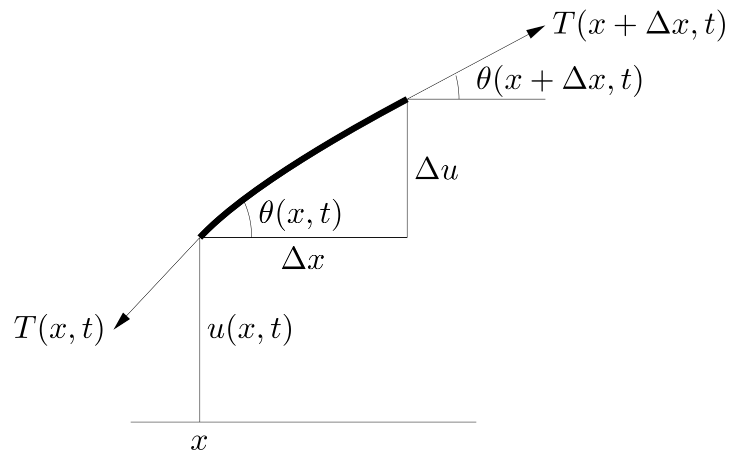
Derivation of the Wave Equation

In these notes we apply Newton's law to an elastic string, concluding that small amplitude transverse vibrations of the string obey the wave equation. Consider a tiny element of the string.



The basic notation is

$$\begin{aligned} u(x,t) &= \text{vertical displacement of the string from the } x \text{ axis at position } x \text{ and time } t\\ \theta(x,t) &= \text{angle between the string and a horizontal line at position } x \text{ and time } t\\ T(x,t) &= \text{tension in the string at position } x \text{ and time } t\\ \rho(x) &= \text{mass density of the string at position } x \end{aligned}$$

The forces acting on the tiny element of string are

(a) tension pulling to the right, which has magnitude $T(x+\Delta x,t)$ and acts at an angle $\theta(x+\Delta x,t)$ above horizontal

(b) tension pulling to the left, which has magnitude $T(x,t)$ and acts at an angle $\theta(x,t)$ below horizontal and, possibly,

(c) various external forces, like gravity. We shall assume that all of the external forces act vertically and we shall denote by $F(x,t)\Delta x$ the net magnitude of the external force acting on the element of string.

The mass of the element of string is essentially $\rho(x)\sqrt{\Delta x^2+\Delta u^2}$ so the vertical component of Newton's law says that

$$\rho(x)\sqrt{\Delta x^2+\Delta u^2}\,\tfrac{\partial^2 u}{\partial t^2}(x,t) = T(x+\Delta x,t)\sin\theta(x+\Delta x,t) - T(x,t)\sin\theta(x,t) + F(x,t)\Delta x$$

Dividing by Δx and taking the limit as $\Delta x \to 0$ gives

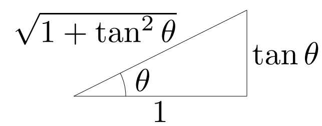
$$\begin{aligned} \rho(x)\sqrt{1+\left(\tfrac{\partial u}{\partial x}\right)^2}\,\tfrac{\partial^2 u}{\partial t^2}(x,t) &= \tfrac{\partial}{\partial x}\big[T(x,t)\sin\theta(x,t)\big] + F(x,t)\\ &= \tfrac{\partial T}{\partial x}(x,t)\sin\theta(x,t) + T(x,t)\cos\theta(x,t)\,\tfrac{\partial \theta}{\partial x}(x,t) + F(x,t) \end{aligned} \tag{1}$$

We can dispose of all the θ's by observing from the figure that

$$\tan\theta(x,t) = \lim_{\Delta x\to 0}\tfrac{\Delta u}{\Delta x} = \tfrac{\partial u}{\partial x}(x,t)$$

which implies, using the figure on the right below, that

$$\sin\theta(x,t) = \frac{\frac{\partial u}{\partial x}(x,t)}{\sqrt{1+\left(\frac{\partial u}{\partial x}(x,t)\right)^2}} \qquad \cos\theta(x,t) = \frac{1}{\sqrt{1+\left(\frac{\partial u}{\partial x}(x,t)\right)^2}}$$



$$\theta(x,t) = \tan^{-1}\tfrac{\partial u}{\partial x}(x,t) \qquad \frac{\partial\theta}{\partial x}(x,t) = \frac{\frac{\partial^2 u}{\partial x^2}(x,t)}{1+\left(\frac{\partial u}{\partial x}(x,t)\right)^2}$$

Substituting these formulae into (1) give a horrendous mess. However, we can get considerable simplification by looking only at small vibrations. By a small vibration, we mean that $|\theta(x,t)| \ll 1$ for all x and t. This implies that $|\tan\theta(x,t)| \ll 1$, hence that $\left|\frac{\partial u}{\partial x}(x,t)\right| \ll 1$ and hence that

$$\sqrt{1+\left(\tfrac{\partial u}{\partial x}\right)^2} \approx 1 \qquad \sin\theta(x,t) \approx \tfrac{\partial u}{\partial x}(x,t) \qquad \cos\theta(x,t)\approx 1 \qquad \tfrac{\partial\theta}{\partial x}(x,t) \approx \tfrac{\partial^2 u}{\partial x^2}(x,t) \tag{2}$$

Substituting these into equation (1) give

$$\rho(x)\tfrac{\partial^2 u}{\partial t^2}(x,t) = \tfrac{\partial T}{\partial x}(x,t)\tfrac{\partial u}{\partial x}(x,t) + T(x,t)\,\tfrac{\partial^2 u}{\partial x^2}(x,t) + F(x,t) \tag{3}$$

which is indeed relatively simple, but still exhibits a problem. This is one equation in the two unknowns u and T.

Fortunately there is a second equation lurking in the background, that we haven't used. Namely, the horizontal component of Newton's law of motion. As a second simplification, we assume that there are only transverse vibrations. Our tiny string element moves only vertically. Then the net horizontal force on it must be zero. That is,

$$T(x+\Delta x,t)\cos\theta(x+\Delta x,t) - T(x,t)\cos\theta(x,t) = 0$$

Dividing by Δx and taking the limit as Δx tends to zero gives

$$\tfrac{\partial}{\partial x}\big[T(x,t)\cos\theta(x,t)\big] = 0$$

For small amplitude vibrations, $\cos\theta$ is very close to one and $\frac{\partial T}{\partial x}(x,t)$ is very close to zero. In other words T is a function of t only, which is determined by how hard you are pulling on the ends of the string at time t. So for small, transverse vibrations, (3) simplifies further to

$$\rho(x)\tfrac{\partial^2 u}{\partial t^2}(x,t) = T(t)\,\tfrac{\partial^2 u}{\partial x^2}(x,t) + F(x,t) \tag{4}$$

In the event that the string density ρ is a constant, independent of x, the string tension $T(t)$ is a constant independent of t (in other words you are not continually playing with the tuning pegs) and there are no external forces F we end up with

$$\boxed{\tfrac{\partial^2 u}{\partial t^2}(x,t) = c^2\,\tfrac{\partial^2 u}{\partial x^2}(x,t)}$$

where

$$c = \sqrt{\tfrac{T}{\rho}}$$

