The Cauchy–Riemann Equations

Let $f(z)$ be defined in a neighbourhood of z_0. Recall that, by definition, f is differentiable at z_0 with derivative $f'(z_0)$ if

$$\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=f'(z_0)$$

Whether or not a function of one real variable is differentiable at some x_0 depends only on how smooth f is at x_0. The following example shows that this is no longer the case for the complex derivative.

Example 1 Let $f(z)=\bar z$. Then, writing Δz in its polar form $re^{i\theta}$,

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}=\frac{\overline{z_0+\Delta z}-\overline{z_0}}{\Delta z}=\frac{\overline{\Delta z}}{\Delta z}=\frac{re^{-i\theta}}{re^{i\theta}}=e^{-2\theta\, i}$$

So

- if we send Δz to zero along the real axis, so that $\theta=0$ or $\theta=\pi$ and hence $e^{-2\theta\, i}=1$, $\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ tends to 1, and
- if we send Δz to 0 along the imaginary axis, so that $\theta=\frac{\pi}{2}$ or $\frac{3\pi}{2}$ and hence $e^{-2\theta\, i}=-1$, $\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ tends to -1.

Thus $\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$ does not exist and $f(z)=\bar z$ is nowhere differentiable. Note that if we write $f(x+iy)=\overline{x+iy}=x-iy=u(x,y)+iv(x,y)$, then all partial derivatives of all orders of $u(x,y)=x$ and $v(x,y)=-y$ exist even though $f'(z)$ does not exist.

This example shows that differentiablility of $u(x,y)$ and $v(x,y)$ does not imply the differentiablility of $f(x+iy)=u(x,y)+iv(x,y)$. These notes explore further the relationship between $f'(z)$ and the partial derivatives of u and v. We shall first ask the question "Suppose that we know that $f'(z_0)$ exists. What does that tell us about $u(x,y)$ and $v(x,y)$?" Here is the answer.

Theorem 2 *Let $f(z)$ be defined in a neighbourhood of z_0. Assume that f is differentiable at z_0. Write $f(x+iy)=u(x,y)+iv(x,y)$. Then all of the partial derivatives $\frac{\partial u}{\partial x}(x_0,y_0)$, $\frac{\partial u}{\partial y}(x_0,y_0)$, $\frac{\partial v}{\partial x}(x_0,y_0)$, and $\frac{\partial v}{\partial y}(x_0,y_0)$ exist and*

$$\frac{\partial u}{\partial x}(x_0,y_0)=\frac{\partial v}{\partial y}(x_0,y_0)\qquad \frac{\partial u}{\partial y}(x_0,y_0)=-\frac{\partial v}{\partial x}(x_0,y_0)\tag{CR}$$

and

$$f'(x_0+iy_0)=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)$$

The equations (CR) are called the Cauchy–Riemann equations.

Proof: By assumption

$$
\begin{aligned}
f'(z_0) &= \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} \\
&= \lim_{\Delta z\to 0} \frac{[u(x_0+\Delta x, y_0+\Delta y)-u(x_0,y_0)]+i[v(x_0+\Delta x, y_0+\Delta y)-v(x_0,y_0)]}{\Delta z}
\end{aligned}
$$

In particular, by sending $\Delta z = \Delta x + i\Delta y$ to zero along the real axis (i.e. setting $\Delta y = 0$ and sending $\Delta x \to 0$), we have

$$
f'(x_0+iy_0) = \lim_{\Delta x\to 0} \frac{[u(x_0+\Delta x, y_0)-u(x_0,y_0)]+i[v(x_0+\Delta x, y_0)-v(x_0,y_0)]}{\Delta x}
$$

and hence

$$
\begin{aligned}
\operatorname{Re} f'(z_0) &= \lim_{\Delta x\to 0} \frac{u(x_0+\Delta x, y_0)-u(x_0,y_0)}{\Delta x} \\
\operatorname{Im} f'(z_0) &= \lim_{\Delta x\to 0} \frac{v(x_0+\Delta x, y_0)-v(x_0,y_0)}{\Delta x}
\end{aligned}
$$

This tells us that the partial derivatives $\frac{\partial u}{\partial x}(x_0,y_0)$, $\frac{\partial v}{\partial x}(x_0,y_0)$ exist and

$$
\tfrac{\partial u}{\partial x}(x_0,y_0) = \operatorname{Re} f'(x_0+iy_0) \qquad \tfrac{\partial v}{\partial x}(x_0,y_0) = \operatorname{Im} f'(x_0+iy_0) \tag{1}
$$

This gives the formula for $f'(x_0+iy_0)$ in the statement of the theorem.

If, instead, we send $\Delta z = \Delta x + i\Delta y$ to zero along the imaginary axis (i.e. set $\Delta x = 0$ and send $\Delta y \to 0$), we have

$$
\begin{aligned}
f'(x_0+iy_0) &= \lim_{\Delta y\to 0} \frac{[u(x_0, y_0+\Delta y)-u(x_0,y_0)]+i[v(x_0, y_0+\Delta y)-v(x_0,y_0)]}{i\Delta y} \\
&= \lim_{\Delta y\to 0} \frac{[v(x_0, y_0+\Delta y)-v(x_0,y_0)]-i[u(x_0, y_0+\Delta y)-u(x_0,y_0)]}{\Delta y}
\end{aligned}
$$

and hence

$$
\begin{aligned}
\operatorname{Re} f'(z_0) &= \lim_{\Delta x\to 0} \frac{v(x_0, y_0+\Delta y)-v(x_0,y_0)}{\Delta y} \\
\operatorname{Im} f'(z_0) &= -\lim_{\Delta x\to 0} \frac{u(x_0, y_0+\Delta y)-u(x_0,y_0)}{\Delta y}
\end{aligned}
$$

This tells us that the partial derivatives $\frac{\partial v}{\partial y}(x_0,y_0)$, $\frac{\partial u}{\partial y}(x_0,y_0)$ exist and

$$
\tfrac{\partial v}{\partial y}(x_0,y_0) = \operatorname{Re} f'(x_0+iy_0) \qquad \tfrac{\partial u}{\partial y}(x_0,y_0) = -\operatorname{Im} f'(x_0+iy_0) \tag{2}
$$

Comparing (1) and (2) gives (CR).

■

Theorem 2 says that it is necessary for $u(x,y)$ and $v(x,y)$ to obey the Cauchy–Riemann equations in order for $f(x+iy) = u(x+iy) + v(x+iy)$ to be differentiable. The following theorem says that, provided the first order partial derivatives of u and v are continuous, the converse is also true — if $u(x,y)$ and $v(x,y)$ obey the Cauchy–Riemann equations then $f(x+iy) = u(x+iy) + v(x+iy)$ is differentiable.

Theorem 3 *Let $z_0 \in \mathbb{C}$ and let G be an open subset of $\mathbb{C}$ that contains z_0. If $f(x+iy) = u(x,y) + iv(x,y)$ is defined on G and*

- *the first order partial derivatives of u and v exist in G and are continuous at (x_0, y_0)*
- *u and v obey the Cauchy–Riemann equations at (x_0, y_0),*

then f is differentiable at $z_0 = x_0 + iy_0$ and $f'(x_0 + iy_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$.

Proof: Write

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = U(\Delta z) + iV(\Delta z)$$

where

$$U(\Delta z) = \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta z}$$
$$V(\Delta z) = \frac{v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)}{\Delta z}$$

Our goal is to prove that $\lim_{\Delta z \to 0} [U(\Delta z) + iV(\Delta z)]$ exists and equals $\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$.

Concentrate on $U(\Delta z)$. The first step is to rewrite $U(\Delta z)$ in terms of expressions that will converge to partial derivatives of u and v. For example $\frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y}$ converges to $u_y(x_0, y_0)$ when $\Delta y \to 0$. We can achieve this by adding and subtracting $u(x_0, y_0 + \Delta y)$:

$$\begin{aligned} U(\Delta z) &= \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)}{\Delta z} \\ &= \frac{u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0 + \Delta y)}{\Delta z} + \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta z} \end{aligned}$$

To express $U(\Delta z)$ in terms of partial derivatives of u, we use the (ordinary first year Calculus) mean value theorem. Recall that it says that, if $F(x)$ is differentiable everywhere between x_0 and $x_0 + \Delta x$, then $F(x_0 + \Delta x) - F(x_0) = F'(x_0^*)\,\Delta x$ for some x_0^* between x_0 and $x_0 + \Delta x$. Applying the mean value theorem with $F(x) = u(x, y_0 + \Delta y)$ to the first half of $U(\Delta z)$ and with $F(y) = u(x_0, y)$ to the second half gives

$$U(\Delta z) = \frac{u_x(x_0^*, y_0 + \Delta y)\Delta x}{\Delta z} + \frac{u_y(x_0, y_0^*)\Delta y}{\Delta z}$$

for some x_0^* between x_0 and $x_0 + \Delta x$ and some y_0^* between y_0 and $y_0 + \Delta y$. Because u_x and u_y are continuous, $u_x(x_0^*, y_0 + \Delta y)$ is almost $u_x(x_0, y_0)$ and $u_y(x_0, y_0^*)$ is almost $u_y(x_0, y_0)$

when Δz is small. So we write

$$U(\Delta z) = \frac{u_x(x_0,y_0)\Delta x}{\Delta z} + \frac{u_y(x_0,y_0)\Delta y}{\Delta z} + E_1(\Delta z) + E_2(\Delta z)$$

where the "error terms" are

$$\begin{aligned} E_1(\Delta z) &= \big[u_x(x_0^*, y_0+\Delta y) - u_x(x_0,y_0)\big]\frac{\Delta x}{\Delta z} \\ E_2(\Delta z) &= \big[u_y(x_0, y_0^*) - u_y(x_0,y_0)\big]\frac{\Delta y}{\Delta z} \end{aligned}$$

Similarly

$$\begin{aligned} V(\Delta z) &= \frac{v_x(x_0^{**}, y_0+\Delta y)\Delta x}{\Delta z} + \frac{v_y(x_0, y_0^{**})\Delta y}{\Delta z} \\ &= \frac{v_x(x_0,y_0)\Delta x}{\Delta z} + \frac{v_y(x_0,y_0)\Delta y}{\Delta z} + E_3(\Delta z) + E_4(\Delta z) \end{aligned}$$

for some x_0^{**} between x_0 and $x_0+\Delta x$, and some y_0^{**} between y_0 and $y_0+\Delta y$. The error terms are

$$\begin{aligned} E_3(\Delta z) &= \big[v_x(x_0^{**}, y_0+\Delta y) - v_x(x_0,y_0)\big]\frac{\Delta x}{\Delta z} \\ E_4(\Delta z) &= \big[v_y(x_0, y_0^{**}) - v_y(x_0,y_0)\big]\frac{\Delta y}{\Delta z} \end{aligned}$$

Now as $\Delta z \to 0$

- both x_0^* and x_0^{**} (both of which are between x_0 and $x_0+\Delta x$) must approach x_0 and
- both y_0^* and y_0^{**} (both of which are between y_0 and $y_0+\Delta y$) must approach y_0 and
- $\left|\frac{\Delta x}{\Delta z}\right| \le 1$ and $\left|\frac{\Delta y}{\Delta z}\right| \le 1$

Recalling that u_x, u_y, v_x and v_y are all assumed to be continuous at (x_0,y_0), we conclude that

$$\lim_{\Delta z\to 0} E_1(\Delta z) = \lim_{\Delta z\to 0} E_2(\Delta z) = \lim_{\Delta z\to 0} E_3(\Delta z) = \lim_{\Delta z\to 0} E_4(\Delta z) = 0$$

and, using the Cauchy–Riemann equations,

$$\begin{aligned} \lim_{\Delta z\to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta z\to 0}\big[U(\Delta z) + iV(\Delta z)\big] \\ &= \lim_{\Delta z\to 0}\Big[\frac{u_x(x_0,y_0)\Delta x}{\Delta z} + \frac{u_y(x_0,y_0)\Delta y}{\Delta z} + i\frac{v_x(x_0,y_0)\Delta x}{\Delta z} + i\frac{v_y(x_0,y_0)\Delta y}{\Delta z}\Big] \\ &= \lim_{\Delta z\to 0}\Big[\frac{u_x(x_0,y_0)\Delta x}{\Delta z} - \frac{v_x(x_0,y_0)\Delta y}{\Delta z} + i\frac{v_x(x_0,y_0)\Delta x}{\Delta z} + i\frac{u_x(x_0,y_0)\Delta y}{\Delta z}\Big] \\ &= \lim_{\Delta z\to 0}\Big[u_x(x_0,y_0)\frac{\Delta x + i\Delta y}{\Delta z} + iv_x(x_0,y_0)\frac{\Delta x + i\Delta y}{\Delta z}\Big] \\ &= u_x(x_0,y_0) + iv_x(x_0,y_0) \end{aligned}$$

as desired. ■

Example 4 The function $f(z) = \bar{z}$ has $f(x + iy) = x - iy$ so that

$$u(x, y) = x \text{ and } v(x, y) = -y$$

The first order partial derivatives of u and v are

$$\begin{aligned} u_x(x, y) &= 1 \quad v_x(x, y) = 0 \\ u_y(x, y) &= 0 \quad v_y(x, y) = -1 \end{aligned}$$

As the Cauchy–Riemann equation $u_x(x, y) = v_y(x, y)$ is satisfied nowhere, the function $f(z) = \bar{z}$ is differentiable nowhere. We have already seen this in Example 1.

Example 5 The function $f(z) = e^z$ has

$$f(x + iy) = e^{x+iy} = e^x \big\{ \cos y + i \sin y \big\} = u(x, y) + iv(x, y)$$

with

$$u(x, y) = e^x \cos y \text{ and } v(x, y) = e^x \sin y$$

The first order partial derivatives of u and v are

$$\begin{aligned} u_x(x, y) &= e^x \cos y \qquad v_x(x, y) = e^x \sin y \\ u_y(x, y) &= -e^x \sin y \quad v_y(x, y) = e^x \cos y \end{aligned}$$

As the Cauchy–Riemann equations $u_x(x, y) = v_y(x, y)$, $u_y(x, y) = -v_x(x, y)$ are satisfied for all (x, y), the function $f(z) = e^z$ is entire and its derivative is

$$f'(z) = f'(x + iy) = u_x(x, y) + iv_x(x, y) = e^x \cos y + ie^x \sin y = e^z$$

Example 6 The function $f(x + iy) = x^2 + y + i(y^2 - x)$ has

$$u(x, y) = x^2 + y \text{ and } v(x, y) = y^2 - x$$

The first order partial derivatives of u and v are

$$\begin{aligned} u_x(x, y) &= 2x \quad v_x(x, y) = -1 \\ u_y(x, y) &= 1 \qquad v_y(x, y) = 2y \end{aligned}$$

As the Cauchy–Riemann equations $u_x(x, y) = v_y(x, y)$, $u_y(x, y) = -v_x(x, y)$ are satisfied only on the line $y = x$, the function f is differentiable on the line $y = x$ and nowhere else. So it is nowhere analytic.

Example 7 The function $f(x+iy) = x^2 - y^2 + 2ixy$ has

$$u(x,y) = x^2 - y^2 \text{ and } v(x,y) = 2xy$$

The first order partial derivatives of u and v are

$$\begin{array}{ll} u_x(x,y) = 2x & v_x(x,y) = 2y \\ u_y(x,y) = -2y & v_y(x,y) = 2x \end{array}$$

As the Cauchy–Riemann equations $u_x(x,y) = v_y(x,y)$, $u_y(x,y) = -v_x(x,y)$ are satisfied for all (x,y), this function is entire. There is another way to see this. It suffices to observe that $f(z) = z^2$, since $(x+iy)^2 = x^2 - y^2 + 2ixy$. So f is a polynomial in z and we already know that all polynomials are differentiable everywhere.

Example 8 The function $f(x+iy) = x^2 + y^2$ has

$$u(x,y) = x^2 + y^2 \text{ and } v(x,y) = 0$$

The first order partial derivatives of u and v are

$$\begin{array}{ll} u_x(x,y) = 2x & v_x(x,y) = 0 \\ u_y(x,y) = 2y & v_y(x,y) = 0 \end{array}$$

As the Cauchy–Riemann equations $u_x(x,y) = v_y(x,y)$, $u_y(x,y) = -v_x(x,y)$ are satisfied only at $x = y = 0$, the function f is differentiable only at the point $z = 0$. So it is nowhere analytic. There is another way to see that $f(z)$ cannot be differentiable at any $z \neq 0$. Just observe that $f(z) = z\bar{z}$. If $f(z)$ were differentiable at some $z_0 \neq 0$, then $\bar{z} = \frac{f(z)}{z}$ would also be differentiable at z_0 and we already know that this is not case.