\mathrm{cn}\ (u|m)\approx\cos u+\frac{1}{4}\ m(u-\sin u\cos u)\sin u$$

16.13.3 $$\mathrm{dn}\ (u|m)\approx 1-\frac{1}{2}\ m\sin^2 u$$

16.13.4 $$\mathrm{am}\ (u|m)\approx u-\frac{1}{4}\ m(u-\sin u\cos u).$$

One way of calculating the Jacobian functions is to use Landen's descending transformation to reduce the parameter sufficiently for the above formulae to become applicable. See also **16.14.**

16.14. Ascending Landen Transformation

To increase the parameter, let

16.14.1 $$\mu=\frac{4m^{1/2}}{(1+m^{1/2})^2},\ \mu_1=\left(\frac{1-m^{1/2}}{1+m^{1/2}}\right)^2,\ v=\frac{u}{1+\mu_1^{1/2}}$$

16.14.2 $$\mathrm{sn}\ (u|m)=(1+\mu_1^{1/2})\ \frac{\mathrm{sn}\ (v|\mu)\ \mathrm{cn}\ (v|\mu)}{\mathrm{dn}\ (v|\mu)}$$

16.14.3 $$\mathrm{cn}\ (u|m)=\frac{1+\mu_1^{1/2}}{\mu}\ \frac{\mathrm{dn}^2(v|\mu)-\mu_1^{1/2}}{\mathrm{dn}\ (v|\mu)}$$

16.14.4 $$\mathrm{dn}\ (u|m)=\frac{1-\mu_1^{1/2}}{\mu}\ \frac{\mathrm{dn}^2\ (v|\mu)+\mu_1^{1/2}}{\mathrm{dn}\ (v|\mu)}$$

Note that, when successive applications are to be made, it is simplest to calculate dn $(u|m)$ since this is expressed always in terms of the same function. The calculation of cn $(u|m)$ leads to that of dn $(v|\mu)$.

The calculation of sn $(u|m)$ necessitates the evaluation of all three functions.

16.15. Approximation in Terms of Hyperbolic Functions

When the parameter m is so close to unity that m_1^2 and higher powers of m_1 can be neglected we have the approximations

16.15.1

$$\mathrm{sn}\,(u|m)\approx\tanh u+\frac{1}{4}\,m_1\,(\sinh u\,\cosh u-u)\,\mathrm{sech}^2\,u$$

16.15.2

$$\mathrm{cn}\,(u|m)\approx\mathrm{sech}\,u$$
$$-\frac{1}{4}\,m_1\,(\sinh u\,\cosh u-u)\,\tanh u\,\mathrm{sech}\,u$$

16.15.3

$$\mathrm{dn}\,(u|m)\approx\mathrm{sech}\,u$$
$$+\frac{1}{4}\,m_1\,(\sinh u\,\cosh u+u)\,\tanh u\,\mathrm{sech}\,u$$

16.15.4

$$\mathrm{am}\,(u|m)\approx\mathrm{gd}\,u+\frac{1}{4}\,m_1\,(\sinh u\,\cosh u-u)\,\mathrm{sech}\,u.$$

Another way of calculating the Jacobian functions is to use Landen's ascending transformation to increase the parameter sufficiently for the above formulae to become applicable. See also **16.13.**

16.16. Derivatives

	Function	Derivative	
16.16.1	sn u	cn u dn u	Pole n
16.16.2	cn u	$-$sn u dn u	
16.16.3	dn u	$-m$ sn u cn u	
16.16.4	cd u	$-m_1$ sd u nd u	Pole d
16.16.5	sd u	cd u nd u	
16.16.6	nd u	m sd u cd u	
16.16.7	dc u	m_1 sc u nc u	Pole c
16.16.8	nc u	sc u dc u	
16.16.9	sc u	dc u nc u	
16.16.10	ns u	$-$ds u cs u	Pole s
16.16.11	ds u	$-$cs u ns u	
16.16.12	cs u	$-$ns u ds u	

Note that the derivative is proportional to the product of the two copolar functions.

16.17. Addition Theorems

16.17.1 $\mathrm{sn}(u+v)$

$$=\frac{\mathrm{sn}\,u\cdot\mathrm{cn}\,v\cdot\mathrm{dn}\,v+\mathrm{sn}\,v\cdot\mathrm{cn}\,u\cdot\mathrm{dn}\,u}{1-m\,\mathrm{sn}^2u\cdot\mathrm{sn}^2v}$$

16.17.2 $\mathrm{cn}(u+v)$

$$=\frac{\mathrm{cn}\,u\cdot\mathrm{cn}\,v-\mathrm{sn}\,u\cdot\mathrm{dn}\,u\cdot\mathrm{sn}\,v\cdot\mathrm{dn}\,v}{1-m\,\mathrm{sn}^2u\,\mathrm{sn}^2v}$$

16.17.3 $$\mathrm{dn}(u+v)=\frac{\mathrm{dn}\,u\cdot\mathrm{dn}\,v-m\,\mathrm{sn}\,u\cdot\mathrm{cn}\,u\cdot\mathrm{sn}\,v\cdot\mathrm{cn}\,v}{1-m\,\mathrm{sn}^2u\cdot\mathrm{sn}^2v}$$

Addition theorems are derivable one from another and are expressible in a great variety of forms. Thus $\mathrm{ns}(u+v)$ comes from $1/\mathrm{sn}(u+v)$ in the form $(1-m\,\mathrm{sn}^2u\,\mathrm{sn}^2v)/(\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v+\mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{dn}\,u)$ from **16.17.1.**

Alternatively $\mathrm{ns}(u+v)=m^{1/2}\mathrm{sn}\,\{(iK'-u)-v\}$ which again from **16.17.1** yields the form $(\mathrm{ns}\,u\,\mathrm{cs}\,v\,\mathrm{ds}\,u-\mathrm{ns}\,v\,\mathrm{cs}\,u\,\mathrm{ds}\,v)/(\mathrm{ns}^2u-\mathrm{ns}^2v)$.

The function $\mathrm{pq}(u+v)$ is a rational function of the four functions pq u, pq v, pq$'u$, pq$'v$.

16.18. Double Arguments

16.18.1 sn $2u$

$$=\frac{2\mathrm{sn}\,u\cdot\mathrm{cn}\,u\cdot\mathrm{dn}\,u}{1-m\,\mathrm{sn}^4u}=\frac{2\mathrm{sn}\,u\cdot\mathrm{cn}\,u\cdot\mathrm{dn}\,u}{\mathrm{cn}^2u+\mathrm{sn}^2u\cdot\mathrm{dn}^2u}$$

16.18.2 cn $2u$

$$=\frac{\mathrm{cn}^2u-\mathrm{sn}^2u\cdot\mathrm{dn}^2u}{1-m\,\mathrm{sn}^4u}=\frac{\mathrm{cn}^2u-\mathrm{sn}^2u\cdot\mathrm{dn}^2u}{\mathrm{cn}^2u+\mathrm{sn}^2u\cdot\mathrm{dn}^2u}$$

16.18.3 dn $2u$

$$=\frac{\mathrm{dn}^2u-m\,\mathrm{sn}^2u\cdot\mathrm{cn}^2\,u}{1-m\,\mathrm{sn}^4u}=\frac{\mathrm{dn}^2u+\mathrm{cn}^2u(\mathrm{dn}^2u-1)}{\mathrm{dn}^2u-\mathrm{cn}^2u(\mathrm{dn}^2u-1)}$$

16.18.4 $$\frac{1-\mathrm{cn}\,2u}{1+\mathrm{cn}\,2u}=\frac{\mathrm{sn}^2u\cdot\mathrm{dn}^2u}{\mathrm{cn}^2u}$$

16.18.5 $$\frac{1-\mathrm{dn}\,2u}{1+\mathrm{dn}\,2u}=\frac{m\,\mathrm{sn}^2u\cdot\mathrm{cn}^2u}{\mathrm{dn}^2u}$$

16.19. Half Arguments

16.19.1 $$\mathrm{sn}^2\tfrac{1}{2}u=\frac{1-\mathrm{cn}\,u}{1+\mathrm{dn}\,u}$$

16.19.2 $$\mathrm{cn}^2\tfrac{1}{2}u=\frac{\mathrm{dn}\,u+\mathrm{cn}\,u}{1+\mathrm{dn}\,u}$$

16.19.3 $$\mathrm{dn}^2\tfrac{1}{2}\,u=\frac{m_1+\mathrm{dn}\,u+m\,\mathrm{cn}\,u}{1+\mathrm{dn}\,u}$$

16.20. Jacobi's Imaginary Transformation

16.20.1 $$\mathrm{sn}(iu|m)=i\,\mathrm{sc}(u|m_1)$$

16.20.2 $$\mathrm{cn}(iu|m)=\mathrm{nc}(u|m_1)$$

16.20.3 $$\mathrm{dn}(iu|m)=\mathrm{dc}(u|m_1)$$

16.21. Complex Arguments

With the abbreviations

16.21.1

$$\mathrm{s}=\mathrm{sn}(x|m),\ \mathrm{c}=\mathrm{cn}(x|m),\ \mathrm{d}=\mathrm{dn}(x|m),\ \mathrm{s}_1=\mathrm{sn}(y|m_1),$$
$$\mathrm{c}_1=\mathrm{cn}(y|m_1),\ \mathrm{d}_1=\mathrm{dn}(y|m_1)$$

16.21.2 $$\mathrm{sn}(x+iy|m)=\frac{\mathrm{s}\cdot\mathrm{d}_1+i\mathrm{c}\cdot\mathrm{d}\cdot\mathrm{s}_1\cdot\mathrm{c}_1}{\mathrm{c}_1^2+m\mathrm{s}^2\cdot\mathrm{s}_1^2}$$

16.21.3 $$\mathrm{cn}(x+iy|m)=\frac{\mathrm{c}\cdot\mathrm{c}_1-i\mathrm{s}\cdot\mathrm{d}\cdot\mathrm{s}_1\cdot\mathrm{d}_1}{\mathrm{c}_1^2+m\mathrm{s}^2\cdot\mathrm{s}_1^2}$$

16.21.4 $$\mathrm{dn}(x+iy|m)=\frac{\mathrm{d}\cdot\mathrm{c}_1\cdot\mathrm{d}_1-im\mathrm{s}\cdot\mathrm{c}\cdot\mathrm{s}_1}{\mathrm{c}_1^2+m\mathrm{s}^2\cdot\mathrm{s}_1^2}$$

16.22. Leading Terms of the Series in Ascending Powers of u

16.22.1

$$\mathrm{sn}(u|m)=u-(1+m)\frac{u^3}{3!}+(1+14m+m^2)\frac{u^5}{5!}$$
$$-(1+135m+135m^2+m^3)\frac{u^7}{7!}+\ldots$$

16.22.2

$$\mathrm{cn}(u|m)=1-\frac{u^2}{2!}+(1+4m)\frac{u^4}{4!}$$
$$-(1+44m+16m^2)\frac{u^6}{6!}+\ldots$$

16.22.3

$$\mathrm{dn}(u|m)=1-m\frac{u^2}{2!}+m(4+m)\frac{u^4}{4!}$$
$$-m(16+44m+m^2)\frac{u^6}{6!}+\ldots$$

No formulae are known for the general coefficients in these series.

16.23. Series Expansions in Terms of the Nome $q=e^{-\pi K'/K}$ and the Argument $v=\pi u/(2K)$

16.23.1 $$\mathrm{sn}\ (u|m)=\frac{2\pi}{m^{1/2}K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1-q^{2n+1}}\sin\ (2n+1)v$$

16.23.2 $$\mathrm{cn}\ (u|m)=\frac{2\pi}{m^{1/2}K}\sum_{n=0}^{\infty}\frac{q^{n+1/2}}{1+q^{2n+1}}\cos\ (2n+1)v$$

16.23.3 $$\mathrm{dn}\ (u|m)=\frac{\pi}{2K}+\frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\cos 2nv$$

16.23.4

$$\mathrm{cd}\ (u|m)=\frac{2\pi}{m^{1/2}K}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n+1/2}}{1-q^{2n+1}}\cos\ (2n+1)v$$

16.23.5

$$\mathrm{sd}\ (u|m)=\frac{2\pi}{(mm_1)^{1/2}K}\sum_{n=0}^{\infty}(-1)^n\frac{q^{n+1/2}}{1+q^{2n+1}}\sin\ (2n+1)v$$

16.23.6

$$\mathrm{nd}\ (u|m)=\frac{\pi}{2m_1^{1/2}K}+\frac{2\pi}{m_1^{1/2}K}\sum_{n=1}^{\infty}(-1)^n\frac{q^n}{1+q^{2n}}\cos 2nv$$

16.23.7

$$\mathrm{dc}\ (u|m)=\frac{\pi}{2K}\sec v$$
$$+\frac{2\pi}{K}\sum_{n=0}^{\infty}(-1)^n\frac{q^{2n+1}}{1-q^{2n+1}}\cos\ (2n+1)v$$

16.23.8

$$\mathrm{nc}\ (u|m)=\frac{\pi}{2m_1^{1/2}K}\sec v$$
$$-\frac{2\pi}{m_1^{1/2}K}\sum_{n=0}^{\infty}(-1)^n\frac{q^{2n+1}}{1+q^{2n+1}}\cos\ (2n+1)v$$

16.23.9

$$\mathrm{sc}\ (u|m)=\frac{\pi}{2m_1^{1/2}K}\tan v$$
$$+\frac{2\pi}{m_1^{1/2}K}\sum_{n=1}^{\infty}(-1)^n\frac{q^{2n}}{1+q^{2n}}\sin 2nv$$

16.23.10

$$\mathrm{ns}\ (u|m)=\frac{\pi}{2K}\csc v-\frac{2\pi}{K}\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{2n+1}}\sin\ (2n+1)v$$

16.23.11

$$\mathrm{ds}\ (u|m)=\frac{\pi}{2K}\csc v-\frac{2\pi}{K}\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1+q^{2n+1}}\sin\ (2n+1)v$$

16.23.12

$$\mathrm{cs}\ (u|m)=\frac{\pi}{2K}\cot v-\frac{2\pi}{K}\sum_{n=1}^{\infty}\frac{q^{2n}}{1+q^{2n}}\sin 2nv$$

16.24. Integrals of the Twelve Jacobian Elliptic Functions

16.24.1 $\int \mathrm{sn}\ u\ du=m^{-1/2}\ln\ (\mathrm{dn}\ u-m^{1/2}\mathrm{cn}\ u)$

16.24.2 $\int \mathrm{cn}\ u\ du=m^{-1/2}\arccos\ (\mathrm{dn}\ u)$

16.24.3 $\int \mathrm{dn}\ u\ du=\arcsin\ (\mathrm{sn}\ u)$

16.24.4 $\int \mathrm{cd}\ u\ du=m^{-1/2}\ln\ (\mathrm{nd}\ u+m^{1/2}\mathrm{sd}\ u)$

16.24.5 $\int \mathrm{sd}\ u\ du=(mm_1)^{-1/2}\arcsin\ (-m^{1/2}\mathrm{cd}\ u)$

16.24.6 $\int \mathrm{nd}\ u\ du=m_1^{-1/2}\arccos\ (\mathrm{cd}\ u)$

16.24.7 $\int \mathrm{dc}\ u\ du=\ln\ (\mathrm{nc}\ u+\mathrm{sc}\ u)$

16.24.8 $\int \mathrm{nc}\ u\ du=m_1^{-1/2}\ln\ (\mathrm{dc}\ u+m_1^{1/2}\mathrm{sc}\ u)$

16.24.9 $\int \mathrm{sc}\ u\ du=m_1^{-1/2}\ln\ (\mathrm{dc}\ u+m_1^{1/2}\mathrm{nc}\ u)$

16.24.10 $\int \mathrm{ns}\ u\ du=\ln\ (\mathrm{ds}\ u-\mathrm{cs}\ u)$

16.24.11 $\int \mathrm{ds}\ u\ du=\ln\ (\mathrm{ns}\ u-\mathrm{cs}\ u)$

16.24.12 $\int \mathrm{cs}\ u\ du=\ln\ (\mathrm{ns}\ u-\mathrm{ds}\ u)$

In numerical use of the above table certain restrictions must be put on u in order to keep the arguments of the logarithms positive and to avoid trouble with many-valued inverse circular functions.

16.25. Notation for the Integrals of the Squares of the Twelve Jacobian Elliptic Functions

16.25.1 $$\mathrm{Pq}\, u=\int_0^u \mathrm{pq}^2 t\, dt \text{ when } \mathrm{q}\neq \mathrm{s}$$

16.25.2 $$\mathrm{Ps}\, u=\int_0^u \left(\mathrm{pq}^2 t-\frac{1}{t^2}\right) dt-\frac{1}{u}$$

Examples

$$\mathrm{Cd}\, u=\int_0^u \mathrm{cd}^2 t\, dt, \mathrm{Ns}\ u=\int_0^u\left(\mathrm{ns}^2 t-\frac{1}{t^2}\right)dt-\frac{1}{u}$$

16.26. Integrals in Terms of the Elliptic Integral of the Second Kind (see 17.4)

16.26.1 $$m\mathrm{Sn}\ u=-E(u)+u$$

16.26.2 $$m\mathrm{Cn}\ u=E(u)-m_1 u$$ Pole n

16.26.3 $$\mathrm{Dn}\ u=E(u)$$

16.26.4 $$m\mathrm{Cd}\ u=-E(u)+u+m\mathrm{sn}\ u\ \mathrm{cd}\ u$$

16.26.5

$$mm_1\mathrm{Sd}\ u=E(u)-m_1 u-m\mathrm{sn}\ u\ \mathrm{cd}\ u$$ Pole d

16.26.6 $$m_1\mathrm{Nd}\ u=E(u)-m\mathrm{sn}\ u\ \mathrm{cd}\ u$$

16.26.7 $$\mathrm{Dc}\ u=-E(u)+u+\mathrm{sn}\ u\ \mathrm{dc}\ u$$

16.26.8

$$m_1\mathrm{Nc}\ u=-E(u)+m_1 u+\mathrm{sn}\ u\ \mathrm{dc}\ u$$ Pole c

16.26.9 $$m_1\mathrm{Sc}\ u=-E(u)+\mathrm{sn}\ u\ \mathrm{dc}\ u$$

16.26.10 $$\mathrm{Ns}\ u=-E(u)+u-\mathrm{cn}\ u\ \mathrm{ds}\ u$$

16.26.11

$$\mathrm{Ds}\ u=-E(u)+m_1 u-\mathrm{cn}\ u\ \mathrm{ds}\ u$$ Pole s

16.26.12 $$\mathrm{Cs}\ u=-E(u)-\mathrm{cn}\ u\ \mathrm{ds}\ u$$

All the above may be expressed in terms of Jacobi's zeta function (see **17.4.27**).

$$Z(u)=E(u)-\frac{E}{K}u, \text{ where } E=E(K)$$

16.27. Theta Functions; Expansions in Terms of the Nome q

16.27.1

$$\vartheta_1(z,\ q)=\vartheta_1(z)=2q^{1/4}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)} \sin (2n+1)z$$

16.27.2

$$\vartheta_2(z,\ q)=\vartheta_2(z)=2q^{1/4}\sum_{n=0}^{\infty} q^{n(n+1)} \cos\ (2n+1)z$$

16.27.3 $$\vartheta_3(z,\ q)=\vartheta_3(z)=1+2\sum_{n=1}^{\infty} q^{n^2} \cos 2nz$$

16.27.4

$$\vartheta_4(z,\ q)=\vartheta_4(z)=1+2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos 2nz$$

Theta functions are important because every one of the Jacobian elliptic functions can be expressed as the ratio of two theta functions. See **16.36.**

The notation shows these functions as depending on the variable z and the nome q, $|q|<1$. In this case, here and elsewhere, the convergence is not dependent on the trigonometrical terms. In their relation to the Jacobian elliptic functions, we note that the nome q is given by

$$q=e^{-\pi K'/K},$$

where K and iK' are the quarter periods. Since $q=q(m)$ is determined when the parameter m is given, we can also regard the theta functions as dependent upon m and then we write

$$\vartheta_a(z,\ q)=\vartheta_a(z|m),\ a=1,\ 2,\ 3,\ 4$$

but when no ambiguity is to be feared, we write $\vartheta_a(z)$ simply.

The above notations are those given in Modern Analysis [16.6].

There is a bewildering variety of notations, for example the function $\vartheta_4(z)$ above is sometimes denoted by $\vartheta_0(z)$ or $\vartheta(z)$; see the table given in Modern Analysis [16.6]. Further the argument $u=2Kz/\pi$ is frequently used so that in consulting books caution should be exercised.

16.28. Relations Between the Squares of the Theta Functions

16.28.1 $$\vartheta_1^2(z)\vartheta_4^2(0)=\vartheta_3^2(z)\vartheta_2^2(0)-\vartheta_2^2(z)\vartheta_3^2(0)$$

16.28.2 $$\vartheta_2^2(z)\vartheta_4^2(0)=\vartheta_4^2(z)\vartheta_2^2(0)-\vartheta_1^2(z)\vartheta_3^2(0)$$

16.28.3 $$\vartheta_3^2(z)\vartheta_4^2(0)=\vartheta_4^2(z)\vartheta_3^2(0)-\vartheta_1^2(z)\vartheta_2^2(0)$$

16.28.4 $$\vartheta_4^2(z)\vartheta_4^2(0)=\vartheta_3^2(z)\vartheta_3^2(0)-\vartheta_2^2(z)\vartheta_2^2(0)$$

16.28.5 $$\vartheta_2^4(0)+\vartheta_4^4(0)=\vartheta_3^4(0)$$

Note also the important relation

16.28.6 $$\vartheta_1'(0)=\vartheta_2(0)\vartheta_3(0)\vartheta_4(0) \text{ or } \vartheta_1'=\vartheta_2\vartheta_3\vartheta_4$$

16.29. Logarithmic Derivatives of the Theta Functions

16.29.1 $$\frac{\vartheta_1'(u)}{\vartheta_1(u)}=\cot\ u+4\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}} \sin 2nu$$

16.29.2

$$\frac{\vartheta_2'(u)}{\vartheta_2(u)} = -\tan u + 4\sum_{n=1}^{\infty} (-1)^n \frac{q^{2n}}{1-q^{2n}} \sin 2nu$$

16.29.3 $$\frac{\vartheta_3'(u)}{\vartheta_3(u)} = 4\sum_{n=1}^{\infty} (-1)^n \frac{q^n}{1-q^{2n}} \sin 2nu$$

16.29.4 $$\frac{\vartheta_4'(u)}{\vartheta_4(u)} = 4\sum_{n=1}^{\infty} \frac{q^n}{1-q^{2n}} \sin 2nu$$

16.30. Logarithms of Theta Functions of Sum and Difference

16.30.1

$$\ln \frac{\vartheta_1(\alpha+\beta)}{\vartheta_1(\alpha-\beta)} = \ln \frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} + 4\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^{2n}}{1-q^{2n}} \sin 2n\alpha \sin 2n\beta$$

16.30.2

$$\ln \frac{\vartheta_2(\alpha+\beta)}{\vartheta_2(\alpha-\beta)} = \ln \frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{q^{2n}}{1-q^{2n}} \sin 2n\alpha \sin 2n\beta$$

16.30.3

$$\ln \frac{\vartheta_3(\alpha+\beta)}{\vartheta_3(\alpha-\beta)} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \frac{q^n}{1-q^{2n}} \sin 2\,n\alpha \sin 2n\beta$$

16.30.4

$$\ln \frac{\vartheta_4(\alpha+\beta)}{\vartheta_4(\alpha-\beta)} = 4\sum_{n=1}^{\infty} \frac{1}{n} \frac{q^n}{1-q^{2n}} \sin 2n\alpha \sin 2n\beta$$

The corresponding expressions when $\beta = i\gamma$ are easily deduced by use of the formulae **4.3.55** and **4.3.56.**

16.31. Jacobi's Notation for Theta Functions

16.31.1 $$\Theta(u|m) = \Theta(u) = \vartheta_4(v), \qquad v = \frac{\pi u}{2K}$$

16.31.2 $$\Theta_1(u|m) = \Theta_1(u) = \vartheta_3(v) = \Theta(u+K)$$

16.31.3 $$\mathrm{H}(u|m) = \mathrm{H}(u) = \vartheta_1(v)$$

16.31.4 $$\mathrm{H}_1(u|m) = \mathrm{H}_1(u) = \vartheta_2(v) = \mathrm{H}(u+K)$$

16.32. Calculation of Jacobi's Theta Function $\Theta(u|m)$ by Use of the Arithmetic-Geometric Mean

Form the A.G.M. scale starting with

16.32.1 $$a_0 = 1,\ b_0 = \sqrt{m_1},\ c_0 = \sqrt{m}$$

terminating with the Nth step when c_N is negligible to the accuracy required. Find φ_N in degrees, where

16.32.2 $$\varphi_N = 2^N a_N u \frac{180°}{\pi}$$

and then compute successively $\varphi_{N-1}, \varphi_{N-2}, \ldots, \varphi_1, \varphi_0$ from the recurrence relation

16.32.3 $$\sin(2\varphi_{n-1} - \varphi_n) = \frac{c_n}{a_n} \sin \varphi_n.$$

Then

16.32.4

$$\ln \Theta(u|m) = \frac{1}{2} \ln \frac{2m_1^{1/2}K(m)}{\pi} + \frac{1}{2} \ln \frac{\cos(\varphi_1 - \varphi_0)}{\cos \varphi_0} + \frac{1}{4} \ln \sec(2\varphi_0 - \varphi_1) + \frac{1}{8} \ln \sec(2\varphi_1 - \varphi_2) + \ldots + \frac{1}{2^{N+1}} \ln \sec(2\varphi_{N-1} - \varphi_N)$$

16.33. Addition of Quarter-Periods to Jacobi's Eta and Theta Functions

u	$-u$	$u+K$	$u+2K$	$u+iK'$	$u+2iK'$	$u+K+iK'$	$u+2K+2iK'$
16.33.1 $\mathrm{H}(u)$	$-\mathrm{H}(u)$	$\mathrm{H}_1(u)$	$-\mathrm{H}(u)$	$iM(u)\Theta(u)$	$-N(u)\mathrm{H}(u)$	$M(u)\Theta_1(u)$	$N(u)\mathrm{H}(u)$
16.33.2 $\mathrm{H}_1(u)$	$\mathrm{H}_1(u)$	$-\mathrm{H}(u)$	$-\mathrm{H}_1(u)$	$M(u)\Theta_1(u)$	$N(u)\mathrm{H}_1(u)$	$-iM(u)\Theta(u)$	$-N(u)\mathrm{H}_1(u)$
16.33.3 $\Theta_1(u)$	$\Theta_1(u)$	$\Theta(u)$	$\Theta_1(u)$	$M(u)\mathrm{H}_1(u)$	$N(u)\Theta_1(u)$	$iM(u)\mathrm{H}(u)$	$N(u)\Theta_1(u)$
16.33.4 $\Theta(u)$	$\Theta(u)$	$\Theta_1(u)$	$\Theta(u)$	$iM(u)\mathrm{H}(u)$	$-N(u)\Theta(u)$	$M(u)\mathrm{H}_1(u)$	$-N(u)\Theta(u)$

where

$$M(u) = \left[\exp\left(-\frac{\pi i u}{2K}\right)\right] q^{-\frac{1}{4}},$$

$$N(u) = \left[\exp\left(-\frac{\pi i u}{K}\right)\right] q^{-1}$$

$\mathrm{H}(u)$ and $\mathrm{H}_1(u)$ have the period $4K$. $\Theta(u)$ and $\Theta_1(u)$ have the period $2K$.

$2iK'$ is a quasi-period for all four functions, that is to say, increase of the argument by $2iK'$ multiplies the function by a factor.

16.34. Relation of Jacobi's Zeta Function to the Theta Functions

$$Z(u)=\frac{\partial}{\partial u}\ln\Theta(u)$$

16.34.1 $$Z(u)=\frac{\pi}{2K}\frac{\vartheta_1'\left(\frac{\pi u}{2K}\right)}{\vartheta_1\left(\frac{\pi u}{2K}\right)}-\frac{\operatorname{cn}u\operatorname{dn}u}{\operatorname{sn}u}$$

16.34.2 $$=\frac{\pi}{2K}\frac{\vartheta_2'\left(\frac{\pi u}{2K}\right)}{\vartheta_2\left(\frac{\pi u}{2K}\right)}+\frac{\operatorname{dn}u\operatorname{sn}u}{\operatorname{cn}u}$$

16.34.3 $$=\frac{\pi}{2K}\frac{\vartheta_3'\left(\frac{\pi u}{2K}\right)}{\vartheta_3\left(\frac{\pi u}{2K}\right)}-m\frac{\operatorname{sn}u\operatorname{cn}u}{\operatorname{dn}u}$$

16.34.4 $$=\frac{\pi}{2K}\frac{\vartheta_4'\left(\frac{\pi u}{2K}\right)}{\vartheta_4\left(\frac{\pi u}{2K}\right)}$$

16.35. Calculation of Jacobi's Zeta Function *Z(u|m)* by Use of the Arithmetic-Geometric Mean

Form the A.G.M. scale **17.6** starting with

16.35.1 $$a_0=1,\ b_0=\sqrt{m_1},\ c_0=\sqrt{m}$$

terminating at the Nth step when c_N is negligible to the accuracy required. Find φ_N in degrees where

16.35.2 $$\varphi_N=2^Na_Nu\,\frac{180^\circ}{\pi}$$

and then compute successively φ_{N-1}, φ_{N-2}, . . ., φ_1, φ_0 from the recurrence relation

16.35.3 $$\sin(2\varphi_{n-1}-\varphi_n)=\frac{c_n}{a_n}\sin\varphi_n.$$

Then

16.35.4

$$Z(u|m)=c_1\sin\varphi_1+c_2\sin\varphi_2+\ .\ .\ .\ +c_N\sin\varphi_N.$$

16.36. Neville's Notation for Theta Functions

These functions are defined in terms of Jacobi's theta functions of **16.31** by

16.36.1 $$\vartheta_s(u)=\frac{\mathrm{H}(u)}{\mathrm{H}'(0)},\ \vartheta_c(u)=\frac{\mathrm{H}(u+K)}{\mathrm{H}(K)}$$

16.36.2 $$\vartheta_d(u)=\frac{\Theta(u+K)}{\Theta(K)},\ \vartheta_n(u)=\frac{\Theta(u)}{\Theta(0)}.$$

If λ, μ are any integers positive, negative, or zero the points $u_0+2\lambda K+2\mu iK'$ are said to be *congruent to* u_0.

$\vartheta_s(u)$ has zeros at the points congruent to 0
$\vartheta_c(u)$ has zeros at the points congruent to K
$\vartheta_n(u)$ has zeros at the points congruent to iK'
$\vartheta_d(u)$ has zeros at the points congruent to $K+iK'$

Thus the suffix secures that the function $\vartheta_p(u)$ has zeros at the points marked p in the introductory diagram in **16.1.2**, and the constant by which Jacobi's function is divided secures that the leading coefficient of $\vartheta_p(u)$ at the origin is unity. Therefore the functions have the fundamentally important property that if p, q are any two of the letters s, c, n, d, the Jacobian elliptic function pq u is given by

16.36.3 $$\operatorname{pq}u=\frac{\vartheta_p(u)}{\vartheta_q(u)}.$$

These functions also have the property

16.36.4 $$m_1^{-1/4}\vartheta_c(K-u)=\vartheta_s(u)$$

16.36.5 $$m_1^{-1/4}\vartheta_d(K-u)=\vartheta_n(u),$$

for complementary arguments u and $K-u$.

In terms of the theta functions defined in **16.27**, let $v=\pi u/(2K)$, then

16.36.6 $$\vartheta_s(u)=\frac{2K\vartheta_1(v)}{\vartheta_1'(0)},\ \vartheta_c(u)=\frac{\vartheta_2(v)}{\vartheta_2(0)}$$

16.36.7 $$\vartheta_d(u)=\frac{\vartheta_3(v)}{\vartheta_3(0)},\ \vartheta_n(u)=\frac{\vartheta_4(v)}{\vartheta_4(0)}.$$

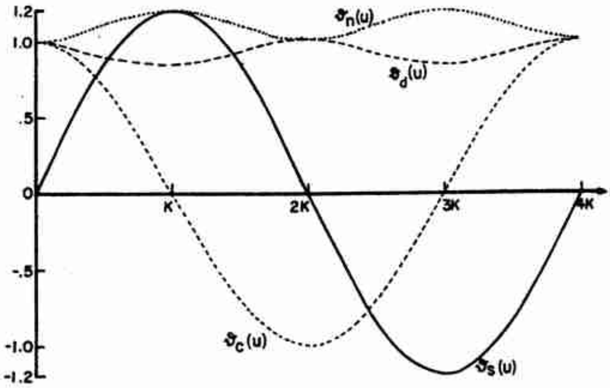

FIGURE 16.4. *Neville's theta functions* $\vartheta_s(u)$, $\vartheta_c(u)$, $\vartheta_d(u)$, $\vartheta_n(u)$
$m=\frac{1}{2}$

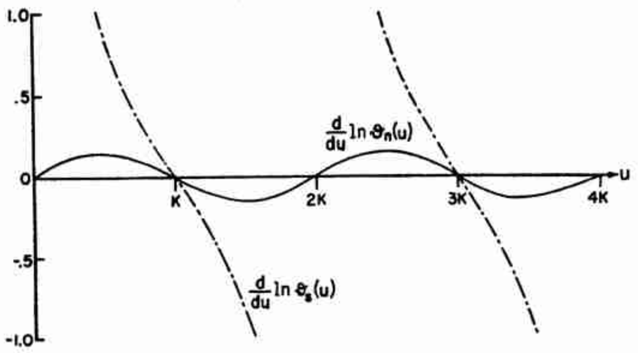

FIGURE 16.5. *Logarithmic derivatives of theta functions*

$$\frac{d}{du}\ln\vartheta_s(u),\ \frac{d}{du}\ln\vartheta_n(u)$$

$$m=\frac{1}{2}$$

16.37. Expression as Infinite Products

$$q=q(m),\ v=\pi u/(2K)$$

16.37.1

$$\vartheta_s(u)=\left(\frac{16q}{mm_1}\right)^{1/6}\sin v\prod_{n=1}^{\infty}(1-2q^{2n}\cos 2v+q^{4n})$$

16.37.2

$$\vartheta_c(u)=\left(\frac{16qm_1^{1/2}}{m}\right)^{1/6}\cos v\prod_{n=1}^{\infty}(1+2q^{2n}\cos 2v+q^{4n})$$

16.37.3

$$\vartheta_d(u)=\left(\frac{mm_1}{16q}\right)^{1/12}\prod_{n=1}^{\infty}(1+2q^{2n-1}\cos 2v+q^{4n-2})$$

16.37.4

$$\vartheta_n(u)=\left(\frac{m}{16qm_1^2}\right)^{1/12}\prod_{n=1}^{\infty}(1-2q^{2n-1}\cos 2v+q^{4n-2})$$

16.38. Expression as Infinite Series

Let $v=\pi u/(2K)$

16.38.1

$$\vartheta_s(u)=\left[\frac{2\pi q^{1/2}}{m^{1/2}m_1^{1/2}K}\right]^{1/2}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin(2n+1)v$$

16.38.2 $$\vartheta_c(u)=\left[\frac{2\pi q^{1/2}}{m^{1/2}K}\right]^{1/2}\sum_{n=0}^{\infty}q^{n(n+1)}\cos(2n+1)v$$

16.38.3 $$\vartheta_d(u)=\left[\frac{\pi}{2K}\right]^{1/2}\{1+2\sum_{n=1}^{\infty}q^{n^2}\cos 2nv\}$$

16.38.4

$$\vartheta_n(u)=\left[\frac{\pi}{2m_1^{1/2}K}\right]^{1/2}\{1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nv\}$$

16.38.5 $$(2K/\pi)^{1/2}=1+2q+2q^4+2q^9+\ldots=\vartheta_3(0,q)$$

16.38.6

$$(2K'/\pi)^{1/2}=1+2q_1+2q_1^4+2q_1^9+\ldots=\vartheta_3(0,q_1)$$

16.38.7

$$(2m^{1/2}K/\pi)^{1/2}=2q^{1/4}(1+q^2+q^6+q^{12}+q^{20}+\ldots)$$
$$=\vartheta_2(0,q)$$

16.38.8

$$(2m_1^{1/2}K/\pi)^{1/2}=1-2q+2q^4-2q^9+\ldots=\vartheta_4(0,q).$$

Numerical Methods

16.39. Use and Extension of the Tables

Example 1. Calculate nc (1.99650|.64) to 4S.

From **Table 17.1**, $1.99650=K+.001$. From the table of principal terms

$$\text{nc } u=-m_1^{-1/2}/(u-K)+\ldots$$

$$\text{nc }(K+.001|.64)=\frac{-(.36)^{-1/2}}{.001}+\ldots$$
$$=-\frac{10000}{6}+\ldots$$
$$=-1667+\ldots$$

and since the next term is of order .001 this value -1667 is correct to at least 4S.

Example 2. Use the descending Landen transformation to calculate dn (.20|.19) to 6D.

Here $m=.19$, $m_1^{1/2}=.9$ and so from **16.12.1**

$$\mu=\left(\frac{1}{19}\right)^2,\ 1+\mu^{1/2}=\frac{20}{19},\ v=.19.$$

Also

$$\mu^2=\left(\frac{1}{19}\right)^4=10^{-6}\times 7.67$$

which is negligible.

From **16.12.4**

$$\text{dn}(.20|.19)=\frac{\text{dn}^2\left[.19\left|\left(\frac{1}{19}\right)^2\right.\right]-\left(1-\frac{1}{19}\right)}{\left(1+\frac{1}{19}\right)-\text{dn}^2\left[.19\left|\left(\frac{1}{19}\right)^2\right.\right]}.$$

Now from **16.13.3**

$$\text{dn}\left[.19\left|\left(\frac{1}{19}\right)^2\right.\right]=.999951$$

whence dn (.20|.19) = .996253.

Example 3. Use the ascending Landen transformation to calculate dn (.20|.81) to 5D.

From **16.14.1**

$$\mu=\frac{4(.9)}{(1.9)^2}=\frac{360}{361},\ \mu_1=\left(\frac{1}{19}\right)^2$$

$$1+\mu_1^{1/2}=\frac{20}{19},\quad v=\frac{19}{20}\times .20=.19,$$

μ_1^2 is negligible to 4D. Thus

Example 7. Use the q-series to compute cs $(.53601\ 62|.09)$.

Here we use the series **16.23.12,** $K=1.60804\ 862$, $q=.00589\ 414$, $v=\frac{\pi u}{2K}=\frac{\pi}{6}$radians or 30°.

Since q^4 is negligible to 8D, we have to 7D cs $(.53601\ 62|.09)$

$$=\frac{\pi}{2K}\cot 30^\circ-\frac{2\pi}{K}\left\{\frac{q^2}{1+q^2}\sin 60^\circ\right\}$$
$$=(.97683\ 3852)(1.73205\ 081)$$
$$-3.90733\ 541[(.00003\ 4740)(.86602\ 5404)]$$
$$=1.69180\ 83.$$

Example 8. Use theta functions to compute sn $(.61802|.5)$ to 5D.

Here $K(\tfrac{1}{2})=1.85407$

$$\epsilon^\circ=\frac{.61802}{1.85407}\times 90^\circ=30^\circ$$
$$\sin^2\alpha=1/2,\ \alpha=45^\circ.$$

Thus

$$\text{sn }(.61802|.5)=\frac{\vartheta_s(30^\circ\backslash 45^\circ)}{\vartheta_n(30^\circ\backslash 45^\circ)}$$
$$=\frac{.59128}{1.04729}=.56458$$

from **Table 16.1.**

Example 9. Use theta functions to compute sc $(.61802|.5)$ to 5D.

As in the preceding example

$$\epsilon^\circ=30^\circ,\ \alpha^\circ=45^\circ$$

so that

$$\text{sc }(.61802|.5)=\frac{\vartheta_s(30^\circ\backslash 45^\circ)}{\vartheta_c(30^\circ\backslash 45^\circ)}.$$

We use **Table 16.1** to give

$$\vartheta_s(30^\circ\backslash 45^\circ)=.59128$$
$$(\sec 45^\circ)^{\frac{1}{2}}\vartheta_c(30^\circ\backslash 45^\circ)=1.02796.$$

Therefore

$$\text{sc }(.61802|.5)=\frac{.59128}{1.02796}(\sec 45^\circ)^{\frac{1}{2}}$$
$$=.68402.$$

Example 10. Find sn $(.75342|.7)$ by inverse interpolation in **Table 17.5.**

This method is explained in chapter **17, Example 7.**

Example 11. Find u, given that cs $(u|.5)=.75$. From **16.9.4** we have

$$\text{sn}^2\,u=\frac{1}{1+\text{cs}^2 u}.$$

Thus

$$\text{sn}^2\,(u|.5)=.64$$

and

$$\text{sn }(u|.5)=.8.$$

We have therefore replaced the problem by that of finding u given sn $(u|m)$, where m is known. If $\varphi=\text{am}\ u$

$$\sin\varphi=\text{sn}\ u \text{ and so}$$
$$\varphi=.9272952 \text{ radians or } 53.13010^\circ.$$

From **Table 17.5,**

$$u=F(53.13010^\circ\backslash 45^\circ)=.99391.$$

Alternatively, starting with the above value of φ we can use the A.G.M. scale to calculate $F(\varphi\backslash\alpha)$ as explained in **17.6.** This method is to be preferred if more figures are required, or if α differs from a tabular value in **Table 17.5.**

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17. Elliptic Integrals

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Contents

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17. Elliptic Integrals

Mathematical Properties

17.1. Definition of Elliptic Integrals

If $R(x, y)$ is a rational function of x and y, where y^2 is equal to a cubic or quartic polynomial in x, the integral

17.1.1
$$\int R(x,y)dx$$

is called an *elliptic integral*.

The elliptic integral just defined can not, in general, be expressed in terms of elementary functions.

Exceptions to this are

(i) when $R(x, y)$ contains no odd powers of y.

(ii) when the polynomial y^2 has a repeated factor.

We therefore exclude these cases.

By substituting for y^2 and denoting by $p_s(x)$ a polynomial in x we get [2]

$$R(x,y)=\frac{p_1(x)+yp_2(x)}{p_3(x)+yp_4(x)}$$
$$=\frac{[p_1(x)+yp_2(x)][p_3(x)-yp_4(x)]y}{\{[p_3(x)]^2-y^2[p_4(x)]^2\}y}$$
$$=\frac{p_5(x)+yp_6(x)}{yp_7(x)}=R_1(x)+\frac{R_2(x)}{y}$$

where $R_1(x)$ and $R_2(x)$ are rational functions of x. Hence, by expressing $R_2(x)$ as the sum of a polynomial and partial fractions

$$\int R(x,y)dx=\int R_1(x)dx+\Sigma_s A_s\int x^s y^{-1}dx$$
$$+\Sigma_s B_s\int[(x-c)^s y]^{-1}dx$$

Reduction Formulae

Let

17.1.2
$$y^2=a_0x^4+a_1x^3+a_2x^2+a_3x+a_4 \qquad (|a_0|+|a_1|\neq 0)$$
$$=b_0(x-c)^4+b_1(x-c)^3+b_2(x-c)^2+b_3(x-c)+b_4$$
$$(|b_0|+|b_1|\neq 0)$$

17.1.3 $I_s=\int x^s y^{-1}dx,\ J_s=\int[y(x-c)^s]^{-1}dx$

By integrating the derivatives of yx^s and $y(x-c)^{-s}$ we get the reduction formulae

17.1.4
$$(s+2)a_0I_{s+3}+\tfrac{1}{2}a_1(2s+3)I_{s+2}+a_2(s+1)I_{s+1}$$
$$+\tfrac{1}{2}a_3(2s+1)I_s+sa_4I_{s-1}=x^sy \quad (s=0,\ 1,\ 2,\ .\ .\ .)$$

17.1.5
$$(2-s)b_0J_{s-3}+\tfrac{1}{2}b_1(3-2s)J_{s-2}+b_2(1-s)J_{s-1}$$
$$+\tfrac{1}{2}b_3(1-2s)J_s-sb_4J_{s+1}=y(x-c)^{-s}$$
$$(s=1,\ 2,\ 3,\ .\ .\ .)$$

By means of these reduction formulae and certain transformations (see **Examples 1** and **2**) every elliptic integral can be brought to depend on the integral of a rational function and on three canonical forms for elliptic integrals.

17.2. Canonical Forms

Definitions

17.2.1

$m=\sin^2\alpha$; m is the parameter,
α is the modular angle

17.2.2 $x=\sin\varphi=\mathrm{sn}\ u$

17.2.3 $\cos\varphi=\mathrm{cn}\ u$

17.2.4

$(1-m\sin^2\varphi)^{\frac{1}{2}}=\mathrm{dn}\ u=\Delta(\varphi)$, the delta amplitude

17.2.5 $\varphi=\arcsin(\mathrm{sn}\ u)=\mathrm{am}\ u$, the amplitude

Elliptic Integral of the First Kind

17.2.6 $F(\varphi\backslash\alpha)=F(\varphi|m)=\int_0^\varphi(1-\sin^2\alpha\sin^2\theta)^{-\frac{1}{2}}d\theta$

17.2.7
$$=\int_0^x[(1-t^2)(1-mt^2)]^{-\frac{1}{2}}dt$$
$$=\int_0^u dw=u$$

Elliptic Integral of the Second Kind

17.2.8 $E(\varphi\backslash\alpha)=E(u|m)=\int_0^x(1-t^2)^{-\frac{1}{2}}(1-mt^2)^{\frac{1}{2}}dt$

17.2.9
$$=\int_0^\varphi(1-\sin^2\alpha\sin^2\theta)^{\frac{1}{2}}d\theta$$

17.2.10
$$=\int_0^u \mathrm{dn}^2 w\,dw$$

17.2.11
$$=m_1u+m\int_0^u \mathrm{cn}^2 w\,dw$$

---

[2] See [17.7] **22.72.**

17.2.12 $E(\varphi\backslash\alpha)=u-m\int_0^u \mathrm{sn}^2\, w\, dw$

17.2.13 $$=\frac{\pi}{2K(m)}\frac{\vartheta_4'(\pi u/2K)}{\vartheta_4(\pi u/2K)}+\frac{E(m)u}{K(m)}$$

(For theta functions, see chapter **16.**)

Elliptic Integral of the Third Kind

17.2.14

$$\Pi(n;\varphi\backslash\alpha)=\int_0^{\varphi}(1-n\sin^2\theta)^{-1}[1-\sin^2\alpha\sin^2\theta]^{-1/2}d\theta$$

If $x=\mathrm{sn}\ (u|m)$,

17.2.15

$$\Pi(n;u|m)=\int_0^x(1-nt^2)^{-1}[(1-t^2)(1-mt^2)]^{-1/2}dt$$

17.2.16 $$=\int_0^u(1-n\,\mathrm{sn}^2\ (w|m))^{-1}dw$$

The Amplitude φ

17.2.17 $\varphi=\mathrm{am}\ u=\arcsin\ (\mathrm{sn}\ u)=\arcsin\ x$

can be calculated from **Tables 17.5** and **4.14.**

The Parameter m

Dependence on the *parameter* m is denoted by a vertical stroke preceding the parameter, e.g., $F(\varphi|m)$.

Together with the parameter we define the *complementary parameter* m_1 by

17.2.18 $$m+m_1=1$$

When the parameter is real, it can always be arranged, see **17.4,** that $0\leq m\leq 1$.

The Modular Angle α

Dependence on the modular angle α, defined in terms of the parameter by **17.2.1,** is denoted by a backward stroke \ preceding the modular angle, thus $E(\varphi\backslash\alpha)$. The *complementary modular* angle is $\pi/2-\alpha$ or $90°-\alpha$ according to the unit and thus $m_1=\sin^2\ (90°-\alpha)=\cos^2\ \alpha$.

The Modulus k

In terms of Jacobian elliptic functions (chapter **16**), the modulus k and the complementary modulus are defined by

17.2.19 $k=\mathrm{ns}\ (K+iK')$, $k'=\mathrm{dn}\ K$.

They are related to the parameter by $k^2=m$, $k'^2=m_1$.

Dependence on the modulus is denoted by a comma preceding it, thus $\Pi(n;u,k)$.

In computation the modulus is of minimal importance, since it is the parameter and its complement which arise naturally. The parameter and the modular angle will be employed in this chapter to the exclusion of the modulus.

The Characteristic n

The elliptic integral of the third kind depends on three variables namely (i) the parameter, (ii) the amplitude, (iii) the characteristic n. When real, the characteristic may be any number in the interval $(-\infty,\infty)$. The properties of the integral depend upon the location of the characteristic in this interval, see **17.7.**

17.3. Complete Elliptic Integrals of the First and Second Kinds

Referred to the canonical forms of **17.2,** the elliptic integrals are said to be *complete* when the amplitude is $\frac{1}{2}\pi$ and so $x=1$. These complete integrals are designated as follows

17.3.1

$$[K(m)]=K=\int_0^1[(1-t^2)(1-mt^2)]^{-1/2}dt$$
$$=\int_0^{\pi/2}(1-m\sin^2\theta)^{-1/2}d\theta$$

17.3.2 $$K=F(\tfrac{1}{2}\pi|m)=F(\tfrac{1}{2}\pi\backslash\alpha)$$

17.3.3

$$E[K(m)]=E=\int_0^1(1-t^2)^{-1/2}(1-mt^2)^{1/2}dt$$
$$=\int_0^{\pi/2}(1-m\sin^2\theta)^{1/2}d\theta$$

17.3.4 $$E=E[K(m)]=E(m)=E(\tfrac{1}{2}\pi\backslash\alpha)$$

We also define

17.3.5

$$K'=K(m_1)=K(1-m)=\int_0^{\pi/2}(1-m_1\sin^2\theta)^{-1/2}d\theta$$

17.3.6 $$K'=F(\tfrac{1}{2}\pi|m_1)=F(\tfrac{1}{2}\pi\backslash\tfrac{1}{2}\pi-\alpha)$$

17.3.7

$$E'=E(m_1)=E(1-m)=\int_0^{\pi/2}(1-m_1\sin^2\theta)^{1/2}d\theta$$

17.3.8 $E'=E[K(m_1)]=E\,(m_1)=E(\tfrac{1}{2}\pi\backslash\tfrac{1}{2}\pi-\alpha)$

K and iK' are the "real" and "imaginary" *quarter-periods* of the corresponding Jacobian elliptic functions (see chapter **16**).

Relation to the Hypergeometric Function

(see chapter 15)

17.3.9 $$K=\tfrac{1}{2}\pi F(\tfrac{1}{2},\tfrac{1}{2};1;m)$$

17.3.10 $$E=\tfrac{1}{2}\pi F(-\tfrac{1}{2},\tfrac{1}{2};1;m)$$

Infinite Series

17.3.11

$$K(m)=\frac{1}{2}\pi\left[1+\left(\frac{1}{2}\right)^2 m+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 m^2+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 m^3+\ldots\right]\qquad(|m|<1)$$

17.3.12

$$E(m)=\frac{1}{2}\pi\left[1-\left(\frac{1}{2}\right)^2 \frac{m}{1}-\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 \frac{m^2}{3}-\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 \frac{m^3}{5}-\ldots\right]\qquad(|m|<1)$$

Legendre's Relation

17.3.13 $$EK'+E'K-KK'=\tfrac{1}{2}\pi$$

Auxiliary Function

17.3.14 $$L(m)=\frac{K'(m)}{\pi}\ln\frac{16}{m_1}-K(m)$$

17.3.15 $$m=1-16\exp\left[-\pi(K(m)+L(m))/K'(m)\right]$$

17.3.16 $$m=16\exp\left[-\pi(K'(m)+L(m_1))/K(m)\right]$$

The function $L(m)$ is tabulated in **Table 17.4.**

q-Series

The Nome q and the Complementary Nome q_1

17.3.17 $$q=q(m)=\exp\left[-\pi K'/K\right]$$

17.3.18 $$q_1=q(m_1)=\exp\left[-\pi K/K'\right]$$

17.3.19 $$\ln\frac{1}{q}\ln\frac{1}{q_1}=\pi^2$$

17.3.20

$$\log_{10}\frac{1}{q}\log_{10}\frac{1}{q_1}=(\pi\log_{10}e)^2=1.86152\ 28349\text{ to 10D}$$

17.3.21

$$q=\exp\left[-\pi K'/K\right]=\frac{m}{16}+8\left(\frac{m}{16}\right)^2+84\left(\frac{m}{16}\right)^3+992\left(\frac{m}{16}\right)^4+\ldots\qquad(|m|<1)$$

17.3.22 $$K=\frac{1}{2}\pi+2\pi\sum_{s=1}^{\infty}\frac{q^s}{1+q^{2s}}$$

17.3.23

$$\frac{E}{K}=\frac{1}{3}(1+m_1)+(\pi/K)^2\left[1/12-2\sum_{s=1}^{\infty}q^{2s}(1-q^{2s})^{-2}\right]$$

17.3.24 $$\operatorname{am}u=v+\sum_{s=1}^{\infty}\frac{2q^s\sin 2sv}{s(1+q^{2s})}\text{ where }v=\pi u/(2K)$$

Limiting Values

17.3.25 $$\lim_{m\to 0}K'(E-K)=0$$

17.3.26 $$\lim_{m\to 1}\left[K-\tfrac{1}{2}\ln(16/m_1)\right]=0$$

17.3.27 $$\lim_{m\to 0}m^{-1}(K-E)=\lim_{m\to 0}m^{-1}(E-m_1K)=\pi/4$$

17.3.28 $$\lim_{m\to 0}q/m=\lim_{m_1\to 1}q_1/m_1=1/16$$

Alternative Evaluations of K and E (see also 17.5)

17.3.29

$$K(m)=2[1+m_1^{1/2}]^{-1}K([(1-m_1^{1/2})/(1+m_1^{1/2})]^2)^*$$

17.3.30

$$E(m)=(1+m_1^{1/2})E([(1-m_1^{1/2})/(1+m_1^{1/2})]^2)-2m_1^{1/2}(1+m_1^{1/2})^{-1}K([(1-m_1^{1/2})/(1+m_1^{1/2})]^2)$$

17.3.31 $$K(\alpha)=2F(\arctan(\sec^{1/2}\alpha)\backslash\alpha)$$

17.3.32 $$E(\alpha)=2E(\arctan(\sec^{1/2}\alpha)\backslash\alpha)-1+\cos\alpha$$

Polynomial Approximations[3] $(0\leq m<1)$

17.3.33

$$K(m)=[a_0+a_1m_1+a_2m_1^2]+[b_0+b_1m_1+b_2m_1^2]\ln(1/m_1)+\epsilon(m)$$

$$|\epsilon(m)|\leq 3\times 10^{-5}$$

$a_0=1.38629\ 44$	$b_0=.5$
$a_1=\ .11197\ 23$	$b_1=.12134\ 78$
$a_2=\ .07252\ 96$	$b_2=.02887\ 29$

17.3.34

$$K(m)=[a_0+a_1m_1+\ldots+a_4m_1^4]+[b_0+b_1m_1+\ldots+b_4m_1^4]\ln(1/m_1)+\epsilon(m)$$

$$|\epsilon(m)|\leq 2\times 10^{-8}$$

$a_0=1.38629\ 436112$	$b_0=.5$
$a_1=\ .09666\ 344259$	$b_1=.12498\ 593597$
$a_2=\ .03590\ 092383$	$b_2=.06880\ 248576$
$a_3=\ .03742\ 563713$	$b_3=.03328\ 355346$
$a_4=\ .01451\ 196212$	$b_4=.00441\ 787012$

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[3] The approximations **17.3.33–17.3.36** are from C. Hastings, Jr., Approximations for Digital Computers, Princeton Univ. Press, Princeton, N. J. (with permission).

*See page II.

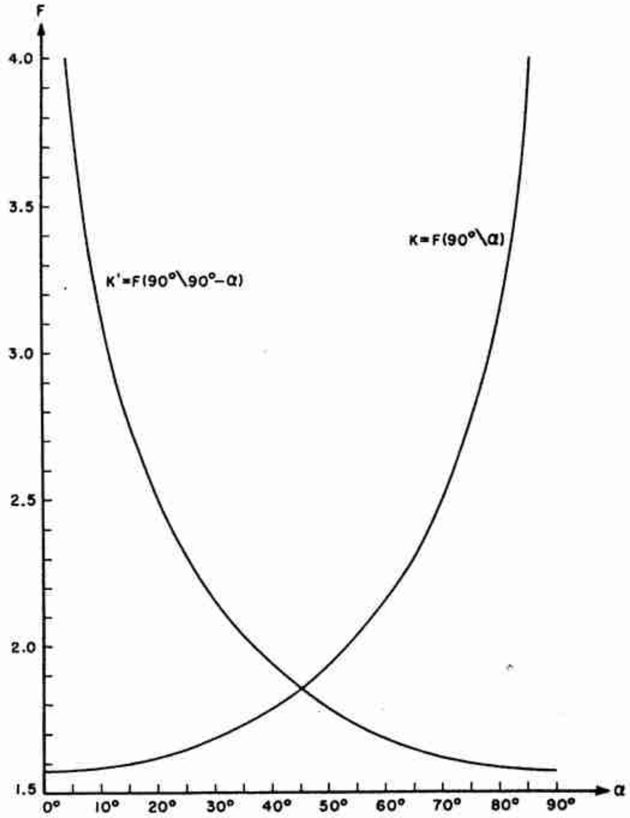

FIGURE 17.1. *Complete elliptic integral of the first kind.*

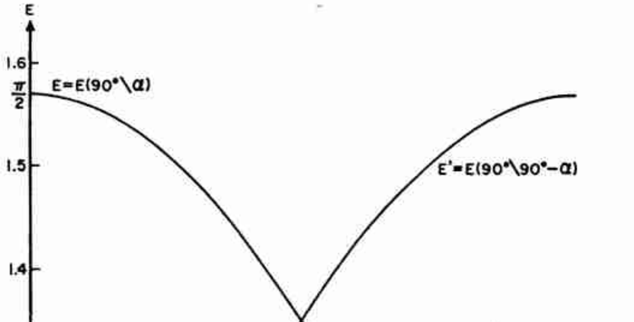

FIGURE 17.2. *Complete elliptic integral of the second kind.*

17.3.35

$$E(m)=[1+a_1m_1+a_2m_1^2]+[b_1m_1+b_2m_1^2]\ln(1/m_1)+\epsilon(m)$$

$$|\epsilon(m)|<4\times10^{-5}$$

$a_1=.46301\ 51$	$b_1=.24527\ 27$
$a_2=.10778\ 12$	$b_2=.04124\ 96$

17.3.36

$$E(m)=[1+a_1m_1+\ldots+a_4m_1^4]+[b_1m_1+\ldots+b_4m_1^4]\ln(1/m_1)+\epsilon(m)$$

$$|\epsilon(m)|<2\times10^{-8}$$

$a_1=.44325\ 141463$	$b_1=.24998\ 368310$
$a_2=.06260\ 601220$	$b_2=.09200\ 180037$
$a_3=.04757\ 383546$	$b_3=.04069\ 697526$
$a_4=.01736\ 506451$	$b_4=.00526\ 449639$

17.4. Incomplete Elliptic Integrals of the First and Second Kinds

Extension of the Tables

Negative Amplitude

17.4.1 $$F(-\varphi|m)=-F(\varphi|m)$$

17.4.2 $$E(-\varphi|m)=-E(\varphi|m)$$

Amplitude of Any Magnitude

17.4.3 $$F(s\pi\pm\varphi|m)=2sK\pm F(\varphi|m)$$

17.4.4 $$E(u+2K)=E(u)+2E$$

17.4.5 $$E(u+2iK')=E(u)+2i(K'-E')$$

17.4.6

$$E(u+2mK+2niK')=E(u)+2mE+2ni(K'-E')$$

17.4.7 $$E(K-u)=E-E(u)+m\,\mathrm{sn}\,u\,\mathrm{cd}\,u$$

Imaginary Amplitude

If $\tan\theta=\sinh\varphi$

17.4.8 $$F(i\varphi\backslash\alpha)=iF(\theta\backslash\tfrac{1}{2}\pi-\alpha)$$

17.4.9

$$E(i\varphi\backslash\alpha)=-iE(\theta\backslash\tfrac{1}{2}\pi-\alpha)+iF(\theta\backslash\tfrac{1}{2}\pi-\alpha)+i\tan\theta(1-\cos^2\alpha\sin^2\theta)^{\frac{1}{2}}$$

Jacobi's Imaginary Transformation

17.4.10

$$E(iu|m)=i[u+\mathrm{dn}(u|m_1)\mathrm{sc}(u|m_1)-E(u|m_1)]$$

Complex Amplitude

17.4.11 $$F(\varphi+i\psi|m)=F(\lambda|m)+iF(\mu|m_1)$$

where $\cot^2 \lambda$ is the positive root of the equation $x^2-[\cot^2 \varphi+m \sinh^2\psi \csc^2\varphi-m_1]x-m_1 \cot^2\varphi=0$ and $m \tan^2 \mu=\tan^2\varphi \cot^2\lambda-1$.

17.4.12

$$E(\varphi+i\psi\backslash\alpha)=E(\lambda\backslash\alpha)-iE(\mu\backslash 90^\circ-\alpha)+iF(\mu\backslash 90^\circ-\alpha)+\frac{b_1+ib_2}{b_3}$$

where

$$b_1=\sin^2 \alpha \sin \lambda \cos \lambda \sin^2 \mu(1-\sin^2 \alpha \sin^2 \lambda)^{\frac{1}{2}}$$

$$b_2=(1-\sin^2 \alpha \sin^2 \lambda)(1-\cos^2 \alpha \sin^2 \mu)^{\frac{1}{2}} \sin \mu \cos \mu$$

$$b_3=\cos^2 \mu+\sin^2 \alpha \sin^2 \lambda \sin^2 \mu$$

Amplitude Near to $\pi/2$ (see also 17.5)

If $\cos \alpha \tan \varphi \tan \psi=1$

17.4.13 $F(\varphi\backslash\alpha)+F(\psi\backslash\alpha)=F(\pi/2\backslash\alpha)=K$

17.4.14

$$E(\varphi\backslash\alpha)+E(\psi\backslash\alpha)=E(\pi/2\backslash\alpha)+\sin^2\alpha \sin\varphi \sin\psi$$

Values when φ is near to $\pi/2$ and m is near to unity can be calculated by these formulae.

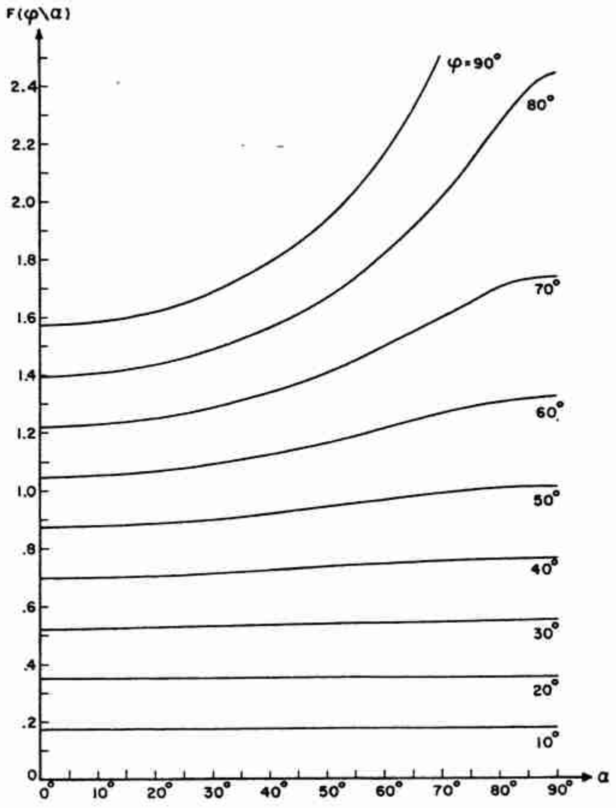

FIGURE 17.3. *Incomplete elliptic integral of the first kind.*

$F(\varphi\backslash\alpha)$, φ constant

Parameter Greater Than Unity

17.4.15 $F(\varphi|m)=m^{-\frac{1}{2}}F(\theta|m^{-1})$, $\sin \theta=m^{\frac{1}{2}} \sin \varphi$

17.4.16 $E(u|m)=m^{\frac{1}{2}}E(um^{\frac{1}{2}}|m^{-1})-(m-1)u$

by which a parameter greater than unity can be replaced by a parameter less than unity.

Negative Parameter

17.4.17

$$F(\varphi|-m)=(1+m)^{-\frac{1}{2}}K(m(1+m)^{-1})-(1+m)^{-\frac{1}{2}}F\left(\frac{\pi}{2}-\varphi \,|\, m(1+m)^{-1}\right)$$

17.4.18

$$E(u|-m)=(1+m)^{\frac{1}{2}}\{E(u(1+m)^{\frac{1}{2}}|m(m+1)^{-1})-m(1+m)^{-\frac{1}{2}}\mathrm{sn}(u(1+m)^{\frac{1}{2}}|m(1+m)^{-1})\,\mathrm{cd}(u(1+m)^{\frac{1}{2}}|m(1+m)^{-1})\}$$

whereby computations can be made for negative parameters, and therefore for pure imaginary modulus.

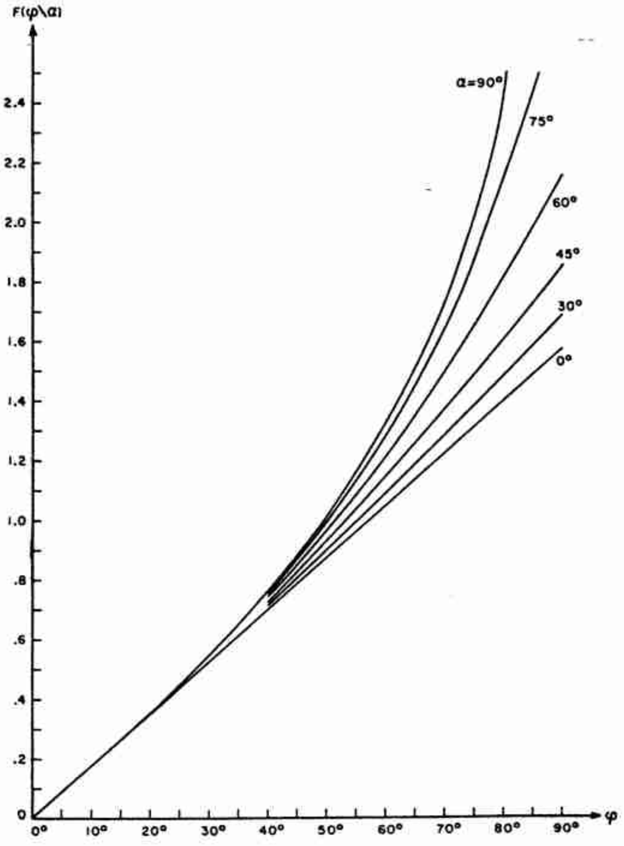

FIGURE 17.4. *Incomplete elliptic integral of the first kind.*

$F(\varphi\backslash\alpha)$, α constant

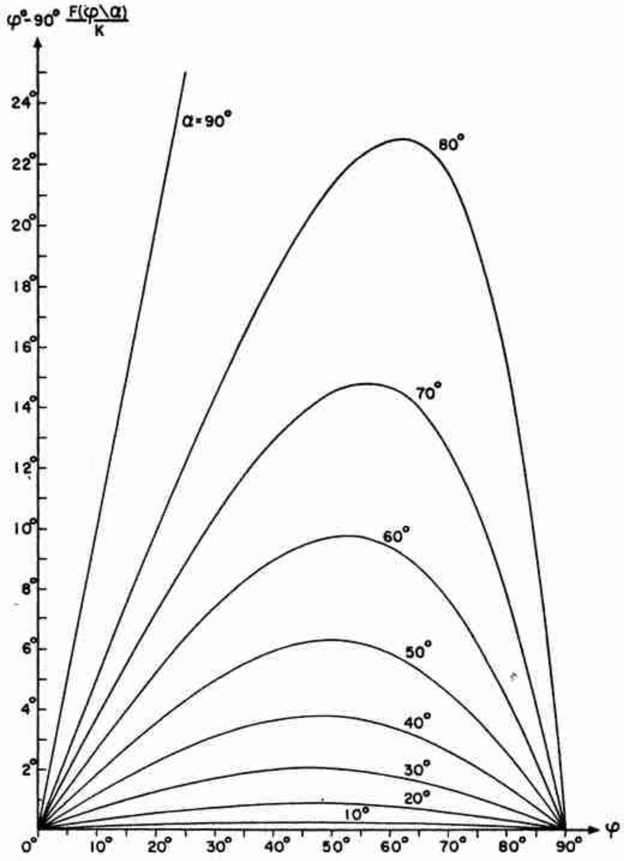

FIGURE 17.5. $\varphi - 90° \frac{F(\varphi\backslash\alpha)}{K}$, *α constant.*

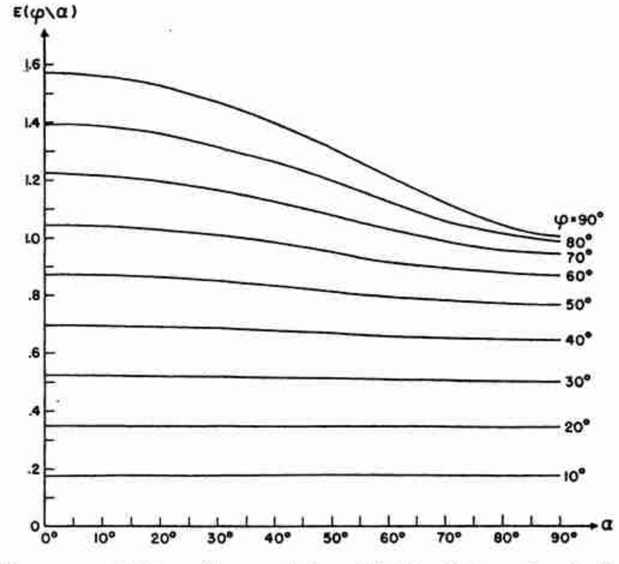

FIGURE 17.6. *Incomplete elliptic integral of the second kind.*

$E(\varphi\backslash\alpha)$, φ constant

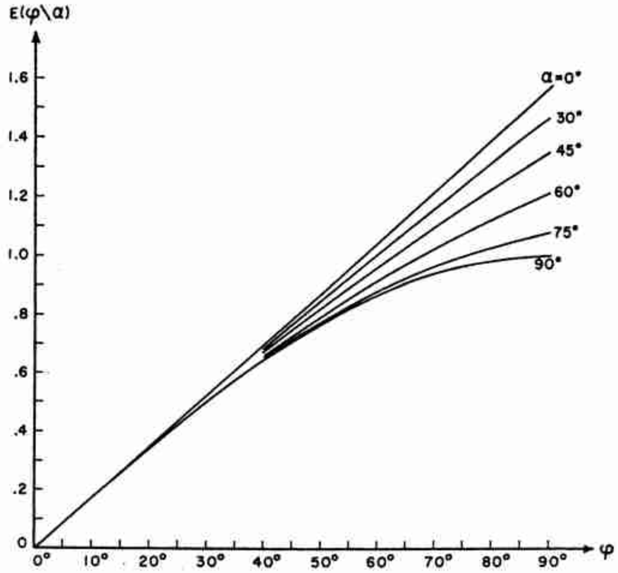

FIGURE 17.7. *Incomplete elliptic integral of the second kind.*

$E(\varphi\backslash\alpha)$, α constant

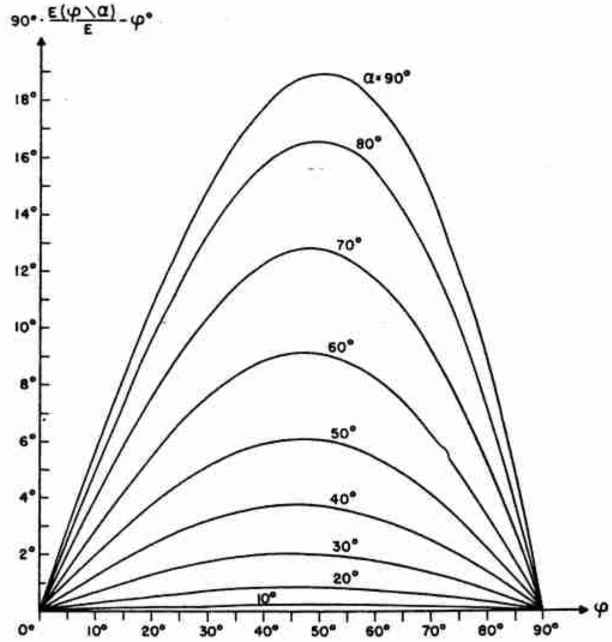

FIGURE 17.8. $90° \frac{E(\varphi\backslash\alpha)}{E} - \varphi$, *α constant.*

Special Cases

17.4.19 $F(\varphi\backslash 0) = \varphi$

17.4.20 $F(i\varphi\backslash 0) = i\varphi$

17.4.21

$$F(\varphi\backslash 90°) = \ln(\sec\varphi + \tan\varphi) = \ln\tan\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)$$

17.4.22 $F(i\varphi\backslash 90°) = i\arctan(\sinh\varphi)$

17.4.23 $E(\varphi\backslash 0) = \varphi$

17.4.24 $E(i\varphi\backslash 0) = i\varphi$

17.4.25 $E(\varphi\backslash 90°) = \sin\varphi$

17.4.26 $E(i\varphi\backslash 90°) = i\sinh\varphi$

Jacobi's Zeta Function

17.4.27 $$Z(\varphi\backslash\alpha)=E(\varphi\backslash\alpha)-E(\alpha)F(\varphi\backslash\alpha)/K(\alpha)$$

17.4.28 $$Z(u|m)=Z(u)=E(u)-uE(m)/K(m)$$

17.4.29 $$Z(-u)=-Z(u)$$

17.4.30 $$Z(u+2K)=Z(u)$$

17.4.31 $$Z(K-u)=-Z(K+u)$$

17.4.32 $$Z(u)=Z(u-K)-m\operatorname{sn}(u-K)\operatorname{cd}(u-K)$$

Special Values

17.4.33 $$Z(u|0)=0$$

17.4.34 $$Z(u|1)=\tanh u$$

Addition Theorem

17.4.35

$$Z(u+v)=Z(u)+Z(v)-m\operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u+v)$$

Jacobi's Imaginary Transformation

17.4.36

$$iZ(iu|m)=Z(u|m_1)+\frac{\pi u}{2KK'}-\operatorname{dn}(u|m_1)\operatorname{sc}(u|m_1)$$

Relation to Jacobi's Theta Function

17.4.37 $$Z(u)=\Theta'(u)/\Theta(u)=\frac{d}{du}\ln\Theta(u)$$

q-Series

17.4.38 $$Z(u)=\frac{2\pi}{K}\sum_{s=1}^{\infty}q^s(1-q^{2s})^{-1}\sin(\pi su/K)$$

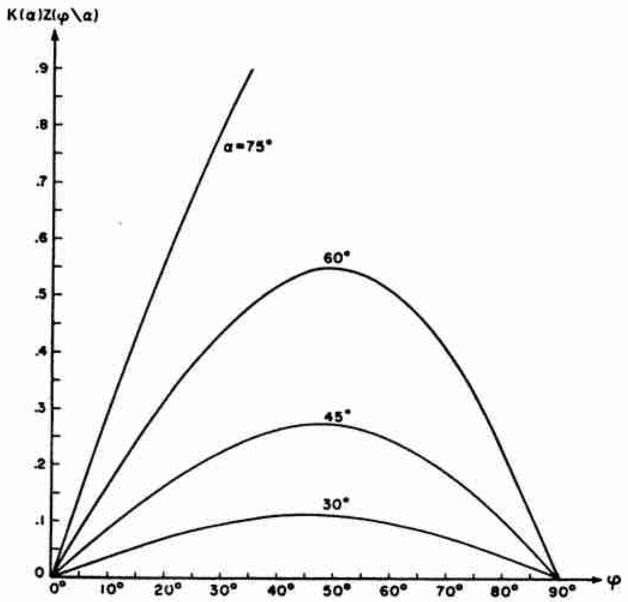

FIGURE 17.9. *Jacobian zeta function* $K(\alpha)Z(\varphi\backslash\alpha)$.

Heuman's Lambda Function

17.4.39

$$\Lambda_0(\varphi\backslash\alpha)=\frac{F(\varphi\backslash 90^\circ-\alpha)}{K'(\alpha)}+\frac{2}{\pi}K(\alpha)Z(\varphi\backslash 90^\circ-\alpha)$$

17.4.40 $$=\frac{2}{\pi}\{K(\alpha)E(\varphi\backslash 90^\circ-\alpha)$$
$$-[K(\alpha)-E(\alpha)]F(\varphi\backslash 90^\circ-\alpha)\}$$

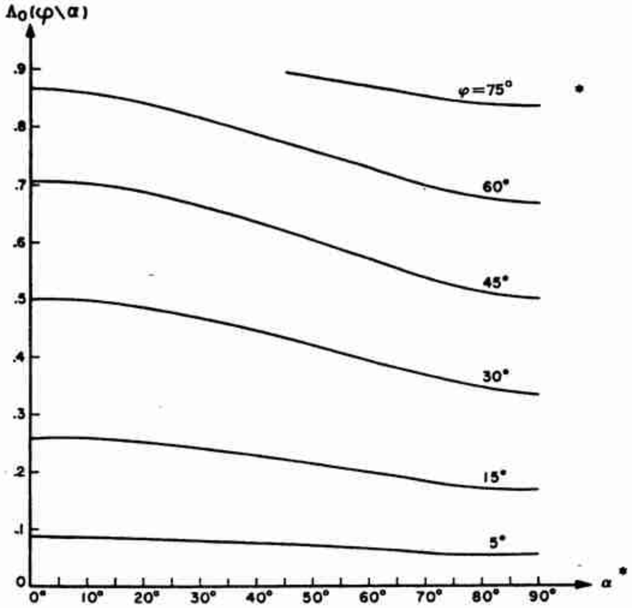

FIGURE 17.10. *Heuman's lambda function* $\Lambda_0(\varphi\backslash\alpha)$.

Numerical Evaluation of Incomplete Integrals of the First and Second Kinds

For the numerical evaluation of an elliptic integral the quartic (or cubic [4]) under the radical should first be expressed in terms of t^2, see **Examples 1 and 2.** In the resulting quartic there are only six possible sign patterns or combinations of the factors namely

$$(t^2+a^2)(t^2+b^2),\quad (a^2-t^2)(t^2-b^2),$$
$$(a^2-t^2)(b^2-t^2),\quad (t^2-a^2)(t^2-b^2),\quad (t^2+a^2)(t^2-b^2),$$
$$(t^2+a^2)(b^2-t^2).$$

The list which follows is then exhaustive for integrals which reduce to $F(\varphi\backslash\alpha)$ or $E(\varphi\backslash\alpha)$.

The value of the elliptic integral of the first kind is also expressed as an *inverse* Jacobian elliptic function. Here, for example, the notation $u=\operatorname{sn}^{-1}x$ means that $x=\operatorname{sn} u$.

The column headed "t substitution" gives the Jacobian elliptic function substitution which is appropriate to reduce every elliptic integral which contains the given quartic.

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*See page II.

[4] For an alternate treatment of cubics see **17.4.61** and **17.4.70.**

	$F(\varphi\backslash\alpha)$	Equivalent Inverse Jacobian Elliptic Function	φ	t Substitution	$E(\varphi\backslash\alpha)$
$\cos\alpha=b/a$ $a>b$ $m=(a^2-b^2)/a^2$	**17.4.41** $a\int_0^x\frac{dt}{[(t^2+a^2)(t^2+b^2)]^{1/2}}$	$\mathrm{sc}^{-1}\left(\frac{x}{b}\middle\vert\frac{a^2-b^2}{a^2}\right)$	$\tan\varphi=\frac{x}{b}$	$t=b\ \mathrm{sc}\ v$	$\frac{b^2}{a}\int_0^x\left(\frac{t^2+a^2}{t^2+b^2}\right)\frac{dt}{[(t^2+a^2)(t^2+b^2)]^{1/2}}$
	17.4.42 $a\int_x^\infty\frac{dt}{[(t^2+a^2)(t^2+b^2)]^{1/2}}$	$\mathrm{cs}^{-1}\left(\frac{x}{a}\middle\vert\frac{a^2-b^2}{a^2}\right)$	$\tan\varphi=\frac{a}{x}$	$t=a\ \mathrm{cs}\ v$	$a\int_x^\infty\left(\frac{t^2+b^2}{t^2+a^2}\right)\frac{dt}{[(t^2+a^2)(t^2+b^2)]^{1/2}}$
	17.4.43 $a\int_b^x\frac{dt}{[(a^2-t^2)(t^2-b^2)]^{1/2}}$	$\mathrm{nd}^{-1}\left(\frac{x}{b}\middle\vert\frac{a^2-b^2}{a^2}\right)$	$\sin^2\varphi=\frac{a^2(x^2-b^2)}{x^2(a^2-b^2)}$	$t=b\ \mathrm{nd}\ v$	$ab^2\int_b^x\frac{1}{t^2}\frac{dt}{[(a^2-t^2)(t^2-b^2)]^{1/2}}$
	17.4.44 $a\int_x^a\frac{dt}{[(a^2-t^2)(t^2-b^2)]^{1/2}}$	$\mathrm{dn}^{-1}\left(\frac{x}{a}\middle\vert\frac{a^2-b^2}{a^2}\right)$	$\sin^2\varphi=\frac{a^2-x^2}{a^2-b^2}$	$t=a\ \mathrm{dn}\ v$	$\frac{1}{a}\int_x^a\frac{t^2dt}{[(a^2-t^2)(t^2-b^2)]^{1/2}}$
$\sin\alpha=b/a$ $a>b$ $m=b^2/a^2$	**17.4.45** $a\int_0^x\frac{dt}{[(a^2-t^2)(b^2-t^2)]^{1/2}}$	$\mathrm{sn}^{-1}\left(\frac{x}{b}\middle\vert\frac{b^2}{a^2}\right)$	$\sin\varphi=\frac{x}{b}$	$t=b\ \mathrm{sn}\ v$	$\frac{1}{a}\int_0^x\frac{(a^2-t^2)dt}{[(a^2-t^2)(b^2-t^2)]^{1/2}}$
	17.4.46 $a\int_x^b\frac{dt}{[(a^2-t^2)(b^2-t^2)]^{1/2}}$	$\mathrm{cd}^{-1}\left(\frac{x}{b}\middle\vert\frac{b^2}{a^2}\right)$	$\sin^2\varphi=\frac{a^2(b^2-x^2)}{b^2(a^2-x^2)}$	$t=b\ \mathrm{cd}\ v$	$a(a^2-b^2)\int_x^b\left(\frac{1}{a^2-t^2}\right)\frac{dt}{[(a^2-t^2)(b^2-t^2)]^{1/2}}$
	17.4.47 $a\int_a^x\frac{dt}{[(t^2-a^2)(t^2-b^2)]^{1/2}}$	$\mathrm{dc}^{-1}\left(\frac{x}{a}\middle\vert\frac{b^2}{a^2}\right)$	$\sin^2\varphi=\frac{x^2-a^2}{x^2-b^2}$	$t=a\ \mathrm{dc}\ v$	$\frac{a^2-b^2}{a}\int_a^x\left(\frac{t^2}{t^2-b^2}\right)\frac{dt}{[(t^2-a^2)(t^2-b^2)]^{1/2}}$
	17.4.48 $a\int_x^\infty\frac{dt}{[(t^2-a^2)(t^2-b^2)]^{1/2}}$	$\mathrm{ns}^{-1}\left(\frac{x}{a}\middle\vert\frac{b^2}{a^2}\right)$	$\sin\varphi=\frac{a}{x}$	$t=a\ \mathrm{ns}\ v$	$a\int_x^\infty\left(\frac{t^2-b^2}{t^2}\right)\frac{dt}{[(t^2-a^2)(t^2-b^2)]^{1/2}}$
$\cot\alpha=\frac{b}{a}$ $m=a^2/(a^2+b^2)$	**17.4.49** $(a^2+b^2)^{1/2}\int_b^x\frac{dt}{[(t^2+a^2)(t^2-b^2)]^{1/2}}$	$\mathrm{nc}^{-1}\left(\frac{x}{b}\middle\vert\frac{a^2}{a^2+b^2}\right)$	$\cos\varphi=\frac{b}{x}$	$t=b\ \mathrm{nc}\ v$	$\frac{b^2}{(a^2+b^2)^{1/2}}\int_b^x\frac{t^2+a^2}{t^2}\frac{dt}{[(t^2+a^2)(t^2-b^2)]^{1/2}}$
	17.4.50 $(a^2+b^2)^{1/2}\int_x^\infty\frac{dt}{[(t^2+a^2)(t^2-b^2)]^{1/2}}$	$\mathrm{ds}^{-1}\left(\frac{x}{(a^2+b^2)^{1/2}}\middle\vert\frac{a^2}{a^2+b^2}\right)$	$\sin^2\varphi=\frac{a^2+b^2}{a^2+x^2}$	$t=(a^2+b^2)^{1/2}\,\mathrm{ds}\,v$	$(a^2+b^2)^{1/2}\int_x^\infty\frac{t^2}{(t^2+a^2)}\frac{dt}{[(t^2+a^2)(t^2-b^2)]^{1/2}}$
$\tan\alpha=\frac{b}{a}$ $m=b^2/(a^2+b^2)$	**17.4.51** $(a^2+b^2)^{1/2}\int_0^x\frac{dt}{[(t^2+a^2)(b^2-t^2)]^{1/2}}$	$\mathrm{sd}^{-1}\left(\frac{x(a^2+b^2)^{1/2}}{ab}\middle\vert\frac{b^2}{a^2+b^2}\right)$	$\sin^2\varphi=\frac{x^2(a^2+b^2)}{b^2(a^2+x^2)}$	$t=\frac{ab}{(a^2+b^2)^{1/2}}\,\mathrm{sd}\,v$	$a^2(a^2+b^2)^{1/2}\int_0^x\frac{1}{(t^2+a^2)}\frac{dt}{[(t^2+a^2)(b^2-t^2)]^{1/2}}$
	17.4.52 $(a^2+b^2)^{1/2}\int_x^b\frac{dt}{[(t^2+a^2)(b^2-t^2)]^{1/2}}$	$\mathrm{cn}^{-1}\left(\frac{x}{b}\middle\vert\frac{b^2}{a^2+b^2}\right)$	$\cos\varphi=\frac{x}{b}$	$t=b\ \mathrm{cn}\ v$	$\frac{1}{(a^2+b^2)^{1/2}}\int_x^b\frac{(t^2+a^2)dt}{[(t^2+a^2)(b^2-t^2)]^{1/2}}$

Some Important Special Cases

$\frac{1}{2}F(\varphi\backslash\alpha)$	$\cos\varphi$	α
17.4.53 $\int_x^{\infty}\frac{dt}{(1+t^4)^{\frac{1}{2}}}$	$\frac{x^2-1}{x^2+1}$	45°
17.4.54 $\int_0^x\frac{dt}{(1+t^4)^{\frac{1}{2}}}$	$\frac{1-x^2}{1+x^2}$	45°
17.4.55 * $\frac{1}{2^{\frac{1}{2}}}\int_1^x\frac{dt}{(t^4-1)^{\frac{1}{2}}}$	$\frac{1}{x}$	45°
17.4.56 * $\frac{1}{2^{\frac{1}{2}}}\int_x^1\frac{dt}{(1-t^4)^{\frac{1}{2}}}$	x	45°

$\frac{1}{3^{1/4}}F(\varphi\backslash\alpha)$	$\cos\varphi$	α
17.4.57 $\int_x^{\infty}\frac{dt}{(t^3-1)^{\frac{1}{2}}}$	$\frac{x-1-\sqrt{3}}{x-1+\sqrt{3}}$	15°
17.4.58 $\int_1^x\frac{dt}{(t^3-1)^{\frac{1}{2}}}$	$\frac{\sqrt{3}+1-x}{\sqrt{3}-1+x}$	15°
17.4.59 $\int_x^1\frac{dt}{(1-t^3)^{\frac{1}{2}}}$	$\frac{\sqrt{3}-1+x}{\sqrt{3}+1-x}$	75°
17.4.60 $\int_{-\infty}^x\frac{dt}{(1-t^3)^{\frac{1}{2}}}$	$\frac{1-\sqrt{3}-x}{1+\sqrt{3}-x}$	75°

Reduction of $\int dt/\sqrt{P}$ where $P=P(t)$ is a cubic polynomial with three real factors $P=(t-\beta_1)(t-\beta_2)(t-\beta_3)$ where $\beta_1>\beta_2>\beta_3$. Write

17.4.61

$$\lambda=\frac{1}{2}(\beta_1-\beta_3)^{1/2},\ m=\sin^2\alpha=\frac{\beta_2-\beta_3}{\beta_1-\beta_3},$$

$$m_1=\cos^2\alpha=\frac{\beta_1-\beta_2}{\beta_1-\beta_3}$$

17.4.62 $\lambda\int_{\beta_3}^x\frac{dt}{\sqrt{P}}$	$F(\varphi\backslash\alpha)$	$\sin^2\varphi=\frac{x-\beta_3}{\beta_2-\beta_3}$
17.4.63 $\lambda\int_x^{\beta_2}\frac{dt}{\sqrt{P}}$	$F(\varphi\backslash\alpha)$	$\cos^2\varphi=\frac{(\beta_1-\beta_2)(x-\beta_3)}{(\beta_2-\beta_3)(\beta_1-x)}$
17.4.64 $\lambda\int_{\beta_1}^x\frac{dt}{\sqrt{P}}$	$F(\varphi\backslash\alpha)$	$\sin^2\varphi=\frac{x-\beta_1}{x-\beta_2}$
17.4.65 $\lambda\int_x^{\infty}\frac{dt}{\sqrt{P}}$	$F(\varphi\backslash\alpha)$	$\cos^2\varphi=\frac{x-\beta_1}{x-\beta_3}$
17.4.66 $\lambda\int_{-\infty}^x\frac{dt}{\sqrt{-P}}$	$F(\varphi\backslash(90°-\alpha°))$	$\sin^2\varphi=\frac{\beta_1-\beta_3}{\beta_1-x}$
17.4.67 $\lambda\int_x^{\beta_3}\frac{dt}{\sqrt{-P}}$	$F(\varphi\backslash(90°-\alpha°))$	$\cos^2\varphi=\frac{\beta_2-\beta_3}{\beta_2-x}$
17.4.68 $\lambda\int_{\beta_2}^x\frac{dt}{\sqrt{-P}}$	$F(\varphi\backslash(90°-\alpha°))$	$\sin^2\varphi=\frac{(\beta_1-\beta_3)(x-\beta_2)}{(\beta_1-\beta_2)(x-\beta_3)}$
17.4.69 $\lambda\int_x^{\beta_1}\frac{dt}{\sqrt{-P}}$	$F(\varphi\backslash(90°-\alpha°))$	$\cos^2\varphi=\frac{x-\beta_2}{\beta_1-\beta_2}$

Reduction of $\int dt/\sqrt{P}$ when $P=P(t)=t^3+a_1t^2+a_2t+a_3$ is a cubic polynomial with only one real root $t=\beta$. We form the first and second derivatives $P'(t)$, $P''(t)$ with respect to t and then write

17.4.70 $\lambda^2=[P'(\beta)]^{1/2},\ m=\sin^2\alpha=\frac{1}{2}-\frac{1}{8}\frac{P''(\beta)}{[P'(\beta)]^{1/2}}$

17.4.71 $\lambda\int_{\beta}^x\frac{dt}{\sqrt{P}}$	$F(\varphi\backslash\alpha)$	$\cos\varphi=\frac{\lambda^2-(x-\beta)}{\lambda^2+(x-\beta)}$
17.4.72 $\lambda\int_x^{\infty}\frac{dt}{\sqrt{P}}$	$F(\varphi\backslash\alpha)$	$\cos\varphi=\frac{(x-\beta)-\lambda^2}{(x-\beta)+\lambda^2}$
17.4.73 $\lambda\int_{-\infty}^x\frac{dt}{\sqrt{(-P)}}$	$F(\varphi\backslash(90°-\alpha°))$	$\cos\varphi=\frac{(\beta-x)-\lambda^2}{(\beta-x)+\lambda^2}$
17.4.74 $\lambda\int_x^{\beta}\frac{dt}{\sqrt{(-P)}}$	$F(\varphi\backslash(90°-\alpha°))$	$\cos\varphi=\frac{\lambda^2-(\beta-x)}{\lambda^2+(\beta-x)}$

17.5. Landen's Transformation

Descending Landen Transformation [5]

Let α_n, α_{n+1} be two modular angles such that

17.5.1 $(1+\sin\alpha_{n+1})(1+\cos\alpha_n)=2 \qquad (\alpha_{n+1}<\alpha_n)$

and let φ_n, φ_{n+1} be two corresponding amplitudes such that

17.5.2 $\tan(\varphi_{n+1}-\varphi_n)=\cos\alpha_n\tan\varphi_n \quad (\varphi_{n+1}>\varphi_n)$

---

*See page II.

[5] The emphasis here is on the modular angle since this is an argument of the Tables. All formulae concerning Landen's transformation may also be expressed in terms of the modulus $k=m^{\frac{1}{2}}=\sin\alpha$ and its complement $k'=m_1^{\frac{1}{2}}=\cos\alpha$.

Thus the step from n to $n+1$ decreases the modular angle but increases the amplitude. By iterating the process we can descend from a given modular angle to one whose magnitude is negligible, when **17.4.19** becomes applicable.

With $\alpha_0=\alpha$ we have

17.5.3

$$F(\varphi\backslash\alpha)=(1+\cos\,\alpha)^{-1}F(\varphi_1\backslash\alpha_1)$$
$$=\tfrac{1}{2}(1+\sin\,\alpha_1)F(\varphi_1\backslash\alpha_1)$$

17.5.4 $$F(\varphi\backslash\alpha)=2^{-n}\prod_{s=1}^{n}(1+\sin\alpha_s)F(\varphi_n\backslash\alpha_n)$$

17.5.5 $$F(\varphi\backslash\alpha)=\Phi\prod_{s=1}^{\infty}(1+\sin\,\alpha_s)$$

17.5.6 $$\Phi=\lim_{n\to\infty}\frac{1}{2^n}F(\varphi_n\backslash\alpha_n)=\lim_{n\to\infty}\frac{\varphi_n}{2^n}$$

17.5.7 $$K=F(\tfrac{1}{2}\pi\backslash\alpha)=\tfrac{1}{2}\pi\prod_{s=1}^{\infty}(1+\sin\,\alpha_s)$$

17.5.8 $$F(\varphi\backslash\alpha)=2\pi^{-1}K\Phi$$

17.5.9

$$E(\varphi\backslash\alpha)=F(\varphi\backslash\alpha)\left[1-\frac{1}{2}\sin^2\alpha\left(1+\frac{1}{2}\sin\,\alpha_1\right.\right.$$
$$\left.\left.+\frac{1}{2^2}\sin\,\alpha_1\sin\,\alpha_2+\ldots\right)\right]+\sin\,\alpha\left[\frac{1}{2}(\sin\,\alpha_1)^{1/2}\sin\,\varphi_1\right.$$
$$\left.+\frac{1}{2^2}(\sin\,\alpha_1\sin\,\alpha_2)^{1/2}\sin\,\varphi_2+\ldots\right]$$

17.5.10

$$E=K\left[1-\frac{1}{2}\sin^2\alpha\left(1+\frac{1}{2}\sin\,\alpha_1+\frac{1}{2^2}\sin\,\alpha_1\sin\,\alpha_2\right.\right.$$
$$\left.\left.+\frac{1}{2^3}\sin\,\alpha_1\sin\,\alpha_2\sin\,\alpha_3+\ldots\right)\right]$$

Ascending Landen Transformation

Let α_n, α_{n+1} be two modular angles such that

17.5.11 $$(1+\sin\,\alpha_n)(1+\cos\,\alpha_{n+1})=2\qquad(\alpha_{n+1}>\alpha_n)$$

and let φ_n, φ_{n+1} be two corresponding amplitudes such that

17.5.12 $$\sin\,(2\varphi_{n+1}-\varphi_n)=\sin\,\alpha_n\sin\,\varphi_n\qquad(\varphi_{n+1}<\varphi_n)$$

Thus the step from n to $n+1$ increases the modular angle but decreases the amplitude. By iterating the process we can ascend from a given modular angle to one whose difference from a right angle is so small that **17.4.21** becomes applicable.

With $\alpha_0=\alpha$ we have

17.5.13 $$F(\varphi\backslash\alpha)=2(1+\sin\,\alpha)^{-1}F(\varphi_1\backslash\alpha_1)$$

17.5.14 $$F(\varphi\backslash\alpha)=2^n\prod_{s=0}^{n=1}(1+\sin\,\alpha_s)^{-1}F(\varphi_n\backslash\alpha_n)$$

17.5.15 $$F(\varphi\backslash\alpha)=\prod_{s=1}^{n}(1+\cos\,\alpha_s)F(\varphi_n\backslash\alpha_n)$$

17.5.16 $$F(\varphi\backslash\alpha)=[\csc\,\alpha\prod_{s=1}^{\infty}\sin\,\alpha_s]^{\frac{1}{2}}\ln\,\tan\,(\tfrac{1}{4}\pi+\tfrac{1}{2}\Phi)$$

17.5.17 $$\Phi=\lim_{n\to\infty}\varphi_n$$

Neighborhood of a Right Angle (see also 17.4.13)

When both φ and α are near to a right angle, interpolation in the table $F(\varphi\backslash\alpha)$ is difficult. Either Landen's transformation can then be used with advantage to increase the modular angle and decrease the amplitude or vice-versa.

17.6. The Process of the Arithmetic-Geometric Mean

Starting with a given number triple $(a_0,\,b_0,\,c_0)$ we proceed to determine number triples $(a_1,\,b_1,\,c_1,)$, $(a_2,\,b_2,\,c_2)$, . . ., $(a_N,\,b_N,\,c_N)$ according to the following scheme of arithmetic and geometric means

17.6.1

$$\begin{array}{lll}
a_0 & b_0 & \\
a_1=\frac{1}{2}(a_0+b_0) & b_1=(a_0b_0)^{\frac{1}{2}} & \\
a_2=\frac{1}{2}(a_1+b_1) & b_2=(a_1b_1)^{\frac{1}{2}} & \\
\vdots & \vdots & \\
a_N=\frac{1}{2}(a_{N-1}+b_{N-1}) & b_N=(a_{N-1}b_{N-1})^{\frac{1}{2}} & \\
 & & c_0 \\
 & & c_1=\frac{1}{2}(a_0-b_0) \\
 & & c_2=\frac{1}{2}(a_1-b_1) \\
 & & \vdots \\
 & & c_N=\frac{1}{2}(a_{N-1}-b_{N-1}).
\end{array}$$

We stop at the Nth step when $a_N=b_N$, i.e., when $c_N=0$ to the degree of accuracy to which the numbers are required.

To determine the complete elliptic integrals $K(\alpha)$, $E(\alpha)$ we start with

17.6.2 $$a_0=1,\ b_0=\cos\,\alpha,\ c_0=\sin\,\alpha$$

whence

17.6.3 $$K(\alpha)=\frac{\pi}{2a_N}$$

17.6.4 $$\frac{K(\alpha)-E(\alpha)}{K(\alpha)}=\frac{1}{2}\,[c_0^2+2c_1^2+2^2c_2^2+\ .\ .\ .\ +2^Nc_N^2]$$

To determine $K'(\alpha)$, $E'(\alpha)$ we start with

17.6.5 $$a_0'=1,\ b_0'=\sin\alpha,\ c_0'=\cos\alpha$$

whence

17.6.6 $$K'(\alpha)=\frac{\pi}{2a_N'}$$

17.6.7

$$\frac{K'(\alpha)-E'(\alpha)}{K'(\alpha)}=\frac{1}{2}\,[c_0'^2+2c_1'^2+2^2c_2'^2+\ .\ .\ .\ +2^Nc_N'^2]$$

To calculate $F(\varphi\backslash\alpha)$, $E(\varphi\backslash\alpha)$ start from **17.5.2** which corresponds to the descending Landen transformation and determine $\varphi_1,\ \varphi_2,\ .\ .\ .\ ,\ \varphi_N$ successively from the relation

17.6.8 $$\tan\,(\varphi_{n+1}-\varphi_n)=(b_n/a_n)\tan\varphi_n,\ \varphi_0=\varphi$$

Then to the prescribed accuracy

17.6.9 $$F(\varphi\backslash\alpha)=\varphi_N/(2^Na_N)\quad *$$

17.6.10

$$Z(\varphi\backslash\alpha)=E(\varphi\backslash\alpha)-(E/K)F(\varphi\backslash\alpha)$$

$$*\qquad =c_1\sin\varphi_1+c_2\sin\varphi_2+\ .\ .\ .\ +c_N\sin\varphi_N$$

17.7. Elliptic Integrals of the Third Kind

17.7.1

$$\Pi(n;\,\varphi\backslash\alpha)=\int_0^\varphi(1-n\sin^2\theta)^{-1}(1-\sin^2\alpha\sin^2\theta)^{-\frac{1}{2}}d\theta$$

17.7.2 $$\Pi\,(n;\tfrac{1}{2}\pi\backslash\alpha)=\Pi\,(n\backslash\alpha)$$

Case (i) Hyperbolic Case $0<n<\sin^2\alpha$

$$\epsilon=\arcsin\,(n/\sin^2\alpha)^{\frac{1}{2}},\qquad 0\leq\epsilon\leq\tfrac{1}{2}\pi$$

$$\beta=\tfrac{1}{2}\pi F(\epsilon\backslash\alpha)/K(\alpha)$$

$$q=q(\alpha)$$

$$v=\tfrac{1}{2}\pi F(\varphi\backslash\alpha)/K(\alpha),$$

$$\delta_1=[n(1-n)^{-1}(\sin^2\alpha-n)^{-1}]^{\frac{1}{2}}$$

17.7.3

$$\Pi(n;\varphi\backslash\alpha)=\delta_1\,[-\tfrac{1}{2}\ln\,[\vartheta_4(v+\beta)/\vartheta_4(v-\beta)]$$

$$+v\vartheta_1'(\beta)/\vartheta_1(\beta)]$$

17.7.4

$$\tfrac{1}{2}\ln\frac{\vartheta_4(v+\beta)}{\vartheta_4(v-\beta)}=2\sum_{s=1}^\infty s^{-1}q^s(1-q^{2s})^{-1}\sin 2sv\sin 2s\beta$$

17.7.5

$$\frac{\vartheta_1'(\beta)}{\vartheta_1(\beta)}=\cot\beta+4\sum_{s=1}^\infty q^{2s}(1-2q^{2s}\cos 2\beta+q^{4s})^{-1}\sin 2\beta$$

In the above we can also use Neville's theta functions **16.36.**

17.7.6 $$\Pi(n\backslash\alpha)=K(\alpha)+\delta_1K(\alpha)Z(\epsilon\backslash\alpha)$$

Case (ii) Hyperbolic Case $n>1$

The case $n>1$ can be reduced to the case $0<N<\sin^2\alpha$ by writing

17.7.7 $$N=n^{-1}\sin^2\alpha,\ p_1=[(n-1)(1-n^{-1}\sin^2\alpha)]^{\frac{1}{2}}$$

17.7.8

$$\Pi(n;\,\varphi\backslash\alpha)=-\Pi(N;\,\varphi\backslash\alpha)+F(\varphi\backslash\alpha)$$

$$+\frac{1}{2p_1}\ln\,[(\Delta(\varphi)+p_1\tan\varphi)(\Delta(\varphi)-p_1\tan\varphi)^{-1}]$$

where $\Delta(\varphi)$ is the delta amplitude, **17.2.4.**

17.7.9 $$\Pi(n\backslash\alpha)=K(\alpha)-\Pi(N\backslash\alpha)$$

Case (iii) Circular Case $\sin^2\alpha<n<1$

$$\epsilon=\arcsin\,[(1-n)/\cos^2\alpha]^{\frac{1}{2}}\qquad 0\leq\epsilon\leq\tfrac{1}{2}\pi$$

$$\beta=\tfrac{1}{2}\pi F(\epsilon\backslash 90^\circ-\alpha)/K(\alpha)$$

$$q=q(\alpha)$$

17.7.10

$$v=\tfrac{1}{2}\pi F(\varphi\backslash\alpha)/K(\alpha),\ \delta_2=[n(1-n)^{-1}(n-\sin^2\alpha)^{-1}]^{\frac{1}{2}}$$

17.7.11 $$\Pi(n;\,\varphi\backslash\alpha)=\delta_2(\lambda-4\mu v)$$

17.7.12

$$\lambda=\arctan\,(\tanh\beta\tan v)$$

$$+2\sum_{s=1}^\infty(-1)^{s-1}s^{-1}q^{2s}(1-q^{2s})^{-1}\sin 2sv\sinh 2s\beta$$

17.7.13

$$\mu=\left[\sum_{s=1}^\infty sq^{s^2}\sinh 2s\beta\right]\left[1+2\sum_{s=1}^\infty q^{s^2}\cosh 2s\beta\right]^{-1}$$

17.7.14 $$\Pi(n\backslash\alpha)=K(\alpha)+\tfrac{1}{2}\pi\delta_2[1-\Lambda_0(\epsilon\backslash\alpha)]$$

where Λ_0 is Heuman's Lambda function, **17.4.39.**

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*See page II.

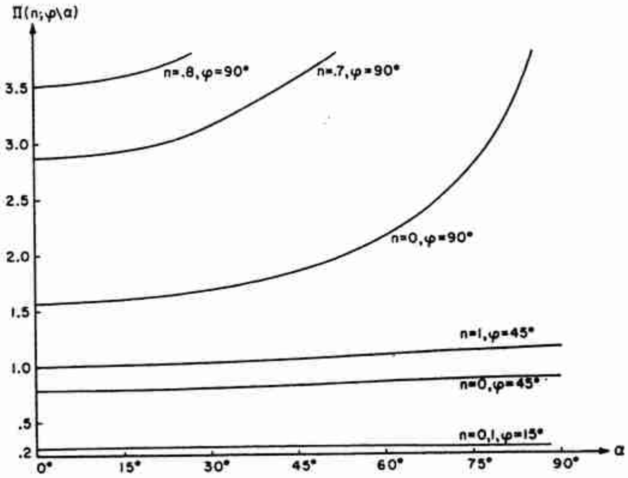

FIGURE 17.11. *Elliptic integral of the third kind* $\Pi(n;\ \varphi\backslash\alpha)$.

Case (iv) Circular Case $n<0$

The case $n<0$ can be reduced to the case $\sin^2\alpha<N<1$ by writing

17.7.15

$$N=(\sin^2\alpha-n)(1-n)^{-1}$$

$$p_2=[-n(1-n)^{-1}(\sin^2\alpha-n)]^{\frac{1}{2}}$$

17.7.16

$$[(1-n)(1-n^{-1}\sin^2\alpha)]^{\frac{1}{2}}\Pi(n;\ \varphi\backslash\alpha)$$
$$=[(1-N)(1-N^{-1}\sin^2\alpha)]^{\frac{1}{2}}\Pi(N;\ \varphi\backslash\alpha)$$
$$+p_2^{-1}\sin^2\alpha F(\varphi\backslash\alpha)+\arctan\left[\tfrac{1}{2}p_2\sin 2\varphi/\Delta(\varphi)\right]$$

17.7.17

$$\Pi(n\backslash\alpha)=(-n\cos^2\alpha)(1-n)^{-1}(\sin^2\alpha-n)^{-1}\Pi(N\backslash\alpha)$$
$$+\sin^2\alpha(\sin^2\alpha-n)^{-1}K(\alpha)$$

Special Cases

17.7.18 $\qquad n=0$

$$\Pi(0;\ \varphi\backslash\alpha)=F(\varphi\backslash\alpha)$$

17.7.19 $\qquad n=0,\ \alpha=0$

$$\Pi(0;\ \varphi\backslash 0)=\varphi$$

17.7.20 $\qquad \alpha=0$

$$\Pi(n;\ \varphi\backslash 0)=(1-n)^{-\frac{1}{2}}\arctan\left[(1-n)^{\frac{1}{2}}\tan\varphi\right],\quad * \quad n<1$$
$$=(n-1)^{-\frac{1}{2}}\operatorname{arctanh}\left[(n-1)^{\frac{1}{2}}\tan\varphi\right],\quad n>1$$
$$=\tan\varphi \qquad n=1$$

17.7.21 $\qquad \alpha=\pi/2$

$$\Pi(n;\ \varphi\backslash\pi/2)=(1-n)^{-1}[\ln(\tan\varphi+\sec\varphi)$$
$$-\tfrac{1}{2}n^{\frac{1}{2}}\ln(1+n^{\frac{1}{2}}\sin\varphi)(1-n^{\frac{1}{2}}\sin\varphi)^{-1}]\qquad n\neq 1$$

17.7.22 $\qquad n=\pm\sin\alpha$

$$(1\mp\sin\alpha)\{2\Pi(\pm\sin\alpha;\ \varphi\backslash\alpha)-F(\varphi\backslash\alpha)\}$$
$$=\arctan\left[(1\mp\sin\alpha)\tan\varphi/\Delta(\varphi)\right]$$

17.7.23 $\qquad n=1\pm\cos\alpha$

$$2\cos\alpha\Pi(1\pm\cos\alpha;\ \varphi\backslash\alpha)=\pm\tfrac{1}{2}\ln[(1+\tan\varphi$$
$$\cdot\Delta(\varphi))(1-\tan\varphi\cdot\Delta(\varphi))^{-1}]+\tfrac{1}{2}\ln[(\Delta(\varphi)$$
$$+\cos\alpha\cdot\tan\varphi)(\Delta(\varphi)-\cos\alpha\tan\varphi)^{-1}]$$
$$\mp(1\mp\cos\alpha)F(\varphi\backslash\alpha)$$

17.7.24 $\qquad n=\sin^2\alpha$

$$\Pi(\sin^2\alpha;\ \varphi\backslash\alpha)=\sec^2\alpha E(\varphi\backslash\alpha)-(\tan^2\alpha\sin 2\varphi)/(2\Delta(\varphi))$$

17.7.25 $\qquad n=1$

$$\Pi(1;\ \varphi\backslash\alpha)=F(\varphi\backslash\alpha)-\sec^2\alpha E(\varphi\backslash\alpha)+\sec^2\alpha\tan\varphi\Delta(\varphi)$$

Numerical Methods

17.8. Use and Extension of the Tables

Example 1. Reduce to canonical form $\int y^{-1}dx$, where

$$y^2=-3x^4+34x^3-119x^2+172x-90$$

By inspection or by solving an equation of the fourth degree we find that

$$y^2=Q_1Q_2 \text{ where } Q_1=3x^2-10x+9,\ Q_2=-x^2+8x-10$$

First Method

$Q_1-\lambda Q_2=(3+\lambda)x^2-(10+8\lambda)x+9+10\lambda$ is a perfect square if the discriminant $(10+8\lambda)^2-4(3+\lambda)(9+10\lambda)=0$; i.e., if $\lambda=-\dfrac{2}{3}$ or $\dfrac{1}{2}$ and then

$$Q_1+\frac{2}{3}Q_2=\frac{7}{3}(x-1)^2,\ Q_1-\frac{1}{2}Q_2=\frac{7}{2}(x-2)^2$$

Solving for Q_1 and Q_2 we get

$$Q_1=(x-1)^2+2(x-2)^2,\ Q_2=2(x-1)^2-3(x-2)^2$$

The substitution $t=(x-1)/(x-2)$ then gives

$$\int y^{-1}dx=\pm\int[(t^2+2)(2t^2-3)]^{-\frac{1}{2}}dt$$

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*See page II.

If the quartic $y^2=0$ has four real roots in x (or in the case of a cubic all three roots are real), we must so combine the factors that no root of $Q_1=0$ lies between the roots of $Q_2=0$ and no root of $Q_2=0$ lies between the roots of $Q_1=0$. Provided this condition is observed the method just described will always lead to real values of λ. These values may, however, be irrational.

Second Method

Write

$$t^2=\frac{Q_1}{Q_2}=\frac{3x^2-10x+9}{-x^2+8x-10}$$

and let the discriminant of $Q_2t^2-Q_1$ be

$$\begin{aligned}4T^2&=(8t^2+10)^2-4(t^2+3)(10t^2+9)\\&=4(3t^2+2)(2t^2-1)\end{aligned}$$

Then

$$\int y^{-1}dx=\pm\int T^{-1}dt=\pm\int[(3t^2+2)(2t^2-1)]^{-\frac{1}{2}}dt$$

This method will succeed if, as here, T^2 as a function of t^2 has real factors. If the coefficients of the given quartic are rational numbers, the factors of T^2 will likewise be rational.

Third Method

Write

$$w=\frac{Q_1}{Q_2}=\frac{3x^2-10x+9}{-x^2+8x-10}$$

and let the discriminant of Q_2w-Q_1 be

$$4W=4(3w+2)(2w-1)=4(Aw^2+Bw+C)$$

Then if

$$z^2=W/w \text{ and } Z^2=(B-z^2)^2-4AC=(z^2-1)^2+48$$

$$\int y^{-1}dx=\pm\int Z^{-1}dz$$

However, in this case the factors of Z are complex and the method fails.

Of the second and third methods one will always succeed where the other fails, and if the coefficients of the given quartic are rational numbers, the factors of T^2 or Z^2, as the case may be, will be rational.

Example 2. Reduce to canonical form $\int y^{-1}dx$ where $y^2=x(x-1)(x-2)$.

We use the third method of **Example 1** taking $Q_1=(x-1)$, $Q_2=x(x-2)$ and writing

$$w=\frac{Q_1}{Q_2}=\frac{x-1}{x^2-2x}$$

The discriminant of $Q_2w-Q_1=x^2w-(2w+1)x+1$ is

$$4W=(2w+1)^2-4w=4w^2+1$$

so that

$$W=Aw^2+Bw+C \text{ where } A=1,\ B=0,\ C=\frac{1}{4}$$

and if we write $z^2=W/w$ and

$$Z^2=(B-z^2)^2-4AC=(z^2)^2-1=(z^2-1)(z^2+1),$$

$$\int y^{-1}dx=\pm\int[(z^2-1)(z^2+1)]^{-1/2}\,dz$$

The first method of **Example 1** fails with the above values of Q_1 and Q_2 since the root of $Q_1=0$ lies between the roots of $Q_2=0$, and we get imaginary values of λ. The method succeeds, however, if we take $Q_1=x$, $Q_2=(x-1)(x-2)$, for then the roots of $Q_1=0$ do not lie between those of $Q_2=0$.

Example 3. Find $K(80/81)$.

First Method

Use **17.3.29** with $m=80/81$, $m_1=1/81$, $m_1^{1/2}=1/9$. Since $[(1-m_1^{1/2})(1+m_1^{1/2})^{-1}]^2=.64$, $K(80/81)=1.8\,K(.64)=3.59154\ 500$ to 8D, taking $K(.64)$ from **Table 17.1.**

Second Method

Table 17.4 giving $L(m)$ is useful for computing $K(m)$ when m is near unity or $K'(m)$ when m is near zero.

$$K(80/81)=\frac{1}{\pi}\,K'(80/81)\ \ln\ (16\times81)-L(80/81).$$

By interpolation in **Tables 17.1** and **17.4,** since $80/81=.98765\ 43210$,

$$K'(80/81)=1.57567\ 8423$$

$$L(80/81)=.00311\ 16543$$

$$\begin{aligned}K(80/81)&=\pi^{-1}(1.57567\ 8423)(7.16703\ 7877)\\&\qquad-.00311\ 16543\\&=3.59154\ 5000 \text{ to 9D.}\end{aligned}$$

Third Method

The polynomial approximation **17.3.34** gives to 8D

$$K(80/81)=3.59154\ 501$$

Fourth Method, Arithmetic-Geometric Mean

Here $\sin^2\alpha=80/81$ and we start with

$$a_0=1,\ b_0=\frac{1}{9},\ c_0=\sqrt{80/81}=.99380\ 79900$$

giving

$$\epsilon=\arcsin\,[(1-n)/\cos^2\alpha]^{\frac{1}{2}}=45°$$

$$\beta=\tfrac{1}{2}\pi F(45°\backslash 60°)/K(30°)=.79317\,74$$

$$v=\tfrac{1}{2}\pi F(45°\backslash 30°)/K(30°)=.74951\,51$$

$$\delta_2=(40/9)^{\frac{1}{2}}$$

$$q=.01797\,24$$

and so from **17.7.11**

$$\Pi\,(\tfrac{5}{8};\,45°\backslash 30°)=(40/9)^{1/2}(\lambda-4\mu v)$$

$$=2.10818\,51\{.55248\,32-4(.03854\,26)(.74951\,51)\}=.921129.$$

Table 17.9 gives .92113 with 4 point Lagrangian interpolation.

Example 18. Evaluate the complete elliptic integral

$$\Pi\,(\tfrac{5}{8}\backslash 30°) \text{ to 5D.}$$

From **17.7.14** we have

$$\Pi\,(\tfrac{5}{8}\backslash 30°)=K(30°)+\frac{\pi}{2}\sqrt{\frac{40}{9}}\,[1-\Lambda_0(\epsilon\backslash 30°)]$$

where $\epsilon=\arcsin\,[(1-n)/\cos^2\alpha]^{1/2}=45°$. Thus using **Table 17.8**

$$\Pi\,(\tfrac{5}{8}\backslash 30°)=2.80099.$$

Table 17.9 gives 2.80126 by 6 point Lagrangian interpolation. The discrepancy results from interpolation with respect to n for $\varphi=90°$ in **Table 17.9.**

Example 19. Evaluate

$$\Pi\,(\tfrac{5}{4};\,45°\backslash 30°)=\int_0^{\pi/4}(1-\tfrac{5}{4}\sin^2\theta)^{-1}(1-\tfrac{1}{4}\sin^2\theta)^{-1/2}\,d\theta$$

to 5D.

Here $n=\frac{5}{4}$, $\varphi=45°$, $\alpha=30°$ and since the characteristic is greater than unity we use **17.7.7**

$$N=n^{-1}\sin^2\alpha=.2,\ p_1=(1/5)^{\frac{1}{2}}$$

$$\Pi\,(\tfrac{5}{4};45°\backslash 30°)=-\Pi(\,2;45°\backslash 30°)+F(45°\backslash 30°)+(\tfrac{1}{2}\sqrt{5})\,\ln\frac{(7/8)^{\frac{1}{2}}+(1/5)^{\frac{1}{2}}}{(7/8)^{\frac{1}{2}}-(1/5)^{\frac{1}{2}}}$$

$$=-.83612+.80437$$

$$+\tfrac{1}{2}\sqrt{5}\,\ln\frac{\sqrt{35}+\sqrt{8}}{\sqrt{35}-\sqrt{8}}$$

$$=1.13214.$$

Numerical quadrature gives the same result.

Example 20. Evaluate

$$\Pi\,(-\tfrac{1}{4};\,45°\backslash 30°)=\int_0^{\pi/4}(1+\tfrac{1}{4}\sin^2\theta)^{-1}(1-\tfrac{1}{4}\sin^2\theta)^{-\frac{1}{2}}d\theta$$

to 5D.

Here the characteristic is negative and we therefore use **17.7.15** with $n=-\frac{1}{4}$, $\sin^2\alpha=\frac{1}{4}$

$$N=(1-n)^{-1}(\sin^2\alpha-n)=.4,\ \ p_2=\sqrt{.1}$$

and therefore

$$(5/2)^{\frac{1}{2}}\,\Pi\,(-\tfrac{1}{4};\,45°\backslash 30°)=(9/40)^{\frac{1}{2}}\Pi\,(\tfrac{3}{8};\,45°\backslash 30°)+\frac{1}{2}(5/2)^{\frac{1}{2}}F(45°\backslash 30°)+\arctan\,(35)^{-\frac{1}{2}}$$

Using **Tables 4.14, 17.5,** and **17.9** we get

$$\Pi\,(-\tfrac{1}{4};\,45°\backslash 30°)=.76987$$

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18. Weierstrass Elliptic and Related Functions

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Contents

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18. Weierstrass Elliptic and Related Functions

Mathematical Properties

18.1. Definitions, Symbolism, Restrictions and Conventions

An elliptic function is a single-valued doubly periodic function of a single complex variable which is analytic except at poles and whose only singularities in the finite plane are poles. If ω and ω' are a pair of (primitive) half-periods of such a function $f(z)$, then $f(z+2M\omega+2N\omega')=f(z)$, M and N being integers. Thus the study of any such function can be reduced to consideration of its behavior in a *fundamental period parallelogram* (FPP). An elliptic function has a finite number of poles (and the same number of zeros) in a FPP; the number of such poles (zeros) (an irreducible set) is the *order* of the function (poles and zeros are counted according to their multiplicity). All other poles (zeros) are called *congruent* to the irreducible set. The simplest (nontrivial) elliptic functions are of order two. One may choose as the standard function of order two either a function with two simple poles (Jacobi's choice) or one double pole (Weierstrass' choice) in a FPP.

Weierstrass' $\wp$-Function. Let ω, ω' denote a pair of complex numbers with $\mathscr{I}(\omega'/\omega)>0$. Then $\wp(z)=\wp(z|\omega, \omega')$ is an elliptic function of order two with periods 2ω, $2\omega'$ and having a double pole at $z=0$, whose principal part is z^{-2}; $\wp(z)-z^{-2}$ is analytic in a neighborhood of the origin and vanishes at $z=0$.

Weierstrass' ζ-Function $\zeta(z)=\zeta(z|\omega, \omega')$ satisfies the condition $\zeta'(z)=-\wp(z)$; further, $\zeta(z)$ has a simple pole at $z=0$ whose principal part is z^{-1}; $\zeta(z)-z^{-1}$ vanishes at $z=0$ and is analytic in a neighborhood of the origin. $\zeta(z)$ is *NOT* an elliptic function, since it is not periodic. However, it is quasi-periodic (see "period" relations), so reduction to FPP is possible.

Weierstrass' σ-Function $\sigma(z)=\sigma(z|\omega, \omega')$ satisfies the condition $\sigma'(z)/\sigma(z)=\zeta(z)$; further, $\sigma(z)$ is an entire function which vanishes at the origin. Like ζ, it is *NOT* an elliptic function, since it is not periodic. However, it is quasi-periodic (see "period" relations), so reduction to FPP is possible.

Invariants g_2 and g_3

Let $W=2M\omega+2N\omega'$, M and N being integers. Then

18.1.1 $\qquad g_2=60\Sigma' W^{-4}$ and $g_3=140\Sigma' W^{-6}$

are the INVARIANTS, summation being over all pairs M, N except $M=N=0$.

Alternate Symbolism Emphasizing Invariants

18.1.2 $\qquad \wp(z)=\wp(z; g_2, g_3)$

18.1.3 $\qquad \wp'(z)=\wp'(z; g_2, g_3)$

18.1.4 $\qquad \zeta(z)=\zeta(z; g_2, g_3)$

18.1.5 $\qquad \sigma(z)=\sigma(z; g_2, g_3)$

Fundamental Differential Equation, Discriminant and Related Quantities

18.1.6 $\qquad \wp'^2(z)=4\wp^3(z)-g_2\wp(z)-g_3$

18.1.7

$$=4(\wp(z)-e_1)(\wp(z)-e_2)(\wp(z)-e_3)$$

18.1.8

$$\Delta=g_2^3-27g_3^2=16(e_2-e_3)^2(e_3-e_1)^2(e_1-e_2)^2$$

18.1.9

$$g_2=-4(e_1e_2+e_1e_3+e_2e_3)=2(e_1^2+e_2^2+e_3^2)$$

18.1.10 $\qquad g_3=4e_1e_2e_3=\frac{4}{3}(e_1^3+e_2^3+e_3^3)$

18.1.11 $\qquad e_1+e_2+e_3=0$

18.1.12 $\qquad e_1^4+e_2^4+e_3^4=g_2^2/8$

18.1.13 $\qquad 4e_i^3-g_2e_i-g_3=0 (i=1, 2, 3)$

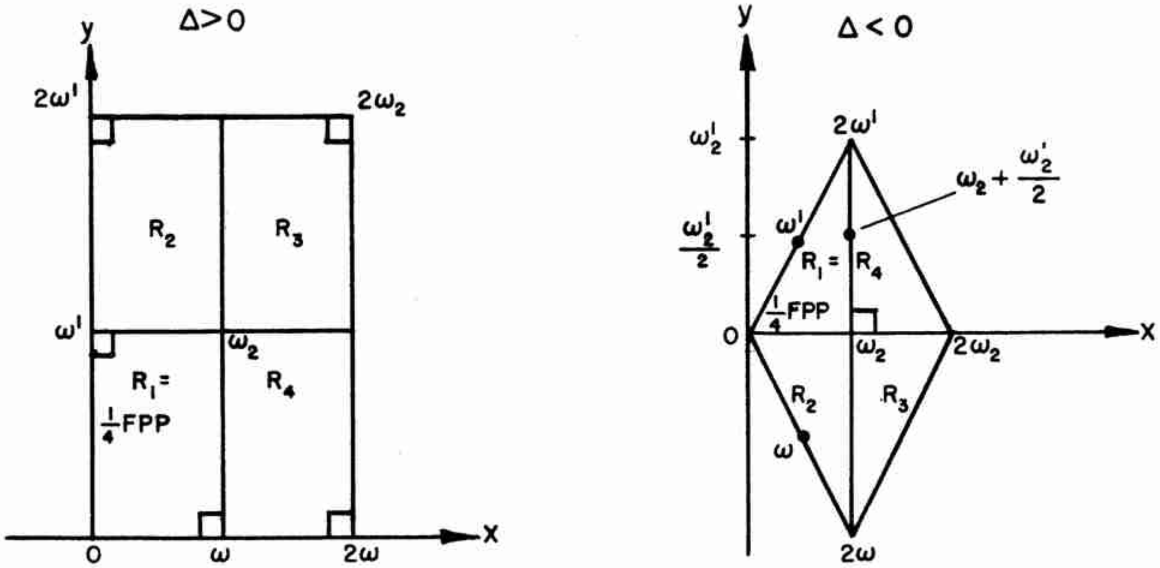
Agreement about Values of Invariants (and Discriminant)

We shall consider, in this chapter, only *real* g_2 and g_3 (this seems to cover most applications)—hence Δ is real. We shall dichotomize most of what follows (either $\Delta>0$ or $\Delta<0$). Homogeneity relations **18.2.1–18.2.15** enable a further restriction to non-negative g_3 (except for one case when $\Delta=0$).

Note on Symbolism for Roots of Complex Numbers and for Conjugate Complex Numbers

In this chapter, $z^{1/n}$ (n a positive integer) is used to denote the principal nth root of z, as in chapter **3**; $\bar{z}$ is used to denote the complex conjugate of z.

FPP's, Symbols for Periods, etc.



RECTANGLE FIGURE 18.1 **RHOMBUS**

$\omega_1=\omega$

$\omega_2=\omega+\omega'$ $\omega_2'=\omega'-\omega$

$\omega_3=\omega'$

ω REAL

ω' PURE IMAG.

$|\omega'|\geqq\omega$, since $g_3\geqq0$

ω_2 REAL

ω_2' PURE IMAG.

$|\omega_2'|\geqq\omega_2$, since $g_3\geqq0$

Fundamental Rectangles

Study of all four functions ($\wp, \wp', \zeta, \sigma$) can be reduced to consideration of their values in a Fundamental Rectangle including the origin (see **18.2** on homogeneity relations, reduction formulas and processes).

$\Delta>0$

Fundamental Rectangle is $\frac{1}{4}$ FPP, which has vertices 0, ω, ω_2 and ω'

$\Delta<0$

Fundamental Rectangle has vertices 0, ω_2, $\omega_2+\frac{\omega_2'}{2}$, $\frac{\omega_2'}{2}$.

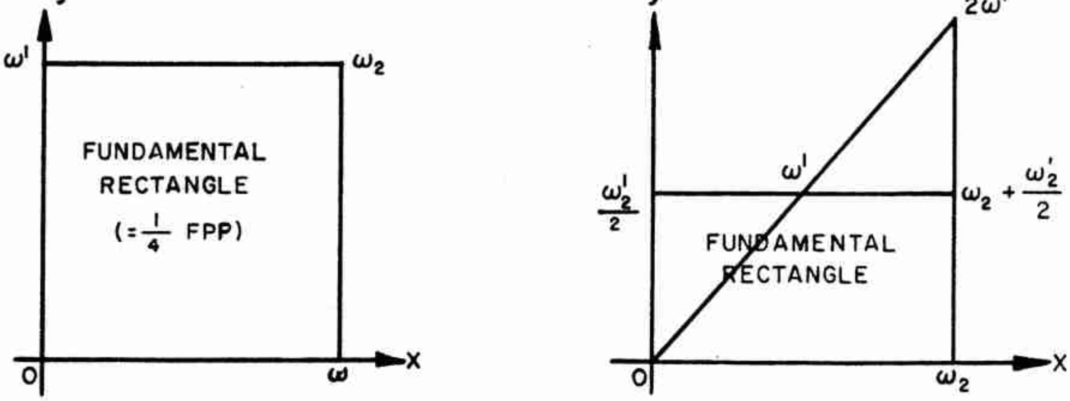


FIGURE 18.2

There is a point on the right boundary of Fundamental Rectangle where $\wp=0$. Denote it by z_0.

18.2. Homogeneity Relations, Reduction Formulas and Processes

Homogeneity Relations (Suppose $t\neq 0$)

Note that Period Ratio is preserved.

18.2.1 $\wp'(tz|t\omega, t\omega')=t^{-3}\wp'(z|\omega, \omega')$

18.2.2 $\wp(tz|t\omega, t\omega')=t^{-2}\wp(z|\omega, \omega')$

18.2.3 $\zeta(tz|t\omega, t\omega')=t^{-1}\zeta(z|\omega, \omega')$

18.2.4 $\sigma(tz|t\omega, t\omega')=t\sigma(z|\omega, \omega')$

18.2.5 $g_2(t\omega, t\omega')=t^{-4}g_2(\omega, \omega')$

18.2.6 $g_3(t\omega, t\omega')=t^{-6}g_3(\omega, \omega')$

18.2.7 $e_i(t\omega, t\omega')=t^{-2}e_i(\omega, \omega'),\ i=1, 2, 3$

18.2.8 $\Delta(t\omega, t\omega')=t^{-12}\Delta(\omega, \omega')$

18.2.9 $H_i(t\omega, t\omega')=t^{-2}H_i(\omega, \omega'),\ i=1, 2, 3$ (See **18.3**)

18.2.10 $q(t\omega, t\omega')=q(\omega, \omega')$ (See **18.10**)

18.2.11 $m(t\omega, t\omega')=m(\omega, \omega')$ (See **18.9**)

18.2.12 $\wp'(tz; t^{-4}g_2, t^{-6}g_3)=t^{-3}\wp'(z; g_2, g_3)$

18.2.13 $\wp(tz; t^{-4}g_2, t^{-6}g_3)=t^{-2}\wp(z; g_2, g_3)$

18.2.14 $\zeta(tz; t^{-4}g_2, t^{-6}g_3)=t^{-1}\zeta(z; g_2, g_3)$

18.2.15 $\sigma(tz; t^{-4}g_2, t^{-6}g_3)=t\sigma(z; g_2, g_3)$

The Case $g_3<0$

Put $t=i$ and obtain, e.g.,

18.2.16 $\wp(z; g_2, g_3)=-\wp(iz; g_2, -g_3)$

Thus the case $g_3<0$ can be reduced to one where $g_3>0$.

"Period" Relations and Reduction to the FPP (M, N integers)

18.2.17 $\wp'(z+2M\omega+2N\omega')=\wp'(z)$

18.2.18 $\wp(z+2M\omega+2N\omega')=\wp(z)$

18.2.19

$$\zeta(z+2M\omega+2N\omega')=\zeta(z)+2M\eta+2N\eta'$$

18.2.20

$$\sigma(z+2M\omega+2N\omega')$$
$$=(-1)^{M+N+MN}\sigma(z)\ \exp\ [(z+M\omega+N\omega')(2M\eta+2N\eta')]$$

18.2.21 where $\eta=\zeta(\omega),\ \eta'=\zeta(\omega')$

"Conjugate" Values

$f(\bar{z})=\bar{f}(z)$, where f is any one of the functions $\wp$, $\wp'$, ζ, σ.

Reduction to ¼ FPP (See Figure 18.1)

($\bar{s}$ denotes conjugate of s)

	$\Delta>0$	$\Delta<0$
Point z_4 in R_4		
18.2.22	$\wp'(z_4)=-\overline{\wp'}(\overline{2\omega-z_4})$	$\wp'(\bar{z}_4)=-\overline{\wp}'\overline{(2\omega_2-z_4)}$
18.2.23	$\wp(z_4)=\overline{\wp}(2\omega-z_4)$	$\wp(z_4)=\overline{\wp}(\overline{2\omega_2-z_4})$
18.2.24	$\zeta(z_4)=-\zeta(\overline{2\omega-z_4})+2\eta$	$\zeta(z_4)=-\bar{\zeta}(\overline{2\omega_2-z_4})+2(\eta+\eta')$
18.2.25	$\sigma(z_4)=\bar{\sigma}(\overline{2\omega-z_4})\ \exp\ [2\eta(z_4-\omega)]$	$\sigma(z_4)=\bar{\sigma}(\overline{2\omega_2-z_4})\ \exp\ [2(\eta+\eta')(z_4-\omega_2)]$
Point z_3 in R_3		
18.2.26	$\wp'(z_3)=-\wp'(2\omega_2-z_3)$	$\wp'(z_3)=-\wp'(2\omega_2-z_3)$
18.2.27	$\wp(z_3)=\wp(2\omega_2-z_3)$	$\wp(z_3)=\wp(2\omega_2-z_3)$
18.2.28	$\zeta(z_3)=-\zeta(2\omega_2-z_3)+2(\eta+\eta')$	$\zeta(z_3)=-\zeta(2\omega_2-z_3)+2(\eta+\eta')$
18.2.29	$\sigma(z_3)=\sigma(2\omega_2-z_3)\ \exp\ [2(\eta+\eta')(z_3-\omega_2)]$	$\sigma(z_3)=\sigma(2\omega_2-z_3)\ \exp\ [2(\eta+\eta')(z_3-\omega_2)]$
Point z_2 in R_2		
18.2.30	$\wp'(z_2)=\overline{\wp}'\overline{(z_2-2\omega')}$	$\wp'(z_2)=\overline{\wp}'(\bar{z}_2)$
18.2.31	$\wp(z_2)=\overline{\wp}\,\overline{(z_2-2\omega')}$	$\wp(z_2)=\overline{\wp}(\bar{z}_2)$
18.2.32	$\zeta(z_2)=\bar{\zeta}(\overline{z_2-2\omega'})+2\eta'$	$\zeta(z_2)=\bar{\zeta}(\bar{z}_2)$
18.2.33	$\sigma(z_2)=-\bar{\sigma}(\overline{z_2-2\omega'})\ \exp\ [2\eta'(z_2-\omega')]$	$\sigma(z_2)=\bar{\sigma}(\bar{z}_2)$

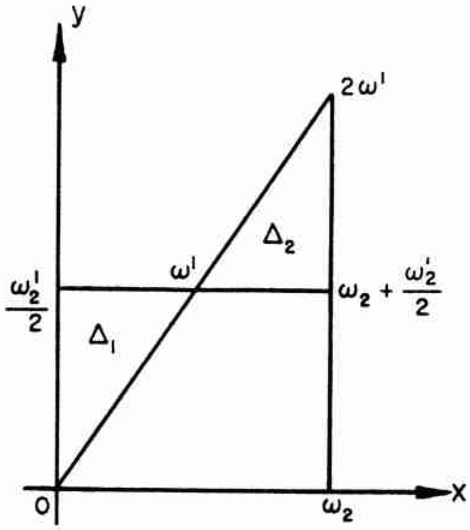


FIGURE 18.3

Reduction from 1/4 FPP to Fundamental Rectangle in Case $\Delta<0$

We need only be concerned with the case when z is in triangle Δ_2 (therefore $2\omega'-z$ is in triangle Δ_1).

18.2.34 $$\wp(z)=\wp(2\omega'-z)$$

18.2.35 $$\wp'(z)=-\wp'(2\omega'-z)$$

18.2.36 $$\zeta(z)=2\eta'-\zeta(2\omega'-z)$$

18.2.37 $$\sigma(z)=\sigma(2\omega'-z)\exp\left[2\eta'(z-\omega')\right]$$

Reduction to Case where Real Half-Period is Unity

(preserving period ratio)

	$\Delta>0$	$\Delta<0$ $(\omega_2=\omega+\omega')$
18.2.38	$\wp'(z\vert\omega,\omega')=\omega^{-3}\wp'\left(z\omega^{-1}\vert 1,\frac{\omega'}{\omega}\right)$	$\wp'(z\vert\omega,\omega')=\omega_2^{-3}\wp'\left(z\omega_2^{-1}\middle\vert\frac{\omega}{\omega_2},\frac{\omega'}{\omega_2}\right)$
18.2.39	$\wp(z\vert\omega,\omega')=\omega^{-2}\wp\left(z\omega^{-1}\vert 1,\frac{\omega'}{\omega}\right)$	$\wp(z\vert\omega,\omega')=\omega_2^{-2}\wp\left(z\omega_2^{-1}\middle\vert\frac{\omega}{\omega_2},\frac{\omega'}{\omega_2}\right)$
18.2.40	$\zeta(z\vert\omega,\omega')=\omega^{-1}\zeta\left(z\omega^{-1}\vert 1,\frac{\omega'}{\omega}\right)$	$\zeta(z\vert\omega,\omega')=\omega_2^{-1}\zeta\left(z\omega_2^{-1}\middle\vert\frac{\omega}{\omega_2},\frac{\omega'}{\omega_2}\right)$
18.2.41	$\sigma(z\vert\omega,\omega')=\omega\sigma\left(z\omega^{-1}\vert 1,\frac{\omega'}{\omega}\right)$	$\sigma(z\vert\omega,\omega')=\omega_2\sigma\left(z\omega_2^{-1}\middle\vert\frac{\omega}{\omega_2},\frac{\omega'}{\omega_2}\right)$
18.2.42	$g_2(\omega,\omega')=\omega^{-4}g_2\left(1,\frac{\omega'}{\omega}\right)$	$g_2(\omega,\omega')=\omega_2^{-4}g_2\left(\frac{\omega}{\omega_2},\frac{\omega'}{\omega_2}\right)$
18.2.43	$g_3(\omega,\omega')=\omega^{-6}g_3\left(1,\frac{\omega'}{\omega}\right)$	$g_3(\omega,\omega')=\omega_2^{-6}g_3\left(\frac{\omega}{\omega_2},\frac{\omega'}{\omega_2}\right)$
18.2.44	$e_i(\omega,\omega')=\omega^{-2}e_i\left(1,\frac{\omega'}{\omega}\right)$ $(i=1,2,3)$	$e_i(\omega,\omega')=\omega_2^{-2}e_i\left(\frac{\omega}{\omega_2},\frac{\omega'}{\omega_2}\right)$ $(i=1,2,3)$
18.2.45	$\Delta(\omega,\omega')=\omega^{-12}\Delta\left(1,\frac{\omega'}{\omega}\right)$	$\Delta(\omega,\omega')=\omega_2{}^{-12}\Delta\left(\frac{\omega}{\omega_2},\frac{\omega'}{\omega_2}\right)$

NOTE: New real half-period is

$$\frac{\omega}{\omega_2}+\frac{\omega'}{\omega_2}=\frac{\omega+\omega'}{\omega_2}=1$$

18.3. Special Values and Relations

Values at Periods

$\wp$, $\wp'$, and ζ are infinite, σ is zero at $z=2\omega_1$, $i=1,2,3$ and at $2\omega_2'(\Delta<0)$.

	$\Delta>0$	$\Delta<0$

Half-Periods

18.3.1 $$\wp(\omega_i)=e_i(i=1,2,3)$$

18.3.2 $$\wp'(\omega_i)=0(i=1,2,3)$$

18.3.3 $$\eta_i=\zeta(\omega_i)(i=1,2,3)$$

18.3.4 $$\eta_1=\eta,\ \eta_2=\eta+\eta',\ \eta_3=\eta'$$

18.3.5 $$H_i^2=2e_i^2+e_je_k\ (i,j,k=1,2,3;\ i\neq j,\ i\neq k,\ j\neq k)$$

18.3.6 $$=(e_i-e_j)(e_i-e_k)=2e_i^2+\frac{g_3}{4e_i}=3e_i^2-\frac{g_2}{4}$$

18.3.7	e_i real	e_2 real and non-negative
18.3.8	$e_1>0\geq e_2>e_3$ (equality when $g_3=0$)	($e_2=0$ when $g_3=0$) $e_1=-\alpha+i\beta$, $e_3=\overline{e}_1$ where $\alpha\geq 0, \beta>0$ (equality when $g_3=0$)
18.3.9	$\eta>0$	$\eta_2'=\zeta(\omega_2')=\eta'-\eta$
18.3.10	$\eta'/i\lesseqgtr 0$ if	$\eta_2>0$
18.3.11	$\lvert\omega'\rvert/\omega\lesseqgtr 1.91014\ 050$ (approx.)	$\eta_2'/i\lesseqgtr 0$ if $\lvert\omega_2'\rvert/\omega_2\lesseqgtr 3.81915\ 447$ (approx.)
18.3.12	$H_1>0$, $H_3>0$	$H_2>0$
18.3.13	$H_2\equiv i\sqrt{-H_2^2}$	$\pi/4<\arg(H_3)\leq\pi/2$ (equality if $g_3=0$); $H_1=\overline{H}_3$
18.3.14	$\sigma(\omega)=e^{\eta\omega/2}/H_1^{1/2}$	$\sigma(\omega_2)=e^{\eta_2\omega_2/2}/H_2^{1/2}$
18.3.15	$\sigma(\omega')=ie^{\eta'\omega'/2}/H_3^{1/2}$	$\sigma(\omega_2')=ie^{\eta_2'\omega_2'/2}/H_2^{1/2}$
18.3.16	$\sigma^2(\omega_2)=e^{\eta_2\omega_2}/(-H_2)$	$\sigma^2(\omega')=e^{\eta'\omega'}/(-H_3)$
18.3.17	$\arg[\sigma(\omega_2)]=\dfrac{\eta'\omega}{i}+\dfrac{\pi}{2}$	$\arg[\sigma(\omega')]=\dfrac{\eta_2'\omega_2}{4i}+\dfrac{\pi}{2}-\dfrac{1}{2}\arg(e_2+H_2-e_i)$

Quarter Periods

18.3.18	$\wp(\omega/2)=e_1+H_1>e_1$	$\wp(\omega_2/2)=e_2+H_2>e_2$
18.3.19	$\wp'(\omega/2)=-2H_1\sqrt{2H_1+3e_1}$	$\wp'(\omega_2/2)=-2H_2\sqrt{2H_2+3e_2}$
18.3.20	$\zeta(\omega/2)=\frac{1}{2}[\eta+\sqrt{2H_1+3e_1}]$	$\zeta(\omega_2/2)=\frac{1}{2}[\eta_2+\sqrt{2H_2+3e_2}]$

$\Delta>0$ $\Delta<0$

18.3.21 $\sigma(\omega/2)=\frac{e^{\eta\omega/8}}{2^{1/4}H_1^{3/8}(2H_1+3e_1)^{1/8}}$ $\sigma(\omega_2/2)=\frac{e^{\eta_2\omega_2/8}}{2^{1/4}H_2^{3/8}(2H_2+3e_2)^{1/8}}$

18.3.22 $\wp(\omega'/2)=e_3-H_3<e_3<0$ $\wp(\omega_2'/2)=e_2-H_2=\wp(\omega_2+\omega_2'/2)<e_2<0$

18.3.23 $\wp'(\omega'/2)=-2H_3i\sqrt{2H_3-3e_3}$ $\wp'(\omega_2'/2)=-2H_2i\sqrt{2H_2-3e_2}=\overline{\wp}'(\omega_2+\omega_2'/2)$

18.3.24 $\zeta(\omega'/2)=\tfrac{1}{2}[\eta'-i\sqrt{2H_3-3\mathrm{e}_3}]$ $\zeta(\omega_2'/2)=\tfrac{1}{2}[\eta_2'-i\sqrt{2H_2-3\mathrm{e}_2}]=-\zeta(\omega_2+\omega_2'/2)+2\eta'$

18.3.25 $\sigma(\omega'/2)=\frac{ie^{\eta'\omega'/8}}{2^{1/4}H_3^{3/8}(2H_3-3e_3)^{1/8}}$ $\sigma(\omega_2'/2)=\frac{ie^{\eta_2'\omega_2'/8}}{2^{1/4}H_2^{3/8}(2H_2-3e_2)^{1/8}}$

$=\sigma(\omega_2+\omega_2'/2)\exp[-\eta'\omega_2]$

18.3.26 $\wp(\omega_2/2)=e_2-H_2$ $\wp(\omega'/2)=e_3-H_3$

18.3.27 $\wp'(\omega_2/2)=-2H_2i(2H_2-3e_2)^{\frac{1}{2}}$ $\wp'(\omega'/2)=-2iH_3(2H_3-3e_3)^{\frac{1}{2}}$

18.3.28 $\zeta(\omega_2/2)=\tfrac{1}{2}[\eta_2-i(2H_2-3e_2)^{\frac{1}{2}}]$ $\zeta(\omega'/2)=\tfrac{1}{2}[\eta'-i(2H_3-3e_3)^{\frac{1}{2}}]$

18.3.29 $\sigma(\omega_2/2)=\frac{e^{\eta_2\omega_2/8}e^{i\pi/4}}{[4H_2^3(2H_2-3e_2)]^{1/8}}$ $\sigma(\omega'/2)=\frac{e^{\eta'\omega'/8}e^{i\pi/4}}{[4H_3^3(2H_3-3e_3)]^{1/8}}$

One-Third Period Relations

At $z=2\omega_i/3\,(i=1,2,3)$ or $2\omega_2'/3, \wp''^2=12\wp\wp'^2$;

equivalently:

18.3.30 $$48\wp^4-24g_2\wp^2-48g_3\wp-g_2^2=0$$

$\Delta>0$ $\Delta<0$

18.3.31 $\zeta(2\omega/3)=\frac{2\eta}{3}+\left[\frac{\wp(2\omega/3)}{3}\right]^{\frac{1}{2}}$ $\zeta(2\omega_2/3)=\frac{2\eta_2}{3}+\left[\frac{\wp(2\omega_2/3)}{3}\right]^{\frac{1}{2}}$

18.3.32 $\zeta(2\omega'/3)=\frac{2\eta'}{3}-\left[\frac{\wp(2\omega'/3)}{3}\right]^{\frac{1}{2}}$ $\zeta(2\omega_2'/3)=\frac{2\eta_2'}{3}-\left[\frac{\wp(2\omega_2'/3)}{3}\right]^{\frac{1}{2}}$

18.3.33 $\zeta(2\omega_2/3)=\frac{2\eta_2}{3}+\left[\frac{\wp(2\omega_2/3)}{3}\right]^{\frac{1}{2}}$ $\zeta(2\omega'/3)=\frac{2\eta'}{3}+\left[\frac{\wp(2\omega'/3)}{3}\right]^{\frac{1}{2}}$

18.3.34 $\sigma(2\omega/3)=\frac{-\exp[2\eta\omega/9]}{\sqrt[3]{\wp'(2\omega/3)}}$ $\sigma(2\omega_2/3)=\frac{-\exp[2\eta_2\omega_2/9]}{\sqrt[3]{\wp'(2\omega_2/3)}}$

18.3.35 $\sigma(2\omega'/3)=\frac{-\exp[2\eta'\omega'/9]}{[\wp'(2\omega'/3)]^{1/3}e^{2\pi i/3}}$ $\sigma(2\omega_2'/3)=\frac{-\exp[2\eta_2'\omega_2'/9]}{[\wp'(2\omega_2'/3)]^{1/3}e^{2\pi i/3}}$

18.3.36 $\sigma(2\omega_2/3)=\frac{-\exp[2\eta_2\omega_2/9]}{[\wp'(2\omega_2/3)]^{1/3}e^{2\pi i/3}}$ $\sigma(2\omega'/3)=\frac{-\exp[2\eta'\omega'/9]}{[\wp'(2\omega'/3)]^{1/3}e^{2\pi i/3}}$

Legendre's Relation

18.3.37 $\eta\omega'-\eta'\omega=\pi i/2$ $\eta_2\omega_2'-\eta_2'\omega_2=\pi i$

(also valid for $\Delta<0$)

Relations Among the H_i

18.3.38 $$H_1^2+H_2^2+H_3^2=3g_2/4$$

18.3.39 $$H_1^2H_2^2+H_2^2H_3^2+H_3^2H_1^2=0$$

18.3.40 $$H_1^2H_2^2H_3^2=-\Delta/16$$

18.3.41 $$16H_i^6-12g_2H_i^4+\Delta=0(i=1, 2, 3)$$

18.4. Addition and Multiplication Formulas

Addition Formulas[2] ($z_1\neq z_2$)

18.4.1 $$\wp(z_1+z_2)=\frac{1}{4}\left[\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}\right]^2-\wp(z_1)-\wp(z_2)$$

18.4.2 $$\wp'(z_1+z_2)=\frac{\wp(z_1+z_2)[\wp'(z_1)-\wp'(z_2)]+\wp(z_1)\wp'(z_2)-\wp'(z_1)\wp(z_2)}{\wp(z_2)-\wp(z_1)}$$

18.4.3 $$\zeta(z_1+z_2)=\zeta(z_1)+\zeta(z_2)+\frac{1}{2}\frac{\wp'(z_1)-\wp'(z_2)}{\wp(z_1)-\wp(z_2)}$$

18.4.4 $$\sigma(z_1+z_2)\sigma(z_1-z_2)=-\sigma^2(z_1)\sigma^2(z_2)[\wp(z_1)-\wp(z_2)]$$

Duplication and Triplication Formulas

$$\left[\text{Note that }\wp''=6\wp^2(z)-\frac{g_2}{2}, \wp'^2(z)=4\wp^3(z)-g_2\wp(z)-g_3 \text{ and } \wp'''(z)=12\wp(z)\wp'(z)\right]$$

18.4.5 $$\wp(2z)=-2\wp(z)+\left[\frac{\wp''(z)}{2\wp'(z)}\right]^2$$

18.4.6 $$\wp'(2z)=\frac{-4\wp'^4(z)+12\wp(z)\wp'^2(z)\wp''(z)-\wp''^3(z)}{4\wp'^3(z)}$$

18.4.7 $$\zeta(2z)=2\zeta(z)+\wp''(z)/2\wp'(z)$$

18.4.8 $$\sigma(2z)=-\wp'(z)\sigma^4(z)$$

18.4.9 $$\zeta(3z)=3\zeta(z)+\frac{4\wp'^3(z)}{\wp'(z)\wp'''(z)-\wp''^2(z)}$$

18.4.10 $$\sigma(3z)=-\wp'^2(z)\sigma^9(z)[\wp(2z)-\wp(z)]$$

18.5. Series Expansions

Laurent Series

18.5.1 $$\wp(z)=z^{-2}+\sum_{k=2}^{\infty}c_kz^{2k-2}$$

18.5.2 where $$c_2=g_2/20,\ c_3=g_3/28$$

and

18.5.3 $$c_k=\frac{3}{(2k+1)(k-3)}\sum_{m=2}^{k-2}c_mc_{k-m},\ k\geqq 4$$

18.5.4 $$\wp'(z)=-2z^{-3}+\sum_{k=2}^{\infty}(2k-2)c_kz^{2k-3}$$

18.5.5 $$\zeta(z)=z^{-1}-\sum_{k=2}^{\infty}c_kz^{2k-1}/(2k-1)$$

18.5.6 $$\sigma(z)=\sum_{m,n=0}^{\infty}a_{m,n}(\tfrac{1}{2}g_2)^m(2g_3)^n\cdot\frac{z^{4m+6n+1}}{(4m+6n+1)!}$$

---

[2] Formulas for ζ and σ are *not* true algebraic addition formulas.

$$\text{where } a_{0,0}=1 \text{ and} \tag{18.5.7}$$

$$a_{m,n}=3(m+1)a_{m+1,n-1}+\frac{16}{3}(n+1)a_{m-2,n+1}-\frac{1}{3}(2m+3n-1)(4m+6n-1)a_{m-1,n}, \tag{18.5.8}$$

it being understood that $a_{m,n}=0$ if either subscript is negative.

(The radius of convergence of the above series for $\wp-z^{-2}$, $\wp'+2z^{-3}$ and $\zeta-z^{-1}$ is equal to the smallest of $|2\omega|$, $|2\omega'|$ and $|2\omega\pm2\omega'|$; series for σ converges for all z.)

Values of Coefficients[3] c_k in Terms of c_2 and c_3

$$c_4=c_2^2/3 \tag{18.5.9}$$

$$c_5=3c_2c_3/11 \tag{18.5.10}$$

$$c_6=[2c_2^3+3c_3^2]/39 \tag{18.5.11}$$

$$c_7=2c_2^2c_3/33 \tag{18.5.12}$$

$$c_8=5c_2(11c_2^3+36c_3^2)/7293 \tag{18.5.13}$$

$$c_9=c_3(29c_2^3+11c_3^2)/2717 \tag{18.5.14}$$

$$c_{10}=(242c_2^5+1455c_2^2c_3^2)/240669 \tag{18.5.15}$$

$$c_{11}=14c_2c_3(389c_2^3+369c_3^2)/3187041 \tag{18.5.16}$$

$$c_{12}=(114950c_2^6+1080000c_2^3c_3^2+166617c_3^4)/891678645 \tag{18.5.17}$$

$$c_{13}=10c_2^2c_3(297c_2^3+530c_3^2)/11685817 \tag{18.5.18}$$

$$c_{14}=\frac{2c_2(528770c_2^6+7164675c_2^3c_3^2+2989602c_3^4)}{(306735)(215441)} \tag{18.5.19}$$

$$c_{15}=\frac{4c_3(62921815c_2^6+179865450c_2^3c_3^2+14051367c_3^4)}{(179685)(38920531)} \tag{18.5.20}$$

$$c_{16}=\frac{c_2^2(58957855c_2^6+1086511320c_2^3c_3^2+875341836c_3^4)}{(5909761)(5132565)} \tag{18.5.21}$$

$$c_{17}=\frac{c_2c_3(30171955c_2^6+126138075c_2^3c_3^2+28151739c_3^4)}{(920205)(6678671)} \tag{18.5.22}$$

$$c_{18}=\frac{1541470\cdot949003c_2^9+30458088737\cdot1155c_2^6c_3^2+122378650673\cdot378c_2^3c_3^4+2348703\cdot887777c_3^6}{(1342211013)(4695105713)} \tag{18.5.23}$$

$$c_{19}=\frac{2c_2^2c_3(3365544215c_2^6+429852433\cdot45c_2^3c_3^2+8527743477c_3^4)}{(91100295)(113537407)} \tag{18.5.24}$$

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[3] *NOTES:*

1. c_4–c_{16} were computed and checked independently by D. H. Lehmer; these were double-checked by substituting $g_2=20\ c_2$, $g_3=28\ c_3$ in values given in [18.10].

2. c_{17}–c_{18} were derived from values in [18.10] by the same substitution. These were checked (numerically) for particular values of g_2, g_3.

3. c_{19} is given incorrectly in [18.12] (factor 13 is missing in denominator of third term of bracket); this value was computed independently.

4. No factors of any of the above integers with more than ten digits are known to the author. This is not necessarily true of smaller integers, which have, in many instances, been arranged for convenient use with a desk calculator.

Value[4] of Coefficients $a_{m,n}$

$n \uparrow$ 8	$-2^{7}\cdot 3^{12}\cdot 5\cdot 59$ $\cdot 107895773$												
7	$-2^{7}\cdot 3^{10}\cdot 5\cdot 23$ $\cdot 257\cdot 18049$	$-2^{7}\cdot 3^{11}\cdot 5\cdot 59$ $\cdot 107895773$											
6	$-2^{6}\cdot 3^{9}\cdot 5$ $\cdot 229\cdot 2683$	$-2^{7}\cdot 3^{9}\cdot 5\cdot 23$ $\cdot 257$ $\cdot 18049$	$-2^{6}\cdot 3^{11}\cdot 5\cdot 7$ $\cdot 181\cdot 1699$ $\cdot 2803$	$-2^{8}\cdot 3^{10}\cdot 5\cdot 7$ $\cdot 41\cdot 6047$ $\cdot 4922497$									
5	$2^{3}\cdot 3^{8}\cdot 5$ $\cdot 9103$	$-2^{6}\cdot 3^{8}\cdot 5\cdot 229$ $\cdot 2683$	$-2^{5}\cdot 3^{10}\cdot 5$ $\cdot 40570423$	$-2^{6}\cdot 3^{9}\cdot 5\cdot 7$ $\cdot 59\cdot 179$ $\cdot 142231$	$-2^{4}\cdot 3^{10}\cdot 5\cdot 7$ $\cdot 1321$ $\cdot 1415535763$								
4	$2^{3}\cdot 3^{5}\cdot 5^{3}$ $\cdot 31$	$2^{3}\cdot 3^{7}\cdot 5\cdot 9103$	$-2^{5}\cdot 3^{8}\cdot 5\cdot 7$ $\cdot 13\cdot 37\cdot 41$	$-2^{5}\cdot 3^{8}\cdot 5\cdot 691$ $\cdot 83609$	$-2^{4}\cdot 3^{8}\cdot 5\cdot 11\cdot 31$ $\cdot 313\cdot 190387$	$-2^{4}\cdot 3^{10}\cdot 5^{2}\cdot 7\cdot 23$ $\cdot 263\cdot 4848953$	$-2^{5}\cdot 3^{9}\cdot 5\cdot 7\cdot 19$ $\cdot 1752686144977$						
3	$2^{3}\cdot 3^{4}\cdot 23$	$2^{3}\cdot 3^{4}\cdot 5^{2}\cdot 31$	$2^{3}\cdot 3^{7}\cdot 5\cdot 17$ $\cdot 109$	$-2^{3}\cdot 3^{6}\cdot 5\cdot 83$ $\cdot 3911$	$-2^{4}\cdot 3^{6}\cdot 5\cdot 503$ $\cdot 156217$	$-2^{4}\cdot 3^{8}\cdot 31$ $\cdot 315989669$	$-2^{4}\cdot 3^{7}\cdot 5\cdot 7\cdot 11\cdot 29$ $\cdot 83\cdot 1129\cdot 9551$	$-2^{4}\cdot 3^{8}\cdot 5\cdot 7\cdot 613$ $\cdot 17605225081$					
2	$-2\cdot 3^{3}$	$2^{3}\cdot 3^{3}\cdot 23$	$2^{3}\cdot 3^{5}\cdot 5\cdot 53$	$2^{3}\cdot 3^{4}\cdot 5\cdot 37$ $\cdot 167$	$-2\cdot 3^{6}\cdot 5\cdot 17$ $\cdot 3037$	$-2^{4}\cdot 3^{7}\cdot 61\cdot 151$ $\cdot 653$	$-2^{3}\cdot 3^{7}$ $\cdot 2387260103$	$-2^{4}\cdot 3^{7}\cdot 5\cdot 17\cdot 53$ $\cdot 2957\cdot 41189$	$-2\cdot 3^{9}\cdot 5\cdot 7^{2}\cdot 17$ $\cdot 67\cdot 195651059$	$2^{3}\cdot 3^{7}\cdot 5\cdot 7^{2}$ $\cdot 35866647631901$			
1	-3	$-2\cdot 3^{2}$	$3^{3}\cdot 19$	$2^{2}\cdot 3^{3}\cdot 311$	$3^{3}\cdot 5\cdot 20807$	$-2\cdot 3^{6}\cdot 11$ $\cdot 2609$	$-3^{5}\cdot 17$ $\cdot 1578257$	$-2^{3}\cdot 3^{5}\cdot 7\cdot 13$ $\cdot 2742587$	-3^{7} $\cdot 248882935409$	$-2\cdot 3^{5}\cdot 5\cdot 7^{2}\cdot 193$ $\cdot 13679\cdot 274973$	$3^{6}\cdot 7$ $\cdot 89555603641079$		
0	1	-1	-3^{2}	$3\cdot 23$	$3\cdot 107$	$3^{3}\cdot 7\cdot 23\cdot 37$	$3^{2}\cdot 313\cdot 503$	$-3^{4}\cdot 7$ $\cdot 685973$	$3^{5}\cdot 11\cdot 37$ $\cdot 257981$	$-3^{4}\cdot 7\cdot 23\cdot 14387$ $\cdot 40763$	$-3^{4}\cdot 71$ $\cdot 176302760639$	$-3^{6}\cdot 7\cdot 11\cdot 23^{2}$ $\cdot 383\cdot 739\cdot 18539$	$3^{5}\cdot 7^{2}\cdot 24733$ $\cdot 198922785511$
	0	1	2	3	4	5	6	7	8	9	10	11	12

$\rightarrow m$

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[4] Values of $a_{m,n}$ in unfactored form for $4m+6n+1\leq 35$ are given in [18.25], p. 7; of $(a_{m,n})3^{-n}$ in factored form in [18.15], Vol. 4, p. 89 for $4m+6n+1\leq 25$. Additional values were computed and checked on desk calculators; primality of large factors was established with the aid of SWAC (National Bureau of Standards Western Automatic Computer).

Reversed Series[5] for Large $|\wp|$

18.5.25

$$z=\frac{1}{2}\Big[2u+c_2u^5+c_3u^7+\frac{\alpha_2^2}{3}u^9+\frac{6\alpha_2\alpha_3}{11}u^{11}$$

$$+\frac{1}{13}(3\alpha_3^2+5\alpha_2^3)u^{13}+\alpha_2^2\alpha_3u^{15}+\frac{5\alpha_2}{68}(12\alpha_3^2+7\alpha_2^3)u^{17}$$

$$+\frac{5\alpha_3}{19}(\alpha_3^2+7\alpha_2^3)u^{19}+\frac{\alpha_2^2}{4}(3\alpha_2^3+10\alpha_3^2)u^{21}$$

$$+\frac{35\alpha_2\alpha_3}{92}(9\alpha_2^3+4\alpha_3^2)u^{23}$$

$$+\frac{7}{200}(33\alpha_2^6+180\alpha_2^3\alpha_3^2+10\alpha_3^4)u^{25}$$

$$+\frac{7\alpha_2^2\alpha_3}{12}(11\alpha_2^3+10\alpha_3^2)u^{27}$$

$$+\frac{3\alpha_2}{2^3\cdot 29}(143\alpha_2^6+1155\alpha_2^3\alpha_3^2+210\alpha_3^4)u^{29}$$

$$+\frac{21\alpha_3}{2^3\cdot 31}(143\alpha_2^6+220\alpha_2^3\alpha_3^2+6\alpha_3^4)u^{31}$$

$$+\frac{3\alpha_2^2}{2^6}(65\alpha_2^6+728\alpha_2^3\alpha_3^2+280\alpha_3^4)u^{33}$$

$$+\frac{33\alpha_2\alpha_3}{2^3\cdot 5\cdot 7}(195\alpha_2^6+455\alpha_2^3\alpha_3^2+42\alpha_3^4)u^{35}$$

$$+\frac{11}{2^6\cdot 37}(1105\alpha_2^9+16380\alpha_2^6\alpha_3^2+10920\alpha_2^3\alpha_3^4$$

$$+168\alpha_3^6)u^{37}+\frac{33\alpha_2^2\alpha_3}{2^6}(85\alpha_2^6+280\alpha_2^3\alpha_3^2+56\alpha_3^4)u^{39}$$

$$+\frac{143\alpha_2}{2^7\cdot 41}(323\alpha_2^9+6120\alpha_2^6\alpha_3^2+6300\alpha_2^3\alpha_3^4+336\alpha_3^6)u^{41}$$

$$+\frac{143\alpha_3}{2^6\cdot 43}(1615\alpha_2^9+7140\alpha_2^6\alpha_3^2+2520\alpha_2^3\alpha_3^4+24\alpha_3^6)u^{43}$$

$$+O(u^{45})\Big],$$

18.5.26 where $\alpha_2=g_2/8$

18.5.27 $\alpha_3=g_3/8$

18.5.28 $u=(\wp^{-1})^{\frac{1}{2}}$

Reversed Series for Large $|\wp'|$

18.5.29 $z=A_1u+A_5u^5+A_7u^7+A_9u^9+\ldots$

18.5.30 where $u=(\wp'^{1/3})^{-1}e^{i\pi/3}$

18.5.31 $A_1=2^{1/3}$

18.5.32 $A_5=-\frac{a_2}{5}A_1^2$

18.5.33 $A_7=\frac{-4a_3A_1}{7}$

18.5.34 $A_9=0$

18.5.35 $A_{11}=8a_2a_3A_1^2/11$

18.5.36 $A_{13}=\frac{10A_1}{39}(a_2^3+6a_3^2)$

18.5.37 $A_{15}=-96a_2^2a_3/175$

18.5.38 $A_{17}=-\frac{14a_2A_1^2}{51}(a_2^3+12a_3^2)$

18.5.39 where $a_2=g_2/6$, $a_3=g_3/6$

Reversed Series for Large $|\zeta|$

18.5.40 $z=u+A_5u^5+A_7u^7+A_9u^9+\ldots$

18.5.41 where $u=\zeta^{-1}$

18.5.42 $A_5=-\delta_2/5$

18.5.43 $A_7=-\delta_3/7$

18.5.44 $A_9=\delta_2^2/7$

18.5.45 $A_{11}=3\delta_2\delta_3/11$

18.5.46 $A_{13}=\frac{17}{1001}(-8\delta_2^3+7\delta_3^2)$

18.5.47 $A_{15}=-41\delta_2^2\delta_3/91$

18.5.48 $A_{17}=\frac{\delta_2}{9163}(1349\delta_2^3-4116\delta_3^2)$

18.5.49 $A_{19}=\frac{2\delta_3}{323323}(115431\delta_2^3-22568\delta_3^2)$

18.5.50 where $\delta_2=g_2/12$

18.5.51 $\delta_3=g_3/20$

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[5] In this and other series a choice of the value of the root has been made so that z will be in the Fundamental Rectangle **(Figure 18.2)**, whenever the value of the given function is appropriate.

Other Series Involving $\wp$

Series near z_0 $[\wp(z_0)=0]$

18.5.52

$$\begin{aligned}\wp=\wp_0'u\Bigl[&1-3c_2u^4-4c_3u^6+\frac{10c_2^2}{3}u^8+\frac{114c_2c_3}{11}u^{10}\\&+\frac{7(12c_3^2-5c_2^3)}{13}u^{12}-\frac{488c_2^2c_3}{33}u^{14}\Bigr]+u^2\Bigl[-5c_2-14c_3u^2\\&+5c_2^2u^4+33c_2c_3u^6+\frac{84c_3^2-10c_2^3}{3}u^8-\frac{1363c_2^2c_3u^{10}}{33}\\&+\frac{5c_2(55c_2^3-2316c_3^2)u^{12}}{143}\Bigr]+\dots\end{aligned}$$

18.5.53 where $u=(z-z_0)$, $\wp_0'\equiv\wp'(z_0)=i\sqrt{g_3}$

18.5.54

$$\begin{aligned}u=\wp_0'[&v+av^2+2a^2v^3+\Bigl(\frac{g_3\wp_0'^2}{2}+5a^3\Bigr)v^4+\frac{a}{5}(3\wp_0'^4\\&+15g_3\wp_0'^2+70a^3)v^5+2a^2(2\wp_0'^4+7g_3\wp_0'^2+21a^3)v^6\\&+\Bigl(\frac{g_3\wp_0'^6}{7}+\{g_3^2+20a^3\}\wp_0'^4+15a^2g_3\wp_0'^2+132a^6\Bigr)v^7\\&+15a\Bigl(\frac{g_3\wp_0'^6}{4}+\Bigl\{\frac{3g_3^2}{4}+6a^3\Bigr\}\wp_0'^4+\frac{33ag_3}{2}\wp_0'^2\\&+\frac{143a^6}{5}\Bigr)v^8+\frac{5a^2}{2}\Bigl(\frac{2}{3}\wp_0'^8+15g_3\wp_0'^6\\&+\{154a^3+33g_3^2\}\wp_0'^4+\frac{2002a^3g_3\wp_0'^2}{5}+572a^6\Bigr)v^9\\&+\frac{1}{4}\Bigl(3\{28a^3+g_3^2\}\wp_0'^8+11g_3\{98a^3+g_3^2\}\wp_0'^6\\&+2002a^3\Bigl\{\frac{16}{5}a^3+g_3^2\Bigr\}\wp_0'^4\\&+16016\,a^6g_3\wp_0'^2+19448\,a^9\Bigr)v^{10}]+\dots\end{aligned}$$

18.5.55 where $v=\wp/(\wp_0')^2$ and $a=g_2/4$

Series near ω_i

18.5.56

$$\begin{aligned}(\wp-e_i)=&(3e_i^2-5c_2)u+(10c_2e_i+21c_3)u^2+(7c_2e_i^2\\&+21c_3e_i+5c_2^2)u^3+(18c_3e_i^2+30c_2^2e_i\\&+33c_2c_3)u^4+\Bigl(22c_2^2e_i^2+92c_2c_3e_i+105c_3^2\\&-\frac{10c_2^3}{3}\Bigr)u^5+\Bigl(\frac{728}{11}c_2c_3e_i^2+\frac{220}{3}c_2^3e_i+84c_3^2e_i\\&+\frac{1214}{11}c_2^2c_3\Bigr)u^6+\Bigl(\frac{635}{13}c_2^3e_i^2+\frac{855}{13}c_3^2e_i^2\\&+\frac{3405}{11}c_2^2c_3e_i+\frac{45750}{143}c_2c_3^2+\frac{25}{13}c_2^4\Bigr)u^7+\dots,\end{aligned}$$

18.5.57 where $u=(z-\omega_i)^2$

Other Series Involving $\wp'$

Series near z_0

18.5.58

$$\begin{aligned}(\wp'-\wp_0')=\Bigl[&-10c_2u-56c_3u^3+30c_2^2u^5+264c_2c_3u^7\\&+\frac{(840c_3^2-100c_2^3)}{3}u^9-\frac{5452c_2^2c_3}{11}u^{11}\\&+\frac{70c_2(55c_2^3-2316c_3^2)}{143}u^{13}\Bigr]\\+\wp_0'\Bigl[&-15c_2u^4-28c_3u^6+30c_2^2u^8+114c_2c_3u^{10}\\&+7(12c_3^2-5c_2^3)u^{12}-\frac{2440c_2^2c_3}{11}u^{14}\Bigr]+\dots\end{aligned}$$

18.5.59 where $u=(z-z_0)$

18.5.60

$$\begin{aligned}(z-z_0)=&A-bA^3-\frac{3\wp_0'}{2}A^4+3(c_2+b^2)A^5\\&+10b\wp_0'A^6-3[36c_3-3\wp_0'+4b^3]A^7\\&-3\wp_0'\Bigl(\frac{25}{2}c_2+21b^2\Bigr)A^8+\frac{5}{12}\Bigl(285b^2c_2\\&+100c_2^2-279\wp_0'^2b+132b^4\Bigr)A^9+\dots\end{aligned}$$

18.5.61 where $A=(\wp'-\wp_0')/(-10c_2)$

18.5.62 and $b=4g_3/g_2$

Series near ω_i

18.5.63

$$\begin{aligned}\wp'=&2(3e_i^2-5c_2)\alpha+4(10c_2e_i+21c_3)\alpha^3+6(7c_2e_i^2\\&+21c_3e_i+5c_2^2)\alpha^5+24(6c_3e_i^2+10c_2^2e_i\\&+11c_2c_3)\alpha^7+10\Bigl(22c_2^2e_i^2+92c_2c_3e_i+105c_3^2\\&-\frac{10c_2^3}{3}\Bigr)\alpha^9+24\Bigl(\frac{364}{11}c_2c_3e_i^2+\frac{110}{3}c_2^3e_i\\&+42c_3^2e_i+\frac{607}{11}c_2^2c_3\Bigr)\alpha^{11}+70\Bigl(\frac{127}{13}c_2^3e_i^2\\&+\frac{171}{13}c_3^2e_i^2+\frac{681}{11}c_2^2c_3e_i+\frac{9150}{143}c_2c_3^2+\frac{5}{13}c_2^4\Bigr)\alpha^{13}\\&+\dots,\end{aligned}$$

18.5.64 where $\alpha=(z-\omega_i)$.

Other Series Involving ζ

Series near z_0 $[\wp(z_0)=0]$

18.5.65

$$\zeta-\zeta_0=\wp_0'\left[-\frac{u^2}{2}+\frac{c_2u^6}{2}+\frac{c_3u^8}{2}-\frac{c_2^2u^{10}}{3}-\frac{19c_2c_3u^{12}}{22}+\frac{(5c_2^3-12c_3^2)}{26}u^{14}+\frac{61c_2^2c_3u^{16}}{66}\right]+\left[\frac{5c_2u^3}{3}+\frac{7c_3u^5}{2}-\frac{5c_2^2u^7}{7}-\frac{11c_2c_3u^9}{3}+\frac{(10c_2^3-84c_3^2)}{33}u^{11}+\frac{1363c_2^2c_3}{429}u^{13}+\frac{c_2(2316c_3^2-55c_2^3)}{429}u^{15}\right]+\ldots,$$

18.5.66 where $u=(z-z_0)$,

18.5.67 $\zeta_0\equiv\zeta(z_0)$

Series near ω_i

18.5.68

$$(\zeta-\eta_i)=-e_i\alpha-\frac{(3e_i^2-5c_2)}{3}\alpha^3-\frac{(10c_2e_i+21c_3)\alpha^5}{5}-\frac{(7c_2e_i^2+21c_3e_i+5c_2^2)\alpha^7}{7}-\frac{(6c_3e_i^2+10c_2^2e_i+11c_2c_3)\alpha^9}{3}-\frac{\left(22c_2^2e_i^2+92c_2c_3e_i+105c_3^2-\frac{10}{3}c_2^3\right)\alpha^{11}}{11}-\frac{2}{13}\left(\frac{364}{11}c_2c_3e_i^2+\frac{110}{3}c_2^3e_i+42c_3^2e_i+\frac{607}{11}c_2^2c_3\right)\alpha^{13}-\frac{1}{3}\left(\frac{127}{13}c_2^3e_i^2+\frac{171}{13}c_3^2e_i^2+\frac{681}{11}c_2^2c_3e_i+\frac{9150}{143}c_2c_3^2+\frac{5}{13}c_2^4\right)\alpha^{15}-\ldots,$$

18.5.69 where $\alpha=(z-\omega_i)$

Reversed Series for Small $|\sigma|$

18.5.70

$$z=\sigma+\frac{\gamma_2}{5}\sigma^5+\frac{\gamma_3}{7}\sigma^7+\frac{3\gamma_2^2}{14}\sigma^9+\frac{19\gamma_2\gamma_3}{55}\sigma^{11}+\frac{3842\gamma_2^3+861\gamma_3^2}{6006}\sigma^{13}+\ldots,$$

18.5.71 where $\gamma_2=g_2/48$

18.5.72 $\gamma_3=g_3/120$

For reversion of Maclaurin series, see **3.6.25** and [18.18].

18.6. Derivatives and Differential Equations

Ordinary $(c_2=g_2/20,\ c_3=g_3/28)$

18.6.1 $\zeta'(z)=-\wp(z)$

18.6.2 $\sigma'(z)/\sigma(z)=\zeta(z)$

18.6.3

$$\wp'^2(z)=4\wp^3(z)-g_2\wp(z)-g_3=4(\wp^3-5c_2\wp-7c_3)$$

18.6.4 $\wp''(z)=6\wp^2(z)-\frac{1}{2}g_2=6\wp^2-10c_2$

18.6.5 $\wp'''(z)=12\wp\wp'$

18.6.6

$$\wp^{(4)}(z)=12(\wp\wp''+\wp'\wp')=5!\left[\wp^3-3c_2\wp-\frac{14c_3}{5}\right]$$

18.6.7

$$\wp^{(5)}(z)=12(\wp\wp'''+2\wp'\wp''+\wp''\wp')=3\cdot5!\,\wp'[\wp^2-c_2]$$

18.6.8

$$\wp^{(6)}(z)=12(\wp\wp^{(4)}+3\wp'\wp'''+3\wp''\wp''+\wp'''\wp')$$

18.6.9 $=7![\wp^4-4c_2\wp^2-4c_3\wp+5c_2^2/7]$

18.6.10 $\wp^{(7)}(z)=4\cdot7!\,\wp'[\wp^3-2c_2\wp-c_3]$

18.6.11

$$\wp^{(8)}(z)=9![\wp^5-5c_2\wp^3-5c_3\wp^2+(10c_2^2\wp+11c_2c_3)/3]$$

18.6.12

$$\wp^{(9)}(z)=5\cdot9!\,\wp'[\wp^4-3c_2\wp^2-2c_3\wp+2c_2^2/3]$$

18.6.13

$$\wp^{(10)}(z)=11![\wp^6-6c_2\wp^4-6c_3\wp^3+7c_2^2\wp^2+(342c_2c_3\wp+84c_3^2-10c_2^3)/33]$$

18.6.14

$$\wp^{(11)}(z)=6\cdot11!\,\wp'[\wp^5-4c_2\wp^3-3c_3\wp^2+(77c_2^2\wp+57c_2c_3)/33]$$

18.6.15

$$\wp^{(12)}(z)=13![\wp^7-7c_2\wp^5-7c_3\wp^4+35c_2^2\wp^3/3+210c_2c_3\wp^2/11+(84c_3^2-35c_2^3)\wp/13-1363c_2^2c_3/429]$$

18.6.16

$$\wp^{(13)}(z)=7\cdot13!\,\wp'[\wp^6-5c_2\wp^4-4c_3\wp^3+5c_2^2\wp^2+60c_2c_3\wp/11+(12c_3^2-5c_2^3)/13]$$

18.6.17

$$\wp^{(14)}(z)=15![\wp^8-8c_2\wp^6-8c_3\wp^5+52c_2^2\wp^4/3+328c_2c_3\wp^3/11+(444c_3^2-328c_2^3)\wp^2/39-488c_2^2c_3\wp/33+c_2(55c_2^3-2316c_3^2)/429]$$

18.6.18

$$\wp^{(15)}(z)=8\cdot15!\,\wp'[\wp^7-6c_2\wp^5-5c_3\wp^4+26c_2^2\wp^3/3+123c_2c_3\wp^2/11+(111c_3^2-82c_2^3)\wp/39-61c_2^2c_3/33]$$

Partial Derivatives with Respect to Invariants

18.6.19

$$\Delta \frac{\partial \wp}{\partial g_3}=\wp'\left(3g_2\zeta-\frac{9}{2}g_3z\right)+6g_2\wp^2-9g_3\wp-g_2^2$$

18.6.20

$$\Delta \frac{\partial \wp}{\partial g_2}=\wp'\left(-\frac{9}{2}g_3\zeta+\frac{g_2^2z}{4}\right)-9g_3\wp^2+\frac{g_2^2}{2}\wp+\frac{3}{2}g_2g_3$$

18.6.21

$$\Delta \frac{\partial \zeta}{\partial g_3}=-3\zeta\left(g_2\wp+\frac{3}{2}g_3\right)$$
$$+\frac{1}{2}z\left(9g_3\wp+\frac{1}{2}g_2^2\right)-\frac{3}{2}g_2\wp'$$

18.6.22

$$\Delta \frac{\partial \zeta}{\partial g_2}=\frac{1}{2}\zeta\left(9g_3\wp+\frac{1}{2}g_2^2\right)$$
$$-\frac{1}{2}g_2z\left(\frac{1}{2}g_2\wp+\frac{3}{4}g_3\right)+\frac{9}{4}g_3\wp'$$

18.6.23 $$\Delta \frac{\partial \sigma}{\partial g_3}=\frac{3}{2}g_2\sigma''+\frac{9}{2}g_3\sigma+\frac{1}{8}g_2^2z^2\sigma-\frac{9}{2}g_2z\sigma'$$

18.6.24

$$\Delta \frac{\partial \sigma}{\partial g_2}=-\frac{9}{4}g_3\sigma''-\frac{1}{4}g_2^2\sigma-\frac{3}{16}g_2g_3z^2\sigma+\frac{1}{4}g_2^2z\sigma'$$

$$\left(\text{here } ' \text{ denotes } \frac{\partial}{\partial z}\right)$$

Differential Equations

	Equation	*Solution*
18.6.25	$y'^3=y^2(y-a)^2$	$y=\frac{a}{2}+\frac{27}{16}\wp'\left(\frac{z}{2};0,-\frac{64a^2}{729}\right)$
18.6.26	$y'^3=(y^3-3ay^2+3y)^2$	$y=\frac{2}{a-3\wp'(z;0,g_3)}$, $g_3=\frac{4-3a^2}{27}$
18.6.27	$y'^4=\frac{128}{3}(y+a)^2(y+b)^3$	$y=6\wp^2(z;g_2,0)-b$, $g_2=-\frac{2}{3}(a-b)$

$y''=[a\wp(z)+b]y$ (Lamé's equation)—see [18.8], 2.26

For other (more specialized) equations (of orders 1–3) involving $\wp(z)$, see [18.8], nos. 1.49, 2.28, 2.72–3, 2.439–440, 3.9–12.

For the use of $\wp(z)$ in solving differential equations of the form $y'^m+A(z,y)=0$, where $A(z,y)$ is a polynomial in y of degree $2m$, with coefficients which are analytic functions of z, see [18.7], p. 312ff.

18.7. Integrals

Indefinite

18.7.1 $$\int \wp^2(z)dz=\frac{1}{6}\wp'(z)+\frac{1}{12}g_2z$$

18.7.2 $$\int \wp^3(z)dz=\frac{1}{120}\wp'''(z)-\frac{3}{20}g_2\zeta(z)+\frac{1}{10}g_3z$$

(formulas for higher powers may be derived by integration of formulas for $\wp^{(2k)}(z)$)

For $\int \wp^n(z)dz$, n any positive integer, see [18.15] vol. 4, pp. 108–9.

If $\wp'(a)\neq 0$

18.7.3

$$\wp'(a)\int \frac{dz}{\wp(z)-\wp(a)}$$
$$=2z\zeta(a)+\ln \sigma(z-a)-\ln \sigma(z+a)$$

For $\int dz/[\wp(z)-\wp(a)]^n$, ($\wp'(a)\neq 0$) n any positive integer, see [18.15], vol. 4, pp. 109–110.

Definite

	$\Delta>0$	$\Delta<0$
18.7.4	$\omega=\int_{e_1}^{\infty}\frac{dt}{\sqrt{s(t)}}$	$\omega_2=\int_{e_2}^{\infty}\frac{dt}{\sqrt{s(t)}}$
18.7.5	$\omega'=i\int_{-\infty}^{e_3}\frac{dt}{\sqrt{\lvert s(t)\rvert}}$	$\omega_2'=i\int_{-\infty}^{e_2}\frac{dt}{\sqrt{\lvert s(t)\rvert}}$

18.7.6 where t is real and

18.7.7 $$s(t)=4t^3-g_2t-g_3$$

18.8 Conformal Mapping

$w=u+iv$

$\Delta>0$

$w=\wp(z)$ maps the Fundamental Rectangle onto the half-plane $v\leq 0$; if $|\omega'|=\omega(g_3=0)$, the isosceles triangle $0\omega\omega_2$ is mapped onto $u\geq 0$, $v\leq 0$.

$w=\wp'(z)$ maps the Fundamental Rectangle onto the w-plane less quadrant III; if $|\omega'|=\omega$, the triangle $0\omega\omega_2$ is mapped onto $v\geq 0$, $v\geq u$.

$\Delta<0$

$w=\wp(z)$ maps the Fundamental Rectangle onto the half-plane $v\leq 0$; if $|\omega_2'|=\omega_2(g_3=0)$, the isosceles triangle $0\omega_2\omega'$ is mapped onto $u\geq 0$, $v\leq 0$.

$w=\wp'(z)$ maps the Fundamental Rectangle onto most of the w-plane less quadrant III; if $|\omega_2'|=\omega_2$, the triangle $0\omega_2\omega'$ is mapped onto $v\geq 0$, $v\geq u$.

(a=period ratio)

$w=\zeta(z)$ maps the Fundamental Rectangle onto the half-plane $u\geq 0$. If $a\leq 1.9$ (approx.), $v\leq 0$; otherwise the image extends into quadrant I. For very large a, the image has a large area in quadrant I.

$w=\sigma(z)$ maps the Fundamental Rectangle onto quadrant I if a <1.9 (approx.), onto quadrants I and II if $1.9\leq a<3.8$ (approx.). For large a, $\arg[\sigma(\omega_2)]\approx\frac{\pi^2 a}{12}$; consequently the image winds around the origin for large a.

Other maps are described in [18.23] arts. 13.7 (square on circle), 13.11 (ring on plane with 2 slits in line) and in [18.24], p. 35 (double half equilateral triangle on half-plane).

$w=\zeta(z)$ maps the Fundamental Rectangle onto the half-plane $u\geq 0$. The image is mostly in quadrant IV for small a, entirely so for (approx.) $1.3\leq a\leq 3.8$. For very large a, the image has a large area in quadrant I.

$w=\sigma(z)$ maps the Fundamental Rectangle onto quadrant I if $a<3.8$ (approx.), onto quadrants I and II if $3.8\leq a<7.6$ (approx.). For large a, $\arg\left[\sigma\left(\omega_2+\frac{\omega_2'}{2}\right)\right]\approx\frac{\pi^2 a}{24}$; consequently the image winds around the origin for large a.

Other maps are described in [18.23] arts. 13.8 (equilateral triangle on half-plane) and 13.9 (isosceles triangle on half-plane).

Obtaining $\wp'$ from $\wp'^2$

Fundamental Rectangle **Fundamental Rectangle**

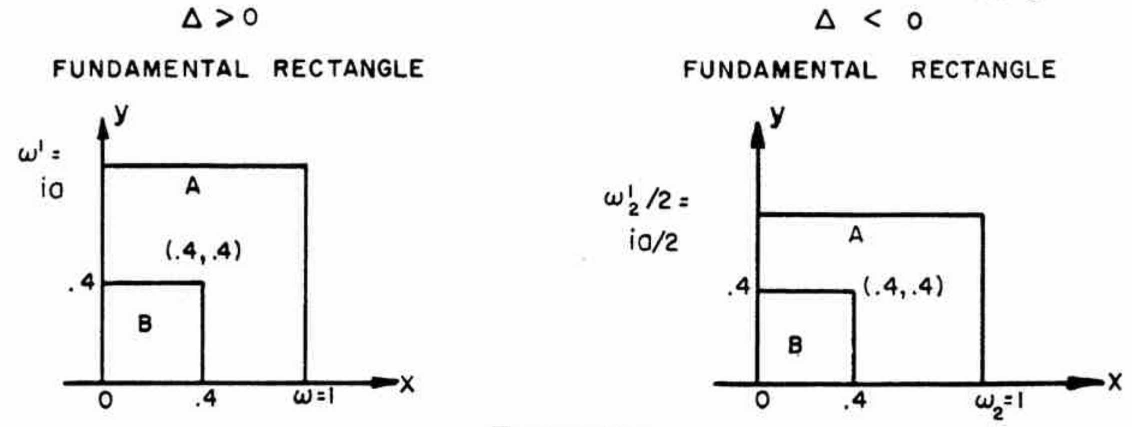


FIGURE 18.4

In region A

$\mathscr{R}(\wp')\geq 0$ if $y\geq .4$ and $x\leq .5$; $\mathscr{I}(\wp')\geq 0$ elsewhere

In region A

(1) If $a\geq 1.05$, use criterion for region A for $\Delta>0$.

(2) If $1\leq a<1.05$: $\mathscr{R}(\wp')\geq 0$ if $y\geq .4$ and $x\leq .4$, $-\pi/4<\arg\ (\wp')<3\pi/4$ if $.4<y\leq .5$ and $.4<x\leq .5$. $\mathscr{I}(\wp')\geq 0$ elsewhere

In region B

The sign (indeed, perhaps one or more significant digits) of $\wp'$ is obtainable from the first term, $-2/z^3$, of the Laurent series for $\wp'$.

In region B

Use the criterion for region B for $\Delta>0$.

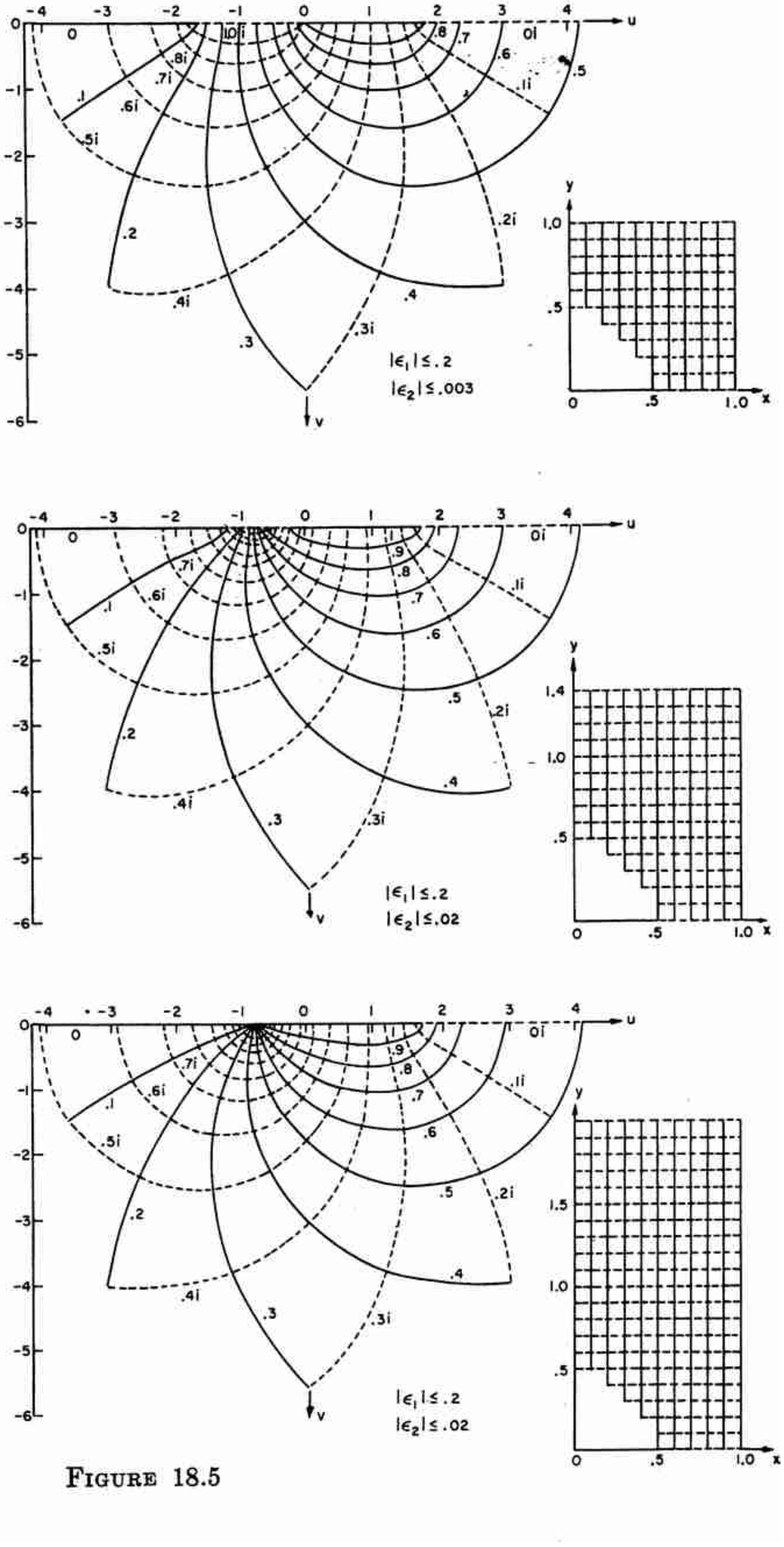
(Precisely similar criteria apply when the real half-period$\neq 1$)

$\Delta>0 \qquad \omega=1$

Map: $\wp(z)=u+iv$

Near zero: $\wp(z)=\frac{1}{z^2}+\epsilon_1$

$$\wp(z)=\frac{1}{z^2}+c_2z^2+\epsilon_2$$



$\omega'=i$

$\omega'=1.4i$

$\omega'=2.0i$

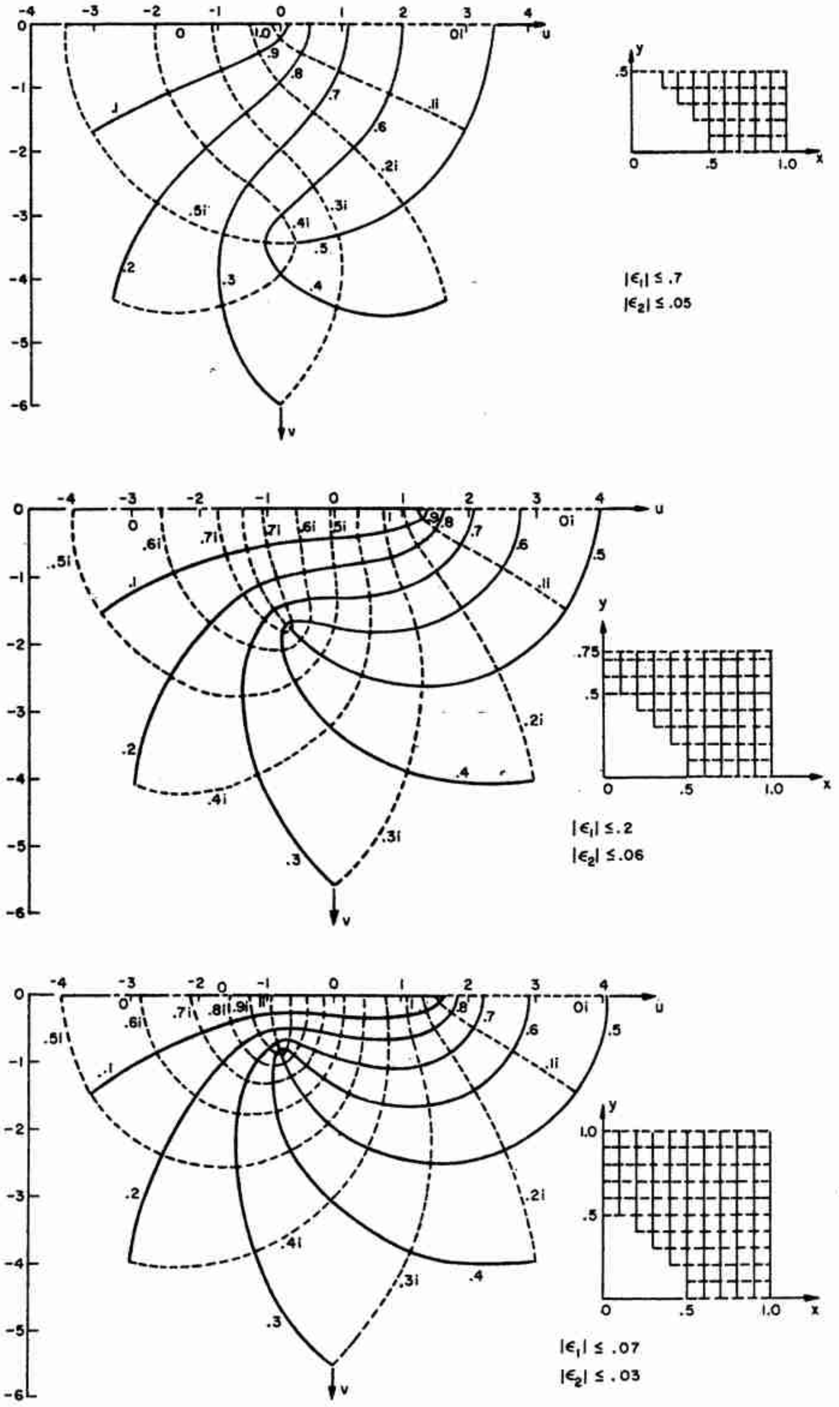
FIGURE 18.5

$$\Delta < 0 \qquad \omega_2 = 1$$

Map: $\wp(z) = u + iv$

Near zero: $\wp(z) = \frac{1}{z^2} + \epsilon_1$

$$\wp(z) = \frac{1}{z^2} + c_2 z^2 + \epsilon_2$$



$\omega_2' = i$

$|\epsilon_1| \le .7$
$|\epsilon_2| \le .05$

$\omega_2' = 1.5i$

$|\epsilon_1| \le .2$
$|\epsilon_2| \le .06$

$\omega_2' = 2.0i$

$|\epsilon_1| \le .07$
$|\epsilon_2| \le .03$

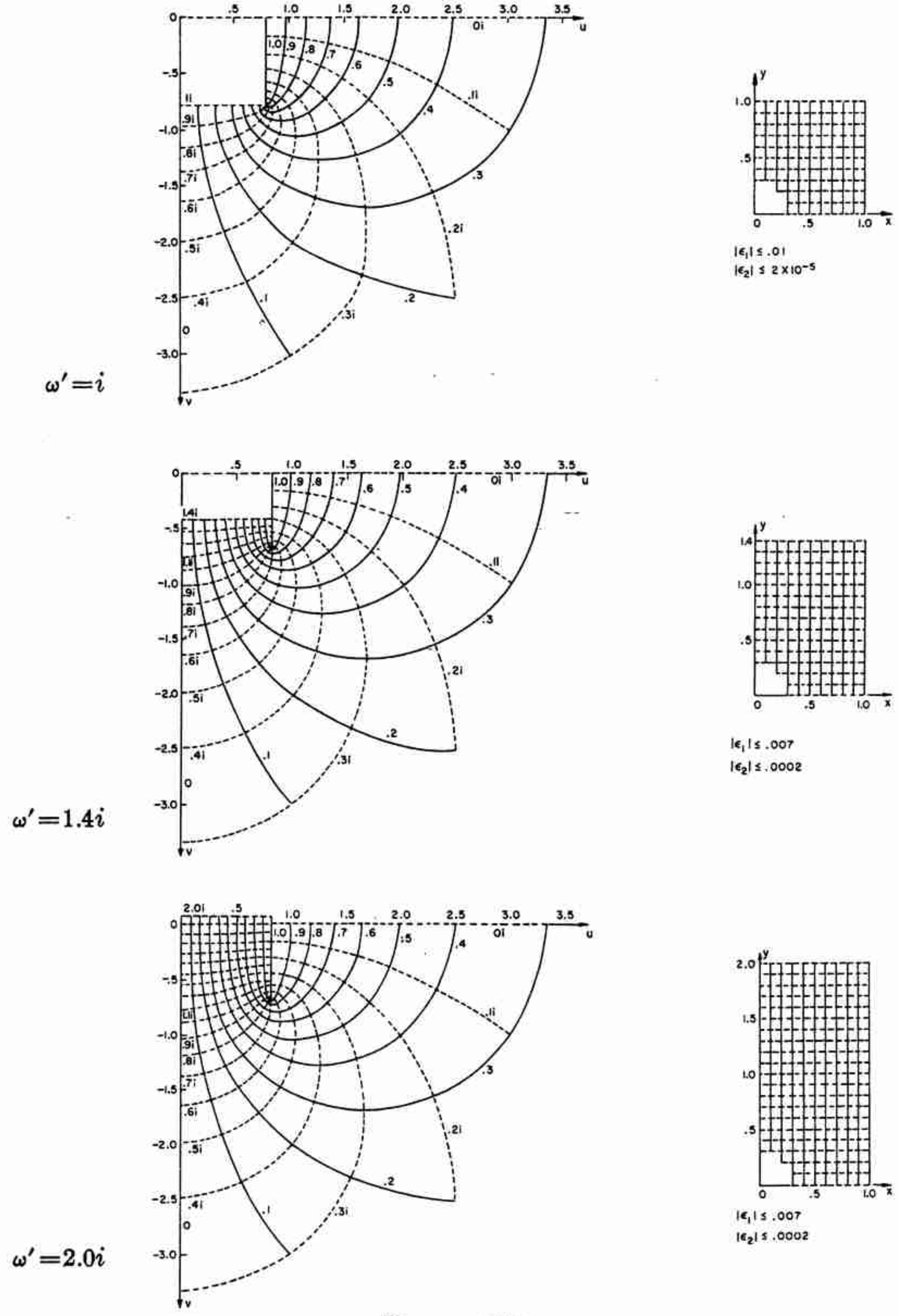
FIGURE 18.6

$$\Delta > 0 \qquad \omega = 1$$

Map: $\zeta(z) = u + iv$

Near zero: $\zeta(z) = \frac{1}{z} + \epsilon_1$

$$\zeta(z) = \frac{1}{z} - \frac{c_2 z^3}{3} + \epsilon_2$$



$\omega' = i$

$|\epsilon_1| \leq .01$
$|\epsilon_2| \leq 2 \times 10^{-5}$

$\omega' = 1.4i$

$|\epsilon_1| \leq .007$
$|\epsilon_2| \leq .0002$

$\omega' = 2.0i$

$|\epsilon_1| \leq .007$
$|\epsilon_2| \leq .0002$

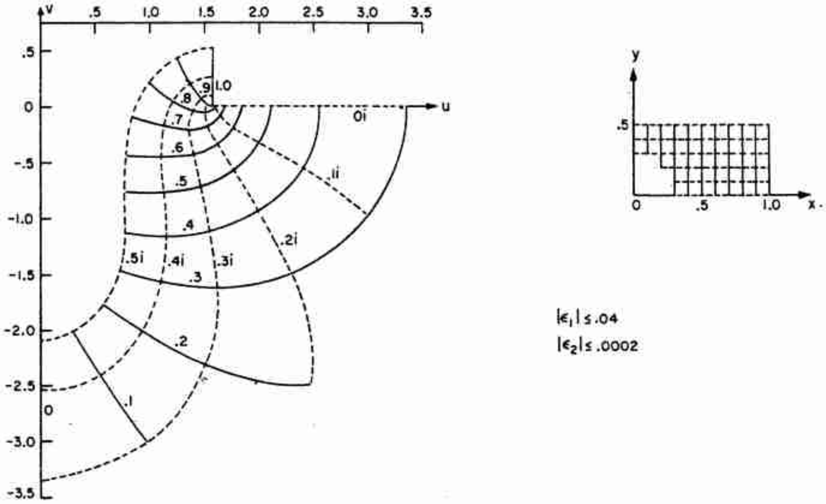
FIGURE 18.7

$\Delta<0 \qquad \omega_2=1$

Map: $\zeta(z)=u+iv$

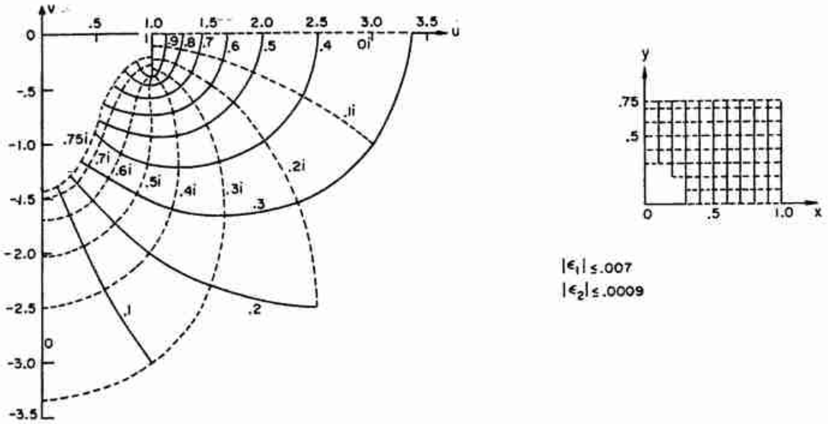
Near zero: $\zeta(z)=\frac{1}{z}+\epsilon_1$

$$\zeta(z)=\frac{1}{z}-\frac{c_2z^3}{3}+\epsilon_2$$



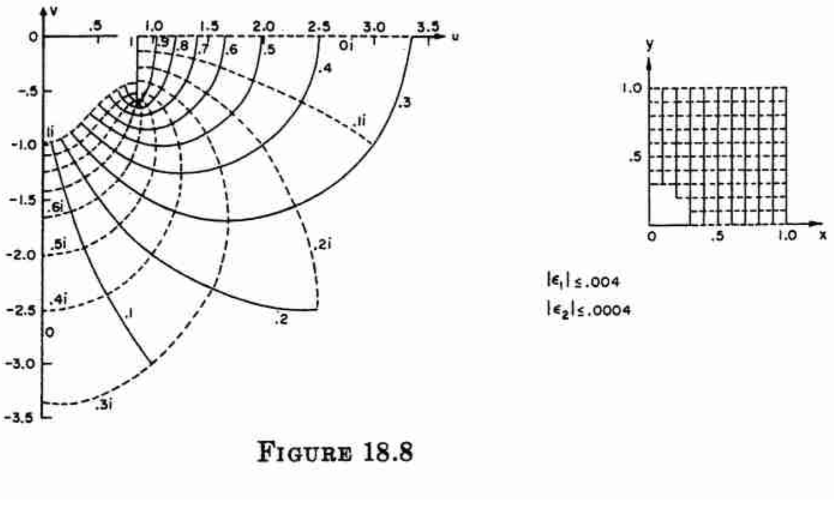
$|\epsilon_1|\leq .04$
$|\epsilon_2|\leq .0002$

$\omega_2'=i$



$|\epsilon_1|\leq .007$
$|\epsilon_2|\leq .0009$

$\omega_2'=1.5i$



$|\epsilon_1|\leq .004$
$|\epsilon_2|\leq .0004$

$\omega_2'=2.0i$

FIGURE 18.8

$\Delta > 0 \qquad \omega = 1$

Map: $\sigma(z) = u + iv$

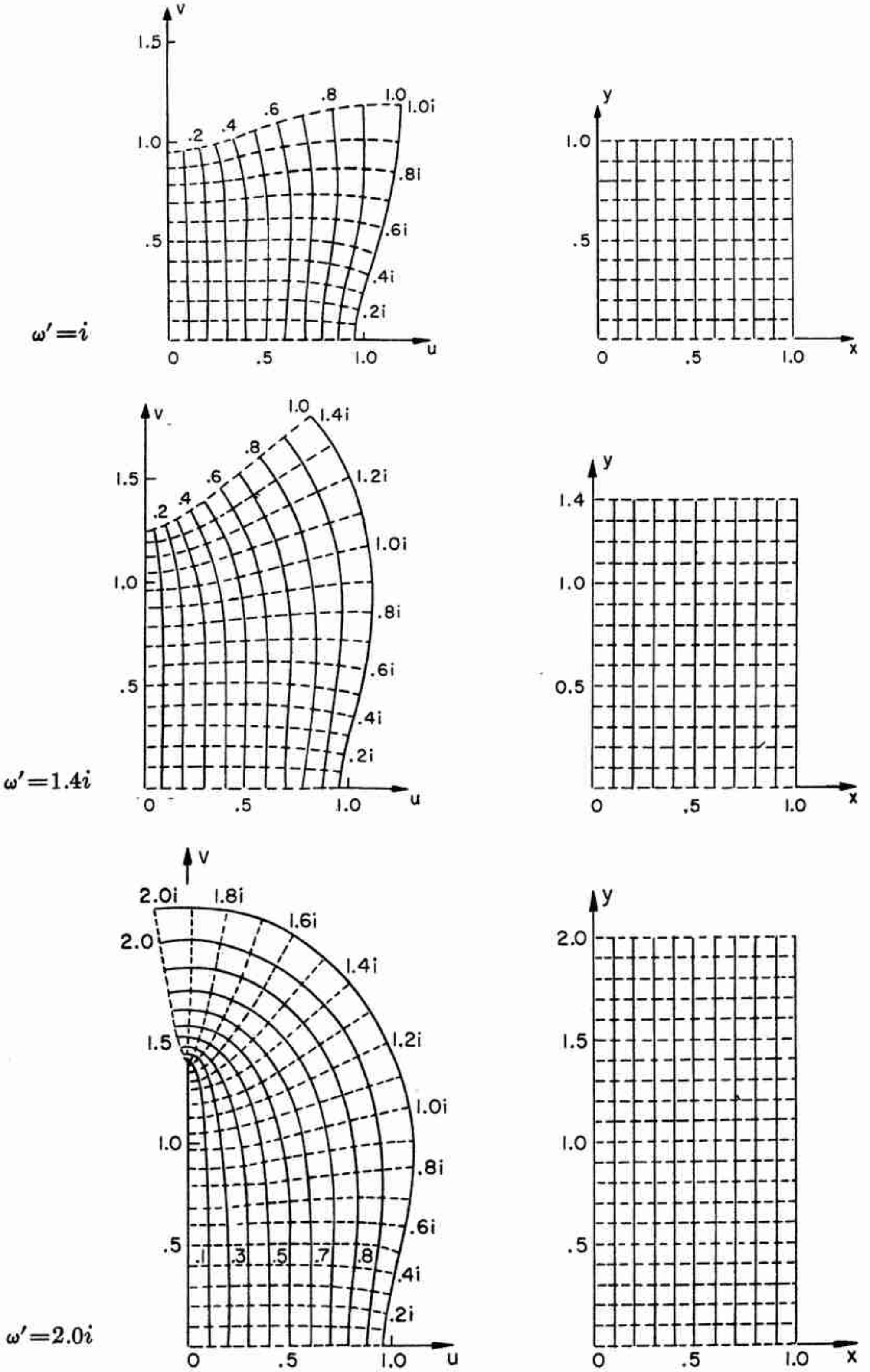


FIGURE 18.9

$\Delta<0 \qquad \omega_2=1$

Map: $\sigma(z)=u+iv$

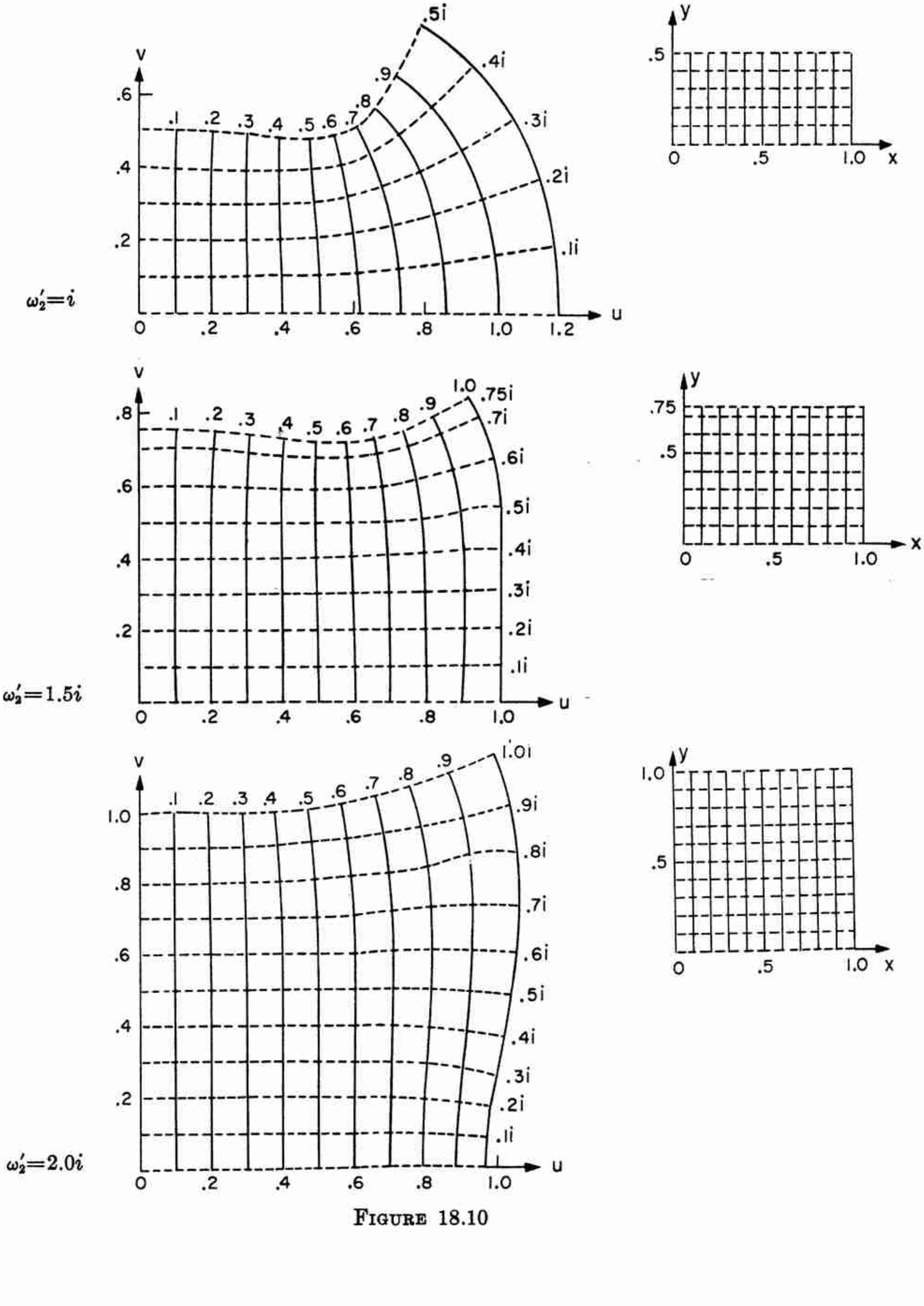


FIGURE 18.10

18.9. Relations with Complete Elliptic Integrals K and K' and Their Parameter m and with Jacobi's Elliptic Functions (see chapter 16)

(Here $K(m)$ and $K'(m)=K(1-m)$ are complete elliptic integrals of the 1st kind; see chapter 17.)

	$\Delta>0$	$\Delta<0$
18.9.1	$e_1=\frac{(2-m)K^2(m)}{3\omega^2}$	$e_1=\frac{(2m-1)+6i\sqrt{m-m^2}}{3\omega_2^2}\cdot K^2(m)$
18.9.2	$e_2=\frac{(2m-1)K^2(m)}{3\omega^2}$	$e_2=\frac{2(1-2m)K^2(m)}{3\omega_2^2}$
18.9.3	$e_3=\frac{-(m+1)K^2(m)}{3\omega^2}$	$e_3=\frac{(2m-1)-6i\sqrt{m-m^2}}{3\omega_2^2}\cdot K^2(m)$
18.9.4	$g_2=\frac{4(m^2-m+1)K^4(m)}{3\omega^4}$	$g_2=\frac{4(16m^2-16m+1)K^4(m)}{3\omega_2^4}$
18.9.5	$g_3=\frac{4(m-2)(2m-1)(m+1)K^6(m)}{27\omega^6}$	$g_3=\frac{8(2m-1)(32m^2-32m-1)K^6(m)}{27\omega_2^6}$
18.9.6	$\Delta=\frac{16m^2(m-1)^2K^{12}(m)}{\omega^{12}}$	$\Delta=\frac{-256(m-m^2)K^{12}(m)}{\omega_2^{12}}$
18.9.7	$\omega'=\frac{iK'(m)\omega}{K(m)}$	$\omega_2'=\frac{iK'(m)\omega_2}{K(m)}$
18.9.8	$\omega=K(m)/(e_1-e_3)^{1/2}$	$\omega_2=K(m)/H_2^{1/2}$
18.9.9	$m=(e_2-e_3)/(e_1-e_3)$	$m=\frac{1}{2}-\frac{3e_2}{4H_2}$
18.9.10		$[0<m\leq\frac{1}{2}$, since $g_3\geq0]$
18.9.11	$\wp(z)=e_3+(e_1-e_3)/\mathrm{sn}^2(z^*\vert m)$	$\wp(z)=e_2+H_2\frac{1+\mathrm{cn}(z'\vert m)}{1-\mathrm{cn}(z'\vert m)}$
18.9.12	$\wp'(z)=-2(e_1-e_3)^{3/2}\cdot\mathrm{cn}(z^*\vert m)\mathrm{dn}(z^*\vert m)/\mathrm{sn}^3(z^*\vert m)$ where $z^*=(e_1-e_3)^{\frac{1}{2}}z$	$\wp'(z)=\frac{-4H_2^{3/2}\mathrm{sn}(z'\vert m)\mathrm{dn}(z'\vert m)}{[1-\mathrm{cn}(z'\vert m)]^2}$ where $z'=2zH_2^{1/2}$
18.9.13	$\eta=\zeta(\omega)=\frac{K(m)}{3\omega}[3E(m)+(m-2)K(m)]$	$\eta_2=\zeta(\omega_2)=\frac{K(m)}{3\omega_2}[6E(m)+(4m-5)K(m)]$
18.9.14	$\eta'=\zeta(\omega')=\frac{\eta\omega'-\frac{1}{2}\pi i}{\omega}$	$\eta_2'=\zeta(\omega_2')=\frac{\eta_2\omega_2'-\pi i}{\omega_2}$

[$E(m)$ is a complete elliptic integral of the 2d kind (see chapter 17).]

18.10. Relations with Theta Functions (chapter 16)

The formal definitions of the four ϑ functions are given by the series **16.27.1–16.27.4** which converge for all complex z and all q defined below. (Some authors use πz, instead of z, as the independent variable.)

These functions depend on z and on a parameter q, which is usually suppressed. Note that

$$\vartheta_1'(0)=\vartheta_2(0)\vartheta_3(0)\vartheta_4(0), \text{ where } \vartheta_i(0)=\vartheta_i(0, q).$$

	$\Delta>0$	$\Delta<0$
18.10.1	$\tau=\omega'/\omega$	$\tau_2=\omega_2'/2\omega_2$
18.10.2	$q=e^{i\pi\tau}=e^{-\pi K'/K}$	$q=iq_2=ie^{i\pi\tau_2}=ie^{-\pi\lvert\omega_2'\rvert/2\omega_2}$
18.10.3	q is real and since $g_3\geq 0(\lvert\omega'\rvert\geq\omega)$, $0<q\leq e^{-\pi}$	q is pure imaginary and since $g_3\geq 0(\lvert\omega_2'\rvert\geq\omega_2)$, $0<\lvert q\rvert\leq e^{-\pi/2}$
18.10.4	$(v=\pi z/2\omega)$	$(v=\pi z/2\omega_2)$
18.10.5	$\wp(z)=e_j+\frac{\pi^2}{4\omega^2}\left[\frac{\vartheta_1'(0)\vartheta_{j+1}(v)}{\vartheta_{j+1}(0)\vartheta_1(v)}\right]^2$ $j=1, 2, 3$	$\wp(z)=e_2+\frac{\pi^2}{4\omega_2^2}\left[\frac{\vartheta_1'(0)\vartheta_2(v)}{\vartheta_2(0)\vartheta_1(v)}\right]^2$
18.10.6	$\wp'(z)=-\frac{\pi^3}{4\omega^3}\frac{\vartheta_2(v)\vartheta_3(v)\vartheta_4(v)\vartheta_1'^3(0)}{\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)\vartheta_1{}^3(v)}$	$\wp'(z)=-\frac{\pi^3}{4\omega_2^3}\frac{\vartheta_2(v)\vartheta_3(v)\vartheta_4(v)\vartheta_1'^3(0)}{\vartheta_2(0)\vartheta_3(0)\vartheta_4(0)\vartheta_1{}^3(v)}$
18.10.7	$\zeta(z)=\frac{\eta z}{\omega}+\frac{\pi\vartheta_1'(v)}{2\omega\vartheta_1(v)}$	$\zeta(z)=\frac{\eta_2 z}{\omega_2}+\frac{\pi\vartheta_1'(v)}{2\omega_2\vartheta_1(v)}$
18.10.8	$\sigma(z)=\frac{2\omega}{\pi}\exp\left(\frac{\eta z^2}{2\omega}\right)\frac{\vartheta_1(v)}{\vartheta_1'(0)}$	$\sigma(z)=\frac{2\omega_2}{\pi}\exp\left(\frac{\eta_2 z^2}{2\omega_2}\right)\frac{\vartheta_1(v)}{\vartheta_1'(0)}$
18.10.9	$12\omega^2 e_1=\pi^2[\vartheta_3^4(0)+\vartheta_4^4(0)]$	$12\omega_2^2 e_1=\pi^2[\vartheta_2^4(0)-\vartheta_4^4(0)]$
18.10.10	$12\omega^2 e_2=\pi^2[\vartheta_2^4(0)-\vartheta_4^4(0)]$	$12\omega_2^2 e_2=\pi^2[\vartheta_3^4(0)+\vartheta_4^4(0)]$
18.10.11	$12\omega^2 e_3=-\pi^2[\vartheta_2^4(0)+\vartheta_3^4(0)]$	$12\omega_2^2 e_3=-\pi^2[\vartheta_2^4(0)+\vartheta_3^4(0)]$
18.10.12	$(e_2-e_3)^{\frac{1}{2}}=-i(e_3-e_2)^{\frac{1}{2}}=\frac{\pi}{2\omega}\vartheta_2^2(0)$	$(e_2-e_3)^{\frac{1}{2}}=i(e_3-e_2)^{\frac{1}{2}}=\frac{\pi}{2\omega_2}\vartheta_3^2(0)$
18.10.13	$(e_1-e_3)^{\frac{1}{2}}=-i(e_3-e_1)^{\frac{1}{2}}=\frac{\pi}{2\omega}\vartheta_3^2(0)$	$(e_1-e_3)^{\frac{1}{2}}=i(e_3-e_1)^{\frac{1}{2}}=\frac{\pi}{2\omega_2}\vartheta_2^2(0)$
18.10.14	$(e_1-e_2)^{\frac{1}{2}}=-i(e_2-e_1)^{\frac{1}{2}}=\frac{\pi}{2\omega}\vartheta_4^2(0)$	$(e_2-e_1)^{\frac{1}{2}}=-i(e_1-e_2)^{\frac{1}{2}}=\frac{\pi}{2\omega_2}\vartheta_4^2(0)$
18.10.15	$g_2=\frac{2}{3}\left(\frac{\pi}{2\omega}\right)^4[\vartheta_2^8(0)+\vartheta_3^8(0)+\vartheta_4^8(0)]$	$g_2=\frac{2}{3}\left(\frac{\pi}{2\omega_2}\right)^4[\vartheta_2^8(0)+\vartheta_3^8(0)+\vartheta_4^8(0)]$
18.10.16	$g_3=4e_1e_2e_3$	$g_3=4e_1e_2e_3$
18.10.17	$\Delta^{\frac{1}{2}}=\frac{\pi^3}{4\omega^3}\vartheta_1'^2(0)$	$(-\Delta)^{\frac{1}{2}}=\frac{\pi^3}{4\omega_2^3}\vartheta_1'^2(0)e^{-i\pi/4}$
18.10.18	$\eta\equiv\zeta(\omega)=-\frac{\pi^2\vartheta_1'''(0)}{12\omega\vartheta_1'(0)}$	$\eta_2\equiv\zeta(\omega_2)=-\frac{\pi^2\vartheta_1'''(0)}{12\omega_2\vartheta_1'(0)}$
18.10.19	$\eta'\equiv\zeta(\omega')=\frac{\eta\omega'-\frac{1}{2}\pi i}{\omega}$	$\eta_2'\equiv\zeta(\omega_2')=\frac{\eta_2\omega_2'-\pi i}{\omega_2}$

Series

18.10.20 $$\vartheta_1(0)=0$$

18.10.21 $$\vartheta_2(0)=2q^{\frac{1}{4}}[1+q^{1\cdot 2}+q^{2\cdot 3}+q^{3\cdot 4}+\ldots+q^{n(n+1)}+\ldots]$$

18.10.22 $$\vartheta_3(0)=1+2[q+q^4+q^9+\ldots+q^{n^2}+\ldots]$$

18.10.23 $$\vartheta_4(0)=1+2[-q+q^4-q^9+\ldots+(-1)^n q^{n^2}+\ldots]$$

Attainable Accuracy

	$\Delta>0$	$\Delta<0$
	Note: $\vartheta_j(0)>0$, $j=2, 3, 4$	Note: $\vartheta_2(0)=Ae^{i\pi/8}$, $A>0$; $\mathcal{R}\vartheta_3(0)>0$; $\vartheta_4(0)=\overline{\vartheta_3(0)}$
$\vartheta_j(0)$:	2 terms give at least 5S	2 terms give at least 3S
$j=2, 3, 4$	3 terms give at least 11S	3 terms give at least 5S
	4 terms give at least 21S	4 terms give at least 10S

18.11 Expressing any Elliptic Function in Terms of $\wp$ and $\wp'$

If $f(z)$ is any elliptic function and $\wp(z)$ has same periods, write

18.11.1 $$f(z)=\tfrac{1}{2}[f(z)+f(-z)]+\tfrac{1}{2}[\{f(z)-f(-z)\}\{\wp'(z)\}^{-1}]\wp'(z).$$

Since both brackets represent even elliptic functions, we ask how to express an even elliptic function $g(z)$ (of order $2k$) in terms of $\wp(z)$. Because of the evenness, an irreducible set of zeros can be denoted by a_i $(i=1, 2, \ldots, k)$ and the set of points congruent to $-a_i$ $(i=1, 2, \ldots, k)$; correspondingly in connection with the poles we consider the points $\pm b_i$, $i=1, 2, \ldots, k$. Then

18.11.2 $$g(z)=A\prod_{i=1}^{k}\left\{\frac{\wp(z)-\wp(a_i)}{\wp(z)-\wp(b_i)}\right\},\quad \text{where } A \text{ is}$$

a constant. If any a_i or b_i is congruent to the origin, the corresponding factor is omitted from the product. Factors corresponding to multiple poles (zeros) are repeated according to the multiplicity.

18.12. Case $\Delta=0(c>0)$

Subcase I

18.12.1 $g_2>0$, $g_3<0$: $(e_1=e_2=c,\ e_3=-2c)$

18.12.2 $$H_1=H_2=0,\ H_3=3c$$

18.12.3 $$\wp(z; 12c^2, -8c^3)=c+3c\{\sinh[(3c)^{\frac{1}{2}}z]\}^{-2}$$

18.12.4 $$\zeta(z; 12c^2, -8c^3)=-cz+(3c)^{\frac{1}{2}}\coth[(3c)^{\frac{1}{2}}z]$$

18.12.5 $$\sigma(z; 12c^2, -8c^3)=(3c)^{-\frac{1}{2}}\sinh[(3c)^{\frac{1}{2}}z]e^{-cz^2/2}$$

18.12.6 $$\omega=\infty,\ \omega'=(12c)^{-\frac{1}{2}}\pi i$$

18.12.7 $$\eta=\zeta(\omega)=-\infty$$

18.12.8 $$\eta'=\zeta(\omega')=-c\omega'$$

18.12.9 $$q=1,\qquad m=1$$

18.12.10 $$\sigma(\omega)=0$$

18.12.11 $$\sigma(\omega')=\frac{2\omega' e^{\pi^2/24}}{\pi}$$

18.12.12 $$\sigma(\omega_2)=0$$

18.12.13 $$\wp(\omega/2)=c$$

18.12.14 $$\wp'(\omega/2)=0$$

18.12.15 $$\zeta(\omega/2)=-\infty$$

18.12.16 $$\sigma(\omega/2)=0$$

18.12.17 $$\wp(\omega'/2)=-5c$$

18.12.18 $$\wp'(\omega'/2)=\frac{-\pi^3}{2\omega'^3}$$

18.12.19 $$\zeta(\omega'/2)=\tfrac{1}{2}(-c\omega'+\pi/\omega')$$

18.12.20 $$\sigma(\omega'/2)=\frac{\omega' e^{\pi^2/96}\sqrt{2}}{\pi}$$

18.12.21 $$\wp(\omega_2/2)=c$$

18.12.22 $$\wp'(\omega_2/2)=0$$

18.12.23 $$\zeta(\omega_2/2)=-\infty-\frac{c\omega'}{2}$$

18.12.24 $$\sigma(\omega_2/2)=0$$

Subcase II

18.12.25

$$g_2>0,\ g_3>0\colon\ (e_1=2c,\ e_2=e_3=-c)$$

18.12.26 $$H_1=3c,\ H_2=H_3=0$$

18.12.27 $$\wp(z;12c^2,8c^3)=-c+3c\{\sin[(3c)^{\frac{1}{2}}z]\}^{-2}$$

18.12.28

$$\zeta(z;12c^2,8c^3)=cz+(3c)^{\frac{1}{2}}\cot[(3c)^{\frac{1}{2}}z]$$

18.12.29

$$\sigma(z;12c^2,8c^3)=(3c)^{-\frac{1}{2}}\sin[(3c)^{\frac{1}{2}}z]e^{cz^2/2}$$

18.12.30 $$\omega=(12c)^{-\frac{1}{2}}\pi,\ \omega'=i\infty$$

18.12.31 $$\eta=\zeta(\omega)=c\omega$$

18.12.32 $$\eta'=\zeta(\omega')=i\infty$$

18.12.33 $$q=0,\qquad m=0$$

18.12.34 $$\sigma(\omega)=\frac{2\omega e^{\pi^2/24}}{\pi}$$

18.12.35 $$\sigma(\omega')=0$$

18.12.36 $$\sigma(\omega_2)=0$$

18.12.37 $$\wp(\omega/2)=5c$$

18.12.38 $$\wp'(\omega/2)=\frac{-\pi^3}{2\omega^3}$$

18.12.39 $$\zeta(\omega/2)=\tfrac{1}{2}(c\omega+\pi/\omega)$$

18.12.40 $$\sigma(\omega/2)=\frac{e^{\pi^2/96}\omega\sqrt{2}}{\pi}$$

18.12.41 $$\wp(\omega'/2)=-c$$

18.12.42 $$\wp'(\omega'/2)=0$$

18.12.43 $$\zeta(\omega'/2)=+i\infty$$

18.12.44 $$\sigma(\omega'/2)=0$$

18.12.45 $$\wp(\omega_2/2)=-c$$

18.12.46 $$\wp'(\omega_2/2)=0$$

18.12.47 $$\zeta(\omega_2/2)=\frac{c\omega}{2}+i\infty$$

18.12.48 $$\sigma(\omega_2/2)=0$$

Subcase III

18.12.49 $$g_2=0,\ g_3=0\,(e_1=e_2=e_3=0)$$

18.12.50 $$\wp(z;0,0)=z^{-2}$$

18.12.51 $$\zeta(z;0,0)=z^{-1}$$

18.12.52 $$\sigma(z;0,0)=z$$

18.12.53 $$\omega=-i\omega'=\infty$$

18.13. Equianharmonic Case ($g_2=0$, $g_3=1$)

If $g_2=0$ and $g_3>0$, homogeneity relations allow us to reduce our considerations of $\wp$ to $\wp(z;0,1)$ ($\wp'$, ζ and σ are handled similarly). Thus $\wp(z;0,g_3)=g_3^{1/3}\wp(zg_3^{1/6};0,1)$. The case $g_2=0$, $g_3=1$ is called the EQUIANHARMONIC case.

$\frac{1}{4}$ FPP; *Reduction to Fundamental Triangle*

$\Delta_1\equiv\Delta 0\omega_2 z_0$ is the Fundamental Triangle

Let ϵ denote $e^{[illegible]\pi/3}$ throughout **18.13.**

$\omega_2\approx 1.5299\ 54037\ 05719\ 28749\ 13194\ 17231$[6]

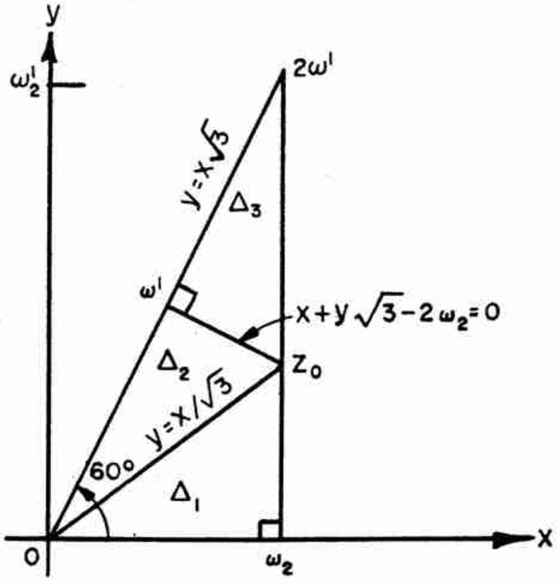


FIGURE 18.11

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[6] This value was computed and checked by multiple precision on a desk calculator and is believed correct to 30S.

Reduction for z_2 *in* Δ_2*:* $z_1=\epsilon\bar{z}_2$ is in Δ_1.

18.13.1 $\wp(z_2)=\epsilon^{-2}\overline{\wp}(z_1)$

18.13.2 $\wp'(z_2)=-\overline{\wp}'(z_1)$

18.13.3 $\zeta(z_2)=\epsilon^{-1}\bar{\zeta}(z_1)$

18.13.4 $\sigma(z_2)=\epsilon\bar{\sigma}(z_1)$

Reduction for z_3 *in* Δ_3*:* $z_1=\epsilon^{-1}(2\omega'-z_3)$ is in Δ_1

18.13.5 $\wp(z_3)=\epsilon^{-2}\wp(z_1)$

18.13.6 $\wp'(z_3)=\wp'(z_1)$

18.13.7 $\zeta(z_3)=-\epsilon^{-1}\zeta(z_1)+2\eta', \qquad \eta'=\zeta(\omega')$

18.13.8 $\sigma(z_3)=\epsilon\sigma(z_1)\exp[(z_3-\omega')(2\eta')]$

Special Values and Formulas

18.13.9

$$\Delta=-27, \qquad H_1=\sqrt{3}(4^{-1/3})\bar{\epsilon},$$

$$H_2=\sqrt{3}(4^{-1/3}), \qquad H_3=\sqrt{3}(4^{-1/3})\epsilon$$

18.13.10 $m=\sin^2 15°=\frac{2-\sqrt{3}}{4}, \qquad q=ie^{-\pi\sqrt{3}/2}$

18.13.11 $\vartheta_2(0)=Ae^{i\pi/8}$

18.13.12 $\vartheta_3(0)=Ae^{i\pi/24}$

18.13.13 $\vartheta_4(0)=Ae^{-i\pi/24}$

18.13.14

where $A=(\omega_2/\pi)^{1/2}2^{1/3}3^{1/8}\approx 1.0086\ 67$

18.13.15 $\omega_2=\frac{K(m)2^{1/3}}{3^{1/4}}=\frac{\Gamma^3(1/3)}{4\pi}$

Values at Half-periods

	$\wp$	$\wp'$	ζ	σ
18.13.16 $\omega\equiv\omega_1$	$e_1=4^{-1/3}\epsilon^2$	0	$\eta=\epsilon\pi/2\omega_2\sqrt{3}$	$\epsilon^{-1}\sigma(\omega_2)$
18.13.17 ω_2	$e_2=4^{-1/3}$	0	$\eta_2=\eta+\eta'=\pi/2\omega_2\sqrt{3}$	$\frac{e^{\pi/4\sqrt{3}}(2^{1/3})}{3^{\frac{1}{4}}}$
18.13.18 $\omega'\equiv\omega_3$	$e_3=4^{-1/3}\epsilon^{-2}$	0	$\eta'=\epsilon^{-1}\pi/2\omega_2\sqrt{3}$	$\epsilon\sigma(w_2)$
18.13.19 ω_2'	$e_2=4^{-1/3}$	0	$\eta_2'=-\pi i/2\omega_2=\eta'-\eta$	$\frac{ie^{3\pi/4\sqrt{3}}(2^{1/3})}{3^{\frac{1}{4}}}$

Values[7] along $(0, \omega_2)$

	$\wp$	$\wp'$	ζ	σ
18.13.20 $2\omega_2/9$	$\frac{\sqrt[3]{\cos 80°}}{\sqrt[3]{\cos 20°}-\sqrt[3]{\cos 40°}}$	$\frac{-\sqrt{3}[\sqrt[3]{\cos 20°}+\sqrt[3]{\cos 40°}]}{\sqrt[3]{\cos 20°}-\sqrt[3]{\cos 40°}}$		
18.13.21 $\omega_2/3$	$1/(2^{1/3}-1)$	$-\sqrt{3}(2^{1/3}+1)/(2^{1/3}-1)$	$\frac{\eta_2}{3}+\frac{\sqrt{3}(2^{2/3}+2+2^{4/3})}{6}$	$\frac{e^{\pi/36\sqrt{3}}}{3^{1/6}}\sqrt[4]{\frac{2^{1/3}-1}{2^{1/3}+1}}$
18.13.22 $4\omega_2/9$	$\frac{\sqrt[3]{\cos 40°}}{\sqrt[3]{\cos 20°}-\sqrt[3]{\cos 80°}}$	$\frac{-\sqrt{3}[\sqrt[3]{\cos 20°}+\sqrt[3]{\cos 80°}]}{\sqrt[3]{\cos 20°}-\sqrt[3]{\cos 80°}}$		
18.13.23 $\omega_2/2$	e_2+H_2	$-3^{3/4}\sqrt{2+\sqrt{3}}$	$(\pi/4\omega_2\sqrt{3})+(3^{1/4}\sqrt{2+\sqrt{3}}/2^{4/3})$	$\frac{e^{\pi/16\sqrt{3}}(2^{1/12})}{3^{1/4}\sqrt[8]{2+\sqrt{3}}}$
18.13.24 $2\omega_2/3$	1	$-\sqrt{3}$	$\frac{2}{3}(\eta_2)+3^{-1/2}$	$e^{\pi/9\sqrt{3}}/3^{1/6}$
18.13.25 $8\omega_2/9$	$\frac{\sqrt[3]{\cos 20°}}{\sqrt[3]{\cos 40°}+\sqrt[3]{\cos 80°}}$	$\frac{-\sqrt{3}[\sqrt[3]{\cos 40°}-\sqrt[3]{\cos 80°}]}{\sqrt[3]{\cos 40°}+\sqrt[3]{\cos 80°}}$		

---

[7] Values at $2\omega_2/9$, $4\omega_2/9$ and $8\omega_2/9$ from [18.14].

Values along (0, z_0)

	$\wp$	$\wp'$	ζ	σ
18.13.26 $z_0/2$	$-2^{1/3}\epsilon^2$	$3i$	$\left[\frac{\eta_2}{\sqrt{3}}+2^{-1/3}\right]e^{-i\pi/6}$	$\frac{e^{\pi/12\sqrt{3}}e^{i\pi/6}}{3^{1/4}}$
18.13.27 $3z_0/4$	$\epsilon^2(e_2-H_2)$	$i(3^{3/4})\sqrt{2-\sqrt{3}}$	$\left[\frac{\pi}{4\omega_2}+\frac{3^{1/4}\sqrt{2-\sqrt{3}}}{2^{4/3}}\right]e^{-i\pi/6}$	$\frac{e^{3\pi/16\sqrt{3}}(2^{1/12})e^{i\pi/6}}{3^{1/4}\sqrt[8]{2-\sqrt{3}}}$
18.13.28 z_0	0	i	$\frac{2\eta_2}{\sqrt{3}}e^{-i\pi/6}$	$e^{\pi/3\sqrt{3}} \cdot e^{i\pi/6}$

Duplication Formulas

18.13.29 $$\wp(2z)=\frac{\wp(z)[\wp^3(z)+2]}{4\wp^3(z)-1}$$

18.13.30 $$\wp'(2z)=\frac{2\wp^6(z)-10\wp^3(z)-1}{[\wp'(z)]^3}$$

18.13.31 $$\zeta(2z)=2\zeta(z)+\frac{3\wp^2(z)}{\wp'(z)}$$

18.13.32 $$\sigma(2z)=-\wp'(z)\sigma^4(z)$$

Trisection Formulas (x real)

18.13.33 $$\wp\left(\frac{x}{3}\right)=\frac{\sqrt[3]{\cos\frac{\phi-\pi}{3}}}{\sqrt[3]{\cos\frac{\phi}{3}}-\sqrt[3]{\cos\frac{\phi+\pi}{3}}}$$

18.13.34 $$\wp'\left(\frac{x}{3}\right)=-\sqrt{3}\,\frac{\sqrt[3]{\cos\frac{\phi}{3}}+\sqrt[3]{\cos\frac{\phi+\pi}{3}}}{\sqrt[3]{\cos\frac{\phi}{3}}-\sqrt[3]{\cos\frac{\phi+\pi}{3}}}$$

where $\tan\phi=\wp'(x)$, $0<x<2\omega_2$ and we must choose ϕ in intervals

$$\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\left(\frac{\pi}{2},\frac{3\pi}{2}\right),\left(\frac{3\pi}{2},\frac{5\pi}{2}\right)\text{ to get}$$

$$\wp\left(\frac{x}{3}\right),\wp\left(\frac{x}{3}+\frac{2\omega_2}{3}\right),\wp\left(\frac{x}{3}+\frac{4\omega_2}{3}\right)\text{, respectively.}$$

Complex Multiplication

18.13.35 $$\wp(\epsilon z)=\epsilon^{-2}\wp(z)$$

18.13.36 $$\wp'(\epsilon z)=-\wp'(z)$$

18.13.37 $$\zeta(\epsilon z)=\epsilon^{-1}\zeta(z)$$

18.13.38 $$\sigma(\epsilon z)=\epsilon\sigma(z)$$

In the above, ϵ denotes (as it does throughout section **18.13**), $e^{i\pi/3}$. The above equations are useful as follows, e.g.:

If z is real, ϵz is on $0\omega'$ **(Figure 18.11)**; if ϵz were purely imaginary, z would be on $0z_0$ **(Figure 18.11).**

Conformal Maps

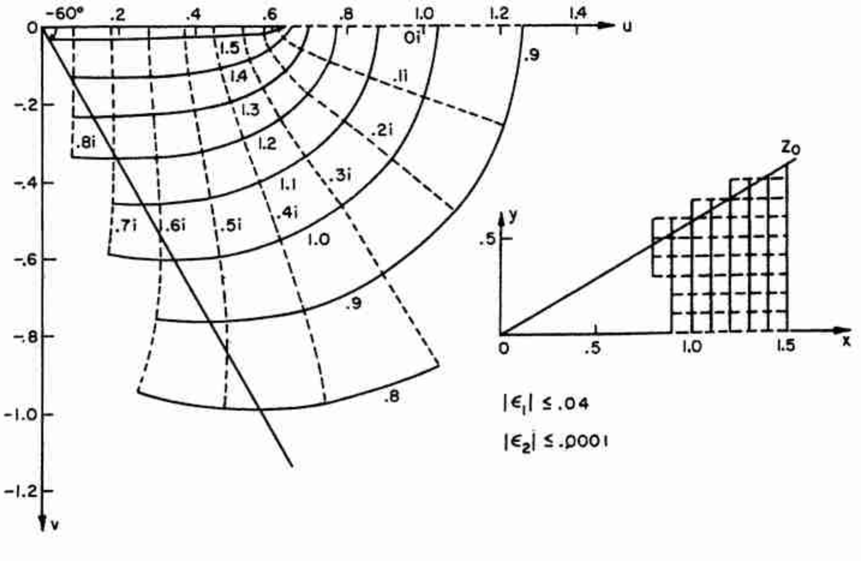
Equianharmonic Case

Map: $f(z)=u+iv$

$\wp(z)$

Near zero: $\wp(z)=\frac{1}{z^2}+\epsilon_1$

$$\wp(z)=\frac{1}{z^2}+\frac{z^4}{28}+\epsilon_2$$



$|\epsilon_1| \leq .04$

$|\epsilon_2| \leq .0001$

$\wp'(z)$

Near zero: $\wp'(z)=\frac{-2}{z^3}+\epsilon_1$

$$\wp'(z)=\frac{-2}{z^3}+\frac{z^3}{7}+\epsilon_2$$

$|\epsilon_1| \leq .2$
$|\epsilon_2| \leq .001$

$\zeta(z)$

Near zero: $\zeta(z)=\frac{1}{z}+\epsilon_1$

$$\zeta(z)=\frac{1}{z}-\frac{z^5}{140}+\epsilon_2$$

$|\epsilon_1| \leq .007$
$|\epsilon_2| \leq 1 \times 10^{-5}$

$\sigma(z)$

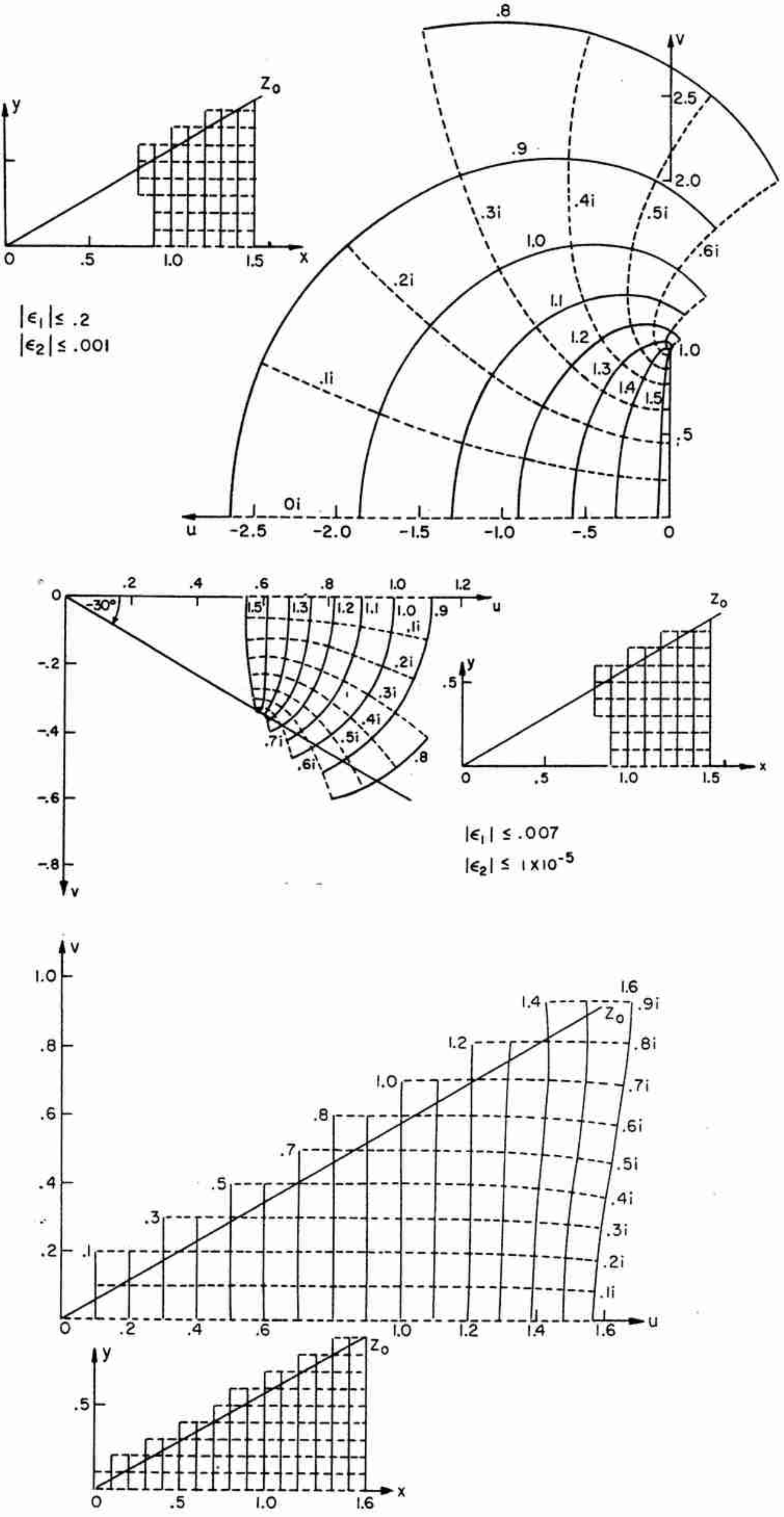


FIGURE 18.12

Coefficients for Laurent Series for $\wp$, $\wp'$ and ζ

($c_m=0$ for $m\neq 3k$)

k	EXACT c_{3k}	APPROXIMATE c_{3k}
1	$1/28$	3. 5714 28571 42857 $\ldots\times 10^{-2}$
2	$1/(13\cdot 28^2)=1/10192$	9. 8116 16954 47409 73312 40188×10^{-5}
3	$1/(13\cdot 19\cdot 28^3)=1/5422144$	1. 8442 88901 21693 55885 78983×10^{-7}
4	$3/(5\cdot 13^2\cdot 19\cdot 28^4)=234375/(7709611\times 10^8)$	3. 0400 36650 35758 61350 20301×10^{-10}
5	$4/(5\cdot 13^2\cdot 19\cdot 31\cdot 28^5)=78125/(16729\ 85587\times 10^8)$	4. 6697 95161 83961 00384 33643×10^{-13}
6	$(7\cdot 43)/(13^3\cdot 19^2\cdot 31\cdot 37\cdot 28^6)$	6. 8662 18676 79393 36788 98×10^{-16}
7	$(6\cdot 431)/(5\cdot 13^3\cdot 19^2\cdot 31\cdot 37\cdot 43\cdot 28^7)$	9. 7990 31742 57961 41839 66×10^{-19}
8	$(3\cdot 7\cdot 313)/(5^2\cdot 13^4\cdot 19^2\cdot 31\cdot 37\cdot 43\cdot 28^8)$	1. 3685 06574 79360 13026 87×10^{-21}
9	$(4\cdot 1201)/(5^2\cdot 13^4\cdot 19^3\cdot 31\cdot 37\cdot 43\cdot 28^9)$	1. 8800 72610 01329 79236 40×10^{-24}
10	$(2^2\cdot 3\cdot 41\cdot 1823)/(5\cdot 13^5\cdot 19^3\cdot 31^2\cdot 37\cdot 43\cdot 61\cdot 28^{10})$	2. 5497 66946 68202 63683×10^{-27}
11	$(3\cdot 79\cdot 733)/(5\cdot 13^4\cdot 19^3\cdot 31^2\cdot 37\cdot 43\cdot 61\cdot 67\cdot 28^{11})$	3. 4222 48599 51463 05316×10^{-30}
12	$\dfrac{3\cdot 1153\cdot 13963\cdot 29059}{5^3\cdot 13^6\cdot 19^4\cdot 31^2\cdot 37^2\cdot 43\cdot 61\cdot 67\cdot 73\cdot 28^{12}}$	4. 5541 38864 99184 30391×10^{-33}
13	$\dfrac{2^2\cdot 3^2\cdot 7\cdot 11\cdot 2647111}{5^3\cdot 13^5\cdot 19^4\cdot 31^2\cdot 37^2\cdot 61\cdot 67\cdot 73\cdot 79\cdot 28^{13}}$	6. 0171 15776 98241 99591×10^{-36}

First 5 approximate values determined from exact values of c_{3k}; subsequent values determined by using exact ratios c_{3k}/c_{3k-3}, using at least double precision arithmetic with a desk calculator. All approximate c's were checked with the use of the recursion relation; c_3-c_{27} are believed correct to at least 21S; $c_{30}-c_{39}$ are believed correct to 20S.

$$c_{3k}\leq\frac{c_3}{13^{k-1}\cdot 28^{k-1}},\ k=2,3,4,\ldots$$

Other Series Involving $\wp$

Reversed Series for Large $|\wp|$

18.13.39

$$z=(\wp^{-1})^{1/2}\left[1+\frac{u}{7}+\frac{3u^2}{26}+\frac{5u^3}{38}+\frac{7u^4}{40}+\frac{63u^5}{248}+\frac{231u^6}{592}+\frac{429u^7}{688}+O(u^8)\right],$$

18.13.40 where $u=\wp^{-3}/8$ and z is in the Fundamental Triangle **(Figure 18.11)** if $\wp$ has an appropriate value.

Series near z_0

18.13.41

$$\wp=iu\left[1-\frac{u^6}{7}+\frac{3u^{12}}{364}\right]+u^4\left[-\tfrac{1}{2}+\frac{u^6}{28}\right]+O(u^{16})$$

18.13.42

$$u=-i\wp\left[1+\frac{\wp^3}{2}+\frac{6\wp^6}{7}+2\wp^9+\frac{70\wp^{12}}{13}+O(\wp^{15})\right],$$

18.13.43 where $u=(z-z_0)$

Series near ω_2

18.13.44

$$(\wp-e_2)=3e_2^2u\left[1+x+x^2+\frac{6}{7}x^3+\frac{5}{7}x^4+\frac{4}{7}x^5+\frac{285}{637}x^6+O(x^7)\right],$$

18.13.45 where $u=(z-\omega_2)^2,\ x=e_2u$

18.13.46

$$u=e_2^{-1}\left[w-w^2+w^3-\frac{6}{7}w^4+\frac{3}{7}w^5+\frac{3}{7}w^6-\frac{1143}{637}w^7+O(w^8)\right],$$

18.13.47 where $w=(\wp-e_2)/3e_2$

Other Series Involving $\wp'$

Reversed Series for Large $|\wp'|$

18.13.48

$$z=2^{1/3}(\wp'^{1/3})^{-1}e^{i\pi/3}\left[1-\frac{2}{21}(\wp')^{-2}+\frac{5}{117}(\wp')^{-4}+O(\wp'^{-6})\right],$$

z being in the Fundamental Triangle **(Figure 18.11)** if $\wp'$ has an appropriate value.

Series near z_0

18.13.49

$$(\wp'-i)=x\left[-2-ix+\frac{5}{14}x^2+\frac{3i}{28}x^3+O(x^4)\right]$$

18.13.50 where $x=(z-z_0)^3$

18.13.51 $x=2\alpha\left[1-i\alpha-\frac{9}{7}\alpha^2+\frac{13i\alpha^3}{7}+O(\alpha^4)\right],$

18.13.52 where $\alpha=(\wp'-i)/(-4)$

Series near ω_2

18.13.53

$$\wp'=6e_2^2(z-\omega_2)\left[1+2v+3v^2+\frac{24}{7}v^3+\frac{25}{7}v^4+\frac{24}{7}v^5+\frac{285}{91}v^6+O(v^7)\right],$$

18.13.54 where $v=e_2(z-\omega_2)^2$

18.13.55

$$(z-\omega_2)=(\wp'/6e_2^2)\left[1-2w+9w^2-\frac{360}{7}w^3+330w^4-2268w^5+\frac{212058}{13}w^6+O(w^7)\right],$$

18.13.56 where $w=\wp'^2/9$

Other Series Involving ζ

Reversed Series for Large $|\zeta|$

18.13.57

$$z=\zeta^{-1}\left[1-\frac{\gamma}{7}+\frac{17\gamma^2}{143}-\frac{496\gamma^3}{3553}+O(\gamma^4)\right],$$

18.13.58 $$\gamma=\zeta^{-6}/20$$

Series near z_0

18.13.59

$$(\zeta-\zeta_0)=i\left[-\frac{u^2}{2}+\frac{u^8}{56}-\frac{3u^{14}}{5096}\right]+\left[\frac{u^5}{8}-\frac{u^{11}}{308}\right]+O(u^{17}),$$

18.13.60 where $u=(z-z_0)$

Series near ω_2

18.13.61

$$(\zeta-\eta_2)=-e_2(z-\omega_2)\left[1+v+\frac{3}{5}v^2+\frac{3}{7}v^3+\frac{2}{7}v^4+\frac{15}{77}v^5+\frac{12}{91}v^6+\frac{57}{637}v^7+O(v^8)\right],$$

18.13.62 $$v=e_2(z-\omega_2)^2$$

18.13.63

$$(z-\omega_2)=\frac{(\zeta-\eta_2)}{-e_2}\left[1-w+\frac{12w^2}{5}-\frac{267w^3}{35}+\frac{139w^4}{5}-\frac{30192w^5}{275}+\frac{1634208}{3575}w^6+O(w^7)\right],$$

18.13.64 $$w=(\zeta-\eta_2)^2/e_2$$

Series Involving σ

18.13.65

$$\sigma=z-\frac{2\cdot 3}{7!}z^7-\frac{2^3\cdot 3^3}{13!}z^{13}+\frac{2^6\cdot 3^4\cdot 23}{19!}z^{19}+\frac{2^7\cdot 3^5\cdot 5^2\cdot 31}{25!}z^{25}+\frac{2^8\cdot 3^8\cdot 5\cdot 9103}{31!}z^{31}-\frac{2^{12}\cdot 3^9\cdot 5\cdot 229\cdot 2683}{37!}z^{37}-\frac{2^{14}\cdot 3^{10}\cdot 5\cdot 23\cdot 257\cdot 18049}{43!}z^{43}-\frac{2^{15}\cdot 3^{12}\cdot 5\cdot 59\cdot 107895773}{49!}z^{49}+O(z^{55})$$

18.13.66

$$z=\sigma+\frac{\sigma^7}{2^3\cdot 3\cdot 5\cdot 7}+\frac{41\sigma^{13}}{2^7\cdot 3^2\cdot 5^2\cdot 11\cdot 13}+\frac{13\cdot 337\sigma^{19}}{2^{10}\cdot 3^4\cdot 5^3\cdot 11\cdot 17\cdot 19}+\frac{31\cdot 101\sigma^{25}}{2^{15}\cdot 3^5\cdot 5\cdot 11^2\cdot 17\cdot 23}+O(\sigma^{31})$$

Economized Polynomials ($0\leq x\leq 1.53$)

18.13.67 $$x^2\wp(x)=\sum_0^6 a_nx^{6n}+\epsilon(x)$$

$$|\epsilon(x)|<2\times 10^{-7}$$

$a_0=(-1)9.99999\ 96$

$a_1=(-2)3.57143\ 20$

$a_2=(-5)9.80689\ 93$

$a_3=(-7)2.00835\ 02$

$a_4=-(-9)2.20892\ 47$

$a_5=(-10)1.74915\ 35$

$a_6=-(-12)4.46863\ 93$

18.13.68 $$x^3\wp'(x)=\sum_0^6 a_nx^{6n}+\epsilon(x)$$

$$|\epsilon(x)|<4\times 10^{-7}$$

$a_0=-2.00000\ 00$

$a_1=(-1)1.42857\ 22$

$a_2=(-4)9.81018\ 03$

$a_3=(-6)3.00511\ 93$

$a_4=-(-9)2.12719\ 66$

$a_5=(-10)6.53654\ 67$

$a_6=-(-11)1.70510\ 78$

18.13.69 $$x\zeta(x)=\sum_0^6 a_nx^{6n}+\epsilon(x)$$

$$|\epsilon(x)|<3\times 10^{-8}$$

$a_0=(-1)9.99999\ 98$

$a_1=-(-3)7.14285\ 86$

$a_2=-(-6)8.91165\ 65$

$a_3=-(-8)1.44381\ 84$

$a_4=(-10)6.12486\ 14$

$a_5=(-11)4.66919\ 85$

$a_6=(-12)1.25014\ 65$

18.14. Lemniscatic Case

$(g_2=1,\ g_3=0)$

If $g_2>0$ and $g_3=0$, homogeneity relations allow us to reduce our consideration of $\wp$ to $\wp\ (z;1,0)$ ($\wp'$, ζ and σ are handled similarly). Thus $\wp(z; g_2,\ 0)=g_2^{\frac{1}{2}}\wp(zg_2^{\frac{1}{4}};\ 1,\ 0)$. The case $g_2=1$, $g_3=0$ is called the LEMNISCATIC case.

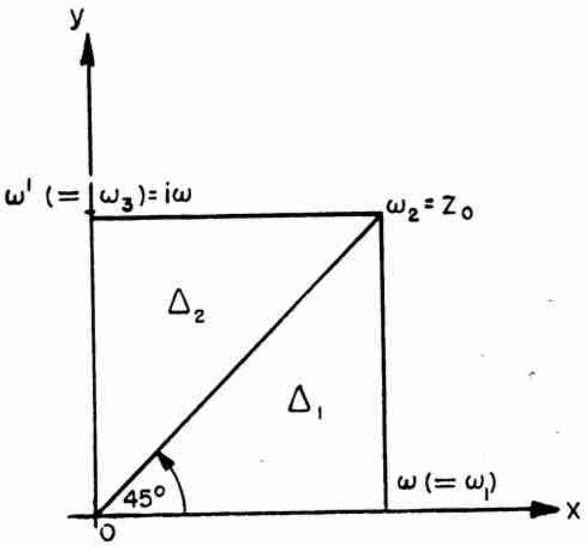


FIGURE 18.13

$\frac{1}{4}$FPP; *Reduction to Fundamental Triangle*

$\Delta_1\equiv\Delta 0\omega\omega_2$ is the Fundamental Triangle

$\omega\approx 1.8540\ 74677\ 30137\ 192$[8]

Reduction for z_2 in Δ_2: $z_1=i\bar{z}_2$ is in Δ_1

18.14.1 $$\wp(z_2)=-\overline{\wp}(z_1)$$

18.14.2 $$\wp'(z_2)=i\overline{\wp'}(z_1)$$

18.14.3 $$\zeta(z_2)=-i\bar{\zeta}(z_1)$$

18.14.4 $$\sigma(z_2)=i\bar{\sigma}(z_1)$$

Special Values and Formulas

18.14.5

$\Delta=1,\ H_1=H_3=2^{-\frac{1}{4}},\ H_2=i/2,$

$m=\sin^2 45°=\frac{1}{2},\ q=e^{-\pi}$

18.14.6 $\vartheta_2(0)=\vartheta_4(0)=(\omega\sqrt{2}/\pi)^{\frac{1}{2}};\ \vartheta_3(0)=(2\omega/\pi)^{\frac{1}{2}}$

18.14.7 $\omega=K(\sin^2 45°)=\frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}=\frac{\tilde{\omega}}{\sqrt{2}}$ where

$\tilde{\omega}\approx 2.62205\ 75542\ 92119\ 81046\ 48395\ 89891\ 11941\ 36827\ 54951\ 43162$ is the Lemniscate constant [18.9]

---

[8] This value was computed and checked by double precision methods on a desk calculator and is believed correct to 18S.

Values at Half-periods

	$\wp$	$\wp'$	ζ	σ
18.14.8 $\omega=\omega_1$	$e_1=\frac{1}{2}$	0	$\eta=\pi/4\omega$	$e^{\pi/8}(2^{1/4})$
18.14.9 $\omega_2=z_0$	$e_2=0$	0	$\eta+\eta'$	$e^{\pi/4}(\sqrt{2})e^{i\pi/4}$
18.14.10 $\omega'=\omega_3$	$e_3=-\frac{1}{2}$	0	$\eta'=-\pi i/4\omega$	$ie^{\pi/8}(2^{1/4})$

Values along (0, ω)

	$\wp$	$\wp'$	ζ	σ
18.14.11 $\omega/4$	$\frac{\sqrt{\alpha}}{2}(\sqrt{\alpha}+2^{1/4})(1+2^{1/4})$			
18.14.12 $\omega/2$	$\alpha/2$	$-\alpha$	$\frac{\pi}{8\omega}+\frac{\alpha}{2\sqrt{2}}$	$\frac{e^{\pi/32}(2^{1/16})}{\alpha^{\frac{1}{2}}}$
18.14.13 $2\omega/3$	$\frac{1}{2}\sqrt{1+\sec 30°}$	$-\frac{\sqrt[4]{2\sqrt{3}+3}}{\sqrt{3}}$	$\frac{2\eta}{3}+\sqrt{\frac{\wp(2\omega/3)}{3}}$	$\frac{e^{\pi/18}(3^{1/8})}{(2+\sqrt{3})^{1/12}}$
18.14.14 $3\omega/4$	$\frac{\sqrt{\alpha}}{2}(\sqrt{\alpha}-2^{\frac{1}{4}})(1+2^{\frac{1}{4}})$			

Values along $(0, z_0)$

	$\wp$	$\wp'$	ζ	σ
18.14.15 $z_0/4$	$-\frac{i}{2}(\alpha+\sqrt{2\alpha})$	$\alpha(\sqrt{\alpha}+\sqrt{2})e^{i\pi/4}$		$\frac{e^{\pi/64}(2^{1/32})}{\alpha^{1/4}(\sqrt{\alpha}+\sqrt{2})^{1/4}}\, e^{i\pi/4}$
18.14.16 $z_0/2$	$-i/2$	$e^{i\pi/4}$	$\left[\frac{\pi}{4\omega\sqrt{2}}+\frac{1}{2}\right]e^{-i\pi/4}$	$e^{\pi/16}(2^{1/8})e^{i\pi/4}$
18.14.17 $2z_0/3$	$\frac{-i}{2}\sqrt{\sec 30^\circ-1}$	$\frac{e^{i\pi/4}\sqrt[4]{2\sqrt{3}-3}}{\sqrt{3}}$	$\frac{2\eta_2}{3}+\left[\frac{\wp(2z_0/3)}{3}\right]^{1/2}$	$\frac{e^{\pi/9}e^{i\pi/4}(3^{1/8})}{\sqrt[12]{2\sqrt{3}-3}}$
18.14.18 $3z_0/4$	$-\frac{i}{2}(\alpha-\sqrt{2\alpha})$	$\alpha(\sqrt{\alpha}-\sqrt{2})e^{i\pi/4}$		$\frac{e^{9\pi/64}(2^{1/32})}{\alpha^{1/4}(\sqrt{\alpha}-\sqrt{2})^{1/4}}\, e^{i\pi/4}$

$\alpha=1+\sqrt{2}$

Duplication Formulas

18.14.19 $\wp(2z)$

$$=[\wp^2(z)+\tfrac{1}{4}]^2/\{\wp(z)[4\wp^2(z)-1]\}$$

18.14.20

$$\wp'(2z)=(\beta+1)(\beta^2-6\beta+1)/[32\wp'^3(z)],\ \beta=4\wp^2(z)$$

18.14.21 $$\zeta(2z)=2\zeta(z)+\frac{6\wp^2(z)-\frac{1}{2}}{2\wp'(z)}$$

18.14.22 $$\sigma(2z)=-\wp'(z)\sigma^4(z)$$

Bisection Formulas $(0<x<2\omega)$

18.14.23

$$\wp\left(\frac{x}{2}\right)$$
$$=[\wp^{\frac{1}{2}}(x)+\{\wp(x)+\tfrac{1}{2}\}^{\frac{1}{2}}]\,[\wp^{\frac{1}{2}}(x)\pm\{\wp(x)-\tfrac{1}{2}\}^{\frac{1}{2}}]$$

[Use + on $0<x\leq\omega$, − on $\omega\leq x<2\omega$]

18.14.24

$$\tfrac{1}{2}\wp'\left(\frac{x}{2}\right)=\wp'(x)\mp[2\wp(x)+\tfrac{1}{2}]\sqrt{\wp(x)-\tfrac{1}{2}}$$
$$-[2\wp(x)-\tfrac{1}{2}]\sqrt{\wp(x)+\tfrac{1}{2}}$$
$$-2\wp^{3/2}(x)\ \text{(See [18.13].)}$$

[Use − on $0<x\leq\omega$, + on $\omega\leq x<2\omega$]

Complex Multiplication

18.14.25 $$\wp(iz)=-\wp(z)$$

18.14.26 $$\wp'(iz)=i\wp'(z)$$

18.14.27 $$\zeta(iz)=-i\zeta(z)$$

18.14.28 $$\sigma(iz)=i\sigma(z)$$

The above equations could be used as follows, e.g.: If z were real, iz would be purely imaginary.

Conformal Maps

Lemniscatic Case

Map: $f(z)=u+iv$

$\wp(z)$

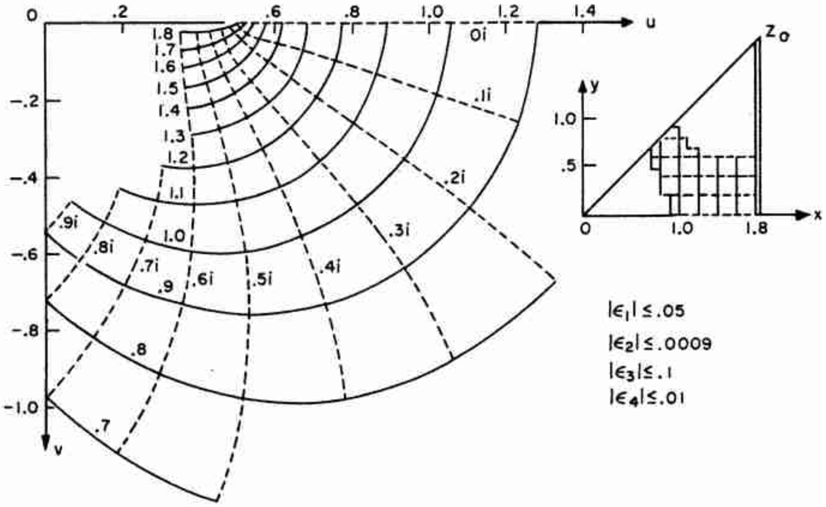
Near zero: $\wp(z)=\frac{1}{z^2}+\epsilon_1$

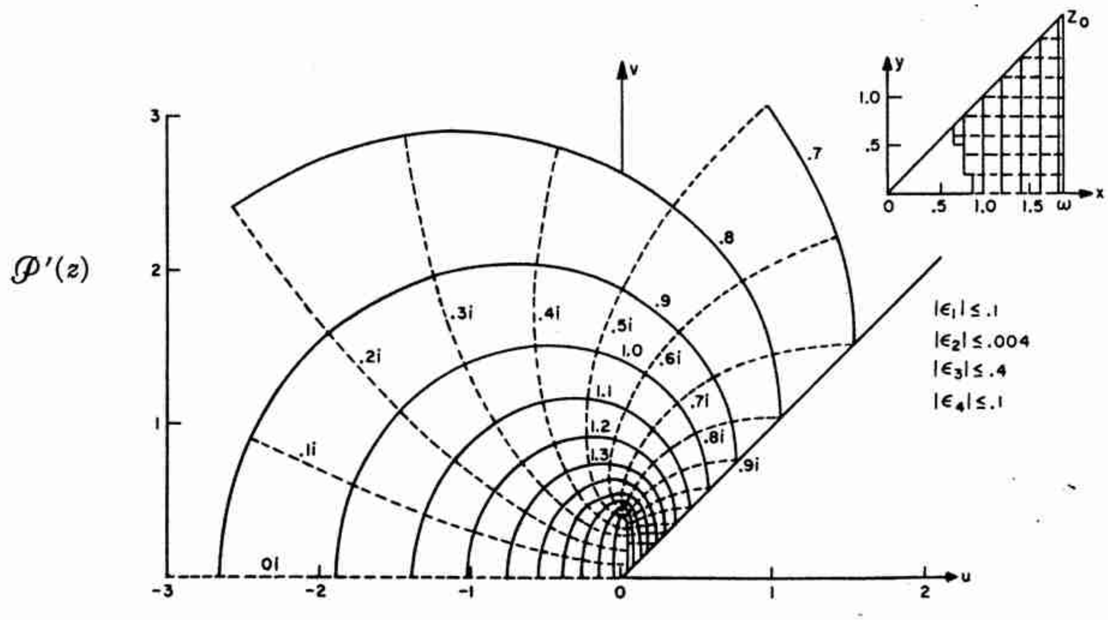
$$\wp(z)=\frac{1}{z^2}+\frac{z^2}{20}+\epsilon_2,\ |z|<1$$

Near z_0: $\wp(z)=\frac{-(z-z_0)^2}{4}+\epsilon_3,$

$$|z-z_0|<\sqrt{2}$$

$$\wp(z)=\frac{-(z-z_0)^2}{4}+\frac{(z-z_0)^6}{80}+\epsilon_4$$





Near zero: $\wp'(z)=\frac{-2}{z^3}+\epsilon_1$

$$\wp'(z)=\frac{-2}{z^3}+\frac{z}{10}+\epsilon_2$$

Near z_0: $\wp'(z)=\frac{-(z-z_0)}{2}+\epsilon_3$

$$\wp'(z)=\frac{-(z-z_0)}{2}+\frac{3(z-z_0)^5}{40}+\epsilon_4$$

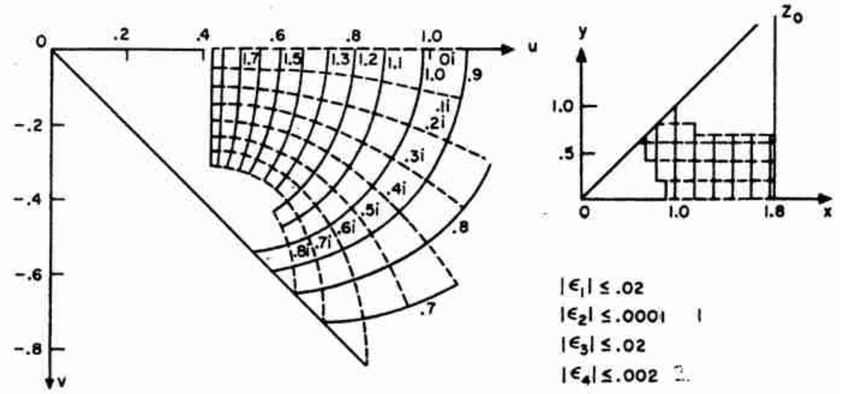
Near zero: $\zeta(z)=\frac{1}{z}+\epsilon_1$

$$\zeta(z)=\frac{1}{z}-\frac{z^3}{60}+\epsilon_2,\quad |z|<1$$

Near z_0: $\zeta(z)=\zeta_0+\frac{(z-z_0)^3}{12}+\epsilon_3,$

$$|z-z_0|<\sqrt{2}$$

$$\zeta(z)=\zeta_0+\frac{(z-z_0)^3}{12}-\frac{(z-z_0)^7}{560}+\epsilon_4$$



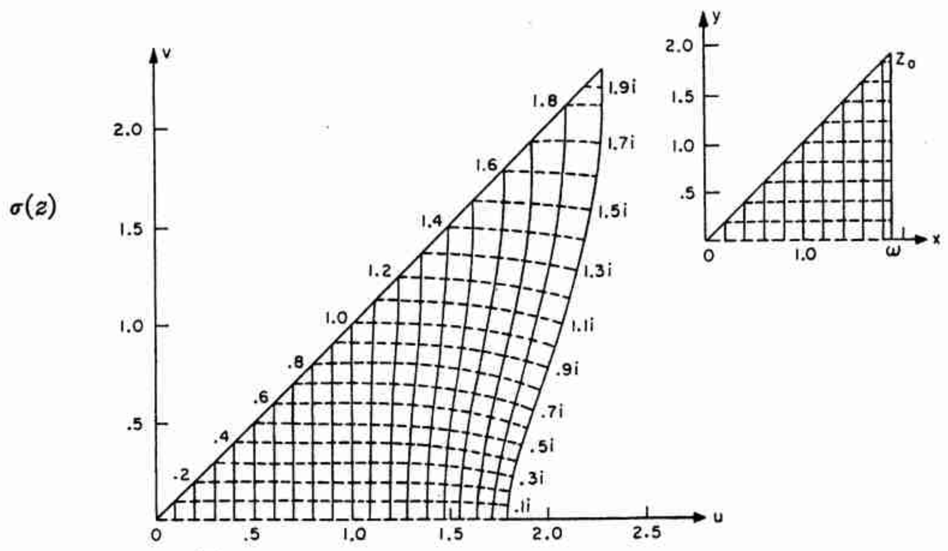


Figure 18.14

Coefficients for Laurent Series for $\wp$, $\wp'$, and ζ

($c_m=0$ for m odd)

k	EXACT c_{2k}	APPROXIMATE c_{2k}
1	$1/20$	.05
2	$1/(3\cdot 20^2)=1/1200$	.8333 . . . $\times 10^{-3}$
3	$2/(3\cdot 13\cdot 20^3)=1/156000$	.641025 641025 . . . $\times 10^{-5}$
4	$5/(3\cdot 13\cdot 17\cdot 20^4)=1/21216000$	.47134 23831 07088 98944 $\times 10^{-7}$
5	$2/(3^2\cdot 13\cdot 17\cdot 20^5)=1/(31824\times 10^5)$	.31422 82554 04725 99296 $\times 10^{-9}$
6	$10/(3^3\cdot 13^2\cdot 17\cdot 20^6)=1/(4964544\times 10^5)$	.20142 83688 49183 32882 $\times 10^{-11}$
7	$4/(3\cdot 13^2\cdot 17\cdot 29\cdot 20^7)=1/(7998432\times 10^7)$	.12502 45048 02941 37651 $\times 10^{-13}$
8	$2453/(3^4\cdot 11\cdot 13^2\cdot 17^2\cdot 29\cdot 20^8)=958203125/(1262002599\times 10^{16})$	.75927 19109 76468 59917 $\times 10^{-16}$
9	$2\cdot 5\cdot 7\cdot 61/(3^3\cdot 13^3\cdot 17^2\cdot 29\cdot 37\cdot 20^9)=833984375/(18394643943\times 10^{17})$	.45338 43533 93461 06092 $\times 10^{-18}$
	$c_{2k}\leq\frac{c_2^k}{3^{k-1}},\ k=1,2,\ldots$	

Other Series Involving $\wp$

Reversed Series for Large $|\wp|$

18.14.29

$$z=(\wp^{-1})^{1/2}\left[1+\frac{w}{5}+\frac{w^2}{6}+\frac{5w^3}{26}+\frac{35w^4}{136}+\frac{3w^5}{8}+\frac{231w^6}{400}+\frac{429w^7}{464}+\frac{195w^8}{128}+\frac{12155w^9}{4736}+\frac{46189w^{10}}{10496}+O(w^{11})\right],$$

18.14.30 $w=\wp^{-2}/8$, and z is in the Fundamental Triangle **(Figure 18.13)** if $\wp$ has an appropriate value.

Series near z_0

18.14.31 $$2\wp=-x+\frac{x^3}{5}-\frac{2x^5}{75}+\frac{x^7}{325}+O(x^9),$$

18.14.32 $$x=(z-z_0)^2/2$$

18.14.33 $$x=-\left[w+\frac{w^3}{5}+\frac{7w^5}{75}+\frac{11w^7}{195}+O(w^9)\right]$$

$$w=2\wp$$

Series near ω

18.14.34

$$(\wp-e_1)=v+v^2+\frac{4v^3}{5}+\frac{3v^4}{5}+\frac{32v^5}{75}+\frac{22v^6}{75}+\frac{64v^7}{325}+O(v^8),$$

18.14.35 $$v=(z-\omega)^2/2$$

18.14.36

$$v=y\left[1-y+\frac{6y^2}{5}-\frac{8y^3}{5}+\frac{172y^4}{75}-\frac{52y^5}{15}+\frac{1064y^6}{195}+O(y^7)\right],$$

18.14.37 $$y=(\wp-e_1)$$

Other Series Involving $\wp'$

Reversed Series for Large $|\wp'|$

18.14.38

$$z=Au\left[1-\frac{v}{5}+\frac{5v^3}{39}-\frac{7v^4}{51}+O(v^5)\right],\quad u=(\wp'^{1/3})^{-1}e^{i\pi/3},$$

18.14.39 $A=2^{1/3}$, $v=Au^4/6$, and z is in the Fundamental Triangle **(Figure 18.13)** if $\wp'$ has an appropriate value.

Series near z_0

18.14.40

$$\wp'=\frac{1}{2}(z-z_0)\left[-1+3w-\frac{10w^2}{3}+\frac{35w^3}{13}+O(w^4)\right],$$

18.14.41 $$w=(z-z_0)^4/20$$

18.14.42

$$(z-z_0)=2\wp'\left[1+\frac{3u}{5}+\frac{5u^2}{3}+\frac{84u^3}{13}+O(u^4)\right],$$

18.14.43 $$u=4\wp'^4$$

Series near ω

18.14.44

$$\wp'=x\left[1+x^2+\frac{3}{5}x^4+\frac{3}{10}x^6+\frac{2}{15}x^8+\frac{11}{200}x^{10}+O(x^{12})\right],$$

18.14.45 $$x=(z-\omega)$$

18.14.46

$$x=\wp'-\wp'^3+\frac{12\wp'^5}{5}-\frac{15\wp'^7}{2}+\frac{80\wp'^9}{3}-\frac{819\wp'^{11}}{8}+O(\wp'^{13})$$

Other Series Involving ζ

Reversed Series for Large $|\zeta|$

18.14.47 $$z=\zeta^{-1}\left[1-\frac{v}{5}+\frac{v^2}{7}-\frac{136v^3}{1001}+\frac{1349v^4}{9163}+O(v^5)\right],$$

18.14.48 $$v=\zeta^{-4}/12$$

Series near z_0

18.14.49

$$(\zeta-\zeta_0)=\frac{1}{4}(z-z_0)^3\left[\frac{1}{3}-\frac{v}{7}+\frac{2v^2}{33}-\frac{v^3}{39}+O(v^4)\right],$$

18.14.50 $$v=(z-z_0)^4/20$$

Series near ω

18.14.51

$$(\zeta-\eta)=-\frac{x}{2}-\frac{x^3}{6}-\frac{x^5}{20}-\frac{x^7}{70}-\frac{x^9}{240}-\frac{x^{11}}{825}-\frac{11x^{13}}{31200}-\frac{x^{15}}{9750}+O(x^{17}),$$

18.14.52 $$x=(z-\omega)$$

18.14.53

$$x=w-\frac{w^3}{3}+\frac{7w^5}{30}-\frac{13w^7}{63}+\frac{929w^9}{4536}-\frac{194w^{11}}{891}+\frac{942883w^{13}}{3891888}+O(w^{15})$$

18.14.54 $$w=-2(\zeta-\eta)$$

Series Involving σ

18.14.55

$$\sigma=z-\frac{z^5}{2\cdot5!}-\frac{3^2z^9}{2^2\cdot9!}+\frac{3\cdot23z^{13}}{2^3\cdot13!}+\frac{3\cdot107z^{17}}{2^4\cdot17!}+\frac{3^3\cdot7\cdot23\cdot37z^{21}}{2^5\cdot21!}+\frac{3^2\cdot313\cdot503z^{25}}{2^6\cdot25!}-\frac{3^4\cdot7\cdot685973z^{29}}{2^7\cdot29!}+O(z^{33})$$

18.14.56

$$z=\sigma+\frac{\sigma^5}{2^4\cdot3\cdot5}+\frac{\sigma^9}{2^9\cdot3\cdot7}+\frac{17\cdot113\sigma^{13}}{2^{13}\cdot3^4\cdot7\cdot11\cdot13}+\frac{122051\sigma^{17}}{2^{19}\cdot3^5\cdot7^2\cdot11\cdot17}+\frac{5\cdot13\sigma^{21}}{2^{23}\cdot3^2\cdot11\cdot19}+O(\sigma^{25})$$

Economized Polynomials ($0\leq x\leq1.86$)

18.14.57 $$x^2\wp(x)=\sum_0^6 a_nx^{4n}+\epsilon(x)$$

$$|\epsilon(x)|<2\times10^{-7}$$

$a_0=(-1)9.99999\ 98$ $\qquad a_4=(-8)4.81438\ 20$

$a_1=(-2)4.99999\ 62$ $\qquad a_5=(-10)2.29729\ 21$

$a_2=(-4)8.33352\ 77$ $\qquad a_6=(-12)4.94511\ 45$

$a_3=(-6)6.40412\ 86$

18.14.58 $$x^3\wp'(x)=\sum_0^6 a_nx^{4n}+\epsilon(x)$$

$$|\epsilon(x)|<4\times10^{-7}$$

$a_0=-2.00000\ 00$ $\qquad a_4=(-7)6.58947\ 52$

$a_1=(-1)1.00000\ 02$ $\qquad a_5=(-9)5.59262\ 49$

$a_2=(-3)4.99995\ 38$ $\qquad a_6=(-11)5.54177\ 69$

$a_3=(-5)6.41145\ 59$

18.14.59 $$x\zeta(x)=\sum_0^6 a_nx^{4n}+\epsilon(x)$$

$$|\epsilon(x)|<3\times10^{-8}$$

$a_0=(-1)9.99999\ 99$ $\qquad a_4=-(-9)2.57492\ 62$

$a_1=-(-2)1.66666\ 74$ $\qquad a_5=-(-11)5.67008\ 00$

$a_2=-(-4)1.19036\ 70$ $\qquad a_6=(-13)9.70015\ 80$

$a_3=-(-7)5.86451\ 63$

18.15. Pseudo-Lemniscatic Case

($g_2=-1,\ g_3=0$)

If $g_2<0$ and $g_3=0$, homogeneity relations allow us to reduce our consideration of $\wp$ to $\wp(z;-1,0)$. Thus

18.15.1 $$\wp(z;g_2,0)=|g_2|^{1/2}\wp(z|g_2|^{1/4};-1,0)$$

[$\wp'$, ζ and σ are handled similarly]. Because of its similarity to the lemniscatic case, we refer to the case $g_2=-1$, $g_3=0$ as the pseudo-lemniscatic case. It plays the same role (period ratio unity) for $\Delta<0$ as does the lemniscatic case for $\Delta>0$.

$\omega_2=\sqrt{2}\times$ (real half-period for lemniscatic case) $=\tilde{\omega}$ (the Lemniscate Constant—see **18.14.7**)

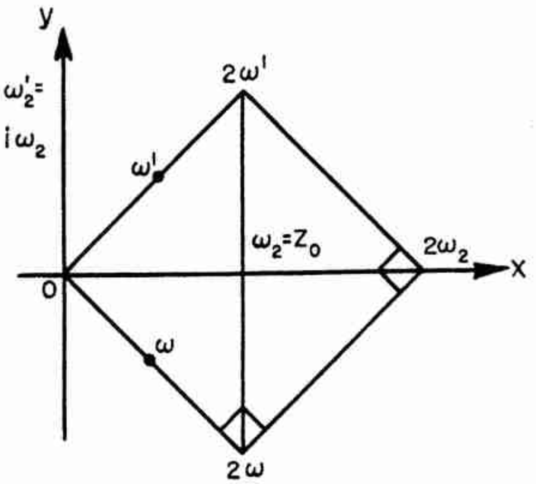


FIGURE 18.15

Special Values and Relations

18.15.2 $\Delta=-1,\ g_2=-1,\ g_3=0$

18.15.3

$H_1=-i/\sqrt{2},\ H_2=\frac{1}{2},\ H_3=i/\sqrt{2},\qquad m=\frac{1}{2},\ q=ie^{-\pi/2}$

18.15.4

$\vartheta_2(0)=R2^{1/4}e^{i\pi/8},\ \vartheta_3(0)=Re^{i\pi/8},\ \vartheta_4(0)=Re^{-i\pi/8},$

18.15.5 where $R=\sqrt{\omega_2\sqrt{2}/\pi}$

Values at Half-Periods

	$\wp$	$\wp'$	ζ	σ
18.15.6 $\omega\equiv\omega_1$	$i/2$	0	$\frac{1}{2}(\eta_2-\eta_2')$	$e^{-i\pi/4}e^{\pi/8}(2^{1/4})$
18.15.7 ω_2	0	0	$\eta_2=\pi/2\omega_2$	$e^{\pi/4}\sqrt{2}$
18.15.8 $\omega'=\omega_3$	$-i/2$	0	$\frac{1}{2}(\eta_2+\eta_2')$	$e^{i\pi/4}e^{\pi/8}(2^{1/4})$
18.15.9 ω_2'	0	0	$\eta_2'=-i\eta_2$	$i\sigma(\omega_2)$

Relations with Lemniscatic Values

18.15.10 $\wp(z;-1,0)=i\wp(ze^{i\pi/4};1,0)$

18.15.11 $\wp'(z;-1,0)=e^{3\pi i/4}\wp'(ze^{i\pi/4};1,0)$

18.15.12 $\zeta(z;-1,0)=e^{i\pi/4}\zeta(ze^{i\pi/4};1,0)$

18.15.13 $\sigma(z;-1,0)=e^{-i\pi/4}\sigma(ze^{i\pi/4};1,0)$

Numerical Methods

18.16. Use and Extension of the Tables

Example 1. Lemniscatic Case

(a) Given $z=x+iy$ in the Fundamental Triangle, find $\wp(\wp',\zeta,\sigma)$ more accurately than can be done with the maps.

σ—Use Maclaurin series throughout the Fundamental Triangle. Five terms give at least six significant figures, six terms at least ten. $\wp$, ζ—Use Laurent's series directly "near" 0, (if $|z|<1$, four terms give at least eight significant figures for $\wp$, nine for ζ; five terms at least ten significant figures for $\wp$, eleven for ζ). Use Taylor's series directly "near" z_0. Elsewhere (unless approximately seven or eight significant figures are insufficient) use economized polynomials to obtain $\wp(x)$, $\wp'(x)$ and/or $\zeta(x)$ as appropriate. To get $\wp(iy)$, $\wp'(iy)$ and/or $\zeta(iy)$, use Laurent's series for "small" y, otherwise use economized polynomials to compute $\wp(y)$, $\wp'(y)$ and/or $\zeta(y)$, then use complex multiplication to obtain $\wp(iy)$, $\wp'(iy)$ and/or $\zeta(iy)$. Finally, use appropriate addition formula to get $\wp(z)$ and/or $\zeta(z)$.

$\wp'$—Use Laurent's series directly "near" 0 (if $|z|<1$, four terms give at least six significant figures, five terms at least eight significant figures). Elsewhere, either use economized polynomials and addition formula as for $\wp$ and ζ, or get $\wp'^2=4\wp^3-\wp$ and extract appropriate square root ($\mathscr{I}\wp'\geq 0$).

(b) Given $\wp(\wp',\zeta,\sigma)$ corresponding to a point in the Fundamental Triangle, compute z more accurately than can be done with the maps. Only a few significant figures are obtainable from the use of any of the given (truncated) reversed series, except in a small neighborhood of the center of the series. For greater accuracy, use inverse interpolation procedures.

Example 2. Equianharmonic Case

(a) Given $z=x+iy$ in the Fundamental Triangle, find $\wp(\wp',\zeta,\sigma)$ more accurately than can be done with the maps.

σ—Use Maclaurin series throughout the Fundamental Triangle. Four terms give at least eleven significant figures, five terms at least twenty one.

$\wp$,ζ—Use Laurent's series directly "near" 0 (if $|z|<1$, four terms give at least 10S for $\wp$, 11S for ζ; five terms at least 13S for $\wp$, 14S for ζ). Elsewhere (unless approximately seven or eight significant figures are insufficient) use economized polynomials to obtain $\wp(x)$, $\wp'(x)$ and/or $\zeta(x)$, as appropriate. To get $\wp(iy)$, $\wp'(iy)$ and/or $\zeta(iy)$, use Laurent's series. Then use appropriate addition formula to get $\wp(z)$ and/or $\zeta(z)$.

$\wp'$—Use Laurent's series directly "near" 0 (if $|z|<1$, four terms give at least 8S, five terms at least 11S). Elsewhere, either proceed as for $\wp$ and ζ, or get $\wp'^2=4\wp^3-1$ and extract appropriate square root ($\mathcal{I}\wp'\geqq 0$).

(b) Given $\wp(\wp',\zeta,\sigma)$ corresponding to a point in the Fundamental Triangle, compute z more accurately than can be done with the maps. Only a few significant figures are obtainable from the use of any of the given (truncated) reversed series, except in a small neighborhood of the center of the series. For greater accuracy, use inverse interpolation procedures.

Example 3. Given period ratio a, find parameters m (of elliptic integrals and Jacobi's functions of chapter **16**) and q (of ϑ functions).

m—In both the cases $\Delta>0$ and $\Delta<0$, the period ratio is equal to $K'(m)/K(m)$ (see **18.9**). Knowing K'/K, if $1<K'/K\leqq 3$, use **Table 17.3** to find m; if $K'/K>3$, use the method of **Example 6** in chapter **17.** An alternative method is to use **Table 18.3** to obtain the necessary entries, thence use

$$m=(e_2-e_3)/(e_1-e_3) \text{ in case } \Delta>0,$$

$$m=\tfrac{1}{2}-3e_2/4H_2 \text{ in case } \Delta<0.$$

q—In both the cases $\Delta>0$ and $\Delta<0$, the period ratio determines the exponent for $q[q=e^{-\pi a}$ if $\Delta>0$, $q=ie^{-\pi a/2}$ if $\Delta<0]$. Hence enter **Table 4.16** [$e^{-\pi x}$, $x=0(.01)1$] and multiply the results as appropriate [e.g., $e^{-4.72\pi}=(e^{-\pi})^4(e^{-.72\pi})$].

Determination of Values at Half-Periods, Invariants and Related Quantities from Given Periods (Table 18.3)

$\Delta>0$

Given ω and ω', form $\omega'/i\omega$ and enter **Table 18.3.** Multiply the results obtained by the appropriate power of ω (see footnotes of **Table 18.3**) to obtain value desired.

Example 4.

Given $\omega=10$, $\omega'=11i$, find e_i, g_i, and Δ.

Here $\omega'/i\omega=1.1$, so that direct reading of **Table 18.3** gives

$$e_1(1)=1.6843\ 041$$

$$e_2(1)=-.2166\ 258\ (=-e_1-e_3)$$

$$e_3(1)=-1.4676\ 783$$

$$g_2(1)=10.0757\ 7364$$

$$g_3(1)=2.1420\ 1000.$$

Multiplying by appropriate powers of $\omega=10$ we obtain

$$e_1=.01684\ 3041$$

$$e_2=-.00216\ 6258$$

$$e_3=-.01467\ 6783$$

$$g_2=1.0075\ 77364 \text{ x } 10^{-3}$$

$$g_3=2.1420\ 1000 \text{ x } 10^{-6}$$

whence

$$\Delta=8.9902\ 3191 \text{ x } 10^{-10}$$

$\Delta<0$

Given ω_2 and ω_2', form $\omega_2'/i\omega_2$ and enter **Table 18.3.** Multiply the results obtained by the appropriate power of ω_2 (see footnotes of **Table 18.3**) to obtain value desired.

Example 4.

Given $\omega_2=10$, $\omega_2'=11i$, find e_i, g_i, and Δ.

Here $\omega_2'/i\omega_2=1.1$, so that direct reading of **Table 18.3** gives

$$e_1(1)=-.2166\ 2576+3.0842\ 589i$$

$$e_2(1)=.4332\ 5152=-2\mathscr{R}(e_1)$$

$$e_3(1)=\bar{e}_1(1)$$

$$g_2(1)=-37.4874\ 912$$

$$g_3(1)=16.5668\ 099.$$

Multiplying by appropriate powers of $\omega_2=10$ we obtain

$$e_1=-.00216\ 62576+.03084\ 2589i$$

$$e_2=.00433\ 25152$$

$$e_3=\bar{e}_1$$

$$g_2=-3.7487\ 4912 \text{ x } 10^{-3}$$

$$g_3=1.6566\ 8099 \text{ x } 10^{-5}$$

whence

$$\Delta=-6.0092\ 019 \text{ x } 10^{-8}$$

Computation of σ for Given z and Arbitrary g_2 and g_3

(or periods from which g_2 and g_3 can be computed—in any case, periods must be known, at least approximately)

First reduce the problem (if necessary) to computation for a point z in the Fundamental Rectangle (see **18.2**). After final reduction let z denote the point obtained.

$\Delta>0$

If $\mathscr{R}z>\omega/2$ or,

$\mathscr{I}z>\omega'/2$, use duplication formula

$$\sigma(z)=-\wp'(z/2)\sigma^4(z/2),$$

obtaining $\sigma(z/2)$ by use of Maclaurin series for σ and $\wp'(z/2)$ by method explained above. Otherwise, simply use Maclaurin series for σ directly.

$\Delta<0$

If $\mathscr{R}z>\omega_2/2$ or

$\mathscr{I}z>\omega_2'/4$, use duplication formula as in case $\Delta>0$. Otherwise, use Maclaurin series for σ directly.

An alternate method is to use theta functions **18.10** first computing q and $\vartheta_i(0)$, $i=2, 3, 4$.

$\Delta>0$

Example 13. Compute $\sigma(.4+1.3i)$ for $g_2=8$, $g_3=4$. From **Example 7**, $\omega=1.009453$ and $\omega'=1.484413i$. Since $\mathscr{I}z>\omega'/2$, the Maclaurin series **18.5.6** is used to obtain $\sigma(z/2)=\sigma(.2+.65i)=.19543\,86+.64947\,28i$, the Laurent series **18.5.4** to obtain $\wp'(.2+.65i)=5.02253\,80-3.56066\,93i$. The duplication formula **18.4.8** gives $\sigma(.4+1.3i)=.278080+1.272785i$.

$\Delta<0$

Example 13. Compute $\sigma(.8+.4i)$ for $g_2=7$, $g_3=6$. From **Example 7**, $\omega_2=.99579\,976$, $\omega_2'=2.33241\,83i$. Since $\mathscr{R}z>\omega_2/2$, the Maclaurin series **18.5.6** is used to obtain $\sigma(z/2)=\sigma(.4+.2i)=.40038\,019+.19962\,017i$, the Laurent series **18.5.4** to obtain $\wp'(.4+.2i)=-3.70986\,70+22.218544i$. The duplication formula **18.4.8** gives $\sigma(.8+.4i)=.81465\,765+.38819\,473i$.

Given $\sigma(\wp, \wp', \zeta)$ corresponding to a point in the Fundamental Rectangle, as well as g_2 and g_3 or the equivalent, find z.

Only a few significant figures are obtainable from the use of any of the given (truncated) reversed series, except in a small neighborhood of the center of the series. For greater accuracy, use inverse interpolation procedures.

If the given function does not correspond to a value of z in the Fundamental Rectangle (see Conformal Maps) the problem can always be reduced to this case by the use of appropriate reduction formulas in **18.2.** This process is relatively simple for $\wp(z)$, more difficult for the other functions (e.g. if $\Delta>0$ and $\wp=a+ib$, where $b>0$, simply consider $\overline{\wp}=a-ib$ and find z_1 in R_1 **[Figure 18.1]**; then compute $z_2=\overline{z}_1+2\omega'$, the point in R_2 corresponding to the given $\wp$).

$\Delta>0$

Example 14. Given $\wp=1-i$, $g_2=10$, $g_3=2$, find z. Using the first three terms of the reversed series **18.5.25** $z_1\approx.727+.423i$. The Laurent series **18.5.1** gives

$$\wp(z_1)=\wp(.727+.423i)=.825-.895i$$

and

$$\wp(z_2)=\wp(.697+.393i)=.938-1.038i.$$

Inverse interpolation gives $z_1^{(1)}=.707+.380i$. Repeated applications of the above procedure yield $z=.706231+.379893i$.

$\Delta<0$

Example 14. Given $\wp=1+i$, $g_2=-10$, $g_3=2$, find z. From **Example 6,** $\omega_2=1.40239\,48$ and $\omega_2'=1.52561\,02i$. Since $b>0$, z exists in R_2 and z is computed with $\overline{\wp}$. Using **18.5.25** with $\alpha_2=-1.25$, $\alpha_3=.25$, $u=[(\overline{\wp})^{-1}]^{1/2}$ and the coefficients c_n from **Example 8**

$$2u=1.55377\,3973+.64359\,42493i$$

$$c_2u^5=\ .08044\,9281-.19422\,17466i$$

$$c_3u^7=-.01961\,9359+.00812\,66047i$$

$$\frac{\alpha_2^2u^9}{3}=-.10115\,7160-.04190\,06673i$$

$\Delta>0$

Example 15. Given $\zeta=10-15i$, $g_2=8$, $g_3=4$, find z. Using the reversed series **18.5.40** with

$$A_5=-.13333\ 333,$$

$$A_7=-.02857\ 14286,$$

$$u=\ .03076\ 923076+.04615\ 384615i$$

$$A_5u^5=-.00000\ 001402+.00000\ 006860i$$

$$A_7u^7=-.00000\ 000004-.00000\ 000003i$$

$$z=\ .03076\ 921670+.04615\ 391472i.$$

$\Delta<0$

Stopping with the term in u^7, $z_1\approx.81+.23i$. Assuming $\Delta z=-.03-.01i$, using **18.5.1,** $\wp(.81+.23i)=.91410\ 95-.86824\ 37i$, $\wp(.78+.22i)=1.03191\ 60-.91795\ 22i$; with inverse interpolation $z_1^{(1)}=.\ 7725+.2404i$. Repeated applications of inverse interpolation yield $z=.\ 772247-.239258i$.

Example 15. Given $\sigma=.4+.1i$, $g_2=7$, $g_3=6$, find z. Using the reversed series **18.5.70** with $\gamma_2=.14583$, $\gamma_3=.05$

$$\sigma=+.40000\ 000+.10000\ 000i$$

$$\frac{\gamma_2\sigma^5}{5}=+.00011\ 783+.00032\ 696i$$

$$\frac{\gamma_3\sigma^7}{7}=-.00000\ 208+.00001\ 432i$$

$$\frac{3\gamma_2^2\sigma^9}{14}=-.00000\ 093+.00000\ 126i$$

$$\frac{19\gamma_2\gamma_3\sigma^{11}}{55}=-.00000\ 013+.00000\ 006i$$

$$z=.40011\ 469+.10034\ 260i$$

Methods of Computation of $\wp$ ($\wp'$, ζ or σ) for Given z and Given g_2, g_3 (or the equivalent), with the Use of Automatic Digital Computing Machinery

(a) Integration of Differential Equation

$\wp$ and $\wp'$ may be generated for any z close enough to a "known point" $z^*(\wp(z^*)$ and $\wp'(z^*)$ being given) by integrating $\wp''=6\wp^2-g_2/2$. A program to do this on SWAC, via a modification of the Hammer-Hollingsworth method (MTAC, July 1955, pp. 92–96) due to Dr. P. Henrici, exists at Numerical Analysis Research, UCLA (code number 00600, written by W. L. Wilson, Jr.). The program has been tested numerically in the equianharmonic case, using integration steps of various sizes. For example, if one starts with $z^*=\omega_2$, using an "integration step" (h,k), where h and k are respectively the horizontal and vertical components of a step, with (h,k) having one of the six values $(\pm2h_0,0)$, $(\pm h_0,\pm k_0)$, $h_0=\omega_2/2000$, $k_0=|\omega_2'|/2000$, one can expect almost 8S in $\wp$ and 7S in $\wp'$ after 1000 steps, unless z is too near a pole.

(b) Use of Series

The process of reducing the computation problem to one in which z is in the Fundamental Rectangle can obviously be mechanized. Inside the Fundamental Rectangle the direct use of Laurent's series is appropriate when the period ratio a is not too large. However, if $a\geqq\sqrt{3}(\Delta>0)$ or $a\geqq2\sqrt{3}(\Delta<0)$, the series will diverge at the far corner of the Fundamental Rectangle, so that use may be made of an appropriate duplication formula. Alternatively, one may compute the functions on $0x$ and $0y$, then use an addition formula. Even so, the series will diverge at $z=ia$ if $a\geqq2(\Delta>0)$ and at $z=ia/2$ if $a\geqq4(\Delta<0)$.

For great accuracy, multiple precision operations might be necessary. Double precision floating point mode has been used in a program, written for SWAC, to compute $\wp$, $\wp'$ and ζ.

For computation of σ, use of the Maclaurin series throughout the Fundamental Rectangle is probably simplest (series converges for all z).

Mention should be made of the possible use of the series defining the ϑ functions. These series converge for all complex v, and the computation of $\wp$, $\wp'$, ζ and σ by **18.10.5–18.10.8** could easily be mechanized. The series involved have the advantage of converging very fast, even in case $\Delta<0$, where $|q|\leqq e^{-\pi/2}(q\leqq e^{-\pi}$ if $\Delta>0)$.

Use of Maps

If the problem (of computing $\wp$, $\wp'$, ζ or σ for given z) is reduced to the case where the real half-period is unity and imaginary half-period is one of those used in the maps in **18.8** inspection of the appropriate figure will give the value of $\wp(z)$ [$\zeta(z)$ or $\sigma(z)$] to 2−3S. If $\wp'$ is wanted instead, get $\wp$, use **18.6.3** to obtain $\wp'^2$ and select sign (*s*) of $\wp'$ appropriately. (See Conformal Mapping **(18.8)** for choice of sign of square root of $\wp'^2$).

Computation of z_0

Given g_2, g_3 (or equivalent)

Since $z_0^2\wp(z_0)=0$, the Laurent's series gives

$$0=1+c_2u^2+c_3u^3+c_4u^4+\ .\ .\ .$$

where $u=z_0^2$. We may solve this equation [by Graeffe's (root-squaring) process or otherwise] for its absolutely smallest root [having found an approximation to $|z_0|$ by Graeffe's process, we may use the fact that $z_0=\omega+iy_0(\Delta>0)$, $z_0=\omega_2+iy_0(\Delta<0)$ to obtain an approximation to z_0].

It is noted that y_0/ω is a monotonic decreasing function of (period ratio) $a\geqq 1$ for $\Delta>0$ and

$$[1\geqq y_0/\omega>\frac{2}{\pi}\text{ arccosh }\sqrt{3}(\approx.7297)].$$

y_0/ω_2 is a monotonic increasing function of a for $\Delta<0$ and

$$[0\leqq y_0/\omega_2<\frac{2}{\pi}\text{ arccosh }\sqrt{3}]$$

Further data is available from **Table 18.2** or from Conformal Maps defined by $\wp(z)$.

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19. Parabolic Cylinder Functions

J. C. P. MILLER[1]

Contents

The author acknowledges permission from H.M. Stationery Office to draw freely from [19.11] the material in the introduction, and the tabular values of $W(a, x)$ for $a=-5(1)5$, $\pm x=0(.1)5$. Other tables of $W(a, x)$ and the tables of $U(a, x)$ and $V(a, x)$ were prepared on EDSAC 2 at the University Mathematical Laboratory, Cambridge, England, using a program prepared by Miss Joan Walsh for solution of general second order linear homogeneous differential equations with quadratic polynomial coefficients. The auxiliary tables were prepared at the Computation Laboratory of the National Bureau of Standards.

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[1] The University Mathematical Laboratory, Cambridge, England. (Prepared under contract with the National Bureau of Standards.)

19. Parabolic Cylinder Functions

Mathematical Properties

19.1. The Parabolic Cylinder Functions

Introductory

These are solutions of the differential equation

19.1.1 $$\frac{d^2y}{dx^2}+(ax^2+bx+c)y=0$$

with two real and distinct standard forms

19.1.2 $$\frac{d^2y}{dx^2}-(\tfrac{1}{4}x^2+a)y=0$$

19.1.3 $$\frac{d^2y}{dx^2}+(\tfrac{1}{4}x^2-a)y=0$$

The functions

19.1.4
$$y(a, x) \qquad y(a, -x) \qquad y(-a, ix) \qquad y(-a, -ix)$$

are all solutions either of **19.1.2** or of **19.1.3** if any one is such a solution.

Replacement of a by $-ia$ and x by $xe^{\frac{1}{4}i\pi}$ converts **19.1.2** into **19.1.3**. If $y(a, x)$ is a solution of **19.1.2**, then **19.1.3** has solutions:

19.1.5
$$y(-ia, xe^{\frac{1}{4}i\pi}) \qquad y(-ia, -xe^{\frac{1}{4}i\pi}) \qquad y(ia, -xe^{-\frac{1}{4}i\pi}) \qquad y(ia, xe^{-\frac{1}{4}i\pi})$$

Both variable x and the parameter a may take on general complex values in this section and in many subsequent sections. Practical applications appear to be confined to real solutions of real equations; therefore attention is confined to such solutions, and, in general, formulas are given for the two equations **19.1.2** and **19.1.3** independently. The principal computational consequence of the remarks above is that reflection in the y-axis produces an independent solution in almost all cases (Hermite functions provide an exception), so that tables may be confined either to positive x or to a single solution of **19.1.2** or **19.1.3.**

The Equation $\frac{d^2y}{dx^2}-\left(\frac{1}{4}x^2+a\right)y=0$

19.2. Power Series in x

Even and odd solutions of **19.1.2** are given by

19.2.1
$$\begin{aligned}y_1&=e^{-\frac{1}{4}x^2}M(\tfrac{1}{2}a+\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}x^2)\\&=e^{-\frac{1}{4}x^2}\left\{1+(a+\tfrac{1}{2})\frac{x^2}{2!}+(a+\tfrac{1}{2})(a+\tfrac{5}{2})\frac{x^4}{4!}+\ldots\right\}\\&=e^{-\frac{1}{4}x^2}{}_1F_1(\tfrac{1}{2}a+\tfrac{1}{4}; \tfrac{1}{2}; \tfrac{1}{2}x^2)\end{aligned}$$

19.2.2
$$\begin{aligned}&=e^{\frac{1}{4}x^2}M(-\tfrac{1}{2}a+\tfrac{1}{4}, \tfrac{1}{2}, -\tfrac{1}{2}x^2)\\&=e^{\frac{1}{4}x^2}\left\{1+(a-\tfrac{1}{2})\frac{x^2}{2!}+(a-\tfrac{1}{2})(a-\tfrac{5}{2})\frac{x^4}{4!}+\ldots\right\}\end{aligned}$$

19.2.3
$$\begin{aligned}y_2&=xe^{-\frac{1}{4}x^2}M(\tfrac{1}{2}a+\tfrac{3}{4}, \tfrac{3}{2}, \tfrac{1}{2}x^2)\\&=e^{-\frac{1}{4}x^2}\left\{x+(a+\tfrac{3}{2})\frac{x^3}{3!}+(a+\tfrac{3}{2})(a+\tfrac{7}{2})\frac{x^5}{5!}+\ldots\right\}\end{aligned}$$

19.2.4
$$\begin{aligned}&=xe^{\frac{1}{4}x^2}M(-\tfrac{1}{2}a+\tfrac{3}{4}, \tfrac{3}{2}, -\tfrac{1}{2}x^2)\\&=e^{\frac{1}{4}x^2}\left\{x+(a-\tfrac{3}{2})\frac{x^3}{3!}+(a-\tfrac{3}{2})(a-\tfrac{7}{2})\frac{x^5}{5!}+\ldots\right\}\end{aligned}$$

these series being convergent for all values of x (see chapter **13** for $M(a, c, z)$).

Alternatively,

19.2.5
$$\begin{aligned}y_1=1&+a\frac{x^2}{2!}+\left(a^2+\frac{1}{2}\right)\frac{x^4}{4!}+\left(a^3+\frac{7}{2}a\right)\frac{x^6}{6!}\\&+\left(a^4+11a^2+\frac{15}{4}\right)\frac{x^8}{8!}+\left(a^5+25a^3+\frac{211}{4}a\right)\frac{x^{10}}{10!}+\ldots\end{aligned}$$

19.2.6
$$\begin{aligned}y_2=x&+a\frac{x^3}{3!}+\left(a^2+\frac{3}{2}\right)\frac{x^5}{5!}+\left(a^3+\frac{13}{2}a\right)\frac{x^7}{7!}\\&+\left(a^4+17a^2+\frac{63}{4}\right)\frac{x^9}{9!}+\left(a^5+35a^3+\frac{531}{4}a\right)\frac{x^{11}}{11!}+\ldots\end{aligned}$$

in which non-zero coefficients a_n of $x^n/n!$ are connected by

19.2.7 $$a_{n+2}=a\cdot a_n+\frac{1}{4}n(n-1)a_{n-2}$$

19.3. Standard Solutions

These have been chosen to have the asymptotic behavior exhibited in **19.8.** The first is Whittaker's function [19.8, 19.9] in a more symmetrical notation.

19.3.1

$$U(a,x)=D_{-a-\frac{1}{2}}(x)=\cos\pi(\tfrac{1}{4}+\tfrac{1}{2}a)\cdot Y_1 \\ -\sin\pi(\tfrac{1}{4}+\tfrac{1}{2}a)\cdot Y_2$$

19.3.2

$$V(a,x)=\frac{1}{\Gamma(\frac{1}{2}-a)}\{\sin\pi(\tfrac{1}{4}+\tfrac{1}{2}a)\cdot Y_1 \\ +\cos\pi(\tfrac{1}{4}+\tfrac{1}{2}a)\cdot Y_2\}$$

in which

19.3.3 $$Y_1=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{1}{4}}}y_1=\sqrt{\pi}\frac{\sec\pi(\frac{1}{4}+\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{1}{4}}\Gamma(\frac{3}{4}+\frac{1}{2}a)}y_1$$

19.3.4 $$Y_2=\frac{1}{\sqrt{\pi}}\frac{\Gamma(\frac{3}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a-\frac{1}{4}}}y_2=\sqrt{\pi}\frac{\csc\pi(\frac{1}{4}+\frac{1}{2}a)}{2^{\frac{1}{2}a-\frac{1}{4}}\Gamma(\frac{1}{4}+\frac{1}{2}a)}y_2$$

19.3.5

$$U(a,0)=\frac{\sqrt{\pi}}{2^{\frac{1}{2}a+\frac{1}{4}}\Gamma(\frac{3}{4}+\frac{1}{2}a)}$$

$$U'(a,0)=-\frac{\sqrt{\pi}}{2^{\frac{1}{2}a-\frac{1}{4}}\Gamma(\frac{1}{4}+\frac{1}{2}a)}$$

19.3.6

$$V(a,0)=\frac{2^{\frac{1}{2}a+\frac{1}{4}}\sin\pi(\frac{3}{4}-\frac{1}{2}a)}{\Gamma(\frac{3}{4}-\frac{1}{2}a)}$$

$$V'(a,0)=\frac{2^{\frac{1}{2}a+\frac{3}{4}}\sin\pi(\frac{1}{4}-\frac{1}{2}a)}{\Gamma(\frac{1}{4}-\frac{1}{2}a)}$$

In terms of the more familiar $D_n(x)$ of Whittaker,

19.3.7 $$U(a,x)=D_{-a-\frac{1}{2}}(x)$$

19.3.8

$$V(a,x)=\frac{1}{\pi}\Gamma(\tfrac{1}{2}+a)\{\sin\pi a\cdot D_{-a-\frac{1}{2}}(x)+D_{-a-\frac{1}{2}}(-x)\}$$

19.4. Wronskian and Other Relations

19.4.1 $$W\{U,V\}=\sqrt{2/\pi}$$

19.4.2

$$\pi V(a,x)=\Gamma(\tfrac{1}{2}+a)\{\sin\pi a\cdot U(a,x)+U(a,-x)\}$$

19.4.3

$$\Gamma(\tfrac{1}{2}+a)U(a,x)=\pi\sec^2\pi a\{V(a,-x) \\ -\sin\pi a\cdot V(a,x)\}$$

19.4.4

$$\frac{\Gamma(\frac{1}{4}-\frac{1}{2}a)\cos\pi(\frac{1}{4}+\frac{1}{2}a)}{\sqrt{\pi}2^{\frac{1}{2}a-\frac{3}{4}}}y_1=2\sin\pi(\tfrac{3}{4}+\tfrac{1}{2}a)\cdot Y_1 \\ =U(a,x)+U(a,-x)$$

19.4.5

$$-\frac{\Gamma(\frac{3}{4}-\frac{1}{2}a)\sin\pi(\frac{1}{4}+\frac{1}{2}a)}{\sqrt{\pi}2^{\frac{1}{2}a-\frac{5}{4}}}y_2=2\cos\pi(\tfrac{3}{4}+\tfrac{1}{2}a)\cdot Y_2 \\ =U(a,x)-U(a,-x)$$

19.4.6

$$\sqrt{2\pi}U(-a,\pm ix)= \\ \Gamma(\tfrac{1}{2}+a)\{e^{-i\pi(\frac{1}{2}a-\frac{1}{4})}U(a,\pm x)+e^{i\pi(\frac{1}{2}a-\frac{1}{4})}U(a,\mp x)\}$$

19.4.7

$$\sqrt{2\pi}U(a,\pm x)= \\ \Gamma(\tfrac{1}{2}-a)\{e^{-i\pi(\frac{1}{2}a+\frac{1}{4})}U(-a,\pm ix)+e^{i\pi(\frac{1}{2}a+\frac{1}{4})}U(-a,\mp ix)\}$$

19.5. Integral Representations

A full treatment is given in [19.11] section 4. Representations are given here for $U(a, z)$ only; others may be derived by use of the relations given in **19.4.**

19.5.1 $$U(a,z)=\frac{\Gamma(\frac{1}{2}-a)}{2\pi i}e^{-\frac{1}{4}z^2}\int_{\alpha}e^{zs-\frac{1}{2}s^2}s^{a-\frac{1}{2}}ds$$

19.5.2 $$=\frac{\Gamma(\frac{1}{2}-a)}{2\pi i}e^{\frac{1}{4}z^2}\int_{\beta}e^{-\frac{1}{2}t^2}(z+t)^{a-\frac{1}{2}}dt$$

where α and β are the contours shown in **Figures 19.1** and **19.2.**

When $a+\frac{1}{2}$ is a positive integer these integrals become indeterminate; in this case

19.5.3 $$U(a,z)=\frac{1}{\Gamma(\frac{1}{2}+a)}e^{-\frac{1}{4}z^2}\int_0^{\infty}e^{-zs-\frac{1}{2}s^2}s^{a-\frac{1}{2}}ds$$

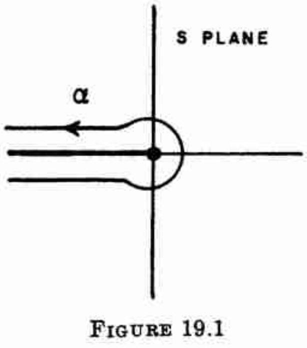

FIGURE 19.1

$-\pi<\arg s<\pi$

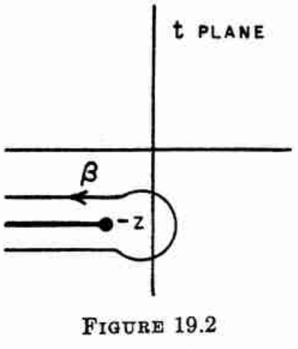

FIGURE 19.2

$-\pi<\arg (z+t)>\pi$

19.5.4 $$U(a,z)=\frac{1}{\sqrt{2\pi i}}\,e^{\frac{1}{4}z^2}\int_{\epsilon}e^{-zs+\frac{1}{2}s^2}s^{-a-\frac{1}{2}}ds$$

19.5.5 $$=\frac{e^{(a-\frac{1}{2})\pi i}}{\sqrt{2\pi i}}\,e^{\frac{1}{4}z^2}\int_{\epsilon_3}e^{zs+\frac{1}{2}s^2}s^{-a-\frac{1}{2}}ds$$

19.5.6 $$=\frac{e^{-(a-\frac{1}{2})\pi i}}{\sqrt{2\pi i}}\,e^{\frac{1}{4}z^2}\int_{\epsilon_4}e^{zs+\frac{1}{2}s^2}s^{-a-\frac{1}{2}}ds$$

where ϵ, ϵ_3 and ϵ_4 are shown in **Figures 19.3** and **19.4.**

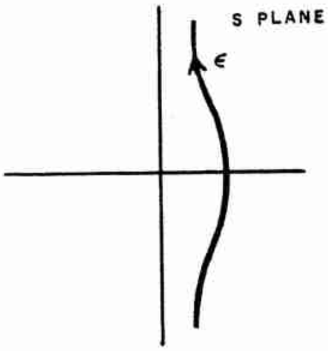

FIGURE 19.3

$-\frac{1}{2}\pi<\arg s<\frac{1}{2}\pi$

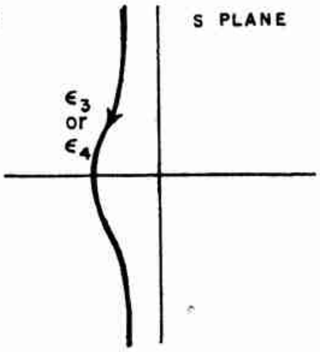

FIGURE 19.4

On ϵ_3 $\frac{1}{2}\pi<\arg s<\frac{3}{2}\pi$

On ϵ_4 $-\frac{3}{2}\pi<\arg s<-\frac{1}{2}\pi$

19.5.7
$$U(a,z)=\frac{\Gamma(\frac{3}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{3}{4}}\pi i}\int_{(\zeta_1)}e^{\frac{1}{4}z^2t}(1+t)^{\frac{1}{2}a-\frac{3}{4}}(1-t)^{-\frac{1}{2}a-\frac{3}{4}}dt$$

19.5.8
$$=\frac{\Gamma(\frac{3}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{3}{4}}\pi i}\int_{\zeta_1}\tfrac{1}{2}ze^{v}(\tfrac{1}{4}z^2+v)^{\frac{1}{2}a-\frac{3}{4}}(\tfrac{1}{4}z^2-v)^{-\frac{1}{2}a-\frac{3}{4}}dv$$

19.5.9
$$U(a,z)$$
$$=\frac{i\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{1}{4}}\pi}\int_{(\eta_1)}\tfrac{1}{2}ze^{-\frac{1}{4}z^2t}(1+t)^{-\frac{1}{2}a-\frac{1}{4}}(1-t)^{\frac{1}{2}a-\frac{1}{4}}dt$$

19.5.10
$$=\frac{i\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{1}{4}}\pi}\int_{\eta_1}e^{-v}(\tfrac{1}{4}z^2+v)^{-\frac{1}{2}a-\frac{1}{4}}(\tfrac{1}{4}z^2-v)^{\frac{1}{2}a-\frac{1}{4}}dv$$

The contour ζ_1 is such that $(\frac{1}{4}z^2+v)$ goes from $\infty e^{-i\pi}$ to $\infty e^{i\pi}$ while $v=\frac{1}{4}z^2$ is *not* encircled; $(\frac{1}{4}z^2-v)^{-\frac{1}{2}a-\frac{3}{4}}$ has its principal value except possibly in the immediate neighborhood of the branch-point when encirclement is being avoided. Likewise η_1 is such that $(\frac{1}{4}z^2-v)$ goes from $\infty e^{i\pi}$ to $\infty e^{-i\pi}$ while encirclement of $v=-\frac{1}{4}z^2$ is similarly avoided. The contours (ζ_1) and (η_1) may be obtained from ζ_1 and η_1 by use of the substitution $v=\frac{1}{4}z^2t$.

The expressions **19.5.7** and **19.5.8** become indeterminate when $a=\frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \ldots$; for these values

19.5.11
$$U(a,z)=\frac{1}{\Gamma(\frac{1}{4}+\frac{1}{2}a)}\,ze^{-\frac{1}{4}z^2}\int_0^\infty e^{-s}s^{\frac{1}{2}a-\frac{3}{4}}(z^2+2s)^{-\frac{1}{2}a-\frac{3}{4}}ds$$

Again **19.5.9** and **19.5.10** become indeterminate when $a=\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots$; for these values

19.5.12
$$U(a,z)=\frac{1}{\Gamma(\frac{3}{4}+\frac{1}{2}a)}\,e^{-\frac{1}{4}z^2}\int_0^\infty e^{-s}s^{\frac{1}{2}a-\frac{1}{4}}(z^2+2s)^{-\frac{1}{2}a-\frac{1}{4}}ds$$

Barnes-Type Integrals

19.5.13 $$U(a,z)=\frac{e^{-\frac{1}{4}z^2}}{2\pi i}\,z^{-a-\frac{1}{2}}\int_{-\infty i}^{+\infty i}\frac{\Gamma(s)\Gamma(\frac{1}{2}+a-2s)}{\Gamma(\frac{1}{2}+a)}\,(\sqrt{2}z)^{2s}ds \qquad (|\arg z|<\tfrac{3}{4}\pi)$$

where the contour separates the zeros of $\Gamma(s)$ from those of $\Gamma(a+\frac{1}{2}-2s)$. Similarly

19.5.14 $$V(a,z)=\sqrt{\frac{2}{\pi}}\frac{e^{\frac{1}{4}z^2}}{2\pi i}\,z^{a-\frac{1}{2}}\int_{-\infty i}^{+\infty i}\frac{\Gamma(s)\Gamma(\frac{1}{2}-a-2s)}{\Gamma(\frac{1}{2}-a)}\,(\sqrt{2}z)^{2s}\cos s\pi\;ds \qquad (|\arg z|<\tfrac{1}{4}\pi)$$

19.6. Recurrence Relations

19.6.1 $U'(a,x)+\frac{1}{2}xU(a,x)+(a+\frac{1}{2})U(a+1,x)=0$

19.6.2 $U'(a,x)-\frac{1}{2}xU(a,x)+U(a-1,x)=0$

19.6.3 $2U'(a,x)+U(a-1,x)+(a+\frac{1}{2})U(a+1,x)=0$

19.6.4 $xU(a,x)-U(a-1,x)+(a+\frac{1}{2})U(a+1,x)=0$

These are also satisfied by $\Gamma(\frac{1}{2}-a)V(a,x)$.

19.6.5 $V'(a,x)-\frac{1}{2}xV(a,x)-(a-\frac{1}{2})V(a-1,x)=0$

19.6.6 $V'(a,x)+\frac{1}{2}xV(a,x)-V(a+1,x)=0$

19.6.7
$$2V'(a,x)-V(a+1,x)-(a-\tfrac{1}{2})V(a-1,x)=0$$

19.6.8
$$xV(a,x)-V(a+1,x)+(a-\tfrac{1}{2})V(a-1,x)=0$$

These are also satisfied by $U(a,x)/\Gamma(\frac{1}{2}-a)$

19.6.9 $y_1'(a,x)+\frac{1}{2}xy_1(a,x)=(a+\frac{1}{2})y_2(a+1,x)$

19.6.10 $y_1'(a,x)-\frac{1}{2}xy_1(a,x)=(a-\frac{1}{2})y_2(a-1,x)$

19.6.11 $$y_2'(a,x)+\tfrac{1}{2}xy_2(a,x)=y_1(a+1,x)$$

19.6.12 $$y_2'(a,x)-\tfrac{1}{2}xy_2(a,x)=y_1(a-1,x)$$

Asymptotic Expansions

19.7. Expressions in Terms of Airy Functions

When a is large and negative, write, for $0\leq x<\infty$

$$x=2\sqrt{|a|}\xi \qquad t=(4|a|)^{\frac{2}{3}}\tau$$

19.7.1

$$\tau=-(\tfrac{3}{2}\vartheta_3)^{\frac{2}{3}}$$

$$\vartheta_3=\tfrac{1}{2}\int_\xi^1\sqrt{1-s^2}ds=\tfrac{1}{4}\arccos\xi-\tfrac{1}{4}\xi\sqrt{1-\xi^2} \quad (\xi\leq 1)$$

19.7.2

$$\tau=+(\tfrac{3}{2}\vartheta_2)^{\frac{2}{3}}$$

$$\vartheta_2=\tfrac{1}{2}\int_1^\xi\sqrt{s^2-1}ds=\tfrac{1}{4}\xi\sqrt{\xi^2-1}-\tfrac{1}{4}\operatorname{arccosh}\xi \quad (\xi\geq 1)$$

Then for $x\geq 0$, $a\to-\infty$

19.7.3

$$U(a,x)\sim 2^{-\frac{1}{4}-\frac{1}{2}a}\Gamma\,(\tfrac{1}{4}-\tfrac{1}{2}a)\left(\frac{t}{\xi^2-1}\right)^{\frac{1}{4}}\mathrm{Ai}(t)$$

19.7.4

$$\Gamma\,(\tfrac{1}{2}-a)\,V(a,x)\sim 2^{-\frac{1}{4}-\frac{1}{2}a}\Gamma\,(\tfrac{1}{4}-\tfrac{1}{2}a)\left(\frac{t}{\xi^2-1}\right)^{\frac{1}{4}}\mathrm{Bi}(t)$$

Table 19.3 gives τ as a function of ξ.
See [19.5] for further developments.

19.8. Expansions for x Large and a Moderate

When $x>>|a|$

19.8.1

$$U(a,x)\sim e^{-\frac{1}{4}x^2}x^{-a-\frac{1}{2}}\left\{1-\frac{(a+\frac{1}{2})(a+\frac{3}{2})}{2x^2}+\frac{(a+\frac{1}{2})(a+\frac{3}{2})(a+\frac{5}{2})(a+\frac{7}{2})}{2\cdot 4x^4}-\cdots\right\}$$
$$(x\to+\infty)$$

19.8.2

$$V(a,x)\sim\sqrt{\frac{2}{\pi}}e^{\frac{1}{4}x^2}x^{a-\frac{1}{2}}\left\{1+\frac{(a-\frac{1}{2})(a-\frac{3}{2})}{2x^2}+\frac{(a-\frac{1}{2})(a-\frac{3}{2})(a-\frac{5}{2})(a-\frac{7}{2})}{2\cdot 4x^4}+\cdots\right\}$$
$$(x\to+\infty)$$

These expansions form the basis for the choice of standard solutions in **19.3.** The former is valid for complex x, with $|\arg x|<\frac{3}{4}\pi$, in the complete sense of Watson [19.6], although valid for a wider range of $|\arg x|$ in Poincaré's sense; the second series is completely valid *only for x real and positive.*

19.9. Expansions for a Large With x Moderate

(i) a positive

When $a>>x^2$, with $p=\sqrt{a}$, then

19.9.1 $$U(a,x)=\frac{\sqrt{\pi}}{2^{\frac{1}{2}a+\frac{1}{4}}\Gamma(\frac{3}{4}+\frac{1}{2}a)}\exp\,(-px+v_1)$$

19.9.2 $$U(a,-x)=\frac{\sqrt{\pi}}{2^{\frac{1}{2}a+\frac{1}{4}}\Gamma(\frac{3}{4}+\frac{1}{2}a)}\exp\,(px+v_2)$$

where

19.9.3 $$v_1,\,v_2\sim\mp\frac{\frac{2}{3}(\frac{1}{2}x)^3}{2p}-\frac{(\frac{1}{2}x)^2}{(2p)^2}\mp\frac{\frac{1}{2}x-\frac{2}{5}(\frac{1}{2}x)^5}{(2p)^3}+\frac{2(\frac{1}{2}x)^4}{(2p)^4}\pm\frac{(\frac{16}{3}\frac{1}{2}x)^3-\frac{4}{7}(\frac{1}{2}x)^7}{(2p)^5}+\cdots$$
$$(a\to+\infty)$$

The upper sign gives the first function, and the lower sign the second function.

(ii) a negative

When $-a>>x^2$, with $p=\sqrt{-a}$, then

19.9.4

$$U(a,x)+i\Gamma\,(\tfrac{1}{2}-a)\cdot V(a,x)=\frac{e^{i\pi(\frac{1}{4}+\frac{1}{2}a)}\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{1}{4}}\sqrt{\pi}}\,e^{ipx}\exp\,(v_r+iv_i)$$

where

19.9.5

$$v_r\sim+\frac{(\frac{1}{2}x)^2}{(2p)^2}+\frac{2(\frac{1}{2}x)^4}{(2p)^4}-\frac{9(\frac{1}{2}x)^2-\frac{16}{3}(\frac{1}{2}x)^6}{(2p)^6}-\cdots$$

$$v_i\sim-\frac{\frac{2}{3}(\frac{1}{2}x)^3}{2p}+\frac{\frac{1}{2}x+\frac{2}{5}(\frac{1}{2}x)^5}{(2p)^3}+\frac{\frac{16}{3}(\frac{1}{2}x)^3-\frac{4}{7}(\frac{1}{2}x)^7}{(2p)^5}-\cdots$$
$$(a\to-\infty)$$

Further expansions of a similar type will be found in [19.11].

19.10. Darwin's Expansions

(i) a positive, x^2+4a large. Write

19.10.1 $$X=\sqrt{x^2+4a}$$

$$\theta=4a\vartheta_1(x/2\sqrt{a})=\tfrac{1}{2}\int_0^x Xdx=\tfrac{1}{4}xX+a\ln\frac{x+X}{2\sqrt{a}}$$
$$=\frac{x}{4}\sqrt{x^2+4a}+a\operatorname{arcsinh}\frac{x}{2\sqrt{a}}$$

(see **Table 19.3** for ϑ_1), then

19.10.2 $$U(a,x)=\frac{(2\pi)^{1/4}}{\sqrt{\Gamma(\frac{1}{2}+a)}}\exp\{-\theta+v(a,x)\}$$

19.10.3 $$U(a,-x)=\frac{(2\pi)^{1/4}}{\sqrt{\Gamma(\frac{1}{2}+a)}}\exp\{\theta+v(a,-x)\}$$

where

19.10.4

$$v(a,x)\sim-\tfrac{1}{2}\ln X+\sum_{s=1}(-1)^s d_{3s}/X^{3s}\qquad(a>0,\ x^2+4a\to+\infty)$$

and d_{3s} is given by **19.10.13.**

(ii) a negative, x^2+4a large and positive. Write

19.10.5 $$X=\sqrt{x^2-4|a|}$$

$$\theta=4|a|\vartheta_2(x/2\sqrt{|a|})=\tfrac{1}{2}\int_{2\sqrt{|a|}}^{x}Xdx=\tfrac{1}{4}xX+a\ln\frac{x+X}{2\sqrt{|a|}}$$

$$=\tfrac{1}{4}x\sqrt{x^2-4|a|}+a\operatorname{arccosh}\frac{x}{2\sqrt{|a|}}$$

(see **Table 19.3** for ϑ_2), then

19.10.6 $$U(a,x)=\frac{\sqrt{\Gamma(\frac{1}{2}-a)}}{(2\pi)^{1/4}}\exp\{-\theta+v(a,x)\}$$

19.10.7

$$V(a,x)=\frac{2}{(2\pi)^{1/4}\sqrt{\Gamma(\frac{1}{2}-a)}}\exp\{\theta+v(a,-x)\}$$

where again

19.10.8

$$v(a,x)\sim-\tfrac{1}{2}\ln X+\sum_{s=1}(-1)^s d_{3s}/X^{3s}\qquad(a<0,\ x^2+4a\to+\infty)$$

and d_{3s} is given by **19.10.13.**

(iii) a large and negative and x moderate. Write

19.10.9 $$Y=\sqrt{4|a|-x^2}$$

$$\theta=4|a|\vartheta_4(x/2\sqrt{|a|})$$

$$=\tfrac{1}{2}\int_0^x Ydx=\tfrac{1}{4}xY+|a|\arcsin\frac{x}{2\sqrt{|a|}}$$

(see **Table 19.3** for $\vartheta_4=\frac{1}{8}\pi-\vartheta_3$), then

19.10.10

$$U(a,x)=\frac{2\sqrt{\Gamma(\frac{1}{2}-a)}}{(2\pi)^{1/4}}e^{v_r}\cos\left\{\tfrac{1}{4}\pi+\tfrac{1}{2}\pi a+\theta+v_i\right\}$$

19.10.11

$$V(a,x)=\frac{2}{(2\pi)^{\frac{1}{4}}\sqrt{\Gamma(\frac{1}{2}-a)}}e^{v_r}\sin\{\tfrac{1}{4}\pi+\tfrac{1}{2}\pi a+\theta+v_i\}$$

where

19.10.12 $$v_r\sim-\tfrac{1}{2}\ln Y-\frac{d_6}{Y^6}+\frac{d_{12}}{Y^{12}}-\ldots$$

$$v_i\sim\frac{d_3}{Y^3}-\frac{d_9}{Y^9}+\ldots\qquad(x^2+4a\to-\infty)$$

In each case the coefficients d_{3r} are given by

19.10.13

$$d_3=\frac{1}{a}\left(\frac{x^3}{48}+\tfrac{1}{2}ax\right)$$

$$d_6=\tfrac{3}{4}x^2-2a$$

$$d_9=\frac{1}{a^3}\left(-\frac{7}{5760}x^9-\frac{7}{320}ax^7-\frac{49}{320}a^2x^5+\frac{31}{12}a^3x^3-19a^4x\right)$$

$$d_{12}=\frac{153}{8}x^4-186ax^2+80a^2$$

See [19.11] for $d_{15},\ldots,d_{24}$, and [19.5] for an alternative form.

19.11. Modulus and Phase

When a is negative and $|x|<2\sqrt{|a|}$, the functions U and V are oscillatory and it is sometimes convenient to write

19.11.1 $$U(a,x)+i\Gamma(\tfrac{1}{2}-a)V(a,x)=F(a,x)e^{i\chi(a,x)}$$

19.11.2 $$U'(a,x)+i\Gamma(\tfrac{1}{2}-a)V'(a,x)=-G(a,x)e^{i\psi(a,x)}$$

Then, when $a<0$ and $|a|>>x^2$,

19.11.3

$$F=\frac{\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{1}{4}}\sqrt{\pi}}e^{v_r},\qquad\chi=(\tfrac{1}{2}a+\tfrac{1}{4})\pi+px+v_i$$

where v_r, v_i are given by **19.9.5** and $p=\sqrt{-a}$.

Alternatively, with $p=\sqrt{|a|}$, and again $-a>>x^2$,

19.11.4

$$F\sim\frac{\Gamma(\frac{1}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a+\frac{1}{4}}\sqrt{\pi}}\left\{1+\frac{x^2}{(4p)^2}+\frac{\frac{5}{2}x^4}{(4p)^4}+\frac{\frac{15}{2}x^6-144x^2}{(4p)^6}+\ldots\right\}$$

19.11.5
$$\chi\sim(\tfrac{1}{2}a+\tfrac{1}{4})\pi+px\left\{1-\frac{\frac{2}{3}x^2}{(4p)^2}-\frac{\frac{2}{5}x^4-16}{(4p)^4}-\frac{\frac{4}{7}x^6-\frac{256}{3}x^2}{(4p)^6}-\ldots\right\}$$

19.11.6
$$G\sim\frac{\Gamma(\frac{3}{4}-\frac{1}{2}a)}{2^{\frac{1}{2}a-\frac{1}{4}}\sqrt{\pi}}\left\{1-\frac{x^2}{(4p)^2}-\frac{\frac{3}{2}x^4}{(4p)^4}-\frac{\frac{7}{2}x^6-176x^2}{(4p)^6}-\ldots\right\}$$

19.11.7
$$\psi\sim(\tfrac{1}{2}a-\tfrac{1}{4})\pi+px\left\{1-\frac{\frac{2}{3}x^2}{(4p)^2}-\frac{\frac{2}{5}x^4+16}{(4p)^4}-\frac{\frac{4}{7}x^6+\frac{320}{3}x^2}{(4p)^6}-\ldots\right\}$$

Again, when x^2+4a is large and negative, with $Y=\sqrt{4|a|-x^2}$, then

19.11.8
$$F=\frac{2\sqrt{\Gamma(\frac{1}{2}-a)}}{(2\pi)^{\frac{1}{4}}}\,e^{v_r}\qquad\chi=\tfrac{1}{4}\pi+\tfrac{1}{2}\pi a+\theta+v_i$$

where θ, v_r and v_i are given by **19.10.9** and **19.10.12.**

Another form is

19.11.9
$$F\sim\frac{2\sqrt{\Gamma(\frac{1}{2}-a)}}{(2\pi)^{\frac{1}{4}}\sqrt{Y}}\left(1+\frac{3}{4Y^4}+\frac{5a}{Y^6}+\frac{621}{32Y^8}+\ldots\right)\qquad(x^2+4a\rightarrow-\infty)$$

19.11.10
$$G\sim\frac{\sqrt{Y}\sqrt{\Gamma(\frac{1}{2}-a)}}{(2\pi)^{\frac{1}{4}}}\left(1-\frac{5}{4Y^4}-\frac{7a}{Y^6}-\frac{835}{32Y^8}-\ldots\right)\qquad(x^2+4a\rightarrow-\infty)$$

while ψ and χ are connected by

19.11.11
$$\psi-\chi\sim-\tfrac{1}{2}\pi-\frac{x}{Y^3}\left(1+\frac{47}{6Y^4}+\frac{214a}{3Y^6}+\frac{14483}{40Y^8}+\ldots\right)\qquad(x^2+4a\rightarrow-\infty)$$

Connections With Other Functions

19.12. Connection With Confluent Hypergeometric Functions (see chapter 13)

19.12.1
$$U(a,\pm x)=\frac{\sqrt{\pi}2^{-\frac{1}{2}a}x^{-\frac{1}{2}}}{\Gamma(\frac{3}{4}+\frac{1}{2}a)}\,M_{-\frac{1}{2}a,\,-\frac{1}{4}}(\tfrac{1}{2}x^2)\mp\frac{\sqrt{\pi}2^{1-\frac{1}{2}a}x^{-\frac{1}{2}}}{\Gamma(\frac{1}{4}+\frac{1}{2}a)}\,M_{-\frac{1}{2}a,\,\frac{1}{4}}(\tfrac{1}{2}x^2)$$

19.12.2 $U(a,x)=2^{-\frac{1}{2}a}x^{-\frac{1}{2}}W_{-\frac{1}{2}a,\,-\frac{1}{4}}(\tfrac{1}{2}x^2)$

19.12.3
$$U(a,\pm x)=\frac{\sqrt{\pi}2^{-\frac{1}{4}-\frac{1}{2}a}e^{-\frac{1}{4}x^2}}{\Gamma(\frac{3}{4}+\frac{1}{2}a)}\,M(\tfrac{1}{2}a+\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{2}x^2)\mp\frac{\sqrt{\pi}2^{\frac{1}{4}-\frac{1}{2}a}xe^{-\frac{1}{4}x^2}}{\Gamma(\frac{1}{4}+\frac{1}{2}a)}\,M(\tfrac{1}{2}a+\tfrac{3}{4},\tfrac{3}{2},\tfrac{1}{2}x^2)$$

19.12.4
$$U(a,x)=2^{-\frac{1}{4}-\frac{1}{2}a}e^{-\frac{1}{4}x^2}U(\tfrac{1}{2}a+\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{2}x^2)=2^{-\frac{3}{4}-\frac{1}{2}a}xe^{-\frac{1}{4}x^2}U(\tfrac{1}{2}a+\tfrac{3}{4},\tfrac{3}{2},\tfrac{1}{2}x^2)$$

Expressions for $V(a, x)$ may be obtained from these by use of **19.4.2.**

19.13. Connection With Hermite Polynomials and Functions

When n is a non-negative integer

19.13.1
$$U(-n-\tfrac{1}{2},x)=e^{-\frac{1}{4}x^2}He_n(x)=2^{-\frac{1}{2}n}e^{-\frac{1}{4}x^2}H_n(x/\sqrt{2})$$

19.13.2
$$V(n+\tfrac{1}{2},x)=\sqrt{2/\pi}e^{\frac{1}{4}x^2}He_n^*(x)=2^{-\frac{1}{2}n}e^{\frac{1}{4}x^2}H_n^*(x/\sqrt{2})$$

in which $H_n(x)$ and $He_n(x)$ are Hermite polynomials (see chapter **22**) while

19.13.3 $He_n^*(x)=e^{-\frac{1}{2}x^2}\dfrac{d^n}{dx^n}\,e^{\frac{1}{2}x^2}=(-i)^nHe_n(ix)$

19.13.4 $H_n^*(x)=e^{-x^2}\dfrac{d^n}{dx^n}\,e^{x^2}=(-i)^nH_n(ix)$

This gives one elementary solution to **19.1.2** whenever $2a$ is an odd integer, positive or negative.

19.14. Connection With Probability Integrals and Dawson's Integral (see chapter 7)

If, as in [19.10]

19.14.1 $Hh_{-1}(x)=e^{-\frac{1}{2}x^2}$

19.14.2
$$Hh_n(x)=\int_x^\infty Hh_{n-1}(t)dt=(1/n!)\int_x^\infty(t-x)^ne^{-\frac{1}{2}t^2}dt\qquad(n\geq 0)$$

then

19.14.3 $U(n+\tfrac{1}{2},x)=e^{\frac{1}{4}x^2}Hh_n(x)\qquad(n\geq -1)$

Correspondingly

19.14.4 $$V(\tfrac12,x)=\sqrt{2/\pi}e^{\frac14 x^2}$$

and

19.14.5

$$V(-n-\tfrac12,x)=e^{-\frac14 x^2}\left\{\int_0^x e^{-\frac14 t^2}V(-n+\tfrac12,t)dt-\frac{\sin\frac12 n\pi}{2^{\frac12 n}\Gamma(\frac12 n+1)}\right\}\ (n\geq 0)$$

Here $V(-\frac12, x)$ is closely related to Dawson's integral

$$\int_0^x e^{t^2}dt$$

These relations give a second solution of **19.1.2** whenever $2a$ is an odd integer, and a second solution is unobtainable from $U(a, x)$ by reflection in the y-axis.

19.15. Explicit Formula in Terms of Bessel Functions When $2a$ Is an Integer

Write

19.15.1 $$I_{-n}-I_n=(2/\pi)\sin n\pi\cdot K_n$$

19.15.2 $$I_{-n}+I_n=\cos n\pi\cdot\mathscr{I}_n$$

where the argument of all modified Bessel functions is $\frac14 x^2$. Then

19.15.3 $$U(1, x)=2\pi^{-\frac12}(\tfrac12 x)^{\frac32}(-K_{\frac14}+K_{\frac34})$$

19.15.4 $$U(2, x)=2\cdot\tfrac23\pi^{-\frac12}(\tfrac12 x)^{\frac52}(2K_{\frac14}-3K_{\frac34}+K_{\frac54})$$

19.15.5

$$U(3, x)=2\cdot\tfrac23\cdot\tfrac25\pi^{-\frac12}(\tfrac12 x)^{\frac72}(-5K_{\frac14}+9K_{\frac34}-5K_{\frac54}+K_{\frac74})$$

19.15.6 $$V(1, x)=\tfrac12(\tfrac12 x)^{\frac32}(\mathscr{I}_{\frac14}-\mathscr{I}_{\frac34})$$

19.15.7 $$V(2, x)=\tfrac12(\tfrac12 x)^{\frac52}(2\mathscr{I}_{\frac14}-3\mathscr{I}_{\frac34}+\mathscr{I}_{\frac54})$$

19.15.8 $$V(3, x)=\tfrac12(\tfrac12 x)^{\frac72}(5\mathscr{I}_{\frac14}-9\mathscr{I}_{\frac34}+5\mathscr{I}_{\frac54}-\mathscr{I}_{\frac74})$$

19.15.9 $$U(0, x)=\pi^{-\frac12}(\tfrac12 x)^{\frac12}K_{\frac14}$$

19.15.10 $$U(-1, x)=\pi^{-\frac12}(\tfrac12 x)^{\frac32}(K_{\frac14}+K_{\frac34})$$

19.15.11

$$U(-2, x)=\pi^{-\frac12}(\tfrac12 x)^{\frac52}(2K_{\frac14}+3K_{\frac34}-K_{\frac54})$$

19.15.12

$$U(-3, x)=\pi^{-\frac12}(\tfrac12 x)^{\frac72}(5K_{\frac14}+9K_{\frac34}-5K_{\frac54}-K_{\frac74})$$

19.15.13 $$V(0, x)=\tfrac12(\tfrac12 x)^{\frac12}\mathscr{I}_{\frac14}$$

19.15.14 $$V(-1, x)=(\tfrac12 x)^{\frac32}(\mathscr{I}_{\frac14}+\mathscr{I}_{\frac34})$$

19.15.15 $$V(-2, x)=\tfrac23(\tfrac12 x)^{\frac52}(2\mathscr{I}_{\frac14}+3\mathscr{I}_{\frac34}-\mathscr{I}_{\frac54})$$

19.15.16

$$V(-3, x)=\tfrac23\cdot\tfrac25(\tfrac12 x)^{\frac72}(5\mathscr{I}_{\frac14}+9\mathscr{I}_{\frac34}-5\mathscr{I}_{\frac54}-\mathscr{I}_{\frac74})$$

19.15.17 $$U(-\tfrac12, x)=\sqrt{2/\pi}(\tfrac12 x)K_{\frac12}$$

19.15.18 $$U(-\tfrac32, x)=\sqrt{2/\pi}(\tfrac12 x)^2 2K_{\frac12}$$

19.15.19 $$U(-\tfrac52, x)=\sqrt{2/\pi}(\tfrac12 x)^3(5K_{\frac12}-K_{\frac32})$$

19.15.20 $$V(\tfrac12, x)=(\tfrac12 x)(I_{\frac12}+I_{-\frac12})$$

19.15.21 $$V(\tfrac32, x)=(\tfrac12 x)^2(2I_{\frac12}+2I_{-\frac12})$$

19.15.22 $$V(\tfrac52, x)=(\tfrac12 x)^3(5I_{\frac12}+5I_{-\frac12}-I_{\frac32}-I_{-\frac32})$$

The Equation $\frac{d^2y}{dx^2}+\left(\frac14 x^2-a\right)y=0$

19.16. Power Series in x

Even and odd solutions are given by **19.2.1** to **19.2.4** with $-ia$ written for a and $xe^{\frac14 i\pi}$ for x; the series involves complex quantities in which the imaginary part of the sum vanishes identically.

Alternatively,

19.16.1

$$y_1=1+a\frac{x^2}{2!}+(a^2-\tfrac12)\frac{x^4}{4!}+(a^3-\tfrac72 a)\frac{x^6}{6!}+(a^4-11a^2+\tfrac{15}{4})\frac{x^8}{8!}+(a^5-25a^3+\tfrac{211}{4}a)\frac{x^{10}}{10!}+\ldots$$

19.16.2

$$y_2=x+a\frac{x^3}{3!}+(a^2-\tfrac32)\frac{x^5}{5!}+(a^3-\tfrac{13}{2}a)\frac{x^7}{7!}+(a^4-17a^2+\tfrac{63}{4})\frac{x^9}{9!}+(a^5-35a^3+\tfrac{531}{4}a)\frac{x^{11}}{11!}+\ldots$$

in which non-zero coefficients a_n of $x^n/n!$ are connected by

19.16.3 $$a_{n+2}=a\cdot a_n-\tfrac14 n(n-1)a_{n-2}$$

19.17. Standard Solutions (see [19.4])

19.17.1 $$W(a,\pm x)=\frac{(\cosh\pi a)^{\frac14}}{2\sqrt{\pi}}(G_1y_1\mp\sqrt{2}G_3y_2)$$

19.17.2 $$=2^{-3/4}\left(\sqrt{\frac{G_1}{G_3}}y_1\mp\sqrt{\frac{2G_3}{G_1}}y_2\right)$$

where

19.17.3 $G_1=|\Gamma(\frac14+\frac12 ia)|$ $\qquad G_3=|\Gamma(\frac34+\frac12 ia)|$

At $x=0$,

19.17.4 $W(a,0)=\frac{1}{2^{\frac{3}{4}}}\left|\frac{\Gamma(\frac{1}{4}+\frac{1}{2}ia)}{\Gamma(\frac{3}{4}+\frac{1}{2}ia)}\right|^{\frac{1}{2}}=\frac{1}{2^{\frac{3}{4}}}\sqrt{\frac{G_1}{G_3}}$

19.17.5

$$W'(a,0)=-\frac{1}{2^{\frac{1}{4}}}\left|\frac{\Gamma(\frac{3}{4}+\frac{1}{2}ia)}{\Gamma(\frac{1}{4}+\frac{1}{2}ia)}\right|^{\frac{1}{2}}=-\frac{1}{2^{\frac{1}{4}}}\sqrt{\frac{G_3}{G_1}}$$

Complex Solutions

19.17.6 $E(a,x)=k^{-\frac{1}{2}}W(a,x)+ik^{\frac{1}{2}}W(a,-x)$

19.17.7 $E^*(a,x)=k^{-\frac{1}{2}}W(a,x)-ik^{\frac{1}{2}}W(a,-x)$

where

19.17.8 $k=\sqrt{1+e^{2\pi a}}-e^{\pi a}$ $\qquad 1/k=\sqrt{1+e^{2\pi a}}+e^{\pi a}$

In terms of $U(a, x)$ of **19.3,**

19.17.9 $E(a,x)=\sqrt{2}e^{\frac{1}{4}\pi a+\frac{1}{8}i\pi+\frac{1}{2}i\phi_2}U(ia,xe^{-\frac{1}{4}i\pi})$

with

19.17.10 $\phi_2=\arg\Gamma(\frac{1}{2}+ia)$

where the branch is defined by $\phi_2=0$ when $a=0$ and by continuity elsewhere.

Also

19.17.11

$$\sqrt{2\pi}U(ia,xe^{-\frac{1}{4}i\pi})=\Gamma(\tfrac{1}{2}-ia)\{e^{\frac{1}{2}\pi a-\frac{1}{4}i\pi}U(-ia,xe^{\frac{1}{4}i\pi})$$
$$+e^{-\frac{1}{2}\pi a+\frac{1}{4}i\pi}U(-ia,-xe^{\frac{1}{4}i\pi})\}$$

19.18. Wronskian and Other Relations

19.18.1 $W\{W(a,x),W(a,-x)\}=1$

19.18.2 $W\{E(a,x),E^*(a,x)\}=-2i$

19.18.3 $\sqrt{1+e^{2\pi a}}E(a,x)=e^{\pi a}E^*(a,x)+iE^*(a,-x)$

19.18.4 $E^*(a,x)=e^{-i(\phi_2+\frac{1}{2}\pi)}E(-a,ix)$

19.18.5

$$\sqrt{\Gamma(\tfrac{1}{2}+ia)}E^*(a,x)=e^{-\frac{1}{4}i\pi}\sqrt{\Gamma(\tfrac{1}{2}-ia)}E(-a,ix)$$

19.19. Integral Representations

These are covered for **19.1.3** as well as for **19.1.2** in **19.5** (general complex argument).

Asymptotic Expansions

19.20. Expressions in Terms of Airy Functions

When a is large and positive, write, for $0\leq x<\infty$

$$x=2\sqrt{a}\xi \qquad t=(4a)^{\frac{2}{3}}\tau$$

19.20.1

$$\tau=-(\tfrac{3}{2}\vartheta_3)^{\frac{2}{3}}$$

$$\vartheta_3=\frac{1}{2}\int_\xi^1\sqrt{1-s^2}\,ds=\tfrac{1}{4}\arccos\xi-\tfrac{1}{4}\xi\sqrt{1-\xi^2}\quad(\xi\leq 1)$$

19.20.2

$$\tau=+(\tfrac{3}{2}\vartheta_2)^{\frac{2}{3}}$$

$$\vartheta_2=\tfrac{1}{2}\int_1^\xi\sqrt{s^2-1}ds=\tfrac{1}{4}\xi\sqrt{\xi^2-1}-\tfrac{1}{4}\operatorname{arccosh}\xi\qquad(\xi\geq 1)$$

Then for $x>0$, $a\rightarrow+\infty$

19.20.3

$$W(a,x)\sim\sqrt{\pi}(4a)^{-\frac{1}{4}}e^{-\frac{1}{2}\pi a}\left(\frac{t}{\xi^2-1}\right)^{\frac{1}{4}}\mathrm{Bi}(-t)$$

19.20.4

$$W(a,-x)\sim 2\sqrt{\pi}(4a)^{-\frac{1}{4}}e^{\frac{1}{2}\pi a}\left(\frac{t}{\xi^2-1}\right)^{\frac{1}{4}}\mathrm{Ai}(-t)$$

Table 19.3 gives τ as a function of ξ. See [19.5] for further developments.

19.21. Expansions for x Large and a Moderate

When $x>>|a|$,

19.21.1

$$E(a,x)=\sqrt{2/x}\exp\{i(\tfrac{1}{4}x^2-a\ln x+\tfrac{1}{2}\phi_2+\tfrac{1}{4}\pi)\}s(a,x)$$

19.21.2

$$W(a,x)=\sqrt{2k/x}\{s_1(a,x)\cos(\tfrac{1}{4}x^2-a\ln x+\tfrac{1}{4}\pi+\tfrac{1}{2}\phi_2)$$
$$-s_2(a,x)\sin(\tfrac{1}{4}x^2-a\ln x+\tfrac{1}{4}\pi+\tfrac{1}{2}\phi_2)\}$$

19.21.3

$$W(a,-x)=\sqrt{2/kx}\{s_1(a,x)\sin(\tfrac{1}{4}x^2-a\ln x+\tfrac{1}{4}\pi+\tfrac{1}{2}\phi_2)$$
$$+s_2(a,x)\cos(\tfrac{1}{4}x^2-a\ln x+\tfrac{1}{4}\pi+\tfrac{1}{2}\phi_2)\}$$

where ϕ_2 is defined by **19.17.10** and

19.21.4 $s(a,x)=s_1(a,x)+is_2(a,x)$

19.21.5

$$s_1(a,x)\sim 1+\frac{v_2}{1!2x^2}-\frac{u_4}{2!2^2x^4}-\frac{v_6}{3!2^3x^6}+\frac{u_8}{4!2^4x^8}+\ldots$$

19.21.6

$$s_2(a,x)\sim-\frac{u_2}{1!2x^2}-\frac{v_4}{2!2^2x^4}+\frac{u_6}{3!2^3x^6}+\frac{v_8}{4!2^4x^8}-\ldots$$

with

$(x\rightarrow+\infty)$

19.21.7 $$u_r+iv_r=\Gamma(r+\tfrac{1}{2}+ia)/\Gamma(\tfrac{1}{2}+ia)$$

or

19.21.8 $$s(a,x)\sim\sum_{r=0}^{\infty}(-i)^r\frac{\Gamma(2r+\frac{1}{2}+ia)}{\Gamma(\frac{1}{2}+ia)}\frac{1}{2^r r!x^{2r}}$$

19.22. Expansions for a Large With x Moderate

(i) a positive

When $a>>x^2$, with $p=\sqrt{a}$, then

19.22.1 $$W(a,x)=W(a,0)\exp(-px+v_1)$$

19.22.2 $$W(a,-x)=W(a,0)\exp(px+v_2)$$

where $W(a,0)$ is given by **19.17.4,** and

19.22.3
$$v_1,v_2\sim\pm\frac{\frac{2}{3}(\frac{1}{2}x)^3}{2p}+\frac{(\frac{1}{2}x)^2}{(2p)^2}\pm\frac{\frac{1}{2}x+\frac{2}{5}(\frac{1}{2}x)^5}{(2p)^3}+\frac{2(\frac{1}{2}x)^4}{(2p)^4}\pm\frac{\frac{16}{3}(\frac{1}{2}x)^3+\frac{4}{7}(\frac{1}{2}x)^7}{(2p)^5}+\dots$$
$$(a\rightarrow+\infty)$$

The upper sign gives the first function, and the lower sign the second function.

(ii) a negative

When $-a>>x^2$, with $p=\sqrt{-a}$, then

19.22.4
$$W(a,x)+iW(a,-x)=\sqrt{2}W(a,0)\exp\{v_r+i(px+\tfrac{1}{4}\pi+v_i)\}$$

where $W(a,0)$ is given by **19.17.4,** and

19.22.5
$$v_r\sim-\frac{(\frac{1}{2}x)^2}{(2p)^2}+\frac{2(\frac{1}{2}x)^4}{(2p)^4}-\frac{9(\frac{1}{2}x)^2+\frac{16}{3}(\frac{1}{2}x)^6}{(2p)^6}+\dots$$
$$v_i\sim\frac{\frac{2}{3}(\frac{1}{2}x)^3}{2p}-\frac{\frac{1}{2}x+\frac{2}{5}(\frac{1}{2}x)^5}{(2p)^3}+\frac{\frac{16}{3}(\frac{1}{2}x)^3+\frac{4}{7}(\frac{1}{2}x)^7}{(2p)^5}-\dots$$
$$(a\rightarrow-\infty)$$

Further expansions of a similar type will be found in [19.3].

19.23. Darwin's Expansions

(i) a positive, $x^2-4a>>0$

Write

19.23.1
$$X=\sqrt{x^2-4a}\qquad\theta=4a\vartheta_2(x/2\sqrt{a})=\tfrac{1}{2}\int_{2\sqrt{a}}^{x}Xdx$$
$$=\tfrac{1}{4}xX-a\ln\frac{x+X}{2\sqrt{a}}$$
$$=\tfrac{1}{4}x\sqrt{x^2-4a}-a\operatorname{arccosh}\frac{x}{2\sqrt{a}}$$

(see **Table 19.3** for ϑ_2), then

19.23.2 $$W(a,x)=\sqrt{2k}e^{v_r}\cos(\tfrac{1}{4}\pi+\theta+v_i)$$

19.23.3 $$W(a,-x)=\sqrt{2/k}e^{v_r}\sin(\tfrac{1}{4}\pi+\theta+v_i)$$

where

19.23.4
$$v_r\sim-\tfrac{1}{2}\ln X-\frac{d_6}{X^6}+\frac{d_{12}}{X^{12}}-\dots$$
$$v_i\sim-\frac{d_3}{X^3}+\frac{d_9}{X^9}-\frac{d_{15}}{X^{15}}+\dots$$
$$(x^2-4a\rightarrow\infty)$$

and d_{3r} is given by **19.23.12.**

(ii) a positive, $4a-x^2>>0$

Write

19.23.5
$$Y=\sqrt{4a-x^2}\qquad\theta=4a\vartheta_4(x/2\sqrt{a})$$
$$=\tfrac{1}{2}\int_0^x Ydx=\tfrac{1}{4}xY+a\arcsin\frac{x}{2\sqrt{a}}$$

(see **Table 19.3** for $\vartheta_4=\frac{1}{8}\pi-\vartheta_3$), then

19.23.6 $$W(a,x)=\exp\{-\theta+v(a,x)\}$$

19.23.7 $$W(a,-x)=\exp\{\theta+v(a,-x)\}$$

where

19.23.8
$$v(a,x)\sim-\tfrac{1}{2}\ln Y+\frac{d_3}{Y^3}+\frac{d_6}{Y^6}+\frac{d_9}{Y^9}+\dots$$
$$(x^2-4a\rightarrow-\infty)$$

and d_{3r} is again given by **19.23.12.**

(iii) a negative, $x^2-4a>>0$

Write

19.23.9
$$X=\sqrt{x^2+4|a|}\qquad\theta=4|a|\vartheta_1(x/2\sqrt{|a|})=\tfrac{1}{2}\int_0^x Xdx$$
$$=\tfrac{1}{4}xX-a\ln\frac{x+X}{2\sqrt{|a|}}$$
$$=\tfrac{1}{4}x\sqrt{x^2+4|a|}-a\operatorname{arcsinh}\frac{x}{2\sqrt{|a|}}$$

(see **Table 19.3** for ϑ_1) then

19.23.10 $$W(a,x)=\sqrt{2k}e^{v_r}\cos(\tfrac{1}{4}\pi+\theta+v_i)$$

19.23.11 $$W(a,-x)=\sqrt{2/k}e^{v_r}\sin(\tfrac{1}{4}\pi+\theta+v_i)$$

where v_r and v_i are again given by **19.23.4.** In each case the coefficients d_{3r} are given by

19.23.12

$$d_3=-\frac{1}{a}\left(\frac{x^3}{48}-\tfrac{1}{2}ax\right)$$

$$d_6=\tfrac{3}{4}x^2+2a$$

$$d_9=\frac{1}{a^3}\left(\frac{7}{5760}x^9-\frac{7}{320}ax^7+\frac{49}{320}a^2x^5+\frac{31}{12}a^3x^3+19a^4x\right)$$

$$d_{12}=\frac{153}{8}x^4+186ax^2+80a^2$$

See [19.11] for $d_{15}, \ldots, d_{24}$, and [19.5] for an alternative form.

19.24. Modulus and Phase

When a is positive, the function $W(a, x)$ is oscillatory when $x<-2\sqrt{a}$ and when $x>2\sqrt{a}$; when a is negative, the function is oscillatory for all x. In such cases it is sometimes convenient to write

19.24.1

$$k^{-\frac{1}{2}}W(a, x)+ik^{\frac{1}{2}}W(a,-x)=E(a, x)=Fe^{i\chi} \qquad (x>0)$$

19.24.2

$$k^{-\frac{1}{2}}\frac{dW(a, x)}{dx}+ik^{\frac{1}{2}}\frac{dW(a,-x)}{dx}=E'(a, x)=-Ge^{i\psi} \qquad (x>0)$$

Then, when $x^2>>|a|$,

19.24.3

$$F\sim\sqrt{\frac{2}{x}}\left(1+\frac{a}{x^2}+\frac{10a^2-3}{4x^4}+\frac{30a^3-47a}{4x^6}+\ldots\right)$$

19.24.4

$$\chi\sim\tfrac{1}{4}x^2-a\ln x+\tfrac{1}{2}\phi_2+\tfrac{1}{4}\pi+\frac{4a^2-3}{8x^2}+\frac{4a^3-19a}{8x^4}+\ldots$$

19.24.5

$$G\sim\sqrt{\frac{x}{2}}\left(1-\frac{a}{x^2}-\frac{6a^2-5}{4x^4}-\frac{14a^3-63a}{4x^6}-\ldots\right)$$

19.24.6

$$\psi\sim\tfrac{1}{4}x^2-a\ln x+\tfrac{1}{2}\phi_2-\tfrac{1}{4}\pi+\frac{4a^2+5}{8x^2}+\frac{4a^3+29a}{8x^4}+\ldots$$

where ϕ_2 is defined by **19.17.10.**

When $a<0$, $|a|>>x^2$

19.24.7 $$F\sim\sqrt{2}W(a, 0)e^{v_r}$$

where v_r is given by **19.22.5** with $p=\sqrt{-a}$. Also

19.24.8

$$F\sim\frac{1}{\sqrt{p}}\left(1-\frac{x^2}{(4p)^2}+\frac{\frac{5}{2}x^4+8}{(4p)^4}-\frac{\frac{15}{2}x^6+152x^2}{(4p)^6}+\ldots\right)$$

19.24.9

$$\chi\sim\tfrac{1}{4}\pi+px\left(1+\frac{\frac{2}{3}x^2}{(4p)^2}-\frac{\frac{2}{5}x^4+16}{(4p)^4}+\frac{\frac{4}{7}x^6+\frac{256}{3}x^2}{(4p)^6}-\ldots\right)$$

19.24.10

$$G\sim\sqrt{p}\left(1+\frac{x^2}{(4p)^2}-\frac{\frac{3}{2}x^4+8}{(4p)^4}+\frac{\frac{7}{2}x^6+168x^2}{(4p)^6}-\ldots\right)$$

19.24.11

$$\psi\sim-\tfrac{1}{4}\pi+px\left(1+\frac{\frac{2}{3}x^2}{(4p)^2}-\frac{\frac{2}{5}x^4-16}{(4p)^4}+\frac{\frac{4}{7}x^6-\frac{320}{3}x^2}{(4p)^6}-\ldots\right)$$

Again, when $a<0$, $x^2-4a\gg0$, with $X=\sqrt{x^2+4|a|}$, then

19.24.12 $$F\sim\sqrt{2}e^{v_r} \qquad \chi=\tfrac{1}{4}\pi+\theta+v_i$$

where θ, v_r and v_i are given by **19.23.4** and **19.23.9.**

Another form also when $a>0$, $x^2-4a\rightarrow\infty$ is

19.24.13

$$F\sim\sqrt{\frac{2}{X}}\left(1-\frac{3}{4X^4}-\frac{5a}{X^6}+\frac{621}{32X^8}+\frac{1371a}{4X^{10}}-\ldots\right)$$

19.24.14

$$G\sim\sqrt{\frac{X}{2}}\left(1+\frac{5}{4X^4}+\frac{7a}{X^6}-\frac{835}{32X^8}-\frac{1729a}{4X^{10}}+\ldots\right)$$

while ψ and χ are connected by

19.24.15

$$\psi-\chi\sim-\frac{1}{2}\pi+\frac{x}{X^3}\left(1-\frac{47}{6X^4}-\frac{214a}{3X^6}+\frac{14483}{40X^8}+\ldots\right)$$

19.25. Connections With Other Functions

Connection With Confluent Hypergeometric and Bessel Functions

19.25.1

$$W(a, \pm x)=2^{-\frac{3}{4}}\left\{\sqrt{\frac{G_1}{G_3}}H(-\tfrac{3}{4}, \tfrac{1}{2}a, \tfrac{1}{4}x^2)\pm\sqrt{\frac{2G_3}{G_1}}xH(-\tfrac{1}{4}, \tfrac{1}{2}a, \tfrac{1}{4}x^2)\right\}$$

where

19.25.2

$$H(m, n, x)=e^{-ix}{}_1F_1(m+1-in; 2m+2; 2ix)$$

19.25.3 $$=e^{-ix}M(m+1-in, 2m+2, 2ix)$$

19.25.4

$$W(0, \pm x)=2^{-\frac{5}{4}}\sqrt{\pi x}\{J_{-\frac{1}{4}}(\tfrac{1}{4}x^2)\pm J_{\frac{1}{4}}(\tfrac{1}{4}x^2)\} \qquad (x\geq0)$$

19.25.5

$$\frac{d}{dx}W(0,\pm x)=-2^{-\frac{3}{4}}x\sqrt{\pi x}\{J_{\frac{3}{4}}(\tfrac{1}{4}x^2)\pm J_{-\frac{3}{4}}(\tfrac{1}{4}x^2)\}\qquad (x\geq 0)$$

19.26. Zeros

Zeros of solutions $U(a,x)$, $V(a,x)$ of **19.1.2** occur only for $|x|<2\sqrt{-a}$ when a is negative. A single exceptional zero is possible, for any a, in the general solution; neither $U(a,x)$ nor $V(a,x)$ has such a zero for $x>0$.

Approximations may be obtained by reverting the series for ψ (or χ for zeros of derivatives) in **19.11**, giving ψ (or χ) values that are multiples of $\frac{1}{2}\pi$, odd multiples for $U(a,x)$, even multiples for $V(a,x)$. Writing

$$\alpha=(\tfrac{1}{2}r-\tfrac{1}{2}a-\tfrac{1}{4})\pi$$

as an approximation to a zero of the function, or

$$\beta=(\tfrac{1}{2}r-\tfrac{1}{2}a+\tfrac{1}{4})\pi$$

as an approximation to a zero of the derivative, we obtain for the corresponding zero c or c', with $-a=p^2$ the expressions

$$\textbf{19.26.1}\quad c\approx\frac{\alpha}{p}+\frac{2\alpha^3-3\alpha}{48p^5}+\frac{52\alpha^5-240\alpha^3+315\alpha}{7680p^9}+\dots$$

$$\textbf{19.26.2}\quad c'\approx\frac{\beta}{p}+\frac{2\beta^3+3\beta}{48p^5}+\frac{52\beta^5+280\beta^3-285\beta}{7680p^9}+\dots$$

These expansions, however, are of little value in the neighborhood of the turning point $x=2\sqrt{-a}$. Here first approximations may be obtained by use of the formulas of **19.7**. If a_n (negative) is a zero of $\mathrm{Ai}(t)$, the corresponding zero c of $U(a,x)$ is obtained approximately by solving

19.26.3

$$\vartheta_3=\tfrac{1}{4}\{\arccos\,\xi-\xi\sqrt{1-\xi^2}\}=\frac{(-a_n)^{\frac{3}{2}}}{6|a|}$$

$$c=2\sqrt{|a|}\xi\qquad (a<<0)$$

This may be done by inverse use of **Table 19.3.** For a zero of $V(a,x)$, a_n must be replaced by b_n, a zero of $\mathrm{Bi}(t)$. For further developments see [19.5].

Zeros of solutions $W(a,x)$, $W(a,-x)$ of **19.1.3** occur for $|x|>2\sqrt{a}$ when a is positive; the general solution may, however, have a single zero between $-2\sqrt{a}$ and $+2\sqrt{a}$. If a is negative, zeros are unrestricted in range.

Approximations may be obtained by reverting the series for ψ (or χ) in **19.24.** With $-a=p^2$, $\alpha=(\frac{1}{2}r-\frac{1}{4})\pi$, $\beta=(\frac{1}{2}r+\frac{1}{4})\pi$, $r\geq 0$ being an odd integer for $W(a,x)$ or its derivative, or an even integer for $W(a,-x)$ or its derivative, the zeros $\pm c$, $\pm c'$ have expansions

$$\textbf{19.26.4}\quad c\approx\frac{\alpha}{p}-\frac{2\alpha^3-3\alpha}{48p^5}+\frac{52\alpha^5-240\alpha^3+315\alpha}{7680p^9}+\dots$$

$$\textbf{19.26.5}\quad c'\approx\frac{\beta}{p}-\frac{2\beta^3+3\beta}{48p^5}+\frac{52\beta^5+280\beta^3-285\beta}{7680p^9}+\dots$$

When x is large and a moderate, we may solve inversely the series **19.24.4** or **19.24.6** with $\alpha=\frac{1}{2}(r\pi-\frac{1}{2}\pi-\phi_2)$, $\beta=\frac{1}{2}(r\pi+\frac{1}{2}\pi-\phi_2)$, r odd or even as above; the presence of the logarithm makes it inconvenient to revert formally.

The expansions **19.26.4** and **19.26.5** fail when x is in the neighborhood of $2\sqrt{|a|}$. When a is positive, a zero c of $W(a,-x)$ is obtained approximately by solving

19.26.6

$$\vartheta_2=\tfrac{1}{4}\{\xi\sqrt{\xi^2-1}-\mathrm{arccosh}\,\xi\}=\frac{(-a_n)^{\frac{3}{2}}}{6a}$$

$$c=2\sqrt{a}\xi\qquad (a>>0)$$

with the aid of **Table 19.3.** For a zero of $W(a,x)$ we replace a_n by b_n. When a is negative we solve, again with the aid of **Table 19.3,**

19.26.7

$$\vartheta_1=\tfrac{1}{4}\{\xi\sqrt{\xi^2+1}+\mathrm{arcsinh}\,\xi\}=\frac{(n-\frac{1}{4})\pi}{4|a|}$$

$$c=2\sqrt{|a|}\xi\qquad (-a>>0)$$

where $n=1, 2, 3, \dots$ for an approximate zero of $W(a,-x)$, and $n=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$ for an approximate zero of $W(a,x)$. Further developments are given in [19.5].

Any of the approximations to zeros obtained above may readily be improved as follows:

Let c be a zero of y, and c' a zero of y', where y is a solution of

$$\textbf{19.26.8}\qquad y''-Iy=0$$

Here $I=a\pm\frac{1}{4}x^2$, $I'=\pm\frac{1}{2}x$, $I''=\pm\frac{1}{2}$; the method is general and the following formulae may be used whenever $I'''=0$. Then if γ, γ' are approximations to the zeros c, c' and

$$\textbf{19.26.9}\quad u=y(\gamma)/y'(\gamma)\qquad v=y'(\gamma')/I^2y(\gamma')$$

with $I\equiv I(\gamma)$ or $I\equiv I(\gamma')$ respectively, then

19.26.10

$$c \sim \gamma - u - \tfrac{1}{3} I u^3 + \tfrac{1}{12} I' u^4 - (\tfrac{1}{60} I'' + \tfrac{1}{5} I^2) u^5 + \tfrac{11}{90} I I' u^6 + \dots$$

19.26.11

$$y'(c) \sim y'(\gamma) \{1 - \tfrac{1}{2} I u^2 + \tfrac{1}{6} I' u^3 - (\tfrac{1}{24} I'' + \tfrac{1}{8} I^2) u^4 + \tfrac{7}{60} I I' u^5 + \dots\}$$

19.26.12

$$c' \sim \gamma' - I v - \tfrac{1}{2} I I' v^2 + (\tfrac{1}{6} I^2 I'' - \tfrac{1}{2} I I'^2 - \tfrac{1}{3} I^4) v^3 + (\tfrac{5}{12} I^2 I' I'' - \tfrac{5}{8} I I'^3 - \tfrac{5}{12} I^4 I') v^4 + \dots$$

19.26.13

$$y(c') \sim y(\gamma') \{1 - \tfrac{1}{2} I^3 v^2 - \tfrac{1}{6} I^3 I' v^3 - (\tfrac{1}{8} I^3 I'^2 - \tfrac{1}{24} I^4 I'' + \tfrac{1}{8} I^6) v^4 + \dots\}$$

The process can be repeated, if necessary, using as many terms at any stage as seems convenient.

Note the relations, holding at zeros,

19.26.14 $$U'(a, c) = -\sqrt{2/\pi}/V(a, c)$$

19.26.15 $$V'(a, c') = \sqrt{2/\pi}/U(a, c')$$

19.26.16 $$W'(a, c) = -1/W(a, -c)$$

19.26.17

$$W(a, c') = 1/\left\{\frac{d}{dx} W(a, -x)\right\}_{x=c'} = -1/W'(a, -c')$$

19.27. Bessel Functions of Order $\pm\frac{1}{4}$, $\pm\frac{3}{4}$ as Parabolic Cylinder Functions

Most applications of these functions refer to cases where parabolic cylinder functions would be more appropriate. We have

19.27.1 $$J_{\pm\frac{1}{4}}(\tfrac{1}{4}x^2) = \frac{2^{\frac{1}{4}}}{\sqrt{\pi x}} \{W(0, -x) \mp W(0, x)\}$$

19.27.2 $$J_{\pm\frac{3}{4}}(\tfrac{1}{4}x^2) = \frac{-2^{\frac{3}{4}}}{x\sqrt{\pi x}} \{W(0, x) \pm W(0, -x)\}$$

Functions of other orders may be obtained by use of the recurrence relation **10.1.22,** which here becomes

19.27.3 $$\tfrac{1}{4}x^2 J_{\nu+1}(\tfrac{1}{4}x^2) - 2\nu J_\nu(\tfrac{1}{4}x^2) + \tfrac{1}{4}x^2 J_{\nu-1}(\tfrac{1}{4}x^2) = 0$$

Again

19.27.4 $$I_{-\frac{1}{4}}(\tfrac{1}{4}x^2) + I_{\frac{1}{4}}(\tfrac{1}{4}x^2) = \frac{2}{\sqrt{x}} V(0, x)$$

19.27.5

$$\frac{\sqrt{2}}{\pi} K_{\frac{1}{4}}(\tfrac{1}{4}x^2) = I_{-\frac{1}{4}}(\tfrac{1}{4}x^2) - I_{\frac{1}{4}}(\tfrac{1}{4}x^2) = \frac{2}{\sqrt{\pi x}} U(0, x)$$

19.27.6 $$I_{-\frac{3}{4}}(\tfrac{1}{4}x^2) + I_{\frac{3}{4}}(\tfrac{1}{4}x^2) = -\frac{4}{x\sqrt{x}} \frac{d}{dx} V(0, x)$$

19.27.7

$$\frac{\sqrt{2}}{\pi} K_{\frac{3}{4}}(\tfrac{1}{4}x^2) = I_{-\frac{3}{4}}(\tfrac{1}{4}x^2) - I_{\frac{3}{4}}(\tfrac{1}{4}x^2) = -\frac{4}{x\sqrt{\pi x}} \frac{d}{dx} U(0, x)$$

As before, Bessel functions of other orders may be obtained by use of the recurrence relation **10.2.23,** which here becomes

19.27.8 $$\tfrac{1}{4}x^2 I_{\nu+1}(\tfrac{1}{4}x^2) + 2\nu I_\nu(\tfrac{1}{4}x^2) - \tfrac{1}{4}x^2 I_{\nu-1}(\tfrac{1}{4}x^2) = 0$$

19.27.9 $$\tfrac{1}{4}x^2 K_{\nu+1}(\tfrac{1}{4}x^2) - 2\nu K_\nu(\tfrac{1}{4}x^2) - \tfrac{1}{4}x^2 K_{\nu-1}(\tfrac{1}{4}x^2) = 0$$

Numerical Methods

19.28. Use and Extension of the Tables

For $U(a, x)$, $V(a, x)$ and $W(a, x)$, interpolation x-wise may be carried out to 5-figure accuracy almost everywhere by using 5-point or 6-point Lagrangian interpolation. For $|a| \leq 1$, comparable accuracy a-wise may be obtained with 5- or 6-point interpolation.

For $|a| > 1$, $U(a, x)$ and $V(a, x)$ may be obtained by use of recurrence relations from two values, possibly obtained by interpolation, with $|a| \leq 1$; such a procedure is not available for $W(a, \pm x)$, $|a| > 1$.

In cases where straightforward use of the a-wise recurrence relation results in loss of accuracy by cancellation of leading digits, it may be worth while to remark that greater accuracy is usually attainable by use of the recurrence relation in the reverse direction, from arbitrary starting values (often 1 and 0) for two values of a somewhat beyond the last value desired. This is because the recurrence relation is a second order homogeneous linear difference equation, and has two independent solutions. Loss of accuracy by cancellation occurs when the solution desired is diminishing as a varies, while the companion solution is increasing. By reversing the direction of progress in a, the roles of the two solutions are interchanged, and the contribution of the desired solution now increases, while the unwanted solution diminishes to the point of negligibility. By starting sufficiently beyond the last value of a for which the function is desired, we can ensure that the unwanted solution is negligible but, because the starting values were arbitrary, we have an unknown multiple of the solution desired. The computation is then carried back until a value of a with $|a|\leq 1$ is reached, when the precise multiple that we have of the desired solution may be determined and hence removed throughout. Compare also **9.12, Example 1.**

Example 1. Evaluate $U(a, 5)$ for $a=5, 6, 7, \ldots$, using **19.6.4.**

$$(a+\tfrac{1}{2})U(a+1, x)+xU(a, x)-U(a-1, x)=0$$

a	Forward Recurrence	Backward Recurrence	Final Values
3	(−6) 5. 2847*	(12) 1. 59035	(−6) 5. 2847**
4	(−7) 9. 172*	(11) 2. 76028	(−7) 9. 1724
5	(−7) 1. 5527	(10) 4. 67131	(−7) 1. 55227
6	(−8) 2. 5609	(9) 7. 72041	(−8) 2. 5655
7	(−9) 4. 1885	(9) 1. 24785	(−9) 4. 1466
8	(−10) 6. 2220	(8) 1. 97488	(−10) 6. 5625
9	(−10)+1. 2676	(7) 3. 06369	(−10) 1. 01806
10	(−11)−0. 1221	(6) 4. 66352	(−11) 1. 5497
11	(−11)+1. 2654	(0) 697082	(−12) 2. 3164
12	(−12)−5. 6079	102444	(−13) 3. 404
13	(−12)+3. 2555	14789	(−14) 4. 91
14		2111	(−15) 7. 01
15		292	(−16) 9. 7
16		42	
17		5	
18		1⁺	
19		0⁺	

*From tables. ⁺Starting values.

**This value was used to obtain the constant multiplier $\frac{d}{k^*}=\frac{(-6)5.2847}{(12)1.59035}=(-18)3.32298$ for converting the previous column into this one.

The second column shows forward recurrence starting with values at $a=3, 4$ from **Table 19.1.** Backward recurrence starts with values 0 and 1 at $a=19$ and 18, containing a multiple $kU(a, 5)$ and a subsequently negligible multiple of the other solution $\Gamma(\frac{1}{2}-a)V(a, 5)$. Rounding errors convert $kU(a, x)$ into $k^*U(a, x)$ *without affecting the values in the last column.* The value of $1/k^*$ is identified from the known value of $U(3, 5)$, and used to obtain the final column by multiplying throughout by $1/k^*$. The improvement in $U(5, 5)$ is evident by comparison with **Table 19.1.**

Derivatives. These are not tabulated here. Since the functions $U(a, x)$, $V(a, x)$ and $W(a, x)$ satisfy differential equations, values of derivatives are often required.

For all these functions the equation is second order with first derivative absent, so that *second derivatives* may be readily obtained from function values by use of the differential equation.

First derivatives can be obtained for $U(a, x)$ and $V(a, x)$ by applying the appropriate recurrence relations **19.6.1–2.** If less accuracy is needed they can be found by use of mean central differences of $U(a, x)$, $V(a, x)$ and also of $W(a, x)$ with the formula

$$hu'=h\frac{du}{dx}=\mu\delta u-\tfrac{1}{6}\mu\delta^3u+\tfrac{1}{30}\mu\delta^5u-\ldots$$

using $h=.1$; this usually gives a 3- or 4-figure value of du/dx.

If greater accuracy is needed for $dW(a, x)/dx$ it may be obtained by evaluating d^2W/dx^2 with the help of the differential equation satisfied by W and integrating this second derivative numerically. This requires one accurate value of dW/dx to start off the integration; we describe two methods for obtaining this, both making use of the difference between two fairly widely separated values of W, for example, separated by 5 or 10 tabular intervals.

(i) Write f_r, f'_r, f''_r for $W(a, x_0+rh)$ and its first two derivatives, then f'_0 may be found from

$$hf'_0=\frac{1}{2n}(f_n-f_{-n})-\frac{h^2}{2n}\sum_1^{n-1}(n-r)(f''_r-f''_{-r})$$

$$-\frac{h^2}{2n}\{\tfrac{1}{12}-\tfrac{1}{240}\delta^2+\tfrac{31}{60480}\delta^4-\ldots\}(f''_n-f''_{-n})$$

$$-h^2\{\tfrac{1}{12}\mu\delta-\tfrac{11}{720}\mu\delta^3+\tfrac{191}{60480}\mu\delta^5-\ldots\}f''_0$$

(ii) Consider a solution y of the differential equation for $W(a, x)$, namely $y''=(-\frac{1}{4}x^2+a)y$. If we are given values y and y' at a particular $x=x_0$ and write $T_n=H^ny^{(n)}/n!$, $T_{-1}=T_{-2}=0$, then we may compute T_2, T_3, T_4, . . . in succession by use of the recurrence relation obtained from the differential equation,

$$T_{n+2}=\frac{H^2}{(n+1)(n+2)}[(-\tfrac{1}{4}x_0^2+a)T_n-\tfrac{1}{2}Hx_0T_{n-1}-\tfrac{1}{4}H^2T_{n-2}]$$

These are computed, to a fixed number of decimals until they become negligible, thus giving

$$y(x_0\pm H)=T_0\pm T_1+T_2\pm T_3+\ldots$$

This may be applied, with $H=rh$, h being the tabular interval, and r a small integer, say $r=5$, to the solutions $y=y_1$, $y=y_2$ having

$$y_1(x_0)=W(a, x_0)\qquad y'_1(x_0)=W^{*\prime}(a, x_0)$$

$$y_2(x_0)=0\qquad y'_2(x_0)=1$$

in which $W^{*\prime}(a, x_0)$ is an approximation to $W'(a, x_0)$, not necessarily a good one; it may be obtained from differences, for example. We thus obtain $y_1(x_0 \pm H)$ and $y_2(x_0 \pm H)$.

Now suppose

$$W'(a, x_0) = W^{*\prime}(a, x_0) + \lambda$$

then, for all x

$$W(a, x) = y_1(x) + \lambda y_2(x)$$

and in particular

$$W(a, x_0 \pm H) = y_1(x_0 \pm H) + \lambda y_2(x_0 \pm H)$$

The values of $W(a, x_0 \pm H)$ may be read from the tables and two independent estimates of λ obtained, whence

$$W'(a, x_0) = W^{*\prime}(a, x_0) + \lambda$$

to a suitable accuracy.

Example 2. Evaluate $W'(-3, 1)$ using $r=5$. From **Table 19.2**

$W(-3, .5) = -.05857 \qquad W(-3, 1) = -.61113$

$W(-3, 1.5) = -.69502$

(i) Using the first method

x	$W(-3, x)$	$W''(-3, x)$	δ	δ^2	δ^3
0. 4	+0. 07298	−0. 22186			
0. 5	−. 05857	+. 17937		+131	
0. 6	−. 18832	. 58191			
0. 7	−. 31226	. 97503			
0. 8	−. 42646	1. 34761			
			34081		
0. 9	−. 52722	1. 68842			
			29775		−1095
1. 0	−. 61113	1. 98617			
			24374		−1032
1. 1	−. 67522	2. 22991			
			17941		
1. 2	−. 71706	2. 40932			
1. 3	−. 73488	2. 51513			
1. 4	−. 72761	2. 53936			
1. 5	−. 69502	2. 47601		−9129	
1. 6	−. 63774	2. 32137			

The fifth decimal in $W''(-3, x)$ is only a guard figure which is hardly needed. Only the differences needed have been computed.

Then

$$\tfrac{1}{10}W'(-3, 1)$$

$$=\tfrac{1}{10}(-.69502+.05857)-\frac{1}{1000}(10.38874)$$

$$-\frac{1}{1000}\left\{\tfrac{1}{12}(2.29664)-\frac{1}{240}(-.09260)\right\}$$

$$-\frac{1}{100}\left\{\frac{1}{24}(.54149)-\frac{11}{1440}(-.02127)\right\}$$

$$=-.0636450-.0103887-.0001918-.0002272$$

$$=-.0744527$$

Thus $W'(-3, 1) = -.74453$. This might have an error up to about $1\frac{1}{2}$ units in the last figure but is, in fact, correct to 5 decimals.

(ii) Using the second method, with

$y_1(1) = W(-3, 1) = -.61113 \qquad$ to 5 decimals

$y_1'(1) = -.745 \qquad$ to about 3 decimals

the following values result, with $H = .5$,

	y_1	y_2	$W(-3, x) = y_1 + \lambda y_2$
T_0	− .61113	.0000	At $x=1.5$
T_1	− .37250	+.5000	$x-.695223+.4323\lambda$
			$=-.69502$
T_2	+ .24827 2	.0000	$\lambda=.000203/.4323$
T_3	+ 5680 9	− 677	$=.000470$
T_4	− 1407 4	− 26	So $W'(-3, 1)$
			$=-.745+\lambda$
			$=-.744530$
T_5	− 279 3	+ 24	At $x=.5$
T_6	+ 13 4	+ 2	$-.058363-.4371\lambda$
			$=-.05857$
T_7	+ 5 4		$\lambda=.000207/.4371$
T_8	+ 5		$=.000474$
$y(1.5)$	−.695223	+.4323	So $W'(-3, 1)$
$y(.5)$	−.058363	−.4371	$=-.745+\lambda$
			$=-.744526$

Thus $W'(-3, 1) = -.74453$ which is correct to 5 decimals.

Example 3. Evaluate the positive zero of $U(-3, x)$.

We use **19.7.3** to obtain a first approximation, see **19.26.3.** The appropriate zero of $\mathrm{Ai}(t)$ is at

$$t = (4|a|)^{\frac{1}{3}}\tau = -2.338$$

whence

$$\tau = -(2.338) \times (12)^{-\frac{1}{3}} = -.4461$$

Hence, from **Table 19.3,** $\xi = .3990$ and the approximate zero is $x = 2\sqrt{|a|}\xi = 1.382$.

We improve this by using **19.26.10,** but take, for convenience, $x = 1.4$ as an approximation, so that the value of U can be read directly from the tables. U' can be obtained as in the section following **Example 1.**

We find

$U(-3, 1.4) = .02627 \qquad U'(-3, 1.4) = 2.0637$

Then **19.26.9** gives

$u = U/U' = .012730 \qquad I = -2.51$

$I' = .7 \qquad I'' = .5$

and

$$c=1.4-.012730+.000002=1.38727$$

which is correct to 5 decimals, while **19.26.11** gives

$$y'(c)=2.0637(1+.000203)=2.0641$$

compared with the correct value 2.06416.

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20. Mathieu Functions

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Contents

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20. Mathieu Functions

Mathematical Properties

20.1. Mathieu's Equation

Canonical Form of the Differential Equation

20.1.1 $$\frac{d^2y}{dv^2}+(a-2q\cos 2v)y=0$$

Mathieu's Modified Differential Equation

20.1.2 $$\frac{d^2f}{du^2}-(a-2q\cosh 2u)f=0 \qquad (v=iu, y=f)$$

Relation Between Mathieu's Equation and the Wave Equation for the Elliptic Cylinder

The wave equation in Cartesian coordinates is

20.1.3 $$\frac{\partial^2W}{\partial x^2}+\frac{\partial^2W}{\partial y^2}+\frac{\partial^2W}{\partial z^2}+k^2W=0$$

A solution W is obtainable by separation of variables in elliptical coordinates. Thus, let

$$x=\rho\cosh u\cos v;\ y=\rho\sinh u\sin v;\ z=z;$$

ρ a positive constant; **20.1.3** becomes

20.1.4

$$* \quad \frac{\partial^2W}{\partial z^2}+\frac{2}{\rho^2(\cosh 2u-\cos 2v)}\left(\frac{\partial^2W}{\partial u^2}+\frac{\partial^2W}{\partial v^2}\right)+k^2W=0$$

Assuming a solution of the form

$$W=\varphi(z)f(u)g(v)$$

and substituting the above into **20.1.4** one obtains, after dividing through by W,

$$\frac{1}{\varphi}\frac{d^2\varphi}{dz^2}+G=0$$

where

$$* \quad G=\frac{2}{\rho^2(\cosh 2u-\cos 2v)}\left\{\frac{d^2f}{du^2}\frac{1}{f}+\frac{d^2g}{dv^2}\frac{1}{g}\right\}+k^2$$

Since z, u, v are independent variables, it follows that

20.1.5 $$\frac{d^2\varphi}{dz^2}+c\varphi=0$$

where c is a constant.

Again, from the fact that $G=c$ and that u, v are independent variables, one sets

20.1.6

$$* \quad a=\frac{d^2f}{du^2}\frac{1}{f}+\frac{(k^2-c)}{2}\rho^2\cosh 2u$$

$$a=-\frac{d^2g}{dv^2}\frac{1}{g}+\frac{(k^2-c)}{2}\rho^2\cos 2v \quad *$$

where a is a constant. The above are equivalent to **20.1.1** and **20.1.2.** The constants c and a are often referred to as *separation constants*, due to the role they play in **20.1.5** and **20.1.6.**

For some physically important solutions, the function g must be periodic, of period π or 2π. It can be shown that there exists a countably infinite set of *characteristic values* $a_r(q)$ which yield even periodic solutions of **20.1.1;** there is another countably infinite sequence of *characteristic values* $b_r(q)$ which yield odd periodic solutions of **20.1.1.**

It is known that there exist periodic solutions of period $k\pi$, where k is any positive integer. In what follows, however, the term *characteristic value* will be reserved for a value associated with solutions of period π or 2π only. These characteristic values are of basic importance to the general theory of the differential equation for arbitrary parameters a and q.

An Algebraic Form of Mathieu's Equation

20.1.7

$$(1-t^2)\frac{d^2y}{dt^2}-t\frac{dy}{dt}+(a+2q-4qt^2)y=0 \qquad (\cos v=t)$$

Relation to Spheroidal Wave Equation

20.1.8 $$(1-t^2)\frac{d^2y}{dt^2}-2(b+1)t\frac{dy}{dt}+(c-4qt^2)y=0 \quad *$$

Thus, Mathieu's equation is a special case of **20.1.8,** with $b=-\frac{1}{2}$, $c=a+2q$.

20.2. Determination of Characteristic Values

A solution of **20.1.1** with v replaced by z, having period π or 2π is of the form

20.2.1 $$y=\sum_{m=0}^{\infty}(A_m\cos mz+B_m\sin mz)$$

where B_0 can be taken as zero. If the above is substituted into **20.1.1** one obtains

20.2.2

$$\sum_{m=-2}^{\infty}[(a-m^2)A_m-q(A_{m-2}+A_{m+2})]\cos mz$$
$$+\sum_{m=-1}^{\infty}[(a-m^2)B_m-q(B_{m-2}+B_{m+2})]\sin mz=0$$
$$A_{-m}, B_{-m}=0 \qquad m>0$$

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*See page II.

Equation **20.2.2** can be reduced to one of four simpler types, given in **20.2.3** and **20.2.4** below

20.2.3 $$y_0=\sum_{m=0}^{\infty} A_{2m+p}\cos(2m+p)z, \qquad p=0 \text{ or } 1$$

20.2.4 $$y_1=\sum_{m=0}^{\infty} B_{2m+p}\sin(2m+p)z, \qquad p=0 \text{ or } 1$$

If $p=0$, the solution is of period π; if $p=1$, the solution is of period 2π.

Recurrence Relations Among the Coefficients

Even solutions of period π:

20.2.5 $$aA_0-qA_2=0$$

20.2.6 $$(a-4)A_2-q(2A_0+A_4)=0$$

20.2.7 $$(a-m^2)A_m-q(A_{m-2}+A_{m+2})=0 \qquad (m\geq 3)$$

Even solutions of period 2π:

20.2.8 $$(a-1)A_1-q(A_1+A_3)=0,$$

along with **20.2.7** for $m\geq 3$.

Odd solutions of period π:

20.2.9 $$(a-4)B_2-qB_4=0$$

* **20.2.10** $$(a-m^2)B_m-q(B_{m-2}+B_{m+2})=0 \qquad (m\geq 3)$$

Odd solutions of period 2π:

20.2.11 $$(a-1)B_1+q(B_1-B_3)=0,$$

along with **20.2.10** for $m\geq 3$.

Let

20.2.12 $$Ge_m=A_m/A_{m-2},\ Go_m=B_m/B_{m-2};$$

$G_m=Ge_m$ or Go_m when the same operations apply to both, and no ambiguity is likely to arise. Further let

20.2.13 $$V_m=(a-m^2)/q.$$

Equations **20.2.5–20.2.7** are equivalent to

20.2.14 $$Ge_2=V_0;\ Ge_4=V_2-\frac{2}{Ge_2}$$

20.2.15 $$G_m=1/(V_m-G_{m+2}) \qquad (m\geq 3),$$

for even solutions of period π.

Similarly

20.2.16 $V_1-1=Ge_3$; for even solutions of period 2π, along with **20.2.15**

20.2.17 $V_1+1=Go_3$, for odd solutions of period 2π, along with **20.2.15**

20.2.18 $V_2=Go_4$, for odd solutions of period π, along with **20.2.15**

These three-term recurrence relations among the coefficients indicate that every G_m can be developed into two types of continued fractions. Thus **20.2.15** is equivalent to

20.2.19

$$G_m=\frac{1}{V_m-G_{m+2}}=\frac{1}{V_m-}\,\frac{1}{V_{m+2}-}\,\frac{1}{V_{m+4}-}\cdots \quad (m\geq 3)$$

20.2.20

$$G_{m+2}=V_m-1/G_m$$
$$=V_m-\frac{1}{V_{m-2}-}\,\frac{1}{V_{m-4}-}\cdots\frac{\varphi_0}{V_{0+d}+\varphi_1} \qquad (m\geq 3)$$

where

$\varphi_1=d=0;\ \varphi_0=2$, if $G_{m+2}=A_{2s}/A_{2s-2}$
$\varphi_1=d=\varphi_0=0$, if $G_{m+2}=B_{2s}/B_{2s-2}$
$\varphi_1=-1;\ \varphi_0=d=1$, if $G_{m+2}=A_{2s+1}/A_{2s-1}$
$\varphi_1=d=\phi_0=1$, if $G_{m+2}=B_{2s+1}/B_{2s-1}$

The four choices of the parameters φ_1, φ_0, d correspond to the four types of solutions **20.2.3–20.2.4.** Hereafter, it will be convenient to separate the characteristic values a into two major subsets:

$a=a_r$, associated with even periodic solutions

$a=b_r$, associated with odd periodic solutions

If **20.2.19** is suitably combined with **20.2.13–20.2.18** there result four types of continued fractions, the roots of which yield the required characteristic values

20.2.21 $$V_0-\frac{2}{V_2-}\,\frac{1}{V_4-}\,\frac{1}{V_6-}\cdots=0 \qquad \text{Roots: } a_{2r}$$

20.2.22

$$V_1-1-\frac{1}{V_3-}\,\frac{1}{V_5-}\,\frac{1}{V_7-}\cdots=0 \qquad \text{Roots: } a_{2r+1}$$

20.2.23 $$V_2-\frac{1}{V_4-}\,\frac{1}{V_6-}\,\frac{1}{V_8-}\cdots=0 \qquad \text{Roots: } b_{2r}$$

20.2.24

$$V_1+1-\frac{1}{V_3-}\,\frac{1}{V_5-}\,\frac{1}{V_7-}\cdots=0 \qquad \text{Roots: } b_{2r+1}$$

If a is a root of **20.2.21–20.2.24,** then the corresponding solution exists and is an entire function of z, for general complex values of q.

If q is real, then the Sturmian theory of second order linear differential equations yields the

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*See page II.

following:

(a) For a fixed real q, characteristic values a_r and b_r are real and distinct, if $q \neq 0$; $a_0 < b_1 < a_1 < b_2 < a_2 < \ldots$, $q > 0$ and $a_r(q)$, $b_r(q)$ approach r^2 as q approaches zero.

(b) A solution of **20.1.1** associated with a_r or b_r has r zeros in the interval $0 \leq z < \pi$, (q real).

(c) The form of **20.2.21** and **20.2.23** shows that if a_{2r} is a root of **20.2.21** and q is different from zero, then a_{2r} cannot be a root of **20.2.23**; similarly, no root of **20.2.22** can be a root of **20.2.24** if $q \neq 0$. It may be shown from other considerations that for a given point (a, q) there can be at most one periodic solution of period π or 2π if $q \neq 0$. This no longer holds for solutions of period $s\pi$, $s \geq 3$; for these all solutions are periodic, if one is.

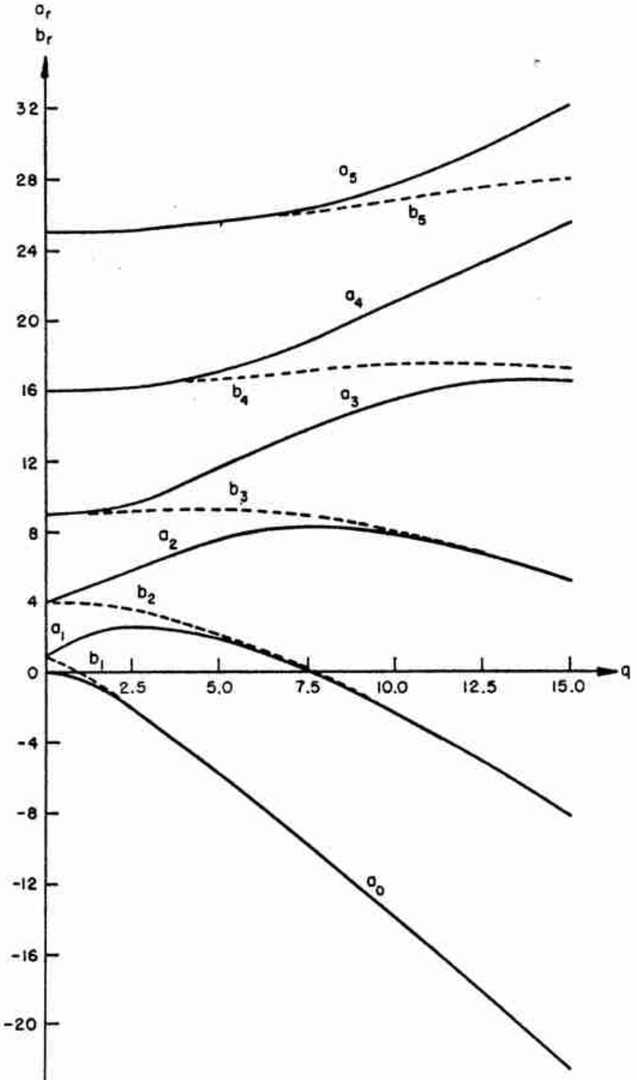

FIGURE 20.1. *Characteristic Values* a_r, b_r $\qquad r=0,1(1)5$

Power Series for Characteristic Values

20.2.25

$$a_0(q) = -\frac{q^2}{2} + \frac{7q^4}{128} - \frac{29q^6}{2304} + \frac{68687q^8}{18874368} + \ldots$$

$$\left.\begin{matrix} a_1(-q) \\ b_1(q) \end{matrix}\right. = 1 - q - \frac{q^2}{8} + \frac{q^3}{64} - \frac{q^4}{1536} - \frac{11q^5}{36864} + \frac{49q^6}{589824} - \frac{55q^7}{9437184} - \frac{83q^8}{35389440} + \ldots$$

$$b_2(q) = 4 - \frac{q^2}{12} + \frac{5q^4}{13824} - \frac{289q^6}{79626240} + \frac{21391q^8}{458647142400} + \ldots$$

$$a_2(q) = 4 + \frac{5q^2}{12} - \frac{763q^4}{13824} + \frac{1002401q^6}{79626240} - \frac{1669068401q^8}{458647142400} + \ldots$$

$$\left.\begin{matrix} a_3(-q) \\ b_3(q) \end{matrix}\right. = 9 + \frac{q^2}{16} - \frac{q^3}{64} + \frac{13q^4}{20480} + \frac{5q^5}{16384} - \frac{1961q^6}{23592960} + \frac{609q^7}{104857600} + \ldots$$

$$b_4(q) = 16 + \frac{q^2}{30} - \frac{317q^4}{864000} + \frac{10049q^6}{2721600000} + \ldots$$

$$a_4(q) = 16 + \frac{q^2}{30} + \frac{433q^4}{864000} - \frac{5701q^6}{2721600000} + \ldots$$

$$\left.\begin{matrix} a_5(-q) \\ b_5(q) \end{matrix}\right. = 25 + \frac{q^2}{48} + \frac{11q^4}{774144} - \frac{q^5}{147456} + \frac{37q^6}{891813888} + \ldots$$

$$b_6(q) = 36 + \frac{q^2}{70} + \frac{187q^4}{43904000} - \frac{5861633q^6}{92935987200000} + \ldots$$

$$a_6(q) = 36 + \frac{q^2}{70} + \frac{187q^4}{43904000} + \frac{6743617q^6}{92935987200000} + \ldots$$

For $r \geq 7$, and $|q|$ not too large, a_r is approximately equal to b_r, and the following approximation may be used

20.2.26

$$\left.\begin{matrix} a_r \\ b_r \end{matrix}\right\} = r^2 + \frac{q^2}{2(r^2-1)} + \frac{(5r^2+7)q^4}{32(r^2-1)^3(r^2-4)} + \frac{(9r^4+58r^2+29)q^6}{64(r^2-1)^5(r^2-4)(r^2-9)} + \ldots$$

The above expansion is not limited to integral values of r, and it is a very good approximation for r of the form $n+\frac{1}{2}$ where n is an integer. In case of integral values of $r=n$, the series holds only up to terms not involving r^2-n^2 in the denominator. Subsequent terms must be derived specially (as shown by Mathieu). Mulholland and Goldstein [20.38] have computed characteristic values for purely imaginary q and found that a_0 and a_2 have a common real value for $|q|$ in the neighborhood of 1.468; Bouwkamp [20.5] has computed this number as $q_0=\pm i$ 1.46876852 to 8 decimals. For values of $-iq>-iq_0$, a_0 and a_2 are conjugate complex numbers. From equation **20.2.25** it follows that the radius of convergence for the series defining a_0 is no greater than $|q_0|$. It is shown in [20.36], section **2.25** that the radius of convergence for $a_{2n}(q)$, $n\geq 2$ is greater than 3. Furthermore

$$a_r-b_r=O(q^r/r^{r-1}),\ r\rightarrow\infty.$$

Power Series in q for the Periodic Functions (for sufficiently small $|q|$)

20.2.27

$$ce_0(z,q)=2^{-\frac{1}{2}}\left[1-\frac{q}{2}\cos 2z+q^2\left(\frac{\cos 4z}{32}-\frac{1}{16}\right)\right.$$
$$\left.-q^3\left(\frac{\cos 6z}{1152}-\frac{11\cos 2z}{128}\right)+\ldots\right]$$

$$ce_1(z,q)=\cos z-\frac{q}{8}\cos 3z$$
$$+q^2\left[\frac{\cos 5z}{192}-\frac{\cos 3z}{64}-\frac{\cos z}{128}\right]$$
$$-q^3\left[\frac{\cos 7z}{9216}-\frac{\cos 5z}{1152}-\frac{\cos 3z}{3072}+\frac{\cos z}{512}\right]+\ldots$$

$$se_1(z,q)=\sin z-\frac{q}{8}\sin 3z$$
$$+q^2\left[\frac{\sin 5z}{192}+\frac{\sin 3z}{64}-\frac{\sin z}{128}\right]$$
$$-q^3\left[\frac{\sin 7z}{9216}+\frac{\sin 5z}{1152}-\frac{\sin 3z}{3072}-\frac{\sin z}{512}\right]+\ldots$$

$$ce_2(z,q)=\cos 2z-q\left(\frac{\cos 4z}{12}-\frac{1}{4}\right)+q^2\left(\frac{\cos 6z}{384}-\frac{19\cos 2z}{288}\right)+\ldots$$

$$se_2(z,q)=\sin 2z-q\frac{\sin 4z}{12}+q^2\left(\frac{\sin 6z}{384}-\frac{\sin 2z}{288}\right)+\ldots$$

20.2.28

$$\begin{matrix}ce_r(z,q)\\ se_r(z,q)\end{matrix}=\cos(rz-p(\pi/2))-q\left\{\frac{\cos\left[(r+2)z-p\frac{\pi}{2}\right]}{4(r+1)}\right.$$
$$\left.-\frac{\cos[(r-2)z-p(\pi/2)]}{4(r-1)}\right\}$$
$$+q^2\left\{\frac{\cos[(r+4)z-p(\pi/2)]}{32(r+1)(r+2)}+\frac{\cos[(r-4)z-p(\pi/2)]}{32(r-1)(r-2)}\right.$$
$$\left.-\frac{\cos[rz-p(\pi/2)]}{32}\left[\frac{2(r^2+1)}{(r^2-1)^2}\right]\right\}+\ldots$$

with $p=0$ for $ce_r(z,q)$, $p=1$ for $se_r(z,q)$, $r\geq 3$.

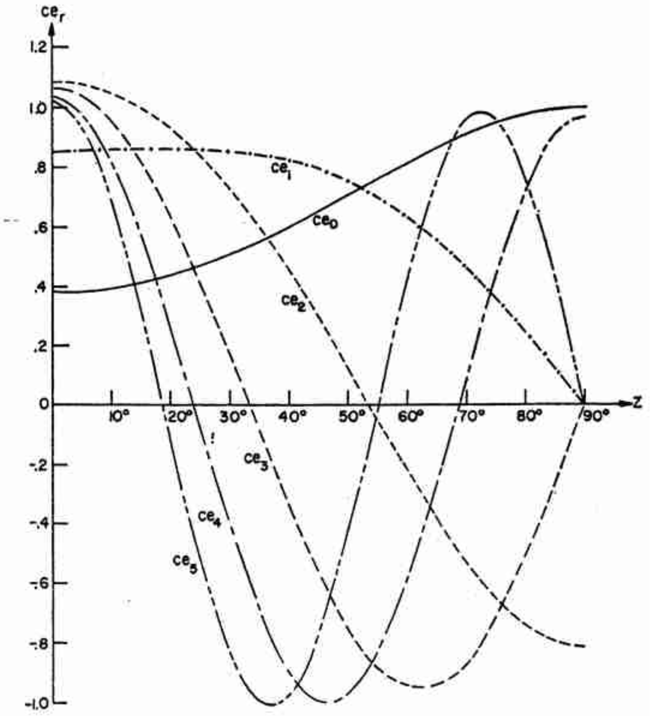

FIGURE 20.2. *Even Periodic Mathieu Functions, Orders 0–5* $q=1$.

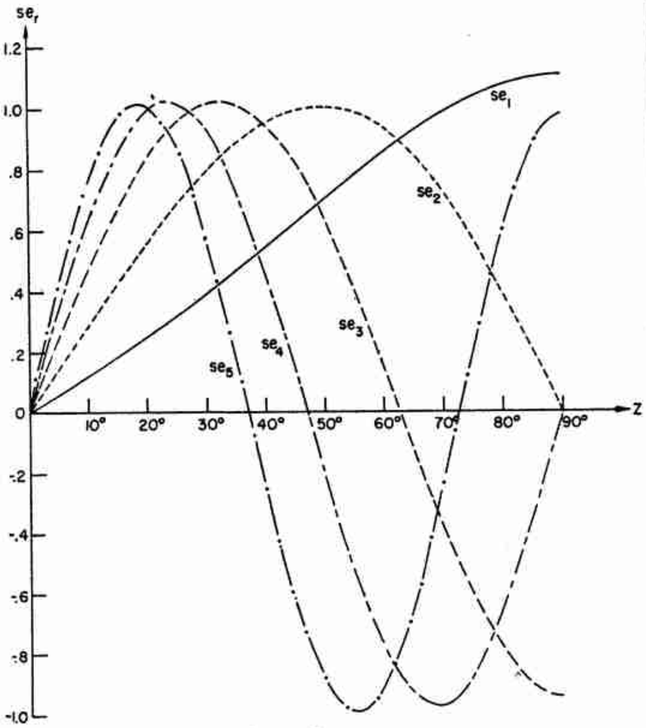

FIGURE 20.3. *Odd Periodic Mathieu Functions, Orders 1–5* $q=1$.

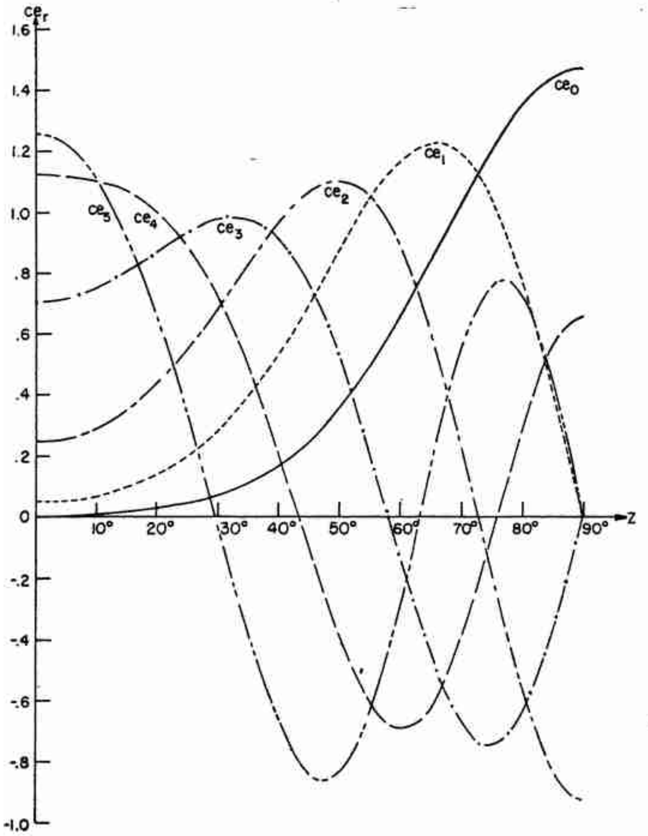

FIGURE 20.4. *Even Periodic Mathieu Functions, Orders 0–5* $q=10$.

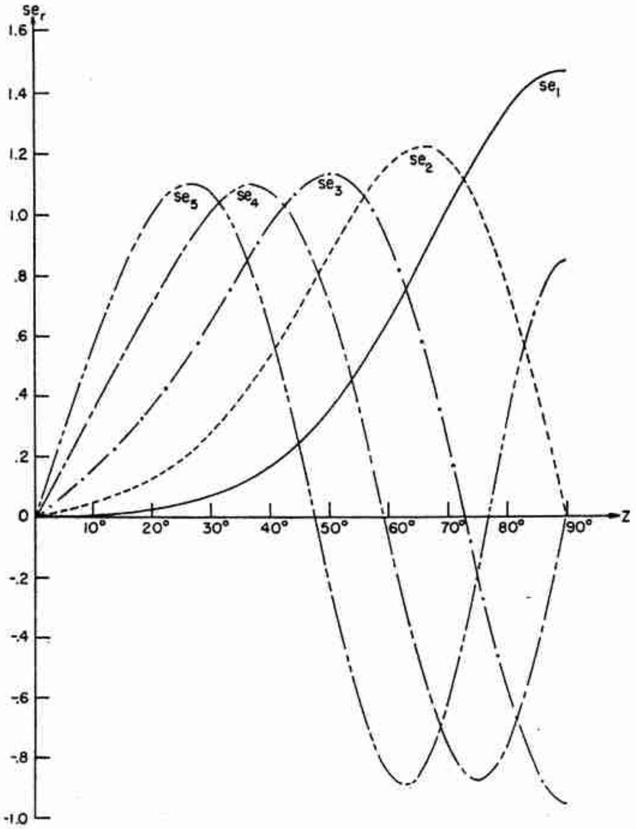

FIGURE 20.5. *Odd Periodic Mathieu Functions, Orders 1–5* $q=10$.

For coefficients associated with above functions

20.2.29

$$A_0^0(0)=2^{-\frac{1}{2}};\ A_r^r(0)=B_r^r(0)=1,\ r>0$$

$$A_{2s}^0=[(-1)^s q^s/s!\,s!\,2^{2s-1}]\,A_0^0+\ .\ .\ .,\ s>0$$

$$\begin{matrix}A_{r+2s}^r\\B_{r+2s}^r\end{matrix}=[(-1)^s r!\ q^s/4^s(r+s)!\ s!]\ C_r^r+\ .\ .\ .$$

$$rs>0,\ C_r^r=A_r^r \text{ or } B_r^r$$

$$A_{r-2s}^r \text{ or } B_{r-2s}^r=\frac{(r-s-1)!}{s!(r-1)!}\frac{q^s}{4^s}C_r^r+\ .\ .\ .$$

Asymptotic Expansion for Characteristic Values, $q\gg 1$

Let $w=2r+1$, $q=w^4\varphi$, φ real. Then

20.2.30

$$a_r\sim b_{r+1}\sim -2q+2w\sqrt{q}-\frac{w^2+1}{8}-\frac{\left(w+\frac{3}{w}\right)}{2^7\sqrt{\varphi}}$$

$$-\frac{d_1}{2^{12}\varphi}-\frac{d_2}{2^{17}\varphi^{3/2}}-\frac{d_3}{2^{20}\varphi^2}-\frac{d_4}{2^{25}\varphi^{5/2}}-\ .\ .\ .$$

where

$$d_1=5+\frac{34}{w^2}+\frac{9}{w^4}$$

$$d_2=\frac{33}{w}+\frac{410}{w^3}+\frac{405}{w^5}$$

$$d_3=\frac{63}{w^2}+\frac{1260}{w^4}+\frac{2943}{w^6}+\frac{486}{w^8}$$

$$d_4=\frac{527}{w^3}+\frac{15617}{w^5}+\frac{69001}{w^7}+\frac{41607}{w^9}$$

20.2.31 $$b_{r+1}-a_r\sim 2^{4r+5}\sqrt{2/\pi}q^{\frac{1}{2}r+\frac{3}{4}}e^{-4\sqrt{q}}/r!, \qquad q\rightarrow\infty$$

(given in [20.36] without proof.)

20.3. Floquet's Theorem and Its Consequences

Since the coefficients of Mathieu's equation

20.3.1 $$y''+(a-2q\cos 2z)y=0$$

are periodic functions of z, it follows from the known theory relating to such equations that there exists a solution of the form

20.3.2 $$F_\nu(z)=e^{i\nu z}P(z),$$

where ν depends on a and q, and $P(z)$ is a periodic function, of the same period as that of the coefficients in **20.3.1,** namely π. (Floquet's theorem; see [20.16] or [20.22] for its more general form.) The constant ν is called the *characteristic exponent.* Similarly

20.3.3 $$F_\nu(-z)=e^{-i\nu z}P(-z)$$

satisfies **20.3.1** whenever **20.3.2** does. Both $F_\nu(z)$ and $F_\nu(-z)$ have the property

20.3.4

$$y(z+k\pi)=C^k y(z),\ y=F_\nu(z) \text{ or } F_\nu(-z),$$
$$C=e^{i\nu\pi} \text{ for } F_\nu(z),\ C=e^{-i\nu\pi} \text{ for } F_\nu(-z)$$

Solutions having the property **20.3.4** will hereafter be termed *Floquet* solutions. Whenever $F_\nu(z)$ and $F_\nu(-z)$ are linearly independent, the general solution of **20.3.1** can be put into the form

20.3.5 $$y=AF_\nu(z)+BF_\nu(-z)$$

If $AB\neq 0$, the above solution will *not be a Floquet solution.* It will be seen later, from the method for determining ν when a and q are given, that there is some ambiguity in the definition of ν; namely, ν can be replaced by $\nu+2k$, where k is an arbitrary integer. This is as it should be, since the addition of the factor exp $(2ikz)$ in **20.3.2** still leaves a periodic function of period π for the coefficient of exp $i\nu z$.

It turns out that when a belongs to the set of characteristic values a_r and b_r of **20.2,** then ν is zero or an integer. It is convenient to associate $\nu=r$ with $a_r(q)$, and $\nu=-r$ with $b_r(q)$; see [20.36]. In the special case when ν is an integer, $F_\nu(z)$ is proportional to $F_\nu(-z)$; the second, independent solution of **20.3.1** then has the form

20.3.6 $$y_2=zce_r(z,q)+\sum_{k=0}^{\infty} d_{2k+p}\sin(2k+p)z,$$

associated with $ce_r(z,q)$

20.3.7 $$y_2=zse_r(z,q)+\sum_{k=0}^{\infty} f_{2k+p}\cos(2k+p)z,$$

associated with $se_r(z,q)$

The coefficients d_{2k+p} and f_{2k+p} depend on the corresponding coefficients A_m and B_m, respectively, of **20.2,** as well as on a and q. See [20.30], section **(7.50)**–**(7.51)** and [20.58], section V, for details.

If ν is not an integer, then the Floquet solutions $F_\nu(z)$ and $F_\nu(-z)$ are linearly independent. It is clear that **20.3.2** can be written in the form

20.3.8 $$F_\nu(z)=\sum_{k=-\infty}^{\infty} c_{2k}e^{i(\nu+2k)z}.$$

From **20.3.8** it follows that if ν is a proper fraction m_1/m_2, then every solution of **20.3.1** is periodic, and of period at most $2\pi m_2$. This agrees with results already noted in **20.2;** i.e., both independent solutions are periodic, if one is, provided the period is different from π and 2π.

Method of Generating the Characteristic Exponent

Define two linearly independent solutions of **20.3.1,** for fixed a, q by

20.3.9
$$y_1(0)=1;\ y_1'(0)=0.$$
$$y_2(0)=0;\ y_2'(0)=1.$$

Then it can be shown that

20.3.10 $$\cos\pi\nu-y_1(\pi)=0$$

20.3.11 $$\cos\pi\nu-1-2y_1'\left(\frac{\pi}{2}\right)y_2\left(\frac{\pi}{2}\right)=0$$

Thus ν may be obtained from a knowledge of $y_1(\pi)$ or from a knowledge of both $y_1'\left(\frac{\pi}{2}\right)$ and $y_2\left(\frac{\pi}{2}\right)$. For numerical purposes **20.3.11** may be more desirable because of the shorter range of integration, and hence the lesser accumulation of round-off errors. Either ν, $-\nu$, or $\pm\nu+2k$ (k an arbitrary integer) can be taken as the solution of **20.3.11.** Once ν has been fixed, the coefficients of **20.3.8** can be determined, except for an arbitrary multiplier which is independent of z.

The characteristic exponent can also be computed from a continued fraction, in a manner analogous to developments in **20.2,** if a sufficiently close first approximation to ν is available. For systematic tabulation, this method is considerably faster than the method of numerical integration. Thus, when **20.3.8** is substituted into **20.3.1,** there result the following recurrence relations:

20.3.12 $$V_{2n}c_{2n}=c_{2n-2}+c_{2n+2}$$

where

20.3.13 $$V_{2n}=[a-(2n+\nu)^2]/q,\ -\infty<n<\infty.$$

When ν is complex, the coefficients V_{2n} may also be complex. As in **20.2,** it is possible to generate the ratios

$$G_m=c_m/c_{m-2}\text{ and }H_{-m}=c_{-m-2}/c_{-m}$$

from the continued fractions

20.3.14

$$G_m=\frac{1}{V_m-}\,\frac{1}{V_{m+2}-\ \cdot\ \cdot\ \cdot},\qquad m\geq0$$

$$H_{-m}=\frac{1}{V_{-m-2}-}\,\frac{1}{V_{-m-4}-\ \cdot\ \cdot\ \cdot},\qquad m\geq0.$$

From the form of **20.3.13** and the known properties of continued fractions it is assured that for sufficiently large values of $|m|$ both $|G_m|$ and $|H_{-m}|$ converge. Once values of G_m and H_{-m} are available for some sufficiently large value of m, then the finite number of ratios $G_{m-2}, G_{m-4}, \ldots, G_0$ can be computed in turn, if they exist. Similarly for $H_{-m+2}, \ldots, H_0$. It is easy to show that ν is the correct characteristic exponent, appropriate for the point (a, q), if and only if $H_0G_0=1$. An iteration technique can be used to improve the value of ν, by the method suggested in [20.3]. One coefficient c_j can be assigned arbitrarily; the rest are then completely determined. After all the c_j become available, a multiplier (depending on q but not on z) can be found to satisfy a prescribed normalization.

It is well known that continued fractions can be converted to determinantal form. Equation **20.3.14** can in fact be written as a determinant with an infinite number of rows—a special case of Hill's determinant. See [20.19], [20.36], [20.15], or [20.30] for details. Although the determinant has actually been used in computations where high-speed computers were available, the direct use of the continued fraction seems much less laborious.

Special Cases (a, q Real)

Corresponding to $q=0$, $y_1=\cos\sqrt{a}z$, $y_2=\sin\sqrt{a}z$; the Floquet solutions are $\exp(iaz)$ and $\exp(-iaz)$. As a, q vary continuously in the $q-a$ plane, ν describes curves; ν is real when (q, a), $q\geq0$ lies in the region between $a_r(q)$ and $b_{r+1}(q)$ and

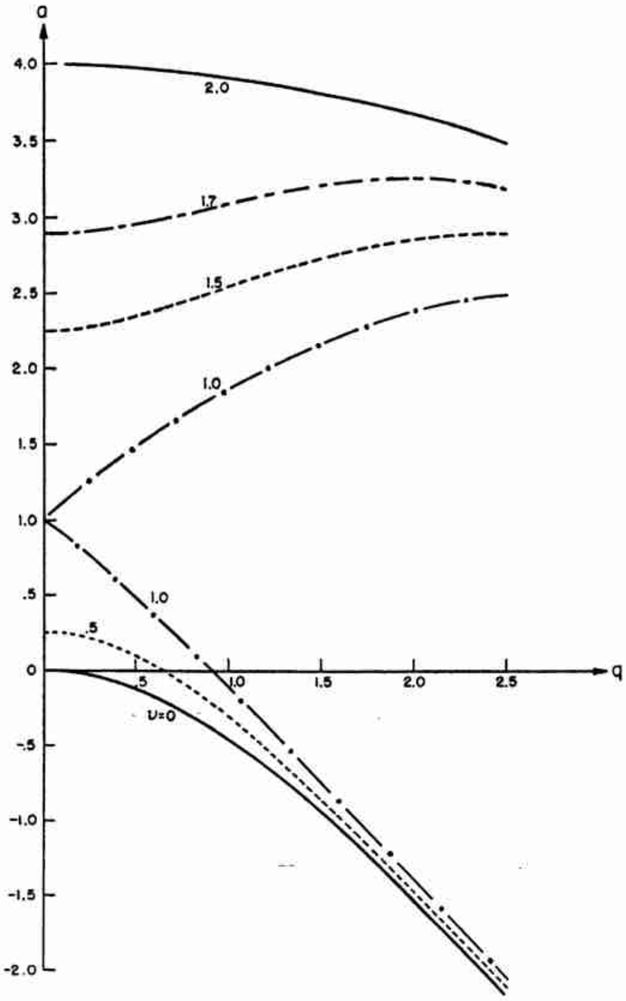

FIGURE 20.6. *Characteristic Exponent-First Two Stable Regions* $y=e^{i\nu x}P(x)$ *where* $P(x)$ *is a periodic function of period* π.

Definition of ν;

In first stable region, $0\leq\nu\leq1$,

In second stable region, $1\leq\nu\leq2$.

(Constructed from tabular values supplied by T. Tamir, Brooklyn Polytechnic Institute)

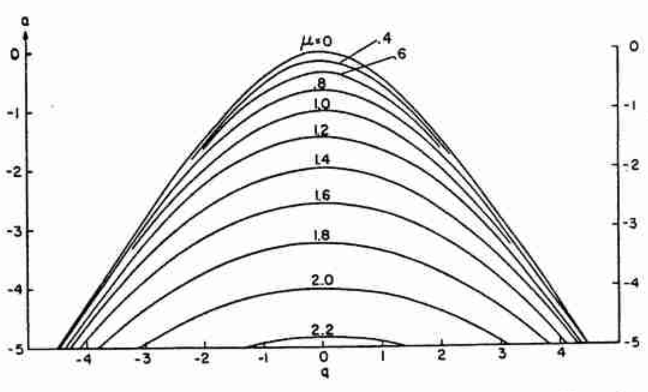

FIGURE 20.7. *Characteristic Exponent in First Unstable Region. Differential equation:* $y''+(a-2q\cos 2x)y=0$. *The Floquet solution* $y=e^{i\nu x}P(x)$, *where* $P(x)$ *is a periodic function of period* π. *In the first unstable region,* $\nu=i\mu$; μ *is given for* $a\geq-5$. (Constructed at NBS.)

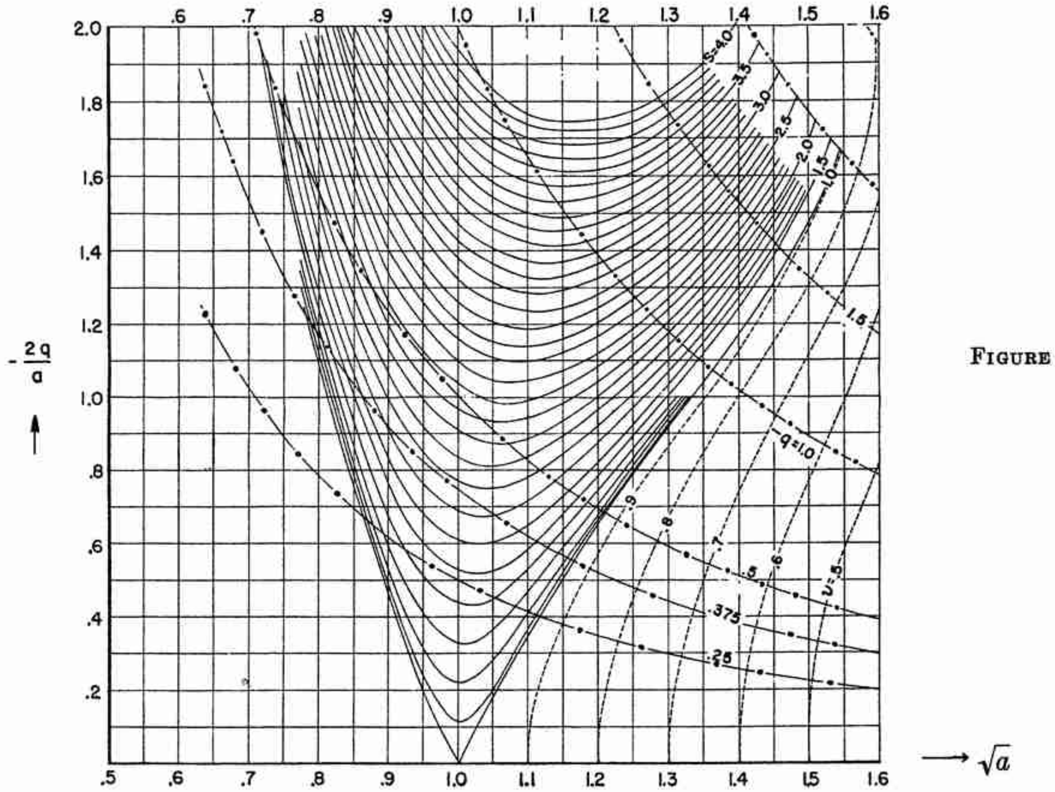

FIGURE 20.8

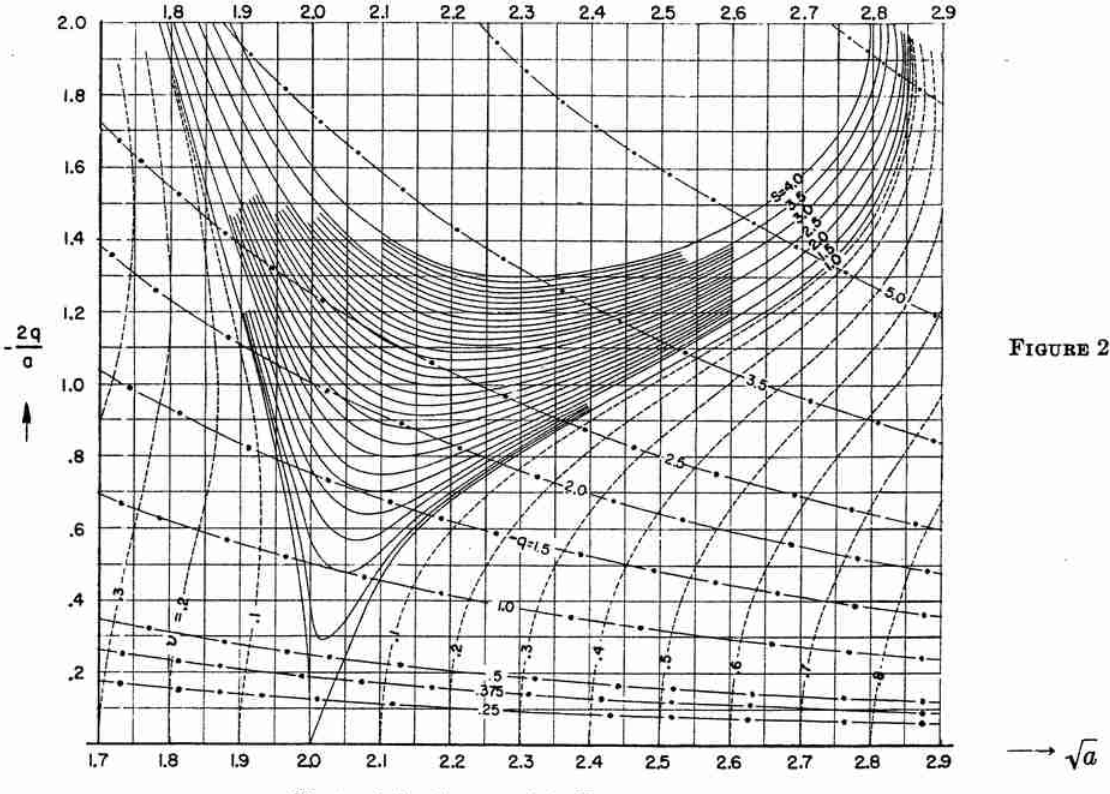

FIGURE 20.9

Charts of the Characteristic Exponent.

(From S. J. Zaroodny, An elementary review of the Mathieu-Hill equation of real variable based on numerical solutions, Ballistic Research Laboratory Memo. Rept. 878, Aberdeen Proving Ground, Md., 1955, with permission.)

——— $s=e^{i\nu\pi}$=*constant; in unstable regions*

– – – – ν=*constant; in stable regions*

–.–.– *Lines of constant values of* $-q$.

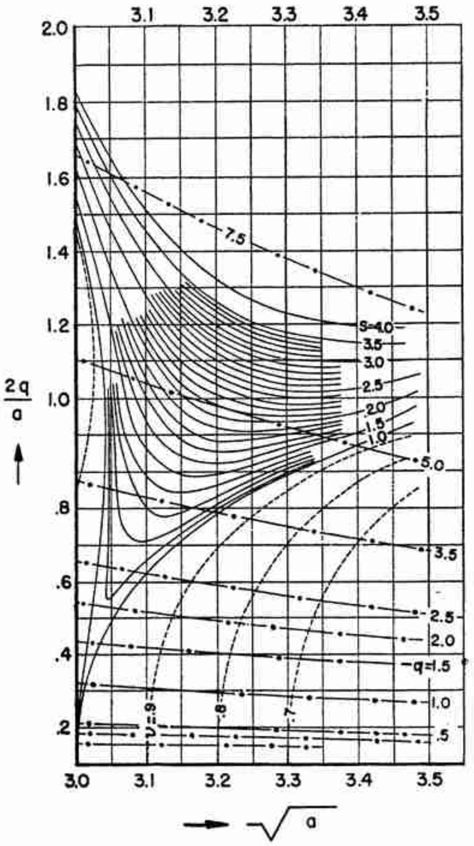

FIGURE 20.10. *Chart of the Characteristic Exponent.*

(From S. J. Zaroodny, An elementary review of the Mathieu-Hill equation of real variable based on numerical solutions, Ballistic Research Laboratory Memo. Rept. 878, Aberdeen Proving Ground, Md., 1955, with permission)

——— *$s=e^{i\nu\pi}$=constant; in unstable regions*
– – – – *ν=constant; in stable regions*
– . – . – *Lines of constant values of $-q$.*

all solutions of **20.1.1** for real z are therefore bounded (stable); ν is complex in regions between b_r and a_r; in these regions every solution becomes infinite at least once; hence these regions are termed "unstable regions". The characteristic curves a_r, b_r separate the regions of stability. For negative q, the stable regions are between b_{2r+1} and b_{2r+2}, a_{2r} and a_{2r+1}; the unstable regions are between a_{2r+1} and b_{2r+1}, a_{2r} and b_{2r}.

In some problems solutions are required for real values of z only. In such cases a knowledge of the characteristic exponent ν and the periodic function $P(z)$ is sufficient for the evaluation of the required functions. For complex values of z, however, the series defining $P(z)$ converges slowly. Other solutions will be determined in the next section; they all have the remarkable property that they depend on the same coefficients c_m developed in connection with Floquet's theorem (except for an arbitrary normalization factor).

Expansions for Small q ([20.36] chapter 2)

If ν, q are fixed:

20.3.15

$$a=\nu^2+\frac{q^2}{2(\nu^2-1)}+\frac{(5\nu^2+7)q^4}{32(\nu^2-1)^3(\nu^2-4)}$$
$$+\frac{(9\nu^4+58\nu^2+29)q^6}{64(\nu^2-1)^5(\nu^2-4)(\nu^2-9)}+\ldots\quad(\nu\neq1,2,3).$$

For the coefficients c_{2j} of **20.3.8**

20.3.16

$$c_2/c_0=\frac{-q}{4(\nu+1)}-\frac{(\nu^2+4\nu+7)q^3}{128(\nu+1)^3(\nu+2)(\nu-1)}+\ldots\quad(\nu\neq1,2)$$

$$c_4/c_0=q^2/32(\nu+1)(\nu+2)+\ldots$$

$$c_{2s}/c_0=(-1)^s q^s\Gamma(\nu+1)/2^{2s}s!\Gamma(\nu+s+1)+\ldots$$

20.3.17

$$F_\nu(z)=c_0\left[e^{i\nu z}-q\left\{\frac{e^{i(\nu+2)z}}{4(\nu+1)}-\frac{e^{i(\nu-2)z}}{4(\nu-1)}\right\}\right]+\ldots$$
(ν not an integer)

For small values of a

20.3.18

$$\cos\nu\pi=\left(1-\frac{a\pi^2}{2}+\frac{a^2\pi^4}{24}+\ldots\right)$$
$$-\frac{q^2\pi^2}{4}\left[1+a\left(1-\frac{\pi^2}{6}\right)+\ldots\right]$$
$$+q^4\left(\frac{\pi^4}{96}-\frac{25\pi^2}{256}+\ldots\right)+\ldots$$

20.4. Other Solutions of Mathieu's Equation

Following Erdélyi [20.14], [20.15], define

20.4.1 $\varphi_k(z)=[e^{i\pi}\cos(z-b)/\cos(z+b)]^{\frac{1}{2}k}J_k(f)$

where

20.4.2 $f=2[q\cos(z-b)\cos(z+b)]^{\frac{1}{2}}$,

and $J_k(f)$ is the Bessel function of order k; b is a fixed, arbitrary complex number. By using the recurrence relations for Bessel functions the following may be verified:

20.4.3

$$\frac{d^2\varphi_k}{dz^2}-2q(\cos 2z)\varphi_k+q(\varphi_{k-2}+\varphi_{k+2})+k^2\varphi_k=0.$$

It follows that a formal solution of **20.1.1** is given by

20.4.4 $$y=\sum_{n=-\infty}^{\infty}c_{2n}\varphi_{2n+\nu}$$

where the coefficients c_{2n} are those associated with Floquet's solution. In the above, ν may be complex. Except for the special case when ν is an integer, the following holds:

$$\frac{\varphi_{2n+\nu-2}}{\varphi_{2n+\nu}}\sim\frac{\varphi_{-2n+\nu}}{\varphi_{-2n+\nu+2}}\sim\frac{-4n^2}{q\,[\cos\,(z-b)]^2}\qquad(n\to\infty)$$

If ν and n are integers, $J_{-2n+\nu}(f)=(-1)^\nu J_{2n-\nu}(f)$.

$$[\varphi_{2n+\nu}/\varphi_{2n+\nu-2}]\sim-[\cos\,(z-b)]^2q/4n^2$$

$$[\varphi_{-2n+\nu}/\varphi_{-2n+\nu+2}]\sim-4n^2/q\,[\cos\,(z-b)]^2$$

On the other hand

$$\frac{c_{2n}}{c_{2n-2}}\sim\frac{c_{-2n}}{c_{-2n+2}}\sim\frac{-q}{4n^2}\qquad(n\to\infty)$$

It follows that **20.4.4** converges absolutely and uniformly in every closed region where

$|\cos\,(z-b)|>d_1>1.$

There are two such disjoint regions:

(I) $\mathscr{I}(z-b)>d_2>0$; $(|\cos\,(z-b)|>d_1>1)$

(II) $\mathscr{I}(z-b)<-d_2<0$; $(|\cos\,(z-b)|>d_1>1)$

If ν is an integer **20.4.4** converges for all values of z. Various representations are found by specializing b.

20.4.5

$$\text{If } b=0,\ y=e^{i\pi\nu/2}\sum_{n=-\infty}^{\infty}c_{2n}(-1)^nJ_{2n+\nu}(2\sqrt{q}\cos z)$$

$$(|\cos z|>1,\ |\arg 2\sqrt{q}\cos z|\leq\pi)$$

20.4.6

$$\text{If } b=\frac{\pi}{2},\ y=\sum_{n=-\infty}^{\infty}c_{2n}J_{2n+\nu}(2i\sqrt{q}\sin z)$$

$$(|\sin z|>1,\ |\arg 2\sqrt{q}\sin z|\leq\pi)$$

If $b\to\infty i$, y reduces to a multiple of the solution **20.3.8.** The fact that **20.3.8, 20.4.5,** and **20.4.6** are special cases of **20.4.4** explains why it is that these apparently dissimilar expansions involve the same set of coefficients c_{2n}.

Since **20.4.4** results from the recurrence properties of Bessel functions, $J_k(f)$ can be replaced by $H_k^{(j)}(f)$, $j=1, 2$, where $H_k^{(j)}$ is the Hankel function, at least formally. Thus let

$$\psi_k^j=[e^{i\pi}\cos\,(z-b)/\cos\,(z+b)]^{\frac{1}{2}k}H_k^{(j)}(f)$$

where f satisfies **20.4.2.** An examination of the ratios $\psi_{2n+\nu}/\psi_{2n+\nu-2}$ shows that

$$y=\sum_{n=-\infty}^{\infty}c_{2n}\psi_{2n+\nu}^{(j)}$$

will be a solution provided

$$|\cos\,(z-b)|>1;\ |\cos\,(z+b)|>1.$$

The above two conditions are necessary even when ν is an integer. Once b is fixed, the regions in which the solutions converge can be readily established.

Following [20.36] let

20.4.7

$$J_p(x)=Z_p^{(1)}(x);\quad Y_p(x)=Z_p^{(2)}(x);$$
$$H_p^{(1)}(x)=Z_p^{(3)}(x);\quad H_p^{(2)}(x)=Z_p^{(4)}(x)$$

If z is replaced by $-iz$ in **20.4.5** and **20.4.6** solutions of **20.1.2** are obtained. Thus

20.4.8

$$y_1^{(j)}(z)=\sum_{n=-\infty}^{\infty}c_{2n}(-1)^nZ_{2n+\nu}^{(j)}(2\sqrt{q}\cosh z)$$

$$(|\cosh z|>1)$$

20.4.9

$$y_2^{(j)}(z)=\sum_{n=-\infty}^{\infty}c_{2n}Z_{2n+\nu}^{(j)}(2\sqrt{q}\sinh z)$$

$$(|\sinh z|>1,\ j=1, 2, 3, 4)$$

The relation between $y_1^{(j)}(z)$ and $y_2^{(j)}(z)$ can be determined from the asymptotic properties of the Bessel functions for large values of argument. It can be shown that

20.4.10

$$y_1^{(j)}(z)/y_2^{(j)}(z)=[F_\nu(0)/F_\nu\left(\frac{\pi}{2}\right)]e^{i\nu\pi/2}\qquad(\mathscr{R}z>0).$$

When ν is not an integer, the above solutions do not vanish identically. See **20.6** for integral values of ν.

Solutions Involving Products of Bessel Functions

20.4.11

$$y_3^{(j)}(z)=\frac{1}{c_{2s}}\sum_{n=-\infty}^{\infty}c_{2n}(-1)^nZ_{n+\nu+s}^{(j)}(\sqrt{q}e^{iz})J_{n-s}(\sqrt{q}e^{-iz})$$

$$(j=1, 2, 3, 4)$$

satisfies **20.1.1,** where $Z_n^{(j)}(u)$ is defined in **20.4.7,** the coefficients c_{2n} belong to the Floquet solution, and s is an arbitrary integer, $c_{2s}\neq 0$. The solution converges over the entire complex z-plane if $q\neq 0$. Written with z replaced by $-iz$, one obtains solutions of **20.1.2.**

20.4.12

$$M_\nu^j(z,q)=\frac{1}{c_{2s}^\nu}\sum_{n=-\infty}^{\infty} c_{2n}^\nu(-1)^n Z_{n+\nu+s}^{(j)}(\sqrt{q}e^z)J_{n-s}(\sqrt{q}e^{-z})$$

It can be verified from **20.4.8** and **20.4.12** that

20.4.13 $$\frac{y_1^{(j)}(z)}{M_j^j(z,q)}=F_\nu(0), \qquad (\mathscr{R}z>0)$$

provided $c_{2s}\neq 0$. If $c_{2s}=0$, the coefficient of $1/c_{2s}$ in **20.4.11** vanishes identically. For details see [20.43], [20.15], [20.36].

If s is chosen so that $|c_{2s}|$ is the largest coefficient of the set $|c_{2j}|$, then rapid convergence of **20.4.12** is obtained, when $\mathscr{R}z>0$. Even then one must be on guard against the possible loss of significant figures in the process of summing the series, especially so when q is large, and $|z|$ small. (If $j\neq 1$, then the phase of the logarithmic terms occurring in **20.4.12** must be defined, to make the functions single-valued.)

20.5. Properties of Orthogonality and Normalization

If $a(\nu+2p,q)$, $a(\nu+2s,q)$ are simple roots of **20.3.10** then

20.5.1 $$\int_0^\pi F_{\nu+2p}(z)F_{\nu+2s}(-z)dz=0, \text{ if } p\neq s.$$

Define

20.5.2 $$ce_\nu(z,q)=\frac{1}{2}[F_\nu(z)+F_\nu(-z)];$$

$$se_\nu(z,q)=-i\,\frac{1}{2}[F_\nu(z)-F_\nu(-z)]$$

$ce_\nu(z,q)$, $se_\nu(z,q)$ are thus even and odd functions of z, respectively, for all ν (when not identically zero).

If ν is an integer, then $ce_\nu(z,q)$, $se_\nu(z,q)$ are either Floquet solutions or identically zero. The solutions $ce_r(z,q)$ are associated with a_r; $se_r(z,q)$ are associated with b_r; r an integer.

Normalization for Integral Values of ν and Real q

20.5.3 $$\int_0^{2\pi}[ce_r(z,q)]^2dz=\int_0^{2\pi}[se_r(z,q)]^2dz=\pi$$

For integral values of ν the summation in **20.3.8** reduces to the simpler forms **20.2.3–20.2.4;** on account of **20.5.3,** the coefficients A_m and B_m (for all orders r) have the property

20.5.4

$$2A_0^2+A_2^2+\ldots=A_1^2+A_3^2+\ldots$$
$$=B_1^2+B_3^2+\ldots=B_2^2+B_4^2+\ldots=1.$$

20.5.5

$$A_0^{2s}=\frac{1}{2\pi}\int_0^{2\pi}ce_{2s}(z,q)dz;\, A_n^r=\frac{1}{\pi}\int_0^{2\pi}ce_r(z,q)\cos nzdz$$

$$B_n^r=\frac{1}{\pi}\int_0^{2\pi}se_r(z,q)\sin nzdz \qquad n\neq 0$$

For integral values of ν, the functions $ce_r(z,q)$ and $se_r(z,q)$ form a complete orthogonal set for the interval $0\leq z\leq 2\pi$. Each of the four systems $ce_{2r}(z)$, $ce_{2r+1}(z)$, $se_{2r}(z)$, $se_{2r+1}(z)$ is complete in the smaller interval $0\leq z\leq \frac{1}{2}\pi$, and each of the systems $ce_r(z)$, $se_r(z)$ is complete in $0\leq z\leq\pi$.

If q is not real, there exist multiple roots of **20.3.10;** for such special values of $a(q)$, the integrals in **20.5.3** vanish, and the normalization is therefore impossible. In applications, the particular normalization adopted is of little importance, except possibly for obtaining quantitative relations between solutions of various types. For this reason the normalization of $F_\nu(z)$, for arbitrary complex values of a, q, will not be specified here. It is worth noting, however, that solutions

$$\alpha ce_r(z,q), \qquad \beta se_r(z,q)$$

defined so that

$$\alpha ce_r(0,q)=1; \qquad \left[\frac{d}{dz}\beta se_r(z,q)\right]_{z=0}=1$$

are always possible. This normalization has in fact been used in [20.59], and also in [20.58], where the most extensive tabular material is available. The tabulated entries in [20.58] supply the conversion factors $A=1/\alpha$, $B=1/\beta$, along with the coefficients. Thus conversion from one normalization to another is rather easy.

In a similar vein, no general normalization will be imposed on the functions defined in **20.4.8.**

20.6. Solutions of Mathieu's Modified Equation 20.1.2 for Integral ν (Radial Solutions)

Solutions of the first kind

20.6.1

$$Ce_{2r+p}(z,q)=ce_{2r+p}(iz,q)$$

$$=\sum_{k=0}^{\infty}A_{2k+p}^{2r+p}(q)\cosh(2k+p)z$$

associated with a_r

20.6.2 $$Se_{2r+p}(z,q)=-ise_{2r+p}(iz,q)=\sum_{k=0}^{\infty} B_{2k+p}^{2r+p}(q)\sinh(2k+p)z, \text{ associated with } b_r$$

writing $A_{2k+p}^{2r+p}(q)=A_{2k+p}$ for brevity; similarly for B_{2k+p}; $p=0, 1,$

20.6.3 $$Ce_{2r}(z,q)=\frac{ce_{2r}\left(\frac{\pi}{2},q\right)}{A_0^{2r}}\sum_{k=0}^{\infty}(-1)^k A_{2k}J_{2k}(2\sqrt{q}\cosh z)=\frac{ce_{2r}(0,q)}{A_0^{2r}}\sum_{k=0}^{\infty}A_{2k}J_{2k}(2\sqrt{q}\sinh z)$$

20.6.4 $$Ce_{2r+1}(z,q)=\frac{ce'_{2r+1}\left(\frac{\pi}{2},q\right)}{\sqrt{q}A_1^{2r+1}}\sum_{k=0}^{\infty}(-1)^{k+1}A_{2k+1}J_{2k+1}(2\sqrt{q}\cosh z)$$

$$=\frac{ce_{2r+1}(0,q)}{\sqrt{q}A_1^{2r+1}}\coth z\sum_{k=0}^{\infty}(2k+1)A_{2k+1}J_{2k+1}(2\sqrt{q}\sinh z)$$

20.6.5 $$Se_{2r}(z,q)=\frac{se'_{2r}\left(\frac{\pi}{2},q\right)\tanh z}{qB_2^{2r}}\sum_{k=1}^{\infty}(-1)^k 2kB_{2k}J_{2k}(2\sqrt{q}\cosh z)$$

$$=\frac{se'_{2r}(0,q)}{qB_2^{2r}}\coth z\sum_{k=1}^{\infty}2kB_{2k}J_{2k}(2\sqrt{q}\sinh z)$$

20.6.6 $$Se_{2r+1}(z,q)=\frac{se_{2r+1}\left(\frac{\pi}{2},q\right)}{\sqrt{q}B_1^{2r+1}}\tanh z\sum_{k=0}^{\infty}(-1)^k(2k+1)B_{2k+1}J_{2k+1}(2\sqrt{q}\cosh z)$$

$$=\frac{se'_{2r+1}(0,q)}{\sqrt{q}B_1^{2r+1}}\sum_{k=0}^{\infty}B_{2k+1}J_{2k+1}(2\sqrt{q}\sinh z)$$

See [20.30] for still other forms.
Solutions of the second kind, as well as solutions of the third and fourth kind (analogous to Hankel functions) are obtainable from **20.4.12.**

20.6.7 $$Mc_{2r}^{(j)}(z,q)=\sum_{k=0}^{\infty}(-1)^{r+k}A_{2k}^{2r}(q)[J_{k-s}(u_1)Z_{k+s}^{(j)}(u_2)+J_{k+s}(u_1)Z_{k-s}^{(j)}(u_2)]/\epsilon_s A_{2s}^{2r}$$

where $\epsilon_0=2$, $\epsilon_s=1$, for $s=1, 2, \ldots$; s arbitrary, associated with a_{2r}

20.6.8 $$Mc_{2r+1}^{(j)}(z,q)=\sum_{k=0}^{\infty}(-1)^{r+k}A_{2k+1}^{2r+1}(q)[J_{k-s}(u_1)Z_{k+s+1}^{(j)}(u_2)+J_{k+s+1}(u_1)Z_{k-s}^{(j)}(u_2)]/A_{2s+1}^{2r+1}$$

associated with a_{2r+1}

20.6.9 $$Ms_{2r}^{(j)}(z,q)=\sum_{k=1}^{\infty}(-1)^{k+r}B_{2k}^{2r}(q)[J_{k-s}(u_1)Z_{k+s}^{(j)}(u_2)-J_{k+s}(u_1)Z_{k-s}^{(j)}(u_2)]/B_{2s}^{2r}, \text{ associated with } b_{2r}$$

20.6.10 $$Ms_{2r+1}^{(j)}(z,q)=\sum_{k=0}^{\infty}(-1)^{k+r}B_{2k+1}^{2r+1}(q)[J_{k-s}(u_1)Z_{k+s+1}^{(j)}(u_2)-J_{k+s+1}(u_1)Z_{k-s}^{(j)}(u_2)]/B_{2s+1}^{2r+1}$$

associated with b_{2r+1}

where

$$u_1=\sqrt{q}e^{-z},\ u_2=\sqrt{q}e^{z},\ B_{2s+p}^{2r+p},\ A_{2s+p}^{2r+p}\neq 0,\ p=0, 1.$$

See **20.4.7** for definition of $Z_m^{(j)}(x)$.

Solutions **20.6.7–20.6.10** converge for all values of z, when $q\neq 0$. If $j=2, 3, 4$ the logarithmic terms entering into the Bessel functions $Y_m(u_2)$ must be defined, to make the functions single-valued. This can be accomplished as follows:

Define (as in [20.58])

20.6.11 $$\ln(\sqrt{q}e^z)=\ln(\sqrt{q})+z$$

See [20.15] and [20.36], section **2.75** for derivation.

Other Expressions for the Radial Functions (Valid Over More Limited Regions)

20.6.12
$$Mc_{2r}^{(j)}(z, q)=[ce_{2r}(0, q)]^{-1} \sum_{k=0}^{\infty} (-1)^{k+r}A_{2k}^{2r}(q)Z_{2k}^{(j)}(2\sqrt{q} \cosh z)$$

$$Mc_{2r+1}^{(j)}(z, q)=[ce_{2r+1}(0, q)]^{-1} \sum_{k=0}^{\infty} (-1)^{k+r}A_{2k+1}^{2r+1}(q)Z_{2k+1}^{(j)}(2\sqrt{q} \cosh z)$$

20.6.13
$$Ms_{2r}^{(j)}(z, q)=[se_{2r}'(0, q)]^{-1} \tanh z \sum_{k=1}^{\infty} (-1)^{k+r}2kB_{2k}^{2r}(q)Z_{2k}^{(j)}(2\sqrt{q} \cosh z)$$

$$Ms_{2r+1}^{(j)}(z, q)=[se_{2r+1}'(0, q)]^{-1} \tanh z \sum_{k=0}^{\infty} (-1)^{k+r}(2k+1)B_{2k+1}^{2r+1}(q)Z_{2k+1}^{(j)}(2\sqrt{q} \cosh z)$$

Valid for $\mathscr{R}z>0$, $|\cosh z|>1$; if $j=1$, valid for all z. They agree with **20.6.7–20.6.10** if the Bessel functions $Y_m(2q^{\frac{1}{2}} \cosh z)$ are made single-valued in a suitable way. For example, let

$$Y_m(u)=\frac{2}{\pi} (\ln u)J_m(u)+\phi(u)$$

where $\phi(u)$ is single-valued for all finite values of u. With $u=2q^{\frac{1}{2}} \cosh z$, define

20.6.14
$$\ln (2q^{\frac{1}{2}} \cosh z)=\ln 2q^{\frac{1}{2}}+z+\ln \tfrac{1}{2}(1+e^{-2z}) \qquad -\frac{\pi}{2} \leq \arg \tfrac{1}{2}(1+e^{-2z}) \leq \frac{\pi}{2}$$

(If q is not positive, the phase of $\ln 2q^{\frac{1}{2}}$ must also be specified, although this specification will not affect continuity with respect to z. If $Y_m(u)$ is defined from some other expression, the definition must be compatible with **20.6.14**.)

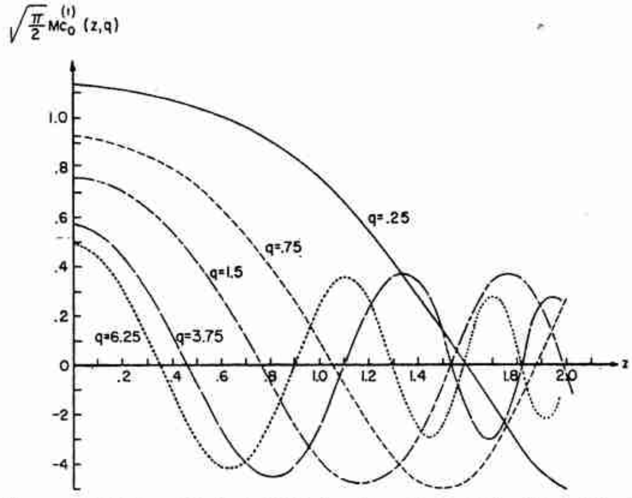

FIGURE 20.11. *Radial Mathieu Function of the First Kind.*

(From J. C. Wiltse and M. J. King, Values of the Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Tech. Rept. AF-53, 1958, with permission)

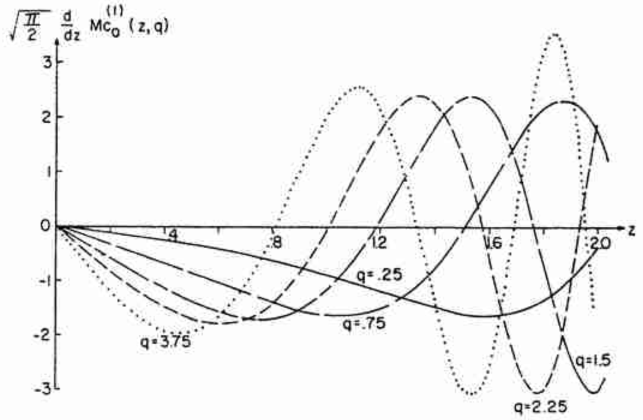

FIGURE 20.12. *Derivative of the Radial Mathieu Function of the First Kind.*

(From J. C. Wiltse and M. J. King, Derivatives, zeros, and other data pertaining to Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Tech. Rept. AF-57, 1958, with permission)

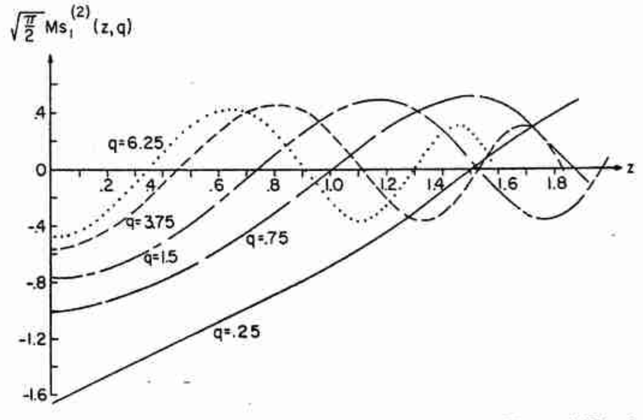

FIGURE 20.13. *Radial Mathieu Function of the Second Kind.*

(From J. C. Wiltse and M. J. King, Values of the Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Tech. Rept. AF-53, 1958, with permission)

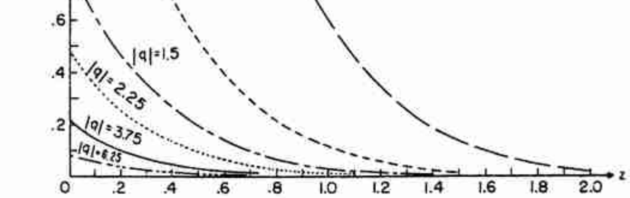

FIGURE 20.14. *Radial Mathieu Function of the Third Kind.*

(From J. C. Wiltse and M. J. King, Values of the Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Tech. Rept. AF-53, 1958, with permission)

If $j=1$, $Mc^{(1)}_{2r+p}$ and $Ms^{(1)}_{2r+p}$, $p=0, 1$ are solutions of the first kind, proportional to Ce_{2r+p} and Se_{2r+p}, respectively.

Thus

20.6.15

$$Ce_{2r}(z,q)=\frac{ce_{2r}\left(\frac{\pi}{2},q\right)ce_{2r}(0,\ q)}{(-1)^r A_0^{2r}}\,Mc_{2r}^{(1)}(z,q)$$

$$Ce_{2r+1}(z,q)=\frac{ce'_{2r+1}\left(\frac{\pi}{2},q\right)ce_{2r+1}(0,\ q)}{(-1)^{r+1}\sqrt{q}A_1^{2r+1}}\,Mc^{(1)}_{2r+1}(z,q)$$

$$Se_{2r}(z,q)=\frac{se'_{2r}\ (0,\ q)se'_{2r}\left(\frac{\pi}{2},\ q\right)}{(-1)^r qB_2^{2r}}\,Ms_{2r}^{(1)}(z,q)$$

$$Se_{2r+1}(z,q)=\frac{se'_{2r+1}(0,\ q)\,se_{2r+1}\left(\frac{\pi}{2},\ q\right)}{(-1)^r\sqrt{q}B_1^{2r+1}}\,Ms^{(1)}_{2r+1}(z,q)$$

The Mathieu-Hankel functions are

20.6.16

$$M_r^{(3)}(z,q)=M_r^{(1)}(z,q)+iM_r^{(2)}(z,q)$$

$$M_r^{(4)}(z,q)=M_r^{(1)}(z,q)-iM_r^{(2)}(z,q)$$

$$M_r^{(j)}=Mc_r^{(j)}\text{ or }Ms_r^{(j)}.$$

From **20.6.7–20.6.11** and the known properties of Bessel functions one obtains

20.6.17

$$M^{(2)}_{2r+p}(z+in\pi,\ q)$$
$$=(-1)^{np}[M^{(2)}_{2r+p}(z,q)+2niM^{(1)}_{2r+p}(z,q)]$$

$$M^{(3)}_{2r+p}(z+in\pi,q)$$
$$=(-1)^{np}[M^{(3)}_{2r+p}(z,q)-\ 2nM^{(1)}_{2r+p}(z,q)]$$

$$M^{(4)}_{2r+p}(z+in\pi,q)$$
$$=(-1)^{np}[M^{(4)}_{2r+p}(z,q)+2nM^{(1)}_{2r+p}(z,q)]$$

where $M=Mc$ or Ms throughout any of the above equations.

Other Properties of Characteristic Functions, q Real (Associated With a_r and b_r)

Consider

20.6.18

$$X_1=Mc_r^{(2)}(z,\ q)+Mc_r^{(2)}(-z,\ q);$$
$$X_2=Ms_r^{(2)}(z,\ q)-Ms_r^{(2)}(-z,\ q)$$

Since X_1 is an even solution it must be proportional to $Mc_r^{(1)}(z,\ q)$; for **20.1.2** admits of only one even solution (aside from an arbitrary constant factor). Similarly, X_2 is proportional to $Ms_r^{(1)}(z,\ q)$. The proportionality factors can be found by considering values of the functions at $z=0$. Define, therefore,

20.6.19

$$Mc_r^{(2)}(-z,\ q)=-Mc_r^{(2)}(z,\ q)-2f_{e,r}Mc_r^{(1)}(z,\ q)$$

20.6.20

$$Ms_r^{(2)}(-z,\ q)=Ms_r^{(2)}(z,\ q)-2f_{o,r}Ms_r^{(1)}(z,\ a)$$

where

20.6.21

$$f_{e,r}=-Mc_r^{(2)}(0,q)/Mc_r^{(1)}(0,q)$$

$$f_{o,r}=\left[\frac{d}{dz}\,Ms_r^{(2)}(z,q)/\frac{d}{dz}\,Ms_r^{(1)}(z,q)\right]_{z=0}$$

See [20.58].

In particular the above equations can be used to extend solutions of **20.6.12–20.6.13** when $\mathscr{R}z<0$. For although the latter converge for $\mathscr{R}z<0$, provided only $|\cosh z|>1$, they do not represent the same functions as **20.6.9–20.6.10.**

20.7. Representations by Integrals and Some Integral Equations

Let

20.7.1 $$G(u)=\oint_C K(u,\ t)V(t)dt$$

be defined for u in a domain U and let the contour C belong to the region T of the complex t-plane, with $t=\gamma_0$ as the starting point of the contour and $t=\gamma_1$ as its end-point. The kernel $K(u,\ t)$ and the function $V(t)$ satisfy **20.7.3** and the hypotheses in **20.7.2.**

20.7.2 $K(u,\ t)$ and its first two partial derivatives with respect to u and t are continuous for t on C and u in U; V and $\frac{dV}{dt}$ are continuous in t.

20.7.3

$$\left[\frac{\partial K}{\partial t}V-\frac{dV}{dt}K\right]_{\gamma_0}^{\gamma_1}=0;\ \frac{d^2V}{dt^2}+(a-2q\cos 2t)V=0.$$

If K satisfies

20.7.4 $$\frac{\partial^2K}{\partial u^2}+\frac{\partial^2K}{\partial t^2}+2q(\cosh 2u-\cos 2t)K=0$$

then $G(u)$ is a solution of Mathieu's modified equation **20.1.2.**

If $K(u,\ t)$ satisfies

20.7.5 $$\frac{\partial^2K}{\partial u^2}+\frac{\partial^2K}{\partial t^2}+2q(\cos 2u-\cos 2t)K=0$$

then $G(u)$ is a solution of Mathieu's equation **20.1.1,** with u replacing v.

Kernels $K_1(z, t)$ and $K_2(z, t)$

20.7.6 $$K_1(z, t)=Z_\nu^{(j)}(u)[M(z, t)]^{-\nu/2}, \qquad (\mathscr{R}z>0)$$

where

20.7.7 $$u=\sqrt{2q(\cosh 2z+\cos 2t)}$$

20.7.8 $$M(z, t)=\cosh (z+it)/\cosh (z-it)$$

To make $M^{-\frac{1}{2}\nu}$ single-valued, define

20.7.9

$$\cosh (z+i\pi)=e^{i\pi} \cosh z$$
$$\cosh (z-i\pi)=e^{-i\pi} \cosh z$$
$$M(z, 0)=1$$
$$[M(z, \pi)]^{-\frac{1}{2}\nu}=e^{-i\nu\pi}M(z, 0)$$

Let

20.7.10 $$G(z, q)=\frac{1}{\pi}\int_0^\pi K_1(u, t)F_\nu(t)dt, \qquad (\mathscr{R}z>0)$$

where $F_\nu(t)$ is defined in **20.3.8.** It may be verified that K_1F_ν satisfies **20.7.3**, K satisfies **20.7.2** and **20.7.4.** Hence G is a solution of **20.1.2** (with z replacing u). It can be shown that K_1 may be replaced by the more general function

20.7.11

$$K_2(z, t)=Z^{(j)}_{\nu+2s}(u)[M(z, t)]^{-\frac{1}{2}\nu+s}, \text{ } s \text{ any integer.}$$

See **20.4.7** for definition of $Z^{(j)}_{\nu+2s}(u)$.

From the known expansions for $Z^{(j)}_{\nu+2s}(u)$ when $\mathscr{R}z$ is large and positive it may be verified that

20.7.12

$$M_\nu^{(j)}(z, q)=$$

$$\frac{(-1)^s}{\pi c_{2s}}\int_0^\pi Z^{(j)}_{\nu+2s}(u)\left[\frac{\cosh z+it}{\cosh z-it}\right]^{-\frac{1}{2}\nu-s}F_\nu(t)dt$$
$$(\mathscr{R}z>0, \mathscr{R}(\nu+\tfrac{1}{2})>0)$$

where $M_\nu^{(j)}(z, q)$ is given by **20.4.12,** $s=0, 1, \ldots,$ $c_{2s}\neq 0$, and $F_\nu(t)$ is the Floquet solution, **20.3.8.**

Kernel $K_3(z, t, a)$.

20.7.13 $$K_3(z, t, a)=e^{2i\sqrt{q}\,w}$$

where

20.7.14 $$w=\cosh z \cos a \cos t+\sinh z \sin a \sin t$$

20.7.15 $$G(z, q, a)=\frac{1}{\pi}\oint_C e^{2i\sqrt{q}\,w}F_\nu(t)dt$$

where $F_\nu(t)$ is the Floquet solution **20.3.8.** The path C is chosen so that $G(z, t, a)$ exists, and **20.7.2, 20.7.3** are satisfied. Then it may be verified that $K_3(z, t, a)$, considered as a function of z and t, satisfies **20.7.4;** also, considered as a function of a and t, K_3 satisfies **20.7.5.** Consequently $G(z, q, a)=Y(z, q)y(a, q)$, where Y and y satisfy **20.1.2** and **20.1.1,** respectively.

Choice of Path C. Three paths will be defined:

20.7.16

Path C_3: from $-d_1+i\infty$ to $d_2-i\infty$, d_1, d_2 real

$$-d_1<\arg [\sqrt{q}\{\cosh (z+ia)\pm 1\}]<\pi-d_1$$
$$-d_2<\arg [\sqrt{q}\{\cosh (z-ia)\pm 1\}]<\pi-d_2$$

20.7.17

Path C_4: from $d_2-i\infty$ to $2\pi+i\infty-d_1$

(same d_1, d_2 as in **20.7.16**)

20.7.18

$$F_\nu(a)M_\nu^j(z, q)=\frac{e^{-i\nu\frac{\pi}{2}}}{\pi}\oint_{C_j}e^{2i\sqrt{q}\,w}F_\nu(t)dt \qquad j=3, 4$$

where $M_\nu^j(z, q)$ is also given by **20.4.12.**

20.7.19 Path C_1: from $-d_1+i\infty$ to $2\pi-d_1+i\infty$

$$F_\nu(a)M_\nu^{(1)}(z, q)=\frac{e^{-i\nu\frac{\pi}{2}}}{2\pi}\oint_{C_1}e^{2i\sqrt{q}\,w}F_\nu(t)dt$$

See [20.36], section **2.68.**

If ν is an integer the paths can be simplified; for in that case $F_\nu(t)$ is periodic and the integrals exist when the path is taken from 0 to 2π. Still further simplifications are possible, if z is also real.

The following are among the more important integral representations for the periodic functions $ce_r(z, q)$, $se_r(z, q)$ and for the associated radial solutions.

Let $r=2s+p$, $p=0$ or 1

20.7.20

$$ce_r(z,q)=\rho_r\int_0^{\pi/2}\cos\left(2\sqrt{q}\cos z\cos t-p\frac{\pi}{2}\right)ce_r(t, q)dt$$

20.7.21
$$ce_r(z, q)=\sigma_r\int_0^{\pi/2}\cosh\,(2\sqrt{q}\sin z\sin t)[(1-p)+p\cos z\cos t]ce_r(t, q)dt$$

20.7.22
$$se_r(z, q)=\rho_r\int_0^{\pi/2}\sin\left(2\sqrt{q}\cos z\cos t+p\,\frac{\pi}{2}\right)\sin z\sin t\; se_r(t, q)dt$$

20.7.23
$$se_r(z, q)=\sigma_r\int_0^{\pi/2}\sinh\,(2\sqrt{q}\sin z\sin t)[(1-p)\cos z\cos t+p]se_r(t, q)dt$$

where

20.7.24
$$\rho_r=\frac{2}{\pi}ce_{2s}\left(\frac{\pi}{2}, q\right)/A_0^{2s}(q)\,;\; p=0\,\rho_r=\frac{-2}{\pi}ce'_{2s+1}\left(\frac{\pi}{2}, q\right)/\sqrt{q}A_1^{2s+1}(q)\text{ if }p=1,\text{ for functions }ce_r(z, q)$$

$$\rho_r=\frac{-4}{\pi}se'_{2s}\left(\frac{\pi}{2}, q\right)/\sqrt{q}B_2^{2s}(q)\,;\;\rho_r=\frac{4}{\pi}se_{2s+1}\left(\frac{\pi}{2}, q\right)/B_1^{2s+1}(q),\text{ for functions }se_r(z, q)$$

$$\sigma_r=\frac{2}{\pi}ce_{2s}(0, q)/A_0^{2s}(q)\text{ if }p=0;\qquad \sigma_r=\frac{4}{\pi}ce_{2s+1}(0, q)/A_1^{2s+1}(q),\text{ if }p=1;\text{ associated with functions }ce_r(z, q)$$

$$\sigma_r=\frac{4}{\pi}se'_{2s}(0,q)/\sqrt{q}B_2^{2s}(q),\text{ if }p=0;\qquad \sigma_r=\frac{2}{\pi}se'_{2s+1}(0, q)/\sqrt{q}B_1^{2s+1}(q),\text{ if }p=1;\text{ associated with }se_r(z, q)$$

Integrals Involving Bessel Function Kernels

Let

20.7.25
$$u=\sqrt{2q(\cosh 2z+\cos 2t)},\;(\mathscr{R}\cosh 2z>1;\text{ if }j=1,\text{ valid also when }z=0)$$

20.7.26

$$Mc_{2r}^{(j)}(z, q)=\frac{(-1)^r 2}{\pi A_0^{2r}}\int_0^{\frac{\pi}{2}}Z_0^{(j)}(u)ce_{2r}(t, q)dt\,;\; Mc_{2r+1}^{(j)}(z, q)=\frac{(-1)^r 8\sqrt{q}\cosh z}{\pi A_1^{2r+1}}\int_0^{\frac{\pi}{2}}\frac{Z_1^{(j)}(u)\cos t}{u}ce_{2r+1}(t, q)dt$$

20.7.27
$$Ms_{2r}^{(j)}(z, q)=\frac{(-1)^{r+1}8q\sinh 2z}{\pi B_2^{2r}}\int_0^{\frac{\pi}{2}}\frac{Z_2^{(j)}(u)\sin 2t\; se_{2r}(t, q)dt}{u^2}$$

$$Ms_{2r+1}^{(j)}(z, q)=\frac{(-1)^r 8\sqrt{q}\sinh z}{\pi B_1^{2r+1}}\int_0^{\frac{\pi}{2}}\frac{Z_1^{(j)}(u)\sin t\; se_{2r+1}(t, q)dt}{u}$$

In the above the j-convention of **20.4.7** applies and the functions Mc, Ms are defined in **20.5.1–20.5.4.** (These solutions are normalized so that they approach the corresponding Bessel-Hankel functions as $\mathscr{R}z\rightarrow\infty$.)

Other Integrals for $Mc_r^{(1)}(z, q)$ and $Ms_r^{(1)}(z, q)$

20.7.28
$$Mc_r^{(1)}(z, q)=\frac{(-1)^s 2}{\pi ce_r(0, q)}\int_0^{\frac{\pi}{2}}\cos\left(2\sqrt{q}\cosh z\cos t-p\,\frac{\pi}{2}\right)ce_r(t, q)dt$$

20.7.29
$$Mc_r^{(1)}(z, q)=\tau_r\int_0^{\frac{\pi}{2}}[(1-p)+p\cosh z\cos t]\cos\,(2\sqrt{q}\sinh z\sin \mathrm{t})ce_r(t, q)dt$$

$$r=2s+p,\; p=0, 1\,;\;\tau_r=\frac{2}{\pi}(-1)^s/ce_{2s}\left(\frac{\pi}{2}, q\right),\text{ if }p=0\,;\;\tau_r=\frac{2}{\pi}(-1)^{s+1}2\sqrt{q}/ce'_{2s+1}\left(\frac{\pi}{2}, q\right)$$

20.7.30
$$Ms_{2r+1}^{(1)}(z, q)=\frac{2}{\pi}\frac{(-1)^r}{se_{2r+1}\left(\frac{\pi}{2}, q\right)}\int_0^{\frac{\pi}{2}}\sin\,(2\sqrt{q}\sinh z\sin t)se_{2r+1}(t, q)dt$$

20.7.31
$$Ms_{2r+1}^{(1)}(z, q)=\frac{4}{\pi}\frac{\sqrt{q}(-1)^r}{se'_{2r+1}(0, q)}\int_0^{\frac{\pi}{2}}\sinh z\sin \mathrm{t}\cos\,(2\sqrt{q}\cosh z\cos t)se_{2r+1}(t, q)dt$$

20.7.32
$$Ms_{2r}^{(1)}(z, q)=\frac{4}{\pi}\sqrt{q}\,\frac{(-1)^{r+1}}{se'_{2r}(0, q)}\int_0^{\frac{\pi}{2}}\sin\,(2\sqrt{q}\cosh z\cos t)\,[\sinh z\sin t\; se_{2r}\,(t, q)]dt$$

20.7.33
$$Ms_{2r}^{(1)}(z, q)=\frac{4}{\pi}\frac{(-1)^r\sqrt{q}}{se'_{2r}\left(\frac{\pi}{2}, q\right)}\int_0^{\frac{\pi}{2}}\sin\,(2\sqrt{q}\sinh z\sin t)[\cosh z\cos t\; se_{2r}(t, q)]dt$$

Further with $w=\cosh z \cos \alpha \cos t+\sinh z \sin \alpha \sin t$

20.7.34
$$ce_r(\alpha, q)Mc_r^{(1)}(z, q)=\frac{(-1)^s(i)^{-p}}{2\pi}\int_0^{2\pi} e^{2i\sqrt{q}\,w}ce_r(t, q)dt$$

20.7.35
$$se_r(\alpha, q)Ms_r^{(1)}(z, q)=\frac{(-1)^s(-i)^p}{2\pi}\int_0^{2\pi} e^{2i\sqrt{q}\,w}se_r(t, q)dt.$$

The above can be differentiated with respect to α, and we obtain

20.7.36
$$ce_r'(\alpha, q)Mc_r^{(1)}(z, q)=\frac{(-1)^s(i)^{-p+1}\sqrt{q}}{\pi}\int_0^{2\pi} e^{2i\sqrt{q}\,w}\frac{\partial w}{\partial \alpha}ce_r(t, q)dt$$

20.7.37
$$se_r'(\alpha, q)Ms_r^{(1)}(z, q)=\frac{(-1)^{s+p}(i)^{-p+1}\sqrt{q}}{\pi}\int_0^{2\pi} e^{2i\sqrt{q}\,w}\frac{\partial w}{\partial \alpha}se_r(t, q)dt$$

Integrals With Infinite Limits

$$r=2s+p$$

In **20.7.38–20.7.41** below, z and q are positive.

20.7.38
$$Mc_r^{(1)}(z, q)=\gamma_r\int_0^{\infty} \sin\left(2\sqrt{q}\cosh z \cosh t+p\frac{\pi}{2}\right)Mc_r^{(1)}(t, q)dt$$

$$\gamma_r=2ce_{2s}\left(\frac{\pi}{2}, q\right)/\pi A_0^{2s}, \text{ if } p=0 \qquad \gamma_r=2ce_{2s+1}'\left(\frac{\pi}{2}, q\right)/\sqrt{q}\,\pi A_1^{2s+1}, \text{ if } p=1$$

20.7.39
$$Ms_r^{(1)}(z, q)=\gamma_r\int_0^{\infty} \sinh z \sinh t\left[\cos\left(2\sqrt{q}\cosh z \cosh t-p\frac{\pi}{2}\right)\right]Ms_r^{(1)}(t, q)dt$$

$$\gamma_r=-4se_{2s}'\left(\frac{\pi}{2}, q\right)/\sqrt{q}\pi B_2^{2s}, \text{ if } p=0 \qquad \gamma_r=-4se_{2s+1}\left(\frac{\pi}{2}, q\right)/\pi B_1^{2s+1}, \text{ if } p=1$$

20.7.40
$$Mc_r^{(2)}(z, q)=\gamma_r\int_0^{\infty} \cos\left(2\sqrt{q}\cosh z \cosh t-p\frac{\pi}{2}\right)Mc_r^{(1)}(t, q)dt$$

$$\gamma_r=-2ce_{2s}(\tfrac{1}{2}\pi, q)/\pi A_0^{2s}, \text{ if } p=0 \qquad \gamma_r=2ce_{2s+1}'(\tfrac{1}{2}\pi, q)/\pi\sqrt{q}A_1^{2s+1}, \text{ if } p=1$$

20.7.41
$$Ms_r^{(2)}(z, q)=\gamma_r\int_0^{\infty} \sin\left(2\sqrt{q}\cosh z \cosh t+p\frac{\pi}{2}\right)\sinh z \sinh t\, Ms_r^{(1)}(t, q)dt$$

$$\gamma_r=-4se_{2s}'(\tfrac{1}{2}\pi, q)/\sqrt{q}\,\pi B_2{}^{2s}, \text{ if } p=0 \qquad \gamma_r=4se_{2s+1}(\tfrac{1}{2}\pi, q)/\pi B_1{}^{2s+1}, \text{ if } p=1$$

Additional forms in [20.30], [20.36], [20.15].

20.8. Other Properties

Relations Between Solutions for Parameters q and $-q$

Replacing z by $\frac{1}{2}\pi-z$ in **20.1.1** one obtains

20.8.1 $y''+(a+2q\cos 2z)y=0$

Hence if $u(z)$ is a solution of **20.1.1** then $u(\frac{1}{2}\pi-z)$ satisfies **20.8.1.** It can be shown that

20.8.2

$$a(-\nu, q)=a(\nu, -q)=a(\nu, q), \nu \text{ not an integer}$$

$$c_{2m}^{\nu}(-q)=\rho(-1)^m c_{2m}^{\nu}(q), \nu \text{ not an integer}$$

(c_{2m} defined in **20.3.8**) and ρ depending on the normalization;

$$F_\nu(z, -q)=\rho e^{-i\nu\pi/2}F_\nu\left(z+\frac{\pi}{2}, q\right)=\rho e^{i\nu\pi/2}F_\nu\left(z-\frac{\pi}{2}, q\right)$$

20.8.3

$$a_{2r}(-q)=a_{2r}(q)\,;\; b_{2r}(-q)=b_{2r}(q), \text{ for integral } \nu$$

$$a_{2r+1}(-q)=b_{2r+1}(q),\; b_{2r+1}(-q)=a_{2r+1}(q)$$

20.8.4

$$ce_{2r}(z,\,-q)=(-1)^r ce_{2r}(\tfrac{1}{2}\pi-z,\,q)$$

$$ce_{2r+1}(z,\,-q)=(-1)^r se_{2r+1}(\tfrac{1}{2}\pi-z,\,q)$$

$$se_{2r+1}(z,\,-q)=(-1)^r ce_{2r+1}(\tfrac{1}{2}\pi-z,\,q)$$

$$se_{2r}(z,\,-q)=(-1)^{r-1} se_{2r}(\tfrac{1}{2}\pi-z,\,q)$$

For the coefficients associated with the above solutions for integral ν:

20.8.5

$$A_{2m}^{2r}(-q)=(-1)^{m-r}A_{2m}^{2r}(q)\,;\qquad B_{2m}^{2r}(-q)=(-1)^{m-r}B_{2m}^{2r}(q)$$

$$A_{2m+1}^{2r+1}(-q)=(-1)^{m-r}B_{2m+1}^{2r+1}(q)\,;\qquad B_{2m+1}^{2r+1}(-q)=(-1)^{m-r}A_{2m+1}^{2r+1}(q).$$

For the corresponding modified equation

20.8.6 $$y''-(a+2q\cosh 2z)y=0$$

20.8.7

$$M_\nu^{(j)}(z,\,-q)=M_\nu^{(j)}\left(z+i\frac{\pi}{2},\,q\right),$$

$M_\nu^{(j)}(z,\,q)$ defined in **20.4.12.**

For integral values of ν let

20.8.8

$$Ie_{2r}(z,\,q)=\sum_{k=0}^{\infty}(-1)^{k+s}A_{2k}[I_{k-s}(u_1)I_{k+s}(u_2)+I_{k+s}(u_1)I_{k-s}(u_2)]/A_{2s}\epsilon_s$$

$$Io_{2r}(z,\,q)=\sum_{k=1}^{\infty}(-1)^{k+s}B_{2k}[I_{k-s}(u_1)I_{k+s}(u_2)-I_{k+s}(u_1)I_{k-s}(u_2)]/B_{2s}$$

$$Ie_{2r+1}(z,\,q)=\sum_{k=0}^{\infty}(-1)^{k+s}B_{2k+1}[I_{k-s}(u_1)I_{k+s+1}(u_2)+I_{k+s+1}(u_1)I_{k-s}(u_2)]/B_{2s+1}$$

$$Io_{2r+1}(z,\,q)=\sum_{k=0}^{\infty}(-1)^{k+s}A_{2k+1}[I_{k-s}(u_1)I_{k+s+1}(u_2)-I_{k+s+1}(u_1)I_{k-s}(u_2)]/A_{2s+1}$$

20.8.9

$$Ke_{2r}(z,\,q)=\sum_{k=0}^{\infty}A_{2k}[I_{k-s}(u_1)K_{k+s}(u_2)+I_{k+s}(u_1)K_{k-s}(u_2)]/A_{2s}\epsilon_s$$

$$\ast\; Ko_{2r}(z,\,q)=\sum_{k=0}^{\infty}B_{2k}[I_{k-s}(u_1)K_{k+s}(u_2)$$
$$\ast\qquad -I_{k+s}(u_1)K_{k-s}(u_2)]/B_{2s}$$

$$Ke_{2r+1}(z,\,q)=\sum_{k=0}^{\infty}B_{2k+1}[I_{k-s}(u_1)K_{k+s+1}(u_2)-I_{k+s+1}(u_1)K_{k-s}(u_2)]/B_{2s+1}$$

$$Ko_{2r+1}(z,\,q)=\sum_{k=0}^{\infty}A_{2k+1}[I_{k-s}(u_1)K_{k+s+1}(u_2)+I_{k+s+1}(u_1)K_{k-s}(u_2)]/A_{2s+1}$$

where $I_m(x)$, $K_m(x)$ are the modified Bessel functions, u_1, u_2 are defined below **20.6.10.** Superscripts are omitted, $\epsilon_s=2$, if $s=0$, $\epsilon_s=1$ if $s\neq 0$.

Then for functions of first kind:

20.8.10

$$Mc_{2r}^{(1)}(z,\,-q)=(-1)^r Ie_{2r}(z,\,q)$$

$$Ms_{2r}^{(1)}(z,\,-q)=(-1)^r Io_{2r}(z,\,q)$$

$$Mc_{2r+1}^{(1)}(z,\,-q)=(-1)^r iIe_{2r+1}(z,\,q)$$

$$Ms_{2r+1}^{(1)}(z,\,-q)=(-1)^r iIo_{2r+1}(z,\,q)$$

For the Mathieu-Hankel function of first kind:

20.8.11

$$Mc_{2r}^{(3)}(z,\,-q)=(-1)^{r+1}i\frac{2}{\pi}Ke_{2r}(z,\,q)$$

$$Ms_{2r}^{(3)}(z,\,-q)=(-1)^{r+1}i\frac{2}{\pi}Ko_{2r}(z,\,q)$$

$$Mc_{2r+1}^{(3)}(z,\,-q)=(-1)^{r+1}\frac{2}{\pi}Ke_{2r+1}(z,\,q)$$

$$Ms_{2r+1}^{(3)}(z,\,-q)=(-1)^{r+1}\frac{2}{\pi}Ko_{2r+1}(z,\,q)$$

For $M_r^{(j)}(z,\,-q)$, $j=2,\,4$, one may use the definitions

$$M_r^{(2)}=-i(M_r^{(3)}-M_r^{(1)})\,;\; M_r=Mc_r \text{ or } Ms_r$$

also

$$M_r^{(4)}(z,\,-q)=2M_r^{(1)}(z,\,-q)-M_r^{(3)}(z,\,-q)$$

$M=Mc$ or Ms; for real z, q, $M_r^{(j)}(z,\,-q)$ are in general complex if $j=2,\,4$.

Zeros of the Functions for Real Values of q.

See [20.36], section **2.8** for further results.

Zeros of $ce_r(z,\,q)$ and $se_r(z,\,q)$, $Mc_r^{(1)}(z,\,q)$, $Ms_r^{(1)}(z,\,q)$.

In $0\leq z<\pi$, $ce_r(z,\,q)$ and $se_r(z,\,q)$ have r real zeros.

There are complex zeros if $q>0$.

If $z_0=x_0+iy_0$ is any zero of $ce_r(z,\,q)$, $se_r(z,\,q)$ in

$$-\frac{\pi}{2}<x_0<\frac{\pi}{2}, \text{ then } k\pi\pm z_0,\; k\pi\pm\bar{z}_0$$

are also zeros, k an integer.

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*See page II.

In the strip $-\frac{\pi}{2}<x_0<\frac{\pi}{2}$, the imaginary zeros of $ce_r(z, q)$, $se_r(z, q)$ are the real zeros of $Ce_r(z, q)$, $Se_r(z, q)$, hence also the real zeros of $Mc_r^{(1)}(z, q)$ and $Ms_r^{(1)}(z, q)$, respectively.

For small q, the large zeros of $Ce_r(z, q)$, $Se_r(z, q)$ approach the zeros of $J_r(2\sqrt{q}\cosh z)$.

Tabulation of Zeros

Ince [20.56] tabulates the first "non-trivial" zero $\left(\text{i.e. different from } 0, \frac{\pi}{2}, \pi\right)$ for $ce_r(z)$, $se_r(z)$, $r=2(1)5$ and for $se_6(z)$ to within $^{\circ}10^{-4}$, for $q=0(1)10(2)40$. He also gives the "turning" points (zeros of the derivative) and also expansions for them for small q. Wiltse and King [20.61,2] tabulate the first two (non-trivial) zeros of $Mc_r^{(1)}(z, q)$ and $Ms_r^{(1)}(z, q)$ and of their derivatives $r=0$, 1, 2 for 6 or 7 values of q between .25 and 10. The graphs reproduced here indicate their location.

Between two real zeros of $Mc_r^{(1)}(z, q)$, $Ms_r^{(1)}(z, q)$ there is a zero of $Mc_r^{(2)}(z, q)$, $Ms_r^{(2)}(z, q)$, respectively. No tabulation of such zeros exists yet.

Available tables are described in the References.

The most comprehensive tabulation of the characteristic values a_r, b_r (in a somewhat different notation) and of the coefficients proportional to A_m and B_m as defined in **20.5.4** and **20.5.5** can be found in [20.58]. In addition, the table contains certain important "joining factors", with the aid of which it is possible to obtain values of $Mc_r^{(j)}(z, q)$ and $Ms_r^{(j)}(z, q)$ as well as their derivatives, at $x=0$. Values of the functions $ce_r(x, q)$ and $se_r(x, q)$ for orders up to five or six can be found in [20.56]. Tabulations of less extensive character, but important in some aspects, are outlined in the other references cited. In this chapter only representative values of the various functions are given, along with several graphs.

Special Values for Arguments 0 and $\frac{\pi}{2}$

20.8.12

$$ce_{2r}\left(\frac{\pi}{2}, q\right)=(-1)^r g_{e,2r}(q) A_0^{2r}(q)\sqrt{\frac{\pi}{2}}$$

$$ce'_{2r+1}\left(\frac{\pi}{2}, q\right)=(-1)^{r+1} g_{e,2r+1}(q) A_1^{2r+1}(q)\sqrt{\frac{\pi}{2}q}$$

$$se'_{2r}\left(\frac{\pi}{2}, q\right)=(-1)^r g_{0,2r}(q) B_2^{2r}(q)\cdot q\sqrt{\frac{\pi}{2}}$$

$$se_{2r+1}\left(\frac{\pi}{2}, q\right)=(-1)^r g_{0,2r+1}(q) B_1^{2r+1}(q)\sqrt{\frac{\pi}{2}q}$$

$$Mc_r^{(1)}(0, q)=\sqrt{\frac{2}{\pi}}\frac{1}{g_{e,r}(q)}$$

$$Mc_r^{(2)}(0, q)=-\sqrt{\frac{2}{\pi}}f_{e,r}(q)/g_{e,r}(q)$$

$$\frac{d}{dz}[Mc_r^{(2)}(z, q)]_{z=0}=\sqrt{\frac{2}{\pi}}\,g_{e,r}(q)$$

$$\frac{d}{dz}[Ms_r^{(1)}(z, q)]_{z=0}=\sqrt{\frac{2}{\pi}}\frac{1}{g_{o,r}(q)}$$

$$\frac{d}{dz}\left[Ms_r^{(2)}(z, q)\right]_{z=0}=\sqrt{\frac{2}{\pi}}f_{o,r}(q)/g_{o,r}(q)$$

$$Ms_r^{(2)}(z, q)=-g_{o,r}(q)\sqrt{\frac{2}{\pi}}$$

The functions $f_{o,r}$, $g_{o,r}$, $f_{e,r}$, $g_{e,r}$ are tabulated in [20.58] for $q\leq 25$.

20.9. Asymptotic Representations

The representations given below are applicable to the *characteristic solutions*, for real values of q, unless otherwise noted. The Floquet exponent ν is defined below, as in [20.36] to be as follows:

In solutions associated with a_r: $\nu=r$
In solutions associated with b_r: $\nu=-r$.

For the functions defined in **20.6.7–20.6.10:**

20.9.1

$$\begin{matrix} Mc_r^{(3)}(z, q) \\ (-1)^r Ms_r^{(3)}(z, q) \end{matrix}$$

$$\sim\frac{e^{i\left(2\sqrt{q}\cosh z-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)}}{\pi^{\frac{1}{2}}q^{1/4}(\cosh z-\sigma)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{D_m}{[-4i\sqrt{q}(\cosh z-\sigma)]^m}$$

where $D_{-1}=D_{-2}=0$; $D_0=1$, and the coefficients D_m are obtainable from the following recurrence formula:

20.9.2

$$(m+1)D_{m+1}+\left[\left(m+\frac{1}{2}\right)^2-\left(m+\frac{1}{4}\right)8i\sqrt{q}\,\sigma\right.$$
$$\left.+2q-a\right]D_m+\left(m-\frac{1}{2}\right)[16q(1-\sigma^2)-8i\sqrt{q}\,\sigma m]D_{m-1}$$
$$+4q(2m-3)(2m-1)(1-\sigma^2)D_{m-2}=0$$

20.9.3

$$\begin{matrix} Mc_r^{(4)}(z, q) \\ (-1)^r Ms_r^{(4)}(z, q) \end{matrix}$$

$$\sim\frac{e^{-i\left[2\sqrt{q}\cosh z-\frac{1}{2}\nu\pi-\frac{1}{4}\pi\right]}}{\pi^{\frac{1}{2}}q^{1/4}(\cosh z-\sigma)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{d_m}{[4i\sqrt{q}(\cosh z-\sigma)]^m}$$

$d_{-1}=d_{-2}=0$; $d_0=1$, and

20.9.4

$$(m+1)d_{m+1}+\left[\left(m+\frac{1}{2}\right)^2+\left(m+\frac{1}{4}\right)8i\sqrt{q}\,\sigma\right.$$
$$\left.+2q-a\right]d_m+\left(m-\frac{1}{2}\right)[16q(1-\sigma^2)+8i\sqrt{q}\,\sigma m]d_{m-1}$$
$$+4q(2m-3)(2m-1)(1-\sigma^2)d_{m-2}=0.$$

In the above

$$-2\pi<\arg\sqrt{q}\cosh z<\pi$$
$$|\cosh z-\sigma|>|\sigma\pm1|,\ \mathscr{R}z>0,$$

but σ is otherwise arbitrary. If $\sigma^2=1$, **20.9.2** and **20.9.4** become three-term recurrence relations.

Formulas **20.9.1** and **20.9.3** are valid for arbitrary a, q, provided ν is also known; they give multiples of **20.4.12,** normalized so as to approach the corresponding Hankel functions $H_\nu^{(1)}(\sqrt{q}e^z)$, $H_\nu^{(2)}(\sqrt{q}e^z)$, as $z\to\infty$. See [20.36], section **2.63.** The formula is especially useful if $|\cosh z|$is large and q is not too large; thus if $\sigma=-1$, the absolute ratio of two successive terms in the expansion is essentially

$$\left|\left(\frac{\sqrt{q}}{m}+\frac{m}{4\sqrt{q}}+2\right)/(\cosh z+1)\right|.$$

If a, q, z, ν are real, the real and imaginary components of $Mc_r^{(3)}(z,q)$ are $Mc_r^{(1)}(z,q)$ and $Mc_r^{(2)}(z,q)$, respectively; similarly for the components of $Ms_r^{(3)}(z,q)$. If the parameters are complex

20.9.5 $$Mc_r^{(1)}(z,q)=\tfrac{1}{2}[Mc_r^{(3)}(z,q)+Mc_r^{(4)}(z,q)]$$

20.9.6 $$Mc_r^{(2)}(z,q)=-\frac{i}{2}[Mc_r^{(3)}(z,q)-Mc_r^{(4)}(z,q)]$$

Replacing c by s in the above will yield corresponding relations among $Ms_r^{(j)}(z,q)$.

Formulas in which the parameter a does not enter explicitly:

Goldstein's Expansions

20.9.7

$$Mc_r^{(3)}(z,q)\sim iMs_{r+.}^{(3)}(z,q)$$
$$\approx[F_0(z)-iF_1(z)]e^{i\phi}/\pi^{\frac{1}{2}}q^{\frac{1}{4}}(\cosh z)^{\frac{1}{2}}$$

where

20.9.8

$$\phi=2\sqrt{q}\sinh z-\frac{1}{2}(2r+1)\arctan\sinh z,$$
$$\mathscr{R}z>0,\ q>>1,\ w=2r+1$$

20.9.9

$$F_0(z)\sim1+\frac{w}{8\sqrt{q}\cosh^2 z}$$
$$+\frac{1}{2048q}\left[\frac{w^4+86w^2+105}{\cosh^4 z}-\frac{w^4+22w^2+57}{\cosh^2 z}\right]$$
$$+\frac{1}{16384q^{3/2}}\left[\frac{-(w^5+14w^3+33w)}{\cosh^2 z}\right.$$
$$\left.-\frac{(2w^5+124w^3+1122w)}{\cosh^4 z}+\frac{3w^5+290w^3+1627w}{\cosh^6 z}\right]+\dots$$

20.9.10

$$F_1(z)\sim\frac{\sinh z}{\cosh^2 z}\left[\frac{w^2+3}{32\sqrt{q}}+\frac{1}{512q}\left(w^3+3w+\frac{4w^3+44w}{\cosh^2 z}\right)\right.$$
$$+\frac{1}{16384q^{3/2}}\left\{5w^4+34w^2+9\right.$$
$$-\frac{(w^6-47w^4+667w^2+2835)}{12\cosh^2 z}$$
$$\left.\left.+\frac{(w^6+505w^4+12139w^2+10395)}{12\cosh^4 z}\right\}\right]+\dots$$

See [20.18] for details and an added term in $q^{-5/2}$; a correction to the latter is noted in [20.58].

The expansions **20.9.7** are especially useful when q is large and z is bounded away from zero. The order of magnitude of $Mc_r^2(0,q)$ cannot be obtained from the expansion. The expansion can also be used, with some success, for $z=ix$, when q is large, if $|\cos x|>>0$; they fail at $x=\frac{1}{2}\pi$. Thus, if q, x are real, one obtains

20.9.11

$$ce_r(x,q)\sim\frac{ce_r(0,q)2^{r-\frac{1}{2}}}{F_0(0)}\{W_1[P_0(x)-P_1(x)]$$
$$+W_2[P_0(x)+P_1(x)]\}$$

20.9.12

$$se_{r+1}(x,q)\sim se'_{r+1}(0,q)\tau_{r+1}\{W_1[P_0(x)-P_1(x)]$$
$$-W_2[P_0(x)+P_1(x)]\}$$

In the above, $P_0(x)$ and $P_1(x)$ are obtainable from $F_0(z)$, $F_1(x)$ in **20.9.9–20.9.10** by replacing $\cosh z$ with $\cos x$ and $\sinh z$ with $\sin x$. Thus $P_0(x)=F_0(ix)$; $P_1(x)=-iF_1(ix)$:

20.9.13

$$W_1=e^{2\sqrt{q}\sin x}[\cos(\tfrac{1}{2}x+\tfrac{1}{4}\pi)]^{2r+1}/(\cos x)^{r+1}$$

$$W_2=e^{-2\sqrt{q}\sin x}[\sin(\tfrac{1}{2}x+\tfrac{1}{4}\pi)]^{2r+1}/(\cos x)^{r+1}$$

20.9.14

$$\tau_{r+1}\sim 2^{r-\frac{1}{2}}\Big/\left[2\sqrt{q}-\tfrac{1}{4}w-\frac{(2w^2+3)}{64\sqrt{q}}-\frac{(7w^3+47w)}{1024q}-\dots\right]$$

See **20.9.23–20.9.24** for expressions relating to $ce_r(0, q)$ and $se_r'(0, q)$. When $|\cos x|>\sqrt{4r+2}/q^{\frac{1}{4}}$, **20.9.11–20.9.12** are useful. The approximations become poorer as r increases.

Expansions in Terms of Parabolic Cylinder Functions

(Good for angles close to $\frac{1}{2}\pi$, for large values of q, especially when $|\cos x|<2^{\frac{1}{2}}/q^{\frac{1}{4}}$.) Due to Sips [20.44–20.46].

20.9.15 $$ce_r(x, q)\sim C_r[Z_0(\alpha)+Z_1(\alpha)]$$

20.9.16

$$se_{r+1}(x, q)\sim S_r[Z_0(\alpha)-Z_1(\alpha)]\sin x, \qquad \alpha=2q^{\frac{1}{4}}\cos x.$$

Let $D_k=D_k(\alpha)=(-1)^k e^{\frac{1}{4}\alpha^2}\dfrac{d^k}{d\alpha^k}e^{-\frac{1}{4}\alpha^2}$.

20.9.17

$$Z_0(\alpha)\sim D_r+\frac{1}{4q^{\frac{1}{4}}}\left[-\frac{D_{r+4}}{16}+\frac{3}{2}\binom{r}{4}D_{r-4}\right]$$
$$+\frac{1}{16q}\left[\frac{D_{r+8}}{512}-\frac{(r+2)D_{r+4}}{16}+\frac{3}{2}(r-1)\binom{r}{4}D_{r-4}\right.$$
$$\left.+\frac{315}{4}\binom{r}{8}D_{r-8}\right]+\dots$$

20.9.18

$$Z_1(\alpha)\sim\frac{1}{4q^{\frac{1}{4}}}\left[-\frac{1}{4}D_{r+2}-\frac{r(r-1)}{4}D_{r-2}\right]$$
$$+\frac{1}{16q}\left[\frac{D_{r+6}}{64}+\frac{(r^2-25r-36)}{64}D_{r+2}\right.$$
$$\left.+\frac{r(r-1)(-r^2-27r+10)}{64}D_{r-2}-\frac{45}{4}\binom{r}{6}D_{r-6}+\dots\right]$$

20.9.19

$$C_r\sim\left(\frac{\pi}{2}\right)^{\frac{1}{4}}q^{\frac{1}{8}}/(r!)^{\frac{1}{2}}\left[1+\frac{2r+1}{8q^{\frac{1}{2}}}\right.$$
$$\left.+\frac{r^4+2r^3+263r^2+262r+108}{2048q}+\frac{f_1}{16384q^{\frac{3}{2}}}+\dots\right]^{-\frac{1}{2}}$$
$$f_1=6r^5+15r^4+1280r^3+1905r^2+1778r+572$$

20.9.20

$$S_r\sim\left(\frac{\pi}{2}\right)^{\frac{1}{4}}q^{\frac{1}{8}}/(r!)^{\frac{1}{2}}\left[1-\frac{2r+1}{8q^{\frac{1}{2}}}\right.$$
$$\left.+\frac{r^4+2r^3-121r^2-122r-84}{2048q}+\frac{f_2}{16384q^{\frac{3}{2}}}+\dots\right]^{-\frac{1}{2}}$$
$$f_2=2r^5+5r^4-416r^3-629r^2-1162r-476$$

It should be noted that **20.9.15** is also valid as an approximation for $se_{r+1}(x, q)$, but **20.9.16** may give slightly better results. See [20.4.]

Explicit Expansions for Orders 0, 1, to Terms in $q^{-3/2}$ (q Large)

20.9.21 For $r=0$:

$$Z_0\sim D_0-\frac{D_4}{64\sqrt{q}}+\frac{1}{16q}\left(-\frac{D_4}{8}+\frac{D_8}{512}\right)\ ^*$$
$$+\frac{1}{64q^{3/2}}\left(-\frac{99D_4}{256}+\frac{3D_8}{256}-\frac{D_{12}}{24576}\right)+\dots$$

$$Z_1\sim\frac{-D_2}{16\sqrt{q}}+\frac{1}{16q}\left(-\frac{9D_2}{16}+\frac{D_6}{64}\right)$$
$$+\frac{1}{64q^{3/2}}\left(-\frac{61D_2}{32}+\frac{25D_6}{256}-\frac{5D_{10}}{10240}\right)+\dots$$

20.9.22 For $r=1$:

$$Z_0\sim D_1-\frac{D_5}{64\sqrt{q}}+\frac{1}{16q}\left(-\frac{3D_5}{16}+\frac{D_9}{512}\right)$$
$$+\frac{1}{64q^{3/2}}\left(-\frac{207D_5}{256}+\frac{D_9}{64}-\frac{D_{13}}{24576}\right)+\dots$$

$$Z_1\sim\frac{-D_3}{16\sqrt{q}}+\frac{1}{16q}\left(-\frac{15D_3}{16}+\frac{D_7}{64}\right)$$
$$+\frac{1}{64q^{3/2}}\left(-\frac{153D_3}{32}+\frac{35D_7}{256}-\frac{D_{11}}{2048}\right)+\dots$$

Formulas Involving $ce_r(0, q)$ and $se_r(0, q)$

20.9.23

$$\frac{ce_0(0, q)}{ce_0(\frac{1}{2}\pi, q)}\sim 2\sqrt{2}\,e^{-2\sqrt{q}}\left(1+\frac{1}{16\sqrt{q}}+\frac{9}{256q}+\dots\right)$$

$$\frac{ce_2(0, q)}{ce_2(\frac{1}{2}\pi, q)}\sim -32q\sqrt{2}\,e^{-2\sqrt{q}}\left(1-\frac{1}{16\sqrt{q}}+\frac{29}{128q}+\dots\right)$$

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*See page II.

$$\frac{ce_1(0,q)}{ce_1'(\frac{1}{2}\pi,q)}\sim -4\sqrt{2}e^{-2\sqrt{q}}\left(1+\frac{3}{16\sqrt{q}}+\frac{45}{256q}+\dots\right)$$

$$\frac{ce_3(0,q)}{ce_3'(\frac{1}{2}\pi,q)}\sim \frac{64}{3}\,q\sqrt{2}\,e^{-2\sqrt{q}}\left(1-\frac{3}{16\sqrt{q}}+\frac{47}{128q}+\dots\right)$$

20.9.24

$$\frac{se_1'(0,q)}{se_1(\frac{1}{2}\pi,q)}\sim 4\,q\sqrt{2}\,e^{-2\sqrt{q}}\left(1-\frac{3}{16\sqrt{q}}-\frac{11}{256q}+\dots\right)$$

$$\frac{se_3'(0,q)}{se_3(\frac{1}{2}\pi,q)}\sim -64\,q\sqrt{2}\,e^{-2\sqrt{q}}\left(1-\frac{21}{16\sqrt{q}}-\frac{17}{128q}+\dots\right)$$

$$\frac{se_2'(0,q)}{se_2'(\frac{1}{2}\pi,q)}\sim -8\,q\sqrt{2}\,e^{-2\sqrt{q}}\left(1-\frac{9}{16\sqrt{q}}-\frac{39}{256q}+\dots\right)$$

$$\frac{se_4'(0,q)}{se_4'(\frac{1}{2}\pi,q)}\sim \frac{128}{3}\,q\sqrt{2}e^{-2\sqrt{q}}\left(1-\frac{31}{16\sqrt{q}}-\frac{15}{128q}+\dots\right)$$

For higher orders, these ratios are increasingly more difficult to obtain. One method of estimating values at the origin is to evaluate both **20.9.11** and **20.9.15** for some x where both expansions are satisfactory, and so to use **20.9.11** as a means to solve for $ce_r(0,q)$; similarly for $se_r'(0,q)$.

Other asymptotic expansions, valid over various regions of the complex z-plane, for real values of a, q, have been given by Langer [20.25]. It is not always easy, however, to determine the linear combinations of Langer's solutions which coincide with those defined here.

20.10. Comparative Notations

	This Volume	[20.58] NBS	[20.59] Stratton-Morse, etc.	[20.36] Meixner and Schäfke	[20.30] McLachlan	[20.15] Bateman Manuscript	Comments
Parameters in **20.1.1**	a	$b=a+2q$	b	λ	a	h	
	q	$s=4q$	$c=2\sqrt{q}$	h^2	q	θ	
	a_r	$be_r=a_r+2q$	$b_r=a_r+2q$	a_r	a_r	a_r	
	b_r	$bo_r=b_r+2q$	$b'_r=b_r+2q$	b_r	b_r	b_r	
Periodic Solutions, of **20.1.1**:							
Even	$ce_r(z,q)$	$A^r\, Se_r(s,x)$ *	$A^r\, Se_r^{(1)}(c,\cos x)$ *	$ce_r(z,h^2)$ *	$ce_r(z,q)$	$ce_r(z,\theta)$	See Note 1.
Odd	$se_r(z,q)$	$B^r\, So_r(s,x)$ *	$A^r\, So_r^{(1)}(c,\cos x)$ *	$se_r(z,h^2)$ *	$se_r(z,q)$	$se_r(z,\theta)$	
Coefficients in Periodic Solutions:							
Even	$A_m^r(q)$	$A^r\, De_m^r(s)$ *	$A^r\, D_m^r$ *	A_m^r	A_m^r	A_m^r	
Odd	$B_m^r(q)$	$B^r\, Do_m^r(s)$ *	$B^r F_m^r$ *	B_m^r	B_m^r	B_m^r	
$\frac{1}{\pi}\int_0^{2\pi} y^2\,dx$, y is the Standard Solution of **20.1.1.**	1	$(A^r)^{-2}$ or $(B^r)^{-2}$	$(A^r)^{-2}$ or $(B^r)^{-2}$	1	1	1	See Note 1.
Floquet's Solutions **20.3.8**	$F_\nu(z)$			$me_\nu(z,h^2)$	$\phi(z)$		
Characteristic Exponent	ν	$\mu=i\nu$		ν	$\mu=i\nu$	$\mu=i\nu$	
Normalizations of Floquet's Solutions.	Unspecified			$\frac{1}{\pi}\int_0^{\pi}(me_\nu(z,h^2)me_{-\nu}(z,h^2)=1$			
Solutions of Modified Equation **20.1.2.**	$Ce_r(z,q)$	$Ag_{e,r}(s)Je_r(s,q)$	$Ag_{e,r}(s)Je_r(c,\cosh x)$	$Ce_r(z,q)$	$Ce_r(z,q)$	$Ce_r(z,\theta)$	
	$Se_r(z,q)$	$Bg_{o,r}(s)Jo_r(s,q)$	$Bg_{o,r}(s)Jo_r(c,\cosh x)$	$Se_r(z,q)$	$Se_r(z,q)$	$Se_r(z,\theta)$	
	$Mc_r^{(1)}(z,q)$	$\sqrt{\frac{2}{\pi}}\, Je_r(s,z)$	$\sqrt{\frac{2}{\pi}}\, Je_r(c,\cosh z)$	$Mc_r^{(1)}(z,h)$	$\sqrt{\frac{2}{\pi}}\, Ce_r(z,q)/Ag_{e,r}(q)$	$\sqrt{\frac{2}{\pi}}\, Ce_r(z,\theta)/Ag_{e,r}(q)$	
	$Ms_r^{(1)}(z,q)$	$\sqrt{\frac{2}{\pi}}\, Jo_r(s,z)$	$\sqrt{\frac{2}{\pi}}\, Jo_r(c,\cosh z)$	$Ms_r^{(1)}(z,h)$	$\sqrt{\frac{2}{\pi}}\, Se_r(z,q)/Bg_{o,r}(q)$	$\sqrt{\frac{2}{\pi}}\, Se_r(z,\theta)/Bg_{o,r}(q)$	
	$Mc_r^{(2)}(z,q)$	$\sqrt{\frac{2}{\pi}}\, Ne_r(s,z)$	$\sqrt{\frac{2}{\pi}}\, Ne_r(c,\cosh z)$	$Mc_r^{(2)}(z,h)$	$\sqrt{\frac{2}{\pi}}\, Fey_r(z,q)/Ag_{e,r}(q)$	$\sqrt{\frac{2}{\pi}}\, Fey_r(z,\theta)/Ag_{e,r}(q)$	
	$Ms_r^{(2)}(z,q)$	$\sqrt{\frac{2}{\pi}}\, No_r(s,z)$	$\sqrt{\frac{2}{\pi}}\, No_r(c,\cosh z)$	$Ms_r^{(2)}(z,h)$	$\sqrt{\frac{2}{\pi}}\, Gey_r(z,q)/Bg_{o,r}(q)$	$\sqrt{\frac{2}{\pi}}\, Gey_r(z,\theta)/Bg_{o,r}(q)$	
Joining Factors	$\sqrt{2/\pi}/Mc_r^{(1)}(0,q)$	$g_{e,r}(s)$	$\sqrt{2\pi}\,\lambda_r^{(e)}$	$\sqrt{2/\pi}/Mc_r^{(1)}(0,h)$	$(-1)^r p_r\sqrt{\frac{2}{\pi}}/A$	Same as [20.30]	See Note 2.
	$\sqrt{2/\pi}/\frac{d}{dz}[Ms_r^{(1)}(z,q)]_{z=0}$	$g_{o,r}(s)$	$\sqrt{2\pi}\,\lambda_r^{(0)}$	$\sqrt{2/\pi}/\frac{d}{dz}[Ms_r^{(1)}(z,h)]_{z=0}$	$(-1)^r s_r\sqrt{\frac{2}{\pi}}/B$		
	$-Mc_r^{(2)}(0,q)/Mc_r^{(1)}(0,q)$	$f_{e,r}(s)$	$-\frac{2}{\pi}\frac{K'_1}{K_1}$	$-Mc_r^{(2)}(0,h)/Mc_r^{(1)}(0,h)$	$\frac{-Fey_r(0,q)}{Ce_r(0,q)}$	Same as [20.30]	See Note 3.
	$\left[\frac{\frac{d}{dz}Ms_r^{(2)}(z,q)}{\frac{d}{dz}Ms_r^{(1)}(z,q)}\right]_{z=0}$	$f_{o,r}(s)$	$\frac{2}{\pi}\frac{K'_1}{K_1}$	Same as this volume	$\left[\frac{\frac{d}{dz}Gey_r(z,q)}{\frac{d}{dz}Se_r(z,q)}\right]_{z=0}$	Same as [20.30]	

NOTE: 1. The conversion factors A^r and B^r are tabulated in [20.58] along with the coefficients.
2. The multipliers p_r and s_r are defined in [20.30], Appendix **1**, section **3**, equations **3, 4, 5, 6.**
3. See [20.59], sections **(5.3)** and **(5.5)**. In eq. **(316)** of **(5.5)**, the first term should have a minus sign.

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*See page II.

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[20.59] J. A. Stratton, P. M. Morse, L. J. Chu and R. A. Hutner, Elliptic cylinder and spheroidal wave functions (John Wiley & Sons, Inc., New York, N.Y., 1941). Theory and tables for b_0, b_1, b_2, b_3, b_4, b_1', b_2', b_3', b_4', and coefficients for $Se_r(s, x)$ and $So_r(s, x)$ for $c=0(.2)4.4$ and $.5(1)4.5$; mostly 5S; $c=2q^{\frac{1}{2}}$, $b_r=a_r+2q$, $b_r'=b_r+2q$.

[20.60] T. Tamir, Characteristic exponents of Mathieu equations, Math. Comp. **16**, 77 (1962). The Floquet exponent ν_r of the first three stable regions; namely $r=0, 1, 2$; $q=.1(.1)2.5$; $a=r(.1)r+1$, 5D. *

[20.61] J. C. Wiltse and M. J. King, Values of the Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Technical Report AF–53, Baltimore, Md. (1958). (Notation of [20.58] used: $ce_n(v, q)/A$, $se_n(v, q)/B$ for 12 values of q between .25 and 10 and from 8 to 14 values of v; $\sqrt{\pi/2}\ Mc_r^j(u, q)$, $\sqrt{\pi/2}\ Ms_r^j(u, q)$, $j=1, 2$ for 6 to 8 values of q between .25 and 10 and about 20 values of u, $r=0, 1, 2$; $\sqrt{\pi/2}\ Mc_r^{(3)}(-|u|, q)$, $\sqrt{\pi/2}\ Ms_r^{(3)}(-|u|, q)$, $r=0, 1, 2$ for about 9 values of u and q, 2 to 4 D in all.

[20.62] J. C. Wiltse and M. J. King, Derivatives, zeros, and other data pertaining to Mathieu functions, The Johns Hopkins Univ. Radiation Laboratory Technical Report AF–57, Baltimore, Md. (1958).

[20.63] S. J. Zaroodny, An elementary review of the Mathieu-Hill equation of real variable based on numerical solutions, Ballistic Research Laboratory Memorandum Report 878, Aberdeen Proving Ground, Md. (1955). Chart of the characteristic exponent.

See also [20.18]. It contains, among other tabulations, values of a_r, b_r and coefficients for $ce_r(x, q)$, $se_r(x, q)$, $q=40(20)100(50)200$; 5D, $r\leq 2$.

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21. Spheroidal Wave Functions

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Contents

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21. Spheroidal Wave Functions

Mathematical Properties

21.1. Definition of Elliptical Coordinates

21.1.1
$$\xi=\frac{r_1+r_2}{2f};\quad \eta=\frac{r_1-r_2}{2f}$$

r_1 and r_2 are the distances to the foci of a family of confocal ellipses and hyperbolas; $2f$ is the distance between foci.

21.1.2
$$a=f\xi,\ b=f\sqrt{\xi^2-1},\qquad e=\frac{f}{a}$$

a=semi-major axis; b=semi-minor axis; e=eccentricity.

Equation of Family of Confocal Ellipses

21.1.3
$$\frac{x^2}{\xi^2}+\frac{y^2}{\xi^2-1}=f^2 \qquad (1<\xi<\infty)$$

Equation of Family of Confocal Hyperbolas

21.1.4
$$\frac{x^2}{\eta^2}-\frac{y^2}{1-\eta^2}=f^2 \qquad (-1<\eta<1)$$

Relations Between Cartesian and Elliptical Coordinates

21.1.5
$$x=f\xi\eta;\ y=f\sqrt{(\xi^2-1)(1-\eta^2)}$$

21.2. Definition of Prolate Spheroidal Coordinates

If the system of confocal ellipses and hyberbolas referred to in **21.1.3** and **21.1.4** revolves around the major axis, then

21.2.1
$$\frac{x^2}{\xi^2}+\frac{r^2}{\xi^2-1}=f^2;\qquad \frac{x^2}{\eta^2}-\frac{r^2}{1-\eta^2}=f^2$$

$$y=r\cos\phi;\ \ z=r\sin\phi;\ \ 0\leq\phi\leq 2\pi$$

where ξ, η and ϕ are prolate spheroidal coordinates.

Relations Between Cartesian and Prolate Spheroidal Coordinates

21.2.2
$$x=f\xi\eta;\ y=f\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi;$$
$$z=f\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi$$

21.3. Definition of Oblate Spheroidal Coordinates

If the system of confocal ellipses and hyperbolas referred to in **21.1.3** and **21.1.4** revolves around the minor axis, then

21.3.1
$$\frac{r^2}{\xi^2}+\frac{y^2}{\xi^2-1}=f^2;\qquad \frac{r^2}{\eta^2}-\frac{y^2}{1-\eta^2}=f^2$$

$$z=r\cos\phi;\ x=r\sin\phi;\ 0\leq\phi\leq 2\pi$$

where ξ, η and ϕ are oblate spheroidal coordinates.

Relations Between Cartesian and Oblate Spheroidal Coordinates

21.3.2
$$x=f\xi\eta\sin\phi;\ y=f\sqrt{(\xi^2-1)(1-\eta^2)};\ z=f\xi\eta\cos\phi$$

21.4. Laplacian in Spheroidal Coordinates

21.4.1
$$\nabla^2=\frac{1}{h_\xi h_\eta h_\phi}\left[\frac{\partial}{\partial\xi}\left(\frac{h_\eta h_\phi}{h_\xi}\frac{\partial}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{h_\xi h_\phi}{h_\eta}\frac{\partial}{\partial\eta}\right)+\frac{\partial}{\partial\phi}\left(\frac{h_\xi h_\eta}{h_\phi}\frac{\partial}{\partial\phi}\right)\right] \qquad *$$

$$h_\xi^2=\left(\frac{\partial x}{\partial\xi}\right)^2+\left(\frac{\partial y}{\partial\xi}\right)^2+\left(\frac{\partial z}{\partial\xi}\right)^2$$

$$h_\eta^2=\left(\frac{\partial x}{\partial\eta}\right)^2+\left(\frac{\partial y}{\partial\eta}\right)^2+\left(\frac{\partial z}{\partial\eta}\right)^2$$

$$h_\phi^2=\left(\frac{\partial x}{\partial\phi}\right)^2+\left(\frac{\partial y}{\partial\phi}\right)^2+\left(\frac{\partial z}{\partial\phi}\right)^2$$

Metric Coefficients for Prolate Spheroidal Coordinates

21.4.2
$$h_\xi=f\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}};\ h_\eta=f\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}};\ h_\phi=f\sqrt{(\xi^2-1)(1-\eta^2)} \qquad *$$

Metric Coefficients for Oblate Spheroidal Coordinates

21.4.3
$$h_\xi=f\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}};\ h_\eta=f\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}};\ h_\phi=f\xi\eta \qquad *$$

21.5. Wave Equation in Prolate and Oblate Spheroidal Coordinates

Wave Equation in Prolate Spheroidal Coordinates

21.5.1
$$\nabla^2\Phi+k^2\Phi=\frac{\partial}{\partial\xi}\left[(\xi^2-1)\,\frac{\partial\Phi}{\partial\xi}\right]+\frac{\partial}{\partial\eta}\left[(1-\eta^2)\,\frac{\partial\Phi}{\partial\eta}\right]$$
$$+\frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\frac{\partial^2\Phi}{\partial\phi^2}+c^2(\xi^2-\eta^2)\Phi=0$$
$$\left(c=\frac{1}{2}fk\right)$$

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*See page II.

Wave Equation in Oblate Spheroidal Coordinates

21.5.2

$$\nabla^2\Phi+k^2\Phi=\frac{\partial}{\partial\xi}\left[(\xi^2+1)\frac{\partial\Phi}{\partial\xi}\right]+\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\Phi}{\partial\eta}\right]$$

$$+\frac{\xi^2+\eta^2}{(\xi^2+1)(1-\eta^2)}\frac{\partial^2\Phi}{\partial\phi^2}+c^2(\xi^2+\eta^2)\Phi=0$$

$$\left(c=\frac{1}{2}fk\right)$$

21.5.2 may be obtained from **21.5.1** by the transformations

$$\xi\rightarrow\pm i\xi,\ c\rightarrow\mp ic.$$

21.6. Differential Equations for Radial and Angular Prolate Spheroidal Wave Functions

If in **21.5.1** we put

$$\Phi=R_{mn}(c,\xi)S_{mn}(c,\eta)\begin{matrix}\cos\\ \sin\end{matrix}m\phi$$

then the "radial solution" $R_{mn}(c,\xi)$ and the "angular solution" $S_{mn}(c,\eta)$ satisfy the differential equations

21.6.1

$$\frac{d}{d\xi}\left[(\xi^2-1)\frac{d}{d\xi}R_{mn}(c,\xi)\right]$$

$$-\left(\lambda_{mn}-c^2\xi^2+\frac{m^2}{\xi^2-1}\right)R_{mn}(c,\xi)=0$$

21.6.2

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_{mn}(c,\eta)\right]$$

$$+\left(\lambda_{mn}-c^2\eta^2-\frac{m^2}{1-\eta^2}\right)S_{mn}(c,\eta)=0$$

where the separation constants (or eigenvalues) λ_{mn} are to be determined so that $R_{mn}(c,\xi)$ and $S_{mn}(c,\eta)$ are finite at $\xi=\pm1$ and $\eta=\pm1$ respectively.

(**21.6.1** and **21.6.2** are identical. Radial and angular prolate spheroidal functions satisfy the same differential equation over different ranges of the variable.)

Differential Equations for Radial and Angular Oblate Spheroidal Functions

21.6.3

$$\frac{d}{d\xi}\left[(\xi^2+1)\frac{d}{d\xi}R_{mn}(c,\xi)\right]$$

$$-\left(\lambda_{mn}-c^2\xi^2-\frac{m^2}{\xi^2+1}\right)R_{mn}(c,\xi)=0$$

21.6.4

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{d}{d\eta}S_{mn}(c,\eta)\right]$$

$$+\left(\lambda_{mn}+c^2\eta^2-\frac{m^2}{1-\eta^2}\right)S_{mn}(c,\eta)=0$$

(**21.6.3** may be obtained from **21.6.1** by the transformations $\xi\rightarrow\pm i\xi$, $c\rightarrow\mp ic$; **21.6.4** may be obtained from **21.6.2** by the transformation $c\rightarrow\mp ic$.)

21.7. Prolate Angular Functions

21.7.1

$$S^{(1)}_{mn}(c,\eta)=\sum_{r=0,1}^{\infty}{}' d^{mn}_r(c)P^m_{m+r}(\eta)$$

=Prolate angular function of the first kind

21.7.2

$$S^{(2)}_{mn}(c,\eta)=\sum_{r=-\infty}^{\infty}{}' d^{mn}_r(c)Q^m_{m+r}(\eta)$$

=Prolate angular function of the second kind

($P^m_n(\eta)$ and $Q^m_n(\eta)$ are associated Legendre functions of the first and second kinds respectively. However, for $-1\leq z\leq1$, $P^m_n(z)=(1-z^2)^{m/2}d^mP_n(z)/dz^m$ (see **8.6.6**). The summation is extended over even values or odd values of r.)

Recurrence Relations Between the Coefficients

21.7.3

$$\alpha_kd_{k+2}+(\beta_k-\lambda_{mn})d_k+\gamma_kd_{k-2}=0$$

$$\alpha_k=\frac{(2m+k+2)(2m+k+1)c^2}{(2m+2k+3)(2m+2k+5)}$$

$$\beta_k=(m+k)(m+k+1)$$

$$+\frac{2(m+k)(m+k+1)-2m^2-1}{(2m+2k-1)(2m+2k+3)}c^2$$

$$\gamma_k=\frac{k(k-1)c^2}{(2m+2k-3)(2m+2k-1)}$$

Transcendental Equation for λ_{mn}

21.7.4

$$U(\lambda_{mn})=U_1(\lambda_{mn})+U_2(\lambda_{mn})=0$$

$$U_1(\lambda_{mn})=\gamma^m_r-\lambda_{mn}-\frac{\beta^m_r}{\gamma^m_{r-2}-\lambda_{mn}-}\ \frac{\beta^m_{r-2}}{\gamma^m_{r-4}-\lambda_{mn}-}\ \cdots$$

$$U_2(\lambda_{mn})=-\frac{\beta^m_{r+2}}{\gamma^m_{r+2}-\lambda_{mn}-}\ \frac{\beta^m_{r+4}}{\gamma^m_{r+4}-\lambda_{mn}-}\ \cdots$$

$$\beta^m_k=\frac{k(k-1)(2m+k)(2m+k-1)c^4}{(2m+2k-1)^2(2m+2k+1)(2m+2k-3)}$$

$$(k\geq2)$$

$$\gamma^m_k=(m+k)(m+k+1)$$

$$+\tfrac{1}{2}c^2\left[1-\frac{4m^2-1}{(2m+2k-1)(2m+2k+3)}\right]\ (k\geq0)$$

(The choice of r in **21.7.4** is arbitrary.)

Power Series Expansion for λ_{mn}

21.7.5

$$\lambda_{mn}=\sum_{k=0}^{\infty} l_{2k}c^{2k}$$

$$l_0=n(n+1)$$

$$l_2=\tfrac{1}{2}\left[1-\frac{(2m-1)(2m+1)}{(2n-1)(2n+3)}\right]$$

$$l_4=\frac{-(n-m+1)(n-m+2)(n+m+1)(n+m+2)}{2(2n+1)(2n+3)^3(2n+5)}+\frac{(n-m-1)(n-m)(n+m-1)(n+m)}{2(2n-3)(2n-1)^3(2n+1)}$$

$$l_6=(4m^2-1)\left[\frac{(n-m+1)(n-m+2)(n+m+1)(n+m+2)}{(2n-1)(2n+1)(2n+3)^5(2n+5)(2n+7)}-\frac{(n-m-1)(n-m)(n+m-1)(n+m)}{(2n-5)(2n-3)(2n-1)^5(2n+1)(2n+3)}\right]$$

$$l_8=2(4m^2-1)^2A+\frac{1}{16}B+\frac{1}{8}C+\frac{1}{2}D$$

$$A=\frac{(n-m-1)(n-m)(n+m-1)(n+m)}{(2n-5)^2(2n-3)(2n-1)^7(2n+1)(2n+3)^2}-\frac{(n-m+1)(n-m+2)(n+m+1)(n+m+2)}{(2n-1)^2(2n+1)(2n+3)^7(2n+5)(2n+7)^2}$$

$$B=\frac{(n-m-3)(n-m-2)(n-m-1)(n-m)(n+m-3)(n+m-2)(n+m-1)(n+m)}{(2n-7)(2n-5)^2(2n-3)^3(2n-1)^4(2n+1)}$$
$$-\frac{(n-m+1)(n-m+2)(n-m+3)(n-m+4)(n+m+1)(n+m+2)(n+m+3)(n+m+4)}{(2n+1)(2n+3)^4(2n+5)^3(2n+7)^2(2n+9)}$$

$$C=\frac{(n-m+1)^2(n-m+2)^2(n+m+1)^2(n+m+2)^2}{(2n+1)^2(2n+3)^7(2n+5)^2}-\frac{(n-m-1)^2(n-m)^2(n+m-1)^2(n+m)^2}{(2n-3)^2(2n-1)^7(2n+1)^2}$$

$$D=\frac{(n-m-1)(n-m)(n-m+1)(n-m+2)(n+m-1)(n+m)(n+m+1)(n+m+2)}{(2n-3)(2n-1)^4(2n+1)^2(2n+3)^4(2n+5)}$$

Asymptotic Expansion for λ_{mn}

21.7.6

$$\lambda_{mn}(c)=cq+m^2-\frac{1}{8}(q^2+5)-\frac{q}{64c}(q^2+11-32m^2)$$
$$-\frac{1}{1024c^2}[5(q^4+26q^2+21)-384m^2(q^2+1)]$$
$$-\frac{1}{c^3}\left[\frac{1}{128^2}(33q^5+1594q^3+5621q)-\frac{m^2}{128}(37q^3+167q)+\frac{m^4}{8}q\right]$$
$$-\frac{1}{c^4}\left[\frac{1}{256^2}(63q^6+4940q^4+43327q^2+22470)-\frac{m^2}{512}(115q^4+1310q^2+735)+\frac{3m^4}{8}(q^2+1)\right]$$
$$-\frac{1}{c^5}\left[\frac{1}{1024^2}(527q^7+61529q^5+1043961q^3+2241599q)-\frac{m^2}{32\cdot 1024}(5739q^5+127550q^3+298951q)+\frac{m^4}{512}(355q^3+1505q)-\frac{m^6q}{16}\right]+O(c^{-6})$$

$$q=2(n-m)+1$$

Refinement of Approximate Values of λ_{mn}

If $\lambda_{mn}^{(1)}$ is an approximation to λ_{mn} obtained either from **21.7.5** or **21.7.6** then

21.7.7

$$\lambda_{mn}=\lambda_{mn}^{(1)}+\delta\lambda_{mn}$$

$$\delta\lambda_{mn}=\frac{U_1(\lambda_{mn}^{(1)})+U_2(\lambda_{mn}^{(1)})}{\Delta_1+\Delta_2}$$

$$\Delta_1=1+\frac{\beta_r^m}{(N_r^m)^2}+\frac{\beta_r^m\beta_{r-2}^m}{(N_r^mN_{r-2}^m)^2}+\frac{\beta_r^m\beta_{r-2}^m\beta_{r-4}^m}{(N_r^mN_{r-2}^mN_{r-4}^m)^2}+\dots$$

$$\Delta_2=\frac{(N_{r+2}^m)^2}{\beta_{r+2}^m}+\frac{(N_{r+2}^mN_{r+4}^m)^2}{\beta_{r+2}^m\beta_{r+4}^m}+\frac{(N_{r+2}^mN_{r+4}^mN_{r+6}^m)^2}{\beta_{r+2}^m\beta_{r+4}^m\beta_{r+6}^m}+\dots$$

$$N_r^m=\frac{(2m+r)(2m+r-1)c^2}{(2m+2r-1)(2m+2r+1)}\frac{d_r}{d_{r-2}}\qquad (r\geq 2)$$

$$\beta_r^m=\frac{r(r-1)(2m+r)(2m+r-1)c^4}{(2m+2r-1)^2(2m+2r+1)(2m+2r-3)}\qquad (r\geq 2)$$

Evaluation of Coefficients

Step 1. Calculate N_r^m's from

21.7.8

$$N_{r+2}^m=\gamma_r^m-\lambda_{mn}-\frac{\beta_r^m}{N_r^m} \qquad (r\geq 2)$$

$$N_2^m=\gamma_0^m-\lambda_{mn};\ N_3^m=\gamma_1^m-\lambda_{mn}$$

$$\gamma_r^m=(m+r)(m+r+1)$$
$$+\frac{1}{2}c^2\left[1-\frac{4m^2-1}{(2m+2r-1)(2m+2r+3)}\right]\ (r\geq 0)$$

Step 2. Calculate ratios $\frac{d_0}{d_{2r}}$ and $\frac{d_1}{d_{2p+1}}$ from

21.7.9 $$\frac{d_0}{d_{2r}}=\left(\frac{d_0}{d_2}\right)\left(\frac{d_2}{d_4}\right)\dots\left(\frac{d_{2r-2}}{d_{2r}}\right)$$

21.7.10 $$\frac{d_1}{d_{2p+1}}=\left(\frac{d_1}{d_3}\right)\left(\frac{d_3}{d_5}\right)\dots\left(\frac{d_{2p-1}}{d_{2p+1}}\right).$$

and the formula for N_r^m in **21.7.7.**

The coefficients d_r^{mn} are determined to within the arbitrary factor d_0 for r even and d_1 for r odd. The choice of these factors depends on the normalization scheme adopted.

Normalization of Angular Functions

Meixner-Schäfke Scheme

21.7.11 $$\int_{-1}^{1}[S_{mn}(c,\eta)]^2d\eta=\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$$

Stratton-Morse-Chu-Little-Corbató Scheme

21.7.12 $$\sum_{r=0,1}^{\prime}\frac{(r+2m)!}{r!}d_r=\frac{(n+m)!}{(n-m)!}$$

(This normalization has the effect that $S_{mn}(c,\eta)\rightarrow P_n^m(\eta)$ as $\eta\rightarrow 1$.)

Flammer Scheme [21.4]

21.7.13

$$S_{mn}(c,0)=P_n^m(0)=\frac{(-1)^{\frac{n-m}{2}}(n+m)!}{2^n\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}$$

$(n-m)$ even

21.7.14

$$S_{mn}'(c,0)=P_n^{m\prime}(0)=\frac{(-1)^{\frac{n-m-1}{2}}(n+m+1)!}{2^n\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!}$$

$(n-m)$ odd

The above lead to the following conditions for d_r^{mn}

21.7.15

$$\sum_{r=0}^{\infty}{}'\frac{(-1)^{r/2}(r+2m)!}{2^r\left(\frac{r}{2}\right)!\left(\frac{r+2m}{2}\right)!}d_r^{mn}=\frac{(-1)^{\frac{n-m}{2}}(n+m)!}{2^{n-m}\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}$$

$(n-m)$ even

21.7.16

$$\sum_{r=1}^{\infty}{}'\frac{(-1)^{\frac{r-1}{2}}(r+2m+1)!}{2^r\left(\frac{r-1}{2}\right)!\left(\frac{r+2m+1}{2}\right)!}d_r^{mn}$$

$$=\frac{(-1)^{\frac{n-m-1}{2}}(n+m+1)!}{2^{n-m}\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!}\qquad (n-m)\text{ odd}$$

(The normalization scheme **21.7.13** and **21.7.14** is also used in [21.10].)

Asymptotic Expansions for $S_{mn}(c,\eta)$

21.7.17

$$S_{mn}(c,\eta)=(1-\eta^2)^{\frac{1}{2}}U_{mn}(c,\eta) \qquad (c\rightarrow\infty)$$

$$U_{mn}(x)=\sum_{r=-\infty}^{\infty}h_r^lD_{l+r}(x) \qquad l=n-m$$

where the $D_r(x)$'s are the parabolic cylinder functions (see chapter **19**).

$$D_r(x)=(-1)^re^{x^2/4}\frac{d^2}{dx^2}e^{-x^2/2}=2^{-r/2}e^{-x^2/4}H_r\left(\frac{x}{\sqrt{2}}\right)$$

and the $H_r(x)$ are the Hermite polynomials (see chapter **22**). (For tables of $h_{\pm r}^l/h_0^l$ see [21.4].)

Expansion of $S_{mn}(c,\eta)$ in Powers of η

21.7.18

$$S_{mn}(c,\eta)=(1-\eta^2)^{m/2}\sum_{r=0,1}^{\infty}{}'p_r^{mn}(c)\eta^r$$

$$(r+1)(r+2)p_{r+2}^{mn}(c)-[r(r+2m+1)+m(m+1)$$
$$-\lambda_{mn}(c)]p_r^{mn}(c)-c^2p_{r-2}^{mn}(c)=0$$

(The derivation of the transcendental equation for λ_{mn} is similar to the derivation of **21.7.4** from **21.7.3**.)

Expansion of $S_{mn}(c,\eta)$ in Powers of $(1-\eta^2)$

21.7.19

$$S_{mn}(c,\eta)=(1-\eta^2)^{m/2}\sum_{k=0}^{\infty}c_{2k}^{mn}(1-\eta^2)^k \qquad (n-m)\text{ even}$$

21.7.20

$$S_{mn}(c,\eta)=\eta(1-\eta^2)^{m/2}\sum_{k=0}^{\infty}c_{2k}^{mn}(1-\eta^2)^k \qquad (n-m)\text{ odd}$$

$$c_{2k}^{mn}=\frac{1}{2^m k!(m+k)!}\sum_{r=k}^{\infty}\frac{(2m+2r)!}{(2r)!}(-r)_k\left(m+r+\frac{1}{2}\right)_k d_{2r}^{mn}$$

$$(n-m)\text{ even}$$

$$c_{2k}^{mn}=$$

$$\frac{1}{2^m k!(m+k)!}\sum_{r=k}^{\infty}\frac{(2m+2r+1)!}{(2r+1)!}(-r)_k\left(m+r+\frac{3}{2}\right)_k d_{2r+1}^{mn}$$

$$(n-m)\text{ odd}$$

$$(\alpha)_k=\alpha(\alpha+1)(\alpha+2)\ .\ .\ .\ (\alpha+k+1)$$

(The d_r^{mn}'s are the coefficients in **21.7.1.**)

Prolate Angular Functions—Second Kind

Expansion **21.7.2** ultimately leads to

21.7.21

$$S_{mn}^{(2)}(c,\eta)=\sum_{r=-2m,\ -2m+1}^{\infty}{}' d_r^{mn}Q_{m+r}^m(\eta)$$

$$+\sum_{r=2m+2,\ 2m+1}^{\infty}{}' d_{\rho|r}^{mn}P_{r-m-1}^m(\eta)$$

(The coefficients d_r^{mn} are the same as in **21.7.1**; the coefficients $d_{\rho|r}^{mn}$ are tabulated in [21.4].)

21.8. Oblate Angular Functions

Power Series Expansion for Eigenvalues

21.8.1 $$\lambda_{mn}=\sum_{k=0}^{\infty}(-1)^k l_{2k}c^{2k}$$

where the l_k's are the same as in **21.7.5.**

Asymptotic Expansion for Eigenvalues [21.4]

21.8.2

$$\lambda_{mn}=-c^2+2c(2\nu+m+1)-2\nu(\nu+m+1)$$

$$-(m+1)+\Lambda_{mn}$$

$$\nu=\frac{1}{2}(n-m)\text{ for }(n-m)\text{ even;}$$

$$\nu=\frac{1}{2}(n-m-1)\text{ for }(n-m)\text{ odd}$$

$$\Lambda_{mn}=\sum_{k=1}^{\infty}\beta_k^{mn}c^{-k}$$

$$\beta_1^{mn}=-2^{-3}q(q^2+1-m^2)$$

$$\beta_2^{mn}=-2^{-6}[5q^4+10q^2+1-2m^2(3q^2+1)+m^4]$$

$$\beta_3^{mn}=-2^{-9}q[33q^4+114q^2+37-2m^2(23q^2+25)$$

$$+13m^4]$$

$$\beta_4^{mn}=-2^{-10}[63q^6+340q^4+239q^2+14$$

$$-10m^2(10q^4+23q^2+3)+m^4(39q^2-18)-2m^6]$$

$$\beta_k^{mn}=\nu(\nu+m)a_k^{-1}+(\nu+1)(\nu+m+1)a_k^{+1}$$

$q=n+1$ for $(n-m)$ even; $q=n$ for $(n-m)$ odd

(For the definition of $a_k^{\pm r}$ see **21.8.3.**)

Asymptotic Expansion for Oblate Angular Functions

21.8.3

$$S_{mn}(-ic,\eta)\sim(1-\eta^2)^{m/2}\sum_{s=-\nu}^{\infty}A_s^{mn}\{e^{-c(1-\eta)}L_{\nu+s}^{(m)}[2c(1-\eta)]$$

$$+(-1)^{n-m}e^{-c(1+\eta)}L_{\nu+s}^{(m)}[2c(1+\eta)]\}$$

where the $L_\nu^{(m)}(x)$ are Laguerre polynomials (see chapter **22**) and

$$\frac{A_{\pm r}^{mn}}{A_0^{mn}}=\sum_{k=r}^{\infty}a_k^{\pm r}(m,n)c^{-k}$$

(Expressions of $a_k{}^{\pm r}$ are given in [21.4].)

21.9. Radial Spheroidal Wave Functions

21.9.1

$$R_{mn}^{(p)}(c,\xi)=\left\{\sum_{r=0,\ 1}^{\infty}{}'\frac{(2m+r)!}{r!}d_r^{mn}\right\}^{-1}\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}$$

$$\cdot\sum_{r=0,\ 1}^{\infty}{}' i^{r+m-n}\frac{(2m+r)!}{r!}d_r^{mn}Z_{m+r}^{(p)}(c\xi)\ {}^{*}$$

$$Z_n^{(p)}(z)=\sqrt{\frac{\pi}{2z}}J_{n+\frac{1}{2}}(z) \qquad (p=1)$$

$$=\sqrt{\frac{\pi}{2z}}Y_{n+\frac{1}{2}}(z) \qquad (p=2)$$

($J_{n+\frac{1}{2}}(z)$ and $Y_{n+\frac{1}{2}}(z)$ are Bessel functions, order $n+\frac{1}{2}$, of the first and second kind respectively (see chapter **10**).)

21.9.2 $$R_{mn}^{(3)}(c,\xi)=R_{mn}^{(1)}(c,\xi)+iR_{mn}^{(2)}(c,\xi)$$

21.9.3 $$R_{mn}^{(4)}(c,\xi)=R_{mn}^{(1)}(c,\xi)-iR_{mn}^{(2)}(c,\xi)$$

Asymptotic Behavior of $R_{mn}^{(1)}(c,\xi)$ and $R_{mn}^{(2)}(c,\xi)$

21.9.4 $$R_{mn}^{(1)}(c,\xi)\xrightarrow[c\xi\to\infty]{}\frac{1}{c\xi}\cos\left[c\xi-\tfrac{1}{2}(n+1)\pi\right]$$

21.9.5 $$R_{mn}^{(2)}(c,\xi)\xrightarrow[c\xi\to\infty]{}\frac{1}{c\xi}\sin\left[c\xi-\tfrac{1}{2}(n+1)\pi\right]$$

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[*] See page II.

21.10. Joining Factors for Prolate Spheroidal Wave Functions

21.10.1

$$S_{mn}^{(1)}(c,\xi)=\kappa_{mn}^{(1)}(c)R_{mn}^{(1)}(c,\xi)$$

$$\kappa_{mn}^{(1)}(c)=\frac{(2m+1)(n+m)!\sum_{r=0}^{\infty}{}' d_r^{mn}(2m+r)!/r!}{2^{n+m}d_0^{mn}(c)c^m m!\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!} \qquad (n-m) \text{ even}$$

$$=\frac{(2m+3)(n+m+1)!\sum_{r=1}^{\infty}{}' d_r^{mn}(2m+r)!/r!}{2^{n+m}d_1^{mn}(c)c^{m+1}m!\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!} \qquad (n-m) \text{ odd}$$

21.10.2

$$S_{mn}^{(2)}(c,\xi)=\kappa_{mn}^{(2)}(c)R_{mn}^{(2)}(c,\xi)$$

$$\kappa_{mn}^{(2)}(c)=\frac{2^{n-m}(2m)!\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!\,d_{-2m}^{mn}(c)}{(2m-1)m!(n+m)!c^{m-1}}\sum_{r=0}^{\infty}{}'\frac{(2m+r)!}{r!}d_r^{mn}(c) \qquad (n-m) \text{ even}$$

$$=-\frac{2^{n-m}(2m)!\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!\,d_{-2m+1}^{mn}(c)}{(2m-3)(2m-1)m!(n+m+1)!\,c^{m-2}}\sum_{r=1}^{\infty}{}'\frac{(2m+r)!}{r!}d_r^{mn}(c) \qquad (n-m) \text{ odd}$$

(The expression for joining factors appropriate to the oblate case may be obtained from the above formulas by the transformation $c\rightarrow -ic$.)

21.11. Notation

Notation for Prolate Spheroidal Wave Functions

	Ang. coord.	Rad. coord.	Independent variable	Ang. wave function	Rad. wave function	Eigenvalue	Normalization of angular functions	Remarks
Stratton, Morse, Chu, Little and Corbató	η	ξ	h	$S_{ml}(h, \eta)$	$je_{ml}(h, \xi)$ $ne_{ml}(h, \xi)$ $he_{ml}(h, \xi)$	$A_{ml}(h)$	$S_{ml}(h, 1)=P_l^m(1)$	$l=$Flammer's n $A_{ml}=\lambda_{mn}$
Flammer and this chapter	η	ξ	c	$S_{mn}(c, \eta)$	$R_{mn}^{(i)}(c, \xi)$	$\lambda_{mn}(c)$	$S_{mn}(c, 0)=P_n^m(0)$ $(n-m)$ even $S'_{mn}(c, 0)=P_n^{m\prime}(0)$ $(n-m)$ odd	
Chu and Stratton	η	ξ	c	$S_{ml}^{(1)}(c, \eta)$	$R_{ml}^{(i)}(c, \xi)$	A_{ml}	$S_{ml}^{(1)}(c, 0)=P_{m+l}^m(0)$ $(l$ even$)$ $S_{ml}^{(1)\prime}(c, 0)=P_{m+l}^{m\prime}(0)$ $(l$ odd$)$	$l=$Flammer's $n-m$ $A_{ml}=-\lambda_{m,\,n-m}$
Meixner and Schäfke	η	ξ	γ	$PS_n^m(\eta, \gamma^2)$	$S_n^{m(i)}(\xi, \gamma^2)$	$\lambda_n^m(\gamma^2)$	$\int_{-1}^{1}[PS_n^m(\eta, \gamma^2)]^2 d\eta$ $=\frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}$	$\lambda_n^m(\gamma^2)=\lambda_{mn}(c)-c^2$
Morse and Feshbach	$\eta=\cos\vartheta$	$\xi=\cosh\mu$	h	$S_{ml}(h, \eta)$	$je_{ml}(h, \xi)$ $ne_{ml}(h, \xi)$ $he_{ml}(h, \xi)$	A_{ml}	$[(1-\eta^2)^{-m/2}S_{ml}(h, \eta)]_{\eta=1}$ $=[(1-\eta^2)^{-m/2}P_l^m(\eta)]_{\eta=1}$	$l=$Flammer's n $A_{ml}=\lambda_{mn}$
Page	ξ	η	ϵ	$U_{lm}(\xi)$	$v_{lm}(\eta)$ $p_{lm}(\eta)$ $q_{lm}(\eta)$	α_{lm}	$[(1-\xi^2)^{-m/2}U_{lm}(\xi)]=1$ $\xi=1$	$l=$Flammer's n $\alpha_{lm}=\lambda_{mn}-c^2$

Notation for Oblate Spheroidal Wave Functions

	Ang. coord.	Rad. coord.	Independent variable	Ang. wave function	Rad. wave function	Eigenvalue	Normalization of angular functions	Remarks
Stratton, Morse, Chu, Little and Corbató	η	ξ	g	$S_{ml}(ig, \eta)$	$je_{ml}(ig, -i\xi)$	A_{ml}	$S_{ml}(ig, 1)=P_l^m(1)$	$l=$Flammer's n $A_{ml}=\lambda_{mn}$
Flammer and this chapter	η	ξ	c	$S_{mn}(-ic, \eta)$	$R_{mn}^{(i)}(-ic, i\xi)$	$\lambda_{mn}(-ic)$	$S_{mn}(-ic,0)=P_n^m(0)$ $(n-m)$ even $S'_{mn}(-ic, 0)=P_n^{m\prime}(0)$ $(n-m)$odd	
Chu and Stratton	η	ξ	c	$S_{ml}^{(1)}(-ic, \eta)$	$R_{ml}^{(1)}(-ic, i\xi)$	B_{ml}	$S_{ml}^{(1)}(-ic, 0)=P_{m+l}^m(0)$ $(l$ even$)$ $S_{ml}^{(1)\prime}(-ic, 0)=P_{m+l}^{m\prime}(0)$ $(l$ odd$)$	$l=$Flammer's $n-m$ $B_{lm}=-\lambda_{m,\,n-m}$
Meixner and Schäfke	η	ξ	γ	$ps_n^m(\eta,-\gamma^2)$	$S_n^{m(j)}(-i\xi, i\gamma^2)$	$\lambda_n^m(-\gamma^2)$	$\int_{-1}^{1}[ps_n^m(\eta, -\gamma^2)]^2 d\eta$ $=\frac{2}{2n+1}\,\frac{(n+m)!}{(n-m)!}$	$\lambda_n^m(-\gamma^2)=\lambda_{mn}(-ic)+c^2$
Morse and Feshbach	$\eta=\cos\vartheta$	$\xi=\sinh\mu$	g	$S_{ml}(ig, \eta)$	$je_{ml}(ig, -i\xi)$ $ne_{ml}(ig, -i\xi)$ $he_{ml}(ig, -i\xi)$	A_{ml}	$[(1-\eta^2)^{-m/2}S_{ml}(ig, \eta)]_{\eta=1}$ $=[(1-\eta^2)^{-m/2}P_l^m(\eta)]_{\eta=1}$	$l=$Flammer's n $A_{ml}=\lambda_{mn}$
Leitner and Spence	η	ξ	ϵ	$U_{lm}(\eta)$	$\overset{(i)}{v}_{lm}(\xi)$	α_{lm}	$[(1-\eta^2)^{-m/2}U_{lm}(\eta)]_{\eta=1}=1$	$l=$Flammer's n $\alpha_{lm}=\lambda_{mn}+c^2$

The notation in this chapter closely follows the notation in [21.4].

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22. Orthogonal Polynomials

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Contents

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22. Orthogonal Polynomials

Mathematical Properties

22.1. Definition of Orthogonal Polynomials

A system of polynomials $f_n(x)$, degree $[f_n(x)]=n$, is called orthogonal on the interval $a\leq x\leq b$, with respect to the weight function $w(x)$, if

22.1.1

$$\int_a^b w(x)f_n(x)f_m(x)dx=0 \qquad (n\neq m;\, n, m=0,1,2,\ldots)$$

The weight function $w(x)[w(x)\geq 0]$ determines the system $f_n(x)$ up to a constant factor in each polynomial. The specification of these factors is referred to as standardization. For suitably standardized orthogonal polynomials we set

22.1.2

$$\int_a^b w(x)f_n^2(x)dx=h_n, f_n(x)=k_nx^n+k_n'x^{n-1}+\ldots \qquad (n=0,1,2,\ldots)$$

These polynomials satisfy a number of relationships of the same general form. The most important ones are:

Differential Equation

22.1.3 $$g_2(x)f_n''+g_1(x)f_n'+a_nf_n=0$$

where $g_2(x)$, $g_1(x)$ are independent of n and a_n a constant depending only on n.

Recurrence Relation

22.1.4 $$f_{n+1}=(a_n+xb_n)f_n-c_nf_{n-1}$$

where

22.1.5

$$b_n=\frac{k_{n+1}}{k_n}, \quad a_n=b_n\left(\frac{k_{n+1}'}{k_{n+1}}-\frac{k_n'}{k_n}\right), \quad c_n=\frac{k_{n+1}k_{n-1}h_n}{k_n^2h_{n-1}}$$

Rodrigues' Formula

22.1.6 $$f_n=\frac{1}{e_nw(x)}\frac{d^n}{dx_n}\{w(x)[g(x)]^n\}$$

where $g(x)$ is a polynomial in x independent of n. The system $\left\{\frac{df_n}{dx}\right\}$ consists again of orthogonal polynomials.

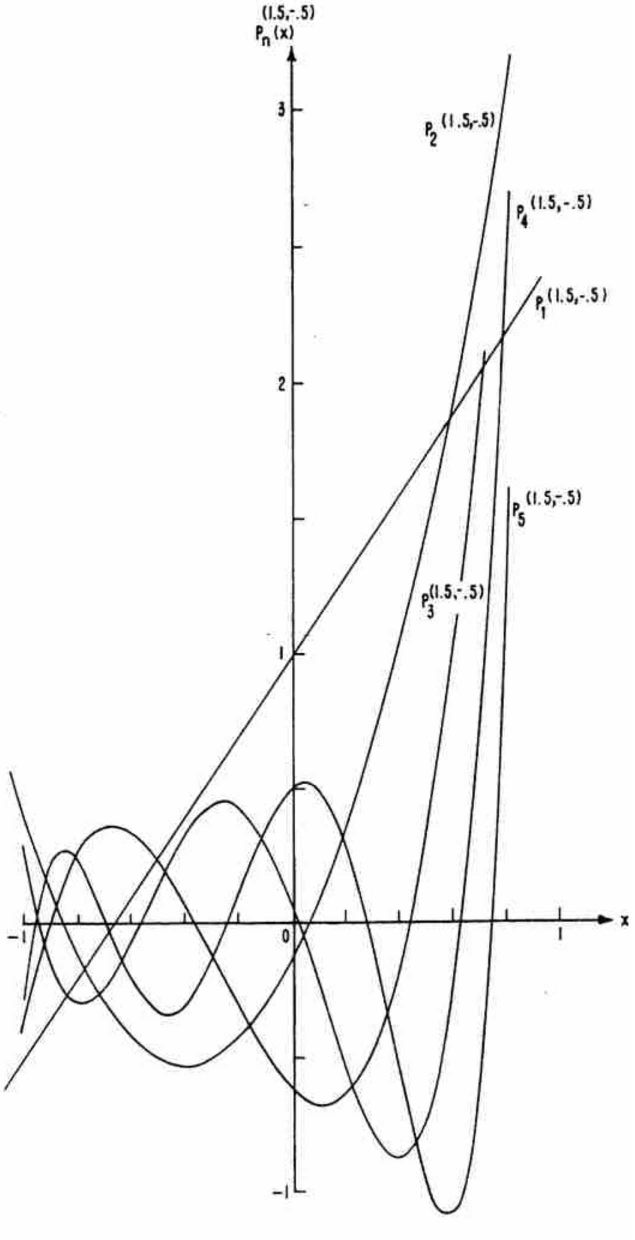

FIGURE 22.1. *Jacobi Polynomials* $P_n^{(\alpha,\beta)}(x)$, $\alpha=1.5$, $\beta=-.5$, $n=1(1)5$.

22.2. Orthogonality Relations

	$f_n(x)$	Name of Polynomial	a	b	$w(x)$	Standardization	h_n	Remarks
22.2.1	$P_n^{(\alpha,\beta)}(x)$	Jacobi	-1	1	$(1-x)^\alpha(1+x)^\beta$	$P_n^{(\alpha,\beta)}(1)=\binom{n+\alpha}{n}$	$\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}$	$\alpha>-1$, $\beta>-1$
22.2.2	$G_n(p, q, x)$	Jacobi	0	1	$(1-x)^{p-q}x^{q-1}$	$k_n=1$	$\frac{n!\Gamma(n+q)\Gamma(n+p)\Gamma(n+p-q+1)}{(2n+p)\Gamma^2(2n+p)}$	$p-q>-1$, $q>0$
22.2.3	$C_n^{(\alpha)}(x)$	Ultraspherical (Gegenbauer)	-1	1	$(1-x^2)^{\alpha-\frac{1}{2}}$	$C_n^{(\alpha)}(1)=\binom{n+2\alpha-1}{n}$ $(\alpha\neq 0)$ $C_n^{(0)}(1)=\frac{2}{n}$, $C_0^{(0)}(1)=1$	$\frac{\pi 2^{1-2\alpha}\Gamma(n+2\alpha)}{n!(n+\alpha)[\Gamma(\alpha)]^2}$ $\alpha\neq 0$ $\frac{2\pi}{n^2}$ $\alpha=0$	$\alpha>-\frac{1}{2}$
22.2.4	$T_n(x)$	Chebyshev of the first kind	-1	1	$(1-x^2)^{-\frac{1}{2}}$	$T_n(1)=1$	$\begin{cases}\frac{\pi}{2} & n\neq 0\\ \pi & n=0\end{cases}$	
22.2.5	$U_n(x)$	Chebyshev of the second kind	-1	1	$(1-x^2)^{\frac{1}{2}}$	$U_n(1)=n+1$	$\frac{\pi}{2}$	∞
* **22.2.6**	$C_n(x)$	Chebyshev of the first kind	-2	2	$\left(1-\frac{x^2}{4}\right)^{-\frac{1}{2}}$	$C_n(2)=2$	$\begin{cases}4\pi & n\neq 0\\ 8\pi & n=0\end{cases}$	
* **22.2.7**	$S_n(x)$	Chebyshev of the second kind	-2	2	$\left(1-\frac{x^2}{4}\right)^{\frac{1}{2}}$	$S_n(2)=n+1$	π	
22.2.8	$T_n^*(x)$	Shifted Chebyshev of the first kind	0	1	$(x-x^2)^{-\frac{1}{2}}$	$T_n^*(1)=1$	$\begin{cases}\frac{\pi}{2} & n\neq 0\\ \pi & n=0\end{cases}$	
22.2.9	$U_n^*(x)$	Shifted Chebyshev of the second kind	0	1	$(x-x^2)^{\frac{1}{2}}$	$U_n^*(1)=n+1$	$\frac{\pi}{8}$ *	
22.2.10	$P_n(x)$	Legendre (Spherical)	-1	1	1	$P_n(1)=1$	$\frac{2}{2n+1}$	
22.2.11	$P_n^*(x)$	Shifted Legendre	0	1	1		$\frac{1}{2n+1}$	

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*See page II.

22.2. Orthogonality Relations—Continued

22.2.12	$L_n^{(\alpha)}(x)$	Generalized Laguerre	0	∞	$e^{-x}x^{\alpha}$	$k_n=\frac{(-1)^n}{n!}$	$\frac{\Gamma(\alpha+n+1)}{n!}$	$\alpha>-1$
22.2.13	$L_n(x)$	Laguerre	0	∞	e^{-x}	$k_n=\frac{(-1)^n}{n!}$	1	
* **22.2.14**	$H_n(x)$	Hermite	$-\infty$	∞	e^{-x^2}	$e_n=(-1)^n$	$\sqrt{\pi}2^n n!$	
* **22.2.15**	$He_n(x)$	Hermite	$-\infty$	∞	$e^{-\frac{x^2}{2}}$	$e_n=(-1)^n$	$\sqrt{2\pi}n!$	

*See page II.

22.3. Explicit Expressions

$$f_n(x)=d_n\sum_{m=0}^{N}c_m g_m(x)$$

	$f_n(x)$	N	d_n	c_m	$g_m(x)$	k_n	Remarks
22.3.1	$P_n^{(\alpha,\beta)}(x)$	n	$\frac{1}{2^n}$	$\binom{n+\alpha}{m}\binom{n+\beta}{n-m}$	$(x-1)^{n-m}(x+1)^m$	$\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}$	$\alpha>-1,\ \beta>-1$
22.3.2	$P_n^{(\alpha,\beta)}(x)$	n	$\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)}$	$\binom{n}{m}\frac{\Gamma(\alpha+\beta+n+m+1)}{2^m\Gamma(\alpha+m+1)}$	$(x-1)^m$	$\frac{1}{2^n}\binom{2n+\alpha+\beta}{n}$	$\alpha>-1,\ \beta>-1$
22.3.3	$G_n(p,q,x)$	n	$\frac{\Gamma(q+n)}{\Gamma(p+2n)}$	$(-1)^m\binom{n}{m}\frac{\Gamma(p+2n-m)}{\Gamma(q+n-m)}$	x^{n-m}	1	$p-q>-1,\ q>0$
22.3.4	$C_n^{(\alpha)}(x)$	$\left[\frac{n}{2}\right]$	$\frac{1}{\Gamma(\alpha)}$	$(-1)^m\frac{\Gamma(\alpha+n-m)}{m!(n-2m)!}$	$(2x)^{n-2m}$	$\frac{2^n}{n!}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$	$\alpha>-\frac{1}{2},\ \alpha\neq 0$
22.3.5	$C_n^{(0)}(x)$	$\left[\frac{n}{2}\right]$	1	$(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	$\frac{2^n}{n}\quad n\neq 0$	$n\neq 0,\ C_0^{(0)}(1)=1$
22.3.6	$T_n(x)$	$\left[\frac{n}{2}\right]$	$\frac{n}{2}$	$(-1)^m\frac{(n-m-1)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^{n-1}	
22.3.7	$U_n(x)$	$\left[\frac{n}{2}\right]$	1	$(-1)^m\frac{(n-m)!}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^n	
22.3.8	$P_n(x)$	$\left[\frac{n}{2}\right]$	$\frac{1}{2^n}$	$(-1)^m\binom{n}{m}\binom{2n-2m}{n}$	x^{n-2m}	$\frac{(2n)!}{2^n(n!)^2}$	
22.3.9	$L_n^{(\alpha)}(x)$	n	1	$(-1)^m\binom{n+\alpha}{n-m}\frac{1}{m!}$	x^m	$\frac{(-1)^n}{n!}$	$\alpha>-1$
22.3.10	$H_n(x)$	$\left[\frac{n}{2}\right]$	$n!$	$(-1)^m\frac{1}{m!(n-2m)!}$	$(2x)^{n-2m}$	2^n	see **22.11**
22.3.11	$He_n(x)$	$\left[\frac{n}{2}\right]$	$n!$	$(-1)^m\frac{1}{m!2^m(n-2m)!}$	x^{n-2m}	1	

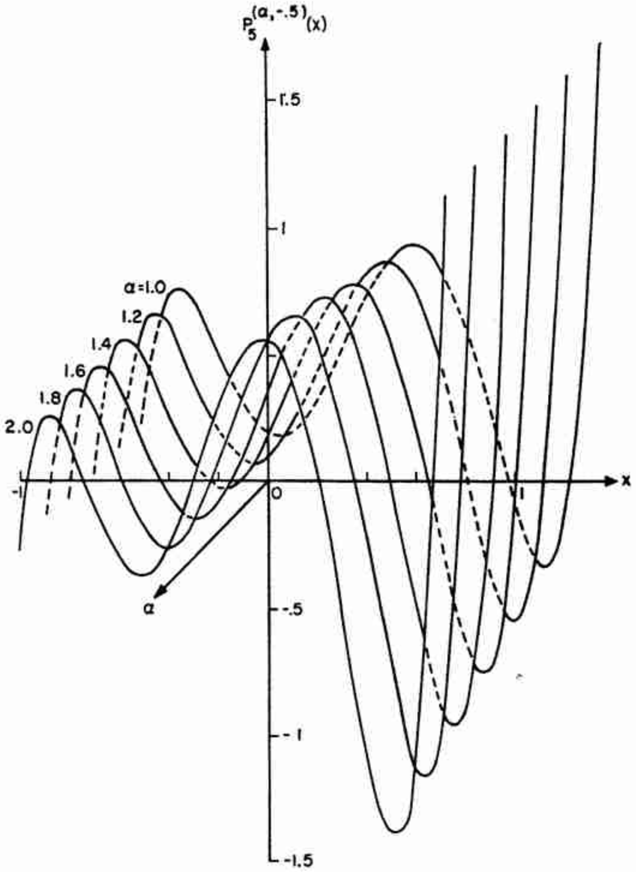

FIGURE 22.2. *Jacobi Polynomials* $P_n^{(\alpha,\beta)}(x)$, $\alpha=1(.2)2$, $\beta=-.5$, $n=5$.

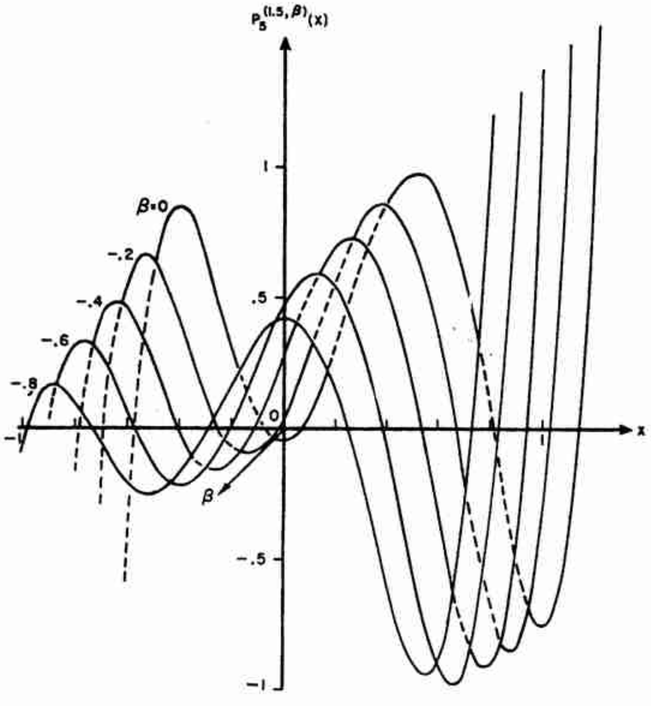

FIGURE 22.3. *Jacobi Polynomials* $P_n^{(\alpha,\beta)}(x)$, $\alpha=1.5$, $\beta=-.8(.2)0$, $n=5$.

Explicit Expressions Involving Trigonometric Functions

$$f_n(\cos\theta)=\sum_{m=0}^{n} a_m \cos(n-2m)\theta$$

	$f_n(\cos\theta)$	a_m	Remarks
22.3.12	$C_n^{(\alpha)}(\cos\theta)$	$\frac{\Gamma(\alpha+m)\Gamma(\alpha+n-m)}{m!(n-m)![\Gamma(\alpha)]^2}$	$\alpha\neq 0$
22.3.13	$P_n(\cos\theta)$	$\frac{1}{4^n}\binom{2m}{m}\binom{2n-2m}{n-m}$	

22.3.14 $$C_n^{(0)}(\cos\theta)=\frac{2}{n}\cos n\theta$$

22.3.15 $$T_n(\cos\theta)=\cos n\theta$$

22.3.16 $$U_n(\cos\theta)=\frac{\sin(n+1)\theta}{\sin\theta}$$

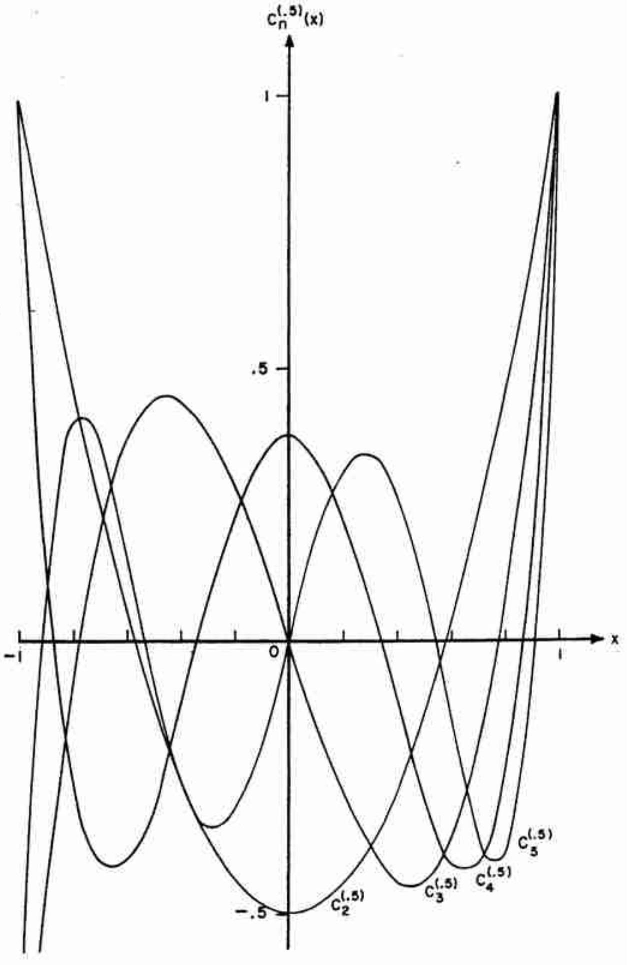

FIGURE 22.4. *Gegenbauer (Ultraspherical) Polynomials* $C_n^{(\alpha)}(x)$, $\alpha=.5$, $n=2(1)5$.

22.4. Special Values

	$f_n(x)$	$f_n(-x)$	$f_n(1)$	$f_n(0)$	$f_0(x)$	$f_1(x)$
22.4.1	$P_n^{(\alpha,\beta)}(x)$	$(-1)^n P_n^{(\beta,\alpha)}(x)$	$\binom{n+\alpha}{n}$ *		1	$\frac{1}{2}[\alpha-\beta+(\alpha+\beta+2)x]$
22.4.2	$C_n^{(\alpha)}(x)$ $\alpha\neq 0$	$(-1)^n C_n^{(\alpha)}(x)$	$\binom{n+2\alpha-1}{n}$	$\begin{cases} 0, & n=2m+1 \\ (-1)^{n/2}\frac{\Gamma(\alpha+n/2)}{\Gamma(\alpha)(n/2)!}, & n=2m \end{cases}$	1	$2\alpha x$
22.4.3	$C_n^{(0)}(x)$	$(-1)^n C_n^{(0)}(x)$	$\frac{2}{n}$, $n\neq 0$	$\begin{cases} \frac{(-1)^m}{m}, & n=2m\neq 0 \\ 0, & n=2m+1 \end{cases}$	1	$2x$
22.4.4	$T_n(x)$	$(-1)^n T_n(x)$	1	$\begin{cases} (-1)^m, & n=2m \\ 0, & n=2m+1 \end{cases}$	1	x
22.4.5	$U_n(x)$	$(-1)^n U_n(x)$	$n+1$	$\begin{cases} (-1)^m, & n=2m \\ 0, & n=2m+1 \end{cases}$	1	$2x$
22.4.6	$P_n(x)$	$(-1)^n P_n(x)$	1	$\begin{cases} \frac{(-1)^m}{4^m}\binom{2m}{m}, & n=2m^* \\ 0, & n=2m+1 \end{cases}$	1	x
22.4.7	$L_n^{(\alpha)}(x)$			$\binom{n+\alpha}{n}$	1	$-x+\alpha+1$
22.4.8	$H_n(x)$	$(-1)^n H_n(x)$		$\begin{cases} (-1)^m\frac{(2m)!}{m!}, & n=2m \\ 0, & n=2m+1 \end{cases}$	1	$2x$

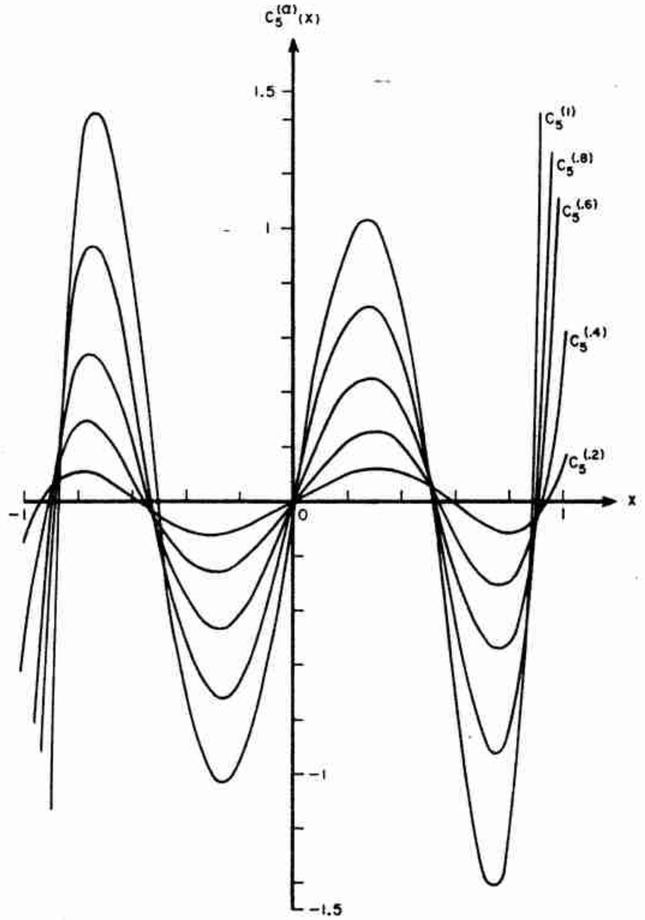

FIGURE 22.5. *Gegenbauer (Ultraspherical) Polynomials* $C_n^{(\alpha)}(x)$, $\alpha=.2(.2)1$, $n=5$.

22.5. Interrelations

Interrelations Between Orthogonal Polynomials of the Same Family

Jacobi Polynomials

22.5.1

$$P_n^{(\alpha,\beta)}(x)=\frac{\Gamma(2n+\alpha+\beta+1)}{n!\Gamma(n+\alpha+\beta+1)}G_n\left(\alpha+\beta+1,\beta+1,\frac{x+1}{2}\right)$$

22.5.2

$$G_n(p,q,x)=\frac{n!\Gamma(n+p)}{\Gamma(2n+p)}P_n^{(p-q,q-1)}(2x-1)$$

(see [22.21]).

22.5.3

$$F_n(p,q,x)=(-1)^n n!\frac{\Gamma(q)}{\Gamma(q+n)}P_n^{(p-q,q-1)}(2x-1)$$

(see [22.13]).

Ultraspherical Polynomials

22.5.4 $$C_n^{(0)}(x)=\lim_{\alpha\to 0}\frac{1}{\alpha}C_n^{(\alpha)}(x)$$

Chebyshev Polynomials

22.5.5 $$T_n(x)=\tfrac{1}{2}C_n(2x)=T_n^*\left(\frac{1+x}{2}\right)$$

22.5.6 $$T_n(x)=U_n(x)-xU_{n-1}(x)$$

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*See page II.

22.5.7 $$T_n(x)=xU_{n-1}(x)-U_{n-2}(x)$$

22.5.8 $$T_n(x)=\tfrac{1}{2}\,[U_n(x)-U_{n-2}(x)]$$

22.5.9 $$U_n(x)=S_n(2x)=U_n^*\left(\frac{1+x}{2}\right)$$

22.5.10 $$U_{n-1}(x)=\frac{1}{1-x^2}\,[xT_n(x)-T_{n+1}(x)]$$

22.5.11 $$C_n(x)=2T_n\left(\frac{x}{2}\right)=2T_n^*\left(\frac{x+2}{4}\right)$$

22.5.12 $$C_n(x)=S_n(x)-S_{n-2}(x)$$

22.5.13 $$S_n(x)=U_n\left(\frac{x}{2}\right)=U_n^*\left(\frac{x+2}{4}\right)$$

22.5.14 $$T_n^*(x)=T_n(2x-1)=\tfrac{1}{2}\,C_n(4x-2)$$

(see [22.22]).

22.5.15 $$U_n^*(x)=S_n(4x-2)=U_n(2x-1)$$

(see [22.22]).

Generalized Laguerre Polynomials

22.5.16 $$L_n^{(0)}(x)=L_n(x)$$

22.5.17 $$L_n^{(m)}(x)=(-1)^m\,\frac{d^m}{dx^m}\,[L_{n+m}(x)]$$

Hermite Polynomials

22.5.18 $$He_n(x)=2^{-n/2}H_n\left(\frac{x}{\sqrt{2}}\right)$$

(see [22.20]).

22.5.19 $$H_n(x)=2^{n/2}He_n(x\sqrt{2})$$

(see [22.13], [22.20]).

Interrelations Between Orthogonal Polynomials of Different Families

Jacobi Polynomials

22.5.20

$$P_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x)=\frac{\Gamma(2\alpha)\,\Gamma(\alpha+n+\frac{1}{2})}{\Gamma(2\alpha+n)\,\Gamma(\alpha+\frac{1}{2})}\;C_n^{(\alpha)}(x)$$

22.5.21

$$P_n^{(\alpha,\frac{1}{2})}(x)=\frac{(\frac{1}{2})_{n+1}}{\sqrt{\frac{x+1}{2}}\;(\alpha+\frac{1}{2})_{n+1}}\;C_{2n+1}^{(\alpha+\frac{1}{2})}\left(\sqrt{\frac{x+1}{2}}\right)$$

22.5.22 $$P_n^{(\alpha,-\frac{1}{2})}(x)=\frac{(\frac{1}{2})_n}{(\alpha+\frac{1}{2})_n}\;C_{2n}^{(\alpha+\frac{1}{2})}\left(\sqrt{\frac{x+1}{2}}\right)$$

22.5.23 $$P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)=\frac{1}{4^n}\binom{2n}{n}\,T_n(x)$$

22.5.24 $$P_n^{(0,0)}(x)=P_n(x)$$

Ultraspherical Polynomials

22.5.25

$$C_{2n}^{(\alpha)}(x)=\frac{\Gamma(\alpha+n)n!2^{2n}}{\Gamma(\alpha)\,(2n)!}\;P_n^{(\alpha-\frac{1}{2},-\frac{1}{2})}\;(2x^2-1)\qquad(\alpha\neq 0)$$

22.5.26

$$C_{2n+1}^{(\alpha)}(x)=\frac{\Gamma(\alpha+n+1)n!2^{2n+1}}{\Gamma(\alpha)\,(2n+1)!}\;xP_n^{(\alpha-\frac{1}{2},\frac{1}{2})}(2x^2-1)\qquad(\alpha\neq 0)$$

22.5.27

$$C_n^{(\alpha)}(x)=\frac{\Gamma(\alpha+\frac{1}{2})\,\Gamma(2\alpha+n)}{\Gamma(2\alpha)\,\Gamma(\alpha+n+\frac{1}{2})}\;P_n^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(x)\qquad(\alpha\neq 0)$$

22.5.28

$$C_n^{(0)}(x)=\frac{2}{n}T_n(x)=2\,\frac{(n-1)!}{\Gamma(n+\frac{1}{2})}\,\sqrt{\pi}P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)$$ *

Chebyshev Polynomials

22.5.29 $$T_{2n+1}(x)=\frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}\;xP_n^{(-\frac{1}{2},\frac{1}{2})}(2x^2-1)$$

22.5.30 $$U_{2n}(x)=\frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}\;P_n^{(\frac{1}{2},-\frac{1}{2})}(2x^2-1)$$

22.5.31 $$T_n(x)=\frac{n!\sqrt{\pi}}{\Gamma(n+\frac{1}{2})}\;P_n^{(-\frac{1}{2},-\frac{1}{2})}(x)$$

22.5.32 $$U_n(x)=\frac{(n+1)!\sqrt{\pi}}{2\Gamma(n+\frac{3}{2})}\;P_n^{(\frac{1}{2},\frac{1}{2})}(x)$$

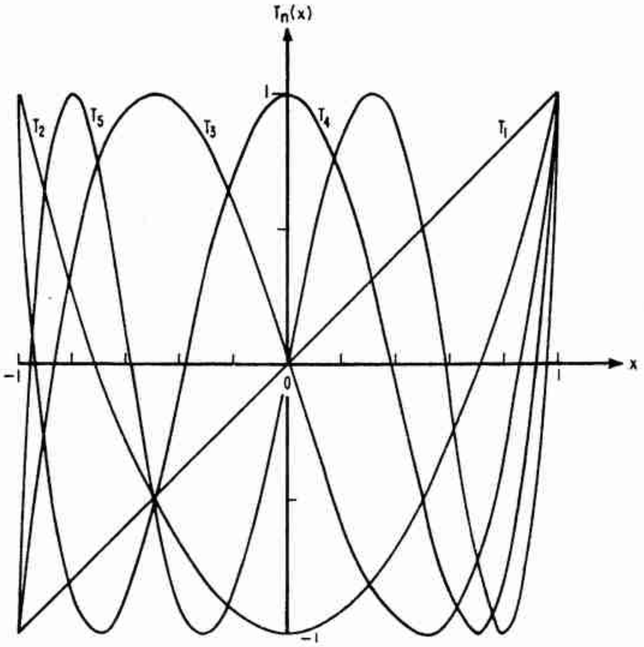

FIGURE 22.6. *Chebyshev Polynomials $T_n(x)$, $n=1(1)5$.*

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*See page II.

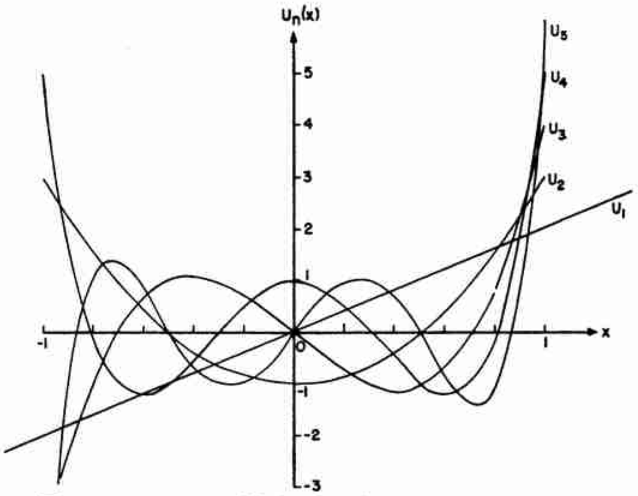

FIGURE 22.7. *Chebyshev Polynomials* $U_n(x)$, $n=1(1)5$.

22.5.33 $$T_n(x)=\frac{n}{2}\,C_n^{(0)}(x)$$

22.5.34 $$U_n(x)=C_n^{(1)}(x)$$

Legendre Polynomials

22.5.35 $$P_n(x)=P_n^{(0,0)}(x)$$

22.5.36 $$P_n(x)=C_n^{(1/2)}(x)$$

22.5.37

$$\frac{d^m}{dx^m}\,[P_n(x)]=1\cdot 3\,\ldots\,(2m-1)\,C_{n-m}^{(m+\frac{1}{2})}(x) \qquad (m\leq n)$$

Generalized Laguerre Polynomials

22.5.38 $$L_n^{(-1/2)}(x)=\frac{(-1)^n}{n!2^{2n}}\,H_{2n}(\sqrt{x})$$

22.5.39 $$L_n^{(1/2)}(x)=\frac{(-1)^n}{n!2^{2n+1}\sqrt{x}}\,H_{2n+1}(\sqrt{x})$$

Hermite Polynomials

22.5.40 $$H_{2m}(x)=(-1)^m 2^{2m} m!\,L_m^{(-1/2)}(x^2)$$

22.5.41 $$H_{2m+1}(x)=(-1)^m 2^{2m+1} m!\,x L_m^{(1/2)}(x^2)$$

Orthogonal Polynomials as Hypergeometric Functions (see chapter 15)

$$f_n(x)=dF(a,\,b;\,c;\,g(x))$$

For each of the listed polynomials there are numerous other representations in terms of hypergeometric functions.

	$f_n(x)$	d	a	b	c	$g(x)$
22.5.42	$P_n^{(\alpha,\beta)}(x)$	$\binom{n+\alpha}{n}$	$-n$	$n+\alpha+\beta+1$	$\alpha+1$	$\frac{1-x}{2}$
22.5.43	$P_n^{(\alpha,\beta)}(x)$	$\binom{2n+\alpha+\beta}{n}\left(\frac{x-1}{2}\right)^n$	$-n$	$-n-\alpha$	$-2n-\alpha-\beta$	$\frac{2}{1-x}$
22.5.44	$P_n^{(\alpha,\beta)}(x)$	$\binom{n+\alpha}{n}\left(\frac{1+x}{2}\right)^n$	$-n$	$-n-\beta$	$\alpha+1$	$\frac{x-1}{x+1}$
22.5.45	$P_n^{(\alpha,\beta)}(x)$	$\binom{n+\beta}{n}\left(\frac{x-1}{2}\right)^n$	$-n$	$-n-\alpha$	$\beta+1$	$\frac{x+1}{x-1}$
22.5.46	$C_n^{(\alpha)}(x)$	$\frac{\Gamma(n+2\alpha)}{n!\Gamma(2\alpha)}$	$-n$	$n+2\alpha$	$\alpha+\frac{1}{2}$	$\frac{1-x}{2}$
22.5.47	$T_n(x)$	1	$-n$	n	$\frac{1}{2}$	$\frac{1-x}{2}$
22.5.48	$U_n(x)$	$n+1$	$-n$	$n+2$ *	$\frac{3}{2}$	$\frac{1-x}{2}$
22.5.49	$P_n(x)$	1	$-n$	$n+1$	1	$\frac{1-x}{2}$
22.5.50	$P_n(x)$	$\binom{2n}{n}\left(\frac{x-1}{2}\right)^n$	$-n$	$-n$	$-2n$	$\frac{2}{1-x}$
22.5.51	$P_n(x)$	$\binom{2n}{n}\left(\frac{x}{2}\right)^n$	$-\frac{n}{2}$	$\frac{1-n}{2}$	$\frac{1}{2}-n$	$\frac{1}{x^2}$
22.5.52	$P_{2n}(x)$	$(-1)^n\frac{(2n)!}{2^{2n}(n!)^2}$	$-n$	$n+\frac{1}{2}$	$\frac{1}{2}$	x^2
22.5.53	$P_{2n+1}(x)$	$(-1)^n\frac{(2n+1)!}{2^{2n}(n!)^2}x$	$-n$	$n+\frac{3}{2}$	$\frac{3}{2}$	x^2

Orthogonal Polynomials as Confluent Hypergeometric Functions (see chapter 13)

22.5.54 $$L_n^{(\alpha)}(x)=\binom{n+\alpha}{n}M(-n,\alpha+1,x)$$

Orthogonal Polynomials as Parabolic Cylinder Functions (see chapter 19)

22.5.55 $$H_n(x)=2^nU\left(\frac{1}{2}-\frac{1}{2}n,\frac{3}{2},x^2\right)$$

22.5.56 $$H_{2m}(x)=(-1)^m\frac{(2m)!}{m!}M\left(-m,\frac{1}{2},x^2\right)$$

22.5.57

* $$H_{2m+1}(x)=(-1)^m\frac{(2m+1)!}{m!}2xM\left(-m,\frac{3}{2},x^2\right)$$

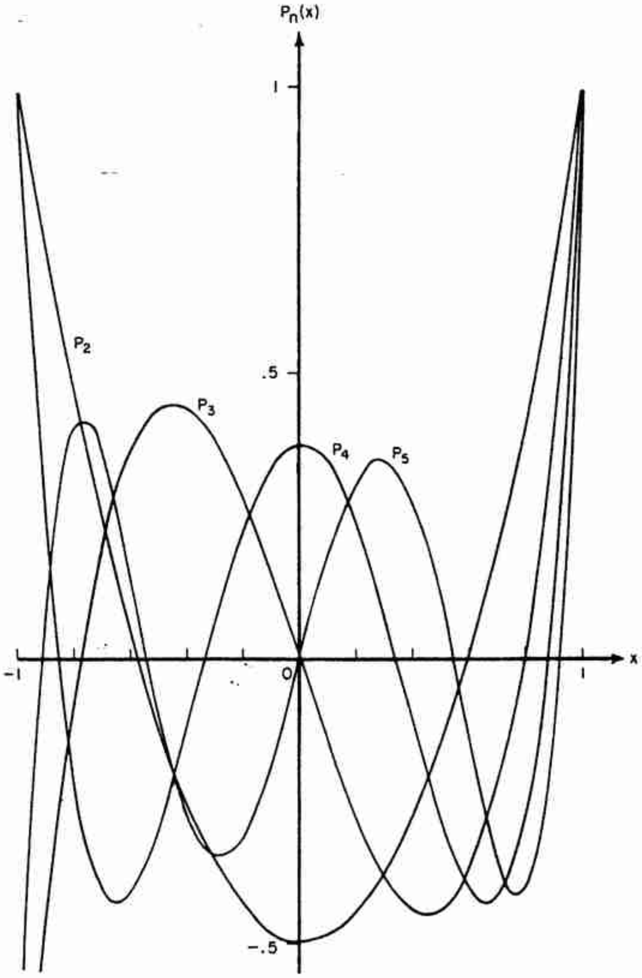

FIGURE 22.8. *Legendre Polynomials* $P_n(x)$, $n=2(1)5$.

22.5.58

$$H_n(x)=2^{n/2}e^{x^2/2}D_n(\sqrt{2}x)=2^{n/2}e^{x^2/2}U\left(-n-\frac{1}{2},\sqrt{2}x\right)$$

22.5.59 $$He_n(x)=e^{x^2/4}D_n(x)=e^{x^2/4}U\left(-n-\frac{1}{2},x\right)$$

Orthogonal Polynomials as Legendre Functions (see chapter 8)

22.5.60

$$C_n^{(\alpha)}(x)=\frac{\Gamma(\alpha+\frac{1}{2})\Gamma(2\alpha+n)}{n!\Gamma(2\alpha)}\left[\frac{1}{4}(x^2-1)\right]^{\frac{1}{4}-\frac{\alpha}{2}}P^{(\frac{1}{2}-\alpha)}_{n+\alpha-\frac{1}{2}}(x)\qquad(\alpha\neq0)$$

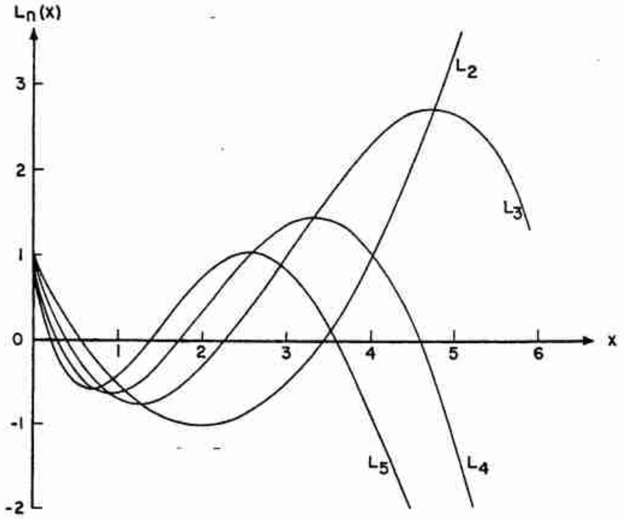

FIGURE 22.9. *Laguerre Polynomials* $L_n(x)$, $n=2(1)5$.

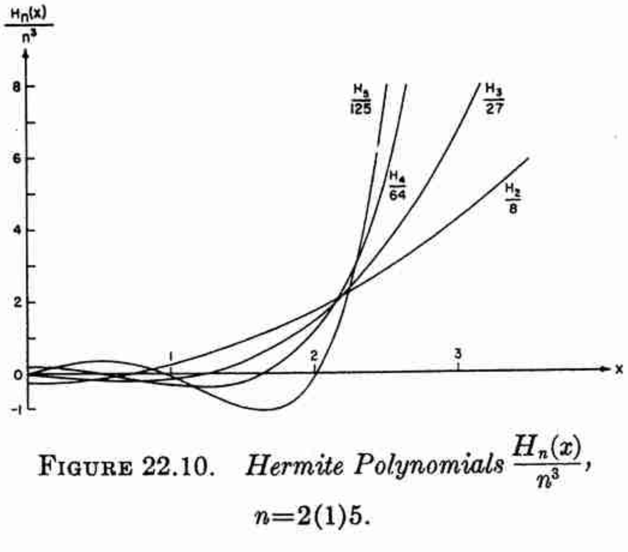

FIGURE 22.10. *Hermite Polynomials* $\frac{H_n(x)}{n^3}$, $n=2(1)5$.

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*See page II.

22.6. Differential Equations

$$g_2(x)y''+g_1(x)y'+g_0(x)y=0$$

	y	$g_2(x)$	$g_1(x)$	$g_0(x)$
22.6.1	$P_n^{(\alpha,\beta)}(x)$	$1-x^2$	$\beta-\alpha-(\alpha+\beta+2)x$	$n(n+\alpha+\beta+1)$
22.6.2	$(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)$	$1-x^2$	$\alpha-\beta+(\alpha+\beta-2)x$	$(n+1)(n+\alpha+\beta)$
22.6.3	$(1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}}P_n^{(\alpha,\beta)}(x)$	1	0	$\frac{1}{4}\frac{1-\alpha^2}{(1-x)^2}+\frac{1}{4}\frac{1-\beta^2}{(1+x)^2}+\frac{2n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)}{2(1-x^2)}$
22.6.4	$\left(\sin\frac{x}{2}\right)^{\alpha+\frac{1}{2}}\left(\cos\frac{x}{2}\right)^{\beta+\frac{1}{2}}P_n^{(\alpha,\beta)}(\cos x)$	1	0	$\frac{1-4\alpha^2}{16\sin^2\frac{x}{2}}+\frac{1-4\beta^2}{16\cos^2\frac{x}{2}}+\left(n+\frac{\alpha+\beta+1}{2}\right)^2$
22.6.5	$C_n^{(\alpha)}(x)$	$1-x^2$	$-(2\alpha+1)x$	$n(n+2\alpha)$
22.6.6	$(1-x^2)^{\alpha-\frac{1}{2}}C_n^{(\alpha)}(x)$	$1-x^2$	$(2\alpha-3)x$	$(n+1)(n+2\alpha-1)$
22.6.7	$(1-x^2)^{\frac{\alpha}{2}+\frac{1}{4}}C_n^{(\alpha)}(x)$	1	0	$\frac{(n+\alpha)^2}{1-x^2}+\frac{2+4\alpha-4\alpha^2+x^2}{4(1-x^2)^2}$
22.6.8	$(\sin x)^\alpha C_n^{(\alpha)}(\cos x)$	1	0	$(n+\alpha)^2+\frac{\alpha(1-\alpha)}{\sin^2 x}$
22.6.9	$T_n(x)$	$1-x^2$	$-x$	n^2
22.6.10	$T_n(\cos x)$	1	0	n^2
22.6.11	$\frac{1}{\sqrt{1-x^2}}T_n(x);\ U_{n-1}(x)$ *	$1-x^2$	$-3x$	n^2-1
22.6.12	$U_n(x)$	$1-x^2$	$-3x$	$n(n+2)$
22.6.13	$P_n(x)$	$1-x^2$	$-2x$	$n(n+1)$
22.6.14	$\sqrt{1-x^2}P_n(x)$	1	0	$\frac{n(n+1)}{1-x^2}+\frac{1}{(1-x^2)^2}$
22.6.15	$L_n^{(\alpha)}(x)$	x	$\alpha+1-x$	n
22.6.16	$e^{-x}x^{\alpha/2}L_n^{(\alpha)}(x)$ *	x	$x+1$	$n+\frac{\alpha}{2}+1-\frac{\alpha^2}{4x}$
22.6.17	$e^{-x/2}x^{(\alpha+1)/2}L_n^{(\alpha)}(x)$	1	0	$\frac{2n+\alpha+1}{2x}+\frac{1-\alpha^2}{4x^2}-\frac{1}{4}$
22.6.18	$e^{-x^2/2}x^{\alpha+\frac{1}{2}}L_n^{(\alpha)}(x^2)$	1	0	$4n+2\alpha+2-x^2+\frac{1-4\alpha^2}{4x^2}$
22.6.19	$H_n(x)$	1	$-2x$	$2n$
22.6.20	$e^{-\frac{x^2}{2}}H_n(x)$	1	0	$2n+1-x^2$
22.6.21	$He_n(x)$	1	$-x$	n

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*See page II.

22.7. Recurrence Relations

Recurrence Relations With Respect to the Degree n

$$a_{1n}f_{n+1}(x)=(a_{2n}+a_{3n}x)f_n(x)-a_{4n}f_{n-1}(x)$$

	f_n	a_{1n}	a_{2n}	a_{3n}	a_{4n}
22.7.1	$P_n^{(\alpha,\beta)}(x)$	$2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)$	$(2n+\alpha+\beta+1)(\alpha^2-\beta^2)$	$(2n+\alpha+\beta)_3$	$2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)$
22.7.2	$G_n(p, q, x)$	$(2n+p-2)_4(2n+p-1)$	$-[2n(n+p)+q(p-1)](2n+p-2)_3$	$(2n+p-2)_4(2n+p-1)$	$n(n+q-1)(n+p-1)(n+p-q)(2n+p+1)$
22.7.3	$C_n^{(\alpha)}(x)$	$n+1$	0	$2(n+\alpha)$	$n+2\alpha-1$
22.7.4	$T_n(x)$	1	0	2	1
22.7.5	$U_n(x)$	1	0	2	1
22.7.6	$S_n(x)$	1	0	1	1
22.7.7	$C_n(x)$	1	0	1	1
22.7.8	$T_n^*(x)$	1	-2	4	1
22.7.9	$U_n^*(x)$	1	-2	4	1
22.7.10	$P_n(x)$	$n+1$	0	$2n+1$	n
22.7.11	$P_n^*(x)$	$n+1$	$-2n-1$	$4n+2$	n
22.7.12	$L_n^{(\alpha)}(x)$	$n+1$	$2n+\alpha+1$	-1	$n+\alpha$
22.7.13	$H_n(x)$	1	0	2	$2n$
22.7.14	$He_n(x)$	1	0	1	n

Miscellaneous Recurrence Relations

Jacobi Polynomials

22.7.15

$$\left(n+\frac{\alpha}{2}+\frac{\beta}{2}+1\right)(1-x)P_n^{(\alpha+1,\beta)}(x)$$
$$=(n+\alpha+1)P_n{}^{(\alpha,\beta)}(x)-(n+1)P_{n+1}^{(\alpha,\beta)}(x)$$

22.7.16

$$\left(n+\frac{\alpha}{2}+\frac{\beta}{2}+1\right)(1+x)P_n^{(\alpha,\beta+1)}(x)$$
$$=(n+\beta+1)P_n^{(\alpha,\beta)}(x)+(n+1)P_{n+1}^{(\alpha,\beta)}(x)$$

22.7.17

$$(1-x)P_n^{(\alpha+1,\beta)}(x)+(1+x)P_n^{(\alpha,\beta+1)}(x)=2P_n^{(\alpha,\beta)}(x)$$

22.7.18

$$(2n+\alpha+\beta)P_n^{(\alpha-1,\beta)}(x)=(n+\alpha+\beta)P_n^{(\alpha,\beta)}(x)$$
$$-(n+\beta)P_{n-1}^{(\alpha,\beta)}(x)$$

22.7.19

$$(2n+\alpha+\beta)P_n^{(\alpha,\beta-1)}(x)=(n+\alpha+\beta)P_n^{(\alpha,\beta)}(x)$$
$$+(n+\alpha)P_{n-1}^{(\alpha,\beta)}(x)$$

22.7.20 $P_n^{(\alpha,\beta-1)}(x)-P_n^{(\alpha-1,\beta)}(x)=P_{n-1}^{(\alpha,\beta)}(x)$

Ultraspherical Polynomials

22.7.21

$$2\alpha(1-x^2)C_{n-1}^{(\alpha+1)}(x)=(2\alpha+n-1)C_{n-1}^{(\alpha)}(x)-nxC_n^{(\alpha)}(x)$$

22.7.22

$$=(n+2\alpha)xC_n^{(\alpha)}(x)$$
$$-(n+1)C_{n+1}^{(\alpha)}(x)$$

22.7.23 $(n+\alpha)C_{n+1}^{(\alpha-1)}(x)=(\alpha-1)[C_{n+1}^{(\alpha)}(x)-C_{n-1}^{(\alpha)}(x)]$

Chebyshev Polynomials

22.7.24

$$2T_m(x)T_n(x)=T_{n+m}(x)+T_{n-m}(x) \qquad (n\geq m) \qquad *$$

22.7.25

$$2(x^2-1)U_{m-1}(x)U_{n-1}(x)=T_{n+m}(x)-T_{n-m}(x) \qquad (n\geq m)$$

22.7.26

$$2T_m(x)U_{n-1}(x)=U_{n+m-1}(x)+U_{n-m-1}(x) \qquad (n>m)$$

22.7.27

$$2T_n(x)U_{m-1}(x)=U_{n+m-1}(x)-U_{n-m-1}(x) \qquad (n>m)$$

22.7.28 $2T_n(x)U_{n-1}(x)=U_{2n-1}(x)$

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*See page II.

Generalized Laguerre Polynomials

22.7.29

$$L_n^{(\alpha+1)}(x)=\frac{1}{x}\left[(x-n)L_n^{(\alpha)}(x)+(\alpha+n)L_{n-1}^{(\alpha)}(x)\right]$$

22.7.30 $$L_n^{(\alpha-1)}(x)=L_n^{(\alpha)}(x)-L_{n-1}^{(\alpha)}(x)$$

22.7.31

$$L_n^{(\alpha+1)}(x)=\frac{1}{x}\left[(n+\alpha+1)L_n^{(\alpha)}(x)-(n+1)L_{n+1}^{(\alpha)}(x)\right]$$

22.7.32

$$L_n^{(\alpha-1)}(x)=\frac{1}{n+\alpha}\left[(n+1)L_{n+1}^{(\alpha)}(x)-(n+1-x)L_n^{(\alpha)}(x)\right]$$

22.8. Differential Relations

$$g_2(x)\frac{d}{dx}f_n(x)=g_1(x)f_n(x)+g_0(x)f_{n-1}(x)$$

	f_n	g_2	g_1	g_0
22.8.1	$P_n^{(\alpha,\beta)}(x)$	$(2n+\alpha+\beta)(1-x^2)$	$n[\alpha-\beta-(2n+\alpha+\beta)x]$	$2(n+\alpha)(n+\beta)$
22.8.2	$C_n^{(\alpha)}(x)$	$1-x^2$	$-nx$	$n+2\alpha-1$
22.8.3	$T_n(x)$	$1-x^2$	$-nx$	n
22.8.4	$U_n(x)$	$1-x^2$	$-nx$	$n+1$
22.8.5	$P_n(x)$	$1-x^2$	$-nx$	n
22.8.6	$L_n^{(\alpha)}(x)$	x	n	$-(n+\alpha)$
22.8.7	$H_n(x)$	1	0	$2n$
22.8.8	$He_n(x)$	1	0	n

22.9. Generating Functions

$$g(x,z)=\sum_{n=0}^{\infty}a_nf_n(x)z^n \qquad R=\sqrt{1-2xz+z^2}$$

	$f_n(x)$	a_n	$g(x,z)$	Remarks
22.9.1	$P_n^{(\alpha,\beta)}(x)$	$2^{-\alpha-\beta}$	$R^{-1}(1-z+R)^{-\alpha}(1+z+R)^{-\beta}$	$\lvert z\rvert<1$
22.9.2	$C_n^{(\alpha)}(x)$	$\dfrac{2^{\frac{1}{2}-\alpha}\Gamma(\alpha+\frac{1}{2}+n)\Gamma(2\alpha)}{\Gamma(\alpha+\frac{1}{2})\Gamma(2\alpha+n)}$	$R^{-1}(1-xz+R)^{\frac{1}{2}-\alpha}$	$\lvert z\rvert<1, \alpha\neq0$
22.9.3	$C_n^{(\alpha)}(x)$	1	$R^{-2\alpha}$	$\lvert z\rvert<1, \alpha\neq0$
22.9.4	$C_n^{(0)}(x)$	1	$-\ln R^2$	$\lvert z\rvert<1$
22.9.5	$C_n^{(\alpha)}(x)$	$\dfrac{\Gamma(2\alpha)}{\Gamma(\alpha+\frac{1}{2})\Gamma(2\alpha+n)}$	$e^{z\cos\theta}\left(\frac{z}{2}\sin\theta\right)^{\frac{1}{2}-\alpha}J_{\alpha-\frac{1}{2}}(z\sin\theta)$	$x=\cos\theta$
22.9.6	$T_n(x)$	2	$\left(\frac{1-z^2}{R^2}+1\right)$	$-1<x<1$ $\lvert z\rvert<1$
22.9.7	$T_n(x)$	$\frac{\sqrt{2}}{4^n}\binom{2n}{n}$	$R^{-1}(1-xz+R)^{1/2}$	$-1<x<1$ $\lvert z\rvert<1$
22.9.8	$T_n(x)$	$\frac{1}{n}$	$1-\frac{1}{2}\ln R^2$	$a_0=1$ $-1<x<1$ $\lvert z\rvert<1$
22.9.9	$T_n(x)$	1	$\frac{1-xz}{R^2}$	$-1<x<1$ $\lvert z\rvert<1$
22.9.10	$U_n(x)$	1	R^{-2}	$-1<x<1$ $\lvert z\rvert<1$
22.9.11	$U_n(x)$	$\frac{\sqrt{2}}{4^{n+1}}\binom{2n+2}{n+1}$	$\frac{1}{R}(1-xz+R)^{-1/2}$ *	$-1<x<1$ $\lvert z\rvert<1$

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*See page II.

22.9. Generating Functions—Continued

$$g(x,z)=\sum_{n=0}^{\infty} a_n f_n(x) z^n \qquad R=\sqrt{1-2xz+z^2}$$

	$f_n(x)$	a_n	$g(x,z)$	Remarks
22.9.12	$P_n(x)$	1	R^{-1}	$-1<x<1$ $\lvert z\rvert<1$
22.9.13	$P_n(x)$	$\frac{1}{n!}$	$e^{z\cos\theta}J_0(z\sin\theta)$	$x=\cos\theta$
22.9.14	$S_n(x)$	1	$(1-xz+z^2)^{-1}$	$-2<x<2$ $\lvert z\rvert<1$
22.9.15	$L_n^{(\alpha)}(x)$	1	$(1-z)^{-\alpha-1}\exp\left(\frac{xz}{z-1}\right)$	$\lvert z\rvert<1$
22.9.16	$L_n^{(\alpha)}(x)$	$\frac{1}{\Gamma(n+\alpha+1)}$	$(xz)^{-\frac{1}{2}\alpha}e^zJ_\alpha[2(xz)^{1/2}]$	
22.9.17	$H_n(x)$	$\frac{1}{n!}$	e^{2xz-z^2}	
22.9.18	$H_{2n}(x)$	$\frac{(-1)^n}{(2n)!}$	$e^z\cos(2x\sqrt{z})$ *	
22.9.19	$H_{2n+1}(x)$	$\frac{(-1)^n}{(2n+1)!}$	$z^{-1/2}e^z\sin(2x\sqrt{z})$ *	

22.10. Integral Representations

Contour Integral Representations

$f_n(x)=\frac{g_0(x)}{2\pi i}\int_C [g_1(z,x)]^n g_2(z,x)dz$ where C is a closed contour taken around $z=a$ in the positive sense

	$f_n(x)$	$g_0(x)$	$g_1(z,x)$	$g_2(z,x)$	a	Remarks
22.10.1	$P_n^{(\alpha,\beta)}(x)$	$\frac{1}{(1-x)^\alpha(1+x)^\beta}$	$\frac{z^2-1}{2(z-x)}$	$\frac{(1-z)^\alpha(1+z)^\beta}{z-x}$	x	±1 outside C
22.10.2	$C_n^{(\alpha)}(x)$	1	$1/z$	$(1-2xz+z^2)^{-\alpha}z^{-1}$	0	Both zeros of $1-2xz+z^2$ outside C, $\alpha>0$
22.10.3	$T_n(x)$	$1/2$	$1/z$	$\frac{1-z^2}{z(1-2xz+z^2)}$	0	Both zeros of $1-2xz+z^2$ outside C
22.10.4	$U_n(x)$	1	$1/z$	$\frac{1}{z(1-2xz+z^2)}$	0	Both zeros of $1-2xz+z^2$ outside C
22.10.5	$P_n(x)$	1	$1/z$	$\frac{1}{z}(1-2xz+z^2)^{-1/2}$	0	Both zeros of $1-2xz+z^2$ outside C
22.10.6	$P_n(x)$	$\frac{1}{2^n}$	$\frac{z^2-1}{z-x}$	$\frac{1}{z-x}$	x	
22.10.7	$L_n^{(\alpha)}(x)$	$e^xx^{-\alpha}$	$\frac{z}{z-x}$	$\frac{z^\alpha}{z-x}e^{-z}$	x	Zero outside C
22.10.8	$L_n^{(\alpha)}(x)$	1	$1+\frac{x}{z}$	$e^{-z}\left(1+\frac{z}{x}\right)^\alpha 1/z$	0	$z=-x$ outside C
22.10.9	$H_n(x)$	$n!$	$1/z$	$\frac{e^{2xz-z^2}}{z}$	0	

Miscellaneous Integral Representations

22.10.10 $$C_n^{(\alpha)}(x)=\frac{2^{(1-2\alpha)}\Gamma(n+2\alpha)}{n![\Gamma(\alpha)]^2}\int_0^\pi [x+\sqrt{x^2-1}\cos\phi]^n(\sin\phi)^{2\alpha-1}d\phi \qquad (\alpha>0)$$

22.10.11 $$C_n^{(\alpha)}(\cos\theta)=\frac{2^{1-\alpha}\Gamma(n+2\alpha)}{n![\Gamma(\alpha)]^2}(\sin\theta)^{1-2\alpha}\int_0^\theta \frac{\cos(n+\alpha)\phi}{(\cos\phi-\cos\theta)^{1-\alpha}}d\phi \qquad (\alpha>0)$$

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*See page II.

22.10.12 $P_n(\cos\theta)=\frac{1}{\pi}\int_0^\pi (\cos\theta+i\sin\theta\cos\phi)^n d\phi$

22.10.13 $P_n(\cos\theta)=\frac{\sqrt{2}}{\pi}\int_\theta^\pi \frac{\sin(n+\frac{1}{2})\phi d\phi}{(\cos\theta-\cos\phi)^{\frac{1}{2}}}$

22.10.14 $L_n^{(\alpha)}(x)=\frac{e^x x^{-\frac{\alpha}{2}}}{n!}\int_0^\infty e^{-t}t^{n+\frac{\alpha}{2}} J_\alpha(2\sqrt{tx})dt$

22.10.15

$$H_n(x)=e^{x^2}\frac{2^{n+1}}{\sqrt{\pi}}\int_0^\infty e^{-t^2}t^n\cos\left(2xt-\frac{n}{2}\pi\right)dt$$

22.11. Rodrigues' Formula

$$f_n(x)=\frac{1}{a_n\rho(x)}\frac{d^n}{dx^n}\{\rho(x)(g(x))^n\}$$

The polynomials given in the following table are the only orthogonal polynomials which satisfy this formula.

	$f_n(x)$	a_n	$\rho(x)$	$g(x)$
22.11.1	$P_n^{(\alpha,\beta)}(x)$	$(-1)^n 2^n n!$	$(1-x)^\alpha(1+x)^\beta$	$1-x^2$
22.11.2	$C_n^{(\alpha)}(x)$	$(-1)^n 2^n n!\frac{\Gamma(2\alpha)\Gamma(\alpha+n+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\Gamma(n+2\alpha)}$	$(1-x^2)^{\alpha-\frac{1}{2}}$	$1-x^2$
22.11.3	$T_n(x)$	$(-1)^n 2^n \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}}$ *	$(1-x^2)^{-\frac{1}{2}}$	$1-x^2$
22.11.4	$U_n(x)$	$(-1)^n 2^{n+1}\frac{\Gamma(n+\frac{3}{2})}{(n+1)\sqrt{\pi}}$	$(1-x^2)^{\frac{1}{2}}$	$1-x^2$
22.11.5	$P_n(x)$	$(-1)^n 2^n n!$	1	$1-x^2$
22.11.6	$L_n^{(\alpha)}(x)$	$n!$	$e^{-x}x^\alpha$	x
22.11.7	$H_n(x)$	$(-1)^n$	e^{-x^2}	1
22.11.8	$He_n(x)$	$(-1)^n$	$e^{-x^2/2}$	1

22.12. Sum Formulas

Christoffel-Darboux Formula

22.12.1

$$\sum_{m=0}^{n}\frac{1}{h_m}f_m(x)f_m(y)=\frac{k_n}{k_{n+1}h_n}\frac{f_{n+1}(x)f_n(y)-f_n(x)f_{n+1}(y)}{x-y}$$

Miscellaneous Sum Formulas (Only a Limited Selection Is Given Here.)

22.12.2 $\sum_{m=0}^{n} T_{2m}(x)=\frac{1}{2}[1+U_{2n}(x)]$

22.12.3 $\sum_{m=0}^{n-1} T_{2m+1}(x)=\frac{1}{2}U_{2n-1}(x)$

22.12.4 $\sum_{m=0}^{n} U_{2m}(x)=\frac{1-T_{2n+2}(x)}{2(1-x^2)}$

22.12.5 $\sum_{m=0}^{n-1} U_{2m+1}(x)=\frac{x-T_{2n+1}(x)}{2(1-x^2)}$

22.12.6 $\sum_{m=0}^{n} L_m^{(\alpha)}(x)L_{n-m}^{(\beta)}(y)=L_n^{(\alpha+\beta+1)}(x+y)$

22.12.7 $\sum_{m=0}^{n}\binom{n+\alpha}{m}\mu^{n-m}(1-\mu)^m L_{n-m}^{(\alpha)}(x)=L_n^{(\alpha)}(\mu x)$

22.12.8

$$H_n(x+y)=\frac{1}{2^{n/2}}\sum_{k=0}^{n}\binom{n}{k}H_k(\sqrt{2}x)H_{n-k}(\sqrt{2}y)$$

22.13. Integrals Involving Orthogonal Polynomials

22.13.1

$$2n\int_0^x (1-y)^\alpha(1+y)^\beta P_n^{(\alpha,\beta)}(y)dy$$
$$=P_{n-1}^{(\alpha+1,\beta+1)}(0)-(1-x)^{\alpha+1}(1+x)^{\beta+1}P_{n-1}^{(\alpha+1,\beta+1)}(x)$$

22.13.2

$$\frac{n(2\alpha+n)}{2\alpha}\int_0^x (1-y^2)^{\alpha-\frac{1}{2}}C_n^{(\alpha)}(y)dy$$
$$=C_{n-1}^{(\alpha+1)}(0)-(1-x^2)^{\alpha+\frac{1}{2}}C_{n-1}^{(\alpha+1)}(x)$$

22.13.3 $\fint_{-1}^{1}\frac{T_n(y)dy}{(y-x)\sqrt{1-y^2}}=\pi U_{n-1}(x)$

22.13.4 $\fint_{-1}^{1}\frac{\sqrt{1-y^2}U_{n-1}(y)dy}{(y-x)}=-\pi T_n(x)$ *

22.13.5 $\int_{-1}^{1}(1-x)^{-1/2}P_n(x)dx=\frac{2^{3/2}}{2n+1}$ *

22.13.6 $\int_0^\pi P_{2n}(\cos\theta)d\theta=\frac{\pi}{16^n}\binom{2n}{n}^2$

22.13.7 $\int_0^\pi P_{2n+1}(\cos\theta)\cos\theta d\theta=\frac{\pi}{4^{2n+1}}\binom{2n}{n}\binom{2n+2}{n+1}$

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*See page II.

22.13.8

$$\int_0^1 x^\lambda P_{2n}(x)dx=\frac{(-1)^n\Gamma\left(n-\frac{\lambda}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\lambda}{2}\right)}{2\Gamma\left(-\frac{\lambda}{2}\right)\Gamma\left(n+\frac{3}{2}+\frac{\lambda}{2}\right)}\quad(\lambda>-1)$$

22.13.9

$$\int_0^1 x^\lambda P_{2n+1}(x)dx=\frac{(-1)^n\Gamma\left(n+\frac{1}{2}-\frac{\lambda}{2}\right)\Gamma\left(1+\frac{\lambda}{2}\right)}{2\Gamma\left(n+2+\frac{\lambda}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\lambda}{2}\right)}\quad(\lambda>-2)$$

22.13.10

$$\int_{-1}^x \frac{P_n(t)dt}{\sqrt{x-t}}=\frac{1}{(n+\frac{1}{2})\sqrt{1+x}}[T_n(x)+T_{n+1}(x)]$$

22.13.11

$$\int_x^1 \frac{P_n(t)dt}{\sqrt{t-x}}=\frac{1}{(n+\frac{1}{2})\sqrt{1-x}}[T_n(x)-T_{n+1}(x)]$$

22.13.12 $$\int_x^\infty e^{-t}L_n^{(\alpha)}(t)dt=e^{-x}[L_n^{(\alpha)}(x)-L_{n-1}^{(\alpha)}(x)]$$

22.13.13

$$\Gamma(\alpha+\beta+n+1)\int_0^x (x-t)^{\beta-1}t^\alpha L_n^{(\alpha)}(t)dt$$
$$=\Gamma(\alpha+n+1)\Gamma(\beta)x^{\alpha+\beta}L_n^{(\alpha+\beta)}(x)$$
$$(\mathscr{R}\alpha>-1,\ \mathscr{R}\beta>0)$$

22.13.14

$$\int_0^x L_m(t)L_n(x-t)dt$$
$$=\int_0^x L_{m+n}(t)dt=L_{m+n}(x)-L_{m+n+1}(x)$$

22.13.15 $$\int_0^x e^{-t^2}H_n(t)dt=H_{n-1}(0)-e^{-x^2}H_{n-1}(x)$$

22.13.16 $$\int_0^x H_n(t)dt=\frac{1}{2(n+1)}[H_{n+1}(x)-H_{n+1}(0)]$$

22.13.17 $$\int_{-\infty}^\infty e^{-t^2}H_{2m}(tx)dt=\sqrt{\pi}\,\frac{(2m)!}{m!}(x^2-1)^m$$

22.13.18

$$\int_{-\infty}^\infty e^{-t^2}tH_{2m+1}(tx)dt=\sqrt{\pi}\,\frac{(2m+1)!}{m!}x(x^2-1)^m$$

22.13.19 $$\int_{-\infty}^\infty e^{-t^2}t^nH_n(xt)dt=\sqrt{\pi}n!P_n(x)$$

22.13.20

$$\int_0^\infty e^{-t^2}[H_n(t)]^2\cos(xt)dt=\sqrt{\pi}2^{n-1}n!e^{-\frac{1}{4}x^2}L_n\left(\frac{x^2}{2}\right)$$

22.14. Inequalities

22.14.1

$$|P_n^{(\alpha,\beta)}(x)|\leqq\begin{cases}\binom{n+q}{n}\approx n^q, \text{ if } q=\max(\alpha,\beta)\geq-1/2\\ \qquad(\alpha>-1,\beta>-1)\\ |P_n^{(\alpha,\beta)}(x')|\approx\sqrt{\frac{1}{n}}, \text{ if } q<-\frac{1}{2}\end{cases}$$

x' maximum point nearest to $\frac{\beta-\alpha}{\alpha+\beta+1}$

22.14.2

$$|C_n^{(\alpha)}(x)|\leq\begin{cases}\binom{n+2\alpha-1}{n} & (\alpha>0)\\ |C_n^{(\alpha)}(x')| & \left(-\frac{1}{2}<\alpha<0\right)\end{cases}$$

$x'=0$ if $n=2m$; x'=maximum point nearest zero if $n=2m+1$

22.14.3

$$|C_n^{(\alpha)}(\cos\theta)|<2^{1-\alpha}\frac{n^{\alpha-1}}{(\sin\theta)^\alpha\Gamma(\alpha)}\quad(0<\alpha<1, 0<\theta<\pi)$$

22.14.4 $$|T_n(x)|\leq1\qquad(-1\leq x\leq1)$$

22.14.5 $$\left|\frac{dT_n(x)}{dx}\right|\leq n^2\qquad(-1\leq x\leq1)$$

22.14.6 $$|U_n(x)|\leq n+1\qquad(-1\leq x\leq1)$$

22.14.7 $$|P_n(x)|\leq1\qquad(-1\leq x\leq1)$$

22.14.8 $$\left|\frac{dP_n(x)}{dx}\right|\leq\frac{1}{2}n(n+1)\qquad(-1\leq x\leq1)$$

22.14.9 $$|P_n(x)|\leq\sqrt{\frac{2}{\pi n}}\frac{1}{\sqrt[4]{1-x^2}}\qquad(-1<x\leq1)^*$$

22.14.10

$$P_n^2(x)-P_{n-1}(x)P_{n+1}(x)<\frac{2n+1}{3n(n+1)}\qquad(-1\leq x\leq1)$$

22.14.11

$$P_n^2(x)-P_{n-1}(x)P_{n+1}(x)\geq\frac{1-P_n^2(x)}{(2n-1)(n+1)}$$
$$(-1\leq x\leq1)$$

22.14.12 $$|L_n(x)|\leq e^{x/2}\qquad(x\geq0)$$

22.14.13 $$|L_n^{(\alpha)}(x)|\leq\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}e^{x/2}\qquad(\alpha\geq0,\ x\geq0)$$

22.14.14

$$|L_n^{(\alpha)}(x)|\leq\left[2-\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)}\right]e^{x/2}\qquad(-1<\alpha<0,\ x\geq0)$$

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*See page II.

22.14.15 $|H_{2m}(x)| \leq e^{x^2/2} 2^{2m} m! \left[2 - \frac{1}{2^{2m}} \binom{2m}{m}\right]$

22.14.16 $|H_{2m+1}(x)| \leq x e^{x^2/2} \frac{(2m+2)!}{(m+1)!} \quad (x \geq 0)$

22.14.17 $|H_n(x)| < e^{x^2/2} k 2^{n/2} \sqrt{n!} \qquad k \approx 1.086435$

22.15. Limit Relations

22.15.1

$$\lim_{n\to\infty} \left[\frac{1}{n^\alpha} P_n^{(\alpha,\beta)} \left(\cos \frac{x}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^\alpha} P_n^{(\alpha,\beta)} \left(1 - \frac{x^2}{2n^2}\right) = \left(\frac{2}{x}\right)^\alpha J_\alpha(x)$$

22.15.2 $\lim_{n\to\infty} \left[\frac{1}{n^\alpha} L_n^{(\alpha)} \left(\frac{x}{n}\right)\right] = x^{-\alpha/2} J_\alpha(2\sqrt{x})$

22.15.3 $\lim_{n\to\infty} \left[\frac{(-1)^n \sqrt{n}}{4^n n!} H_{2n} \left(\frac{x}{2\sqrt{n}}\right)\right] = \frac{1}{\sqrt{\pi}} \cos x$

22.15.4 $\lim_{n\to\infty} \left[\frac{(-1)^n}{4^n n!} H_{2n+1} \left(\frac{x}{2\sqrt{n}}\right)\right] = \frac{2}{\sqrt{\pi}} \sin x$

22.15.5 $\lim_{\beta\to\infty} P_n^{(\alpha,\beta)} \left(1 - \frac{2x}{\beta}\right) = L_n^{(\alpha)}(x)$

22.15.6 $\lim_{\alpha\to\infty} \frac{1}{\alpha^{n/2}} C_n^{(\alpha)} \left(\frac{x}{\sqrt{\alpha}}\right) = \frac{1}{n!} H_n(x)$

For asymptotic expansions, see [22.5] and [22.17].

22.16. Zeros

For tables of the zeros and associated weight factors necessary for the Gaussian-type quadrature formulas see chapter **25**. All the zeros of the orthogonal polynomials are real, simple and located in the interior of the interval of orthogonality.

Explicit and Asymptotic Formulas and Inequalities

Notations:

$x_m^{(n)}$ mth zero of $f_n(x)$ $(x_1^{(n)} < x_2^{(n)} < \ldots < x_n^{(n)})$

$\theta_m^{(n)} = \arccos x_{n-m+1}^{(n)} (0 < \theta_1^{(n)} < \theta_2^{(n)} < \ldots < \theta_n^{(n)} < \pi)$

$j_{\alpha,m}$, mth positive zero of the Bessel function $J_\alpha(x)$

$0 < j_{\alpha,1} < j_{\alpha,2} < \ldots$

	$f_n(x)$	Relation
22.16.1	$P_n^{(\alpha,\beta)}(\cos\theta)$	$\lim_{n\to\infty} n\theta_m^{(n)} = j_{\alpha,m} \qquad (\alpha > -1, \beta > -1)$
22.16.2	$C_n^{(\alpha)}(x)$	$x_m^{(n)} = 1 - \frac{j^2_{\alpha-\frac{1}{2},m}}{2n^2} \left[1 - \frac{2\alpha}{n} + O\left(\frac{1}{n^2}\right)\right]$
22.16.3	$C_n^{(\alpha)}(\cos\theta)$	$\frac{(m+\alpha-1)\pi}{n+\alpha} \leq \theta_m^{(n)} \leq \frac{m\pi}{n+\alpha} \qquad (0 \leq \alpha \leq 1)$
22.16.4	$T_n(x)$	$x_m^{(n)} = \cos \frac{2m-1}{2n} \pi$
22.16.5	$U_n(x)$	$x_m^{(n)} = \cos \frac{m}{n+1} \pi$
22.16.6	$P_n(\cos\theta)$	$\begin{cases} \frac{2m-1}{2n+1} \pi \leq \theta_m^{(n)} \leq \frac{2m}{2n+1} \pi \\ \theta_m^{(n)} = \frac{4m-1}{4n+2} \pi + \frac{1}{8n^2} \cot \frac{4m-1}{4n+2} \pi + O(n^{-3}) \end{cases}$
22.16.7	$P_n(x)$	$\begin{cases} x_m^{(n)} = 1 - \frac{j^2_{0,m}}{2n^2} [1 - \frac{1}{n} + O(n^{-2})] \\ x_m^{(n)} = 1 - \frac{4\xi_m^{(n)}}{2n+1+\xi_m^{(n)}};\ \xi_m^{(n)} = \frac{j^2_{0,m}}{4n+2} \left[1 + \frac{j^2_{0,m-2}}{12(2n+1)^2}\right] + O\left(\frac{1}{n^5}\right) \end{cases}$
22.16.8	$L_n^{(\alpha)}(x)$	$\left.\begin{cases} x_m^{(n)} > \frac{j^2_{\alpha,m}}{4k_n} \\ x_m^{(n)} < \frac{k_m}{k_n} (2k_m + \sqrt{4k_m^2 + \frac{1}{4} - \alpha^2}) \\ x_m^{(n)} = \frac{j^2_{\alpha,m}}{4k_n} \left(1 + \frac{2(\alpha^2-1) + j^2_{\alpha,m}}{48k_n^2}\right) + O(n^{-5}) \end{cases}\right\} k_\tau = \tau + \frac{\alpha+1}{2}$

For error estimates see [22.6].

22.17. Orthogonal Polynomials of a Discrete Variable

In this section some polynomials $f_n(x)$ are listed which are orthogonal with respect to the scalar product

22.17.1 $$(f_n, f_m)=\sum_i w^*(x_i)f_n(x_i)f_m(x_i).$$

The x_i are the integers in the interval $a \le x_i \le b$ and $w^*(x_i)$ is a positive function such that $\sum_i w^*(x_i)$ is finite. The constant factor which is still free in each polynomial when only the orthogonality condition is given is defined here by the explicit representation (which corresponds to the Rodrigues' formula)

22.17.2 $$f_n(x)=\frac{1}{r_n w^*(x)}\,\Delta^n[w^*(x)g(x,n)]$$

where $g(x, n)=g(x)g(x-1)\ .\ .\ .\ g(x-n+1)$ and $g(x)$ is a polynomial in x independent of n.

Name	a	b	$w^*(x)$	r_n	$g(x, n)$	Remarks
Chebyshev	0	$N-1$	1	$1/n!$	$\binom{x}{n}\binom{x-N}{n}$	
Krawtchouk	0	N	$p^x q^{N-x}\binom{N}{x}$	$(-1)^n n!$	$\frac{q^n x!}{(x-n)!}$	$p, q>0;$ $p+q=1$
Charlier	0	∞	$\frac{e^{-a}a^x}{x!}$	$(-1)^n\sqrt{a^n n!}$	$\frac{x!}{(x-n)!}$	$a>0$
Meixner	0	∞	$\frac{c^x\Gamma(b+x)}{\Gamma(b)x!}$	c^n	$\frac{x!}{(x-n)!}$	$b>0,\ 0<c<1$
Hahn	0	∞	$\frac{\Gamma(b)\Gamma(c+x)\Gamma(d+x)}{x!\Gamma(b+x)\Gamma(c)\Gamma(d)}$	$n!$	$\frac{x!\Gamma(b+x)}{(x-n)!\Gamma(b+x-n)}$	

For a more complete list of the properties of these polynomials see [22.5] and [22.17].

Numerical Methods

22.18. Use and Extension of the Tables

Evaluation of an orthogonal polynomial for which the coefficients are given numerically.

Example 1. Evaluate $L_6(1.5)$ and its first and second derivative using **Table 22.10** and the Horner scheme.

1	−36	450	−2400	5400	−4320	720
$x=1.5$	1. 5	−51. 75	597. 375	−2703. 9375	4044. 09375	−413. 859375
1	−34. 5	398. 25	−1802. 625	2696. 0625	−275. 90625	306. 140625
1.5	1. 5	−49. 5	523. 125	−1919. 25	1165. 21875	$L_6=\frac{306.140625}{720}$ $=.42519\ 53$
1	−33. 0	348. 75	−1279. 500	776. 8125	889. 3125	
1.5	1. 5	−47. 25	452. 250	−1240. 875		$L_6'=\frac{889.3125}{720}$ $=1.23515\ 625$
1	−31. 5	301. 50	−827. 250	−464. 0625		$L_6''=2\frac{[-464.0625]}{720}$ $=-1.28906\ 25$

Evaluation of an orthogonal polynomial using the explicit representation when the coefficients are not given numerically.

If an isolated value of the orthogonal polynomial $f_n(x)$ is to be computed, use the proper explicit expression rewritten in the form

$$f_n(x)=d_n(x)a_0(x)$$

and generate $a_0(x)$ recursively, where

$$a_{m-1}(x)=1-\frac{b_m}{c_m}f(x)a_m(x) \qquad (m=n,\ n-1,\ .\ .\ .,\ 2,\ 1,\ a_n(x)=1).$$

The $d_n(x)$, b_m, c_m, $f(x)$ for the polynomials of this chapter are listed in the following table:

$f_n(x)$	$d_n(x)$	b_m	c_m	$f(x)$
$P_n^{(\alpha,\beta)}$	$\binom{n+\alpha}{n}$	$(n-m+1)(\alpha+\beta+n+m)$	$2m(\alpha+m)$	$1-x$
$C_{2n}^{(\alpha)}$	$(-1)^n\frac{(\alpha)_n}{n!}$	$2(n-m+1)(\alpha+n+m-1)$	$m(2m-1)$	x^2
$C_{2n+1}^{(\alpha)}$	$(-1)^n\frac{(\alpha)_{n+1}}{n!}2x$	$2(n-m+1)(\alpha+n+m)$	$m(2m+1)$	x^2
T_{2n}	$(-1)^n$	$2(n-m+1)(n+m-1)$	$m(2m-1)$	x^2
T_{2n+1}	$(-1)^n(2n+1)x$	$2(n-m+1)(n+m)$	$m(2m+1)$	x^2
U_{2n}	$(-1)^n$	$2(n-m+1)(n+m)$	$m(2m-1)$	x^2
U_{2n+1}	$(-1)^n2(n+1)x$	$2(n-m+1)(n+m+1)$	$m(2m+1)$	x^2
P_{2n}	$\frac{(-1)^n}{4^n}\binom{2n}{n}$	$(n-m+1)(2n+2m-1)$	$m(2m-1)$	x^2
P_{2n+1}	$\frac{(-1)^n}{4^n}\binom{2n+1}{n}(n+1)x$	$(n-m+1)(2n+2m+1)$	$m(2m+1)$	x^2
$L_n^{(\alpha)}$	$\binom{n+\alpha}{n}$	$n-m+1$	$m(\alpha+m)$	x
H_{2n}	$(-1)^n\frac{(2n)!}{n!}$	$2(n-m+1)$	$m(2m-1)$	x^2
H_{2n+1}	$(-1)^n\frac{(2n+1)!}{n!}2x$	$2(n-m+1)$	$m(2m+1)$	x^2

Example 2. Compute $P_8^{(1/2,3/2)}(2)$. Here $d_8=\binom{8.5}{8}=3.33847$, $f(2)=-1$.

m	8	7	6	5	4	3	2	1	0
a_m	1	1. 132353	1. 366667	1. 841026	3. 008392	6. 849651	26. 44156	223. 1091	6545. 533
b_m	18	34	48	60	70	78	84	88	90
c_m	136	105	78	55	36	21	10	3	0

$P_8^{(1/2,3/2)}(2)=d_8a_0(2)=(3.33847)(6545.533)=21852.07$

Evaluation of orthogonal polynomials by means of their recurrence relations

Example 3. Compute $C_n^{(\frac{1}{4})}(2.5)$ for $n=2,3,4,5,6$.

From **Table 22.2** $C_0^{(\frac{1}{4})}=1$, $C_1^{(\frac{1}{4})}=1.25$ and from **22.7** the recurrence relation is

$$C_{n+1}^{(\frac{1}{4})}(2.5)=[5(n+\tfrac{1}{4})C_n^{(\frac{1}{4})}(2.5)-(n-\tfrac{1}{2})C_{n-1}^{(\frac{1}{4})}(2.5)]\frac{1}{n+1}.$$

n	2	3	4	5	6
$C_n^{(\frac{1}{4})}(2.5)$	3. 65625	13. 08594	50. 87648	207. 0649	867. 7516

Check: Compute $C_6^{(\frac{1}{4})}(2.5)$ by the method of **Example 2.**

Change of Interval of Orthogonality

In some applications it is more convenient to use polynomials orthogonal on the interval [0, 1]. One can obtain the new polynomials from the ones given in this chapter by the substitution $x=2\bar{x}-1$. The coefficients of the new polynomial can be computed from the old by the following recursive scheme, provided the standardization is not changed. If

$$f_n(x)=\sum_{m=0}^{n} a_m x^m, \quad f_n^*(x)=f_n(2x-1)=\sum_{m=0}^{n} a_m^* x^m$$

then the a_m^* are given recursively by the a_m through the relations

$a_m^{(j)}=2a_m^{(j-1)}-a_{m+1}^{(j)};\ m=n-1,\ n-2,\ .\ .\ .,\ j;\ j=0,\ 1,\ 2,\ .\ .\ .,\ n$

$a_m^{(-1)}=a_m/2,\ m=0,\ 1,\ 2,\ .\ .\ .,\ n$

$a_n^{(j)}=2^j a_n,\ j=0,\ 1,\ 2,\ .\ .\ .,\ n$ and $a_m^{(m)}=a_m^*;\ m=0,\ 1,\ 2,\ .\ .\ .,\ n.$

Example 4. Given $T_5(x)=5x-20x^3+16x^5$, find $T_5^*(x)$.

m j	5	4	3	2	1	0
-1	$8=a_5^{(-1)}$	0	$-10=a_3^{(-1)}$	0	$2.5=a_1^{(-1)}$	0
0	16	-16	-4	4	1	$-1=a_0^*$
1	32	-64	56	-48	$50=a_1^*$	
2	64	-192	304	$-400=a_2^*$		
3	128	-512	$1120=a_3^*$			
4	256	$-1280=a_4^*$				
5	$512=a_5^*$					

Hence, $T_5^*(x)=512x^5-1280x^4+1120x^3-400x^2+50x-1$.

22.19. Least Square Approximations

Problem: Given a function $f(x)$ (analytically or in form of a table) in a domain D (which may be a continuous interval or a set of discrete points).[2] Approximate $f(x)$ by a polynomial $F_n(x)$ of given degree n such that a weighted sum of the squares of the errors in D is least.

Solution: Let $w(x)\geq 0$ be the weight function chosen according to the relative importance of the errors in different parts of D. Let $f_m(x)$ be orthogonal polynomials in D relative to $w(x)$, i.e. $(f_m, f_n)=0$ for $m\neq n$, where

$$(f,g)=\begin{cases} \int_D w(x)f(x)g(x)dx & \text{if } D \text{ is a continuous interval} \\ \sum_{m=1}^{N} w(x_m)f(x_m)g(x_m) & \text{if } D \text{ is a set of } N \text{ discrete points } x_m. \end{cases}$$

Then

$$F_n(x)=\sum_{m=0}^{n} a_m f_m(x)$$

where

$$a_m=(f,f_m)/(f_m,f_m).$$

D a Continuous Interval

Example 5. Find a least square polynomial of degree 5 for $f(x)=\frac{1}{1+x}$, in the interval $2\leq x\leq 5$, using the weight function

$$w(x)=\frac{1}{\sqrt{(x-2)(5-x)}}$$

which stresses the importance of the errors at the ends of the interval.

Reduction to interval $[-1,1]$, $t=\frac{2x-7}{3}$

$$w(x(t))=\frac{2}{3}\frac{1}{\sqrt{1-t^2}}$$

From **22.2**, $f_m(t)=T_m(t)$ and

$$a_m=\frac{4}{3\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\frac{1}{t+3}T_m(t)dt \qquad (m\neq 0)$$

$$a_0=\frac{2}{3\pi}\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}\frac{dt}{t+3}$$

*
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[2] $f(x)$ has to be square integrable, see e.g. [22.17].

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*See page II.

Example 7. Economize $f(x)=1+x/2+x^2/3+x^3/4+x^4/5+x^5/6$ with $R=.05$.

From **Table 22.3**

$$f(x)=\frac{1}{120}[149T_0(x)+32T_2(x)+3T_4(x)]+\frac{1}{96}[76T_1(x)+11T_3(x)+T_5(x)]$$

so

$$\bar{f}(x)=\frac{1}{120}[149T_0(x)+32T_2(x)]+\frac{1}{96}[76T_1(x)+11T_3(x)]$$

since

$$|\bar{f}(x)-f(x)|\leq\frac{1}{40}+\frac{1}{96}<.05$$

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23. Bernoulli and Euler Polynomials—Riemann Zeta Function

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Contents

The authors acknowledge the assistance of Ruth E. Capuano in the preparation and checking of the tables.

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23. Bernoulli and Euler Polynomials—Riemann Zeta Function

Mathematical Properties

23.1. Bernoulli and Euler Polynomials and the Euler-Maclaurin Formula

Generating Functions

23.1.1
$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!} \qquad |t|<2\pi$$
$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty} E_n(x)\,\frac{t^n}{n!} \qquad |t|<\pi$$

Bernoulli and Euler Numbers

23.1.2
$$B_n=B_n(0) \qquad n=0, 1, \ldots$$
$$E_n=2^nE_n\left(\frac{1}{2}\right)=\text{ integer} \qquad n=0, 1, \ldots$$

23.1.3
$$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_4=-\frac{1}{30}$$
$$E_0=1,\ E_2=-1,\ E_4=5$$

(For occurrence of B_n and E_n in series expansions of circular functions, see chapter **4.**)

Sums of Powers

23.1.4
$$\sum_{k=1}^{m} k^n=\frac{B_{n+1}(m+1)-B_{n+1}}{n+1} \qquad m, n=1, 2, \ldots$$
$$\sum_{k=1}^{m} (-1)^{m-k}k^n=\frac{E_n(m+1)+(-1)^mE_n(0)}{2} \qquad m, n=1, 2, \ldots$$

Derivatives and Differences

23.1.5
$$B_n'(x)=nB_{n-1}(x) \qquad n=1, 2, \ldots$$
$$E_n'(x)=nE_{n-1}(x) \qquad n=1, 2, \ldots$$

23.1.6
$$B_n(x+1)-B_n(x)=nx^{n-1} \qquad n=0, 1, \ldots$$
$$E_n(x+1)+E_n(x)=2x^n \qquad n=0, 1, \ldots$$

Expansions

23.1.7
$$B_n(x+h)=\sum_{k=0}^{n} \binom{n}{k} B_k(x)h^{n-k} \qquad n=0, 1, \ldots$$
$$E_n(x+h)=\sum_{k=0}^{n} \binom{n}{k} E_k(x)h^{n-k} \qquad n=0, 1, \ldots$$
$$E_n(x)=\sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k} \qquad n=0, 1, \ldots$$

Symmetry

23.1.8
$$B_n(1-x)=(-1)^nB_n(x) \qquad n=0, 1, \ldots$$
$$E_n(1-x)=(-1)^nE_n(x) \qquad n=0, 1, \ldots$$

23.1.9
$$(-1)^nB_n(-x)=B_n(x)+nx^{n-1} \qquad n=0,1,\ldots$$
$$(-1)^{n+1}E_n(-x)=E_n(x)-2x^n \qquad n=0, 1, \ldots$$

Multiplication Theorem

23.1.10
$$B_n(mx)=m^{n-1}\sum_{k=0}^{m-1} B_n\left(x+\frac{k}{m}\right) \qquad \begin{array}{l} n=0, 1, \ldots \\ m=1, 2, \ldots \end{array}$$
$$E_n(mx)=m^n\sum_{k=0}^{m-1} (-1)^kE_n\left(x+\frac{k}{m}\right) \qquad \begin{array}{l} n=0, 1, \ldots \\ m=1, 3, \ldots \end{array}$$
$$E_n(mx)=-\frac{2}{n+1}\, m^n \sum_{k=0}^{m-1} (-1)^kB_{n+1}\left(x+\frac{k}{m}\right) \qquad \begin{array}{l} n=0, 1, \ldots \\ m=2, 4, \ldots \end{array}$$

Integrals

23.1.11

$$\int_a^x B_n(t)dt=\frac{B_{n+1}(x)-B_{n+1}(a)}{n+1}$$

$$\int_a^x E_n(t)dt=\frac{E_{n+1}(x)-E_{n+1}(a)}{n+1}$$

23.1.12

$$\int_0^1 B_n(t)B_m(t)dt=(-1)^{n-1}\frac{m!n!}{(m+n)!}B_{m+n} \qquad m,n=1,2,\ldots$$

$$\int_0^1 E_n(t)E_m(t)dt =(-1)^n 4(2^{m+n+2}-1)\frac{m!n!}{(m+n+2)!}B_{m+n+2} \qquad m,n=0,1,\ldots$$

(The polynomials are orthogonal for $m+n$ odd.)

Inequalities

23.1.13

$$|B_{2n}|>|B_{2n}(x)| \qquad n=1,2,\ldots, \quad 1>x>0$$

$$4^{-n}|E_{2n}|>(-1)^n E_{2n}(x)>0 \qquad n=1,2,\ldots, \quad \tfrac{1}{2}>x>0$$

23.1.14

$$\frac{2(2n+1)!}{(2\pi)^{2n+1}}\left(\frac{1}{1-2^{-2n}}\right)>(-1)^{n+1}B_{2n+1}(x)>0 \qquad n=1,2,\ldots, \quad \tfrac{1}{2}>x>0$$

$$\frac{4(2n-1)!}{\pi^{2n}}\left(1+\frac{1}{2^{2n}-2}\right)>(-1)^n E_{2n-1}(x)>0 \qquad n=1,2,\ldots, \quad \tfrac{1}{2}>x>0$$

23.1.15

$$\frac{2(2n)!}{(2\pi)^{2n}}\left(\frac{1}{1-2^{1-2n}}\right)>(-1)^{n+1}B_{2n}>\frac{2(2n)!}{(2\pi)^{2n}} \qquad n=1,2,\ldots$$

$$\frac{4^{n+1}(2n)!}{\pi^{2n+1}}>(-1)^n E_{2n}>\frac{4^{n+1}(2n)!}{\pi^{2n+1}}\left(\frac{1}{1+3^{-1-2n}}\right) \qquad n=0,1,\ldots$$

Fourier Expansions

23.1.16

$$B_n(x)=-2\frac{n!}{(2\pi)^n}\sum_{k=1}^{\infty}\frac{\cos(2\pi kx-\frac{1}{2}\pi n)}{k^n} \qquad \begin{matrix} n>1, 1\geq x\geq 0 \\ n=1, 1>x>0 \end{matrix}$$

$$E_n(x)=4\frac{n!}{\pi^{n+1}}\sum_{k=0}^{\infty}\frac{\sin((2k+1)\pi x-\frac{1}{2}\pi n)}{(2k+1)^{n+1}} \qquad \begin{matrix} n>0, 1\geq x\geq 0 \\ n=0, 1>x>0 \end{matrix}$$

23.1.17

$$B_{2n-1}(x)=\frac{(-1)^n 2(2n-1)!}{(2\pi)^{2n-1}}\sum_{k=1}^{\infty}\frac{\sin 2k\pi x}{k^{2n-1}} \qquad \begin{matrix} n>1, 1\geq x\geq 0 \\ n=1, 1>x>0 \end{matrix}$$

$$E_{2n-1}(x)=\frac{(-1)^n 4(2n-1)!}{\pi^{2n}}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\pi x}{(2k+1)^{2n}} \qquad n=1,2,\ldots, \quad 1\geq x\geq 0$$

23.1.18

$$B_{2n}(x)=\frac{(-1)^{n-1}2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{\cos 2k\pi x}{k^{2n}} \qquad n=1,2,\ldots, \quad 1\geq x\geq 0$$

$$E_{2n}(x)=\frac{(-1)^n 4(2n)!}{\pi^{2n+1}}\sum_{k=0}^{\infty}\frac{\sin(2k+1)\pi x}{(2k+1)^{2n+1}} \qquad \begin{matrix} n>0, 1\geq x\geq 0 \\ n=0, 1>x>0 \end{matrix}$$

Special Values

23.1.19

$$B_{2n+1}=0 \qquad n=1,2,\ldots$$

$$E_{2n+1}=0 \qquad n=0,1,\ldots$$

23.1.20

$$B_n(0)=(-1)^n B_n(1) =B_n \qquad n=0,1,\ldots$$

$$E_n(0)=-E_n(1) =-2(n+1)^{-1}(2^{n+1}-1)B_{n+1} \qquad n=1,2,\ldots$$

23.1.21

$$B_n(\tfrac{1}{2})=-(1-2^{1-n})B_n \qquad n=0,1,\ldots$$

$$E_n(\tfrac{1}{2})=2^{-n}E_n \qquad n=0,1,\ldots$$

23.1.22
$$B_n(\tfrac{1}{4})=(-1)^nB_n(\tfrac{3}{4})$$
$$=-2^{-n}(1-2^{1-n})B_n-n4^{-n}E_{n-1}\qquad n=1,2,\ldots$$

$$E_{2n-1}(\tfrac{1}{3})=-E_{2n-1}(\tfrac{2}{3})$$
$$=-(2n)^{-1}(1-3^{1-2n})(2^{2n}-1)B_{2n}\qquad n=1,2,\ldots$$

23.1.23
$$B_{2n}(\tfrac{1}{3})=B_{2n}(\tfrac{2}{3})$$
$$=-2^{-1}(1-3^{1-2n})B_{2n}\qquad n=0,1,\ldots$$

23.1.24
$$B_{2n}(\tfrac{1}{6})=B_{2n}(\tfrac{5}{6})$$
$$=2^{-1}(1-2^{1-2n})(1-3^{1-2n})B_{2n}\qquad n=0,1,\ldots$$

Symbolic Operations

23.1.25
$$p(B(x)+1)-p(B(x))=p'(x)\qquad\qquad p(E(x)+1)+p(E(x))=2p(x)$$

23.1.26
$$B_n(x+h)=(B(x)+h)^n\qquad n=0,\ 1,\ \ldots\qquad\qquad E_n(x+h)=(E(x)+h)^n\qquad n=0,\ 1,\ \ldots$$

Here $p(x)$ denotes a polynomial in x and after expanding we set $\{B(x)\}^n=B_n(x)$ and $\{E(x)\}^n=E_n(x)$.

Relations Between the Polynomials

23.1.27
$$E_{n-1}(x)=\frac{2^n}{n}\left\{B_n\left(\frac{x+1}{2}\right)-B_n\left(\frac{x}{2}\right)\right\}$$
$$=\frac{2}{n}\left\{B_n(x)-2^nB_n\left(\frac{x}{2}\right)\right\}\qquad n=1,2,\ldots$$

23.1.28
$$E_{n-2}(x)=2\binom{n}{2}^{-1}\sum_{k=0}^{n-2}\binom{n}{k}(2^{n-k}-1)B_{n-k}B_k(x)$$
$$n=2,3,\ldots$$

23.1.29
$$B_n(x)=2^{-n}\sum_{k=0}^{n}\binom{n}{k}B_{n-k}E_k(2x)\qquad n=0,1,\ldots$$

Euler-Maclaurin Formulas

Let $F(x)$ have its first $2n$ derivatives continuous on an interval $(a,\ b)$. Divide the interval into m equal parts and let $h=(b-a)/m$. Then for some θ, $1>\theta>0$, depending on $F^{(2n)}(x)$ on $(a,\ b)$, we have

23.1.30
$$\sum_{k=0}^{m}F(a+kh)=\frac{1}{h}\int_a^b F(t)dt+\frac{1}{2}\{F(b)+F(a)\}$$
$$+\sum_{k=1}^{n-1}\frac{h^{2k-1}}{(2k)!}B_{2k}\{F^{(2k-1)}(b)-F^{(2k-1)}(a)\}$$
$$+\frac{h^{2n}}{(2n)!}B_{2n}\sum_{k=0}^{m-1}F^{(2n)}(a+kh+\theta h)$$

Equivalent to this is

23.1.31
$$\frac{1}{h}\int_x^{x+h}F(t)dt=\frac{1}{2}\{F(x+h)+F(x)\}$$
$$-\sum_{k=1}^{n-1}\frac{h^{2k-1}}{(2k)!}B_{2k}\{F^{(2k-1)}(x+h)-F^{(2k-1)}(x)\}$$
$$-\frac{h^{2n}}{(2n)!}B_{2n}F^{(2n)}(x+\theta h)\qquad b-h\geq x\geq a$$

Let $\hat{B}_n(x)=B_n(x-[x])$. The Euler Summation Formula is

23.1.32
$$\sum_{k=0}^{m-1}F(a+kh+\omega h)=\frac{1}{h}\int_a^b F(t)dt$$
$$+\sum_{k=1}^{p}\frac{h^{k-1}}{k!}B_k(\omega)\{F^{(k-1)}(b)-F^{(k-1)}(a)\}$$
$$-\frac{h^p}{p!}\int_0^1\hat{B}_p(\omega-t)\left\{\sum_{k=0}^{m-1}F^{(p)}(a+kh+th)\right\}dt$$
$$p\leq 2n,\ 1\geq\omega\geq 0$$

23.2. Riemann Zeta Function and Other Sums of Reciprocal Powers

23.2.1 $$\zeta(s)=\sum_{k=1}^{\infty} k^{-s} \qquad \mathscr{R}s>1$$

23.2.2 $$=\prod_{p} (1-p^{-s})^{-1} \qquad \mathscr{R}s>1$$

(product over all primes p).

23.2.3 $$=\frac{1}{s-1}+\frac{1}{2}+\sum_{k=1}^{n} \frac{B_{2k}}{2k}\binom{s+2k-2}{2k-1} -\binom{s+2n}{2n+1}\int_{1}^{\infty}\frac{B_{2n+1}(x-[x])}{x^{s+2n+1}}\,dx$$

$$s\neq 1,\ n=1, 2, \ldots, \quad \mathscr{R}s>-2n$$

* **23.2.4** $$=-\frac{\Gamma(1-s)}{2\pi i}\int_{C}\frac{(-z)^{s-1}}{e^{z}-1}\,dz$$

23.2.5 $$=\frac{1}{s-1}+\sum_{n=0}^{\infty}\frac{(-1)^{n}}{n!}\,\gamma_{n}(s-1)^{n}$$

where

$$\gamma_{n}=\lim_{m\rightarrow\infty}\left\{\sum_{k=1}^{m}\frac{(\ln k)^{n}}{k}-\frac{(\ln m)^{n+1}}{n+1}\right\}$$

$$\mathscr{R}s>0$$

23.2.6 $$=2^{s}\pi^{s-1}\sin\left(\tfrac{1}{2}\pi s\right)\Gamma(1-s)\zeta(1-s)$$

23.2.7 $$=\frac{1}{\Gamma(s)}\int_{0}^{\infty}\frac{x^{s-1}}{e^{x}-1}\,dx \qquad \mathscr{R}s>1$$

23.2.8 $$=\frac{1}{(1-2^{1-s})\Gamma(s)}\int_{0}^{\infty}\frac{x^{s-1}}{e^{x}+1}\,dx$$

23.2.9 $$=\sum_{k=1}^{n}k^{-s}+(s-1)^{-1}n^{1-s}-s\int_{n}^{\infty}\frac{x-[x]}{x^{s+1}}\,dx$$

$$n=1, 2, \ldots, \mathscr{R}s>0$$

23.2.10 $$=\frac{\exp\left(\ln 2\pi-1-\tfrac{1}{2}\gamma\right)s}{2(s-1)\Gamma\left(\tfrac{1}{2}s+1\right)}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$$

product over all zeros ρ of $\zeta(s)$ with $\mathscr{R}\rho>0$.

The contour C in the fourth formula starts at infinity on the positive real axis, circles the origin once in the positive direction excluding the points $\pm 2n i\pi$ for $n=1, 2, \ldots$, and returns to the starting point. Therefore $\zeta(s)$ is regular for all values of s except for a simple pole at $s=1$ with residue 1.

Special Values

23.2.11 $$\zeta(0)=-\tfrac{1}{2}$$

23.2.12 $$\zeta(1)=\infty$$

23.2.13 $$\zeta'(0)=-\tfrac{1}{2}\ln 2\pi$$

23.2.14 $$\zeta(-2n)=0 \qquad n=1, 2, \ldots$$

23.2.15 $$\zeta(1-2n)=-\frac{B_{2n}}{2n} \qquad n=1, 2, \ldots$$

23.2.16 $$\zeta(2n)=\frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}| \qquad n=1, 2, \ldots$$

23.2.17

$$\zeta(2n+1)=\frac{(-1)^{n+1}(2\pi)^{2n+1}}{2(2n+1)!}\int_{0}^{1}B_{2n+1}(x)\cot(\pi x)\,dx$$

$$n=1, 2, \ldots$$

Sums of Reciprocal Powers

The sums referred to are

23.2.18 $$\zeta(n)=\sum_{k=1}^{\infty}k^{-n} \qquad n=2, 3, \ldots$$

23.2.19

$$\eta(n)=\sum_{k=1}^{\infty}(-1)^{k-1}k^{-n}=(1-2^{1-n})\zeta(n) \qquad n=1, 2, \ldots$$

23.2.20

$$\lambda(n)=\sum_{k=0}^{\infty}(2k+1)^{-n}=(1-2^{-n})\zeta(n) \qquad n=2, 3, \ldots$$

23.2.21

$$\beta(n)=\sum_{k=0}^{\infty}(-1)^{k}(2k+1)^{-n} \qquad n=1, 2, \ldots$$

These sums can be calculated from the Bernoulli and Euler polynomials by means of the last two formulas for special values of the zeta function (note that $\eta(1)=\ln 2$), and

23.2.22 $$\beta(2n+1)=\frac{(\pi/2)^{2n+1}}{2(2n)!}|E_{2n}| \qquad n=0, 1, \ldots$$

23.2.23

$$\beta(2n)=\frac{(-1)^{n}\pi^{2n}}{4(2n-1)!}\int_{0}^{1}E_{2n-1}(x)\sec(\pi x)\,dx$$

$$n=1, 2, \ldots$$

$\beta(2)$ is known as Catalan's constant. Some other special values are

23.2.24 $$\zeta(2)=1+\frac{1}{2^{2}}+\frac{1}{3^{2}}+\ldots=\frac{\pi^{2}}{6}$$

23.2.25 $$\zeta(4)=1+\frac{1}{2^{4}}+\frac{1}{3^{4}}+\ldots=\frac{\pi^{4}}{90}$$

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*See page II.

23.2.26 $$\eta(2)=1-\frac{1}{2^2}+\frac{1}{3^2}-\ldots=\frac{\pi^2}{12}$$

23.2.27 $$\eta(4)=1-\frac{1}{2^4}+\frac{1}{3^4}-\ldots=\frac{7\pi^4}{720}$$

23.2.28 $$\lambda(2)=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}$$

23.2.29 $$\lambda(4)=1+\frac{1}{3^4}+\frac{1}{5^4}+\ldots=\frac{\pi^4}{96}$$

23.2.30 $$\beta(1)=1-\frac{1}{3}+\frac{1}{5}-\ldots=\frac{\pi}{4}$$

23.2.31 $$\beta(3)=1-\frac{1}{3^3}+\frac{1}{5^3}-\ldots=\frac{\pi^3}{32}$$

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24. Combinatorial Analysis

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24. Combinatorial Analysis

Mathematical Properties

In each sub-section of this chapter we use a fixed format which emphasizes the use and methods of extending the accompanying tables. The format follows this form:

I. Definitions
- A. Combinatorial
- B. Generating functions
- C. Closed form

II. Relations
- A. Recurrences
- B. Checks in computing
- C. Basic use in numerical analysis

III. Asymptotic and Special Values

In general the notations used are standard. This includes the difference operator Δ defined on functions of x by $\Delta f(x)=f(x+1)-f(x)$, $\Delta^{n+1}f(x)=\Delta(\Delta^n f(x))$, the Kronecker delta δ_{ij}, the Riemann zeta function $\zeta(s)$ and the greatest common divisor symbol (m, n). The range of the summands for a summation sign without limits is explained to the right of the formula.

The notations which are not standard are those for the multinomials which are arbitrary shorthand for use in this chapter, and those for the Stirling numbers which have never been standardized. A short table of various notations for these numbers follows:

Notations for the Stirling Numbers

Reference	*First Kind*	*Second Kind*
This chapter	$S_n^{(m)}$	$\mathfrak{S}_n^{(m)}$
[24.2] Fort	$S_n^{(m)}$	$\mathcal{S}_n^{(m)}$ *
[24.7] Jordan	S_n^m	$\mathfrak{S}_n^m$ *
[24.10] Moser and Wyman	S_n^m	σ_n^m
[24.9] Milne-Thomson	$\binom{n-1}{m-1} B_{n-m}^{(n)}$	$\binom{n}{m} B_{n-m}^{(-m)}$
[24.15] Riordan	$s(n, m)$	$S(n, m)$
[24.1] Carlitz [24.3] Gould	$(-1)^{n-m}S_1(n-1, n-m)$	$S_2(m, n-m)$
Miksa (Unpublished tables)	$S(n-m+1, n)$	${}_mS_n$
[24.17] Gupta		$u(n, m)$

We feel that a capital S is natural for Stirling numbers of the first kind; it is infrequently used for other notation in this context. But once it is used we have difficulty finding a suitable symbol for Stirling numbers of the second kind. The numbers are sufficiently important to warrant a special and easily recognizable symbol, and yet that symbol must be easy to write. We have settled on a script capital $\mathfrak{S}$ without any certainty that we have settled this question permanently.

We feel that the subscript-superscript notation emphasizes the generating functions (which are powers of mutually inverse functions) from which most of the important relations flow.

24.1. Basic Numbers

24.1.1 Binomial Coefficients

I. Definitions

A. $\binom{n}{m}$ is the number of ways of choosing m objects from a collection of n distinct objects without regard to order.

B. Generating functions

$$* \quad (1+x)^n=\sum_{m=0}^{n}\binom{n}{m}x^m \qquad n=0, 1, \ldots$$

$$(1-x)^{-m-1}=\sum_{n=m}^{\infty}\binom{n}{m}x^{n-m} \qquad |x|<1$$

C. Closed form

$$\binom{n}{m}=\frac{n!}{m!(n-m)!}=\binom{n}{n-m} \qquad n\geq m$$

$$=\frac{n(n-1)\ldots(n-m+1)}{m!}$$

II. Relations

A. Recurrences

$$\binom{n+1}{m}=\binom{n}{m}+\binom{n}{m-1} \qquad n\geq m\geq 1$$

$$=\binom{n}{m}+\binom{n-1}{m-1}+\ldots+\binom{n-m}{0} \qquad n\geq m$$

B. Checks

$$\sum_{m=0}^{n}\binom{r}{m}\binom{s}{n-m}=\binom{r+s}{n} \qquad r+s\geq n$$

$$\sum_{m=0}^{n}(-1)^{n-m}\binom{r}{m}=\binom{r-1}{n} \qquad r\geq n+1$$

$$\binom{n}{m}\equiv\binom{n_0}{m_0}\binom{n_1}{m_1}\ldots(\text{mod } p) \qquad p \text{ a prime}$$

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*See page II.

where

$$n=\sum_{k=0}^{\infty} n_k p^k, \qquad m=\sum_{k=0}^{\infty} m_k p^k \qquad p>m_k,\ n_k \geq 0$$

C. Numerical analysis

$$\Delta^n f(x)=\sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} f(x+m)$$

$$=\sum_{k=0}^{r} \binom{r}{k} \Delta^{n+k} f(x-r)$$

$$\sum_{m=0}^{s} (-1)^m \binom{n}{m} f(x-m)$$

$$=\sum_{k=0}^{s} (-1)^{s-k} \binom{n-k-1}{s-k} \Delta^k f(x-s) \qquad s<n$$

III. Special Values

$$\binom{n}{0}=\binom{n}{n}=1$$

$$\binom{2n}{n}=\frac{2^n(2n-1)(2n-3)\ .\ .\ .\ 3\cdot 1}{n!}$$

24.1.2 Multinomial Coefficients

I. Definitions

A. $(n;\ n_1,\ n_2,\ .\ .\ .,\ n_m)$ is the number of ways of putting $n=n_1+n_2+\ .\ .\ .\ +n_m$ different objects into m different boxes with n_k in the k-th box, $k=1,\ 2,\ .\ .\ .,\ m$.

$(n;\ a_1,\ a_2,\ .\ .\ .,\ a_n)^*$ is the number of permutations of $n=a_1+2a_2+\ .\ .\ .\ +na_n$ symbols composed of a_k cycles of length k for $k=1,\ 2,\ .\ .\ .,\ n$.

$(n;\ a_1,\ a_2,\ .\ .\ .,\ a_n)'$ is the number of ways of partitioning a set of $n=a_1+2a_2+\ .\ .\ .\ +na_n$ different objects into a_k subsets containing k objects for $k=1,\ 2,\ .\ .\ .,\ n$.

B. Generating functions

$$(x_1+x_2+\ .\ .\ .\ +x_m)^n=\Sigma(n;\ n_1,\ n_2,\ .\ .\ .,\ n_m)x_1^{n_1}x_2^{n_2}\ .\ .\ .\ x_m^{n_m} \qquad \text{summed over } n_1+n_2+\ .\ .\ .\ +n_m=n$$

$$\left(\sum_{k=1}^{\infty} \frac{x_k}{k} t^k\right)^m=m!\ \sum_{n=m}^{\infty} \frac{t^n}{n!} \Sigma(n;\ a_1,\ a_2,\ .\ .\ .,\ a_n)^* x_1^{a_1}x_2^{a_2}\ .\ .\ .\ x_n^{a_n}$$

summed over $a_1+2a_2+\ .\ .\ .\ +na_n=n$
and $a_1+a_2+\ .\ .\ .\ +a_n=m$

$$\left(\sum_{k=1}^{\infty} \frac{x_k}{k!} t^k\right)^m=m!\ \sum_{n=m}^{\infty} \frac{t^n}{n!} \Sigma(n;\ a_1,\ a_2,\ .\ .\ .,\ a_n)' x_1^{a_1}x_2^{a_2}\ .\ .\ .\ x_n^{a_n}$$

C. Closed forms

$$(n;n_1,n_2,\ .\ .\ .,\ n_m)=n!/n_1!n_2!\ .\ .\ .\ n_m! \qquad n_1+n_2+.\ .\ .+n_m=n$$

$$(n;a_1,a_2,\ .\ .\ .,a_n)^*=n!/1^{a_1}a_1!2^{a_2}a_2!\ .\ .\ .\ n^{a_n}a_n! \qquad a_1+2a_2+\ .\ .\ .+na_n=n$$

$$(n;a_1,a_2,\ .\ .\ .,\ a_n)'=n!/(1!)^{a_1}a_1!(2!)^{a_2}a_2!\ .\ .\ .\ (n!)^{a_n}a_n! \qquad a_1+2a_2+\ .\ .\ .\ +na_n=n$$

II. Relations

A. Recurrence

$$(n+m;n_1+1,n_2+1,\ .\ .\ .,n_m+1)=\sum_{k=1}^{m} (n+m-1;n_1+1,\ .\ .\ .,n_{k-1}+1,n_k,n_{k+1}+1,\ .\ .\ .\ ,n_m+1)$$

B. Checks

$$* \quad \Sigma(n;n_1,n_2,\ .\ .\ .,n_m)=\begin{Bmatrix} m^n & \text{all } n_i \geqslant 1 \\ m!\ \mathfrak{S}_n^{(m)} & \end{Bmatrix} \qquad \text{summed over } n_1+n_2+\ .\ .\ .\ +n_m=n$$

$$\Sigma(n;a_1,a_2,\ .\ .\ .,a_n)^*=(-1)^{n-m}S_n^{(m)}$$

$$\Sigma(n;a_1,a_2,\ .\ .\ .,a_n)'=\mathfrak{S}_n^{(m)}$$

summed over $a_1+2a_2+\ .\ .\ .\ +na_n=n$ and $a_1+a_2+\ .\ .\ .\ +a_n=m$

C. Numerical analysis (Faà di Bruno's formula)

$$\frac{d^n}{dx^n}f(g(x))=\sum_{m=0}^{n} f^{(m)}(g(x))\Sigma(n;a_1,a_2,\ .\ .\ .,a_n)'\{g'(x)\}^{a_1}\{g''(x)\}^{a_2}\ .\ .\ .\ \{g^{(n)}(x)\}^{a_n}$$

summed over $a_1+2a_2+\ .\ .\ .\ +na_n=n$ and $a_1+a_2+\ .\ .\ .\ +a_n=m$.

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*See page II.

$$\begin{vmatrix} P_1 & 1 & 0 & \dots & 0 \\ P_2 & P_1 & 2 & \dots & . \\ P_3 & P_2 & P_1 & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & 0 \\ . & . & . & \dots & n-1 \\ P_n & P_{n-1} & P_{n-2} & \dots & P_1 \end{vmatrix} = \Sigma(-1)^{n-\Sigma a_i}(n; a_1, a_2, \dots, a_n)^* P_1^{a_1} P_2^{a_2} \dots P_n^{a_n}$$

summed over $a_1+2a_2+ \dots +na_n=n$; e.g. if $P_k=\Sigma_{j=1}^{r} x_j^k$ for $k=1, 2, \dots, n$ then the determinant and sum equal $n!\Sigma x_1 x_2 \dots x_n$, the latter sum denoting the n-th elementary symmetric function of $x_1, x_2, \dots, x_r$.

24.1.3 Stirling Numbers of the First Kind

I. Definitions

A. $(-1)^{n-m}S_n^{(m)}$ is the number of permutations of n symbols which have exactly m cycles.

B. Generating functions

$$x(x-1)\dots(x-n+1)=\sum_{m=0}^{n} S_n^{(m)} x^m$$

$$\{\ln(1+x)\}^m = m! \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!} \qquad |x|<1$$

C. Closed form (see closed form for $\mathfrak{S}_n^{(m)}$)

$$S_n^{(m)}=\sum_{k=0}^{n-m} (-1)^k \binom{n-1+k}{n-m+k}\binom{2n-m}{n-m-k} \mathfrak{S}_{n-m+k}^{(k)}$$

II. Relations

A. Recurrences

$$S_{n+1}^{(m)}=S_n^{(m-1)}-nS_n^{(m)} \qquad n\geq m\geq 1$$

$$\binom{m}{r} S_n^{(m)}=\sum_{k=m-r}^{n-r}\binom{n}{k} S_{n-k}^{(r)} S_k^{(m-r)} \qquad n\geq m\geq r$$

B. Checks

$$\sum_{m=1}^{n} S_n^{(m)}=0 \qquad n>1$$

$$\sum_{m=0}^{n}(-1)^{n-m}S_n^{(m)}=n!$$

$$\sum_{k=m}^{n} S_{n+1}^{(k+1)} n^{k-m}=S_n^{(m)}$$

C. Numerical analysis

$$\frac{d^m}{dx^m}f(x)=m!\sum_{n=m}^{\infty}\frac{S_n^{(m)}}{n!}\Delta^n f(x)$$

if convergent.

III. Asymptotics and Special Values

$$|S_n^{(m)}|\sim(n-1)!(\gamma+\ln n)^{m-1}/(m-1)! \qquad \text{for } m=o(\ln n)$$

$$\lim_{m\to\infty}\frac{S_{n+m}^{(m)}}{m^{2n}}=\frac{(-1)^n}{2^n n!}$$

$$\lim_{n\to\infty}\frac{S_{n+1}^{(m)}}{nS_n^{(m)}}=-1$$

$$S_n^{(0)}=\delta_{0n}$$

$$S_n^{(1)}=(-1)^{n-1}(n-1)!$$

$$S_n^{(n-1)}=-\binom{n}{2}$$

$$S_n^{(n)}=1$$

24.1.4 Stirling Numbers of the Second Kind

I. Definitions

A. $\mathfrak{S}_n^{(m)}$ is the number of ways of partitioning a set of n elements into m non-empty subsets.

B. Generating functions

$$x^n=\sum_{m=0}^{n}\mathfrak{S}_n^{(m)}x(x-1)\dots(x-m+1)$$

$$(e^x-1)^m=m!\sum_{n=m}^{\infty}\mathfrak{S}_n^{(m)}\frac{x^n}{n!}$$

$$(1-x)^{-1}(1-2x)^{-1}\dots(1-mx)^{-1}=\sum_{n=m}^{\infty}\mathfrak{S}_n^{(m)}x^{n-m} \qquad |x|<m^{-1}$$

C. Closed form

$$\mathfrak{S}_n^{(m)}=\frac{1}{m!}\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}k^n$$

II. Relations

A. Recurrences

$$\mathfrak{S}_{n+1}^{(m)}=m\,\mathfrak{S}_n^{(m)}+\mathfrak{S}_n^{(m-1)} \qquad n\geq m\geq 1$$

$$\binom{m}{r}\,\mathfrak{S}_n^{(m)}=\sum_{k=m-r}^{n-r}\binom{n}{k}\,\mathfrak{S}_{n-k}^{(r)}\,\mathfrak{S}_k^{(m-r)} \qquad n\geq m\geq r$$

B. Checks

$$\sum_{m=0}^{n}(-1)^{n-m}m!\,\mathfrak{S}_n^{(m)}=1$$

$$\sum_{k=m}^{n}\mathfrak{S}_{k-1}^{(m-1)}m^{n-k}=\mathfrak{S}_n^{(m)}$$

$$\mathfrak{S}_n^{(m)}=\sum_{k=0}^{n-m}(-1)^k\binom{n-1+k}{n-m+k}\binom{2n-m}{n-m-k}S_{n-m+k}^{(k)}$$

$$\sum_{k=m}^{n}S_k^{(m)}\,\mathfrak{S}_n^{(k)}=\sum_{k=m}^{n}S_n^{(k)}\,\mathfrak{S}_k^{(m)}=\delta_{mn}$$

C. Numerical analysis

$$\Delta^m f(x)=m!\sum_{n=m}^{\infty}\frac{\mathfrak{S}_n^{(m)}}{n!}f^{(n)}(x) \qquad \text{if convergent}$$

$$\sum_{k=0}^{n}k^m=\sum_{k=0}^{m}k!\,\mathfrak{S}_m^{(k)}\binom{n+1}{k+1}$$

$$\sum_{k=0}^{n}k^m x^k=\sum_{j=0}^{m}\mathfrak{S}_m^{(j)}x^j\frac{d^j}{dx^j}\left\{\frac{1-x^{n+1}}{1-x}\right\}$$

III. Asymptotics and Special Values

* $$\lim_{n\to\infty}m^{-n}\,\mathfrak{S}_n^{(m)}=(m!)^{-1}$$

$$\mathfrak{S}_{n+m}^{(m)}\sim\frac{m^{2n}}{2^n n!} \qquad \text{for } n=o(m^{\frac{1}{2}})$$

$$\lim_{n\to\infty}\frac{\mathfrak{S}_{n+1}^{(m)}}{\mathfrak{S}_n^{(m)}}=m$$

$$\mathfrak{S}_n^{(0)}=\delta_{0n}$$

$$\mathfrak{S}_n^{(1)}=\mathfrak{S}_n^{(n)}=1$$

$$\mathfrak{S}_n^{(n-1)}=\binom{n}{2}$$

24.2. Partitions

24.2.1 Unrestricted Partitions

I. Definitions

A. $p(n)$ is the number of decompositions of n into integer summands without regard to order. E.g., $5=1+4=2+3=1+1+3=1+2+2=1+1+1+2=1+1+1+1+1$ so that $p(5)=7$.

B. Generating function

$$\sum_{n=0}^{\infty}p(n)x^n=\prod_{n=1}^{\infty}(1-x^n)^{-1}=\left\{\sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{3n^2+n}{2}}\right\}^{-1} \qquad |x|<1$$

C. Closed form

$$p(n)=\frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty}\sqrt{k}A_k(n)\,\frac{d}{dn}\frac{\sinh\left\{\frac{\pi}{k}\sqrt{\frac{2}{3}}\sqrt{n-\frac{1}{24}}\right\}}{\sqrt{n-\frac{1}{24}}}$$

where

$$A_k(n)=\sum_{\substack{0<h\leq k\\(h,k)=1}}e^{\pi i s(h,k)}e^{-\frac{2\pi i h n}{k}}$$

$$s(h,k)=\sum_{j=1}^{k-1}\frac{j}{k}\left(\left(\frac{hj}{k}\right)\right)$$

$$\begin{aligned}((x))&=x-[x]-\tfrac{1}{2} &&\text{if } x \text{ is not an integer}\\ &=0 &&\text{if } x \text{ is an integer}\end{aligned}$$

II. Relations

A. Recurrence

$$p(n)=\sum_{1\leq\frac{3k^2\pm k}{2}\leq n}(-1)^{k-1}p\left(n-\frac{3k^2\pm k}{2}\right) \qquad p(0)=1$$

$$=\frac{1}{n}\sum_{k=1}^{n}\sigma_1(k)p(n-k)$$

B. Check

$$p(n)+\sum_{1\leq\frac{3k^2\pm k}{2}\leq n}(-1)^k\frac{3k^2\pm k}{2}p\left(n-\frac{3k^2\pm k}{2}\right)=\sigma_1(n)$$

III. Asymptotics

$$p(n)\sim\frac{1}{4n\sqrt{3}}e^{\pi\sqrt{2/3}\sqrt{n}}$$

24.2.2 Partitions Into Distinct Parts

I. Definitions

A. $q(n)$ is the number of decompositions of n into distinct integer summands without regard to order. E.g., $5=1+4=2+3$ so that $q(5)=3$.

B. Generating function

$$\sum_{n=0}^{\infty}q(n)x^n=\prod_{n=1}^{\infty}(1+x^n)=\prod_{n=1}^{\infty}(1-x^{2n-1})^{-1} \qquad |x|<1$$

C. Closed form

$$q(n)=\frac{1}{\sqrt{2}}\sum_{k=1}^{\infty}A_{2k-1}(n)\,\frac{d}{dn}J_0\left(\frac{\pi i}{2k-1}\sqrt{\frac{1}{3}}\sqrt{n+\frac{1}{24}}\right)$$

where $J_0(x)$ is the Bessel function of order 0 and $A_{2k-1}(n)$ was defined in part I.C. of the previous subsection.

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*See page II.

II. Relations

A. Recurrences

$$\sum_{0\leq\frac{3k^2\pm k}{2}\leq n}(-1)^k q\left(n-\frac{3k^2\pm k}{2}\right)=(-1)^r \text{ if } n=3r^2\pm r \qquad q(0)=1$$

$$=0 \text{ otherwise}$$

$$q(n)=\frac{1}{n}\sum_{k=1}^{n}\left\{\sigma_1(k)-2\sigma_1\left(\frac{k}{2}\right)\right\}q(n-k)$$

B. Check

$$\sum_{0\leq 3k^2\pm k\leq n}(-1)^k q(n-(3k^2\pm k))=1 \text{ if } n=\frac{r^2-r}{2}$$

$$=0 \text{ otherwise.}$$

III. Asymptotics

$$q(n)\sim\frac{1}{4\cdot 3^{1/4}\cdot n^{3/4}}e^{\pi\sqrt{1/3}\sqrt{n}}$$

24.3. Number Theoretic Functions

24.3.1 The Möbius Function

I. Definitions

A. $\mu(n)=1$ if $n=1$

$=(-1)^k$ if n is the product of k distinct primes

$=0$ if n is divisible by a square >1.

B. Generating functions

$$\sum_{n=1}^{\infty}\mu(n)n^{-s}=1/\zeta(s) \qquad \mathscr{R}s>1$$

$$\sum_{n=1}^{\infty}\frac{\mu(n)x^n}{1-x^n}=x \qquad |x|<1$$

II. Relations

A. Recurrence

$$\mu(mn)=\mu(m)\mu(n) \text{ if } (m,n)=1$$

$$=0 \qquad \text{if } (m,n)>1$$

B. Check

$$\sum_{d|n}\mu(d)=\delta_{n1}$$

C. Numerical analysis

$g(n)=\sum_{d|n}f(d)$ for all n if and only if

$$f(n)=\sum_{d|n}\mu(d)g(n/d) \text{ for all } n$$

$g(n)=\prod_{d|n}f(d)$ for all n if and only if

$$f(n)=\prod_{d|n}g(n/d)^{\mu(d)} \text{ for all } n$$

$g(x)=\sum_{n=1}^{[x]}f(x/n)$ for all $x>0$ if and only if

$$f(x)=\sum_{n=1}^{[x]}\mu(n)g(x/n) \text{ for all } x>0$$

$g(x)=\sum_{n=1}^{\infty}f(nx)$ for all $x>0$ if and only if

$$f(x)=\sum_{n=1}^{\infty}\mu(n)g(nx) \text{ for all } x>0$$

and if $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|f(mnx)|=\sum_{n=1}^{\infty}\sigma_0(n)|f(nx)|$ converges.

The cyclotomic polynomial of order n is $\prod_{d|n}(x^d-1)^{\mu(n/d)}$

III. Asymptotics

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}=0$$

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\ln n=-1$$

$$\sum_{n\leq x}\mu(n)=0(xe^{-c\sqrt{\ln x}})$$

24.3.2 The Euler Totient Function

I. Definitions

A. $\varphi(n)$ is the number of integers not exceeding and relatively prime to n.

B. Generating functions

$$\sum_{n=1}^{\infty}\varphi(n)n^{-s}=\frac{\zeta(s-1)}{\zeta(s)} \qquad \mathscr{R}s>2$$

$$\sum_{n=1}^{\infty}\frac{\varphi(n)x^n}{1-x^n}=\frac{x}{(1-x)^2} \qquad |x|<1$$

C. Closed form

$$\varphi(n)=n\prod_{p|n}\left(1-\frac{1}{p}\right)$$

over distinct primes p dividing n.

II. Relations

A. Recurrence

$$\varphi(mn)=\varphi(m)\varphi(n) \qquad (m,n)=1$$

B. Checks

$$\sum_{d|n}\varphi(d)=n$$

$$\varphi(n)=\sum_{d|n}\mu\left(\frac{n}{d}\right)d$$

$$a^{\varphi(n)}\equiv 1 \pmod{n} \qquad (a,n)=1$$

III. Asymptotics

$$\frac{1}{n^2}\sum_{k=1}^{n}\varphi(k)=\frac{3}{\pi^2}+O\left(\frac{\ln n}{n}\right)$$

24.3.3 Divisor Functions

I. Definitions

A. $\sigma_k(n)$ is the sum of the k-th powers of the divisors of n. Often $\sigma_0(n)$ is denoted by $d(n)$, and $\sigma_1(n)$ by $\sigma(n)$.

B. Generating functions

$$\sum_{n=1}^{\infty} \sigma_k(n) n^{-s} = \zeta(s)\zeta(s-k) \qquad \mathscr{R}s > k+1$$

$$\sum_{n=1}^{\infty} \sigma_k(n) x^n = \sum_{n=1}^{\infty} \frac{n^k x^n}{1-x^n} \qquad |x|<1$$

C. Closed form

$$\sigma_k(n) = \sum_{d|n} d^k = \prod_{i=1}^{s} \frac{p_i^{k(a_i+1)}-1}{p_i^k-1} \qquad n = p_1^{a_1} p_2^{a_2} \ldots p_s^{a_s}$$

II. Relations

A. Recurrences

$$\sigma_k(mn) = \sigma_k(m)\sigma_k(n) \qquad (m, n)=1$$

$$\sigma_k(np) = \sigma_k(n)\sigma_k(p) - p^k \sigma_k(n/p) \qquad p \text{ prime}$$

III. Asymptotics

$$\frac{1}{n}\sum_{m=1}^{n} \sigma_0(m) = \ln n + 2\gamma - 1 + O(n^{-\frac{1}{2}})$$

$$(\gamma = \text{Euler's constant})$$

$$\frac{1}{n^2}\sum_{m=1}^{n} \sigma_1(m) = \frac{\pi^2}{12} + O\left(\frac{\ln n}{n}\right)$$

24.3.4 Primitive Roots

I. Definitions

The integers not exceeding and relatively prime to a fixed integer n form a group; the group is cyclic if and only if $n=2$, 4 or n is of the form p^k or $2p^k$ where p is an odd prime. Then g is a primitive root of n if it generates that group; i.e., if $g, g^2, \ldots, g^{\varphi(n)}$ are distinct modulo n. There are $\varphi(\varphi(n))$ primitive roots of n.

II. Relations

A. Recurrences. If g is a primitive root of a prime p and $g^{p-1} \not\equiv 1 \pmod{p^2}$ then g is a primitive root of p^k for all k. If $g^{p-1} \equiv 1 \pmod{p^2}$ then $g+p$ is a primitive root of p^k for all k.

If g is a primitive root of p^k then either g or $g+p^k$, whichever is odd, is a primitive root of $2p^k$.

B. Checks. If g is a primitive root of n then g^k is a primitive root of n if and only if $(k, \varphi(n))=1$, and each primitive root of n is of this form.

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25. Numerical Interpolation, Differentiation, and Integration

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Contents

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25. Numerical Interpolation, Differentiation, and Integration

Numerical analysts have a tendency to accumulate a multiplicity of tools each designed for highly specialized operations and each requiring special knowledge to use properly. From the vast stock of formulas available we have culled the present selection. We hope that it will be useful. As with all such compendia, the reader may miss his favorites and find others whose utility he thinks is marginal.

We would have liked to give examples to illuminate the formulas, but this has not been feasible. Numerical analysis is partially a science and partially an art, and short of writing a textbook on the subject it has been impossible to indicate where and under what circumstances the various formulas are useful or accurate, or to elucidate the numerical difficulties to which one might be led by uncritical use. The formulas are therefore issued together with a caveat against their blind application.

Formulas

Notation: Abscissas: $x_0 < x_1 < \ldots$; functions: $f, g, \ldots$; values: $f(x_i) = f_i, f'(x_i) = f'_i$. $f', f^{(2)}, \ldots$ indicate 1st, 2d, . . . derivatives. If abscissas are equally spaced, $x_{i+1} - x_i = h$ and $f_p = f(x_0 + ph)$ (p not necessarily integral). R, R_n indicate remainders.

25.1. Differences

Forward Differences

25.1.1

$$\Delta(f_n) = \Delta_n = \Delta_n^1 = f_{n+1} - f_n$$

$$\Delta_n^2 = \Delta_{n+1}^1 - \Delta_n^1 = f_{n+2} - 2f_{n+1} + f_n$$

$$\Delta_n^3 = \Delta_{n+1}^2 - \Delta_n^2 = f_{n+3} - 3f_{n+2} + 3f_{n+1} - f_n$$

$$\Delta_n^k = \Delta_{n+1}^{k-1} - \Delta_n^{k-1} = \sum_{j=0}^{k} (-1)^j \binom{k}{j} f_{n+k-j}$$

Central Differences

25.1.2

$$\delta(f_{n+\frac{1}{2}}) = \delta_{n+\frac{1}{2}} = \delta_{n+\frac{1}{2}}^1 = f_{n+1} - f_n$$

$$\delta_n^2 = \delta_{n+\frac{1}{2}}^1 - \delta_{n-\frac{1}{2}}^1 = f_{n+1} - 2f_n + f_{n-1}$$

$$\delta_{n+\frac{1}{2}}^3 = \delta_{n+1}^2 - \delta_n^2 = f_{n+2} - 3f_{n+1} + 3f_n - f_{n-1}$$

$$\delta_n^{2k} = \sum_{j=0}^{2k} (-1)^j \binom{2k}{j} f_{n+k-j}$$

$$\delta_{n+\frac{1}{2}}^{2k+1} = \sum_{j=0}^{2k+1} (-1)^j \binom{2k+1}{j} f_{n+k+1-j}$$

$\delta_{\frac{1}{2}n}^k = \Delta_{\frac{1}{2}(n-k)}^k$ if n and k are of same parity.

Forward Differences

$$\begin{array}{cccccc}
x_0 & f_0 & & & \\
 & & \Delta_0 & & \\
x_1 & f_1 & & \Delta_0^2 & \\
 & & \Delta_1 & & \Delta_0^3 \\
x_2 & f_2 & & \Delta_1^2 & \\
 & & \Delta_2 & & \\
x_3 & f_3 & & &
\end{array}$$

Central Differences

$$\begin{array}{cccccc}
x_{-1} & f_{-1} & & & \\
 & & \delta_{-\frac{1}{2}} & & \\
x_0 & f_0 & & \delta_0^2 & \\
 & & \delta_{\frac{1}{2}} & & \delta_{\frac{1}{2}}^3 \\
x_1 & f_1 & & \delta_1^2 & \\
 & & \delta_{3/2} & & \\
x_2 & f_2 & & &
\end{array}$$

Mean Differences

25.1.3 $$\mu(f_n) = \tfrac{1}{2}(f_{n+\frac{1}{2}} + f_{n-\frac{1}{2}})$$

Divided Differences

25.1.4 $$[x_0, x_1] = \frac{f_0 - f_1}{x_0 - x_1} = [x_1, x_0]$$

$$[x_0, x_1, x_2] = \frac{[x_0, x_1] - [x_1, x_2]}{x_0 - x_2}$$

$$[x_0, x_1, \ldots, x_k] = \frac{[x_0, \ldots, x_{k-1}] - [x_1, \ldots, x_k]}{x_0 - x_k}$$

Divided Differences in Terms of Functional Values

25.1.5 $$[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} \frac{f_k}{\pi'_n(x_k)}$$

25.1.6 where $\pi_n(x)=(x-x_0)\ (x-x_1)\ .\ .\ .\ (x-x_n)$ and $\pi_n'(x)$ is its derivative:

25.1.7

$$\pi_n'(x_k)=(x_k-x_0)\ .\ .\ .\ (x_k-x_{k-1})(x_k-x_{k+1})\ .\ .\ .\ (x_k-x_n)$$

Let D be a simply connected domain with a piecewise smooth boundary C and contain the points $z_0, . . ., z_n$ in its interior. Let $f(z)$ be analytic in D and continuous in $D+C$. Then,

25.1.8 $$[z_0, z_1, . . ., z_n]=\frac{1}{2\pi i}\int_C \frac{f(z)}{\prod\limits_{k=0}^{n}(z-z_k)}\,dz$$

25.1.9 $$\Delta_0^n=h^n f^{(n)}(\xi) \qquad (x_0<\xi<x_n)$$

25.1.10

$$[x_0, x_1, . . ., x_n]=\frac{\Delta_0^n}{n!h^n}=\frac{f^{(n)}(\xi)}{n!} \qquad (x_0<\xi<x_n)$$

25.1.11

$$[x_{-n}, x_{-n+1}, . . ., x_0, . . ., x_n]=\frac{\delta_0^{2n}}{h^{2n}(2n)!}$$

Reciprocal Differences

25.1.12

$$\rho(x_0, x_1)=\frac{x_0-x_1}{f_0-f_1}$$

$$\rho_2(x_0, x_1, x_2)=\frac{x_0-x_2}{\rho(x_0, x_1)-\rho(x_1, x_2)}+f_1$$

$$\rho_3(x_0, x_1, x_2, x_3)=\frac{x_0-x_3}{\rho_2(x_0, x_1, x_2)-\rho_2(x_1, x_2, x_3)}+\rho(x_1, x_2)$$

$$\vdots$$

$$\rho_n(x_0, x_1, . . ., x_n)=\frac{x_0-x_n}{\rho_{n-1}(x_0, . . ., x_{n-1})-\rho_{n-1}(x_1, . . ., x_n)}+\rho_{n-2}(x_1, . . ., x_{n-1})$$

25.2. Interpolation

Lagrange Interpolation Formulas

25.2.1 $$f(x)=\sum_{i=0}^{n} l_i(x)f_i+R_n(x)$$

25.2.2

$$l_i(x)=\frac{\pi_n(x)}{(x-x_i)\pi_n'(x_i)}$$

$$=\frac{(x-x_0)\ .\ .\ .\ (x-x_{i-1})(x-x_{i+1})\ .\ .\ .\ (x-x_n)}{(x_i-x_0)\ .\ .\ .\ (x_i-x_{i-1})(x_i-x_{i+1})\ .\ .\ .\ (x_i-x_n)}$$

Remainder in Lagrange Interpolation Formula

25.2.3

$$R_n(x)=\pi_n(x)\cdot[x_0, x_1, . . ., x_n, x]$$

$$=\pi_n(x)\cdot\frac{f^{n+1}(\xi)}{(n+1)!} \qquad (x_0<\xi<x_n)$$

25.2.4

$$|R_n(x)|\leq\frac{(x_n-x_0)^{n+1}}{(n+1)!}\max_{x_0\leq x\leq x_n}|f^{(n+1)}(x)|$$

25.2.5

$$R_n(z)=\frac{\pi_n(z)}{2\pi i}\int_C\frac{f(t)}{(t-z)(t-z_0)\dots(t-z_n)}\,dt$$

The conditions of **25.1.8** are assumed here.

Lagrange Interpolation, Equally Spaced Abscissas

n Point Formula

25.2.6 $$f(x_0+ph)=\sum_k A_k^n(p)f_k+R_{n-1}$$

For n even, $\left(-\frac{1}{2}(n-2)\leq k\leq\frac{1}{2}n\right)$.

For n odd, $\left(-\frac{1}{2}(n-1)\leq k\leq\frac{1}{2}(n-1)\right)$.

25.2.7

$$A_k^n(p)=\frac{(-1)^{\frac{1}{2}n+k}}{\left(\frac{n-2}{2}+k\right)!\,(\frac{1}{2}n-k)!\,(p-k)}\prod_{t=1}^{n}(p+\tfrac{1}{2}n-t)$$

n even.

$$A_k^n(p)=\frac{(-1)^{\frac{1}{2}(n-1)+k}}{\left(\frac{n-1}{2}+k\right)!\left(\frac{n-1}{2}-k\right)!\,(p-k)}$$

$$\prod_{t=0}^{n-1}\left(p+\frac{n-1}{2}-t\right), \qquad n \text{ odd.}$$

25.2.8

$$R_{n-1}=\frac{1}{n!}\prod_k(p-k)h^n f^{(n)}(\xi)$$

$$\approx\frac{1}{n!}\prod_k(p-k)\Delta_0^n \qquad (x_0<\xi<x_n)$$

k has the same range as in **25.2.6.**

Lagrange Two Point Interpolation Formula (Linear Interpolation)

25.2.9 $$f(x_0+ph)=(1-p)f_0+pf_1+R_1$$

25.2.10 $$R_1(p)\approx .125h^2f^{(2)}(\xi)\approx .125\Delta^2$$

Lagrange Three Point Interpolation Formula

25.2.11

$$f(x_0+ph)=A_{-1}f_{-1}+A_0f_0+A_1f_1+R_2$$
$$\approx\frac{p(p-1)}{2}f_{-1}+(1-p^2)f_0+\frac{p(p+1)}{2}f_1$$

25.2.12

$$R_2(p)\approx .065h^3f^{(3)}(\xi)\approx .065\Delta^3 \qquad (|p|\leq 1)$$

Lagrange Four Point Interpolation Formula

25.2.13

$$f(x_0+ph)=A_{-1}f_{-1}+A_0f_0+A_1f_1+A_2f_2+R_3$$
$$\approx\frac{-p(p-1)(p-2)}{6}f_{-1}+\frac{(p^2-1)(p-2)}{2}f_0$$
$$-\frac{p(p+1)(p-2)}{2}f_1+\frac{p(p^2-1)}{6}f_2$$

25.2.14 $R_3(p)\approx$

$.024h^4f^{(4)}(\xi)\approx .024\Delta^4 \qquad (0<p<1)$

$.042h^4f^{(4)}(\xi)\approx .042\Delta^4 \qquad (-1<p<0,\ 1<p<2)$

$(x_{-1}<\xi<x_2)$

Lagrange Five Point Interpolation Formula

25.2.15

$$f(x_0+ph)=\sum_{i=-2}^{2}A_if_i+R_4$$
$$\approx\frac{(p^2-1)p(p-2)}{24}f_{-2}-\frac{(p-1)p(p^2-4)}{6}f_{-1}$$
$$+\frac{(p^2-1)(p^2-4)}{4}f_0-\frac{(p+1)p(p^2-4)}{6}f_1$$
$$+\frac{(p^2-1)p(p+2)}{24}f_2$$

25.2.16 $R_4(p)\approx$

$.012h^5f^{(5)}(\xi)\approx .012\Delta^5 \qquad (|p|<1)$

$.031h^5f^{(5)}(\xi)\approx .031\Delta^5 \qquad (1<|p|<2) \qquad (x_{-2}<\xi<x_2)$

Lagrange Six Point Interpolation Formula

25.2.17

$$f(x_0+ph)=\sum_{i=-2}^{3}A_if_i+R_5$$
$$\approx\frac{-p(p^2-1)(p-2)(p-3)}{120}f_{-2}$$
$$+\frac{p(p-1)(p^2-4)(p-3)}{24}f_{-1}$$
$$-\frac{(p^2-1)(p^2-4)(p-3)}{12}f_0$$
$$+\frac{p(p+1)(p^2-4)(p-3)}{12}f_1-\frac{p(p^2-1)(p+2)(p-3)}{24}f_2$$
$$+\frac{p(p^2-1)(p^2-4)}{120}f_3$$

25.2.18 $R_5(p)\approx$

$.0049h^6f^{(6)}(\xi)\approx .0049\Delta^6 \qquad (0<p<1)$

$.0071h^6f^{(6)}(\xi)\approx .0071\Delta^6 \qquad (-1<p<0,\ 1<p<2)$

$.024h^6f^{(6)}(\xi)\approx .024\Delta^6 \qquad (-2<p<-1,\ 2<p<3)$

$(x_{-2}<\xi<x_3)$

Lagrange Seven Point Interpolation Formula

25.2.19 $$f(x_0+ph)=\sum_{i=-3}^{3}A_if_i+R_6$$

25.2.20

$$R_6(p)\approx\begin{cases}.0025h^7f^{(7)}(\xi)\approx .0025\Delta^7 & (|p|<1)\\ .0046h^7f^{(7)}(\xi)\approx .0046\Delta^7 & (1<|p|<2)\\ .019h^7f^{(7)}(\xi)\approx .019\Delta^7 & (2<|p|<3)\end{cases}$$

$(x_{-3}<\xi<x_3)$

Lagrange Eight Point Interpolation Formula

25.2.21 $$f(x_0+ph)=\sum_{i=-3}^{4}A_if_i+R_7$$

25.2.22

$$R_7(p)\approx\begin{cases}.0011h^8f^{(8)}(\xi)\approx .0011\Delta^8 & (0<p<1)\\ .0014h^8f^{(8)}(\xi)\approx .0014\Delta^8 & (-1<p<0)\\ & (1<p<2)\\ .0033h^8f^{(8)}(\xi)\approx .0033\Delta^8 & (-2<p<-1)\\ & (2<p<3)\\ .016h^8f^{(8)}(\xi)\approx .016\Delta^8 & (-3<p<-2)\\ & (3<p<4)\end{cases}$$

$(x_{-3}<\xi<x_4)$

Aitken's Iteration Method

Let $f(x|x_0, x_1, \ldots, x_k)$ denote the unique polynomial of k^{th} degree which coincides in value with $f(x)$ at $x_0, \ldots, x_k$.

25.2.23

$$f(x|x_0,x_1)=\frac{1}{x_1-x_0}\begin{vmatrix}f_0 & x_0-x\\ f_1 & x_1-x\end{vmatrix}$$

$$f(x|x_0,x_2)=\frac{1}{x_2-x_0}\begin{vmatrix}f_0 & x_0-x\\ f_2 & x_2-x\end{vmatrix}$$

$$f(x|x_0,x_1,x_2)=\frac{1}{x_2-x_1}\begin{vmatrix}f(x|x_0,x_1) & x_1-x\\ f(x|x_0,x_2) & x_2-x\end{vmatrix}$$

$$f(x|x_0,x_1,x_2,x_3)=\frac{1}{x_3-x_2}\begin{vmatrix}f(x|x_0,x_1,x_2) & x_2-x\\ f(x|x_0,x_1,x_3) & x_3-x\end{vmatrix}$$

Taylor Expansion

25.2.24

$$f(x)=f_0+(x-x_0)f_0'+\frac{(x-x_0)^2}{2!}f_0^{(2)}+\ldots+\frac{(x-x_0)^n}{n!}f_0^{(n)}+R_n$$

25.2.25
$$R_n=\int_{x_0}^{x} f^{(n+1)}(t)\,\frac{(x-t)^n}{n!}\,dt=\frac{(x-x_0)^{n+1}}{(n+1)!}f^{(n+1)}(\xi)\qquad (x_0<\xi<x)$$

Newton's Divided Difference Interpolation Formula

25.2.26

$$f(x)=f_0+\sum_{k=1}^{n}\pi_{k-1}(x)\,[x_0,x_1,\ldots,x_k]+R_n$$

$$\begin{array}{cccccc}
x_0 & f_0 & & & \\
 & & [x_0,x_1] & & \\
x_1 & f_1 & & [x_0,x_1,x_2] & \\
 & & [x_1,x_2] & & [x_0,x_1,x_2,x_3] \\
x_2 & f_2 & & [x_1,x_2,x_3] & \\
 & & [x_2,x_3] & & \\
x_3 & f_3 & & &
\end{array}$$

25.2.27

$$R_n(x)=\pi_n(x)\,[x_0,\ldots,x_n,x]=\pi_n(x)\frac{f^{(n+1)}(\xi)}{(n+1)!}\qquad (x_0<\xi<x_n)$$

(For π_n see **25.1.6.**)

Newton's Forward Difference Formula

25.2.28

$$f(x_0+ph)=f_0+p\Delta_0+\binom{p}{2}\Delta_0^2+\ldots+\binom{p}{n}\Delta_0^n+R_n$$

$$\begin{array}{ccccc}
x_0 & f_0 & & & \\
 & & \Delta_0 & & \\
x_1 & f_1 & & \Delta_0^2 & \\
 & & \Delta_1 & & \Delta_0^3 \\
x_2 & f_2 & & \Delta_1^2 & \\
 & & \Delta_2 & & \\
x_3 & f_3 & & &
\end{array}$$

25,2.29

$$R_n=h^{n+1}\binom{p}{n+1}f^{(n+1)}(\xi)\approx\binom{p}{n+1}\Delta_0^{n+1}\qquad (x_0<\xi<x_n)$$

Relation Between Newton and Lagrange Coefficients

25.2.30

$$\binom{p}{2}=A_{-1}^{3}(p)\qquad \binom{p}{3}=-A_{-1}^{4}(p)\qquad \binom{p}{4}=A_2^5(1-p)$$
$$\binom{p}{5}=A_3^6(2-p)$$

Everett's Formula

25.2.31

$$f(x_0+ph)=(1-p)f_0+pf_1-\frac{p(p-1)(p-2)}{3!}\delta_0^2+\frac{(p+1)p(p-1)}{3!}\delta_1^2+\ldots-\binom{p+n-1}{2n+1}\delta_0^{2n}+\binom{p+n}{2n+1}\delta_1^{2n}+R_{2n}$$
$$=(1-p)f_0+pf_1+E_2\delta_0^2+F_2\delta_1^2+E_4\delta_0^4+F_4\delta_1^4+\ldots+R_{2n}$$

$$\begin{array}{ccccc}
x_0 & f_0 & \delta_0^2 & & \delta_0^4 \\
 & & \delta_{\frac{1}{2}} & \delta_{\frac{1}{2}}^3 & \\
x_1 & f_1 & \delta_1^2 & & \delta_1^4
\end{array}$$

25.2.32

$$R_{2n}=h^{2n+2}\binom{p+n}{2n+2}f^{(2n+2)}(\xi)\approx\binom{p+n}{2n+2}\left[\frac{\Delta_{-n-1}^{2n+2}+\Delta_{-n}^{2n+2}}{2}\right]\qquad (x_{-n}<\xi<x_{n+1})$$

Relation Between Everett and Lagrange Coefficients

25.2.33

$$E_2=A_{-1}^{4}\qquad E_4=A_{-2}^{6}\qquad E_6=A_{-3}^{8}$$
$$F_2=A_2^4\qquad F_4=A_3^6\qquad F_6=A_4^8$$

Everett's Formula With Throwback
(Modified Central Difference)

25.2.34

$$f(x_0+ph)=(1-p)f_0+pf_1+E_2\delta_{m,0}^2+F_2\delta_{m,1}^2+R$$

25.2.35 $\delta_m^2=\delta^2-.184\delta^4$

25.2.36 $R\approx .00045|\mu\delta_{\frac{1}{2}}^4|+.00061|\delta_{\frac{1}{2}}^5|$

25.2.37

$$f(x_0+ph)=(1-p)f_0+pf_1+E_2\delta_0^2+F_2\delta_1^2+E_4\delta_{m,0}^4+F_4\delta_{m,1}^4+R$$

25.2.38 $\delta_m^4=\delta^4-.207\delta^6+\ldots$

25.2.39 $R\approx .000032|\mu\delta_{\frac{1}{2}}^6|+.000052|\delta_{\frac{1}{2}}^7|$

25.2.40

$$f(x_0+ph)=(1-p)f_0+pf_1+E_2\delta_0^2+F_2\delta_1^2+E_4\delta_0^4+F_4\delta_1^4+E_6\delta_{m,0}^6+F_6\delta_{m,1}^6+R$$

25.2.41 $\delta_m^6=\delta^6-.218\delta^8+.049\delta^{10}+\ldots$

25.2.42 $R\approx .0000037|\mu\delta_{\frac{1}{2}}^8|+\ldots$

Simultaneous Throwback

25.2.43

$$f(x_0+ph)=(1-p)f_0+pf_1+E_2\delta^2_{m,0}+F_2\delta^2_{m,1}+E_4\delta^4_{m,0}+F_4\delta^4_{m,1}+R$$

25.2.44 $\delta^2_m=\delta^2-.01312\delta^6+.0043\delta^8-.001\delta^{10}$

25.2.45 $\delta^4_m=\delta^4-.27827\delta^6+.0685\delta^8-.016\delta^{10}$

25.2.46 $R\approx.00000083|\mu\delta^6_{\frac{1}{2}}|+.0000094\delta^7$

Bessel's Formula With Throwback

25.2.47

$$f(x_0+ph)=(1-p)f_0+pf_1+B_2(\delta^2_{m,0}+\delta^2_{m,1})+B_3\delta^3_{\frac{1}{2}}+R,\ B_2=\frac{p(p-1)}{4},\ B_3=\frac{p(p-1)\ (p-\frac{1}{2})}{6}$$

25.2.48 $\delta^2_m=\delta^2-.184\delta^4$

25.2.49 $R\approx.00045|\mu\delta^4_{\frac{1}{2}}|+.00087|\delta^5_{\frac{1}{2}}|$

Thiele's Interpolation Formula

25.2.50

$$f(x)=f(x_1)+\cfrac{x-x_1}{\rho(x_1,x_2)+\cfrac{x-x_2}{\rho_2(x_1,x_2,x_3)-f(x_1)+\cfrac{x-x_3}{\begin{pmatrix}\rho_3(x_1,x_2,x_3,x_4)\\ -\rho(x_1,x_2)+\ldots\end{pmatrix}}}}$$

(For reciprocal differences, ρ, see **25.1.12.**)

Trigonometric Interpolation

Gauss' Formula

25.2.51 $f(x)\approx\sum_{k=0}^{2n}f_k\zeta_k(x)=t_n(x)$

25.2.52

$$\zeta_k(x)=\frac{\sin\frac{1}{2}(x-x_0)\ \ldots\ \sin\frac{1}{2}(x-x_{k-1})}{\sin\frac{1}{2}(x_k-x_0)\ \ldots\ \sin\frac{1}{2}(x_k-x_{k-1})}\ \frac{\sin\frac{1}{2}(x-x_{k+1})\ \ldots\ \sin\frac{1}{2}(x-x_{2n})}{\sin\frac{1}{2}(x_k-x_{k+1})\ \ldots\ \sin\frac{1}{2}(x_k-x_{2n})}$$

$t_n(x)$ is a trigonometric polynomial of degree n such that $t_n(x_k)=f_k \quad (k=0,1,\ .\ .\ .,2n)$

Harmonic Analysis

Equally spaced abscissas

$$x_0=0,\qquad x_1,\ .\ .\ .,x_{m-1},x_m=2\pi$$

25.2.53

$$f(x)\approx\frac{1}{2}a_0+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx)$$

25.2.54 $m=2n+1$

$$a_k=\frac{2}{2n+1}\sum_{r=0}^{2n}f_r\cos kx_r;\qquad b_k=\frac{2}{2n+1}\sum_{r=0}^{2n}f_r\sin kx_r$$

$$(k=0,1,\ .\ .\ .,n)$$

25.2.55 $m=2n$

$$a_k=\frac{1}{n}\sum_{r=0}^{2n-1}f_r\cos kx_r;\qquad b_k=\frac{1}{n}\sum_{r=0}^{2n-1}f_r\sin kx_r$$

$(k=0,1,\ .\ .\ .,n) \qquad (k=0,1,\ .\ .\ .,n-1)$

b_n is arbitrary.

Subtabulation

Let $f(x)$ be tabulated initially in intervals of width h. It is desired to subtabulate $f(x)$ in intervals of width h/m. Let Δ and $\overline{\Delta}$ designate differences with respect to the original and the final intervals respectively. Thus $\overline{\Delta}_0=f\left(x_0+\frac{h}{m}\right)-f(x_0)$. Assuming that the original 5th order differences are zero,

25.2.56

$$\overline{\Delta}_0=\frac{1}{m}\Delta_0+\frac{1-m}{2m^2}\Delta^2_0+\frac{(1-m)(1-2m)}{6m^3}\Delta^3_0+\frac{(1-m)(1-2m)(1-3m)}{24m^4}\Delta^4_0$$

$$\overline{\Delta}^2_0=\frac{1}{m^2}\Delta^2_0+\frac{1-m}{m^3}\Delta^3_0+\frac{(1-m)(7-11m)}{12m^4}\Delta^4_0$$

$$\overline{\Delta}^3_0=\frac{1}{m^3}\Delta^3_0+\frac{3(1-m)}{2m^4}\Delta^4_0$$

$$\overline{\Delta}^4_0=\frac{1}{m^4}\Delta^4_0$$

From this information we may construct the final tabulation by addition. For $m=10$,

25.2.57

$$\overline{\Delta}_0=.1\Delta_0-.045\Delta^2_0+.0285\Delta^3_0-.02066\Delta^4_0$$

$$\overline{\Delta}^2_0=.01\Delta^2_0-.009\Delta^3_0+.007725\Delta^4_0$$

$$\overline{\Delta}^3_0=.001\Delta^3_0-.00135\Delta^4_0$$

$$\overline{\Delta}^4_0=.0001\Delta^4_0$$

Linear Inverse Interpolation

Find p, given $f_p(=f(x_0+ph))$.

Linear

25.2.58 $p\approx\frac{f_p-f_0}{f_1-f_0}$

Quadratic Inverse Interpolation

25.2.59

$$(f_1-2f_0+f_{-1})p^2+(f_1-f_{-1})p+2(f_0-f_p)\approx 0$$

Inverse Interpolation by Reversion of Series

25.2.60 Given $f(x_0+ph)=f_p=\sum_{k=0}^{\infty} a_k p^k$

25.2.61

$$p=\lambda+c_2\lambda^2+c_3\lambda^3+\ .\ .\ .,\ \lambda=(f_p-a_0)/a_1$$

25.2.62

$$c_2=-a_2/a_1$$

$$c_3=\frac{-a_3}{a_1}+2\left(\frac{a_2}{a_1}\right)^2$$

$$c_4=\frac{-a_4}{a_1}+\frac{5a_2a_3}{a_1^2}-\frac{5a_2^3}{a_1^3}$$

$$c_5=\frac{-a_5}{a_1}+\frac{6a_2a_4}{a_1^2}+\frac{3a_3^2}{a_1^2}-\frac{21a_2^2a_3}{a_1^3}+\frac{14a_2^4}{a_1^4}$$

Inversion of Newton's Forward Difference Formula

25.2.63

$$a_0=f_0$$

$$a_1=\Delta_0-\frac{\Delta_0^2}{2}+\frac{\Delta_0^3}{3}-\frac{\Delta_0^4}{4}+\ .\ .\ .$$

$$a_2=\frac{\Delta_0^2}{2}-\frac{\Delta_0^3}{2}+\frac{11\Delta_0^4}{24}+\ .\ .\ .$$

$$a_3=\frac{\Delta_0^3}{6}-\frac{\Delta_0^4}{4}+\ .\ .\ .$$

$$a_4=\frac{\Delta_0^4}{24}+\ .\ .\ .$$

(Used in conjunction with **25.2.62.**)

Inversion of Everett's Formula

25.2.64

$$a_0=f_0$$

$$a_1=\delta_{\frac{1}{2}}-\frac{\delta_0^2}{3}-\frac{\delta_1^2}{6}+\frac{\delta_0^4}{20}+\frac{\delta_1^4}{30}+\ .\ .\ .$$

$$a_2=\frac{\delta_0^2}{2}-\frac{\delta_0^4}{24}+\ .\ .\ .$$

$$a_3=\frac{-\delta_0^2+\delta_1^2}{6}-\frac{\delta_0^4+\delta_1^4}{24}+\ .\ .\ .$$

$$a_4=\frac{\delta_0^4}{24}+\ .\ .\ .$$

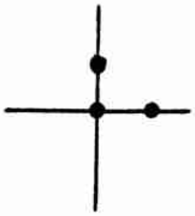
$$a_5=\frac{-\delta_0^4+\delta_1^4}{120}+\ .\ .\ .$$

(Used in conjunction with **25.2.62.**)

Bivariate Interpolation

Three Point Formula (Linear)

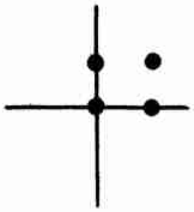
25.2.65



$$f(x_0+ph,y_0+qk)=(1-p-q)f_{0,0}+pf_{1,0}+qf_{0,1}+O(h^2)$$

Four Point Formula

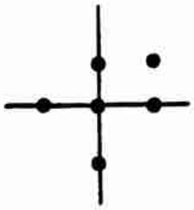
25.2.66



$$f(x_0+ph,y_0+qk)=(1-p)(1-q)f_{0,0}+p(1-q)f_{1,0}+q(1-p)f_{0,1}+pqf_{1,1}+O(h^2)$$

Six Point Formula

25.2.67



$$f(x_0+ph,y_0+qk)=\frac{q(q-1)}{2}f_{0,-1}+\frac{p(p-1)}{2}f_{-1,0}+(1+pq-p^2-q^2)f_{0,0}+\frac{p(p-2q+1)}{2}f_{1,0}+\frac{q(q-2p+1)}{2}f_{0,1}+pqf_{1,1}+O(h^3)$$

25.3. Differentiation

Lagrange's Formula

25.3.1 $$f'(x)=\sum_{k=0}^{n} l_k'(x)f_k+R_n'(x)$$

(See **25.2.1.**)

25.3.2 $$l_k'(x)=\sum_{\substack{j=0\\ j\neq k}}^{n}\frac{\pi_n(x)}{(x-x_k)(x-x_j)\pi_n'(x_k)}$$

25.3.3

$$R_n'(x)=\frac{f^{(n+1)}}{(n+1)!}(\xi)\pi_n'(x)+\frac{\pi_n(x)}{(n+1)!}\frac{d}{dx}f^{(n+1)}(\xi)$$

$$\xi=\xi(x)\quad(x_0<\xi<x_n)$$

Equally Spaced Abscissas

Three Points

25.3.4

$$f_p'=f'(x_0+ph)$$

$$=\frac{1}{h}\{(p-\tfrac{1}{2})f_{-1}-2pf_0+(p+\tfrac{1}{2})f_1\}+R_2'$$

Four Points

25.3.5

$$f_p'=f'(x_0+ph)=\frac{1}{h}\Bigg\{-\frac{3p^2-6p+2}{6}f_{-1}+\frac{3p^2-4p-1}{2}f_0-\frac{3p^2-2p-2}{2}f_1+\frac{3p^2-1}{6}f_2\Bigg\}+R_3'$$

Five Points

25.3.6

$$f_p'=f'(x_0+ph)=\frac{1}{h}\Bigg\{\frac{2p^3-3p^2-p+1}{12}f_{-2}-\frac{4p^3-3p^2-8p+4}{6}f_{-1}+\frac{2p^3-5p}{2}f_0-\frac{4p^3+3p^2-8p-4}{6}f_1+\frac{2p^3+3p^2-p-1}{12}f_2\Bigg\}+R_4'$$

For numerical values of differentiation coefficients see **Table 25.2.**

Markoff's Formulas

(Newton's Forward Difference Formula Differentiated)

25.3.7

$$f'(a_0+ph)=\frac{1}{h}\left[\Delta_0+\frac{2p-1}{2}\Delta_0^2+\frac{3p^2-6p+2}{6}\Delta_0^3+\ldots+\frac{d}{dp}\binom{p}{n}\Delta_0^n\right]+R_n'$$

25.3.8

$$R_n'=h^nf^{(n+1)}(\xi)\frac{d}{dp}\binom{p}{n+1}+h^{n+1}\binom{p}{n+1}\frac{d}{dx}f^{(n+1)}(\xi)$$

$$(a_0<\xi<a_n)$$

25.3.9 $$hf_0'=\Delta_0-\frac{1}{2}\Delta_0^2+\frac{1}{3}\Delta_0^3-\frac{1}{4}\Delta_0^4+\ldots$$

25.3.10 $$h^2f_0^{(2)}=\Delta_0^2-\Delta_0^3+\frac{11}{12}\Delta_0^4-\frac{5}{6}\Delta_0^5+\ldots$$

25.3.11

$$h^3f_0^{(3)}=\Delta_0^3-\frac{3}{2}\Delta_0^4+\frac{7}{4}\Delta_0^5-\frac{15}{8}\Delta_0^6+\ldots$$

25.3.12

$$h^4f_0^{(4)}=\Delta_0^4-2\Delta_0^5+\frac{17}{6}\Delta_0^6-\frac{7}{2}\Delta_0^7+\ldots$$

25.3.13

$$h^5f_0^{(5)}=\Delta_0^5-\frac{5}{2}\Delta_0^6+\frac{25}{6}\Delta_0^7-\frac{35}{6}\Delta_0^8+\ldots$$

Everett's Formula

25.3.14

$$hf'(x_0+ph)\approx-f_0+f_1-\frac{3p^2-6p+2}{6}\delta_0^2+\frac{3p^2-1}{6}\delta_1^2-\frac{5p^4-20p^3+15p^2+10p-6}{120}\delta_0^4+\frac{5p^4-15p^2+4}{120}\delta_1^4+\ldots-\left[\binom{p+n-1}{2n+1}\right]'\delta_0^{2n}+\left[\binom{p+n}{2n+1}\right]'\delta_1^{2n}$$

25.3.15

$$hf_0'\approx-f_0+f_1-\frac{1}{3}\delta_0^2-\frac{1}{6}\delta_1^2+\frac{1}{20}\delta_0^4+\frac{1}{30}\delta_1^4$$

Differences in Terms of Derivatives

25.3.16

$$\Delta_0\approx hf_0'+\frac{h^2}{2!}f_0^{(2)}+\frac{h^3}{3!}f_0^{(3)}+\frac{h^4}{4!}f_0^{(4)}+\frac{h^5}{5!}f_0^{(5)}$$

25.3.17

$$\Delta_0^2\approx h^2f_0^{(2)}+h^3f_0^{(3)}+\frac{7}{12}h^4f_0^{(4)}+\frac{1}{4}h^5f_0^{(5)}$$

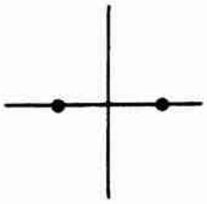
25.3.18 $$\Delta_0^3\approx h^3f_0^{(3)}+\frac{3}{2}h^4f_0^{(4)}+\frac{5}{4}f_0^{(5)}$$

25.3.19 $$\Delta_0^4\approx h^4f_0^{(4)}+2h^5f_0^{(5)}$$

25.3.20 $$\Delta_0^5\approx h^5f_0^{(5)}$$

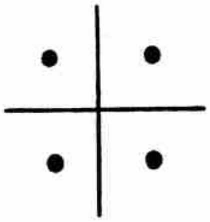
Partial Derivatives

25.3.21



$$\frac{\partial f_{0,0}}{\partial x}=\frac{1}{2h}(f_{1,0}-f_{-1,0})+O(h^2)$$

25.3.22

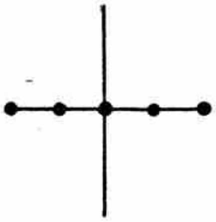


$$\frac{\partial f_{0,0}}{\partial x}=\frac{1}{4h}\left(f_{1,1}-f_{-1,1}+f_{1,-1}-f_{-1,-1}\right)+O(h^2)$$

25.3.23

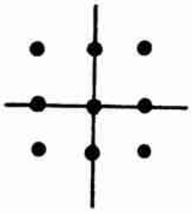
$$\frac{\partial^2 f_{0,0}}{\partial x^2}=\frac{1}{h^2}\left(f_{1,0}-2f_{0,0}+f_{-1,0}\right)+O(h^2)$$

25.3.24



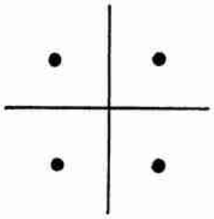
$$\frac{\partial^2 f_{0,0}}{\partial x^2}=\frac{1}{12h^2}\left(-f_{2,0}+16f_{1,0}-30f_{0,0}+16f_{-1,0}-f_{-2,0}\right)+O(h^4)$$

25.3.25



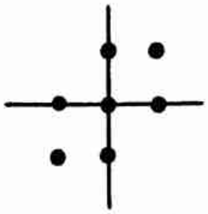
$$\frac{\partial^2 f_{0,0}}{\partial x^2}=\frac{1}{3h^2}\left(f_{1,1}-2f_{0,1}+f_{-1,1}+f_{1,0}-2f_{0,0}+f_{-1,0}+f_{1,-1}-2f_{0,-1}+f_{-1,-1}\right)+O(h^2)$$

25.3.26



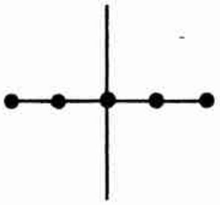
$$\frac{\partial^2 f_{0,0}}{\partial x\partial y}=\frac{1}{4h^2}\left(f_{1,1}-f_{1,-1}-f_{-1,1}+f_{-1,-1}\right)+O(h^2)$$

25.3.27



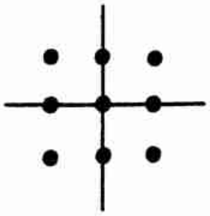
$$\frac{\partial^2 f_{0,0}}{\partial x\partial y}=\frac{-1}{2h^2}\left(f_{1,0}+f_{-1,0}+f_{0,1}+f_{0,-1}-2f_{0,0}-f_{1,1}-f_{-1,-1}\right)+O(h^2)$$

25.3.28



$$\frac{\partial^4 f_{0,0}}{\partial x^4}=\frac{1}{h^4}\left(f_{2,0}-4f_{1,0}+6f_{0,0}-4f_{-1,0}+f_{-2,0}\right)+O(h^2)$$

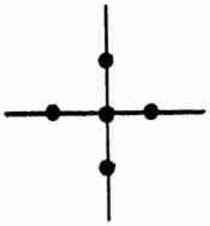
25.3.29



$$\frac{\partial^4 f_{0,0}}{\partial x^2\partial y^2}=\frac{1}{h^4}\left(f_{1,1}+f_{-1,1}+f_{1,-1}+f_{-1,-1}-2f_{1,0}-2f_{-1,0}-2f_{0,1}-2f_{0,-1}+4f_{0,0}\right)+O(h^2)$$

Laplacian

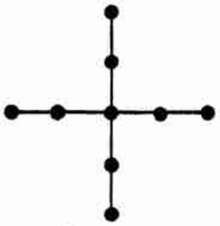
25.3.30



$$\nabla^2 u_{0,0}=\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)_{0,0}$$

$$=\frac{1}{h^2}\,(u_{1,0}+u_{0,1}+u_{-1,0}+u_{0,-1}-4u_{0,0})+O(h^2)$$

25.3.31

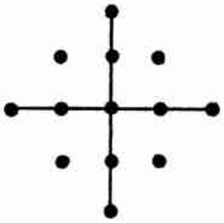


$$\nabla^2 u_{0,0}=\frac{1}{12h^2}\,[-60u_{0,0}+16(u_{1,0}+u_{0,1}+u_{-1,0}+u_{0,-1})$$

$$-(u_{2,0}+u_{0,2}+u_{-2,0}+u_{0,-2})]+O(h^4)$$

Biharmonic Operator

25.3.32



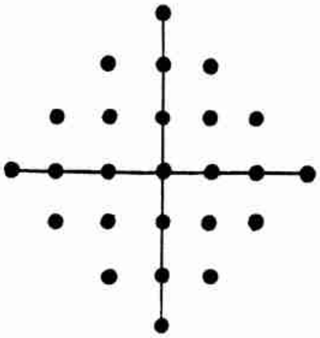
$$\nabla^4 u_{0,0}=\left(\frac{\partial^4 u}{\partial x^4}+2\,\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4}\right)_{0,0}$$

$$=\frac{1}{h^4}[20u_{0,0}-8(u_{1,0}+u_{0,1}+u_{-1,0}+u_{0,-1})$$

$$+2(u_{1,1}+u_{1,-1}+u_{-1,1}+u_{-1,-1})$$

$$+(u_{0,2}+u_{2,0}+u_{-2,0}+u_{0,-2})]+O(h^2)$$

25.3.33



$$\nabla^4 u_{0,0}=\frac{1}{6h^4}\,[-(u_{0,3}+u_{0,-3}+u_{3,0}+u_{-3,0})$$

$$+14(u_{0,2}+u_{0,-2}+u_{2,0}+u_{-2,0})$$

$$-77(u_{0,1}+u_{0,-1}+u_{1,0}+u_{-1,0})$$

$$+184u_{0,0}+20(u_{1,1}+u_{1,-1}+u_{-1,1}+u_{-1,-1})$$

$$-(u_{1,2}+u_{2,1}+u_{1,-2}+u_{2,-1}+u_{-1,2}+u_{-2,1}$$

$$+u_{-1,-2}+u_{-2,-1})]+O(h^4)$$

25.4. Integration

Trapezoidal Rule

25.4.1

$$\int_{x_0}^{x_1} f(x)dx=\frac{h}{2}\,(f_0+f_1)-\frac{1}{2}\int_{x_0}^{x_1}(t-x_0)(x_1-t)f''(t)\,dt$$

$$=\frac{h}{2}\,(f_0+f_1)-\frac{h^3}{12}f''(\xi)\qquad(x_0<\xi<x_1)$$

Extended Trapezoidal Rule

25.4.2

$$\int_{x_0}^{x_m} f(x)dx=h\left[\frac{f_0}{2}+f_1+\ \ldots\ +f_{m-1}+\frac{f_m}{2}\right]$$

$$-\frac{mh^3}{12}f''(\xi)$$

Error Term in Trapezoidal Formula for Periodic Functions

If $f(x)$ is periodic and has a continuous k^{th} derivative, and if the integral is taken over a period, then

25.4.3 $$|\text{Error}|\leq\frac{\text{constant}}{m^k}$$

Modified Trapezoidal Rule

25.4.4

$$\int_{x_0}^{x_m} f(x)dx=h\left[\frac{f_0}{2}+f_1+\ \ldots\ +f_{m-1}+\frac{f_m}{2}\right]$$

$$+\frac{h}{24}\,[-f_{-1}+f_1+f_{m-1}-f_{m+1}]+\frac{11m}{720}\,h^5f^{(4)}(\xi)$$

Simpson's Rule

25.4.5

$$\int_{x_0}^{x_2} f(x)dx=\frac{h}{3}[f_0+4f_1+f_2]$$
$$+\frac{1}{6}\int_{x_0}^{x_1}(x_0-t)^2(x_1-t)f^{(3)}(t)dt$$
$$+\frac{1}{6}\int_{x_1}^{x_2}(x_2-t)^2(x_1-t)f^{(3)}(t)dt$$
$$=\frac{h}{3}[f_0+4f_1+f_2]-\frac{h^5}{90}f^{(4)}(\xi)$$

Extended Simpson's Rule

25.4.6

$$\int_{x_0}^{x_{2n}} f(x)dx=\frac{h}{3}[f_0+4(f_1+f_3+\ \dots\ +f_{2n-1})$$
$$+2(f_2+f_4+\ \dots\ +f_{2n-2})+f_{2n}]-\frac{nh^5}{90}f^{(4)}(\xi)$$

Euler-Maclaurin Summation Formula

25.4.7

$$\int_{x_0}^{x_n} f(x)dx=h\left[\frac{f_0}{2}+f_1+f_2+\ \dots\ +f_{n-1}+\frac{f_n}{2}\right]$$
$$-\frac{B_2}{2!}h^2(f'_n-f'_0)-\ \dots\ -\frac{B_{2k}h^{2k}}{(2k)!}[f_n^{(2k-1)}-f_0^{(2k-1)}]+R_{2k}$$
$$R_{2k}=\frac{\theta nB_{2k+2}h^{2k+3}}{(2k+2)!}\max_{x_0\leq x\leq x_n}|f^{(2k+2)}(x)|,\qquad(-1\leq\theta\leq 1)$$

(For B_{2k}, Bernoulli numbers, see chapter **23.**)

If $f^{(2k+2)}(x)$ and $f^{(2k+4)}(x)$ do not change sign for $x_0<x<x_n$ then $|R_{2k}|$ is less than the first neglected term. If $f^{(2k+2)}(x)$ does not change sign for $x_0<x<x_n$, $|R_{2k}|$ is less than twice the first neglected term.

Lagrange Formula

25.4.8

$$\int_a^b f(x)dx=\sum_{i=0}^{n}(L_i^{(n)}(b)-L_i^{(n)}(a))f_i+R_n$$

(See **25.2.1.**)

25.4.9

$$L_i^{(n)}(x)=\frac{1}{\pi'_n(x_i)}\int_{x_0}^{x}\frac{\pi_n(t)}{t-x_i}dt=\int_{x_0}^{x}l_i(t)dt$$

25.4.10 $$R_n=\frac{1}{(n+1)!}\int_a^b\pi_n(x)f^{(n+1)}(\xi(x))dx$$

Equally Spaced Abscissas

25.4.11

$$\int_{x_0}^{x_k} f(x)dx=\frac{1}{h^n}\sum_{i=0}^{n}f_i\frac{(-1)^{n-i}}{i!(n-i)!}\int_{x_0}^{x_k}\frac{\pi_n(x)}{x-x_i}dx+R_n$$

25.4.12 $$\int_{x_m}^{x_{m+1}} f(x)dx=h\sum_{i=-\left[\frac{n-1}{2}\right]}^{\left[\frac{n}{2}\right]}A_i(m)f_i+R_n$$ *

(See **Table 25.3** for $A_i(m)$.)

Newton-Cotes Formulas (Closed Type)

(For Trapezoidal and Simpson's Rules see **25.4.1–25.4.6.**)

25.4.13 **(Simpson's $\frac{3}{8}$ rule)**

$$\int_{x_0}^{x_3} f(x)dx=\frac{3h}{8}(f_0+3f_1+3f_2+f_3)-\frac{3f^{(4)}(\xi)h^5}{80}$$

25.4.14 **(Bode's rule)**

$$\int_{x_0}^{x_4} f(x)dx=\frac{2h}{45}(7f_0+32f_1+12f_2$$
$$+32f_3+7f_4)-\frac{8f^{(6)}(\xi)h^7}{945}$$

25.4.15

$$\int_{x_0}^{x_5} f(x)dx=\frac{5h}{288}(19f_0+75f_1+50f_2+50f_3$$
$$+75f_4+19f_5)-\frac{275f^{(6)}(\xi)h^7}{12096}$$

25.4.16

$$\int_{x_0}^{x_6} f(x)dx=\frac{h}{140}(41f_0+216f_1+27f_2+272f_3$$
$$+27f_4+216f_5+41f_6)-\frac{9f^{(8)}(\xi)h^9}{1400}$$

25.4.17

$$\int_{x_0}^{x_7} f(x)dx=\frac{7h}{17280}(751f_0+3577f_1+1323f_2$$
$$+2989f_3+2989f_4+1323f_5+3577f_6$$
$$+751f_7)-\frac{8183f^{(8)}(\xi)h^9}{518400}$$

25.4.18

$$\int_{x_0}^{x_8} f(x)dx=\frac{4h}{14175}(989f_0+5888f_1-928f_2$$
$$+10496f_3-4540f_4+10496f_5-928f_6+5888f_7$$
$$+989f_8)-\frac{2368}{467775}f^{(10)}(\xi)h^{11}$$

25.4.19

$$\int_{x_0}^{x_9} f(x)dx=\frac{9h}{89600}\{2857(f_0+f_9)$$
$$+15741(f_1+f_8)+1080(f_2+f_7)+19344(f_3+f_6)$$
$$+5778(f_4+f_5)\}-\frac{173}{14620}f^{(10)}(\xi)h^{11}$$

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*See page II.

25.4.20

$$\int_{x_0}^{x_{10}} f(x)dx=\frac{5h}{299376}\{16067(f_0+f_{10})$$
$$+106300(f_1+f_9)-48525(f_2+f_8)+272400(f_3+f_7)$$
$$-260550(f_4+f_6)+427368f_5\}$$
$$-\frac{1346350}{326918592}f^{(12)}(\xi)h^{13}$$

Newton-Cotes Formulas (Open Type)

25.4.21

$$\int_{x_0}^{x_3} f(x)dx=\frac{3h}{2}(f_1+f_2)+\frac{f^{(2)}(\xi)h^3}{4}$$

25.4.22

$$\int_{x_0}^{x_4} f(x)dx=\frac{4h}{3}(2f_1-f_2+2f_3)+\frac{28f^{(4)}(\xi)h^5}{90}$$

25.4.23

$$\int_{x_0}^{x_5} f(x)dx=\frac{5h}{24}(11f_1+f_2+f_3+11f_4)+\frac{95f^{(4)}(\xi)h^5}{144}$$

25.4.24

$$\int_{x_0}^{x_6} f(x)dx=\frac{6h}{20}(11f_1-14f_2+26f_3-14f_4+11f_5)$$
$$+\frac{41f^{(6)}(\xi)h^7}{140}$$

25.4.25

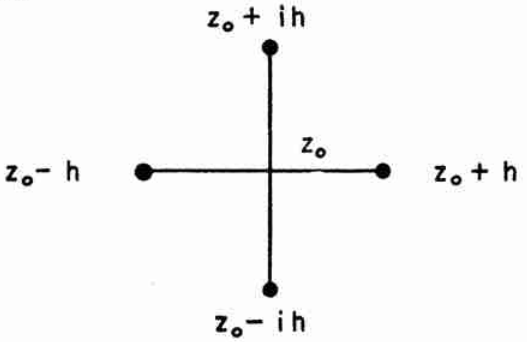
$$\int_{x_0}^{x_7} f(x)dx=\frac{7h}{1440}(611f_1-453f_2+562f_3+562f_4$$
$$-453f_5+611f_6)+\frac{5257}{8640}f^{(6)}(\xi)h^7$$

25.4.26

$$\int_{x_0}^{x_8} f(x)dx=\frac{8h}{945}(460f_1-954f_2+2196f_3-2459f_4$$
$$+2196f_5-954f_6+460f_7)+\frac{3956}{14175}f^{(8)}(\xi)h^9$$

Five Point Rule for Analytic Functions

25.4.27



$$\int_{z_0-h}^{z_0+h} f(z)dz=\frac{h}{15}\{24f(z_0)+4[f(z_0+h)+f(z_0-h)]$$
$$-[f(z_0+ih)+f(z_0-ih)]\}+R$$

$|R|\leq\frac{|h|^7}{1890}\operatorname*{Max}_{z\epsilon S}|f^{(6)}(z)|$, S designates the square with vertices $z_0+i^kh(k=0,1,2,3)$; h can be complex.

Chebyshev's Equal Weight Integration Formula

25.4.28 $$\int_{-1}^{1} f(x)dx=\frac{2}{n}\sum_{i=1}^{n} f(x_i)+R_n$$

Abscissas: x_i is the i^{th} zero of the polynomial part of

$$x^n \exp\left[\frac{-n}{2\cdot 3x^2}-\frac{n}{4\cdot 5x^3}-\frac{n}{6\cdot 7x^4}-\ldots\right]$$

(See **Table 25.5** for x_i.)

For $n=8$ and $n\geq 10$ some of the zeros are complex.

Remainder:

$$R_n=\int_{-1}^{+1}\frac{x^{n+1}}{(n+1)!}f^{(n+1)}(\xi)dx$$
$$-\frac{2}{n(n+1)!}\sum_{i=1}^{n} x_i^{n+1}f^{(n+1)}(\xi_i)$$

where $\xi=\xi(x)$ satisfies $0\leq\xi\leq x$ and $0\leq\xi_i\leq x_i$

$$(i=1,\ldots,n)$$

Integration Formulas of Gaussian Type

(For Orthogonal Polynomials see chapter **22**)

Gauss' Formula

25.4.29 $$\int_{-1}^{1} f(x)dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials: Legendre polynomials $P_n(x)$, $P_n(1)=1$

Abscissas: x_i is the i^{th} zero of $P_n(x)$

Weights: $w_i=2/(1-x_i^2)[P_n'(x_i)]^2$ *
(See **Table 25.4** for x_i and w_i.)

$$R_n=\frac{2^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3}f^{(2n)}(\xi)\qquad(-1<\xi<1)$$

Gauss' Formula, Arbitrary Interval

25.4.30 $$\int_a^b f(y)dy=\frac{b-a}{2}\sum_{i=1}^{n} w_i f(y_i)+R_n$$

$$y_i=\left(\frac{b-a}{2}\right)x_i+\left(\frac{b+a}{2}\right)$$

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*See page II.

Related orthogonal polynomials: $P_n(x)$, $P_n(1)=1$

Abscissas: x_i is the i^{th} zero of $P_n(x)$

* Weights: $w_i=2/(1-x_i^2)\,[P'_n(x_i)]^2$

*
$$R_n=\frac{(b-a)^{2n+1}(n!)^4}{(2n+1)[(2n)!]^3}f^{(2n)}(\xi)$$

Radau's Integration Formula

25.4.31

$$\int_{-1}^{1} f(x)dx=\frac{2}{n^2}f_{-1}+\sum_{i=1}^{n-1} w_i f(x_i)+R_n$$

Related polynomials:

$$\frac{P_{n-1}(x)+P_n(x)}{x+1}$$

Abscissas: x_i is the i^{th} zero of

$$\frac{P_{n-1}(x)+P_n(x)}{x+1}$$

Weights:

$$w_i=\frac{1}{n^2}\frac{1-x_i}{[P_{n-1}(x_i)]^2}=\frac{1}{1-x_i}\frac{1}{[P'_{n-1}(x_i)]^2}$$

Remainder:

$$R_n=\frac{2^{2n-1}\cdot n}{[(2n-1)!]^3}[(n-1)!]^4 f^{(2n-1)}(\xi)\qquad(-1<\xi<1)$$

Lobatto's Integration Formula

25.4.32

$$\int_{-1}^{1} f(x)dx=\frac{2}{n(n-1)}[f(1)+f(-1)]+\sum_{i=2}^{n-1} w_i f(x_i)+R_n$$

Related polynomials: $P'_{n-1}(x)$

Abscissas: x_i is the $(i-1)^{\text{st}}$ zero of $P'_{n-1}(x)$

Weights:

$$w_i=\frac{2}{n(n-1)[P_{n-1}(x_i)]^2}\qquad(x_i\neq\pm 1)$$

(See **Table 25.6** for x_i and w_i.)

Remainder:

$$R_n=\frac{-n(n-1)^3 2^{2n-1}[(n-2)!]^4}{(2n-1)[(2n-2)!]^3}f^{(2n-2)}(\xi)\qquad(-1<\xi<1)$$

25.4.33
$$\int_0^1 x^k f(x)dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials:

$$q_n(x)=\sqrt{k+2n+1}\,P_n^{(k,0)}(1-2x)$$

(For the Jacobi polynomials $P_n^{(k,0)}$ see chapter **22.**)

Abscissas:

x_i is the i^{th} zero of $q_n(x)$

Weights:

$$w_i=\left\{\sum_{j=0}^{n-1}[q_j(x_i)]^2\right\}^{-1}$$ *

(See **Table 25.8** for x_i and w_i.)

Remainder:

$$R_n=\frac{f^{(2n)}(\xi)}{(k+2n+1)(2n)!}\left[\frac{n!(k+n)!}{(k+2n)!}\right]^2\qquad(0<\xi<1)$$

25.4.34

$$\int_0^1 f(x)\sqrt{1-x}\,dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{1-x}}P_{2n+1}(\sqrt{1-x}),\ P_{2n+1}(1)=1$$

Abscissas: $x_i=1-\xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n+1}(x)$.

Weights: $w_i=2\xi_i^2 w_i^{(2n+1)}$ where $w_i^{(2n+1)}$ are the Gaussian weights of order $2n+1$.

Remainder:

$$R_n=\frac{2^{4n+3}[(2n+1)!]^4}{(2n)!(4n+3)[(4n+2)!]^2}f^{(2n)}(\xi)\qquad(0<\xi<1)$$

25.4.35

$$\int_a^b f(y)\sqrt{b-y}\,dy=(b-a)^{3/2}\sum_{i=1}^{n} w_i f(y_i)$$

$$y_i=a+(b-a)x_i$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{1-x}}P_{2n+1}(\sqrt{1-x}),\ P_{2n+1}(1)=1$$

Abscissas: $x_i=1-\xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n+1}(x)$.

Weights: $w_i=2\xi_i^2 w_i^{(2n+1)}$ where $w_i^{(2n+1)}$ are the Gaussian weights of order $2n+1$.

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*See page II.

25.4.36 $$\int_0^1 \frac{f(x)}{\sqrt{1-x}}\,dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials:

$$P_{2n}(\sqrt{1-x}),\ P_{2n}(1)=1$$

Abscissas: $x_i=1-\xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n}(x)$.
Weights: $w_i=2w_i^{(2n)}$, $w_i^{(2n)}$ are the Gaussian weights of order $2n$.
Remainder:

$$R_n=\frac{2^{4n+1}}{4n+1}\frac{[(2n)!]^3}{[(4n)!]^2}f^{(2n)}(\xi)\qquad(0<\xi<1)$$

25.4.37 $$\int_a^b \frac{f(y)}{\sqrt{b-y}}\,dy=\sqrt{b-a}\sum_{i=1}^{n} w_i f(y_i)+R_n$$

$$y_i=a+(b-a)x_i$$

Related orthogonal polynomials:

$$P_{2n}(\sqrt{1-x}),P_{2n}(1)=1$$

Abscissas:
$x_i=1-\xi_i^2$ where ξ_i is the i^{th} positive zero of $P_{2n}(x)$.

Weights: $w_i=2w_i^{(2n)}$, $w_i^{(2n)}$ are the Gaussian weights of order $2n$.

25.4.38 $$\int_{-1}^{+1} \frac{f(x)}{\sqrt{1-x^2}}\,dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials: Chebyshev Polynomials of First Kind

$$T_n(x),T_n(1)=\frac{1}{2^{n-1}}$$

Abscissas:

$$x_i=\cos\frac{(2i-1)\pi}{2n}$$

Weights:

$$w_i=\frac{\pi}{n}$$

Remainder:

$$R_n=\frac{\pi}{(2n)!2^{2n-1}}f^{(2n)}(\xi)\quad(-1<\xi<1)$$

25.4.39

$$\int_a^b \frac{f(y)dy}{\sqrt{(y-a)(b-y)}}=\sum_{i=1}^{n} w_i f(y_i)+R_n$$

$$y_i=\frac{b+a}{2}+\frac{b-a}{2}x_i$$

Related orthogonal polynomials:

$$T_n(x),T_n(1)=\frac{1}{2^{n-1}}$$

Abscissas:

$$x_i=\cos\frac{(2i-1)\pi}{2n}$$

Weights:

$$w_i=\frac{\pi}{n}$$

25.4.40

$$\int_{-1}^{+1} f(x)\sqrt{1-x^2}dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials: Chebyshev Polynomials of Second Kind

$$U_n(x)=\frac{\sin\,[(n+1)\arccos x]}{\sin\,(\arccos x)}$$ *

Abscissas:

$$x_i=\cos\frac{i}{n+1}\pi$$ *

Weights:

$$w_i=\frac{\pi}{n+1}\sin^2\frac{i}{n+1}\pi$$ *

Remainder:

$$R_n=\frac{\pi}{(2n)!2^{2n+1}}f^{(2n)}(\xi)\qquad(-1<\xi<1)$$

25.4.41

$$\int_a^b \sqrt{(y-a)(b-y)}f(y)dy=\left(\frac{b-a}{2}\right)^2\sum_{i=1}^{n} w_i f(y_i)+R_n$$

$$y_i=\frac{b+a}{2}+\frac{b-a}{2}x_i$$

Related orthogonal polynomials:

$$U_n(x)=\frac{\sin\,[(n+1)\arccos x]}{\sin\,(\arccos x)}$$ *

Abscissas:

$$x_i=\cos\frac{i}{n+1}\pi$$ *

Weights:

$$w_i=\frac{\pi}{n+1}\sin^2\frac{i}{n+1}\pi$$ *

25.4.42 $$\int_0^1 f(x)\sqrt{\frac{x}{1-x}}\,dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{x}}T_{2n+1}(\sqrt{x})$$

Abscissas:

$$x_i=\cos^2\frac{2i-1}{2n+1}\cdot\frac{\pi}{2}$$

Weights:

$$w_i=\frac{2\pi}{2n+1}x_i$$

Remainder:

$$R_n=\frac{\pi}{(2n)!2^{4n+1}}f^{(2n)}(\xi) \qquad (0<\xi<1)$$

25.4.43

$$\int_a^b f(x)\sqrt{\frac{x-a}{b-x}}\,dx=(b-a)\sum_{i=1}^{n} w_i f(y_i)+R_n$$

$$y_i=a+(b-a)x_i$$

Related orthogonal polynomials:

$$\frac{1}{\sqrt{x}}\,T_{2n+1}(\sqrt{x})$$

Abscissas:

$$x_i=\cos^2\frac{2i-1}{2n+1}\cdot\frac{\pi}{2}$$

Weights:

$$w_i=\frac{2\pi}{2n+1}\,x_i$$

25.4.44 $$\int_0^1 \ln x\, f(x)dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials: polynomials orthogonal with respect to the weight function $-\ln x$
Abscissas: See **Table 25.7**
Weights: See **Table 25.7**

25.4.45

$$\int_0^\infty e^{-x}f(x)dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials: Laguerre polynomials $L_n(x)$.
Abscissas: x_i is the i^{th} zero of $L_n(x)$
Weights:

* $$w_i=\frac{x_i}{(n+1)^2[L_{n+1}(x_i)]^2}$$

(See **Table 25.9** for x_i and w_i.)
Remainder:

$$R_n=\frac{(n!)^2}{(2n)!}f^{(2n)}(\xi) \qquad (0<\xi<\infty)$$

25.4.46

$$\int_{-\infty}^{\infty} e^{-x^2}f(x)dx=\sum_{i=1}^{n} w_i f(x_i)+R_n$$

Related orthogonal polynomials: Hermite polynomials $H_n(x)$.
Abscissas: x_i is the i^{th} zero of $H_n(x)$
Weights:

$$\frac{2^{n-1}n!\sqrt{\pi}}{n^2[H_{n-1}(x_i)]^2}$$

(See **Table 25.10** for x_i and w_i.)

Remainder:

$$R_n=\frac{n!\sqrt{\pi}}{2^n(2n)!}f^{(2n)}(\xi) \qquad (-\infty<\xi<\infty)$$

Filon's Integration Formula [3]

25.4.47

$$\int_{x_0}^{x_{2n}} f(x)\cos tx\,dx=h\Big[\alpha(th)(f_{2n}\sin tx_{2n}-f_0\sin tx_0)+\beta(th)\cdot C_{2n}+\gamma(th)\cdot C_{2n-1}+\frac{2}{45}\,th^4S'_{2n-1}\Big]-R_n$$

25.4.48

$$C_{2n}=\sum_{i=0}^{n} f_{2i}\cos(tx_{2i})-\tfrac{1}{2}[f_{2n}\cos tx_{2n}+f_0\cos tx_0]$$

25.4.49

$$C_{2n-1}=\sum_{i=1}^{n} f_{2i-1}\cos tx_{2i-1}$$

25.4.50

$$S'_{2n-1}=\sum_{i=1}^{n} f^{(3)}_{2i-1}\sin tx_{2i-1}$$

25.4.51

$$R_n=\frac{1}{90}nh^5f^{(4)}(\xi)+O(th^7)$$

25.4.52

$$\alpha(\theta)=\frac{1}{\theta}+\frac{\sin 2\theta}{2\theta^2}-\frac{2\sin^2\theta}{\theta^3}$$

$$\beta(\theta)=2\left(\frac{1+\cos^2\theta}{\theta^2}-\frac{\sin 2\theta}{\theta^3}\right)$$

$$\gamma(\theta)=4\left(\frac{\sin\theta}{\theta^3}-\frac{\cos\theta}{\theta^2}\right)$$

For small θ we have

25.4.53

$$\alpha=\frac{2\theta^3}{45}-\frac{2\theta^5}{315}+\frac{2\theta^7}{4725}-\cdots$$

$$\beta=\frac{2}{3}+\frac{2\theta^2}{15}-\frac{4\theta^4}{105}+\frac{2\theta^6}{567}-\cdots$$

$$\gamma=\frac{4}{3}-\frac{2\theta^2}{15}+\frac{\theta^4}{210}-\frac{\theta^6}{11340}+\cdots$$

25.4.54

$$\int_{x_0}^{x_{2n}} f(x)\sin tx\,dx=h\Big[\alpha(th)(f_0\cos tx_0-f_{2n}\cos tx_{2n})+\beta S_{2n}+\gamma S_{2n-1}+\frac{2}{45}\,th^4C'_{2n-1}\Big]-R_n$$

25.4.55

$$S_{2n}=\sum_{i=0}^{n} f_{2i}\sin(tx_{2i})-\frac{1}{2}[f_{2n}\sin(tx_{2n})+f_0\sin(tx_0)]$$

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[3] For certain difficulties associated with this formula, see the article by J. W. Tukey, p. 400, "On Numerical Approximation," Ed. R. E. Langer, Madison, 1959.

*See page II.

25.4.56 $$S_{2n-1}=\sum_{i=1}^{n} f_{2i-1}\sin(tx_{2i-1})$$

25.4.57 $$C'_{2n-1}=\sum_{i=1}^{n} f^{(3)}_{2i-1}\cos(tx_{2i-1})$$

(See **Table 25.11** for α, β, γ.)

Iterated Integrals

25.4.58

$$\int_0^x dt_n \int_0^{t_n} dt_{n-1} \ldots \int_0^{t_3} dt_2 \int_0^{t_2} f(t_1)dt_1 = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)dt$$

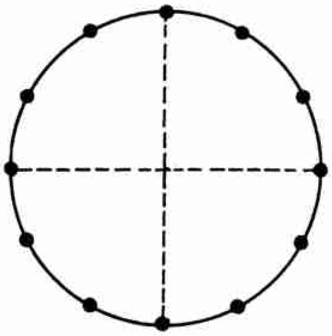
25.4.59

$$\int_a^x dt_n \int_a^{t_n} dt_{n-1} \ldots \int_a^{t_3} dt_2 \int_a^{t_2} f(t_1)dt_1 = \frac{(x-a)^n}{(n-1)!}\int_0^1 t^{n-1} f(x-(x-a)t)dt$$

Multidimensional Integration

Circumference of Circle Γ: $x^2+y^2=h^2$.

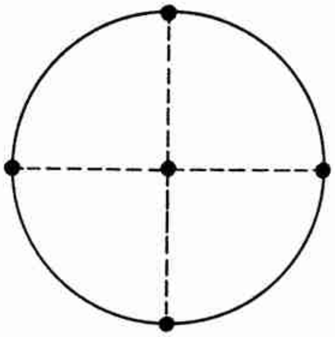
25.4.60



$$\frac{1}{2\pi h}\int_\Gamma f(x,y)ds=\frac{1}{2m}\sum_{n=1}^{2m} f\left(h\cos\frac{\pi n}{m}, h\sin\frac{\pi n}{m}\right)+O(h^{2m-2})$$

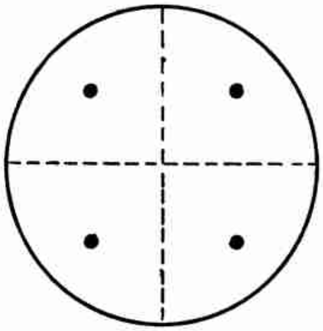
Circle C: $x^2+y^2\leq h^2$.

25.4.61

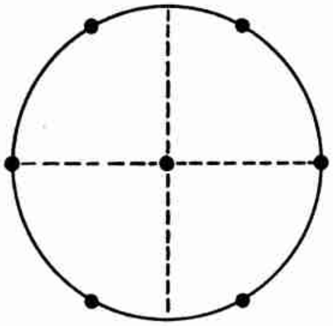


$$\frac{1}{\pi h^2}\iint_C f(x,y)dxdy=\sum_{i=1}^{n} w_i f(x_i,y_i)+R$$

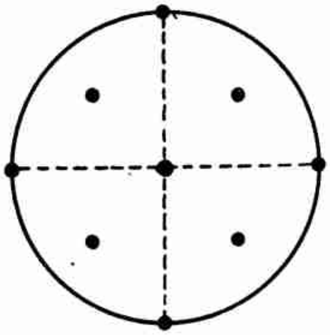
(x_i,y_i)	w_i	
$(0,0)$	1/2	$R=O(h^4)$
$(\pm h,0)$, $(0,\pm h)$	1/8	



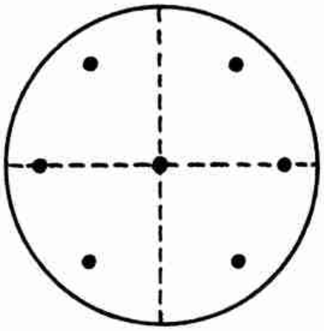
(x_i,y_i)	w_i	
$\left(\pm\frac{h}{2},\pm\frac{h}{2}\right)$	1/4	$R=O(h^4)$



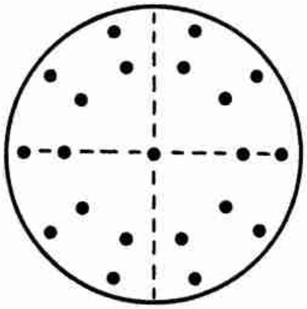
(x_i,y_i)	w_i	
$(0,0)$	1/2	
$(\pm h,0)$	1/12	$R=O(h^4)$
$\left(\pm\frac{h}{2},\pm\frac{h}{2}\sqrt{3}\right)$	1/12	



(x_i, y_i)	w_i	
$(0,0)$	$1/6$	
$(\pm h, 0)$	$1/24$	$R=O(h^6)$
$(0, \pm h)$	$1/24$	
$\left(\pm\frac{h}{2}, \pm\frac{h}{2}\right)$	$1/6$	



(x_i, y_i)	w_i	
$(0,0)$	$1/4$	
$\left(\pm\sqrt{\frac{2}{3}}\,h, 0\right)$	$1/8$	$R=O(h^6)$
$\left(\pm\sqrt{\frac{1}{6}}\,h, \pm\frac{h}{2}\sqrt{2}\right)$	$1/8$	

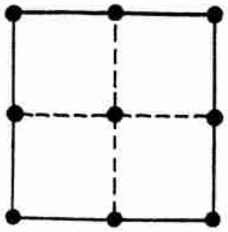


(x_i, y_i)	w_i
$(0,0)$	$1/9$
$\left(\sqrt{\frac{6-\sqrt{6}}{10}}\,h \cos\frac{2\pi k}{10}, \sqrt{\frac{6-\sqrt{6}}{10}}\,h \sin\frac{2\pi k}{10}\right)$	$\frac{16+\sqrt{6}}{360}$
$(k=1, \ldots, 10)$	
$\left(\sqrt{\frac{6+\sqrt{6}}{10}}\,h \cos\frac{2\pi k}{10}, \sqrt{\frac{6+\sqrt{6}}{10}}\,h \sin\frac{2\pi k}{10}\right)$	$\frac{16-\sqrt{6}}{360}$
	$R=O(h^{10})$

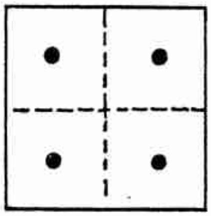
Square[4] S: $|x|\leq h, |y|\leq h$

25.4.62

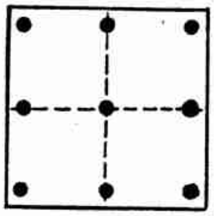
$$\frac{1}{4h^2}\iint_S f(x,y)\,dx\,dy=\sum_{i=1}^{n} w_i f(x_i, y_i)+R$$



(x_i, y_i)	w_i	
$(0,0)$	$4/9$	
$(\pm h, \pm h)$	$1/36$	$R=O(h^4)$
$(\pm h, 0)$	$1/9$	
$(0, \pm h)$	$1/9$	



(x_i, y_i)	w_i	
$\left(\pm h\sqrt{\frac{1}{3}}, \pm h\sqrt{\frac{1}{3}}\right)$	$1/4$	$R=O(h^4)$



(x_i, y_i)	w_i
$(0,0)$	$16/81$

---

[4] For regions, such as the square, cube, cylinder, etc., which are the Cartesian products of lower dimensional regions, one may always develop integration rules by "multiplying together" the lower dimensional rules. Thus if

$$\int_0^1 f(x)\,dx \approx \sum_{i=1}^{n} w_i f(x_i)$$

is a one dimensional rule, then

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy \approx \sum_{i,j=1}^{n} w_i w_j f(x_i, x_j)$$

becomes a two dimensional rule. Such rules are not necessarily the most "economical".

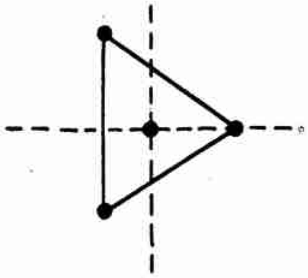
$\left(\pm\sqrt{\frac{3}{5}}\,h,\pm\sqrt{\frac{3}{5}}\,h\right)$	25/324	$R=O(h^6)$
$\left(0,\pm\sqrt{\frac{3}{5}}\,h\right)$	10/81	
$\left(\pm\sqrt{\frac{3}{5}}\,h,0\right)$	10/81	

Equilateral Triangle T

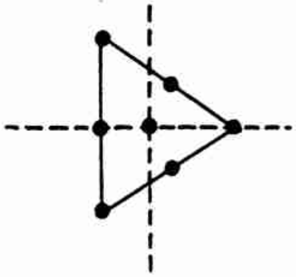
Radius of Circumscribed Circle$=h$

25.4.63

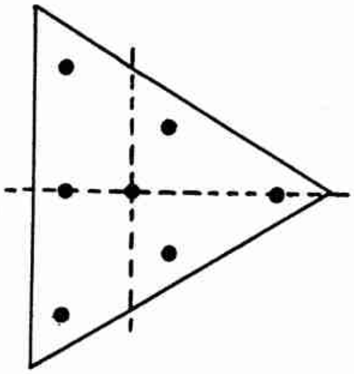
$$\frac{1}{\frac{3}{4}\sqrt{3}h^2}\iint_T f(x,y)dxdy=\sum_{i=1}^{n} w_i f(x_i,y_i)+R$$



(x_i,y_i)	w_i	
$(0,0)$	3/4	
$(h,0)$	1/12	$R=O(h^3)$
$\left(-\frac{h}{2},\pm\frac{h}{2}\sqrt{3}\right)$	1/12	



(x_i,y_i)	w_i	
$(0,0)$	27/60	
$(h,0)$	3/60	
$\left(-\frac{h}{2},\pm\frac{h}{2}\sqrt{3}\right)$	3/60	$R=O(h^4)$
$\left(-\frac{h}{2},0\right)$	8/60	
$\left(\frac{h}{4},\pm\frac{h}{4}\sqrt{3}\right)$	8/60	



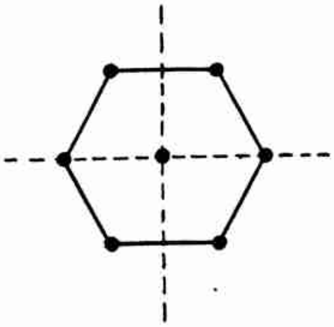
(x_i,y_i)	w_i	
$(0,0)$	270/1200	
$\left(\left(\frac{\sqrt{15}+1}{7}\right)h,0\right)$	$\frac{155-\sqrt{15}}{1200}$	$R=O(h^6)$
$\left(\left(\frac{-\sqrt{15}+1}{14}\right)h,\pm\left(\frac{\sqrt{15}+1}{14}\right)\sqrt{3}h\right)$		
$\left(\left(-\frac{\sqrt{15}-1}{7}\right)h,0\right)$	$\frac{155+\sqrt{15}}{1200}$	
$\left(\left(\frac{\sqrt{15}-1}{14}\right)h,\pm\left(\frac{\sqrt{15}-1}{14}\right)\sqrt{3}h\right)$		

Regular Hexagon H

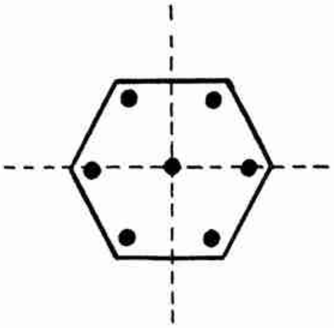
Radius of Circumscribed Circle$=h$

25.4.64

$$\frac{1}{\frac{3}{2}\sqrt{3}h^2}\iint_H f(x,y)dxdy=\sum_{i=1}^{n} w_i f(x_i,y_i)+R$$



(x_i,y_i)	w_i	
$(0,0)$	21/36	
$\left(\pm\frac{h}{2},\pm\frac{h}{2}\sqrt{3}\right)$	5/72	$R=O(h^4)$
$(\pm h,0)$	5/72	

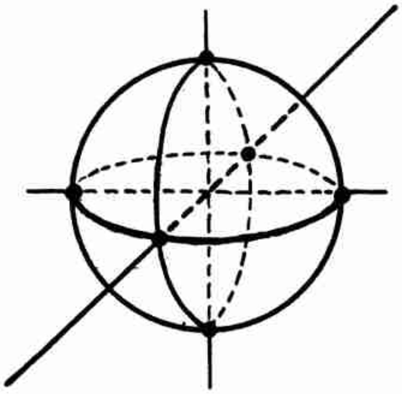


(x_i,y_i)	w_i	
$(0,0)$	258/1008	
$\left(\pm\frac{h}{10}\sqrt{14},\pm\frac{h}{10}\sqrt{42}\right)$	125/1008	$R=O(h^6)$
$\left(\pm h\frac{\sqrt{14}}{5},0\right)$	125/1008	

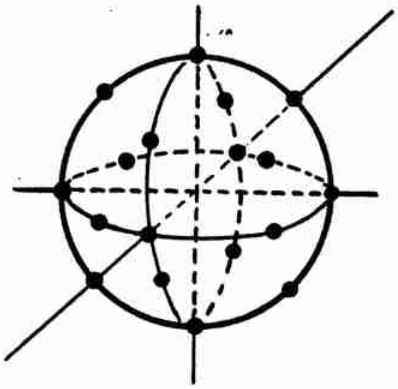
Surface of Sphere Σ: $x^2+y^2+z^2=h^2$

25.4.65

$$\frac{1}{4\pi h^2}\int_\Sigma\int f(x,y,z)d\sigma=\sum_{i=1}^{n} w_i f(x_i,y_i,z_i)+R$$



(x_i,y_i,z_i)	w_i	
$(\pm h,0,0)$	1/6	$R=O(h^4)$
$(0,\pm h,0)$	1/6	
$(0,0,\pm h)$	1/6	



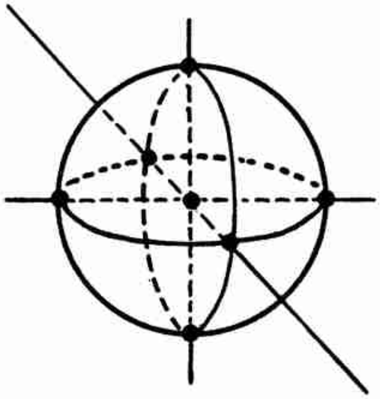
(x_i,y_i,z_i)	w_i	
$\left(\pm\sqrt{\frac{1}{2}}h,\pm\sqrt{\frac{1}{2}}h,0\right)$		
$\left(\pm\sqrt{\frac{1}{2}}h,0,\pm\sqrt{\frac{1}{2}}h\right)$	1/15	
$\left(0,\pm\sqrt{\frac{1}{2}}h,\pm\sqrt{\frac{1}{2}}h\right)$		
		$R=O(h^6)$
$(\pm h,0,0)$		
$(0,\pm h,0)$	1/30	
$(0,0,\pm h)$		

(x_i,y_i,z_i)	w_i	
$\left(\pm\sqrt{\frac{1}{3}}h,\pm\sqrt{\frac{1}{3}}h,\pm\sqrt{\frac{1}{3}}h\right)$	27/840	
$\left(\pm\sqrt{\frac{1}{2}}h,\pm\sqrt{\frac{1}{2}}h,0\right)$		
$\left(\pm\sqrt{\frac{1}{2}}h,0,\pm\sqrt{\frac{1}{2}}h\right)$	32/840	$R=O(h^8)$
$\left(0,\pm\sqrt{\frac{1}{2}}h,\pm\sqrt{\frac{1}{2}}h\right)$		
$(\pm h,0,0)$		
$(0,\pm h,0)$	40/840	
$(0,0,\pm h)$		

Sphere S: $x^2+y^2+z^2\leq h^2$

25.4.66

$$\frac{1}{\frac{4}{3}\pi h^3}\iiint_S f(x,y,z)dxdydz=\sum_{i=1}^{n} w_i f(x_i,y_i,z_i)+R$$



(x_i, y_i, z_i)	w_i
$(0,0,0)$	2/5
$(\pm h,0,0)$	1/10
$(0,\pm h,0)$	1/10
$(0,0,\pm h)$	1/10

$R=O(h^4)$

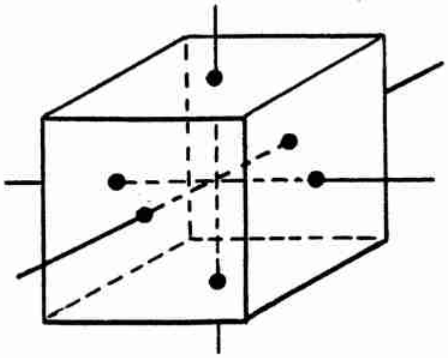
Cube[5] C: $|x|\leq h$

$|y|\leq h$

$|z|\leq h$

25.4.67

$$\frac{1}{8h^3}\iiint_C f(x,y,z)dxdydz=\sum_{i=1}^{n} w_i f(x_i,y_i,z_i)+R$$



(x_i, y_i, z_i)	w_i
$(\pm h,0,0)$	1/6
$(0,\pm h,0)$	1/6
$(0,0,\pm h)$	1/6

$R=O(h^4)$

25.4.68

$$\frac{1}{8h^3}\iiint_C f(x,y,z)dxdydz$$

$$=\frac{1}{360}\left[-496f_m+128\sum f_r+8\sum f_f+5\sum f_v\right]+O(h^6)$$

25.4.69

$$=\frac{1}{450}\left[91\sum f_f-40\sum f_e+16\sum f_d\right]+O(h^6)$$

where $f_m=f(0,0,0)$.

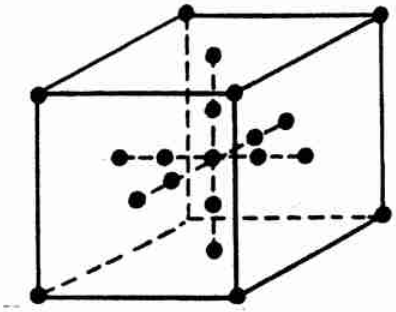
$\sum f_r$=sum of values of f at the 6 points midway from the center of C to the 6 faces.

$\sum f_f$=sum of values of f at the 6 centers of the faces of C.

$\sum f_v$=sum of values of f at the 8 vertices of C.

$\sum f_e$=sum of values of f at the 12 midpoints of edges of C.

$\sum f_d$=sum of values of f at the 4 points on the diagonals of each face at a distance of $\frac{1}{2}\sqrt{5}h$ from the center of the face.



Tetrahedron: $\mathscr{T}$

25.4.70

$$\frac{1}{V}\iiint_{\mathscr{T}} f(x,y,z)dxdydz=\frac{1}{40}\sum f_v+\frac{9}{40}\sum f_f$$

+terms of 4th order

$$=\frac{32}{60}f_m+\frac{1}{60}\sum f_v+\frac{4}{60}\sum f_e$$

+terms of 4th order

where

V: Volume of $\mathscr{T}$

$\sum f_v$: Sum of values of the function at the vertices of $\mathscr{T}$.

$\sum f_e$: Sum of values of the function at midpoints of the edges of $\mathscr{T}$.

$\sum f_f$: Sum of values of the function at the center of gravity of the faces of $\mathscr{T}$.

f_m: Value of function at center of gravity of $\mathscr{T}$.

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[5] See footnote to **25.4.62.**

25.5. Ordinary Differential Equations[6]

First Order: $y'=f(x, y)$

Point Slope Formula

25.5.1 $$y_{n+1}=y_n+hy'_n+O(h^2)$$

25.5.2 $$y_{n+1}=y_{n-1}+2hy'_n+O(h^3)$$

Trapezoidal Formula

25.5.3 $$y_{n+1}=y_n+\frac{h}{2}(y'_{n+1}+y'_n)+O(h^3)$$

Adams' Extrapolation Formula

25.5.4

$$y_{n+1}=y_n+\frac{h}{24}(55y'_n-59y'_{n-1}+37y'_{n-2}-9y'_{n-3})+O(h^5)$$

Adams' Interpolation Formula

25.5.5

$$y_{n+1}=y_n+\frac{h}{24}(9y'_{n+1}+19y'_n-5y'_{n-1}+y'_{n-2})+O(h^5)$$

Runge-Kutta Methods

Second Order

25.5.6

$$y_{n+1}=y_n+\frac{1}{2}(k_1+k_2)+O(h^3)$$

$$k_1=hf(x_n,y_n),k_2=hf(x_n+h,y_n+k_1)$$

25.5.7

$$y_{n+1}=y_n+k_2+O(h^3)$$

$$k_1=hf(x_n,y_n),k_2=hf\left(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_1\right)$$

Third Order

25.5.8

$$y_{n+1}=y_n+\frac{1}{6}k_1+\frac{2}{3}k_2+\frac{1}{6}k_3+O(h^4)$$

$$k_1=hf(x_n,y_n),k_2=hf\left(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_1\right)$$

$$k_3=hf(x_n+h,y_n-k_1+2k_2)$$

25.5.9

$$y_{n+1}=y_n+\frac{1}{4}k_1+\frac{3}{4}k_3+O(h^4)$$

$$k_1=hf(x_n,y_n),k_2=hf\left(x_n+\frac{1}{3}h,y_n+\frac{1}{3}k_1\right)$$

$$k_3=hf\left(x_n+\frac{2}{3}h,y_n+\frac{2}{3}k_2\right)$$

Fourth Order

25.5.10

$$y_{n+1}=y_n+\frac{1}{6}k_1+\frac{1}{3}k_2+\frac{1}{3}k_3+\frac{1}{6}k_4+O(h^5)$$

$$k_1=hf(x_n,y_n),k_2=hf\left(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_1\right)$$

$$k_3=hf\left(x_n+\frac{1}{2}h,y_n+\frac{1}{2}k_2\right),k_4=hf(x_n+h,y_n+k_3)$$

25.5.11

$$y_{n+1}=y_n+\frac{1}{8}k_1+\frac{3}{8}k_2+\frac{3}{8}k_3+\frac{1}{8}k_4+O(h^5)$$

$$k_1=hf(x_n,y_n),k_2=hf\left(x_n+\frac{1}{3}h,y_n+\frac{1}{3}k_1\right)$$

$$k_3=hf\left(x_n+\frac{2}{3}h,y_n-\frac{1}{3}k_1+k_2\right),$$

$$k_4=hf(x_n+h,y_n+k_1-k_2+k_3)$$

Gill's Method

25.5.12

$$y_{n+1}=y_n+\frac{1}{6}\left(k_1+2\left(1-\sqrt{\frac{1}{2}}\right)k_2\right.$$

$$\left.+2\left(1+\sqrt{\frac{1}{2}}\right)k_3+k_4\right)+O(h^5)$$

$$k_1=hf(x_n, y_n)$$

$$k_2=hf\left(x_n+\frac{1}{2}h, y_n+\frac{1}{2}k_1\right)$$

$$k_3=hf\left(x_n+\frac{1}{2}h, y_n+\left(-\frac{1}{2}+\sqrt{\frac{1}{2}}\right)k_1\right.$$

$$\left.+\left(1-\sqrt{\frac{1}{2}}\right)k_2\right)$$

$$k_4=hf\left(x_n+h, y_n-\sqrt{\frac{1}{2}}k_2+\left(1+\sqrt{\frac{1}{2}}\right)k_3\right)$$

Predictor-Corrector Methods

Milne's Methods

25.5.13

$$\text{P:}\quad y_{n+1}=y_{n-3}+\frac{4h}{3}(2y'_n-y'_{n-1}+2y'_{n-2})+O(h^5)$$

$$\text{C:}\quad y_{n+1}=y_{n-1}+\frac{h}{3}(y'_{n-1}+4y'_n+y'_{n+1})+O(h^5)$$

---

[6] The reader is cautioned against possible instabilities especially in formulas **25.5.2** and **25.5.13**. See, e.g. [25.11], [25.12].

25.5.14

$$\text{P:}\quad y_{n+1}=y_{n-5}+\frac{3h}{10}\,(11y'_n-14y'_{n-1}+26y'_{n-2}-14y'_{n-3}+11y'_{n-4})+O(h^7)$$

$$\text{C:}\quad y_{n+1}=y_{n-3}+\frac{2h}{45}\,(7y'_{n+1}+32y'_n+12y'_{n-1}+32y'_{n-2}+7y'_{n-3})+O(h^7)$$

Formulas Using Higher Derivatives

25.5.15

$$\text{P:}\quad y_{n+1}=y_{n-2}+3(y_n-y_{n-1})+h^2(y''_n-y''_{n-1})+O(h^5)$$

$$\text{C:}\quad y_{n+1}=y_n+\frac{h}{2}\,(y'_{n+1}+y'_n)-\frac{h^2}{12}\,(y''_{n+1}-y''_n)+O(h^5)$$

25.5.16

$$\text{P:}\quad y_{n+1}=y_{n-2}+3(y_n-y_{n-1})+\frac{h^3}{2}\,(y'''_n+y'''_{n-1})+O(h^7)$$

$$\text{C:}\quad y_{n+1}=y_n+\frac{h}{2}\,(y'_{n+1}+y'_n)-\frac{h^2}{10}\,(y''_{n+1}-y''_n)+\frac{h^3}{120}\,(y'''_{n+1}+y'''_n)+O(h^7)$$

Systems of Differential Equations

First Order: $y'=f(x,y,z),\ z'=g(x,y,z)$.

Second Order Runge-Kutta

25.5.17

$$y_{n+1}=y_n+\frac{1}{2}\,(k_1+k_2)+O(h^3),\qquad z_{n+1}=z_n+\frac{1}{2}\,(l_1+l_2)+O(h^3)$$

$$k_1=hf(x_n,y_n,z_n),\qquad l_1=hg(x_n,y_n,z_n)$$

$$k_2=hf(x_n+h,y_n+k_1,z_n+l_1),\qquad l_2=hg(x_n+h,y_n+k_1,z_n+l_1)$$

Fourth Order Runge-Kutta

25.5.18

$$y_{n+1}=y_n+\frac{1}{6}\,(k_1+2k_2+2k_3+k_4)+O(h^5),\qquad z_{n+1}=z_n+\frac{1}{6}\,(l_1+2l_2+2l_3+l_4)+O(h^5)$$

$$k_1=hf(x_n,y_n,z_n)\qquad l_1=hg(x_n,y_n,z_n)$$

$$k_2=hf\left(x_n+\frac{1}{2}\,h,y_n+\frac{1}{2}\,k_1,z_n+\frac{1}{2}\,l_1\right)\qquad l_2=hg\left(x_n+\frac{h}{2},y_n+\frac{k_1}{2},z_n+\frac{l_1}{2}\right)$$

$$k_3=hf\left(x_n+\frac{1}{2}\,h,y_n+\frac{1}{2}\,k_2,z_n+\frac{1}{2}\,l_2\right)\qquad l_3=hg\left(x_n+\frac{h}{2},y_n+\frac{k_2}{2},z_n+\frac{l_2}{2}\right)$$

$$k_4=hf(x_n+h,y_n+k_3,z_n+l_3)\qquad l_4=hg(x_n+h,y_n+k_3,z_n+l_3)$$

Second Order: $y''=f(x,y,y')$

Milne's Method

25.5.19

$$\text{P:}\quad y'_{n+1}=y'_{n-3}+\frac{4h}{3}\,(2y''_{n-2}-y''_{n-1}+2y''_n)+O(h^5)$$

$$\text{C:}\quad y'_{n+1}=y'_{n-1}+\frac{h}{3}\,(y''_{n-1}+4y''_n+y''_{n+1})+O(h^5)$$

Runge-Kutta Method

25.5.20

$$y_{n+1}=y_n+h\left[y'_n+\frac{1}{6}\,(k_1+k_2+k_3)\right]+O(h^5)$$

$$y'_{n+1}=y'_n+\frac{1}{6}\,(k_1+2k_2+2k_3+k_4)$$

$$k_1=hf(x_n,y_n,y'_n)$$

$$k_2=hf\left(x_n+\frac{1}{2}\,h,y_n+\frac{h}{2}\,y'_n+\frac{h}{8}\,k_1,y'_n+\frac{k_1}{2}\right)$$

$$k_3=hf\left(x_n+\frac{1}{2}\,h,y_n+\frac{h}{2}\,y'_n+\frac{h}{8}\,k_1,y'_n+\frac{k_2}{2}\right)\qquad *$$

$$k_4=hf\left(x_n+h,y_n+hy'_n+\frac{h}{2}\,k_3,y'_n+k_3\right)$$

Second Order: $y''=f(x,y)$

Milne's Method

25.5.21

$$\text{P:}\quad y_{n+1}=y_n+y_{n-2}-y_{n-3}+\frac{h^2}{4}\,(5y''_n+2y''_{n-1}+5y''_{n-2})+O(h^6)$$

$$\text{C:}\quad y_n=2y_{n-1}-y_{n-2}+\frac{h^2}{12}\,(y''_n+10y''_{n-1}+y''_{n-2})+O(h^6)$$

Runge-Kutta Method

25.5.22 $\quad y_{n+1}=y_n+h\left(y'_n+\frac{1}{6}\,(k_1+2k_2)\right)+O(h^4)$

$$y'_{n+1}=y'_n+\frac{1}{6}\,k_1+\frac{2}{3}\,k_2+\frac{1}{6}\,k_3$$

$$k_1=hf(x_n,y_n)$$

$$k_2=hf\left(x_n+\frac{h}{2},y_n+\frac{h}{2}\,y'_n+\frac{h}{8}\,k_1\right)$$

$$k_3=hf\left(x_n+h,y_n+hy'_n+\frac{h}{2}\,k_2\right).$$

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26. Probability Functions

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Contents

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26. Probability Functions

Mathematical Properties[3]

26.1. Probability Functions: Definitions and Properties

Univariate Cumulative Distribution Functions

A real-valued function $F(x)$ is termed a (univariate) cumulative distribution function (c.d.f.) or simply distribution function if

i) $F(x)$ is non-decreasing, i.e., $F(x_1) \leq F(x_2)$ for $x_1 \leq x_2$

ii) $F(x)$ is everywhere continuous from the right, i.e., $F(x) = \lim_{\epsilon \to 0+} F(x+\epsilon)$

iii) $F(-\infty)=0$, $F(\infty)=1$.

The function $F(x)$ signifies the probability of the event "$X \leq x$" where X is a random variable, i.e., $Pr\{X \leq x\} = F(x)$, and thus describes the c.d.f. of X. The two principal types of distribution functions are termed *discrete* and *continuous*.

Discrete Distributions: Discrete distributions are characterized by the random variable X taking on an enumerable number of values $\ldots, x_{-1}, x_0, x_1, \ldots$ with point probabilities

$$p_n = Pr\{X = x_n\} \geq 0$$

which need only be subject to the restriction

$$\sum_n p_n = 1.$$

The corresponding distribution function can then be written

26.1.1 $$F(x) = Pr\{X \leq x\} = \sum_{x_n \leq x} p_n$$

where the summation is over all values of x for which $x_n \leq x$. The set $\{x_n\}$ of values for which $p_n > 0$ is termed the domain of the random variable X. A discrete distribution of a random variable is called a *lattice distribution* if there exist numbers a and $b \neq 0$ such that every possible value of X can be represented in the form $a+bn$ where n takes on only integral values. A summary of some properties of certain discrete distributions is presented in **26.1.19–26.1.24.**

Continuous Distributions. Continuous distributions are characterized by $F(x)$ being absolutely continuous. Hence $F(x)$ possesses a derivative $F'(x) = f(x)$ and the c.d.f. can be written

26.1.2 $$F(x) = Pr\{X \leq x\} = \int_{-\infty}^{x} f(t)dt.$$

The derivative $f(x)$ is termed the *probability density function* (p.d.f.) or *frequency function*, and the values of x for which $f(x) > 0$ make up the domain of the random variable X. A summary of some properties of certain selected continuous distributions is presented in **26.1.25–26.1.34.**

Multivariate Probability Functions

The real-valued function $F(x_1, x_2, \ldots x_n)$ defines an n-variate cumulative distribution function if

i) $F(x_1, x_2, \ldots x_n)$ is a non-decreasing function for each x_i

ii) $F(x_1, x_2, \ldots x_n)$ is continuous from the right in each x_i; i.e., $F(x_1, x_2, \ldots x_n) = \lim_{\epsilon \to 0+} F(x_1, \ldots, x_i+\epsilon, \ldots, x_n)$

iii) $F(x_1, x_2, \ldots x_n) = 0$ when any $x_i = -\infty$; $F(\infty, \infty, \ldots, \infty) = 1$.

iv) $F(x_1, x_2, \ldots, x_n)$ assigns nonnegative probability to the event $x_1 < X_1 \leq x_1+h_1$, $x_2 < X_2 \leq x_2+h_2, \ldots, x_n < X_n \leq x_n+h_n$ for all $x_1, x_2, \ldots, x_n$ and all nonnegative $h_1, h_2, \ldots, h_n$, e.g., for $n=2$, $F(x_1+h_1, x_2+h_2) - F(x_1, x_2+h_2) - F(x_1+h_1, x_2) + F(x_1, x_2) \geq 0$ and in general for $x_i < X_i \leq x_i + h_i$ $(i=1, 2, \ldots, n)$, the kth order difference $\Delta_k F(x_1, x_2, \ldots, x_n) > 0$ for $k = 1, 2, \ldots, n$.

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[3] *Comment on notation and conventions.*

a. We follow the customary convention of denoting a random variable by a capital letter, i.e., X, and using the corresponding lower case letter, i.e., x, for a particular value that the random variable assumes.

b. For statistical applications it is often convenient to have tabulated the "upper tail area," $1-F(x)$, or the c.d.f. for $|X|$, $F(x)-F(-x)$, instead of simply the c.d.f. $F(x)$. We use the notation P to indicate the c.d.f. of X, $Q=1-P$ to indicate the "upper tail area" and $A = P-Q$ to denote the c.d.f. of $|X|$. In particular we use $P(x)$, $Q(x)$, and $A(x)$ to denote the corresponding functions for the normal or Gaussian probability function, see **26.2.2–26.2.4.** When these distributions depend on other parameters, say θ_1 and θ_2, we indicate this by writing $P(x|\theta_1, \theta_2)$, $Q(x|\theta_1, \theta_2)$, or $A(x|\theta_1, \theta_2)$. For example the chi-square distribution **26.4** depends on the parameter ν and the tabulated function is written $Q(\chi^2|\nu)$.

*See page II.

The joint probability of the event $X_1 \leq x_1$, $X_2 \leq x_2, \ldots, X_n \leq x_n$ is $F(x_1, x_2, \ldots x_n)$. Analogous to the one-dimensional case, *discrete* distributions assign all probability to an enumerable set of vectors $(x_1, x_2, \ldots, x_n)$ and *continuous* distributions are characterized by absolute continuity of $F(x_1, x_2, \ldots, x_n)$.

Characteristics of distribution functions: Moments, characteristic functions, cumulants

		Continuous distributions	Discrete distributions
26.1.3	n^{th} moment about origin	$\mu_n'=\int_{-\infty}^{\infty} x^n f(x)dx$	$\mu_n'=\sum_s x_s^n p_s$
26.1.4	mean	$m=\mu_1'=\int_{-\infty}^{\infty} xf(x)dx$	$m=\mu_1'=\sum_s x_s p_s$
26.1.5	variance	$\sigma^2=\mu_2'-m^2=\int_{-\infty}^{\infty}(x-m)^2 f(x)dx$	$\sigma^2=\mu_2'-m^2=\sum_s (x_s-m)^2 p_s$
26.1.6	n^{th} central moment	$\mu_n=\int_{-\infty}^{\infty}(x-m)^n f(x)dx$	$\mu_n=\sum_s (x_s-m)^n p_s$
26.1.7	expected value operator for the function $g(x)$	$E[g(X)]=\int_{-\infty}^{\infty} g(x)f(x)dx$	$E[g(X)]=\sum_s g(x_s)p_s$
26.1.8	characteristic function of X	$\phi(t)=E(e^{itX})=\int_{-\infty}^{\infty} e^{itx}f(x)dx$	$\phi(t)=E(e^{itX})=\sum_s e^{itx_s}p_s$
26.1.9	characteristic function of $g(X)$	$\phi_g(t)=E(e^{itg(X)})=\int_{-\infty}^{\infty} e^{itg(x)}f(x)dx$	$\phi_g(t)=E(e^{itg(X)})=\sum_s e^{itg(x_s)}p_s$
26.1.10	inversion formula	$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\phi(t)dt$	$p_n=\frac{b}{2\pi}\int_{-\pi/b}^{\pi/b} e^{-itx_n}\phi(t)dt$ (lattice distributions only)

Relation of the Characteristic Function to Moments About the Origin

26.1.11
$$\phi^{(n)}(0)=\left[\frac{d^n}{dt^n}\phi(t)\right]_{t=0}=i^n\mu_n'$$

Cumulant Function

26.1.12
$$\ln \phi(t)=\sum_{n=0}^{\infty}\kappa_n\frac{(it)^n}{n!}$$

κ_n is called the n^{th} cumulant.

26.1.13 $\kappa_1=m,\ \kappa_2=\sigma^2,\ \kappa_3=\mu_3,\ \kappa_4=\mu_4-3\mu_2^2$

Relation of Central Moments to Moments About the Origin

26.1.14
$$\mu_n=\sum_{j=0}^{n}\binom{n}{j}(-1)^{n-j}\mu_j'm^{n-j}$$

Coefficients of Skewness and Excess

26.1.15
$$\gamma_1=\frac{\kappa_3}{\kappa_2^{3/2}}=\frac{\mu_3}{\sigma^3} \qquad \text{(skewness)}$$

26.1.16
$$\gamma_2=\frac{\kappa_4}{\kappa_2^2}=\frac{\mu_4}{\sigma^4}-3 \qquad \text{(excess)}$$

Occasionally coefficients of skewness and excess (or kurtosis) are given by

26.1.17
$$\beta_1=\gamma_1^2=\left(\frac{\mu_3}{\sigma^3}\right)^2 \qquad \text{(skewness)}$$

26.1.18
$$\beta_2=\gamma_2+3=\frac{\mu_4}{\sigma^4}$$

(excess or kurtosis)

Some one-dimensional discrete distribution functions

Name	Domain	Point Probabilities	Restrictions on Parameters	Mean	Variance	Skewness γ_1	Excess γ_2	Characteristic function	Cumulants
26.1.19 Single point or degenerate	$x=c$ (c a constant)	$p=1$	$-\infty<c<+\infty$	c	0	----------	----------	$e^{i\lambda t}$	$\kappa_1=\lambda,\ \kappa_r=0$ for $r>1$
26.1.20 Binomial	$x_s=s$, for $s=0, 1, 2, \ldots, n$	$\binom{n}{s} p^s(1-p)^{n-s}$	$0<p<1$ $(q=1-p)$	np	npq	$\frac{q-p}{\sqrt{npq}}$	$\frac{1-6pq}{npq}$	$(q+pe^{it})^n$	$\kappa_1=np$ $\kappa_{r+1}=pq\,\frac{d\kappa_r}{dp}$ for $r\geq1$
26.1.21 Hypergeometric	$x_s=s$, for $s=0, 1, \ldots \min(n, N_1)$	$\frac{\binom{N_1}{s}\binom{N_2}{n-s}}{\binom{N_1+N_2}{n}}$	N_1 and N_2 integers, and $n\leq N_1+N_2$ $(N=N_1+N_2$, $p=N_1/N$ and $q=1-p=N_2/N)$	np	$npq\left(\frac{N-n}{N-1}\right)$	$\frac{q-p}{\sqrt{npq}}\left(\frac{N-1}{N-n}\right)^{\frac{1}{2}}\left(\frac{N-2n}{N-2}\right)$	Complicated	$\frac{\binom{N_2}{n}}{\binom{N}{n}} F(-n, -N_1; N_2-n+1; e^{it})$	Complicated
26.1.22 Poisson	$x_s=s$, for $s=0, 1, 2, \ldots, \infty$	$\frac{e^{-m}m^s}{s!}$	$0<m<\infty$	m	m	$m^{-\frac{1}{2}}$	m^{-1}	$e^{m(e^{it}-1)}$	$\kappa_r=m$ for $r=1, 2, \ldots$
26.1.23 Negative binomial	$x_s=s$, for $s=0, 1, 2, \ldots, \infty$	$\binom{n+s-1}{s} p^n(1-p)^s$	$n\geq0$ and $0<p<1$ $(p=1/Q$, and $1-p=P/Q)$	nP	nPQ	$\frac{Q+P}{\sqrt{nPQ}}$	$\frac{1+6PQ}{nPQ}$	$(Q-Pe^{it})^{-n}$	$\kappa_1=nP$ $\kappa_{r+1}=PQ\,\frac{d\kappa_r}{dQ}$ for $r\geq1$
26.1.24 Geometric	$x_s=s$, for $s=0, 1, 2, \ldots, \infty$	$p(1-p)^s$	$0<p<1$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{2-p}{\sqrt{1-p}}$	$6+\frac{p^2}{1-p}$	$p[1-(1-p)e^{it}]^{-1}$	$\kappa_1=\frac{1-p}{p}$, $\kappa_{r+1}=-(1-p)\,\frac{d\kappa_r}{dp}$, $r\geq1$

Some one-dimensional continuous distribution functions

	Name	Domain	Probability Density Function $f(x)$	Restrictions on Parameters	Mean	Variance	Skewness γ_1	Excess γ_2	Characteristic function	Cumulants
26.1.25	Error function	$-\infty<x<\infty$	$\frac{h}{\sqrt{\pi}}e^{-h^2x^2}$	$0<h<\infty$	0	$\frac{1}{2h^2}$	0	0	$e^{\frac{-t^2}{4h^2}}$	$\kappa_1=0,\ \kappa_2=\frac{1}{2h^2}$ $\kappa_n=0$ for $n>2$
26.1.26	Normal	$-\infty<x<\infty$	$\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$	$-\infty<m<\infty$ $0<\sigma<\infty$	m	σ^2	0	0	$e^{imt-\frac{\sigma^2t^2}{2}}$	$\kappa_1=m,\ \kappa_2=\sigma^2,\ \kappa_n=0$ for $n>2$
26.1.27	Cauchy	$-\infty<x<\infty$	$\frac{1}{\pi\beta}\frac{1}{1+\left(\frac{x-\alpha}{\beta}\right)^2}$	$-\infty<\alpha<\infty$ $0<\beta<\infty$	not defined	not defined	not defined	not defined	$e^{i\alpha t-\beta\|t\|}$	not defined
26.1.28	Exponential	$\alpha\leq x<\infty$	$\frac{1}{\beta}e^{-\left(\frac{x-\alpha}{\beta}\right)}$	$-\infty<\alpha<\infty$ $0<\beta<\infty$	$\alpha+\beta$	β^2	2	6	$e^{i\alpha t}(1-i\beta t)^{-1}$	$\kappa_1=\alpha+\beta,\ \kappa_n=\beta^n\Gamma(n)$ for $n>1$
26.1.29	Laplace, or double exponential	$-\infty<x<\infty$	$\frac{1}{2\beta}e^{-\left\|\frac{x-\alpha}{\beta}\right\|}$	$-\infty<\alpha<\infty$ $0<\beta<\infty$	α	$2\beta^2$	0	3	$e^{i\alpha t}(1+\beta^2t^2)^{-1}$	$\kappa_1=\alpha,\ \kappa_2=2\beta^2$ $\kappa_{2n+1}=0,\ \kappa_{2n}=\frac{(2n)!}{n}\beta^{2n}$ for $n=1, 2, \ldots$
26.1.30	Extreme-Value,[4] (Fisher-Tippett Type I or doubly exponential)	$-\infty<x<\infty$	$\frac{1}{\beta}\exp\ (-y-e^{-y})$ with $y=\frac{x-\alpha}{\beta}$	$-\infty<\alpha<\infty$ $0<\beta<\infty$	$\alpha+\gamma\beta$	$\frac{(\pi\beta)^2}{6}$	1.3	2.4	$\Gamma(1-i\beta t)e^{i\alpha t}$	$\kappa_1=\gamma,\ \kappa_2=\frac{(\pi\beta)^2}{6}$ $\kappa_n=\beta^n\Gamma(n)\sum_{r=1}^{\infty}\frac{1}{r^n}$ for $n>2$
26.1.31	Pearson Type III	$\alpha\leq x<\infty$	$\frac{1}{\beta\Gamma(p)}y^{p-1}e^{-y}$ with $y=\frac{x-\alpha}{\beta}$	$-\infty<\alpha<\infty$ $0<\beta<\infty$ $0<p<\infty$	$\alpha+p\beta$	$p\beta^2$	$\frac{2}{\sqrt{p}}$	$6/p$	$e^{i\alpha t}(1-i\beta t)^{-p}$	$\kappa_1=\alpha+\beta p,\ \kappa_n=\beta^n p\Gamma(n)$ for $n>1$
26.1.32	Gamma distribution	$0\leq x<\infty$	$\frac{1}{\Gamma(p)}x^{p-1}e^{-x}$	$0<p<\infty$	p	p	$\frac{2}{\sqrt{p}}$	$6/p$	$(1-it)^{-p}$	$\kappa_1=p,\ \kappa_n=p\Gamma(n)$ for $n>1$
26.1.33	Beta distribution	$0\leq x\leq 1$	$\frac{1}{B(a,\ b)}x^{a-1}(1-x)^{b-1}$	$1\leq a<\infty$ $1\leq b<\infty$	$\frac{a}{a+b}$	$\frac{ab}{(a+b)^2(a+b+1)}$	$\frac{2(a-b)}{(a+b+2)}$	See footnote 5.	$M(a,\ a+b,\ it)$	
26.1.34	Rectangular, or uniform	$m-\frac{h}{2}\leq x\leq m+\frac{h}{2}$	$\frac{1}{h}$	$-\infty<m<\infty$ $0<h<\infty$	m	$\frac{h^2}{12}$	0	-1.2	$\frac{2}{ht}\sin\left(\frac{ht}{2}\right)e^{imt}$	$\kappa_1=m,\ \kappa_{2n+1}=0$ $\kappa_{2n}=\frac{h^{2n}B_{2n}}{2n}$ * B_{2n} (Bernoulli numbers), $B_2=\frac{1}{6},\ B_4=-\frac{1}{30},\ \ldots$

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[4] γ (Euler's constant) $=.57721\ 56649\ \ldots$.

[5] $\gamma_2=\sqrt{\frac{a+b+1}{ab}}\left\{\frac{3(a+b+1)[2(a+b)^2+ab(a+b-6)]}{ab(a+b+2)(a+b+3)}-3\right\}.$

*See page II.

Inequalities for distribution functions

($F(x)$ denotes the c.d.f. of the random variable X and t denotes a positive constant; further m is always assumed to be finite and all expectations are assumed to exist.)

Inequality	Conditions
26.1.35 $Pr\{g(X)\geq t\}\leq E[g(X)]/t$	(i) $g(X)\geq 0$
26.1.36 $Pr\{X\geq t\}\leq m/t$ $F(t)\geq 1-\frac{m}{t}$	(i) $Pr\{X<0\}=0$ (ii) $E(X)=m$
26.1.37 $Pr\{\lvert X-m\rvert\geq t\sigma\}\leq 1/t^2$ $F(m+t\sigma)-F(m-t\sigma)\geq 1-\frac{1}{t^2}$	(i) $E(X)=m$ * (ii) $E(X-m)^2=\sigma^2$
26.1.38 $Pr\{\lvert\overline{X}-\overline{m}\rvert\geq t\overline{\sigma}\}\leq\frac{1}{nt^2}$	(i) $E(X_i)=m_i$ (ii) $E(X_i-m_i)^2=\sigma_i^2$ (iii) $E([X_i-m_i][X_j-m_j])=0(i\neq j)$ (iv) $\overline{X}=\sum_{i=1}^{n}\frac{X_i}{n}$, $\overline{m}=\sum_{i=1}^{n}\frac{m_i}{n}$, $\overline{\sigma}=\left[\sum_{i=1}^{n}\frac{\sigma_i^2}{n}\right]^{\frac{1}{2}}$
26.1.39 $Pr\{\lvert X-m\rvert\geq t\sigma\}\leq\frac{4}{9}\left\{\frac{1+\left(\frac{m-x_0}{\sigma}\right)^2}{\left(t-\left\lvert\frac{m-x_0}{\sigma}\right\rvert\right)^2}\right\}$ $F(m+t\sigma)-F(m-t\sigma)\geq 1-\frac{4}{9}\left\{\frac{1+\left(\frac{m-x_0}{\sigma}\right)^2}{\left(t-\left\lvert\frac{m-x_0}{\sigma}\right\rvert\right)^2}\right\}$	(i) $E(X-m)^2=\sigma^2$ (ii) $F(x)$ is a continuous c.d.f. (iii) $F(x)$ is unimodal at x_0[6]
26.1.40 $Pr\{\lvert X-m\rvert\geq t\sigma\}\leq 4/9t^2$ $F(m+t\sigma)-F(m-t\sigma)\geq 1-\frac{4}{9t^2}$	(i) $E(X-m)^2=\sigma^2$ (ii) $F(x)$ is a continuous c.d.f. (iii) $F(x)$ is unimodal at x_0[6] (iv) $m=x_0$
26.1.41 $Pr\{\lvert X-m\rvert\geq t\sigma\}\leq\frac{\mu_4-\sigma^4}{\mu_4+t^4\sigma^4-2t^2\sigma^4}$ $F(m+t\sigma)-F(m-t\sigma)\geq 1-\frac{\mu_4-\sigma^4}{\mu_4+t^4\sigma^4-2t^2\sigma^4}$	(i) $E(X-m)^2=\sigma^2$ (ii) $E(X-m)^4=\mu_4$

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[6] x_0 is such that $F'(x_0)>F'(x)$ for $x\neq x_0$.

26.2. Normal or Gaussian Probability Function

26.2.1
$$Z(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

26.2.2
$$P(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-t^2/2}dt=\int_{-\infty}^{x}Z(t)dt$$

26.2.3
$$Q(x)=\frac{1}{\sqrt{2\pi}}\int_{x}^{\infty}e^{-t^2/2}dt=\int_{x}^{\infty}Z(t)dt$$

26.2.4
$$A(x)=\frac{1}{\sqrt{2\pi}}\int_{-x}^{x}e^{-t^2/2}dt=\int_{-x}^{x}Z(t)dt$$

26.2.5
$$P(x)+Q(x)=1$$

26.2.6
$$P(-x)=Q(x)$$

26.2.7
$$A(x)=2P(x)-1$$

Probability Integral with Mean m and Variance σ^2

A random variable X is said to be normally distributed with mean m and variance σ^2 if the probability that X is less than or equal to x is given by

26.2.8
$$Pr\{X\leq x\}=\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{(t-m)^2}{2\sigma^2}}dt$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(x-m)/\sigma}e^{-t^2/2}dt=P\left(\frac{x-m}{\sigma}\right).$$

The corresponding probability density function is

26.2.9
$$\frac{\partial}{\partial x}P\left(\frac{x-m}{\sigma}\right)=\frac{1}{\sigma}Z\left(\frac{x-m}{\sigma}\right)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-m)^2}{2\sigma^2}}$$

and is symmetric around m, i.e.

$$Z\left(\frac{m+x}{\sigma}\right)=Z\left(\frac{m-x}{\sigma}\right).$$

The inflexion points of the probability density function are at $m\pm\sigma$.

Power Series ($x\geq 0$)

26.2.10 $$P(x)=\frac{1}{2}+\frac{1}{\sqrt{2\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!2^n(2n+1)}$$

26.2.11
$$P(x)=\frac{1}{2}+Z(x)\sum_{n=0}^{\infty}\frac{x^{2n+1}}{1\cdot 3\cdot 5\ldots(2n+1)}$$

Asymptotic Expansions ($x>0$)

26.2.12

$$Q(x)=\frac{Z(x)}{x}\left\{1-\frac{1}{x^2}+\frac{1\cdot 3}{x^4}+\ldots\right.$$
$$\left.+\frac{(-1)^n 1\cdot 3\ldots(2n-1)}{x^{2n}}\right\}+R_n$$

where

$$R_n=(-1)^{n+1}1\cdot 3\ldots(2n+1)\int_x^{\infty}\frac{Z(t)}{t^{2n+2}}\,dt$$

which is less in absolute value than the first neglected term.

26.2.13

$$Q(x)\sim\frac{Z(x)}{x}\left\{1-\frac{a_1}{x^2+2}+\frac{a_2}{(x^2+2)(x^2+4)}\right.$$
$$\left.-\frac{a_3}{(x^2+2)(x^2+4)(x^2+6)}+\ldots\right\}$$

where $a_1=1$, $a_2=1$, $a_3=5$, $a_4=9$, $a_5=129$ and the general term is

$$a_n=c_0 1\cdot 3\ldots(2n-1)+2c_1 1\cdot 3\ldots(2n-3)$$
$$+2^2c_2 1\cdot 3\ldots(2n-5)+\ldots+2^{n-1}c_{n-1}$$

and c_s is the coefficient of t^{n-s} in the expansion of $t(t-1)\ldots(t-n+1)$.

Continued Fraction Expansions

26.2.14

$$Q(x)=Z(x)\left\{\frac{1}{x+}\,\frac{1}{x+}\,\frac{2}{x+}\,\frac{3}{x+}\,\frac{4}{x+}\cdots\right\}\quad(x>0)$$

26.2.15

$$Q(x)=\frac{1}{2}-Z(x)\left\{\frac{x}{1-}\,\frac{x^2}{3+}\,\frac{2x^2}{5-}\,\frac{3x^2}{7+}\,\frac{4x^2}{9-}\cdots\right\}\quad(x\geq 0)$$

Polynomial and Rational Approximations[7] for $P(x)$ and $Z(x)$

$$0\leq x<\infty$$

26.2.16

$$P(x)=1-Z(x)(a_1t+a_2t^2+a_3t^3)+\epsilon(x),\qquad t=\frac{1}{1+px}$$

$$|\epsilon(x)|<1\times 10^{-5}$$

$p=.33267$ $\quad a_1=.43618\ 36$

$a_2=-.12016\ 76$

$a_3=.93729\ 80$

26.2.17

$$P(x)=1-Z(x)(b_1t+b_2t^2+b_3t^3+b_4t^4+b_5t^5)+\epsilon(x),$$
$$t=\frac{1}{1+px}$$

$$|\epsilon(x)|<7.5\times 10^{-8}$$

$p=.23164\ 19$

$b_1=\ .31938\ 1530$ $\quad b_4=-1.82125\ 5978$

$b_2=-.35656\ 3782$ $\quad b_5=\ 1.33027\ 4429$

$b_3=\ 1.78147\ 7937$

26.2.18

$$P(x)=1-\frac{1}{2}(1+c_1x+c_2x^2+c_3x^3+c_4x^4)^{-4}+\epsilon(x)$$

$$|\epsilon(x)|<2.5\times 10^{-4}$$

$c_1=.196854$ $\quad c_3=.000344$

$c_2=.115194$ $\quad c_4=.019527$

26.2.19

$$P(x)=1-\frac{1}{2}(1+d_1x+d_2x^2+d_3x^3$$
$$+d_4x^4+d_5x^5+d_6x^6)^{-16}+\epsilon(x)$$

$$|\epsilon(x)|<1.5\times 10^{-7}$$

$d_1=.04986\ 73470$ $\quad d_4=.00003\ 80036$

$d_2=.02114\ 10061$ $\quad d_5=.00004\ 88906$

$d_3=.00327\ 76263$ $\quad d_6=.00000\ 53830$

26.2.20 $$Z(x)=(a_0+a_2x^2+a_4x^4+a_6x^6)^{-1}+\epsilon(x)$$

$$|\epsilon(x)|<2.7\times 10^{-3}$$

$a_0=2.490895$ $\quad a_4=-.024393$

$a_2=1.466003$ $\quad a_6=\ .178257$

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[7] Based on approximations in C. Hastings, Jr., Approximations for digital computers. Princeton Univ. Press, Princeton, N.J., 1955 (with permission).

26.2.21

$$Z(x)=(b_0+b_2x^2+b_4x^4+b_6x^6+b_8x^8+b_{10}x^{10})^{-1}+\epsilon(x)$$

$$|\epsilon(x)|<2.3\times10^{-4}$$

$$b_0=2.50523\ 67 \qquad b_6=\ .13064\ 69$$
$$b_2=1.28312\ 04 \qquad b_8=-.02024\ 90$$
$$b_4=\ .22647\ 18 \qquad b_{10}=\ .00391\ 32$$

Rational Approximations [7] for x_p where $Q(x_p)=p$

$$0<p\leq .5$$

26.2.22

$$x_p=t-\frac{a_0+a_1t}{1+b_1t+b_2t^2}+\epsilon(p), \qquad t=\sqrt{\ln\frac{1}{p^2}}$$

$$|\epsilon(p)|<3\times10^{-3}$$

$$a_0=2.30753 \qquad b_1=.99229$$
$$a_1=\ .27061 \qquad b_2=.04481$$

26.2.23

$$x_p=t-\frac{c_0+c_1t+c_2t^2}{1+d_1t+d_2t^2+d_3t^3}+\epsilon(p), \qquad t=\sqrt{\ln\frac{1}{p^2}}$$

$$|\epsilon(p)|<4.5\times10^{-4}$$

$$c_0=2.515517 \qquad d_1=1.432788$$
$$c_1=\ .802853 \qquad d_2=\ .189269$$
$$c_2=\ .010328 \qquad d_3=\ .001308$$

Bounds Useful as Approximations to the Normal Distribution Function

26.2.24

$$P(x)\leq\begin{cases}P_1(x)=\frac{1}{2}+\frac{1}{2}\,(1-e^{-2x^2/\pi})^{\frac{1}{2}} & (x>0)\\[2ex] P_2(x)=1-\frac{(4+x^2)^{\frac{1}{2}}-x}{2}\,(2\pi)^{-\frac{1}{2}}e^{-x^2/2} & \\ & (x>1.4)\end{cases}$$

26.2.25

$$P(x)\geq\begin{cases}P_3(x)=\frac{1}{2}+\frac{1}{2}\left(1-e^{-2x^2/\pi}-\frac{2(\pi-3)}{3\pi^2}\,x^4e^{-x^2/2}\right)^{\frac{1}{2}} & \\ & (x>0)\\[2ex] P_4(x)=1-\frac{1}{x}\,(2\pi)^{-\frac{1}{2}}e^{-x^2/2} & (x>2.2)\end{cases}$$

See **Figure 26.1** for error curves.

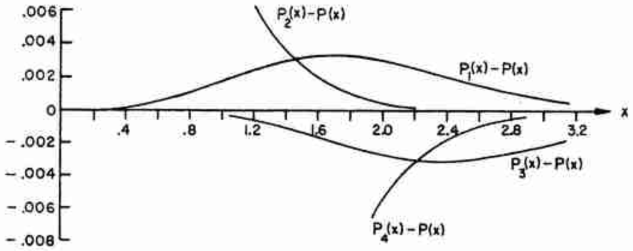

FIGURE 26.1. *Error curves for bounds on normal distribution.*

Derivatives of the Normal Probability Density Function

26.2.26

$$Z^{(m)}(x)=\frac{d^m}{dx^m}\,Z(x)$$

Differential Equation

26.2.27

$$Z^{(m+2)}(x)+xZ^{(m+1)}(x)+(m+1)Z^{(m)}(x)=0$$

Value at $x=0$

26.2.28

$$Z^{(m)}(0)=\begin{cases}\dfrac{(-1)^{m/2}m!}{\sqrt{2\pi}2^{m/2}\left(\dfrac{m}{2}\right)!} & \text{for } m=2r, r=0, 1, \ldots\\[2ex] 0 & \text{for odd } m>0\end{cases}$$

Relation of $P(x)$ and $Z^{(m)}(x)$ to Other Functions

	Function	*Relation*	
26.2.29	Error function	$\operatorname{erf} x=2P(x\sqrt{2})-1$	$(x\geq 0)$
26.2.30	Incomplete gamma function (special case)	$\dfrac{\gamma\left(\frac{1}{2}, x\right)}{\Gamma\left(\frac{1}{2}\right)}=[2P(\sqrt{2x})-1]$	$(x\geq 0)$
26.2.31	Hermite polynomial	$He_n(x)=(-1)^n \dfrac{Z^{(n)}(x)}{Z(x)}$	
26.2.32	“	$H_n(x)=(-1)^n 2^{n/2} \dfrac{Z^{(n)}(x\sqrt{2})}{Z(x\sqrt{2})}$	
26.2.33	*Hh* function	$Hh_{-n}(x)=(-1)^{n-1}\sqrt{2\pi}Z^{(n-1)}(x)$	$(n>0)$
26.2.34	“	$Hh_n(x)=\dfrac{(-1)^n}{n!} Hh_{-1}(x) \dfrac{d^n}{dx^n}\left(\dfrac{Q(x)}{Z(x)}\right)$ *	$(n>0)$
26.2.35	Tetrachoric function	$\tau_n(x)=\dfrac{(-1)^{n-1}}{\sqrt{n!}} Z^{(n-1)}(x)$	
26.2.36	Confluent hypergeometric function (special case)	$M\left(\frac{1}{2}, \frac{3}{2}, -\frac{x^2}{2}\right)=\dfrac{\sqrt{2\pi}}{x}\left\{P(x)-\frac{1}{2}\right\}$	$(x>0)$
26.2.37	“	$M\left(1, \frac{3}{2}, \frac{x^2}{2}\right)=\dfrac{1}{xZ(x)}\left\{P(x)-\frac{1}{2}\right\}$	$(x>0)$
26.2.38	“	$M\left(\frac{2m+1}{2}, \frac{1}{2}, -\frac{x^2}{2}\right)=\dfrac{Z^{(2m)}(x)}{Z^{(2m)}(0)}$	$(x\geq 0)$
26.2.39	“	$M\left(\frac{2m+2}{2}, \frac{3}{2}, -\frac{x^2}{2}\right)=\dfrac{Z^{(2m-1)}(x)}{xZ^{(2m)}(0)}$	$(x\geq 0)$
26.2.40	Parabolic cylinder function	$U\left(-n-\frac{1}{2}, x\right)=e^{-\frac{1}{4}x^2}(-1)^n \dfrac{Z^{(n)}(x)}{Z(x)}$	$(n>0)$

Repeated Integrals of the Normal Probability Integral

26.2.41 $$I_n(x)=\int_x^\infty I_{n-1}(t)dt \qquad (n\geq 0)$$

where $I_{-1}(x)=Z(x)$

26.2.42

$$I_{-n}(x)=\left(-\frac{d}{dx}\right)^{n-1} Z(x)=(-1)^{n-1}Z^{(n-1)}(x)$$

$$(n\geq -1)$$

26.2.43 $$\left(\frac{d^2}{dx^2}+x\frac{dx}{dn}-n\right)I_n(x)=0$$

26.2.44

$$(n+1)I_{n+1}(x)+xI_n(x)-I_{n-1}(x)=0 \qquad (n>-1)$$

---

*See page II.

26.2.45

$$I_n(x)=\int_x^\infty \frac{(t-x)^n}{n!}\,Z(t)dt=e^{-x^2/2}\int_0^\infty \frac{t^n}{n!}\,Z(t)dt \qquad (n>-1)$$

26.2.46 $$I_n(0)=I_{-n}(0)=\frac{1}{\left(\frac{n}{2}\right)!2^{\frac{n+2}{2}}} \qquad (n \text{ even})$$

Asymptotic Expansions of an Arbitrary Probability Density Function and Distribution Function

Let Y_i $(i=1,2, \ldots, n)$ be n independent random variables with mean m_i, variance σ_i^2, and higher cumulants $\kappa_{r,\,i}$. Then asymptotic expansions with respect to n for the probability density and cumulative distribution function of

$$X=\frac{\sum\limits_{i=1}^{m}(Y_i-m_i)}{\left(\sum\limits_{i=1}^{m}\sigma_i^2\right)^{\frac{1}{2}}} \text{ are}$$

26.2.47

$$f(x)\sim Z(x)-\left[\frac{\gamma_1}{6}\,Z^{(3)}(x)\right]+\left[\frac{\gamma_2}{24}\,Z^{(4)}(x)+\frac{\gamma_1^2}{72}\,Z^{(6)}(x)\right]$$
$$-\left[\frac{\gamma_3}{120}\,Z^{(5)}(x)+\frac{\gamma_1\gamma_2}{144}\,Z^{(7)}(x)+\frac{\gamma_1^3}{1296}\,Z^{(9)}(x)\right]$$
$$+\left[\frac{\gamma_4}{720}\,Z^{(6)}(x)+\frac{\gamma_2^2}{1152}\,Z^{(8)}(x)+\frac{\gamma_1\gamma_3}{720}\,Z^{(8)}(x)\right.$$
$$\left.+\frac{\gamma_1^2\gamma_2}{1728}\,Z^{(10)}(x)+\frac{\gamma_1^4}{31104}\,Z^{(12)}(x)\right]+\ldots$$

26.2.48

$$F(x)\sim P(x)-\left[\frac{\gamma_1}{6}\,Z^{(2)}(x)\right]+\left[\frac{\gamma_2}{24}\,Z^{(3)}(x)+\frac{\gamma_1^2}{72}\,Z^{(5)}(x)\right]$$
$$-\left[\frac{\gamma_3}{120}\,Z^{(4)}(x)+\frac{\gamma_1\gamma_2}{144}\,Z^{(6)}(x)+\frac{\gamma_1^3}{1296}\,Z^{(8)}(x)\right]$$
$$+\left[\frac{\gamma_4}{720}\,Z^{(5)}(x)+\frac{\gamma_2^2}{1152}\,Z^{(7)}(x)+\frac{\gamma_1\gamma_3}{720}\,Z^{(7)}(x)\right.$$
$$\left.+\frac{\gamma_1^2\gamma_2}{1728}\,Z^{(9)}(x)+\frac{\gamma_1^4}{31104}\,Z^{(11)}(x)\right]+\ldots$$

where

$$\gamma_{r-2}=\frac{1}{n^{\frac{r}{2}-1}}\,\frac{\left(\frac{1}{n}\sum\limits_{i=1}^{n}\kappa_{r,\,i}\right)}{\left(\frac{1}{n}\sum\limits_{i=1}^{n}\sigma_i^2\right)^{r/2}}$$

Terms in brackets are terms of the same order with respect to n. When the Y_i have the same distribution, then $m_i=m$, $\sigma_i^2=\sigma^2$, $\kappa_{r,\,i}=\kappa_r$ and

$$\gamma_{r-2}=\frac{1}{n^{\frac{1}{2}r-1}}\left(\frac{\kappa_r}{\sigma^r}\right)$$

Asymptotic Expansion for the Inverse Function of an Arbitrary Distribution Function

Let the cumulative distribution function of $Y=\sum\limits_{i=1}^{n}Y_i$ be denoted by $F(y)$. Then the (Cornish-Fisher) asymptotic expansion with respect to n for the value of y_p such that $F(y_p)=1-p$ is

26.2.49 $$y_p\sim m+\sigma w$$

where

$$w=x+[\gamma_1h_1(x)]$$
$$+[\gamma_2h_2(x)+\gamma_1^2h_{11}(x)]$$
$$+[\gamma_3h_3(x)+\gamma_1\gamma_2h_{12}(x)+\gamma_1^3h_{111}(x)]$$
$$+[\gamma_4h_4(x)+\gamma_2^2h_{22}(x)+\gamma_1\gamma_3h_{13}(x)+\gamma_1^2\gamma_2h_{112}(x)$$
$$+\gamma_1^4h_{1111}(x)]+\ldots$$

and

$$Q(x)=p, \qquad \gamma_{r-2}=\frac{\kappa_r}{\kappa_2^{r/2}}, \qquad r=3,4,\ldots$$

26.2.50

$$h_1(x)=\frac{1}{6}\,He_2(x)$$

$$h_2(x)=\frac{1}{24}\,He_3(x)$$

$$h_{11}(x)=-\frac{1}{36}\,[2He_3(x)+He_1(x)]$$

$$h_3(x)=\frac{1}{120}\,[He_4(x)]$$

$$h_{12}(x)=-\frac{1}{24}\,[He_4(x)+He_2(x)]$$

$$h_{111}(x)=\frac{1}{324}\,[12He_4(x)+19He_2(x)]$$

$$h_4(x)=\frac{1}{720}\,He_5(x)$$

$$h_{22}(x)=-\frac{1}{384}\,[3He_5(x)+6He_3(x)+2He_1(x)]$$

$$h_{13}(x)=-\frac{1}{180}\,[2He_5(x)+3He_3(x)]$$

$$h_{112}(x)=\frac{1}{288}\,[14He_5(x)+37He_3(x)+8He_1(x)]$$

$$h_{1111}(x)=-\frac{1}{7776}\,[252He_5(x)+832He_3(x)+227He_1(x)]$$

Terms in brackets in **26.2.49** are terms of the same order with respect to n. The $He_n(x)$ are the Hermite polynomials. (See chapter **22**.)

26.2.51 $$He_n(x)=(-1)^n\frac{Z^{(n)}(x)}{Z(x)}=n!\sum_{m=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^m}{2^m m!(n-2m)!}x^{n-2m}$$

In the following auxiliary table, the polynomial functions $h_1(x), h_2(x)\ .\ .\ .\ h_{1111}(x)$ are tabulated for

$$p=.25,\ .1,\ .05,\ .025,\ .01,\ .005,\ .0025,\ .001,\ .0005.$$

Auxiliary coefficients[8] *for use with Cornish-Fisher asymptotic expansion.* **26.2.49**

	p								
	.25	.10	.05	.025	.01	.005	.0025	.001	.0005
x	.67449	1.28155	1.64485	1.95996	2.32635	2.57583	2.80703	3.09022	3.29053
$h_1(x)$	−.09084	.10706	.28426	.47358	.73532	.93915	1.14657	1.42491	1.63793
$h_2(x)$	−.07153	−.07249	−.02018	.06872	.23379	.39012	.57070	.84331	1.07320
$h_{11}(x)$	.07663	.06106	−.01878	−.14607	−.37634	−.59171	−.83890	−1.21025	−1.52234
$h_3(x)$	.00398	−.03464	−.04928	−.04410	−.00152	.06010	.14841	.30746	.46059
$h_{12}(x)$	.00282	.14644	.17532	.10210	−.17621	−.53531	−1.02868	−1.89355	−2.71243
$h_{111}(x)$	−.01428	−.11629	−.11900	−.02937	.25195	.59757	1.06301	1.86787	2.62337
$h_4(x)$	.00998	.00227	−.01082	−.02357	−.03176	−.02621	−.00666	.04591	.10950
$h_{22}(x)$	−.03285	.00776	.05985	.09659	.07888	−.01226	−.19116	−.59060	−1.03555
$h_{13}(x)$	−.05126	.01086	.09462	.16106	.16058	.05366	−.17498	−.70464	−1.30531
$h_{112}(x)$	.14764	−.10858	−.39517	−.55856	−.32621	.35696	1.60445	4.29304	7.23307
$h_{1111}(x)$	−.06898	.09585	.25623	.31624	.07286	−.46534	−1.39199	−3.32708	−5.40702

[8] From R. A. Fisher, Contributions to mathematical statistics, Paper 30 (with E. A. Cornish) Extrait de la Revue de l'Institute International de Statistique 4, 1–14 (1937) (with permission).

26.3. Bivariate Normal Probability Function

26.3.1

$$g(x,y,\rho)=[2\pi\sqrt{1-\rho^2}]^{-1}\exp-\frac{1}{2}\left(\frac{x^2-2\rho xy+y^2}{1-\rho^2}\right)$$

26.3.2 $$g(x,y,\rho)=(1-\rho^2)^{-\frac{1}{2}}Z(x)Z\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)$$

26.3.3

$$L(h,k,\rho)=\int_h^\infty dx\int_k^\infty g(x,y,\rho)dy$$

$$=\int_h^\infty Z(x)dx\int_w^\infty Z(w)\,dw,\qquad w=\left(\frac{k-\rho x}{\sqrt{1-\rho^2}}\right)$$

26.3.4 $$L(-h,-k,\rho)=\int_{-\infty}^h dx\int_{-\infty}^k g(x,y,\rho)dy$$

26.3.5 $$L(-h,k,-\rho)=\int_{-\infty}^h dx\int_k^\infty g(x,y,\rho)dy$$

26.3.6 $$L(h,-k,-\rho)=\int_h^\infty dx\int_{-\infty}^k g(x,y,\rho)dy$$

26.3.7 $$L(h,k,\rho)=L(k,h,\rho)$$

26.3.8 $$L(-h,k,\rho)+L(h,k,-\rho)=Q(k)$$

26.3.9 $$L(-h,-k,\rho)-L(h,k,\rho)=P(k)-Q(h)$$

26.3.10

* $$2[L(h,\ k,\ \rho)+L(h,\ k,\ -\rho)+P(h)-Q(k)]-1$$

$$=\int_{-h}^h dx\int_{-k}^k g(x,y,\rho)dy$$

Probability Function With Means m_x, m_y, Variances σ_x^2, σ_y^2, and Correlation ρ

The random variables X, Y are said to be distributed as a bivariate Normal distribution with means and variances $(m_x,\ m_y)$ and $(\sigma_x^2,\ \sigma_y^2)$ and correlation ρ if the joint probability that X is less than or equal to h and Y less than or equal to k is given by

26.3.11

$$Pr\{X\leq h,\ Y\leq k\}=\frac{1}{\sigma_x\sigma_y}\int_{-\infty}^{\frac{h-m_x}{\sigma_x}}\int_{-\infty}^{\frac{k-m_y}{\sigma_y}}g(s,t,\rho)ds\ dt$$

$$=L\left(-\left(\frac{h-m_x}{\sigma_x}\right),-\left(\frac{k-m_y}{\sigma_y}\right),\rho\right)$$

The probability density function is

26.3.12

$$\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\exp\frac{-Q}{2(1-\rho^2)}=\frac{1}{\sigma_x\sigma_y}g\left(\frac{x-m_x}{\sigma_x},\frac{y-m_y}{\sigma_y},\rho\right)$$

where

$$Q=\frac{(x-m_x)^2}{\sigma_x^2}-\frac{2\rho(x-m_x)(y-m_y)}{\sigma_x\sigma_y}+\frac{(y-m_y)^2}{\sigma_y^2}$$

Circular Normal Probability Density Function

26.3.13

$$\frac{1}{\sigma^2}g\left(\frac{x-m_x}{\sigma},\frac{y-m_y}{\sigma},0\right)=$$

$$\frac{1}{2\pi\sigma^2}\exp-\frac{(x-m_x)^2+(y-m_y)^2}{2\sigma^2}$$

Special Values of $L(h, k, \rho)$

26.3.14 $$L(h,k,0)=Q(h)Q(k)$$

26.3.15 $$L(h,k,-1)=0 \qquad (h+k\geq 0)$$

26.3.16 $$L(h,k,-1)=P(h)-Q(k) \qquad (h+k\leq 0)$$

26.3.17 $$L(h,k,1)=Q(h) \qquad (k\leq h)$$

26.3.18 $$L(h,k,1)=Q(k) \qquad (k\geq h)$$

26.3.19 $$L(0,0,\rho)=\frac{1}{4}+\frac{\text{arc sin }\rho}{2\pi}$$

$L(h, k, \rho)$ as a Function of $L(h, 0, \rho)$

26.3.20

$$L(h,k,\rho)=L\left(h,0,\frac{(\rho h-k)(\text{sgn } h)}{\sqrt{h^2-2\rho hk+k^2}}\right)+L\left(k,0,\frac{(\rho k-h)(\text{sgn } k)}{\sqrt{h^2-2\rho hk+k^2}}\right)-\begin{cases}0 & \text{if } hk>0 \text{ or } hk=0 \\ & \text{and } h+k\geq 0 \\ \frac{1}{2} & \text{otherwise}\end{cases}$$

where sgn $h=1$ if $h\geq 0$ and sgn $h=-1$ if $h<0$.

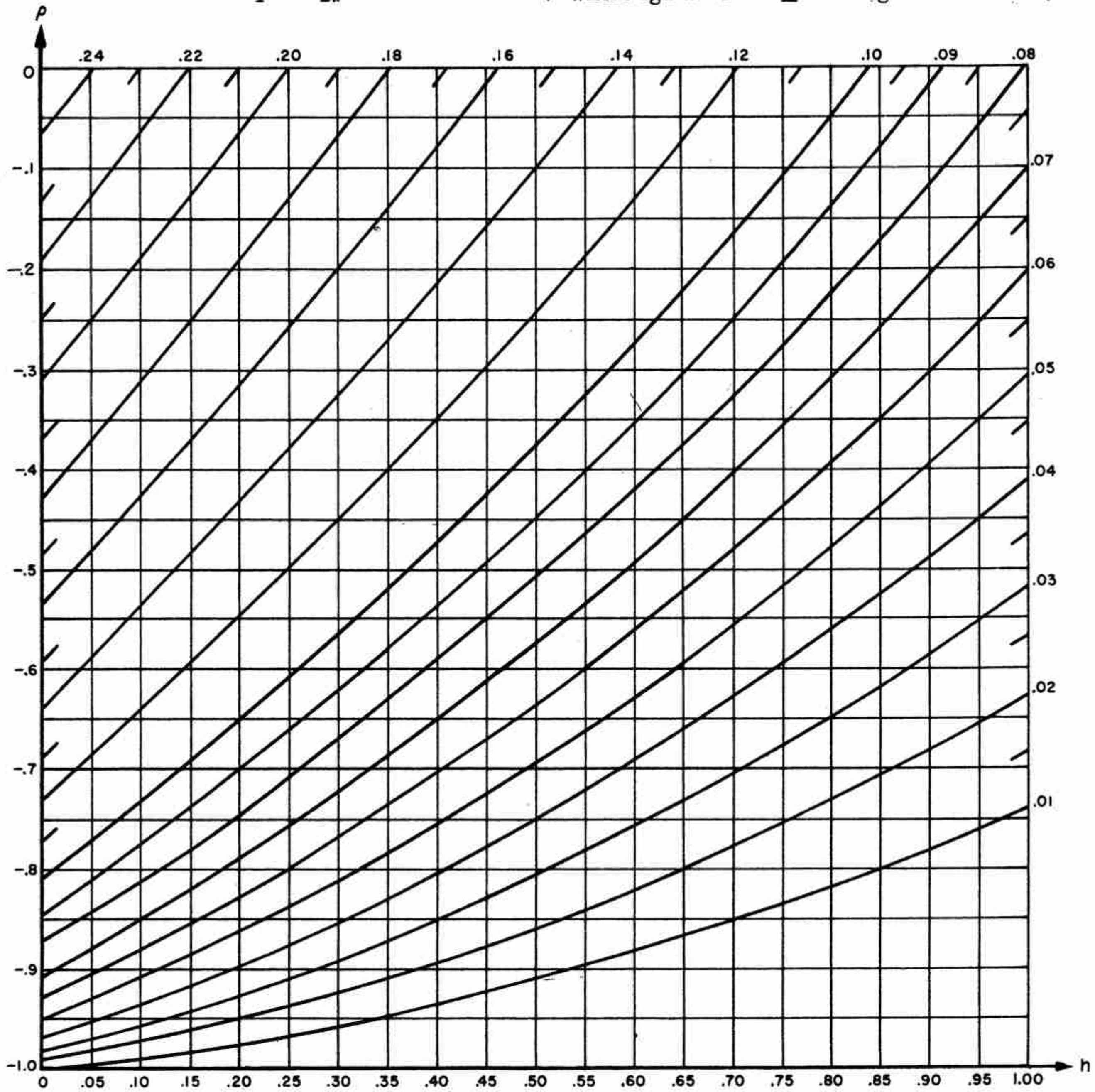

FIGURE 26.2. $L(h,\ 0,\ \rho)$ *for* $0\leq h\leq 1$ *and* $-1\leq\rho\leq 0$.

Values for $h<0$ can be obtained using $L(h,0,-\rho)=\frac{1}{2}-L(-h,0,\rho)$.

Integral Over an Ellipse With Center at (m_x, m_y)

26.3.21

$$\iint_A (\sigma_x\sigma_y)^{-1} g\left(\frac{x-m_x}{\sigma_x}, \frac{y-m_y}{\sigma_y}, \rho\right) dxdy = 1-e^{-a^2/2}$$

where A is the area enclosed by the ellipse

$$\left(\frac{x-m_x}{\sigma_x}\right)^2 - \frac{2\rho(x-m_x)(y-m_y)}{\sigma_x\sigma_y} + \left(\frac{y-m_y}{\sigma_y}\right)^2 = a^2(1-\rho^2)$$

Integral Over an Arbitrary Region

26.3.22

$$\iint_{A(x,y)} (\sigma_x\sigma_y)^{-1} g\left(\frac{x-m_x}{\sigma_x}, \frac{y-m_y}{\sigma_y}, \rho\right) dxdy = \iint_{A^*(s,t)} g(s, t, o) dsdt$$

where $A^*(s, t)$ is the transformed region obtained from the transformation

$$s = \frac{1}{\sqrt{2+2\rho}}\left(\frac{x-m_x}{\sigma_x} + \frac{y-m_y}{\sigma_y}\right)$$

$$t = \frac{-1}{\sqrt{2-2\rho}}\left(\frac{x-m_x}{\sigma_x} - \frac{y-m_y}{\sigma_y}\right)$$

Integral of the Circular Normal Probability Function With Parameters $m_x = m_y = 0$, $\sigma = 1$ Over the Triangle Bounded by $y=0$, $y=ax$, $x=h$

26.3.23

$$V(h, ah) = \frac{1}{2\pi}\int_0^h \int_0^{ax} e^{-\frac{1}{2}(x^2+y^2)} dxdy$$

$$= \frac{1}{4} + L(h, 0, \rho) - L(0, 0, \rho) - \frac{1}{2} Q(h)$$

where

$$\rho = -\frac{a}{\sqrt{1+a^2}}$$

Integral of Circular Normal Distribution Over an Offset Circle With Radius $R\sigma$ and Center a Distance $r\sigma$ From (m_x, m_y)

26.3.24

$$\int_A \int \sigma^{-2} g\left(\frac{x-m_x}{\sigma}, \frac{y-m_y}{\sigma}, 0\right) dxdy = P(R^2|2, r^2)$$

where $P(R^2|2, r^2)$ is the c.d.f. of the non-central χ^2 distribution (see **26.4.25**) with $\nu=2$ degrees of freedom and noncentrality parameter r^2.

Approximation to $P(R^2|2, r^2)$

	Approximation	*Condition*
26.3.25	$\frac{2R^2}{4+R^2} \exp - \frac{2r^2}{4+R^2}$	$R<1$
26.3.26	$P(x_1)$	$R>1$
26.3.27	$P(x_2)$	$R>5$

$$x_1 = \frac{[R^2/(2+r^2)]^{1/3} - \left[1 - \frac{2}{9}\frac{2+2r^2}{(2+r^2)^2}\right]}{\left[\frac{2}{9}\frac{2+2r^2}{(2+r^2)^2}\right]^{\frac{1}{2}}}$$

$x_2 = R - \sqrt{r^2-1}$ $\quad$ R, r both large $\quad$ *

Inequality

26.3.28

$$Q(h) - \frac{1-\rho^2}{\rho h - k} Z(k)\left[Q\left(\frac{h-\rho k}{\sqrt{1-\rho^2}}\right)\right] < L(h, k, \rho) < Q(h)$$

where

$$\rho h - k > 0, \qquad 0 < \rho < 1.$$

Series Expansion

26.3.29

$$L(h, k, \rho) = Q(h)Q(k) + \sum_{n=0}^{\infty} \frac{Z^{(n)}(h) Z^{(n)}(k)}{(n+1)!} \rho^{n+1}$$

26.4. Chi-Square Probability Function

26.4.1

$$P(\chi^2|\nu) = \left[2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\right]^{-1} \int_0^{\chi^2} (t)^{\frac{\nu}{2}-1} e^{-\frac{t}{2}} dt \qquad (0 \leq \chi^2 < \infty)$$

26.4.2

$$Q(\chi^2|\nu) = 1 - P(\chi^2|\nu) \qquad (0 \leq \chi^2 < \infty)$$

$$= \left[2^{\nu/2}\Gamma\left(\frac{\nu}{2}\right)\right]^{-1} \int_{\chi^2}^{\infty} (t)^{\frac{\nu}{2}-1} e^{-\frac{t}{2}} dt$$

Relation to Normal Distribution

Let $X_1, X_2, \ldots, X_\nu$ be independent and identically distributed random variables each following a normal distribution with mean zero and unit variance. Then $X^2 = \sum_{i=1}^{\nu} X_i^2$ is said to follow the chi-square distribution with ν degrees of freedom and the probability that $X^2 \leq \chi^2$ is given by $P(\chi^2|\nu)$.

Cumulants

26.4.3 $\quad$ $\kappa_{n+1} = 2^n n! \nu \qquad (n=0, 1, \ldots)$

Series Expansions

26.4.4

$$Q(\chi^2|\nu)=2Q(\chi)+2Z(\chi)\sum_{r=1}^{\frac{\nu-1}{2}}\frac{\chi^{2r-1}}{1\cdot3\cdot5\ldots(2r-1)}$$

(ν odd) and $\chi=\sqrt{\chi^2}$

26.4.5

$$Q(\chi^2|\nu)=\sqrt{2\pi}Z(\chi)\left\{1+\sum_{r=1}^{\frac{\nu-2}{2}}\frac{\chi^{2r}}{2\cdot4\ldots(2r)}\right\}$$

(ν even)

26.4.6

$$P(\chi^2|\nu)=\left(\frac{1}{2}\chi^2\right)^{\nu/2}\frac{e^{-\chi^2/2}}{\Gamma\left(\frac{\nu+2}{2}\right)}$$

$$* \qquad \left\{1+\sum_{r=1}^{\infty}\frac{\chi^{2r}}{(\nu+2)(\nu+4)\cdots(\nu+2r)}\right\}$$

26.4.7 $$P(\chi^2|\nu)=\frac{1}{\Gamma\left(\frac{\nu}{2}\right)}\sum_{n=0}^{\infty}\frac{(-1)^n(\chi^2/2)^{\frac{\nu}{2}+n}}{n!\left(\frac{\nu}{2}+n\right)}$$

Recurrence and Differential Relations

26.4.8 $$Q(\chi^2|\nu+2)=Q(\chi^2|\nu)+\frac{(\chi^2/2)^{\nu/2}e^{-\chi^2/2}}{\Gamma\left(\frac{\nu}{2}+1\right)}$$

26.4.9 $$\frac{\partial^m Q(\chi^2|\nu)}{\partial(\chi^2)^m}=\frac{1}{2^m}\sum_{j=0}^{m}\binom{m}{j}(-1)^{m+j}Q(\chi^2|\nu-2j)$$

Continued Fraction

26.4.10 $$^*Q(\chi^2|\nu)=\frac{(\chi^2)^{\nu/2}e^{-\chi^2/2}}{2^{\nu/2}\Gamma(\nu/2)}$$

$$\left\{\frac{1}{\chi^2/2+}\,\frac{1-\nu/2}{1+}\,\frac{1}{\chi^2/2+}\,\frac{2-\nu/2}{1+}\,\frac{2}{\chi^2/2+}\ldots\right\}$$

Asymptotic Distribution for Large ν

26.4.11 $P(\chi^2|\nu)\sim P(x)$ where $x=\frac{\chi^2-\nu}{\sqrt{2\nu}}$

Asymptotic Expansions for Large χ^2

26.4.12

$$Q(\chi^2|\nu)\sim\frac{(\chi^2)^{\frac{\nu}{2}-1}e^{-\chi^2/2}}{2^{\nu/2}\Gamma(\nu/2)}\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma\left(1-\frac{\nu}{2}+j\right)}{\Gamma\left(1-\frac{\nu}{2}\right)}\frac{2^{j+1}}{(\chi^2)^j}$$

Approximations to the Chi-Square Distribution for Large ν

26.4.13

Approximation		*Condition*
$Q(\chi^2\|\nu)\approx Q(x_1)$,	$x_1=\sqrt{2\chi^2}-\sqrt{2\nu-1}$	$(\nu>100)$

26.4.14

$$Q(\chi^2|\nu)\approx Q(x_2),\qquad x_2=\frac{(\chi^2/\nu)^{1/3}-\left(1-\frac{2}{9\nu}\right)}{\sqrt{2/9\nu}}\qquad(\nu>30)$$

26.4.15

$$Q(\chi^2|\nu)\approx Q(x_2+h_\nu),\qquad h_\nu=\frac{60}{\nu}h_{60}\qquad(\nu>30)$$

Values of h_{60}

x	h_{60}	x	h_{60}	x	h_{60}
−3.5	−.0118	−1.0	+.0006	+1.5	−.0005
−3.0	−.0067	−.5	.0006	2.0	+.0002
−2.5	−.0033	.0	+.0002	2.5	.0017
−2.0	−.0010	+.5	−.0003	3.0	.0043
−1.5	+.0001	1.0	−.0006	3.5	.0082

Approximations for the Inverse Function for Large ν

If $Q(\chi_p^2|\nu)=p$ and $Q(x_p)=1-P(x_p)=p$, then

Approximation *Condition*

26.4.16 $$\chi_p^2\approx\frac{1}{2}\left\{x_p+\sqrt{2\nu-1}\right\}^2\qquad(\nu>100)$$

26.4.17 $$\chi_p^2\approx\nu\left\{1-\frac{2}{9\nu}+x_p\sqrt{\frac{2}{9\nu}}\right\}^3\qquad(\nu>30)$$

26.4.18 $$\chi_p^2\approx\nu.\left\{1-\frac{2}{9\nu}+(x_p-h_\nu)\sqrt{\frac{2}{9\nu}}\right\}^3\qquad(\nu>30)$$

where h_ν is given by **26.4.15.**

Relation to Other Functions

26.4.19 Incomplete gamma function

$$\frac{\gamma(a,x)}{\Gamma(a)}=P(\chi^2|\nu),\qquad \nu=2a,\ \chi^2=2x$$

$$\frac{\Gamma(a,x)}{\Gamma(a)}=Q(\chi^2|\nu)$$

26.4.20 Pearson's incomplete gamma function

$$I(u,p)=\frac{1}{\Gamma(p+1)}\int_0^{u\sqrt{p+1}}t^pe^{-t}dt=P(\chi^2|\nu)$$

$$\nu=2(p+1),\ \chi^2=2u\sqrt{p+1}$$

26.4.21 Poisson distribution

$$Q(\chi^2|\nu)=\sum_{j=0}^{c-1}e^{-m}\frac{m^j}{j!},\qquad c=\frac{\nu}{2},\ m=\frac{\chi^2}{2},\ (\nu\text{ even})$$

$$Q(\chi^2|\nu)-Q(\chi^2|\nu-2)=e^{-m}\frac{m^{c-1}}{(c-1)!}$$

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*See page II.

26.4.22 Pearson Type III

$$\left[\frac{ab}{e}\right]^{ab}\int_{-a}^{x}\left(1+\frac{t}{a}\right)^{ab}e^{-bt}dt=P(\chi^2|\nu)$$

$$\nu=2ab+2,\ \chi^2=2b(x+a)$$

26.4.23 Incomplete moments of Normal distribution

$$\int_0^x t^n Z(t)dt=\begin{cases}(n-1)!!\,\dfrac{P(\chi^2|\nu)}{2} & (n\text{ even})\\[2ex] \dfrac{(n-1)!!}{\sqrt{2\pi}}P(\chi^2|\nu) & (n\text{ odd})\end{cases}$$

$$\chi^2=x^2,\ \nu=n+1$$

26.4.24 Generalized Laguerre Polynomials

$$n!L_n^{(\alpha)}(x)=\frac{\sum\limits_{j=0}^{n+1}(-1)^{n+j}\binom{n+1}{j}Q(\chi^2|\nu+2-2j)}{2^n[Q(\chi^2|\nu+2)-Q(\chi^2|\nu)]}$$

$$x=\chi^2/2,\ \alpha=\nu/2$$

Non-Central χ^2 Distribution Function

26.4.25

$$P(\chi'^2|\nu,\ \lambda)=\sum_{j=0}^{\infty}e^{-\lambda/2}\frac{(\lambda/2)^j}{j!}P(\chi'^2|\nu+2j)$$

where $\lambda\geq 0$ is termed the non-centrality parameter.

Relation of Non-Central χ^2 Distribution With $\nu=2$ to the Integral of Circular Normal Distribution ($\sigma^2=1$) Over an Offset Circle Having Radius R and Center a Distance $r=\sqrt{\lambda}$ From the Origin. (See **26.3.24–26.3.27.**)

26.4.26

$$\iint_A g(x,\ y,\ 0)\,dxdy=P(\chi^2=R^2|\nu=2,\ \lambda)$$

$$=1-\sum_{j=0}^{\infty}\frac{e^{-\lambda/2}\lambda^j}{2^j j!}Q(R^2|2+2j)$$

Approximations to the Non-Central χ^2 Distribution

$$a=\nu+\lambda \qquad b=\frac{\lambda}{\nu+\lambda}$$

Approximating Function	*Approximation*
26.4.27 χ^2 distribution	$P(\chi'^2\|\nu,\ \lambda)\approx P\left(\frac{\chi^2}{1+b}\middle\|\nu^*\right),\qquad \nu^*=\frac{a}{1+b}$
26.4.28 Normal distribution	$P(\chi'^2\|\nu,\ \lambda)\approx P(x),\qquad x=\frac{(\chi'^2/a)^{1/3}-\left[1-\frac{2}{9}\left(\frac{1+b}{a}\right)\right]}{\sqrt{\frac{2}{9}\left(\frac{1+b}{a}\right)}}$
26.4.29 Normal distribution	$P(\chi'^2\|\nu,\ \lambda)\approx P(x),\qquad x=\left[\frac{2\chi'^2}{1+b}\right]^{\frac{1}{2}}-\left[\frac{2a}{1+b}-1\right]^{\frac{1}{2}}$

Approximations to the Inverse Function of Non-Central χ^2 Distribution

If $Q(\chi_p'^2|\nu,\ \lambda)=p,\ Q(\chi_p^2|\nu^*)=p,$ and $Q(x_p)=p$ then

Approximating Variable	*Approximation to the Inverse Function*
26.4.30 χ^2	$\chi_p'^2\approx(1+b)\chi_p^2$
26.4.31 Normal	$\chi_p'^2\approx\frac{1+b}{2}\left[x_p+\sqrt{\frac{2a}{1+b}-1}\right]^2$
26.4.32 Normal	$\chi_p'^2\approx a\left[x_p\sqrt{\frac{2}{9}\left(\frac{1+b}{a}\right)}+1-\frac{2}{9}\left(\frac{1+b}{a}\right)\right]^3$

Properties of Chi-Square, Non-Central Chi-Square, and Related Quantities

$$a=\nu+\lambda \qquad b=\frac{\lambda}{\nu+\lambda}$$

$$\psi(z)=\frac{d}{dz}\ln\Gamma(z), \qquad \psi^{s}(z)=\frac{d^{s}}{dz^{s}}\psi(z)$$

	Variable	*Mean*	*Variance*	*Coefficient of skewness* (γ_1)	*Coefficient of excess* (γ_2)
26.4.33	χ^2	ν	2ν	$\frac{2^{3/2}}{\sqrt{\nu}}$	$12\nu^{-1}$
26.4.34	$\sqrt{2\chi^2}$	$(2\nu-1)^{\frac{1}{2}}\{1+[16\nu(\nu-1)]^{-1}\}+O(\nu^{-7/2})$	$1-\frac{1}{4\nu}-\frac{1}{8\nu^2}+\frac{5}{64\nu^3}-O(\nu^{-4})$	$\frac{1}{\sqrt{2\nu}}\left[1+\frac{5}{8\nu}-\frac{1}{128\nu^2}\right]+O(\nu^{-7/2})$	$\frac{3}{2^2}\frac{1}{\nu^2}\left[1+\frac{3}{2\nu}\right]+O(\nu^{-4})$
26.4.35	$(\chi^2/\nu)^{1/3}$	$1-\frac{2}{3^2\nu}+\frac{80}{3^7\nu^3}+O(\nu^{-4})$	$\frac{2}{3^2\nu}-\frac{104}{3^7\nu^3}+O(\nu^{-4})$	$\frac{2^{7/2}}{3^3\nu^{3/2}}\left[1+\frac{8}{3^2\nu}\right]+O(\nu^{-7/2})$	$-\frac{4}{9\nu}\left[1+\frac{16}{9\nu}\right]+O(\nu^{-3})$
26.4.36	$\ln(\chi^2/\nu)$	$\psi\left(\frac{\nu}{2}\right)-\ln\left(\frac{\nu}{2}\right)=-\frac{1}{\nu}-\frac{1}{3\nu^2}+O(\nu^{-4})$	$\psi'\left(\frac{\nu}{2}\right)=\frac{2}{\nu-1}\left[1-\frac{1}{3(\nu-1)^2}\right]+O((\nu-1)^{-5})$	$\frac{\psi''\left(\frac{\nu}{2}\right)}{\psi'\left(\frac{\nu}{2}\right)^{3/2}}=-\sqrt{\frac{2}{\nu-1}}\left[1-\frac{1}{2(\nu-1)^2}\right]+O((\nu-1)^{-9/2})$	$\frac{\psi^{(3)}\left(\frac{\nu}{2}\right)}{\psi'\left(\frac{\nu}{2}\right)^2}=\frac{4}{\nu-1}\left[1+\frac{4}{3(\nu-1)^2}\right]+O((\nu-1)^{-5})$
26.4.37	χ'^2	a	$2a(1+b)$	$\left(\frac{2}{1+b}\right)^{3/2}(1+2b)a^{-\frac{1}{2}}$	$\frac{12}{a}\frac{(1+3b)}{(1+b)^2}$
26.4.38	$\sqrt{2\chi'^2}$	$[2a-(1+b)]^{\frac{1}{2}}+O(a^{-3/2})$	$(1+b)-\frac{a^{-1}}{4}[8b+(1+b)\ (1-7b)]+O(a^{-2})$	$\frac{a^{-\frac{1}{2}}(1-b)\ (1+3b)}{2^{\frac{1}{2}}(1+b)^{3/2}}+O(a^{-1})$	$\frac{3b(b+2)}{(1+b)^2}+O(a^{-2})$
26.4.39	$(\chi'^2/a)^{1/3}$	$1-\frac{2}{3^2}\frac{1+b}{a}-\frac{40}{3^4}\frac{b^2}{a^2}+O(a^{-3})$	$\frac{2}{9}a^{-1}(1+b)+\frac{16}{27}a^{-2}b^2+O(a^{-3})$	$\left(\frac{2}{1+b}\right)^{3/2}b^2a^{-\frac{1}{2}}+O(a^{-3/2})$	$-\frac{4}{3^2}\frac{(1+3b+12b^2-44b^3)}{a(1+b)^2}-O(a^{-2})$

26.5. Incomplete Beta Function

26.5.1

$$I_x(a,b)=\frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}dt \qquad (0\leq x\leq 1)$$

26.5.2

$$I_x(a,b)=1-I_{1-x}(b,a)$$

Relation to the Chi-Square Distribution

If X_1^2 and X_2^2 are independent random variables following chi-square distributions **26.4.1** with ν_1 and ν_2 degrees of freedom respectively, then $\frac{X_1^2}{X_1^2+X_2^2}$ is said to follow a beta distribution with ν_1 and ν_2 degrees of freedom and has the distribution function

26.5.3

$$P\left\{\frac{X_1^2}{X_1^2+X_2^2}\leq x\right\}=\frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}dt$$
$$=I_x(a,b) \qquad a=\frac{\nu_1}{2},\ b=\frac{\nu_2}{2}$$

Series Expansions ($0<x<1$)

26.5.4

$$* \quad I_x(a,b)=\frac{x^a(1-x)^b}{aB(a,b)}\left\{1+\sum_{n=0}^{\infty}\frac{B(a+1,n+1)}{B(a+b,n+1)}x^{n+1}\right\}$$

26.5.5

$$I_x(a,b)=\frac{x^a(1-x)^{b-1}}{aB(a,b)}\left\{1+\sum_{n=0}^{\infty}\frac{B(a+1,n+1)}{B(b-n-1,n+1)}\left(\frac{x}{1-x}\right)^{n+1}\right\}$$
$$=\frac{x^a(1-x)^{b-1}}{aB(a,b)}\left\{1+\sum_{n=0}^{s-2}\frac{B(a+1,n+1)}{B(b-n-1,n+1)}\left(\frac{x}{1-x}\right)^{n+1}\right\}+I_x(a+s,b-s)$$

26.5.6

$$1-I_x(a,b)=I_{1-x}(b,a)$$
$$=\frac{(1-x)^b}{B(a,b)}\sum_{i=0}^{a-1}(-1)^i\binom{a-1}{i}\frac{(1-x)^i}{b+i} \text{ (integer a)}$$

26.5.7

$$1-I_x(a,b)=I_{1-x}(b,a)$$
$$=(1-x)^{a+b-1}\sum_{i=0}^{a-1}\binom{a+b-1}{i}\left(\frac{x}{1-x}\right)^i \text{ (integer a)}$$

Continued Fractions

26.5.8

$$I_x(a,b)=\frac{x^a(1-x)^b}{aB(a,b)}\left\{\frac{1}{1+}\,\frac{d_1}{1+}\,\frac{d_2}{1+}\cdots\right\} \quad *$$

$$d_{2m+1}=-\frac{(a+m)(a+b+m)}{(a+2m)(a+2m+1)}x$$

$$d_{2m}=\frac{m(b-m)}{(a+2m-1)(a+2m)}x$$

Best results are obtained when $x<\frac{a-1}{a+b-2}$. Also the $4m$ and $4m+1$ convergents are less than $I_x(a,b)$ and the $4m+2$, $4m+3$ convergents are greater than $I_x(a,b)$.

26.5.9

$$I_x(a,b)=\frac{x^a(1-x)^{b-1}}{aB(a,b)}\left[\frac{e_1}{1+}\,\frac{e_2}{1+}\,\frac{e_3}{1+}\cdots\right]$$

$$* \quad x<1 \qquad e_1=1$$

$$e_{2m}=-\frac{(a+m-1)(b-m)}{(a+2m-2)(a+2m-1)}\,\frac{x}{1-x}$$

$$e_{2m+1}=\frac{m(a+b-1+m)}{(a+2m-1)(a+2m)}\,\frac{x}{1-x}$$

Recurrence Relations

26.5.10

$$I_x(a,b)=xI_x(a-1,b)+(1-x)I_x(a,b-1)$$

26.5.11

$$I_x(a,b)=\frac{1}{x}\{I_x(a+1,b)-(1-x)I_x(a+1,b-1)\}$$

26.5.12

$$\left[I_x(a,b)=\right]\frac{1}{a(1-x)+b}\{bI_x(a,b+1)+a(1-x)I_x(a+1,b-1)\} \quad *$$

26.5.13

$$I_x(a,b)=\frac{1}{a+b}\{aI_x(a+1,b)+bI_x(a,b+1)\}$$

26.5.14

$$I_x(a,a)=\frac{1}{2}I_{1-x'}\left(a,\frac{1}{2}\right), \qquad x'=4\left(x-\frac{1}{2}\right)^2\left[, x\leq\frac{1}{2}^*\right]$$

26.5.15

$$I_x(a,b)=\frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)}x^a(1-x)^{b-1}+I_x(a+1,b-1)$$

26.5.16

$$I_x(a,b)=\frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)}x^a(1-x)^b+I_x(a+1,b)$$

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*See page II.

Asymptotic Expansions

26.5.17

$$1-I_x(a,b)=I_{1-x}(b,a)\sim\frac{\Gamma(b,y)}{\Gamma(b)}$$

$$-\frac{1}{24N^2}\left\{\frac{y^be^{-y}}{(b-2)!}(b+1+y)\right\}$$

$$+\frac{1}{5760N^4}\left\{\frac{y^be^{-y}}{(b-2)!}[(b-3)(b-2)(5b+7)(b+1+y)\right.$$

$$\left.-(5b-7)(b+3+y)y^2]\right\}$$

$$y=-N\ln x,\qquad N=a+\frac{b}{2}-\frac{1}{2}$$

26.5.18

$$I_x(a,b)\sim\frac{\Gamma(a,w)}{\Gamma(a)}+\frac{e^{-w}w^a}{\Gamma(a)}\left\{\frac{(a-1-w)}{2b}\right.$$

$$+\frac{1}{(2b)^2}\left(\frac{a^3}{2}-\frac{5}{3}a^2+\frac{3}{2}a-\frac{1}{3}-w\left[\frac{3}{2}a^2-\frac{11}{6}a+\frac{1}{3}\right]\right.$$

$$\left.\left.+w^2\left(\frac{3}{2}a-\frac{1}{6}\right)-\frac{1}{2}w^3\right)\right\}$$

$$w=b\left(\frac{x}{1-x}\right)$$

26.5.19

$$I_x(a,b)\sim P(y)-Z(y)\left[a_1+\frac{a_2(y-a_1)}{1+a_2}\right.$$

$$\left.+\frac{a_3(1+y^2/2)}{1+a_2}+\cdots\right]$$

$$a_1=\frac{2}{3}(b-a)[(a+b-2)(a-1)(b-1)]^{-1/2}$$

$$a_2=\frac{1}{12}\left[\frac{1}{a-1}+\frac{1}{b-1}-\frac{13}{a+b-1}\right]$$

$$a_3=-\frac{8}{15}\left[a_1\left(a_2+\frac{3}{a+b-2}\right)\right]$$

$$y^2=2\left[(a+b-1)\ln\frac{a+b-1}{a+b-2}+(a-1)\ln\frac{a-1}{(a+b-1)x}\right.$$

$$\left.+(b-1)\ln\frac{b-1}{(a+b-1)(1-x)}\right]$$

and y is taken negative when $x<\frac{a-1}{a+b-2}$

Approximations

26.5.20 If $(a+b-1)(1-x)\leq .8$

$$I_x(a,b)=Q(\chi^2|\nu)+\epsilon,$$

$$|\epsilon|<5\times10^{-3}\text{ if } a+b>6$$

$$\chi^2=(a+b-1)(1-x)(3-x)-(1-x)(b-1),$$

$$\nu=2b$$

26.5.21 If $(a+b-1)(1-x)\geq .8$

$$I_x(a,b)=P(y)+\epsilon,$$

$$|\epsilon|<5\times10^{-3}\text{ if } a+b>6$$

$$y=\frac{3\left[w_1\left(1-\frac{1}{9b}\right)-w_2\left(1-\frac{1}{9a}\right)\right]}{\left[\frac{w_1^2}{b}+\frac{w_2^2}{a}\right]^{\frac{1}{2}}},$$

$$w_1=(bx)^{1/3},\ w_2=[a(1-x)]^{1/3}$$

Approximation to the Inverse Function

26.5.22 If $I_{x_p}(a,b)=p$ and $Q(y_p)=p$ then

$$x_p\approx\frac{a}{a+be^{2w}}$$

$$w=\frac{y_p(h+\lambda)^{\frac{1}{2}}}{h}-\left(\frac{1}{2b-1}-\frac{1}{2a-1}\right)\left(\lambda+\frac{5}{6}-\frac{2}{3h}\right)$$

$$h=2\left(\frac{1}{2a-1}+\frac{1}{2b-1}\right)^{-1},\qquad\lambda=\frac{y_p^2-3}{6}$$

Relations to Other Functions and Distributions

	Function	*Relation*
26.5.23	Hypergeometric function	$\frac{1}{B(a,b)}\frac{x^a}{a}F(a,1-b;a+1;x)=I_x(a,b)$
26.5.24	Binomial distribution	$\sum_{s=a}^{n}\binom{n}{s}p^s(1-p)^{n-s}=I_p(a,n-a+1)$
26.5.25	"	$\binom{n}{a}p^a(1-p)^{n-a}=I_p(a,n-a+1)-I_p(a+1,n-a)$ *
26.5.26	Negative binomial distribution	$\sum_{s=a}^{n}\binom{n+s-1}{s}p^nq^s=I_q(a,n)$
26.5.27	Student's distribution	$\frac{1}{2}[1-A(t\|\nu)]=\frac{1}{2}I_x\left(\frac{\nu}{2},\frac{1}{2}\right),\qquad x=\frac{\nu}{\nu+t^2}$ *
26.5.28	F-(variance-ratio) distribution	$Q(F\|\nu_1,\nu_2)=I_x\left(\frac{\nu_2}{2},\frac{\nu_1}{2}\right),\qquad x=\frac{\nu_2}{\nu_2+\nu_1F}$

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*See page II.

26.6. *F*-(Variance-Ratio) Distribution Function

26.6.1

$$P(F|\nu_1,\nu_2)=\frac{\nu_1^{\frac{1}{2}\nu_1}\nu_2^{\frac{1}{2}\nu_2}}{B\left(\frac{1}{2}\nu_1,\frac{1}{2}\nu_2\right)}\int_0^F t^{\frac{1}{2}(\nu_1-2)}(\nu_2+\nu_1 t)^{-\frac{1}{2}(\nu_1+\nu_2)}dt \qquad (F\geq 0)$$

26.6.2

$$Q(F|\nu_1,\nu_2)=1-P(F|\nu_1,\nu_2)=I_x\left(\frac{\nu_2}{2},\frac{\nu_1}{2}\right)$$

where

$$x=\frac{\nu_2}{\nu_2+\nu_1 F}$$

Relation to the Chi-Square Distribution

If X_1^2 and X_2^2 are independent random variables following chi-square distributions **26.4.1** with ν_1 and ν_2 degrees of freedom respectively, then the distribution of $F=\frac{X_1^2/\nu_1}{X_2^2/\nu_2}$ is said to follow the variance ratio or *F*-distribution with ν_1 and ν_2 degrees of freedom. The corresponding distribution function is $P(F|\nu_1,\nu_2)$.

Statistical Properties

26.6.3

mean: $$m=\frac{\nu_2}{\nu_2-2} \qquad (\nu_2>2)$$

variance: $$\sigma^2=\frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)} \qquad (\nu_2>4)$$

third central moment:

$$\mu_3=\left(\frac{\nu_2}{\nu_1}\right)^3\frac{8\nu_1(\nu_1+\nu_2-2)(2\nu_1+\nu_2-2)}{(\nu_2-2)^3(\nu_2-4)(\nu_2-6)} \qquad (\nu_2>6)$$

moments about the origin:

$$\mu_n'=\left(\frac{\nu_2}{\nu_1}\right)^n\frac{\Gamma\left(\frac{\nu_1+2n}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)}\frac{\Gamma\left(\frac{\nu_1-2n}{2}\right)}{\Gamma\left(\frac{\nu_2}{2}\right)} \qquad (\nu_2>2n)$$

characteristic function:

$$\phi(t)=E(e^{iFt})=M\left(\frac{\nu_1}{2},-\frac{\nu_2}{2},-\frac{\nu_2}{\nu_1}it\right)$$

Series Expansions

$$x=\frac{\nu_2}{\nu_2+\nu_1 F}$$

26.6.4

* $$Q(F|\nu_1,\nu_2)=x^{\nu_2/2}\left[1+\frac{\nu_2}{2}(1-x)+\frac{\nu_2(\nu_2+2)}{2\cdot 4}(1-x)^2+\ldots+\frac{\nu_2(\nu_2+2)\ldots(\nu_2+\nu_1-4)}{2\cdot 4\ldots(\nu_1-2)}(1-x)^{\frac{\nu_1-2}{2}}\right] \qquad (\nu_1 \text{ even})$$

26.6.5

$$Q(F|\nu_1,\nu_2)=1-(1-x)^{\nu_1/2}\left[1+\frac{\nu_1}{2}x+\frac{\nu_1(\nu_1+2)}{2\cdot 4}x^2+\ldots+\frac{\nu_1(\nu_1+2)\ldots(\nu_2+\nu_1-4)}{2\cdot 4\ldots(\nu_2-2)}x^{\frac{\nu_2-2}{2}}\right] \qquad (\nu_2 \text{ even})$$

26.6.6

$$Q(F|\nu_1,\nu_2)=x^{\frac{\nu_1+\nu_2-2}{2}}\left[1+\frac{\nu_1+\nu_2-2}{2}\left(\frac{1-x}{x}\right)+\frac{(\nu_1+\nu_2-2)(\nu_1+\nu_2-4)}{2\cdot 4}\left(\frac{1-x}{x}\right)^2+\ldots+\frac{(\nu_1+\nu_2-2)\ldots(\nu_2+2)}{2\cdot 4\ldots(\nu_1-2)}\left(\frac{1-x}{x}\right)^{\frac{\nu_1-2}{2}}\right] \qquad (\nu_1 \text{ even})$$

26.6.7

$$Q(F|\nu_1,\nu_2)=1-(1-x)^{\frac{\nu_1+\nu_2-2}{2}}\left[1+\frac{\nu_1+\nu_2-2}{2}\left(\frac{x}{1-x}\right)+\ldots+\frac{(\nu_1+\nu_2-2)\ldots(\nu_1+2)}{2\cdot 4\ldots(\nu_2-2)}\left(\frac{x}{1-x}\right)^{\frac{\nu_2-2}{2}}\right] \qquad (\nu_2 \text{ even})$$

26.6.8

$$Q(F|\nu_1,\nu_2)=1-A(t|\nu_2)+\beta(\nu_1,\nu_2) \qquad (\nu_1, \nu_2 \text{ odd})$$

$$A(t|\nu_2)=\begin{cases}\frac{2}{\pi}\left\{\theta+\sin\theta[\cos\theta+\frac{2}{3}\cos^3\theta+\ldots+\frac{2\cdot 4\ldots(\nu_2-3)}{3.5\ldots(\nu_2-2)}\cos^{\nu_2-2}\theta]\right\} & \text{for } \nu_2>1\\ \frac{2\theta}{\pi} \text{ for } \nu_2=1\end{cases}$$

$$\beta(\nu_1,\nu_2)=\begin{cases}\frac{2}{\sqrt{\pi}}\frac{\left(\frac{\nu_2-1}{2}\right)!}{\left(\frac{\nu_2-2}{2}\right)!}\sin\theta\cos^{\nu_2}\theta\left\{1+\frac{\nu_2+1}{3}\sin^2\theta+\cdots+\frac{(\nu_2+1)(\nu_2+3)\ldots(\nu_1+\nu_2-4)\sin^{\nu_1-3}\theta}{3\cdot 5\ldots(\nu_1-2)}\right\} & \text{for } \nu_2>1\\ 0 \text{ for } \nu_1=1 \quad * \end{cases}$$

where

$$\theta=\arctan\sqrt{\frac{\nu_1}{\nu_2}F}$$

Reflexive Relation

If $F_p(\nu_1,\nu_2)$ and $F_{1-p}(\nu_2,\nu_1)$ satisfy

$$Q(F_p(\nu_1,\nu_2)|\nu_1,\nu_2)=p$$

$$Q(F_{1-p}(\nu_2,\nu_1)|\nu_2,\nu_1)=1-p$$

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*See page II.

26.6.9 then

$$F_p(\nu_1,\nu_2)=\frac{1}{F_{1-p}(\nu_2,\nu_1)}$$

Relation to Student's *t*-Distribution Function (See **26.7**)

26.6.10 $Q(F|\nu_1=1,\nu_2)=1-A(t|\nu_2)\qquad t=\sqrt{F}$

Limiting Forms

26.6.11

$$\lim_{\nu_2\to\infty}Q(F|\nu_1,\nu_2)=Q(\chi^2|\nu_1),\qquad \chi^2=\nu_1F$$

26.6.12

$$\lim_{\nu_1\to\infty}Q(F|\nu_1,\nu_2)=P(\chi^2|\nu_2),\qquad \chi^2=\frac{\nu_2}{F}$$

Approximations

26.6.13

$$Q(F|\nu_1,\nu_2)\approx Q(x),\qquad x=\frac{F-\dfrac{\nu_2}{\nu_2-2}}{\dfrac{\nu_2}{\nu_2-2}\sqrt{\dfrac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}}}$$

(ν_1 and ν_2 large)

26.6.14

$$Q(F|\nu_1,\nu_2)\approx Q(x),\qquad x=\frac{\sqrt{(2\nu_2-1)\,\frac{\nu_1}{\nu_2}\,F}-\sqrt{2\nu_1-1}}{\sqrt{1+\frac{\nu_1}{\nu_2}\,F}}$$

26.6.15

$$Q(F|\nu_1,\nu_2)\approx Q(x),\qquad x=\frac{F^{1/3}\left(1-\dfrac{2}{9\nu_2}\right)-\left(1-\dfrac{2}{9\nu_1}\right)}{\sqrt{\dfrac{2}{9\nu_1}+F^{2/3}\,\dfrac{2}{9\nu_2}}}$$

Approximation to the Inverse Function

26.6.16 If $Q(F_p|\nu_1,\nu_2)=p$, then

$F_p\approx e^{2w}$ where w is given by **26.5.22,** with

$$\nu_1=2b,\ \nu_2=2a$$

Non-Central *F*-Distribution Function

26.6.17

$$P(F'|\nu_1,\nu_2,\lambda)=\int_0^{F'}p(t|\nu_1,\nu_2,\lambda)dt=1-Q(F'|\nu_1,\nu_2,\lambda)$$

where

$$p(t|\nu_1,\nu_2,\lambda)=\sum_{j=0}^{\infty}e^{-\lambda/2}\,\frac{(\lambda/2)^j}{j!}\,\frac{(\nu_1+2j)^{\frac{\nu_1+2j}{2}}\,\nu_2{}^{\nu_2/2}}{B\left(\frac{\nu_1+2j}{2},\frac{\nu_2}{2}\right)}$$

$$\times t^{\frac{\nu_1+2j-2}{2}}\,[\nu_2+(\nu_1+2j)t]^{-(\nu_1+2j+\nu_2)/2}$$

and $\lambda\geq0$ is termed the non-centrality parameter.

Relation of Non-Central *F*-Distribution Function to Other Functions

Function	*Relation*
26.6.18 *F*-distribution	$P(F'\|\nu_1,\nu_2,\lambda)=\sum_{j=0}^{\infty}e^{-\lambda/2}\,\frac{(\lambda/2)^j}{j!}\,P(F'\|\nu_1+2j,\nu_2)$ $P(F'\|\nu_1,\nu_2,\lambda=0)=P(F'\|\nu_1,\nu_2)$
26.6.19 Non-central *t*-distribution	$P(F'\|\nu_1=1,\nu_2,\lambda)=P(t'\|\nu,\delta),\ t'=\sqrt{F'},\ \nu=\nu_2,\ \delta=\sqrt{\lambda}$
26.6.20 Incomplete Beta function	$P(F'\|\nu_1,\nu_2)=\sum_{j=0}^{\infty}e^{-\lambda/2}\,\frac{(\lambda/2)^j}{j!}\,I_x\left(\frac{\nu_1}{2}+j,\frac{\nu_2}{2}\right),$ $x=\frac{\nu_1F'}{\nu_1F'+\nu_2}$ *
26.6.21 Confluent hypergeometric function	$P(F'\|\nu_1,\nu_2,\lambda)=\sum_{i=0}^{\frac{\nu_2}{2}-1}\frac{2e^{-\lambda/2}}{(\nu_1+\nu_2)B\left(\frac{\nu_1}{2}+i+1,\frac{\nu_2}{2}-i\right)}\times$ $x^{\frac{\nu_1}{2}+1}(1-x)^{\frac{\nu_2}{2}-i-1}M\left(\frac{\nu_1+\nu_2}{2},\frac{\nu_1}{2}+i+1,\frac{\lambda x}{2}\right)$ $\left(\nu_2\text{ even and }x=\frac{\nu_2}{\nu_1F'+\nu_2}\right)$

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*See page II.

Series Expansion

26.6.22

$$P(F'|\nu_1,\nu_2,\lambda)=e^{-\frac{\lambda}{2}(1-x)}\,x^{\frac{1}{2}(\nu_1+\nu_2-2)}\sum_{i=0}^{\frac{\nu_2}{2}-1}T_i\qquad(\nu_2\text{ even})$$

where

$$T_0=1$$

$$T_1=\frac{1}{2}(\nu_1+\nu_2-2+\lambda x)\frac{1-x}{x}$$

$$T_i=\frac{1-x}{2i}[(\nu_1+\nu_2-2i+\lambda x)T_{i-1}+\lambda(1-x)T_{i-2}]$$

$$x=\frac{\nu_2}{\nu_1F'+\nu_2}$$

Limiting Forms

26.6.23

$$\lim_{\nu_2\to\infty}P(F'|\nu_1,\nu_2,\lambda)=P(\chi'^2|\nu,\lambda),\qquad\chi'^2=\nu_1F',\ \nu=\nu_1$$

26.6.24

$$\lim_{\nu_1\to\infty}P(F'|\nu_1,\nu_2,\lambda)=Q(\chi^2|\nu),\qquad\chi^2=\frac{\nu_2(1+c^2)}{F'}$$

where $\lambda/\nu_1\to c^2$ as $\nu_1\to\infty$.

Approximations to the Non-Central *F*-Distribution

26.6.25 $P(F'|\nu_1,\nu_2,\lambda)\approx P(x_1)$, ($\nu_1$ and ν_2 large)

where

$$x_1=\frac{F'-\frac{\nu_2(\nu_1+\lambda)}{\nu_1(\nu_2-2)}}{\frac{\nu_2}{\nu_1}\left[\frac{2}{(\nu_2-2)(\nu_2-4)}\left\{\frac{(\nu_1+\lambda)^2}{\nu_2-2}+\nu_1+2\lambda\right\}\right]^{\frac{1}{2}}}$$

26.6.26

$$P(F'|\nu_1,\nu_2,\lambda)\approx P(F|\nu_1^*,\nu_2),$$

$$F=\frac{\nu_1}{\nu_1+\lambda}F',\ \nu_1^*=\frac{(\nu_1+\lambda)^2}{\nu_1+2\lambda}$$

26.6.27

$$P(F'|\nu_1,\nu_2,\lambda)\approx P(x_2),$$

$$x_2=\frac{\left[\frac{\nu_1F'}{(\nu_1+\lambda)}\right]^{1/3}\left[1-\frac{2}{9\nu_2}\right]-\left[1-\frac{2(\nu_1+2\lambda)}{9(\nu_1+\lambda)^2}\right]}{\left[\frac{2}{9}\frac{\nu_1+2\lambda}{(\nu_1+\lambda)^2}+\frac{2}{9\nu_2}\left(\frac{\nu_1}{\nu_1+\lambda}F'\right)^{2/3}\right]^{\frac{1}{2}}}$$

26.7. Student's *t*-Distribution

If X is a random variable following a normal distribution with mean zero and variance unity, and χ^2 is a random variable following an independent chi-square distribution with ν degrees of freedom, then the distribution of the ratio $\frac{X}{\sqrt{\chi^2/\nu}}$ is called Student's *t*-distribution with ν degrees of freedom. The probability that $\frac{X}{\sqrt{\chi^2/\nu}}$ will be less in absolute value than a fixed constant t is

26.7.1

$$A(t|\nu)=P_r\left\{\left|\frac{X}{\sqrt{\chi^2/\nu}}\right|\leq t\right\}$$

$$=\left[\sqrt{\nu}B\left(\frac{1}{2},\frac{\nu}{2}\right)\right]^{-1}\int_{-t}^{t}\left(1+\frac{x^2}{\nu}\right)^{-\frac{\nu+1}{2}}dx$$

$$=1-I_x\left(\frac{\nu}{2},\frac{1}{2}\right),\qquad(0\leq t<\infty)\quad *$$

where

$$x=\frac{\nu}{\nu+t^2}$$

Statistical Properties

26.7.2

mean: $m=0$

variance: $\sigma^2=\frac{\nu}{\nu-2}$ $(\nu>2)$

skewness: $\gamma_1=0$

excess: $\gamma_2=\frac{6}{\nu-4}$ $(\nu>4)$

moments:

$$\mu_{2n}=\frac{1\cdot3\ldots(2n-1)\nu^n}{(\nu-2)(\nu-4)\ldots(\nu-2n)}\qquad(\nu>2n)$$

$$\mu_{2n+1}=0$$

characteristic function:

$$\phi(t)=E\left[\exp\left(it\frac{X}{\sqrt{\chi^2/\nu}}\right)\right]=\frac{\left(\frac{|t|}{2\sqrt{\nu}}\right)^{\nu/2}}{\pi\Gamma(\nu/2)}Y_{\frac{\nu}{2}}\left(\frac{|t|}{\sqrt{\nu}}\right)$$

Series Expansions

$$\left(\theta=\arctan\frac{t}{\sqrt{\nu}}\right)$$

26.7.3 .

$$A(t|\nu)=\begin{cases}\frac{2}{\pi}\left\{\theta+\sin\theta\left[\cos\theta+\frac{2}{3}\cos^3\theta+\ldots\right.\right.\\\qquad\left.\left.+\frac{2\cdot4\ldots(\nu-3)}{1\cdot3\ldots(\nu-2)}\cos^{\nu-2}\theta\right]\right\}\quad(\nu>1\text{ and odd})\\\frac{2}{\pi}\theta\qquad(\nu=1)\end{cases}\quad *$$

26.7.4

$$A(t|\nu)=\sin\theta\left\{1+\frac{1}{2}\cos^2\theta+\frac{1\cdot3}{2\cdot4}\cos^4\theta+\ldots\right.$$

$$\left.+\frac{1\cdot3\cdot5\ldots(\nu-3)}{2\cdot4\cdot6\ldots(\nu-2)}\cos^{\nu-2}\theta\right\}\qquad(\nu\text{ even})\quad *$$

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*See page II.

Asymptotic Expansion for the Inverse Function

If $A(t_p|\nu)=1-2p$ and $Q(x_p)=p$, then

26.7.5

$$t_p \sim x_p+\frac{g_1(x_p)}{\nu}+\frac{g_2(x_p)}{\nu^2}+\frac{g_3(x_p)}{\nu^3}+\ldots$$

$$g_1(x)=\frac{1}{4}\,(x^3+x)$$

$$g_2(x)=\frac{1}{96}\,(5x^5+16x^3+3x)$$

$$g_3(x)=\frac{1}{384}\,(3x^7+19x^5+17x^3-15x)$$

$$g_4(x)=\frac{1}{92160}\,(79x^9+776x^7+1482x^5-1920x^3-945x)$$

Limiting Distribution

26.7.6

$$\lim_{\nu\to\infty} A(t|\nu)=\frac{1}{\sqrt{2\pi}}\int_{-t}^{t} e^{-x^2/2}dx=A(t)$$

Approximation for Large Values of t and $\nu\leq 5$

26.7.7 $$A(t|\nu)\approx 1-2\left\{\frac{a_\nu}{t^\nu}+\frac{b_\nu}{t^{\nu+1}}\right\}$$

ν	*1*	*2*	*3*	*4*	*5*
a_ν	.3183	.4991	1.1094	3.0941	9.948
b_ν	.0000	.0518	−.0460	−2.756	−14.05

Approximation for Large ν

26.7.8 $$A(t|\nu)\approx 2P(x)-1, \qquad x=\frac{t\left(1-\frac{1}{4\nu}\right)}{\sqrt{1+\frac{t^2}{2\nu}}}$$

Non-Central t-Distribution

26.7.9

$$P(t'|\nu,\delta)=$$

$$\frac{1}{\sqrt{\nu}B\left(\frac{1}{2},\frac{\nu}{2}\right)}\int_{-\infty}^{t'}\left(\frac{\nu}{\nu+x^2}\right)^{\frac{\nu+1}{2}} e^{-\frac{1}{2}\frac{\nu\delta^2}{\nu+x^2}}Hh_\nu\left(\frac{-\delta x}{\sqrt{\nu+x^2}}\right)dx$$

$$=1-\sum_{j=0}^{\infty} e^{-\delta^2/2}\,\frac{(\delta^2/2)^j}{2j!}\,I_x\left(\frac{\nu}{2},\frac{1}{2}+j\right), \qquad x=\frac{\nu}{\nu+t'^2} \; *$$

where δ is termed the non-centrality parameter.

Approximation to the Non-Central t-Distribution

26.7.10

$$P(t'|\nu,\delta)\approx P(x) \qquad \text{where } x=\frac{t'\left(1-\frac{1}{4\nu}\right)-\delta}{\left(1+\frac{t'^2}{2\nu}\right)^{\frac{1}{2}}}$$

Numerical Methods

26.8. Methods of Generating Random Numbers and Their Applications [9]

Random digits are digits generated by repeated independent drawings from the population 0, 1, 2, . . ., 9 where the probability of selecting any digit is one-tenth. This is equivalent to putting 10 balls, numbered from 0 to 9, into an urn and drawing one ball at a time, replacing the ball after each drawing. The recorded set of numbers forms a collection of random digits. Any group of n successive random digits is known as a *random number*.

Several lengthy tables of random digits are available (see references). However, the use of random numbers in electronic computers has resulted in a need for random numbers to be generated in a completely deterministic way. The numbers so generated are termed pseudo-random numbers. The quality of pseudo-random numbers is determined by subjecting the numbers to several statistical tests, see [26.55], [26.56]. The purpose of these statistical tests is to detect any properties of the pseudo-random numbers which are different from the (conceptual) properties of random numbers.

Experience has shown that the congruence method is the most preferable device for generating random numbers on a computer. Let the sequence of pseudo-random numbers be denoted by $\{X_n\}$, $n=0, 1, 2, \ldots$. Then the congruence method of generating pseudo-random numbers is

$$X_{n+1}=aX_n+b \pmod{T}$$

where b and T are relatively prime. The choice of T is determined by the capacity and base of the computer; a and b are chosen so that: (1) the resulting sequence $\{X_n\}$ possesses the desired statistical properties of random numbers, (2) the period of the sequence is as long as possible, and (3) the speed of generation is fast. A guide for choosing a and b is to make the correlation between the numbers be near zero, e.g., the correlation between X_n and X_{n+s} is

$$\rho_s=\frac{1-6\frac{b_s}{T}\left(1-\frac{b_s}{T}\right)}{a_s}+e$$

where

$$a_s=a^s \pmod{T}$$
$$b_s=(1+a+a^2+\ldots+a^{s-1})b \pmod{T}$$
$$|e|<a_s/T$$

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[9] The authors wish to express their appreciation to Professor J. W. Tukey who made many penetrating and helpful suggestions in this section.

*See page II.

which occur in

$$X_{n+s}=a_sX_n+b_s \pmod{T}$$

When a is chosen so that $a\approx T^{1/2}$, the correlation $\rho_1\approx T^{-1/2}$.

The sequence defined by the multiplicative congruence method will have a full period of T numbers if

(i) b is relatively prime to T
(ii) $a=1 \pmod{p}$ if p is a prime factor of T
(iii) $a=1 \pmod{4}$ if 4 is a factor of T.

Consequently if $T=2^q$, b need only be odd, and $a=1 \pmod{4}$. When $T=10^q$, b need only be not divisible by 2 or 5, and $a=1 \pmod{20}$. The most convenient choices for a are of the form $a=2^s+1$ (for binary computers) and $a=10^s+1$ (for decimal computers). This results in the fastest generation of random numbers as the operations only require a shift operation plus two additions. Also any number can serve as the starting point to generate a sequence of random digits. A good summary of generating pseudo-random numbers is [26.51].

Below are listed various congruence schemes and their properties.

Congruence methods for generating random numbers

$X_{n+1}=aX_n+b \pmod{T}$, T and b relatively prime

	a	b	T	Period	X_0	Special cases for which random numbers have passed statistical tests for randomness[10]
26.8.1	$1+t^s$	odd	$T=t^q$	t^q	$0\leq X_0<T$	$T=2^{35}$, X_0 unknown; $a=2^7+1$, $b=1$; $T=2^{47}$, $a=2^9+1$, $b=29741\ 09625\ 8473$, $X_0=76293\ 94531\ 25$.
26.8.2	$r2^s\pm1$ (r odd, $s\geq2$)	0	$T=t^q$	t^{q-s}	relatively prime to T	$T=2^{40}, 2^{43}$, $X_0=1$; $a=5^{17}(s=2)$ $T=2^{36}$, $X_0=1$; $T=2^{39}$, $X_0=1-2^{-39}$, .5478126193; $a=5^{13}(s=2)$ $T=2^{35}$, $X_0=1$; $a=5^{15}(s=2)$
26.8.3	$r2^s\pm1$ (r odd, $s\geq2$)	0	$T=t^q\pm1$	(varies)	relatively prime to T	$T=2^{35}+1$, $X_0=10{,}987{,}654{,}321$; $a=23$; period $\approx10^8$ $T=10^8+1$, $X_0=47{,}594{,}118$; $a=23$; period $\approx5.8\times10^6$
26.8.4	7^{4s+1}	0	$T=10^q$	$5\cdot10^{q-3}$	relatively prime to T	$T=10^{10}$, $X_0=1$; $a=7$ $T=10^{11}$, $X_0=1$; $a=7^{13}$
26.8.5	3^{4s+1} $(s=0, 2, 3, 4)$	0	$T=10^q$	$5\cdot10^{q-2}$	relatively prime to T	

[10] X_0 given is the starting point for random numbers when statistical tests were made.

When the numbers are generated using a congruence scheme, the least significant digits have short periods. Hence the entire word length cannot be used. If one desired random numbers with as many digits as possible, one would have to modify the congruence schemes. One way is to generate the numbers mod $T\pm1$. This unfortunately reduces the period. *

Generation of Random Deviates

Let $\{X\}$ be a generated sequence of independent random numbers having the domain $(0, T)$. Then $\{U\}=\{T^{-1}X\}$ is a sequence of random deviates (numbers) from a uniform distribution on the interval $(0, 1)$. This is usually a necessary preliminary step in the generation of random deviates having a given cumulative distribution function $F(y)$ or probability density function $f(y)$. Below are summarized some general techniques for producing arbitrary random deviates. (In what follows $\{U\}$ will always denote a sequence of random deviates from a uniform distribution on the interval $(0, 1)$.)

1. Inverse Method

The solutions $\{y\}$ of the equations $\{u=F(y)\}$ form a sequence of independent random deviates with cumulative distribution function $F(y)$. (If $F(y)$ has a discontinuity at $y=y_0$, then whenever u is such that $F(y_0-0)<u<F(y_0)$, select y_0 as the corresponding deviate.) Generally the inverse method is not practical unless the inverse function $y=F^{-1}(u)$ can be obtained explicitly or can be conveniently approximated.

2. Generating a Discrete Random Variable

Let Y be a discrete random variable with point probabilities $p_i=Pr\{Y=y_i\}$ for $i=1, 2, \ldots.$

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*See page II.

The direct way to generate Y is to generate $\{U\}$ and put $Y=y_i$ if

$$p_1+p_2+\ \dots\ +p_{i-1}<U<p_1+p_2+\ \dots\ +p_i.$$

However, this method requires complicated machine programs that take too long.

An alternative way due to Marsaglia [26.53] is simple, fast, and seems to be well suited to high-speed computations. Let p_i for $i=1, 2, \dots, n$ be expressed by k decimal digits as $p_i=.\delta_{1i}\delta_{2i}\ \dots\ \delta_{ki}$ where the δ's are the decimal digits. (If the domain of the random variable is infinite, it is necessary to truncate the probability distribution at p_n.) Define

$$P_0=0,\ P_r=10^{-r}\sum_{i=1}^{n}\delta_{ri} \text{ for } r=1, 2, \dots, k, \text{ and}$$

$$\Pi_s=\sum_{r=0}^{s}10^rP_r,\ s=1, 2, \dots, k.$$

Number the computer memory locations by $0, 1, 2, \dots, \Pi_k-1$. The memory locations are divided into k mutually exclusive sets such that the sth set consists of memory locations $\Pi_{s-1}, \Pi_{s-1}+1, \dots, \Pi_s-1$. The information stored in the memory locations of the sth set consists of y_1 in δ_{s1} locations, y_2 in δ_{s2} locations, $\dots$, y_n in δ_{sn} locations.

Denote the decimal expansion of the uniform deviates generated by the computer by $u=\cdot d_1d_2d_3\ \dots$ and finally let $a\{m\}$ be the contents of memory location m. Then if

$$\sum_{i=0}^{s-1}P_i\leq U<\sum_{i=0}^{s}P_i$$

put

$$y=a\left\{d_1d_2\ \dots\ d_s+\Pi_{s-1}-10^s\sum_{i=1}^{s-1}P_i\right\}.$$

This method is perhaps the best all-around method for generating random deviates from a discrete distribution. In order to illustrate this method consider the problem of generating deviates from the binomial distribution with point probabilities

$$p_i=\binom{n}{i}p^i(1-p)^{n-i}$$

for $n=5$ and $p=.20$. The point probabilities to 4 D are

Value of Random Variable	*Point Probabilities*
0	$p_0=0.3277$
1	$p_1=\ .4096$
2	$p_2=\ .2048$
3	$p_3=\ .0512$
4	$p_4=\ .0064$
5	$p_5=\ .0003$

and thus $P_0=0$, $P_1=.9$, $P_2=.07$, $P_3=.027$, $P_4=.0030$ from which $\Pi_0=0$, $\Pi_1=9$, $\Pi_2=16$, $\Pi_3=43$, $\Pi_4=73$. The 73 memory locations are divided into 4 mutually exclusive sets such that

Set	*Memory Locations*
1	$0, 1, \dots, 8$
2	$9, 10, \dots, 15$
3	$16, \dots, 42$
4	$43, \dots, 72$

Among the nine memory locations of set 1, zero is stored $\delta_{10}=3$ times, 1 is stored $\delta_{11}=4$ times, 2 is stored $\delta_{12}=2$ times; the seven locations of set 2 store 0 $\delta_{20}=2$ times and 3 $\delta_{23}=5$ times; etc. A summary of the memory locations is set out below:

	Value of Random Variable					
	0	1	2	3	4	5
Frequency (set 1)	3	4	2	0	0	0
Frequency (set 2)	2	0	0	5	0	0
Frequency (set 3)	7	9	4	1	6	0
Frequency (set 4)	7	6	8	2	4	3

Then to generate the random variables if

$0\leq u<.9$	put	$y=a\{d_1\}$
$.9\leq u<.97$		$y=a\{d_1d_2-81\}$
$.97\leq u<.997$		$y=a\{d_1d_2d_3-954\}$
$.997\leq u<1.000$		$y=a\{d_1d_2d_3d_4-9927\}$

3. Generating a Continuous Random Variable

The method for generating deviates from a discrete distribution can be adapted to random variables having a continuous distribution. Let $F(y)$ be the cumulative distribution function and assume that the domain of the random variable is (a,b) where the interval is finite. (If the domain is infinite, it must be truncated at (say) the points a and b.) Divide the interval $(b-a)$ into n sub-intervals of length Δ $(n\Delta=b-a)$ such that the boundary of the ith interval is (y_{i-1}, y_i) where $y_i=a+i\Delta$ for $i=0, 1, \dots, n$. Now define a discrete distribution having domain

$$\left\{z_i=\frac{y_i+y_{i-1}}{2}\right\}$$

with point probabilities $p_i=F(y_i)-F(y_{i-1})$. Finally, let W be a random variable having a uniform distribution on $\left(-\frac{\Delta}{2}, \frac{\Delta}{2}\right)$. This can be done by setting $W=\Delta\left(U-\frac{1}{2}\right)$. Then random deviates from the distribution function $F(y)$, can be generated (approximately) by setting $y=z+w=z+\Delta\left(u-\frac{1}{2}\right)$. This is simply an approximate decomposition of the continuous random variable into the sum of a discrete and continuous random variable. The discrete variable can be generated quickly by the method described previously. The smaller the value of Δ the better will be the approximation. Each number can be generated by using the leading digits of U to generate the discrete random variable Z and the remaining digits forming a uniformly distributed deviate having (0,1) domain.

4. Acceptance-Rejection Methods

In what follows the random variable Y will be assumed to have finite domain (a, b). If the domain is infinite, it must be truncated for computational purposes at (say) the points a and b. Then the resulting random deviates will only have this truncated domain.

a) Let f be the maximum of $f(y)$. Then the procedure for generating random deviates is: (1) generate a pair of uniform deviates U_1, U_2; (2) compute a point $y=a+(b-a)u_2$ in (a, b); (3) if $u_1<f(y)/f$ accept y as the random deviate, otherwise reject the pair (u_1, u_2) and start again. The acceptance ratio of deviates actually produced is $[(b-a)f]^{-1}$. Hence the acceptance ratio decreases as the domain increases. One way to increase the acceptance ratio is to divide the interval (a, b) into mutually exclusive sub-intervals and then carry out the acceptance-rejection process. For this purpose let the interval (a, b) be divided into k sub-intervals such that the end points of the jth interval are (ξ_{j-1}, ξ_j) with $\xi_0=a$, $\xi_k=b$ and $\int_{\xi_{j-1}}^{\xi_j} f(y)dy=p_j$; further let the maximum of $f(y)$ in the jth interval be f_j. Then to generate random deviates from $f(y)$, generate n pairs of deviates $(u_{1s}, u_{2s})s=1, 2, \ldots, n$. Assign $[np_j]$ such pairs to the jth interval and compute $y_j=\xi_{j-1}+(\xi_j-\xi_{j-1})u_{2s}$. If $u_{1s}<f(y_j)/f_j$ accept y_j as a deviate. The acceptance ratio of this method is

$$\sum_{j=1}^{k} p_j\,[(\xi_j-\xi_{j-1})f_j]^{-1}$$

b) Let $F(y)$ be such that $f(y)=f_1(y)f_2(y)$ where the domain of y is (a, b). Let f_1 and f_2 be the maximum of $f_1(y)$ and $f_2(y)$ respectively. Then the procedure for generating random deviates having the probability density function $f(y)$ is: (1) generate U_1, U_2, U_3; (2) define $z=a+(b-a)u_3$; (3) if both $u_1<\frac{f_1(z)}{f_1}$ and $u_2<\frac{f_2(z)}{f_2}$, take z as the random deviate; otherwise take another sample of three uniform deviates. The acceptance ratio of this method is $[(b-a)f_1f_2]^{-1}$ and can be increased by dividing (a, b) into sub-intervals as in the previous case.

c) Let the probability density function of Y be

$$f(y)=\int_\alpha^\beta g(y, t)dt,\ (\alpha\leq t\leq\beta),\ (a\leq y\leq b).$$

Let g be the maximum of $g(y, t)$. Then the procedure for generating random deviates having the probability density function $f(y)$ is: (1) generate U_1, U_2, U_3; (2) define $s=\alpha+(\beta-\alpha)u_2$; $z=a+(b-a)u_3$; (3) if $u_1<\frac{g(z, s)}{g}$, take z as the random deviate; otherwise take another sample of three. The acceptance ratio for this method is $[(b-a)g]^{-1}$ and can be increased by dividing the domain of t and y into sub-domains.

5. Composition Method

Let $g_z(y)$ be a probability density function which depends on the parameter z; further let $H(z)$ be the cumulative distribution function for z. In order to generate random deviates Y having the frequency function

$$f(y)=\int_{-\infty}^{\infty} g_z(y)dH(z)$$

one draws a deviate having the cumulative distribution function $H(z)$; then draws a second sample having the probability density function $g_z(y)$.

6. Generation of Random Deviates From Well Known Distributions

a. Normal distribution

(1) *Inverse method:* The inverse method depends on having a convenient approximation to the inverse function $x=P^{-1}(u)$ where

$$u=(2\pi)^{-1/2}\int_{-\infty}^{x} e^{-t^2/2}dt.$$

Two ways of performing this operation are to (i) use **26.2.23** with $t=\left(\ln\frac{1}{u^2}\right)^{1/2}$ or (ii) approximate $x=P^{-1}(u)$ piecewise using Chebyshev polynomials, see [26.54].

(2) *Sum of uniform deviates:* Let U_1, U_2, . . ., U_n be a sequence of n uniform deviates. Then

$$X_n=\left(\sum_{i=1}^{n} U_i-\frac{n}{2}\right)\left(\frac{n}{12}\right)^{-1/2}$$

will be distributed asymptotically as a normal random deviate. When $n=12$, the maximum errors made in the normal deviate are 9×10^{-3} for $|X|<2$, 9×10^{-1} for $2<|X|<3$. An improvement can be made by taking a polynomial function of X_n (say)

$$X_n^*=X_n\sum_{s=0}^{k} a_{2s}X_n^{2s}$$

as the normal deviate where a_{2s} are suitable coefficients. These coefficients may be calculated using (say) Chebyshev polynomials or simply by making the asymptotic random deviate agree with the correc normal deviate at certain specified points. When $n=12$, the maximum error in the normal deviate is 8×10^{-4} using the coefficients

* $a_0=9.8746$ * $a_6=(-7)-5.102$
* $a_2=(-3)3.9439$ * $a_8=(-7)1.141$
* $a_4=(-5)7.474$

(3) *Direct method:* Generate a pair of uniform deviates (U_1, U_2). Then

$$X_1=(-2\ln U_1)^{1/2}\cos 2\pi U_2,$$

$X_2=(-2\ \ln\ U_1)^{1/2}\sin\ 2\pi U_2$ will be a pair of independent normal random deviates with mean zero and unit variance. This procedure can be modified by calculating $\cos 2\pi U$ and $\sin 2\pi U$ using an acceptance rejection method; e.g., (1) generate (U_1, U_2); (2) if $(2U_1-1)^2+(2U_2-1)^2\leq1$ generate a third uniform deviate U_3, otherwise reject the pair and start over; (3) calculate $y_1=(-\ln u_3)^{1/2}\frac{u_1^2-u_2^2}{u_1^2+u_2^2}$, $y_2=\pm2(-\ln u_3)^{1/2}\frac{u_1u_2}{u_1^2+u_2^2}$ ($\pm$ random). Both y_1 and y_2 are the desired random deviates.

(4) *Acceptance-rejection method:* 1) Generate a pair of uniform deviates (U_1, U_2); 2) compute * $x=-\ln u_1$; 3) if $e^{-\frac{1}{2}(x-1)^2}\geq u_2$ (or equivalently $(x-1)^2\leq-2(\ln u_2)$ accept x, otherwise reject the pair and start over. The quantity will be the required normal deviate with mean zero and unit variance.

b. Bivariate normal distribution

Let $\{X_1, X_2\}$ be a pair of independent normal deviates with mean zero and unit variance. Then $\{X_1, \rho X_1+(1-\rho^2)^{1/2}X_2\}$ represent a pair of deviates from a bivariate normal distribution with zero means, unit variances, and correlation coefficient ρ.

c. Exponential distribution

(1) *Inverse method:* Since $F(x)=e^{-x/\theta}$, $X=-\theta\ln U$ will be a deviate from the exponential distribution with parameter θ.

(2) *Acceptance-rejection method:* 1) Generate a pair of independent uniform deviates (U_0, U_1); 2) if $U_1<U_0$ generate a third value U_2; 3) if $U_1+U_2<U_0$ generate a fourth value U_3, etc.; 4) continue generating uniform deviates until an n is obtained such that $U_1+U_2+\ .\ .\ .\ +U_{n-1}<U_0<U_1+\ .\ .\ .\ +U_n$; 5) if n is even reject the procedure and start a fresh trial with a new value of U_0, otherwise if n is odd take $X=\theta U_0$ as the desired deviate; 6) in general if t is the number of trials until an acceptable sequence is obtained $X=\theta(t+U_0)$. The random deviates produced in this way follow an exponential distribution with parameter θ. One can expect to generate approximately six uniform deviates for every exponential deviate.

(3) *Discrete Distribution Method:* Let Y and n be discrete random variables with point probabilities

* $Pr\{Y=r\}=(e-1)e^{-(r+1)}\quad r=0, 1, 2, \ .\ .\ .$
$Pr\{n=s\}=[s!(e-1)]^{-1}\quad s=1, 2, 3, \ .\ .\ .\ .$

Then $X=Y+\min(U_1, U_2, \ .\ .\ .\ , U_n)$ will follow an exponential distribution. The average value of n is 1.58 so that one needs, on the average, only 1.58 u's from which the minimum is selected.

26.9. Use and Extension of the Tables

Use of Probability Function Inequalities

Example 1. Let X be a random variable with finite mean and variance equal to m and σ^2, respectively. Use the inequalities for probability functions **26.1.37, 40, 41** to place lower bounds on

$$A(t)=F(t)-F(-t)=P\left\{\frac{|X-m|}{\sigma}\leq t\right\}$$

for $t=1(1)4$.

Lower bounds on $A(t)=F(t)-F(-t)$

$t=1$	2	3	4	Remarks
0	.7500	.8889	.9375	no knowledge of $F(t)$; **26.1.37**
.5556	.8889	.9506	.9722	$F(t)$ is unimodal and continuous; **26.1.40**
0	.8182	.9697	.9912	$F(t)$ is such that $\mu_4=3$; **26.1.41**

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*See page II.

It is of interest to note that the standard normal distribution is unimodal, has mean zero, unit variance $\mu_4=3$, is continuous, and such that

$$A(t)=P(t)-P(-t)$$
$$=.6827,\ .9545,\ .9973, \text{ and } .9999$$

for $t=1$, 2, 3 and 4 respectively.

Interpolation for $P(x)$ in Table 26.1

Example 2. Compute $P(x)$ for $x=2.576$ to fifteen decimal places using a Taylor expansion.

Writing $x=x_0+\theta$ we have

$$P(x)=P(x_0)+Z(x_0)\theta+Z^{(1)}(x_0)\frac{\theta^2}{2!}$$
$$+Z^{(2)}(x_0)\frac{\theta^3}{3!}+Z^{(3)}(x_0)\frac{\theta^4}{4!}+\ \ldots$$

Taking $x_0=2.58$ and $\theta=-4\times10^{-3}$ we calculate the successive terms to 16D

+.99505	99842	42230		
−	5	72204	35976	6
−		2952	57449	6
−		8	63097	8
−			1439	4
−				9
.99500	24676	84265	7	

The result correct to 17D is

$$P(2.576)=.99500\ \ 24676\ \ 84264\ \ 98$$

Calculation for Arbitrary Mean and Variance

Example 3. Find the value to 5D of

$$P\{X\leq .50\}=\frac{1}{2\sqrt{2\pi}}\int_{-\infty}^{.5}e^{-1/2\left(\frac{t-1}{2}\right)^2}dt$$

using **26.2.8** and **Table 26.1.**

This represents the probability of the random variable being less than or equal to .5 for a normal distribution with mean $m=1$ and variance $\sigma^2=4$. Using **26.2.8** we have

$$P\{X\leq .5\}=P\left(\frac{.5-1}{2}\right)=P(-.25)$$

Since $P(-x)=1-P(x)$, we have

$$P(-.25)=1-P(.25)=1-.59871=.40129$$

where a two-term Taylor series was used for interpolation. Note that when interpolating for $P(x)$ for a value of x midway between the tabulated values we can write $x=x_0+.01$ and a two-term Taylor series is $P(x)=P(x_0)+Z(x_0)10^{-2}$. Thus one need only multiply $Z(x_0)$ by 10^{-2} and add the result to $P(x_0)$.

Calculation of $P(x)$ for x Approximate

Example 4. Using **Table 26.1,** find $P(x)$ for $x=1.96$, when there is a possible error in x of $\pm5\times10^{-3}$.

This is an example where the argument is only known approximately. The question arises as to how many decimal places one should retain in $P(x)$. If Δx and $\Delta P(x)$ denote the error in x and the resulting error in $P(x)$, respectively, then

$$\Delta P(x)\approx Z(x)\Delta x$$

Hence $\Delta P(1.960)=3\times10^{-4}$ which indicates that $P(1.960)$ need only be calculated to 4D. Therefore $P(1.960)=.9750$.

Inverse Interpolation for $P(x)$

Example 5. Find the value of x for which $P(x)=.97500\ 00000\ 00000$ using **Table 26.1** and determining as many decimal places as is consistent with the tabulated function.

For inverse interpolation the tabulated function $P(x)$ may be regarded as having a possible error of $.5\times10^{-15}$. Hence

$$\Delta x\approx\frac{\Delta P(x)}{Z(x)}=\frac{.5\times10^{-15}}{Z(x)}$$

Let $P(x_0)$ correspond to the closest tabulated value of $P(x)$. Then a convenient formula for inverse interpolation is

$$x=x_0+t+\frac{x_0t^2}{2}+\frac{2x_0^2+1}{6}\ t^3$$

where

$$t=\frac{P(x)-P(x_0)}{Z(x_0)}$$

If only the first two terms (i.e., $x=x_0+t$) are used, the error in x will be bounded by $\frac{x}{8}\times10^{-4}$ and the true value will always be greater than the value thus calculated.

With respect to this example, $\Delta x\approx10^{-14}$ and thus the interpolated value of x may be in error by one unit in the fourteenth place. The closest value to $P(x)=.97500\ 00000\ 00000$ is $P(x_0)=.97500\ 21048\ 51780$ with $x_0=1.96$. Hence using the preceding inverse interpolation formulas with

$$t=-.00003\ 60167\ 31129$$

and carrying fifteen decimals we have the successive terms

$$\begin{array}{rrr}
+1.96000 & 00000 & 00000 \\
-\ \ .00003 & 60167 & 31129 \\
+\qquad\quad & 12 & 71261 \\
-\qquad\quad & & 68 \\
 & & 0 \\
\hline
+1.95996 & 39845 & 40064
\end{array}$$

Edgeworth Asymptotic Expansion

Example 6. Find the Edgeworth asymptotic expansion **26.2.49** for the c.d.f. of chi-square.

Method 1. Expansion for χ^2

Let

$$Q(\chi^2|\nu)=1-F(t)$$

where

$$t=\frac{\chi^2-\nu}{(2\nu)^{\frac{1}{2}}}$$

Since the values of γ_1 and γ_2 **26.4.33** are

$$\gamma_1=2\sqrt{2}/\nu^{\frac{1}{2}}$$

$$\gamma_2=12/\nu,$$

we obtain, by using the first two bracketed terms of **26.2.49**

$$F(t)\sim P(t)-\frac{1}{\nu^{\frac{1}{2}}}\left[\frac{\sqrt{2}}{3}Z^{(2)}(t)\right]$$
$$+\frac{1}{\nu}\left[\frac{1}{2}Z^{(3)}(t)+\frac{1}{9}Z^{(5)}(t)\right]$$

The Edgeworth expansion is an asymptotic expansion in terms of derivatives of the normal distribution function. It is often possible to transform a random variable so that the distribution of the transformed random variable more closely approximates the normal distribution function than does the distribution of the original random variable. Hence for the same number of terms, greater accuracy may be achieved by using the transformed variable in the expansion. Since the distribution of $\sqrt{2\chi^2}$ is more closely approximated by a normal distribution than χ^2 itself (as judged by a comparison of the values of γ_1 and γ_2), we would expect that the Edgeworth asymptotic expansion of $\sqrt{2\chi^2}$ would be superior to that of χ^2.

Method 2. Expansion for $\sqrt{2\chi^2}$. Let

$$Q(\chi^2|\nu)=1-F(t)=1-F\left(\frac{\sqrt{2\chi^2}-(2\nu-1)^{\frac{1}{2}}}{\left(1-\frac{1}{4\nu}\right)^{\frac{1}{2}}}\right)$$

where $(2\nu-1)^{\frac{1}{2}}$ and $1-\frac{1}{4\nu}$ are the mean and variance to terms of order ν^{-2} of $\sqrt{2\chi^2}$ (see **26.4.34**). The values of γ_1 and γ_2 for $\sqrt{2\chi^2}$ are

$$\gamma_1\approx\frac{1}{\sqrt{2\nu}}\left[1+\frac{5}{8\nu}\right]\qquad\qquad\gamma_2\approx\frac{3}{4\nu^2}$$

Thus we obtain

$$F(t)\sim P(t)-\frac{1}{\nu^{\frac{1}{2}}}\left[\frac{\sqrt{2}}{12}\left(1+\frac{5}{8\nu}\right)Z^{(2)}(t)\right]$$
$$+\frac{1}{\nu}\left[\frac{1}{32\nu}Z^{(3)}(t)+\frac{1}{144}\left(1+\frac{5}{8\nu}\right)^2Z^{(5)}(t)\right]$$

For numerical examples using these expansions see **Example 12.**

Calculation of $L(h, k, \rho)$

Example 7. Find $L(.5, .4, .8)$. Using **26.3.20**

$$\sqrt{h^2-2\rho hk+k^2}=\sqrt{.09}=.3$$

$$L(.5, .4, .8)=L(.5, 0, 0)+L(.4, 0, -.6)$$

Reference to **Figure 26.2** yields

$$L(.5, 0, 0)+L(.4, 0, -.6)=.16+.08=.24$$

The answer to 3D is $L(.5, .4, .8)=.250$.

Calculation of the Bivariate Normal Probability Function

Example 8. Let X and Y follow a bivariate normal distribution with parameters $m_x=3$, $m_y=2$, $\sigma_x=4$, $\sigma_y=2$, and $\rho=-.125$. Find the value of $P_r\{X\geq 2,\ Y\geq 4\}$ using **26.3.20** and **Figures 26.2, 26.3.**

Since $P_r\{X\geq h, Y\geq k\}=L\left(\frac{h-m_x}{\sigma_x}, \frac{k-m_y}{\sigma_y}, \rho\right)$ we have $P\{X\geq 2,\ Y\geq 4\}=L(-.25, 1, -.125)$. Using **26.3.20**

$$L(-.25, 1, -.125)=L(-.25, 0, .969)$$
$$+L(1, 0, .125)-\frac{1}{2}$$

Figure 26.2 only gives values for $h>0$, however, using the relationship **26.3.8** with $k=0$, $L(-h, 0, \rho)$ $=\frac{1}{2}-L(\mathrm{h}, 0, -\rho)$ and thus $L(-.25, 0, .969)$ $=\frac{1}{2}-L(.25, 0, -.969)$. Therefore $L(-.25, 1, -.125)$ $=-L(.25, 0, -.969)+L(1, 0, .125)=-.01+.09=.08.$

The answer to 3D is $L(-.25,\ 1.,\ -.125)=.080.$

Approximating $Q(F|\nu_1, \nu_2)$

Example 23. Calculate $Q(1.714|10, 40)$ using **26.6.15.**

The approximation given by **26.6.15** will result in a maximum error of .0005. For this example we have

$$x=\frac{(1.714)^{1/3}\left(1-\frac{2}{9(40)}\right)-\left(1-\frac{2}{9(10)}\right)}{\left[\frac{2}{9(10)}+(1.714)^{2/3}\frac{2}{9(40)}\right]^{\frac{1}{2}}}=1.2222$$

Interpolating in **Table 26.1** results in

$$Q(1.714|10, 40)\approx Q(1.2222)=1-P(1.2222)=.1108$$

The correct value to 5D is $Q(1.714|10, 40)=.11108$.

On the other hand the approximation given by **26.6.14** which is usually less accurate results in

$$x=\frac{\sqrt{[2(40)-1]\left(\frac{10}{40}\right)(1.714)}-\sqrt{2(10)-1}}{\sqrt{1+\frac{10}{40}(1.714)}}=1.2210$$

and interpolating in **Table 26.1** gives

$$Q(1.714|10, 40)\approx Q(1.2210)=1-P(1.2210)=.1112$$

Calculation of F Outside the Range of Table 26.9

Example 24. Find the value of F for which $Q(F|10, 20)\approx .0001$ using **26.6.16** and **26.5.22.**

For this problem we have $a=\frac{\nu_2}{2}=10$, $b=\frac{\nu_1}{2}=5$, $p=.0001$. The value of the normal deviate which cuts off .0001 in the tail of the distribution is $y=3.7190$ (i.e., $Q(3.7190)=.0001$). Hence substituting in **26.5.22** gives

$$h=2\left[\frac{1}{19}+\frac{1}{9}\right]^{-1}=12.2143$$

$$\lambda=\frac{3.7190^2-3}{6}=1.8052$$

$$w=3.7190\,\frac{(12.2143+1.8052)^{\frac{1}{2}}}{12.2143}-\left(\frac{1}{9}-\frac{1}{19}\right)\left[1.8052+.8333-\frac{2}{3(12.2143)}\right]$$

$$w=.9889$$

and thus $F\approx e^{2w}=7.23$. The correct answer is $F=7.180$.

Approximating the Non-Central F-Distribution

Example 25. Compute $P(3.71|3, 10, 4)$ using the approximation **26.6.27** to the non-central F-distribution.

Using **26.6.27** with $\nu_1=3$, $\nu_2=10$, $\lambda=4$, $F'=3.71$ we have

$$x=\frac{\left[\left(\frac{3}{3+4}\right)(3.71)\right]^{1/3}\left[1-\frac{2}{9(10)}\right]-\left[1-\frac{2}{9}\frac{(3+8)}{(3+4)^2}\right]}{\left[\frac{2}{9}\frac{3+8}{(3+4)^2}+\frac{2}{9(10)}\left[\left(\frac{3}{3+4}\right)(3.71)\right]^{2/3}\right]^{\frac{1}{2}}}=.675$$

and interpolating in **Table 26.1** gives

$$P(3.71|3,10,4)\approx P(.675)=.750$$

The exact answer is $P(3.71|3, 10, 4)=.745$.

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Normal Probability Integral and Derivatives

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Chi-Square, Non-Central Chi-Square, Probability Integral, Incomplete Gamma Function, Poisson Distribution

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Incomplete Beta Function, Binomial Distribution

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F (Variance-Ratio) and Non-Central F Distribution

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27. Miscellaneous Functions

IRENE A. STEGUN [1]

Contents

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[1] National Bureau of Standards.

27. Miscellaneous Functions

27.1. Debye Functions

Series Representations

27.1.1

$$\int_0^x \frac{t^n dt}{e^t-1}=x^n[\frac{1}{n}-\frac{x}{2(n+1)}+\sum_{k=1}^{\infty}\frac{B_{2k}x^{2k}}{(2k+n)(2k)!}]$$

$$(|x|<2\pi, n\geq 1)$$

(For Bernoulli numbers B_{2k}, see chapter **23.**)

27.1.2

$$\int_x^{\infty}\frac{t^n dt}{e^t-1}=\sum_{k=1}^{\infty}e^{-kx}[\frac{x^n}{k}+\frac{nx^{n-1}}{k^2}+\frac{(n)(n-1)x^{n-2}}{k^3}+\ldots+\frac{n!}{k^{n+1}}](x>0, n\geq 1)$$

Relation to Riemann Zeta Function (see chapter 23)

27.1.3

$$\int_0^{\infty}\frac{t^n dt}{e^t-1}=n!\zeta(n+1).$$

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$\frac{3}{x^3}\int_0^x\frac{y^3dy}{e^y-1}, \frac{12}{x^3}[\int_0^x\frac{y^3dy}{e^y-1}-\frac{3x}{e^x-1}]$, $x=0(.01)24$, 6S.

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$\frac{20}{x^4}\int_0^x\frac{t^4dt}{e^t-1}-\frac{4x}{e^x-1}$,

$x=0(.1)13(.2)18(1)20(2)52(4)80$, 4S.

Table 27.1 **Debye Functions**

x	$\frac{1}{x}\int_0^x\frac{tdt}{e^t-1}$	$\frac{2}{x^2}\int_0^x\frac{t^2dt}{e^t-1}$	$\frac{3}{x^3}\int_0^x\frac{t^3dt}{e^t-1}$	$\frac{4}{x^4}\int_0^x\frac{t^4dt}{e^t-1}$
0. 0	1. 000000	1. 000000	1. 000000	1. 000000
0. 1	0. 975278	0. 967083	0. 963000	0. 960555
0. 2	0. 951111	0. 934999	0. 926999	0. 922221
0. 3	0. 927498	0. 903746	0. 891995	0. 884994
0. 4	0. 904437	0. 873322	0. 857985	0. 848871
0. 5	0. 881927	0. 843721	0. 824963	0. 813846
0. 6	0. 859964	0. 814940	0. 792924	0. 779911
0. 7	0. 838545	0. 786973	0. 761859	0. 747057
0. 8	0. 817665	0. 759813	0. 731759	0. 715275
0. 9	0. 797320	0. 733451	0. 702615	0. 684551
1. 0	0. 777505	0. 707878	0. 674416	0. 654874
1. 1	0. 758213	0. 683086	0. 647148	0. 626228
1. 2	0. 739438	0. 659064	0. 620798	0. 598598
1. 3	0. 721173	0. 635800	0. 595351	0. 571967
1. 4	0. 703412	0. 613281	0. 570793	0. 546317
1. 6	0. 669366	0. 570431	0. 524275	0. 497882
1. 8	0. 637235	0. 530404	0. 481103	0. 453131
2. 0	0. 606947	0. 493083	0. 441129	0. 411893
2. 2	0. 578427	0. 458343	0. 404194	0. 373984
2. 4	0. 551596	0. 426057	0. 370137	0. 339218
2. 6	0. 526375	0. 396095	0. 338793	0. 307405
2. 8	0. 502682	0. 368324	0. 309995	0. 278355
3. 0	0. 480435	0. 342614	0. 283580	0. 251879
3. 2	0. 459555	0. 318834	0. 259385	0. 227792
3. 4	0. 439962	0. 296859	0. 237252	0. 205915
3. 6	0. 421580	0. 276565	0. 217030	0. 186075
3. 8	0. 404332	0. 257835	0. 198571	0. 168107
4. 0	0. 388148	0. 240554	0. 181737	0. 151855
4. 2	0. 372958	0. 224615	0. 166396	0. 137169
4. 4	0. 358696	0. 209916	0. 152424	0. 123913
4. 6	0. 345301	0. 196361	0. 139704	0. 111957
4. 8	0. 332713	0. 183860	0. 128129	0. 101180
5. 0	0. 320876	0. 172329	0. 117597	0. 091471
5. 5	0. 294240	0. 147243	0. 095241	0. 071228
6. 0	0. 271260	0. 126669	0. 077581	0. 055677
6. 5	0. 251331	0. 109727	0. 063604	0. 043730
7. 0	0. 233948	0. 095707	0. 052506	0. 034541
7. 5	0. 218698	0. 084039	0. 043655	0. 027453
8. 0	0. 205239	0. 074269	0. 036560	0. 021968
8. 5	0. 193294	0. 066036	0. 030840	0. 017702
9. 0	0. 182633	0. 059053	0. 026200	0. 014368
9. 5	0. 173068	0. 053092	0. 022411	0. 011747
10. 0	0. 164443	0. 047971	0. 019296	0. 009674
	$\left[{(-4)5 \atop 5}\right]$	$\left[{(-4)6 \atop 5}\right]$	$\left[{(-4)6 \atop 5}\right]$	$\left[{(-4)6 \atop 5}\right]$

Planck's Radiation Function

Table 27.2

$$f(x)=x^{-5}(e^{1/x}-1)^{-1}$$

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0. 050	0. 007	0. 10	4. 540	0. 20	21. 199	0. 40	8. 733	0. 9	0. 831
0. 055	0. 025	0. 11	6. 998	0. 22	20. 819	0. 45	6. 586	1. 0	0. 582
0. 060	0. 074	0. 12	9. 662	0. 24	19. 777	0. 50	5. 009	1. 1	0. 419
0. 065	0. 179	0. 13	12. 296	0. 26	18. 372	0. 55	3. 850	1. 2	0. 309
0. 070	0. 372	0. 14	14. 710	0. 28	16. 809	0. 60	2. 995	1. 3	0. 233
0. 075	0. 682	0. 15	16. 780	0. 30	15. 224	0. 65	2. 356	1. 4	0. 178
0. 080	1. 137	0. 16	18. 446	0. 32	13. 696	0. 70	1. 875	1. 5	0. 139
0. 085	1. 752	0. 17	19. 692	0. 34	12. 270	0. 75	1. 508	2. 0	0. 048
0. 090	2. 531	0. 18	20. 539	0. 36	10. 965	0. 80	1. 225	2. 5	0. 021
0. 095	3. 466	0. 19	21. 025	0. 38	9. 787	0. 85	1. 005	3. 0	0. 010
0. 100	4. 540	0. 20	21. 199	0. 40	8. 733	0. 90	0. 831	3. 5	0. 006
	$\begin{bmatrix}(-2)2\\4\end{bmatrix}$		$\begin{bmatrix}(-2)5\\5\end{bmatrix}$		$\begin{bmatrix}(-2)8\\5\end{bmatrix}$		$\begin{bmatrix}(-2)7\\5\end{bmatrix}$		$\begin{bmatrix}(-2)1\\4\end{bmatrix}$

$x_{\max}=.20140\ 52353$ $f(x_{\max})=21.20143\ 58.$

[27.3] Miscellaneous Physical Tables, Planck's radiation functions and electronic functions, MT 17 (U.S. Government Printing Office, Washington, D.C., 1941).

$$R_\lambda=c_1\lambda^{-5}(e^{c_2/\lambda T}-1)^{-1},\ R_{0-\lambda}=\int_0^\lambda R_\lambda d\lambda,$$

$$N_\lambda=2\pi c\lambda^{-4}(e^{c_2/\lambda T}-1)^{-1},\ N_{0-\lambda}=\int_0^\lambda N_\lambda d\lambda$$

Table I: $\frac{R_\lambda}{R_{\lambda\ \max}}$, $\frac{R_{0-\lambda}}{R_{0-\infty}}$, $\frac{N_\lambda}{N_{\lambda\ \max}}$, $\frac{N_{0-\lambda}}{N_{0-\infty}}$ for $\lambda T=[.05(.001).1(.005).4(.01).6(.02)1(.05)2]$cm $K°$.

Table II: R_λ, $R_{0-\lambda}$, N_λ, $N_{0-\lambda}$ ($T=1000°$ K) for $\lambda=[.5(.01)1(.05)4(.1)6(.2)10(.5)20]$ microns.

Table III: N_λ for $\lambda=[.25(.05)1.6(.2)3(1)10]$ microns, $T=[1000°(500°)3500°$ K and $6000°$ K].

Einstein Functions

Table 27.3

x	$\frac{x^2e^x}{(e^x-1)^2}$	$\frac{x}{e^x-1}$	$\ln(1-e^{-x})$	$\frac{x}{e^x-1}$ $-\ln(1-e^{-x})$
0. 00	1. 00000	1. 00000	$-\infty$	∞
0. 05	0. 99979	0. 97521	−3. 02063	3. 99584
0. 10	0. 99917	0. 95083	−2. 35217	3. 30300
0. 15	0. 99813	0. 92687	−1. 97118	2. 89806
0. 20	0. 99667	0. 90333	−1. 70777	2. 61110
0. 25	0. 99481	0. 88020	−1. 50869	2. 38888
0. 30	0. 99253	0. 85749	−1. 35023	2. 20771
0. 35	0. 98985	0. 83519	−1. 21972	2. 05491
0. 40	0. 98677	0. 81330	−1. 10963	1. 92293
0. 45	0. 98329	0. 79182	−1. 01508	1. 80690
0. 50	0. 97942	0. 77075	−0. 93275	1. 70350
0. 55	0. 97517	0. 75008	−0. 86026	1. 61035
0. 60	0. 97053	0. 72982	−0. 79587	1. 52569
0. 65	0. 96552	0. 70996	−0. 73824	1. 44820
0. 70	0. 96015	0. 69050	−0. 68634	1. 37684
0. 75	0. 95441	0. 67144	−0. 63935	1. 31079
0. 80	0. 94833	0. 65277	−0. 59662	1. 24939
0. 85	0. 94191	0. 63450	−0. 55759	1. 19209
0. 90	0. 93515	0. 61661	−0. 52184	1. 13844
0. 95	0. 92807	0. 59910	−0. 48897	1. 08809
1. 00	0. 92067	0. 58198	−0. 45868	1. 04065
1. 05	0. 91298	0. 56523	−0. 43069	0. 99592
1. 10	0. 90499	0. 54886	−0. 40477	0. 95363
1. 15	0. 89671	0. 53285	−0. 38073	0. 91358
1. 20	0. 88817	0. 51722	−0. 35838	0. 87560
1. 25	0. 87937	0. 50194	−0. 33758	0. 83952
1. 30	0. 87031	0. 48702	−0. 31818	0. 80520
1. 35	0. 86102	0. 47245	−0. 30008	0. 77253
1. 40	0. 85151	0. 45824	−0. 28315	0. 74139
1. 45	0. 84178	0. 44436	−0. 26732	0. 71168
1. 50	0. 83185	0. 43083	−0. 25248	0. 68331
	$\begin{bmatrix}(-5)5\\3\end{bmatrix}$	$\begin{bmatrix}(-5)5\\3\end{bmatrix}$		

Table 27.3

Einstein Functions

x	$\frac{x^2e^x}{(e^x-1)^2}$	$\frac{x}{e^x-1}$	$\ln(1-e^{-x})$	$\frac{x}{e^x-1}$ $-\ln(1-e^{-x})$
1. 6	0. 81143	0. 40475	−0. 22552	0. 63027
1. 7	0. 79035	0. 37998	−0. 20173	0. 58171
1. 8	0. 76869	0. 35646	−0. 18068	0. 53714
1. 9	0. 74657	0. 33416	−0. 16201	0. 49617
2. 0	0. 72406	0. 31304	−0. 14541	0. 45845
2. 1	0. 70127	0. 29304	−0. 13063	0. 42367
2. 2	0. 67827	0. 27414	−0. 11744	0. 39158
2. 3	0. 65515	0. 25629	−0. 10565	0. 36194
2. 4	0. 63200	0. 23945	−0. 09510	0. 33455
2. 5	0. 60889	0. 22356	−0. 08565	0. 30921
2. 6	0. 58589	0. 20861	−0. 07718	0. 28578
2. 7	0. 56307	0. 19453	−0. 06957	0. 26410
2. 8	0. 54049	0. 18129	−0. 06274	0. 24403
2. 9	0. 51820	0. 16886	−0. 05659	0. 22545
3. 0	0. 49627	0. 15719	−0. 05107	0. 20826
3. 2	0. 45363	0. 13598	−0. 04162	0. 17760
3. 4	0. 41289	0. 11739	−0. 03394	0. 15133
3. 6	0. 37429	0. 10113	−0. 02770	0. 12883
3. 8	0. 33799	0. 08695	−0. 02262	0. 10958
4. 0	0. 30409	0. 07463	−0. 01849	0. 09311
4. 2	0. 27264	0. 06394	−0. 01511	0. 07905
4. 4	0. 24363	0. 05469	−0. 01235	0. 06705
4. 6	0. 21704	0. 04671	−0. 01010	0. 05681
4. 8	0. 19277	0. 03983	−0. 00826	0. 04809
5. 0	0. 17074	0. 03392	−0. 00676	0. 04068
5. 2	0. 15083	0. 02885	−0. 00553	0. 03438
5. 4	0. 13290	0. 02450	−0. 00453	0. 02903
5. 6	0. 11683	0. 02078	−0. 00370	0. 02449
5. 8	0. 10247	0. 01761	−0. 00303	0. 02065
6. 0	0. 08968	0. 01491	−0. 00248	0. 01739
	$\left[\begin{smallmatrix}(-4)3\\4\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-4)3\\4\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-4)4\\4\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-4)6\\4\end{smallmatrix}\right]$

[27.4] H. L. Johnston, L. Savedoff and J. Belzer, Contributions to the thermodynamic functions by a Planck-Einstein oscillator in one degree of freedom, NAVEXOS p. 646, Office of Naval Research, Department of the Navy, Washington, D.C. (1949). Values of $x^2e^x(e^x-1)^{-2}$, $x(e^x-1)^{-1}$, $-\ln(1-e^{-x})$ and $x(e^x-1)^{-1}-\ln(1-e^{-x})$ for $x=0(.001)3(.01)\ 14.99$, 5D with first differences.

27.4. Sievert Integral

$$\int_0^\theta e^{-x\sec\phi}d\phi$$

Relation to the Error Function

27.4.1

$$\int_0^\theta e^{-x\sec\phi}d\phi\sim\sqrt{\frac{\pi}{2x}}\,e^{-x}\operatorname{erf}\left(\sqrt{\frac{x}{2}}\,\theta\right)\qquad(x\to\infty)$$

(For erf, see chapter **7.**)

Representation in Terms of Exponential Integrals

27.4.2

$$\int_0^\theta e^{-x\sec\phi}d\phi=\int_0^{\frac{\pi}{2}}e^{-x\sec\phi}d\phi-\sum_{k=0}^{\infty}\alpha_k(\cos\theta)^{2k+1}E_{2k+2}\left(\frac{x}{\cos\theta}\right)$$

$$\left(x\geq 0, 0<\theta<\frac{\pi}{2}\right)$$

$$\alpha_0=1,\alpha_k=\frac{1\cdot 3\cdot 5\ldots(2k-1)}{2\cdot 4\cdot 6\ldots(2k)}$$

(For $E_{2k+2}(x)$, see chapter **5.**)

Relation to the Integral of the Bessel Function $K_0(x)$

27.4.3

$$\int_0^{\frac{\pi}{2}}e^{-x\sec\phi}d\phi=\mathrm{Ki}_1(x)=\int_x^\infty K_0(t)dt\text{ where}$$

$$x^{\frac{1}{2}}e^x\mathrm{Ki}_1(x)\sim(\tfrac{1}{2}\pi)^{\frac{1}{2}}\left\{1-\frac{5}{8x}+\frac{129}{128x^2}-\frac{2655}{1024x^3}+\frac{301035}{32768x^4}-\ldots\right\}$$

(For $\mathrm{Ki}_r(x)$, see chapter **11.**)

[27.5] National Bureau of Standards, Table of the Sievert integral, Applied Math. Series— (U.S. Government Printing Office, Washington, D.C. In press).

$x=0(.01)2(.02)5(.05)10,\ \theta=0°(1°)90°,$ 9D.

[27.6] R. M. Sievert, Die v-Strahlungsintensität an der Oberfläche und in der nächsten Umgebung von Radiumnadeln, Acta Radiologica **11**, 239–301 (1930).

$\int_0^\phi e^{-A\sec\phi}d\phi,\ \phi=30°(1°)90°,\ A=0(.01).5,$ 3D.

Sievert Integral $\int_0^\theta e^{-x\sec\phi}\,d\phi$ **Table 27.4**

$x\backslash\theta$	10°	20°	30°	40°	50°	60°	75°	90°
0. 0	0. 174533	0. 349066	0. 523599	0. 698132	0. 872665	1. 047198	1. 308997	1. 570796
0. 1	0. 157843	0. 315187	0. 471456	0. 625886	0. 777323	0. 923778	1. 123611	1. 228632
0. 2	0. 142749	0. 284598	0. 424515	0. 561159	0. 692565	0. 815477	0. 968414	1. 023680
0. 3	0. 129099	0. 256978	0. 382255	0. 503165	0. 617194	0. 720366	0. 837712	0. 868832
0. 4	0. 116754	0. 232040	0. 344209	0. 451198	0. 550154	0. 636769	0. 727031	0. 745203
0. 5	0. 105589	0. 209522	0. 309957	0. 404629	0. 490508	0. 563236	0. 632830	0. 643694
0. 6	0. 095492	0. 189191	0. 279118	0. 362893	0. 437428	0. 498504	0. 552287	0. 558890
0. 7	0. 086361	0. 170833	0. 251353	0. 325486	0. 390178	0. 441478	0. 483134	0. 487198
0. 8	0. 078103	0. 154256	0. 226354	0. 291957	0. 348109	0. 391204	0. 423535	0. 426062
0. 9	0. 070634	0. 139289	0. 203845	0. 261901	0. 310642	0. 346851	0. 371996	0. 373579
1. 0	0. 063880	0. 125775	0. 183579	0. 234956	0. 277267	0. 307694	0. 327288	0. 328286
1. 2	0. 052247	0. 102553	0. 148899	0. 189138	0. 221027	0. 242523	0. 254485	0. 254889
1. 4	0. 042733	0. 083620	0. 120780	0. 152298	0. 176336	0. 191533	0. 198885	0. 199051
1. 6	0. 034951	0. 068183	0. 097979	0. 122667	0. 140792	0. 151541	0. 156087	0. 156156
1. 8	0. 028587	0. 055597	0. 079488	0. 098829	0. 112497	0. 120105	0. 122932	0. 122961
2. 0	0. 023381	0. 045335	0. 064492	0. 079644	0. 089954	0. 095342	0. 097108	0. 097121
2. 2	0. 019123	0. 036967	0. 052329	0. 064201	0. 071979	0. 075797	0. 076905	0. 076911
2. 4	0. 015641	0. 030145	0. 042463	0. 051766	0. 057635	0. 060342	0. 061040	0. 061043
2. 6	0. 012793	0. 024582	0. 034460	0. 041750	0. 046179	0. 048100	0. 048541	0. 048542
2. 8	0. 010463	0. 020045	0. 027968	0. 033680	0. 037024	0. 038387	0. 038667	0. 038668
3. 0	0. 008558	0. 016347	0. 022700	0. 027177	0. 029702	0. 030670	0. 030848	0. 030848
3. 5	0. 005178	0. 009817	0. 013477	0. 015912	0. 017164	0. 017576	0. 017634	0. 017634
4. 0	0. 003132	0. 005896	0. 008005	0. 009330	0. 009951	0. 010128	0. 010147	0. 010147
4. 5	0. 001895	0. 003542	0. 004756	0. 005478	0. 005787	0. 005862	0. 005869	0. 005869
5. 0	0. 001147	0. 002127	0. 002828	0. 003221	0. 003374	0. 003407	0. 003409	0. 003409
5. 5	0. 000694	0. 001278	0. 001682	0. 001896	0. 001972	0. 001986	0. 001987	0. 001987
6. 0	0. 000420	0. 000768	0. 001001	0. 001117	0. 001155	0. 001162	0. 001162	0. 001162
6. 5	0. 000254	0. 000461	0. 000596	0. 000659	0. 000678	0. 000681	0. 000681	0. 000681
7. 0	0. 000154	0. 000277	0. 000355	0. 000389	0. 000399	0. 000400	0. 000400	0. 000400
7. 5	0. 000093	0. 000167	0. 000211	0. 000230	0. 000235	0. 000235	0. 000235	0. 000235
8. 0	0. 000056	0. 000100	0. 000126	0. 000136	0. 000139	0. 000139	0. 000139	0. 000139
8. 5	0. 000034	0. 000060	0. 000075	0. 000081	0. 000082	0. 000082	0. 000082	0. 000082
9. 0	0. 000021	0. 000036	0. 000045	0. 000048	0. 000048	0. 000048	0. 000048	0. 000048
9. 5	0. 000012	0. 000022	0. 000027	0. 000028	0. 000029	0. 000029	0. 000029	0. 000029
10. 0	0. 000008	0. 000013	0. 000016	0. 000017	0. 000017	0. 000017	0. 000017	0. 000017
	$\left[\begin{smallmatrix}(-3)2\\6\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-4)5\\6\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-4)8\\6\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-3)1\\7\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-3)1\\7\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-3)2\\7\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-3)4\\7\end{smallmatrix}\right]$	$\left[\begin{smallmatrix}(-2)2\\11\end{smallmatrix}\right]$

27.5. $f_m(x)=\int_0^\infty t^m e^{-t^2-\frac{x}{t}}\,dt$ and Related Integrals

$m=0, 1, 2\ldots$

Differential Equations

27.5.1 $$xf_m'''-(m-1)f_m''+2f_m=0$$

27.5.2 $$f_m'=-f_{m-1}\qquad (m=1, 2, \ldots)$$

Recurrence Relation

27.5.3 $$2f_m=(m-1)f_{m-2}+xf_{m-3}\qquad (m\geq 3)$$

Power Series Representations

27.5.4 $$2f_1(x)=\sum_{k=0}^{\infty}(a_k\ln x+b_k)x^k$$

$$a_k=\frac{-2a_{k-2}}{k(k-1)(k-2)}\qquad b_k=\frac{-2b_{k-2}-(3k^2-6k+2)a_k}{k(k-1)(k-2)}$$

$$a_0=a_1=0\qquad a_2=-b_0$$

$$b_0=1\qquad b_1=-\sqrt{\pi}\qquad b_2=\frac{3}{2}(1-\gamma)$$

(For γ, see chapter 6.)

27.5.5

$$2f_1(x)=1-\sqrt{\pi}x+.6342x^2+.5908x^3-.1431x^4$$
$$-.01968x^5+.00324x^6+.000188x^7\ldots$$
$$-x^2\ln x(1-.08333x^2+.001389x^4-.0000083x^6+\ldots)$$

27.5.6

$$2f_2(x)=\frac{\sqrt{\pi}}{2}-x+\frac{\sqrt{\pi}}{2}x^2-.3225x^3-.1477x^4+.03195x^5$$
$$+.00328x^6-.000491x^7-.0000235x^8\ldots$$
$$+x^3\ln x(\tfrac{1}{3}-.01667x^2+.000198x^4-\ldots)$$

27.5.7

$$2f_3(x)=1-\frac{\sqrt{\pi}}{2}x+\frac{x^2}{2}-.2954x^3+.1014x^4+.02954x^5$$
$$-.00578x^6-.00047x^7+.000064x^8\ldots$$
$$-x^4\ln x(.0833-.00278x^2+.000025x^4-\ldots)$$

Asymptotic Representation

27.5.8

$$f_m(x)\sim\sqrt{\frac{\pi}{3}}\,3^{-\frac{m}{2}}v^{\frac{m}{2}}\,e^{-v}\left(a_0+\frac{a_1}{v}+\frac{a_2}{v^2}+\ldots+\frac{a_k}{v^k}+\ldots\right)\qquad(x\rightarrow\infty)$$

$$v=3\left(\frac{x}{2}\right)^{2/3}$$

$$a_0=1,\ a_1=\frac{1}{12}\left(3m^2+3m-1\right)$$

$$12(k+2)a_{k+2}=-(12k^2+36k-3m^2-3m+25)a_{k+1}$$
$$+\tfrac{1}{2}(m-2k)(2k+3-m)(2k+3+2m)a_k\qquad(k=0,1,2\ldots)$$

27.5.9 $$g_1(x)+ig_2(x)=\int_0^\infty t^3e^{-t^2+i\frac{x}{t}}dt$$

27.5.10

$$g_1(x)=\mathscr{R}f_3(ix)\qquad g_2(x)=-\mathscr{I}f_3(ix)$$

Asymptotic Representation

27.5.11

$$g_1(x)=\left(\frac{\pi}{3}\right)^{1/2}\frac{x}{2}\exp\left[-\frac{3}{2}\left(\frac{x}{2}\right)^{2/3}\right](A\sin\theta+B\cos\theta)$$

27.5.12

$$g_2(x)=-\left(\frac{\pi}{3}\right)^{1/2}\frac{x}{2}\exp\left[-\frac{3}{2}\left(\frac{x}{2}\right)^{2/3}\right](A\cos\theta-B\sin\theta)$$

$$\theta=\frac{3}{2}\sqrt{3}\left(\frac{x}{2}\right)^{2/3}$$

$$A\sim a_0-a_3\left(\frac{2}{x}\right)^2+\frac{1}{2}\left[a_1\left(\frac{2}{x}\right)^{2/3}-a_2\left(\frac{2}{x}\right)^{4/3}-a_4\left(\frac{2}{x}\right)^{8/3}+a_5\left(\frac{2}{x}\right)^{10/3}-\ldots\right]\qquad(x\rightarrow\infty)$$

$$B\sim\frac{\sqrt{3}}{2}\left[a_1\left(\frac{2}{x}\right)^{2/3}+a_2\left(\frac{2}{x}\right)^{4/3}-a_4\left(\frac{2}{x}\right)^{8/3}-a_5\left(\frac{2}{x}\right)^{10/3}+\ldots\right]\qquad(x\rightarrow\infty)$$

$$a_0=1\qquad a_1=.972222\qquad a_2=.148534$$
$$a_3=-.017879\qquad a_4=.004594\qquad a_5=-.000762$$

[27.7] M. Abramowitz, Evaluation of the integral $\int_0^\infty e^{-u^2-x/u}du$, J. Math. Phys. **32**, 188–192 (1953).

[27.8] H. Faxén, Expansion in series of the integral $\int_y^\infty \exp\left[-x(t\pm t^{-n})\right]t^p dt$, Ark. Mat., Astr., Fys. **15**, 13, 1–57 (1921).

[27.9] J. E. Kilpatrick and M. F. Kilpatrick, Discrete energy levels associated with the Lennard-Jones potential, J. Chem. Phys. **19**, 7, 930–933 (1951).

[27.10] U. E. Kruse and N. F. Ramsey, The integral $\int_0^\infty y^3\exp\left(-y^2+i\frac{x}{y}\right)dy$, J. Math. Phys. **30**, 40 (1951).

[27.11] O. Laporte, Absorption coefficients for thermal neutrons, Phys. Rev. **52**, 72–74 (1937).

[27.12] H. C. Torrey, Notes on intensities of radio frequency spectra, Phys. Rev. **59**, 293 (1941).

[27.13] C. T. Zahn, Absorption coefficients for thermal neutrons, Phys. Rev. **52**, 67–71 (1937).
$\int_0^\infty y^ne^{-y-x/\sqrt{y}}dy$ for $n=0,\ \frac{1}{2},\ 1;\ x=0(.01).1(.1)1.$

$$f_m(x)=\int_0^\infty t^m e^{-t^2-\frac{x}{t}}\,dt$$

Table 27.5

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	x	$f_1(x)$	$f_2(x)$	$f_3(x)$	x	$f_1(x)$	$f_2(x)$	$f_3(x)$
0.00	0.5000	0.4431	0.5000	0.1	0.4263	0.3970	0.4580	0.6	0.2255	0.2415	0.3025
0.01	0.4914	0.4382	0.4956	0.2	0.3697	0.3573	0.4204	0.7	0.2015	0.2202	0.2793
0.02	0.4832	0.4333	0.4912	0.3	0.3238	0.3227	0.3864	0.8	0.1807	0.2011	0.2584
0.03	0.4753	0.4285	0.4869	0.4	0.2855	0.2923	0.3557	0.9	0.1626	0.1839	0.2392
0.04	0.4676	0.4238	0.4826	0.5	0.2531	0.2654	0.3278	1.0	0.1466	0.1685	0.2215
0.05	0.4602	0.4191	0.4784								
	$\left[{(-5)5 \atop 2}\right]$	$\left[{(-5)5 \atop 2}\right]$	$\left[{(-5)5 \atop 2}\right]$		$\left[{(-3)1 \atop 4}\right]$	$\left[{(-4)7 \atop 3}\right]$	$\left[{(-4)5 \atop 3}\right]$		$\left[{(-4)6 \atop 3}\right]$	$\left[{(-4)4 \atop 3}\right]$	$\left[{(-4)4 \atop 3}\right]$

x	$\mathscr{R}f_3(ix)$	$-\mathscr{I}f_3(ix)$	x	$\mathscr{R}f_3(ix)$	$-\mathscr{I}f_3(ix)$	x	$\mathscr{R}f_3(ix)$	$-\mathscr{I}f_3(ix)$
0.0	0.50000	0.00000	4.0	−0.2626	0.0430	8.0	0.06078	−0.09808
0.2	0.49019	0.08754	4.2	−0.2552	+0.0094	8.5	0.07562	−0.07131
0.4	0.46229	0.16933	4.4	−0.2441	−0.0214	9.0	0.08221	−0.04496
0.6	0.41950	0.24139	4.6	−0.2299	−0.0490	9.5	0.08191	−0.02082
0.8	0.36543	0.30136	4.8	−0.2132	−0.0734	10.0	0.07626	−0.00010
1.0	0.30366	0.34805	5.0	−0.1945	−0.0944	10.5	0.06684	+0.01654
1.2	0.23746	0.38122	5.2	−0.1745	−0.1120	11.0	0.05507	0.02889
1.4	0.16972	0.40127	5.4	−0.1536	−0.1263	11.5	0.04224	0.03707
1.6	0.10288	0.40910	5.6	−0.1322	−0.1374	12.0	0.02937	0.04146
1.8	+0.03892	0.40592	5.8	−0.1108	−0.1455	12.5	0.01727	0.04259
2.0	−0.02062	0.39314	6.0	−0.0896	−0.1507	13.0	+0.00650	0.04109
2.2	−0.0746	0.3722	6.2	−0.0691	−0.1533	13.5	−0.00259	0.03758
2.4	−0.1221	0.3448	6.4	−0.0493	−0.1535	14.0	−0.00982	0.03268
2.6	−0.1629	0.3122	6.6	−0.0307	−0.1515	14.5	−0.01517	0.02696
2.8	−0.1966	0.2759	6.8	−0.0132	−0.1476	15.0	−0.01872	0.02089
3.0	−0.2233	0.2371	7.0	+0.00286	−0.14211	16.0	−0.02118	+0.00921
3.2	−0.2432	0.1971	7.2	0.01749	−0.13518	17.0	−0.01906	−0.00022
3.4	−0.2565	0.1569	7.4	0.03061	−0.12709	18.0	−0.01435	−0.00650
3.6	−0.2639	0.1173	7.6	0.04220	−0.11805	19.0	−0.00879	−0.00965
3.8	−0.2657	0.0792	7.8	0.05224	−0.10830	20.0	−0.00360	−0.01021
	$\left[{(-3)2 \atop 6}\right]$	$\left[{(-3)2 \atop 5}\right]$		$\left[{(-4)5 \atop 3}\right]$	$\left[{(-4)4 \atop 4}\right]$		$\left[{(-3)1 \atop 5}\right]$	$\left[{(-4)7 \atop 5}\right]$

Compiled from U. E. Kruse and N. F. Ramsey, The integral $\int_0^\infty y^3 \exp\left(-y^2+i\frac{x}{y}\right)dy$, J. Math. Phys. **30**, 40 (1951) (with permission).

27.6. $f(x)=\int_0^\infty \frac{e^{-t^2}}{t+x}\,dt$

Power Series Representation

27.6.1

$$f(x)=-e^{-x^2}\ln x+e^{-x^2}\Big[\sqrt{\pi}\sum_{k=0}^{\infty}\frac{x^{2k+1}}{k!(2k+1)}-\sum_{k=1}^{\infty}\frac{x^{2k}}{k!\,2k}-\frac{\gamma}{2}\Big]$$

27.6.2

$$=-e^{-x^2}\ln x+\frac{1}{2}\sum_{k=0}^{\infty}\frac{(-1)^k\psi(k+1)x^{2k}}{k!}+\sqrt{\pi}\sum_{k=0}^{\infty}\frac{(-2)^k x^{2k+1}}{1\cdot 3\cdot 5\ .\ .\ .\ (2k+1)}$$

(For γ and the digamma function $\psi(x)$, see chapter **6.**)

Relation to the Exponential Integral

27.6.3 $f(x)=-\frac{1}{2}e^{-x^2}\operatorname{Ei}(x^2)+\sqrt{\pi}e^{-x^2}\int_0^x e^{t^2}dt$

(For Ei (x) see chapter **5**; $e^{-x^2}\int_0^x e^{t^2}\,dt$, see chapter **7.**

Asymptotic Representation

27.6.4

$$f(x)\sim\frac{\sqrt{\pi}}{2}\Big[\frac{1}{x}+\frac{1}{2x^3}+\frac{1\cdot 3}{4x^5}+\frac{1\cdot 3\cdot 5}{8x^7}+\ .\ .\ .\Big]-\frac{1}{2}\Big[\frac{1}{x^2}+\frac{1}{x^4}+\frac{2!}{x^6}+\frac{3!}{x^8}+\ .\ .\ .\Big]\qquad(x\to\infty)$$

[27.14] A. Erdélyi, Note on the paper "On a definite integral" by R. H. Ritchie, Math. Tables Aids Comp. **4,** 31, 179 (1950).

[27.15] E. T. Goodwin and J. Staton, Table of $\int_0^\infty \frac{e^{-u^2}}{u+x}du$, Quart. J. Mech. Appl. Math. **1,** 319 (1948). x=0(.02)2(.05)3(.1)10. Auxiliary function for x=0(.01)1.

[27.16] R. H. Ritchie, On a definite integral, Math. Tables Aids Comp. **4,** 30, 75 (1950).

Table 27.6

$$f(x)=\int_0^\infty \frac{e^{-t^2}}{t+x}\,dt$$

x	$f(x)+\ln x$	x	$f(x)+\ln x$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0. 00	−0. 2886	0. 50	0. 2704	1. 0	0. 6051	2. 0	0. 3543	3. 0	0. 2519
0. 05	−0. 2081	0. 55	0. 3100	1. 1	0. 5644	2. 1	0. 3404	3. 5	0. 2203
0. 10	−0. 1375	0. 60	0. 3479	1. 2	0. 5291	2. 2	0. 3276	4. 0	0. 1958
0. 15	−0. 0735	0. 65	0. 3842	1. 3	0. 4980	2. 3	0. 3157	4. 5	0. 1762
0. 20	−0. 0146	0. 70	0. 4192	1. 4	0. 4705	2. 4	0. 3046	5. 0	0. 1602
0. 25	+0. 0402	0. 75	0. 4529	1. 5	0. 4460	2. 5	0. 2944	5. 5	0. 1468
0. 30	0. 0915	0. 80	0. 4854	1. 6	0. 4239	2. 6	0. 2848	6. 0	0. 1356
0. 35	0. 1398	0. 85	0. 5168	1. 7	0. 4040	2. 7	0. 2758	6. 5	0. 1259
0. 40	0. 1856	0. 90	0. 5472	1. 8	0. 3860	2. 8	0. 2673	7. 0	0. 1175
0. 45	0. 2290	0. 95	0. 5766	1. 9	0. 3695	2. 9	0. 2594	7. 5	0. 1102
0. 50	0. 2704	1. 00	0. 6051	2. 0	0. 3543	3. 0	0. 2519	8. 0	0. 1037
	$\begin{bmatrix}(-3)1\\4\end{bmatrix}$		$\begin{bmatrix}(-4)2\\3\end{bmatrix}$		$\begin{bmatrix}(-4)7\\4\end{bmatrix}$		$\begin{bmatrix}(-4)1\\3\end{bmatrix}$		$\begin{bmatrix}(-4)9\\4\end{bmatrix}$

Compiled from E. T. Goodwin and J. Staton, Table of $\int_0^\infty \frac{e^{-u^2}}{u+x}\,du$, Quart. J. Mech. Appl. Math. **1**, 319 (1948) (with permission).

27.7. Dilogarithm

(Spence's Integral for $n=2$)

27.7.1 $$f(x)=-\int_1^x \frac{\ln t}{t-1}\,dt$$

Series Expansion

27.7.2 $$f(x)=\sum_{k=1}^\infty (-1)^k \frac{(x-1)^k}{k^2} \qquad (2\geq x\geq 0)$$

Functional Relationships

27.7.3

$$f(x)+f(1-x)=-\ln x\ln(1-x)+\frac{\pi^2}{6} \qquad (1\geq x\geq 0)$$

27.7.4

$$f(1-x)+f(1+x)=\frac{1}{2}f(1-x^2) \qquad (1\geq x>0)$$

27.7.5 $$f(x)+f\left(\frac{1}{x}\right)=-\frac{1}{2}(\ln x)^2 \qquad (0\leq x\leq 1)$$

27.7.6

$$f(x+1)-f(x)=-\ln x\ln(x+1)-\frac{\pi^2}{12}-\frac{1}{2}f(x^2) \qquad (2\geq x\geq 0)$$

Relation to Debye Functions

27.7.7 $$f(e^{-t})=-f(e^t)-\frac{t^2}{2}=\int_0^t \frac{t\,dt}{e^t-1}$$

[27.17] L. Lewin, Dilogarithms and associated functions (Macdonald, London, England, 1958).

[27.18] K. Mitchell, Tables of the function $\int_0^z \frac{-\log|1-y|}{y}\,dy$, with an account of some properties of this and related functions, Phil. Mag. **40**, 351–368 (1949). $x=-1(.01)1$; $x=0(.001).5$, 9D.

[27.19] E. O. Powell, An integral related to the radiation integrals, Phil. Mag. **7**, 34, 600–607 (1943). $\int_1^x \frac{\log y}{y-1}\,dy$, $x=0(.01)2(.02)6$, 7D.

[27.20] A. van Wijngaarden, Polylogarithms, by the Staff of the Computation Department, Report R24, Mathematisch Centrum, Amsterdam, Holland (1954). $F_n(z)=\sum_{h=1}^\infty h^{-n}z^h$ for $z=x=-1(.01)1$; $z=ix$, for $x=0(.01)1$; $z=e^{i\pi\alpha/2}$ for $\alpha=0(.01)2$, 10D.

Dilogarithm

Table 27.7

$$f(x)=-\int_1^x \frac{\ln t}{t-1}\,dt$$

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0.00	1.64493 4067	0.10	1.29971 4723	0.20	1.07479 4600	0.30	0.88937 7624	0.40	0.72758 6308
0.01	1.58862 5448	0.11	1.27452 9160	0.21	1.05485 9830	0.31	0.87229 1733	0.41	0.71239 5042
0.02	1.54579 9712	0.12	1.25008 7584	0.22	1.03527 7934	0.32	0.85542 7404	0.42	0.69736 1058
0.03	1.50789 9041	0.13	1.22632 0101	0.23	1.01603 0062	0.33	0.83877 6261	0.43	0.68247 9725
0.04	1.47312 5860	0.14	1.20316 7961	0.24	0.99709 9088	0.34	0.82233 0471	0.44	0.66774 6644
0.05	1.44063 3797	0.15	1.18058 1124	0.25	0.97846 9393	0.35	0.80608 2689	0.45	0.65315 7631
0.06	1.40992 8300	0.16	1.15851 6487	0.26	0.96012 6675	0.36	0.79002 6024	0.46	0.63870 8705
0.07	1.38068 5041	0.17	1.13693 6560	0.27	0.94205 7798	0.37	0.77415 3992	0.47	0.62439 6071
0.08	1.35267 5161	0.18	1.11580 8451	0.28	0.92425 0654	0.38	0.75846 0483	0.48	0.61021 6108
0.09	1.32572 8728	0.19	1.09510 3088	0.29	0.90669 4053	0.39	0.74293 9737	0.49	0.59616 5361
0.10	1.29971 4723	0.20	1.07479 4600	0.30	0.88937 7624	0.40	0.72758 6308	0.50	0.58224 0526
	$\begin{bmatrix}(-3)2\end{bmatrix}$		$\begin{bmatrix}(-4)1\end{bmatrix}$		$\begin{bmatrix}(-5)5\\7\end{bmatrix}$		$\begin{bmatrix}(-5)3\\6\end{bmatrix}$		$\begin{bmatrix}(-5)2\\5\end{bmatrix}$

From K. Mitchell, Tables of the function $\int_0^z \frac{-\log|1-y|}{y}dy$, with an account of some properties of this and related functions, Phil. Mag. **40**, 351–368 (1949) (with permission).

27.8. Clausen's Integral and Related Summations

27.8.1

$$f(\theta)=-\int_0^\theta \ln\left(2\sin\frac{t}{2}\right)dt=\sum_{k=1}^\infty \frac{\sin k\theta}{k^2} \qquad (0\leq\theta\leq\pi)$$

Series Representation

27.8.2

$$f(\theta)=-\theta\ln|\theta|+\theta+\sum_{k=1}^\infty \frac{(-1)^{k-1}}{(2k)!}B_{2k}\frac{\theta^{2k+1}}{2k(2k+1)} \qquad \left(0\leq\theta<\frac{\pi}{2}\right)$$

27.8.3

$$f(\pi-\theta)=\theta\ln 2-\sum_{k=1}^\infty \frac{(-1)^{k-1}}{(2k)!}B_{2k}(2^{2k}-1)\frac{\theta^{2k+1}}{2k(2k+1)} \qquad (\pi/2<\theta<\pi)$$

Functional Relationship

27.8.4 $f(\pi-\theta)=f(\theta)-\frac{1}{2}f(2\theta) \qquad \left(0\leq\theta\leq\frac{\pi}{2}\right)$

Relation to Spence's Integral

27.8.5

$$if(\theta)=g(e^{i\theta})+\frac{\theta^2}{4} \text{ where } g(x)=\int_1^x \frac{dt}{t}\ln|1+t|$$

Summable Series

27.8.6

$$\sum_{n=1}^\infty \frac{\cos n\theta}{n}=-\ln\left(2\sin\frac{\theta}{2}\right) \qquad (0<\theta<2\pi)$$

$$\sum_{n=1}^\infty \frac{\cos n\theta}{n^2}=\frac{\pi^2}{6}-\frac{\pi\theta}{2}+\frac{\theta^2}{4} \qquad (0\leq\theta\leq2\pi)$$

$$\sum_{n=1}^\infty \frac{\cos n\theta}{n^4}=\frac{\pi^4}{90}-\frac{\pi^2\theta^2}{12}+\frac{\pi\theta^3}{12}-\frac{\theta^4}{48} \qquad (0\leq\theta\leq2\pi)$$

$$\sum_{n=1}^\infty \frac{\sin n\theta}{n}=\frac{1}{2}(\pi-\theta) \qquad (0<\theta<2\pi)$$

$$\sum_{n=1}^\infty \frac{\sin n\theta}{n^3}=\frac{\pi^2\theta}{6}-\frac{\pi\theta^2}{4}+\frac{\theta^3}{12} \qquad (0\leq\theta\leq2\pi)$$

$$\sum_{n=1}^\infty \frac{\sin n\theta}{n^5}=\frac{\pi^4\theta}{90}-\frac{\pi^2\theta^3}{36}+\frac{\pi\theta^4}{48}-\frac{\theta^5}{240} \qquad (0\leq\theta\leq2\pi)$$

[27.21] A. Ashour and A. Sabri, Tabulation of the function $\psi(\theta)=\sum_{n=1}^\infty \frac{\sin n\theta}{n^2}$, Math. Tables Aids Comp. **10**, 54, 57–65 (1956).

[27.22] T. Clausen, Über die Zerlegung reeller gebrochener Funktionen, J. Reine Angew. Math. **8**, 298–300 (1832). $x=0°(1°)180°$, 16D.

[27.23] L. B. W. Jolley, Summation of series (Chapman Publishing Co., London, England, 1925).

[27.24] A. D. Wheelon, A short table of summable series, Report No. SM-14642, Douglas Aircraft Co., Inc., Santa Monica, Calif. (1953).

Table 27.8

Clausen's Integral

$$f(\theta)=-\int_0^{\theta} \ln \left(2 \sin \frac{t}{2}\right) dt$$

$\theta°$	$f(\theta)+\theta \ln \theta$	$\theta°$	$f(\theta)$	$\theta°$	$f(\theta)$	$\theta°$	$f(\theta)$	$\theta°$	$f(\theta)$
0	0. 000000	15	0. 612906	30	0. 864379	60	1. 014942	90	0. 915966
1	0. 017453	16	0. 635781	32	0. 886253	62	1. 014421	95	0. 883872
2	0. 034908	17	0. 657571	34	0. 906001	64	1. 012886	100	0. 848287
3	0. 052362	18	0. 678341	36	0. 923755	66	1. 010376	105	0. 809505
4	0. 069818	19	0. 698149	38	0. 939633	68	1. 006928	110	0. 767800
5	0. 087276	20	0. 717047	40	0. 953741	70	1. 002576	115	0. 723427
6	0. 104735	21	0. 735080	42	0. 966174	72	0. 997355	120	0. 676628
7	0. 122199	22	0. 752292	44	0. 977020	74	0. 991294	125	0. 627629
8	0. 139664	23	0. 768719	46	0. 986357	76	0. 984425	130	0. 576647
9	0. 157133	24	0. 784398	48	0. 994258	78	0. 976776	135	0. 523889
10	0. 174607	25	0. 799360	50	1. 000791	80	0. 968375	140	0. 469554
11	0. 192084	26	0. 813635	52	1. 006016	82	0. 959247	145	0. 413831
12	0. 209567	27	0. 827249	54	1. 009992	84	0. 949419	150	0. 356908
13	0. 227055	28	0. 840230	56	1. 012773	86	0. 938914	160	0. 240176
14	0. 244549	29	0. 852599	58	1. 014407	88	0. 927755	170	0. 120755
15	0. 262049	30	0. 864379	60	1. 014942	90	0. 915966	180	0. 000000
	$\left[{(-7)8 \atop 3}\right]$		$\left[{(-4)1 \atop 4}\right]$		$\left[{(-4)3 \atop 4}\right]$		$\left[{(-4)1 \atop 4}\right]$		$\left[{(-4)4 \atop 6}\right]$

Compiled from A. Ashour and A. Sabri, Tabulation of the function $\psi(\theta)=\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$, Math. Tables Aids Comp. **10**, 54, 57–65 (1956) (with permission).

27.9. Vector-Addition Coefficients

(Wigner coefficients or Clebsch-Gordan coefficients)

Definition

27.9.1

$$(j_1j_2m_1m_2|j_1j_2jm)=\delta(m,\, m_1+m_2)\cdot\sqrt{\frac{(j_1+j_2-j)!(j+j_1-j_2)!(j+j_2-j_1)!(2j+1)}{(j+j_1+j_2+1)!}}$$

$$\cdot\sum_k \frac{(-1)^k\sqrt{(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j+m)!(j-m)!}}{k!(j_1+j_2-j-k)!(j_1-m_1-k)!(j_2+m_2-k)!(j-j_2+m_1+k)!(j-j_1-m_2+k)!}$$

$$\delta(i,\, k)=\begin{cases}1, & i=k\\ 0, & i\neq k\end{cases}$$

Conditions

27.9.2 $j_1,\, j_2,\, j=+n$ or $+\frac{n}{2}$ (n=integer)

27.9.3 $j_1+j_2+j=n$

27.9.4 j_1+j_2-j

27.9.5 j_1-j_2+j

27.9.6 $-j_1+j_2+j$

$$\left.\begin{matrix} j_1+j_2-j \\ j_1-j_2+j \\ -j_1+j_2+j\end{matrix}\right\}\geq 0$$

27.9.7 $m_1,\, m_2,\, m=\pm n$ or $\pm\frac{n}{2}$

27.9.8 $|m_1|\leq j_1,\ |m_2|\leq j_2,\ |m|\leq j$

27.9.9 $(j_1j_2m_1m_2|j_1j_2jm)=0 \qquad m_1+m_2\neq m$

Special Values

27.9.10 $(j_10m_10|j_10jm)=\delta(j_1,\, j)\delta(m_1,\, m)$

27.9.11 $(j_1j_200|j_1j_2j0)=0 \qquad j_1+j_2+j=2n+1$

27.9.12 $(j_1j_1m_1m_1|j_1j_1jm)=0 \qquad 2j_1+j=2n+1$

Symmetry Relations

27.9.13

$$(j_1j_2m_1m_2|j_1j_2jm)$$
$$=(-1)^{j_1+j_2-j}(j_1j_2-m_1-m_2|j_1j_2j-m)$$

27.9.14 $$=(j_2j_1-m_2-m_1|j_2j_1j-m)$$

27.9.15 $$=(-1)^{j_1+j_2-j}(j_2j_1m_1m_2|j_2j_1jm)$$

27.9.16

$$=\sqrt{\frac{2j+1}{2j_1+1}}(-1)^{j_2+m_2}(jj_2-mm_2|jj_2j_1-m_1)$$

27.9.17

$$=\sqrt{\frac{2j+1}{2j_1+1}}(-1)^{j_1-m_1+j-m}(jj_2m-m_2|jj_2j_1m_1)$$

27.9.18

$$=\sqrt{\frac{2j+1}{2j_1+1}}(-1)^{j-m+j_1-m_1}(j_2jm_2-m|j_2jj_1-m_1)$$

27.9.19

$$=\sqrt{\frac{2j+1}{2j_2+1}}(-1)^{j_1-m_1}(j_1jm_1-m|j_1jj_2-m_2)$$

27.9.20

$$=\sqrt{\frac{2j+1}{2j_2+1}}(-1)^{j_1-m_1}(jj_1m-m_1|jj_1j_2m_2)$$

Table 27.9.1

$(j_1\ \tfrac{1}{2}\ m_1\ m_2\,|\,j_1\ \tfrac{1}{2}\ j\ m)$

$j=$	$m_2=\tfrac{1}{2}$	$m_2=-\tfrac{1}{2}$
$j_1+\tfrac{1}{2}$	$\sqrt{\frac{j_1+m+\frac{1}{2}}{2j_1+1}}$	$\sqrt{\frac{j_1-m+\frac{1}{2}}{2j_1+1}}$
$j_1-\tfrac{1}{2}$	$-\sqrt{\frac{j_1-m+\frac{1}{2}}{2j_1+1}}$	$\sqrt{\frac{j_1+m+\frac{1}{2}}{2j_1+1}}$

Table 27.9.2

$(j_1\ 1\ m_1\ m_2\,|\,j_1\ 1\ j\ m)$

$j=$	$m_2=1$	$m_2=0$	$m_2=-1$
j_1+1	$\sqrt{\frac{(j_1+m)(j_1+m+1)}{(2j_1+1)(2j_1+2)}}$	$\sqrt{\frac{(j_1-m+1)(j_1+m+1)}{(2j_1+1)(j_1+1)}}$	$\sqrt{\frac{(j_1-m)(j_1-m+1)}{(2j_1+1)(2j_1+2)}}$
j_1	$-\sqrt{\frac{(j_1+m)(j_1-m+1)}{2j_1(j_1+1)}}$	$\frac{m}{\sqrt{j_1(j_1+1)}}$	$\sqrt{\frac{(j_1-m)(j_1+m+1)}{2j_1(j_1+1)}}$
j_1-1	$\sqrt{\frac{(j_1-m)(j_1-m+1)}{2j_1(2j_1+1)}}$	$-\sqrt{\frac{(j_1-m)(j_1+m)}{j_1(2j_1+1)}}$	$\sqrt{\frac{(j_1+m+1)(j_1+m)}{2j_1(2j_1+1)}}$

Table 27.9.3 $(j_1\ \frac{3}{2}\ m_1\ m_2 | j_1\ \frac{3}{2}\ j\ m)$

$j=$	$m_2=\frac{3}{2}$	$m_2=\frac{1}{2}$
$j_1+\frac{3}{2}$	$\sqrt{\frac{(j_1+m-\frac{1}{2})(j_1+m+\frac{1}{2})(j_1+m+\frac{3}{2})}{(2j_1+1)(2j_1+2)(2j_1+3)}}$	$\sqrt{\frac{3(j_1+m+\frac{1}{2})(j_1+m+\frac{3}{2})(j_1-m+\frac{3}{2})}{(2j_1+1)(2j_1+2)(2j_1+3)}}$
$j_1+\frac{1}{2}$	$-\sqrt{\frac{3(j_1+m-\frac{1}{2})(j_1+m+\frac{1}{2})(j_1-m+\frac{3}{2})}{2j_1(2j_1+1)(2j_1+3)}}$	$-(j_1-3m+\frac{3}{2})\sqrt{\frac{j_1+m+\frac{1}{2}}{2j_1(2j_1+1)(2j_1+3)}}$
$j_1-\frac{1}{2}$	$\sqrt{\frac{3(j_1+m-\frac{1}{2})(j_1-m+\frac{1}{2})(j_1-m+\frac{3}{2})}{(2j_1-1)(2j_1+1)(2j_1+2)}}$	$-(j_1+3m-\frac{1}{2})\sqrt{\frac{j_1-m+\frac{1}{2}}{(2j_1-1)(2j_1+1)(2j_1+2)}}$
$j_1-\frac{3}{2}$	$-\sqrt{\frac{(j_1-m-\frac{1}{2})(j_1-m+\frac{1}{2})(j_1-m+\frac{3}{2})}{2j_1(2j_1-1)(2j_1+1)}}$	$\sqrt{\frac{3(j_1+m-\frac{1}{2})(j_1-m-\frac{1}{2})(j_1-m+\frac{1}{2})}{2j_1(2j_1-1)(2j_1+1)}}$

$j=$	$m_2=-\frac{1}{2}$	$m_2=-\frac{3}{2}$
$j_1+\frac{3}{2}$	$\sqrt{\frac{3(j_1+m+\frac{3}{2})(j_1-m+\frac{1}{2})(j_1-m+\frac{3}{2})}{(2j_1+1)(2j_1+2)(2j_1+3)}}$	$\sqrt{\frac{(j_1-m-\frac{1}{2})(j_1-m+\frac{1}{2})(j_1-m+\frac{3}{2})}{(2j_1+1)(2j_1+2)(2j_1+3)}}$
$j_1+\frac{1}{2}$	$(j_1+3m+\frac{3}{2})\sqrt{\frac{j_1-m+\frac{1}{2}}{2j_1(2j_1+1)(2j_1+3)}}$	$\sqrt{\frac{3(j_1+m+\frac{3}{2})(j_1-m-\frac{1}{2})(j_1-m+\frac{1}{2})}{2j_1(2j_1+1)(2j_1+3)}}$
$j_1-\frac{1}{2}$	$-(j_1-3m-\frac{1}{2})\sqrt{\frac{j_1+m+\frac{1}{2}}{(2j_1-1)(2j_1+1)(2j_1+2)}}$	$\sqrt{\frac{3(j_1+m+\frac{1}{2})(j_1+m+\frac{3}{2})(j_1-m-\frac{1}{2})}{(2j_1-1)(2j_1+1)(2j_1+2)}}$
$j_1-\frac{3}{2}$	$-\sqrt{\frac{3(j_1+m-\frac{1}{2})(j_1+m+\frac{1}{2})(j_1-m-\frac{1}{2})}{2j_1(2j_1-1)(2j_1+1)}}$	$\sqrt{\frac{(j_1+m-\frac{1}{2})(j_1+m+\frac{1}{2})(j_1+m+\frac{3}{2})}{2j_1(2j_1-1)(2j_1+1)}}$

Table 27.9.4 $(j_1\ 2\ m_1\ m_2 | j_1\ 2\ j\ m)$

$j=$	$m_2=2$	$m_2=1$	$m_2=0$	$m_2=-1$	$m_2=-2$
j_1+2	$\sqrt{\frac{(j_1+m-1)(j_1+m)(j_1+m+1)(j_1+m+2)}{(2j_1+1)(2j_1+2)(2j_1+3)(2j_1+4)}}$	$\sqrt{\frac{(j_1-m+2)(j_1+m+2)(j_1+m+1)(j_1+m)}{(2j_1+1)(j_1+1)(2j_1+3)(j_1+2)}}$	$\sqrt{\frac{3(j_1-m+2)(j_1-m+1)(j_1+m+2)(j_1+m+1)}{(2j_1+1)(2j_1+2)(2j_1+3)(j_1+2)}}$	$\sqrt{\frac{(j_1-m+2)(j_1-m+1)(j_1-m)(j_1+m+2)}{(2j_1+1)(j_1+1)(2j_1+3)(j_1+2)}}$	$\sqrt{\frac{(j_1-m-1)(j_1-m)(j_1-m+1)(j_1-m+2)}{(2j_1+1)(2j_1+2)(2j_1+3)(2j_1+4)}}$
j_1+1	$-\sqrt{\frac{(j_1+m-1)(j_1+m)(j_1+m+1)(j_1-m+2)}{2j_1(j_1+1)(j_1+2)(2j_1+1)}}$	$-(j_1-2m+2)\sqrt{\frac{(j_1+m+1)(j_1+m)}{2j_1(2j_1+1)(j_1+1)(j_1+2)}}$	$m\sqrt{\frac{3(j_1-m+1)(j_1+m+1)}{j_1(2j_1+1)(j_1+1)(j_1+2)}}$	$(j_1+2m+2)\sqrt{\frac{(j_1-m+1)(j_1-m)}{j_1(2j_1+1)(2j_1+2)(j_1+2)}}$	$\sqrt{\frac{(j_1-m-1)(j_1-m)(j_1-m+1)(j_1+m+2)}{j_1(2j_1+1)(j_1+1)(2j_1+4)}}$
j_1	$\sqrt{\frac{3(j_1+m-1)(j_1+m)(j_1-m+1)(j_1-m+2)}{(2j_1-1)2j_1(j_1+1)(2j_1+3)}}$	$(1-2m)\sqrt{\frac{3(j_1-m+1)(j_1+m)}{(2j_1-1)j_1(2j_1+2)(2j_1+3)}}$	$\frac{3m^2-j_1(j_1+1)}{\sqrt{(2j_1-1)j_1(j_1+1)(2j_1+3)}}$	$(2m+1)\sqrt{\frac{3(j_1-m)(j_1+m+1)}{(2j_1-1)j_1(2j_1+2)(2j_1+3)}}$	$\sqrt{\frac{3(j_1-m-1)(j_1-m)(j_1+m+1)(j_1+m+2)}{(2j_1-1)j_1(2j_1+2)(2j_1+3)}}$
j_1-1	$-\sqrt{\frac{(j_1+m-1)(j_1-m)(j_1-m+1)(j_1-m+2)}{2(j_1-1)j_1(j_1+1)(2j_1+1)}}$	$(j_1+2m-1)\sqrt{\frac{(j_1-m+1)(j_1-m)}{(j_1-1)j_1(2j_1+1)(2j_1+2)}}$	$-m\sqrt{\frac{3(j_1-m)(j_1+m)}{(j_1-1)j_1(2j_1+1)(j_1+1)}}$	$-(j_1-2m-1)\sqrt{\frac{(j_1+m+1)(j_1+m)}{(j_1-1)j_1(2j_1+1)(2j_1+2)}}$	$\sqrt{\frac{(j_1-m+1)(j_1+m)(j_1+m+1)(j_1+m+2)}{(j_1-1)j_1(2j_1+1)(2j_1+2)}}$
j_1-2	$\sqrt{\frac{(j_1-m-1)(j_1-m)(j_1-m+1)(j_1-m+2)}{(2j_1-2)(2j_1-1)2j_1(2j_1+1)}}$	$-\sqrt{\frac{(j_1-m+1)(j_1-m)(j_1-m-1)(j_1+m-1)}{(j_1-1)(2j_1-1)j_1(2j_1+1)}}$	$\sqrt{\frac{3(j_1-m)(j_1-m-1)(j_1+m)(j_1+m-1)}{(2j_1-2)(2j_1-1)j_1(2j_1+1)}}$	$-\sqrt{\frac{(j_1-m-1)(j_1+m+1)(j_1+m)(j_1+m-1)}{(j_1-1)(2j_1-1)j_1(2j_1+1)}}$	$\sqrt{\frac{(j_1+m-1)(j_1+m)(j_1+m+1)(j_1+m+2)}{(2j_1-2)(2j_1-1)2j_1(2j_1+1)}}$

Table 27.9.5 [By use of symmetry relations, coefficients may be put in standard form $j_1 \leq j_2 \leq j$ and $m \geq 0$]

m_2	m	j_1	j	$(j_1 j_2 m_1 m_2 \mid j_1 j_2 j m)$	
$j_2=\frac{1}{2}$					
$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\sqrt{\frac{1}{2}}$	0.70711
$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\sqrt{\frac{1}{2}}$	0.70711
$\frac{1}{2}$	1	$\frac{1}{2}$	1		1.00000
$j_2=1$					
-1	0	1	1	$\sqrt{\frac{1}{2}}$	0.70711
0	0	1	1		0.00000
1	0	1	1	$-\sqrt{\frac{1}{2}}$	-0.70711
0	1	1	1	$\sqrt{\frac{1}{2}}$	0.70711
1	1	1	1	$-\sqrt{\frac{1}{2}}$	-0.70711
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\sqrt{\frac{2}{3}}$	0.81650
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\sqrt{\frac{1}{3}}$	0.57735
1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$		1.00000 *
-1	0	1	2	$\sqrt{\frac{1}{6}}$	0.40825
0	0	1	2	$\sqrt{\frac{2}{3}}$	0.81650
1	0	1	2	$\sqrt{\frac{1}{6}}$	0.40825
0	1	1	2	$\sqrt{\frac{1}{2}}$	0.70711
1	1	1	2	$\sqrt{\frac{1}{2}}$	0.70711
1	2	1	2		1.00000
$j_2=\frac{3}{2}$					
$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$\sqrt{\frac{8}{15}}$	0.73030
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$-\sqrt{\frac{1}{15}}$	-0.25820
$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$-\sqrt{\frac{2}{5}}$	-0.63246
$\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{3}{2}$	$\sqrt{\frac{2}{5}}$	0.63246
$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{3}{2}$	$-\sqrt{\frac{3}{5}}$	-0.77460
$-\frac{1}{2}$	0	$\frac{1}{2}$	2	$\sqrt{\frac{1}{2}}$	0.70711
$\frac{1}{2}$	0	$\frac{1}{2}$	2	$\sqrt{\frac{1}{2}}$	0.70711
$\frac{1}{2}$	1	$\frac{1}{2}$	2	$\frac{1}{2}\sqrt{3}$	0.86603
$\frac{3}{2}$	1	$\frac{1}{2}$	2		0.50000
$\frac{3}{2}$	2	$\frac{1}{2}$	2		1.00000
$-\frac{3}{2}$	0	$\frac{3}{2}$	2		0.50000
$-\frac{1}{2}$	0	$\frac{3}{2}$	2		0.50000
$\frac{1}{2}$	0	$\frac{3}{2}$	2		-0.50000
$\frac{3}{2}$	0	$\frac{3}{2}$	2		-0.50000
$-\frac{1}{2}$	1	$\frac{3}{2}$	2	$\sqrt{\frac{1}{2}}$	0.70711
$\frac{1}{2}$	1	$\frac{3}{2}$	2		0.00000
$\frac{3}{2}$	1	$\frac{3}{2}$	2	$-\sqrt{\frac{1}{2}}$	-0.70711
$\frac{1}{2}$	2	$\frac{3}{2}$	2	$\sqrt{\frac{1}{2}}$	0.70711
$\frac{3}{2}$	2	$\frac{3}{2}$	2	$-\sqrt{\frac{1}{2}}$	-0.70711
$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{3}{10}}$	0.54772
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{3}{5}}$	0.77460
$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{1}{10}}$	0.31623
$\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{3}{5}}$	0.77460
$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{2}{5}}$	0.63246
$\frac{3}{2}$	$\frac{5}{2}$	1	$\frac{5}{2}$		1.00000

Compiled from A. Simon, Numerical tables of the Clebsch-Gordan coefficients, Oak Ridge National Laboratory Report 1718, Oak Ridge, Tenn. (1954) (with permission).

[27.25] E. U. Condon and G. A. Shortley, Theory of atomic spectra (Cambridge Univ. Press, Cambridge, England, 1935).

[27.26] M. E. Rose, Elementary theory of angular momemtum (John Wiley & Sons, Inc., New York, N.Y., 1955).

[27.27] A. Simon, Numerical tables of the Clebsch-Gordan coefficients, Oak Ridge National Laboratory Report 1718, Oak Ridge, Tenn. (1954).
$C(j_1 j_2 j;\ m_1 m_2 m)$ for all angular moments $<\frac{9}{2}$, 10D.

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*See page II.

29. Laplace Transforms

Contents

29. Laplace Transforms

29.1. Definition of the Laplace Transform

One-dimensional Laplace Transform

29.1.1 $$f(s)=\mathscr{L}\{F(t)\}=\int_0^\infty e^{-st}F(t)dt$$

$F(t)$ is a function of the real variable t and s is a complex variable. $F(t)$ is called the original function and $f(s)$ is called the image function. If the integral in **29.1.1** converges for a real $s=s_0$, i.e.,

$$\lim_{\substack{A\to 0\\ B\to\infty}}\int_A^B e^{-s_0t}F(t)dt$$

exists, then it converges for all s with $\mathscr{R}s>s_0$, and the image function is a single valued analytic function of s in the half-plane $\mathscr{R}s>s_0$.

Two-dimensional Laplace Transform

29.1.2
$$f(u,v)=\mathscr{L}\{F(x,y)\}=\int_0^\infty\int_0^\infty e^{-ux-vy}F(x,y)dxdy$$

Definition of the Unit Step Function

29.1.3 $$u(t)=\begin{cases}0 & (t<0)\\ \frac{1}{2} & (t=0)\\ 1 & (t>0)\end{cases}$$

In the following tables the factor $u(t)$ is to be understood as multiplying the original function $F(t)$.

29.2. Operations for the Laplace Transform[1]

	Original Function F(t)	*Image Function f(s)*
29.2.1	$F(t)$	$\int_0^\infty e^{-st}F(t)dt$
	Inversion Formula	
29.2.2	$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}f(s)ds$	$f(s)$
	Linearity Property	
29.2.3	$AF(t)+BG(t)$	$Af(s)+Bg(s)$
	Differentiation	
29.2.4	$F'(t)$	$sf(s)-F(+0)$
29.2.5	$F^{(n)}(t)$	$s^nf(s)-s^{n-1}F(+0)-s^{n-2}F'(+0)-\ldots-F^{(n-1)}(+0)$
	Integration	
29.2.6	$\int_0^t F(\tau)d\tau$	$\frac{1}{s}f(s)$
29.2.7	$\int_0^t\int_0^\tau F(\lambda)d\lambda d\tau$	$\frac{1}{s^2}f(s)$
	Convolution (Faltung) Theorem	
29.2.8	$\int_0^t F_1(t-\tau)F_2(\tau)d\tau=F_1*F_2$	$f_1(s)f_2(s)$
		Differentiation
29.2.9	$-tF(t)$	$f'(s)$
29.2.10	$(-1)^nt^nF(t)$	$f^{(n)}(s)$

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[1] Adapted by permission from R. V. Churchill, Operational mathematics, 2d ed., McGraw-Hill Book Co., Inc., New York, N.Y., 1958.

	Original Function F(t)	*Image Function f(s)*
		Integration
29.2.11	$\frac{1}{t}F(t)$	$\int_s^\infty f(x)dx$
		Linear Transformation
29.2.12	$e^{at}F(t)$	$f(s-a)$
29.2.13	$\frac{1}{c}F\left(\frac{t}{c}\right) \qquad (c>0)$	$f(cs)$
29.2.14	$\frac{1}{c}e^{(b/c)t}F\left(\frac{t}{c}\right) \qquad (c>0)$	$f(cs-b)$
	Translation	
29.2.15	$F(t-b)u(t-b) \qquad (b>0)$	$e^{-bs}f(s)$
	Periodic Functions	
29.2.16	$F(t+a)=F(t)$	$\frac{\int_0^a e^{-st}F(t)dt}{1-e^{-as}}$
29.2.17	$F(t+a)=-F(t)$	$\frac{\int_0^a e^{-st}F(t)dt}{1+e^{-as}}$
	Half-Wave Rectification of $F(t)$ in 29.2.17	
29.2.18	$F(t)\sum_{n=0}^{\infty}(-1)^n u(t-na)$	$\frac{f(s)}{1-e^{-as}}$
	Full-Wave Rectification of $F(t)$ in 29.2.17	
29.2.19	$\lvert F(t)\rvert$	$f(s)\coth\frac{as}{2}$
		Heaviside Expansion Theorem
29.2.20	$\sum_{n=1}^{m}\frac{p(a_n)}{q'(a_n)}e^{a_n t}$	$\frac{p(s)}{q(s)},\ q(s)=(s-a_1)(s-a_2)\ .\ .\ .\ (s-a_m)$ $p(s)$ a polynomial of degree$<m$
29.2.21	$e^{at}\sum_{n=1}^{r}\frac{p^{(r-n)}(a)}{(r-n)!}\frac{t^{n-1}}{(n-1)!}$	$\frac{p(s)}{(s-a)^r}$ $p(s)$ a polynomial of degree$<r$

29.3. Table of Laplace Transforms[2,3]

For a comprehensive table of Laplace and other integral transforms see [29.9]. For a table of two-dimensional Laplace transforms see [29.11].

	$f(s)$	$F(t)$
29.3.1	$\frac{1}{s}$	1
29.3.2	$\frac{1}{s^2}$	t

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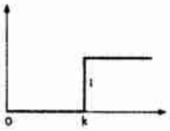
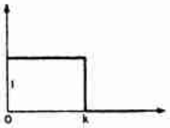
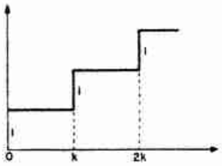
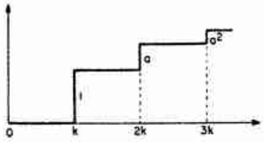
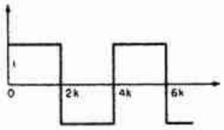
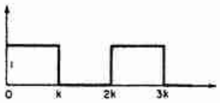
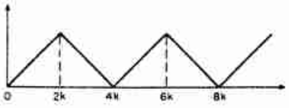
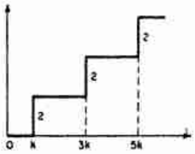
[2] The numbers in bold type in the $f(s)$ and $F(t)$ columns indicate the chapters in which the properties of the respective higher mathematical functions are given.

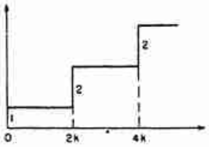
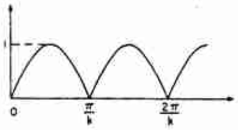
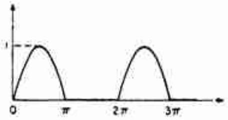
[3] Adapted by permission from R. V. Churchill, Operational mathematics, 2d. ed., McGraw-Hill Book Co., Inc., New York, N. Y., 1958.

	$f(s)$		$F(t)$
29.3.3	$\frac{1}{s^n} \quad (n=1, 2, 3, \ldots)$		$\frac{t^{n-1}}{(n-1)!}$
29.3.4	$\frac{1}{\sqrt{s}}$		$\frac{1}{\sqrt{\pi t}}$
29.3.5	$s^{-3/2}$		$2\sqrt{t/\pi}$
29.3.6	$s^{-(n+\frac{1}{2})} \quad (n=1, 2, 3, \ldots)$		$\frac{2^n t^{n-\frac{1}{2}}}{1 \cdot 3 \cdot 5 \ldots (2n-1)\sqrt{\pi}}$
29.3.7	$\frac{\Gamma(k)}{s^k} \quad (k>0)$	**6**	t^{k-1}
29.3.8	$\frac{1}{s+a}$		e^{-at}
29.3.9	$\frac{1}{(s+a)^2}$		te^{-at}
29.3.10	$\frac{1}{(s+a)^n} \quad (n=1, 2, 3, \ldots)$		$\frac{t^{n-1}e^{-at}}{(n-1)!}$
29.3.11	$\frac{\Gamma(k)}{(s+a)^k} \quad (k>0)$	**6**	$t^{k-1}e^{-at}$
29.3.12	$\frac{1}{(s+a)(s+b)} \quad (a \neq b)$		$\frac{e^{-at}-e^{-bt}}{b-a}$
29.3.13	$\frac{s}{(s+a)(s+b)} \quad (a \neq b)$		$\frac{ae^{-at}-be^{-bt}}{a-b}$
29.3.14	$\frac{1}{(s+a)(s+b)(s+c)}$		$-\frac{(b-c)e^{-at}+(c-a)e^{-bt}+(a-b)e^{-ct}}{(a-b)(b-c)(c-a)}$
	(a, b, c distinct constants)		
29.3.15	$\frac{1}{s^2+a^2}$		$\frac{1}{a} \sin at$
29.3.16	$\frac{s}{s^2+a^2}$		$\cos at$
29.3.17	$\frac{1}{s^2-a^2}$		$\frac{1}{a} \sinh at$
29.3.18	$\frac{s}{s^2-a^2}$		$\cosh at$
29.3.19	$\frac{1}{s(s^2+a^2)}$		$\frac{1}{a^2}(1-\cos at)$
29.3.20	$\frac{1}{s^2(s^2+a^2)}$		$\frac{1}{a^3}(at-\sin at)$
29.3.21	$\frac{1}{(s^2+a^2)^2}$		$\frac{1}{2a^3}(\sin at-at \cos at)$

	$f(s)$	$F(t)$	
29.3.22	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a}\sin at$	
29.3.23	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a}(\sin at+at\cos at)$	
29.3.24	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$	
29.3.25	$\frac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2\neq b^2)$	$\frac{\cos at-\cos bt}{b^2-a^2}$	
29.3.26	$\frac{1}{(s+a)^2+b^2}$	$\frac{1}{b}e^{-at}\sin bt$	
29.3.27	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$	
29.3.28	$\frac{3a^2}{s^3+a^3}$	$e^{-at}-e^{\frac{1}{2}at}\left(\cos\frac{at\sqrt{3}}{2}-\sqrt{3}\sin\frac{at\sqrt{3}}{2}\right)$	
29.3.29	$\frac{4a^3}{s^4+4a^4}$	$\sin at\cosh at-\cos at\sinh at$	
29.3.30	$\frac{s}{s^4+4a^4}$	$\frac{1}{2a^2}\sin at\sinh at$	
29.3.31	$\frac{1}{s^4-a^4}$	$\frac{1}{2a^3}(\sinh at-\sin at)$	
29.3.32	$\frac{s}{s^4-a^4}$	$\frac{1}{2a^2}(\cosh at-\cos at)$	
29.3.33	$\frac{8a^3s^2}{(s^2+a^2)^3}$	$(1+a^2t^2)\sin at-at\cos at$	
29.3.34	$\frac{1}{s}\left(\frac{s-1}{s}\right)^n$	$L_n(t)$	**22**
29.3.35	$\frac{s}{(s+a)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{\pi t}}e^{-at}(1-2at)$	
29.3.36	$\sqrt{s+a}-\sqrt{s+b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{-bt}-e^{-at})$	
29.3.37	$\frac{1}{\sqrt{s}+a}$	$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}\operatorname{erfc} a\sqrt{t}$	**7**
29.3.38	$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}}+ae^{a^2t}\operatorname{erf} a\sqrt{t}$	**7**
29.3.39	$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}d\lambda$	**7**
29.3.40	$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a}e^{a^2t}\operatorname{erf} a\sqrt{t}$	**7**

	$f(s)$	$F(t)$	
29.3.41	$\frac{1}{\sqrt{s}(s+a^2)}$	$\frac{2}{a\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}d\lambda$	**7**
29.3.42	$\frac{b^2-a^2}{(s-a^2)(b+\sqrt{s})}$	$e^{a^2t}[b-a\text{ erf }a\sqrt{t}]-be^{b^2t}\text{ erfc }b\sqrt{t}$	**7**
29.3.43	$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$	$e^{a^2t}\text{ erfc }a\sqrt{t}$	**7**
29.3.44	$\frac{1}{(s+a)\sqrt{s+b}}$	$\frac{1}{\sqrt{b-a}}e^{-at}\text{ erf }(\sqrt{b-a}\sqrt{t})$	**7**
29.3.45	$\frac{b^2-a^2}{\sqrt{s}(s-a^2)(\sqrt{s}+b)}$	$e^{a^2t}\left[\frac{b}{a}\text{ erf }(a\sqrt{t})-1\right]+e^{b^2t}\text{ erfc }b\sqrt{t}$	**7**
29.3.46	$\frac{(1-s)^n}{s^{n+\frac{1}{2}}}$	$\frac{n!}{(2n)!\sqrt{\pi t}}H_{2n}(\sqrt{t})$	**22**
29.3.47	$\frac{(1-s)^n}{s^{n+\frac{3}{2}}}$	$\frac{n!}{(2n+1)!\sqrt{\pi}}H_{2n+1}(\sqrt{t})$	**22**
29.3.48	$\frac{\sqrt{s+2a}}{\sqrt{s}}-1$	$ae^{-at}[I_1(at)+I_0(at)]$	**9**
29.3.49	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-\frac{1}{2}(a+b)t}I_0\left(\frac{a-b}{2}t\right)$	**9**
29.3.50	$\frac{\Gamma(k)}{(s+a)^k(s+b)^k}\quad(k>0)$ **6**	$\sqrt{\pi}\left(\frac{t}{a-b}\right)^{k-\frac{1}{2}}e^{-\frac{1}{2}(a+b)t}I_{k-\frac{1}{2}}\left(\frac{a-b}{2}t\right)$	**10**
29.3.51	$\frac{1}{(s+a)^{\frac{1}{2}}(s+b)^{\frac{3}{2}}}$	$te^{-\frac{1}{2}(a+b)t}\left[I_0\left(\frac{a-b}{2}t\right)+I_1\left(\frac{a-b}{2}t\right)\right]$	**9**
29.3.52	$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$	$\frac{1}{t}e^{-at}I_1(at)$	**9**
29.3.53	$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}}\quad(k>0)$	$\frac{k}{t}e^{-\frac{1}{2}(a+b)t}I_k\left(\frac{a-b}{2}t\right)$	**9**
29.3.54	$\frac{(\sqrt{s+a}+\sqrt{s})^{-2\nu}}{\sqrt{s}\sqrt{s+a}}\quad(\nu>-1)$	$\frac{1}{a^\nu}e^{-\frac{1}{2}at}I_\nu(\frac{1}{2}at)$	**9**
29.3.55	$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	**9**
29.3.56	$\frac{(\sqrt{s^2+a^2}-s)^\nu}{\sqrt{s^2+a^2}}\quad(\nu>-1)$	$a^\nu J_\nu(at)$	**9**
29.3.57	$\frac{1}{(s^2+a^2)^k}\quad(k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}}J_{k-\frac{1}{2}}(at)$	**6, 10**

	$f(s)$	$F(t)$		
29.3.58	$(\sqrt{s^2+a^2}-s)^k \quad (k>0)$	$\frac{ka^k}{t}J_k(at)$		**9**
29.3.59	$\frac{(s-\sqrt{s^2-a^2})^\nu}{\sqrt{s^2-a^2}} \quad (\nu>-1)$	$a^\nu I_\nu(at)$		**9**
29.3.60	$\frac{1}{(s^2-a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}} I_{k-\frac{1}{2}}(at)$		**6, 10**
29.3.61	$\frac{1}{s}e^{-ks}$	$u(t-k)$		
29.3.62	$\frac{1}{s^2}e^{-ks}$	$(t-k)u(t-k)$		
29.3.63	$\frac{1}{s^\mu}e^{-ks} \quad (\mu>0)$	$\frac{(t-k)^{\mu-1}}{\Gamma(\mu)}u(t-k)$		**6**
29.3.64	$\frac{1-e^{-ks}}{s}$	$u(t)-u(t-k)$		
29.3.65	$\frac{1}{s(1-e^{-ks})}=\frac{1+\coth \frac{1}{2}ks}{2s}$	$\sum_{n=0}^{\infty} u(t-nk)$		
29.3.66	$\frac{1}{s(e^{ks}-a)}$	$\sum_{n=1}^{\infty} a^{n-1}u(t-nk)$		
29.3.67	$\frac{1}{s}\tanh ks$	$u(t)+2\sum_{n=1}^{\infty}(-1)^n u(t-2nk)$		
29.3.68	$\frac{1}{s(1+e^{-ks})}$	$\sum_{n=0}^{\infty}(-1)^n u(t-nk)$		
29.3.69	$\frac{1}{s^2}\tanh ks$	$tu(t)+2\sum_{n=1}^{\infty}(-1)^n(t-2nk)u(t-2nk)$		
29.3.70	$\frac{1}{s \sinh ks}$	$2\sum_{n=0}^{\infty} u[t-(2n+1)k]$		
29.3.71	$\frac{1}{s \cosh ks}$	$2\sum_{n=0}^{\infty}(-1)^n u[t-(2n+1)k]$		

	$f(s)$	$F(t)$	
29.3.72	$\frac{1}{s}\coth ks$	$u(t)+2\sum_{n=1}^{\infty} u(t-2nk)$ 	
29.3.73	$\frac{k}{s^2+k^2}\coth\frac{\pi s}{2k}$	$\lvert\sin kt\rvert$ 	
29.3.74	$\frac{1}{(s^2+1)(1-e^{-\pi s})}$	$\sum_{n=0}^{\infty}(-1)^n u(t-n\pi)\sin t$ 	*
29.3.75	$\frac{1}{s}e^{-\frac{k}{s}}$	$J_0(2\sqrt{kt})$	**9**
29.3.76	$\frac{1}{\sqrt{s}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
29.3.77	$\frac{1}{\sqrt{s}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cosh 2\sqrt{kt}$	
29.3.78	$\frac{1}{s^{3/2}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}}\sin 2\sqrt{kt}$	
29.3.79	$\frac{1}{s^{3/2}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
29.3.80	$\frac{1}{s^\mu}e^{-\frac{k}{s}}\quad(\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}}J_{\mu-1}(2\sqrt{kt})$	**9**
29.3.81	$\frac{1}{s^\mu}e^{\frac{k}{s}}\quad(\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}}I_{\mu-1}(2\sqrt{kt})$	**9**
29.3.82	$e^{-k\sqrt{s}}\quad(k>0)$	$\frac{k}{2\sqrt{\pi t^3}}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.83	$\frac{1}{s}e^{-k\sqrt{s}}\quad(k\geq 0)$	$\operatorname{erfc}\frac{k}{2\sqrt{t}}$	**7**
29.3.84	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}\quad(k\geq 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.85	$\frac{1}{s^{\frac{3}{2}}}e^{-k\sqrt{s}}\quad(k\geq 0)$	$2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{k^2}{4t}\right)-k\operatorname{erfc}\frac{k}{2\sqrt{t}}=2\sqrt{t}\ \mathrm{i}\operatorname{erfc}\frac{k}{2\sqrt{t}}$	**7**
29.3.86	$\frac{1}{s^{1+\frac{1}{2}n}}e^{-k\sqrt{s}}\quad(n=0,1,2,\ldots;k\geq 0)$	$(4t)^{\frac{1}{2}n}\mathrm{i}^n\operatorname{erfc}\frac{k}{2\sqrt{t}}$	**7**
29.3.87	$s^{\frac{n-1}{2}}e^{-k\sqrt{s}}\quad(n=0,1,2,\ldots;k>0)$	$\frac{\exp\left(-\frac{k^2}{4t}\right)}{2^n\sqrt{\pi t^{n+1}}}H_n\left(\frac{k}{2\sqrt{t}}\right)$	**22**
29.3.88	$\frac{e^{-k\sqrt{s}}}{a+\sqrt{s}}\quad(k\geq 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)-ae^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$	**7**

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*See page II.

	$f(s)$	$F(t)$	
29.3.89	$\frac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})} \quad (k\geq 0)$	$-e^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)+\operatorname{erfc}\frac{k}{2\sqrt{t}}$	**7**
29.3.90	$\frac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})} \quad (k\geq 0)$	$e^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$	**7**
29.3.91	$\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}} \quad (k\geq 0)$	$e^{-\frac{1}{2}at}I_0(\frac{1}{2}a\sqrt{t^2-k^2})u(t-k)$	**9**
29.3.92	$\frac{e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}} \quad (k\geq 0)$	$J_0(a\sqrt{t^2-k^2})u(t-k)$	**9**
29.3.93	$\frac{e^{-k\sqrt{s^2-a^2}}}{\sqrt{s^2-a^2}} \quad (k\geq 0)$	$I_0(a\sqrt{t^2-k^2})u(t-k)$	**9**
29.3.94	$\frac{e^{-k(\sqrt{s^2+a^2}-s)}}{\sqrt{s^2+a^2}} \quad (k\geq 0)$	$J_0(a\sqrt{t^2+2kt})$	**9**
29.3.95	$e^{-ks}-e^{-k\sqrt{s^2+a^2}} \quad (k>0)$	$\frac{ak}{\sqrt{t^2-k^2}}J_1(a\sqrt{t^2-k^2})u(t-k)$	**9**
29.3.96	$e^{-k\sqrt{s^2-a^2}}-e^{-ks} \quad (k>0)$	$\frac{ak}{\sqrt{t^2-k^2}}I_1(a\sqrt{t^2-k^2})u(t-k)$	**9**
29.3.97	$\frac{a^\nu e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)^\nu} \quad (\nu>-1, k\geq 0)$	$\left(\frac{t-k}{t+k}\right)^{\frac{1}{2}\nu}J_\nu(a\sqrt{t^2-k^2})u(t-k)$	**9**
29.3.98	$\frac{1}{s}\ln s$	$-\gamma-\ln t$ ($\gamma=.57721\ 56649\ldots$ Euler's constant)	
29.3.99	$\frac{1}{s^k}\ln s \quad (k>0)$	$\frac{t^{k-1}}{\Gamma(k)}[\psi(k)-\ln t]$	**6**
29.3.100	$\frac{\ln s}{s-a} \quad (a>0)$	$e^{at}[\ln a+E_1(at)]$	**5**
29.3.101	$\frac{\ln s}{s^2+1}$	$\cos t\ \mathrm{Si}\ (t)-\sin t\ \mathrm{Ci}\ (t)$	**5**
29.3.102	$\frac{s\ln s}{s^2+1}$	$-\sin t\ \mathrm{Si}\ (t)-\cos t\ \mathrm{Ci}\ (t)$	**5**
29.3.103	$\frac{1}{s}\ln(1+ks) \quad (k>0)$	$E_1\left(\frac{t}{k}\right)$	**5**
29.3.104	$\ln\frac{s+a}{s+b}$	$\frac{1}{t}(e^{-bt}-e^{-at})$	
29.3.105	$\frac{1}{s}\ln(1+k^2s^2) \quad (k>0)$	$-2\ \mathrm{Ci}\left(\frac{t}{k}\right)$	**5**
29.3.106	$\frac{1}{s}\ln(s^2+a^2) \quad (a>0)$	$2\ln a-2\ \mathrm{Ci}\ (at)$	**5**

	$f(s)$		$F(t)$	
29.3.107	$\frac{1}{s^2}\ln(s^2+a^2) \qquad (a>0)$		$\frac{2}{a}[at\ln a+\sin at-at\,\mathrm{Ci}\,(at)]$	**5**
29.3.108	$\ln\frac{s^2+a^2}{s^2}$		$\frac{2}{t}(1-\cos at)$	
29.3.109	$\ln\frac{s^2-a^2}{s^2}$		$\frac{2}{t}(1-\cosh at)$	
29.3.110	$\arctan\frac{k}{s}$		$\frac{1}{t}\sin kt$	
29.3.111	$\frac{1}{s}\arctan\frac{k}{s}$		$\mathrm{Si}\,(kt)$	**5**
29.3.112	$e^{k^2s^2}\operatorname{erfc} ks \qquad (k>0)$	**7**	$\frac{1}{k\sqrt{\pi}}\exp\left(-\frac{t^2}{4k^2}\right)$	
29.3.113	$\frac{1}{s}e^{k^2s^2}\operatorname{erfc} ks \qquad (k>0)$	**7**	$\operatorname{erf}\frac{t}{2k}$	**7**
29.3.114	$e^{ks}\operatorname{erfc}\sqrt{ks} \qquad (k>0)$	**7**	$\frac{\sqrt{k}}{\pi\sqrt{t}(t+k)}$	
29.3.115	$\frac{1}{\sqrt{s}}\operatorname{erfc}\sqrt{ks} \qquad (k\geq 0)$	**7**	$\frac{1}{\sqrt{\pi t}}u(t-k)$	
29.3.116	$\frac{1}{\sqrt{s}}e^{ks}\operatorname{erfc}\sqrt{ks} \qquad (k\geq 0)$	**7**	$\frac{1}{\sqrt{\pi(t+k)}}$	
29.3.117	$\operatorname{erf}\frac{k}{\sqrt{s}}$	**7**	$\frac{1}{\pi t}\sin 2k\sqrt{t}$	
29.3.118	$\frac{1}{\sqrt{s}}e^{\frac{k^2}{s}}\operatorname{erfc}\frac{k}{\sqrt{s}}$	**7**	$\frac{1}{\sqrt{\pi t}}e^{-2k\sqrt{t}}$	
29.3.119	$K_0(ks) \qquad (k>0)$	**9**	$\frac{1}{\sqrt{t^2-k^2}}u(t-k)$	
29.3.120	$K_0(k\sqrt{s}) \qquad (k>0)$	**9**	$\frac{1}{2t}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.121	$\frac{1}{s}e^{ks}K_1(ks) \qquad (k>0)$	**9**	$\frac{1}{k}\sqrt{t(t+2k)}$	
29.3.122	$\frac{1}{\sqrt{s}}K_1(k\sqrt{s}) \qquad (k>0)$	**9**	$\frac{1}{k}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.123	$\frac{1}{\sqrt{s}}e^{\frac{k}{s}}K_0\left(\frac{k}{s}\right) \qquad (k>0)$	**9**	$\frac{2}{\sqrt{\pi t}}K_0(2\sqrt{2kt})$	**9**
29.3.124	$\pi e^{-ks}I_0(ks) \qquad (k>0)$	**9**	$\frac{1}{\sqrt{t(2k-t)}}[u(t)-u(t-2k)]$	
29.3.125	$e^{-ks}I_1(ks) \qquad (k>0)$	**9**	$\frac{k-t}{\pi k\sqrt{t(2k-t)}}[u(t)-u(t-2k)]$	

	$f(s)$		$F(t)$
29.3.126	$e^{as}E_1(as) \qquad (a>0)$	**5**	$\frac{1}{t+a}$
29.3.127	$\frac{1}{a}-se^{as}E_1(as) \qquad (a>0)$	**5**	$\frac{1}{(t+a)^2}$
29.3.128	$a^{1-n}e^{as}E_n(as) \qquad (a>0;n=0,1,2,\ldots)$	**5**	$\frac{1}{(t+a)^n}$
29.3.129	$\left[\frac{\pi}{2}-\mathrm{Si}(s)\right]\cos s+\mathrm{Ci}(s)\sin s$	**5**	$\frac{1}{t^2+1}$

29.4. Table of Laplace-Stieltjes Transforms [4]

	$\phi(s)$	$\Phi(t)$
29.4.1	$\int_0^\infty e^{-st}d\Phi(t)$	$\Phi(t)$
29.4.2	$e^{-ks} \qquad (k>0)$	$u(t-k)$
29.4.3	$\frac{1}{1-e^{-ks}} \qquad (k>0)$	$\sum_{n=0}^{\infty} u(t-nk)$
29.4.4	$\frac{1}{1+e^{-ks}} \qquad (k>0)$	$\sum_{n=0}^{\infty} (-1)^n u(t-nk)$
29.4.5	$\frac{1}{\sinh ks} \qquad (k>0)$	$2\sum_{n=0}^{\infty} u[t-(2n+1)k]$
29.4.6	$\frac{1}{\cosh ks} \qquad (k>0)$	$2\sum_{n=0}^{\infty} (-1)^n u[t-(2n+1)k]$
29.4.7	$\tanh ks \qquad (k>0)$	$u(t)+2\sum_{n=1}^{\infty} (-1)^n u(t-2nk)$
29.4.8	$\frac{1}{\sinh (ks+a)} \qquad (k>0)$	$2\sum_{n=0}^{\infty} e^{-(2n+1)a}u[t-(2n+1)k]$
29.4.9	$\frac{e^{-hs}}{\sinh (ks+a)} \qquad (k>0,\, h>0)$	$2\sum_{n=0}^{\infty} e^{-(2n+1)a}u[t-h-(2n+1)k]$
29.4.10	$\frac{\sinh (hs+b)}{\sinh (ks+a)} \qquad (0<h<k)$	$\sum_{n=0}^{\infty} e^{-(2n+1)a}\{e^{b}u[t+h-(2n+1)k]$ $-e^{-b}u[t-h-(2n+1)k]\}$
29.4.11	$\sum_{n=0}^{\infty} a_n e^{-k_n s} \qquad (0<k_0<k_1<\ldots)$	$\sum_{n=0}^{\infty} a_n u(t-k_n)$

For the definition of the Laplace-Stieltjes transform see [29.7]. In practice, Laplace-Stieltjes transforms are often written as ordinary Laplace transforms involving Dirac's delta function $\delta(t)$. This "function" may formally be considered as the derivative of the unit step function, $du(t)=\delta(t)$ dt, so that $\int_{-\infty}^{x} du(t)=\int_{-\infty}^{x} \delta(t)dt=\begin{cases}0 & (x<0)\\ 1 & (x>0).\end{cases}$ The correspondence **29.4.2**, for instance, then assumes the form $e^{-ks}=\int_0^\infty e^{-st}\delta(t-k)dt$.

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[4] Adapted by permission from P. M. Morse and H. Feshbach, Methods of theoretical physics, vols. 1, 2, McGraw-Hill Book Co., Inc., New York, N.Y., 1953.

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$i \quad (=\sqrt{-1})$	70
$\binom{n}{k}$ binomial coefficient	10
$n!$ factorial function	255
$(2n)!! \quad =2\cdot 4\cdot 6 \ldots (2n)=2^n n!$	258
(m,n) greatest common divisor	822
$(n, k) \quad =\frac{\Gamma(\frac{1}{2}+n+k)}{k!\Gamma(\frac{1}{2}+n-k)}$ (Hankel's symbol)	437
$(n; n_1, n_2, \ldots, n_m)$ multinomial coefficient	823
$[x]$ largest integer $\leq x$	66

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$<x>$ nearest integer to x	222
$\bar{z}$ complex conjugate of z $(=x-iy)$	16
$z=x+iy$ complex number (Cartesian form)	16
$=re^{i\theta}$ (polar form)	16
$\|z\|$ absolute value or modulus of z	16
Σ overall summation	822
Σ' restricted summation	755
$\sum_p \prod_p$ sum or product taken over all prime numbers p	807
$\sum_{d\|n} \prod_{d\|n}$ sum or product overall positive divisors d of n	826
$⨍$ Cauchy's principal value of the integral	228
$\approx$ approximately equal	14
$\sim$ asymptotically equal	15
$<, >, \leq \geq$ inequality, inclusion	10
$\neq$ unequal	12

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