Divergence Theorem and Variations

Theorem. *If V is a solid with surface ∂V*

$$\iint_{\partial V} \vec v\cdot\hat{\mathbf n}\,dS = \iiint_V \vec\nabla\cdot\vec v\ dV$$
$$\iint_{\partial V} \psi\hat{\mathbf n}\,dS = \iiint_V \vec\nabla\psi\ dV$$
$$\iint_{\partial V} \hat{\mathbf n}\times\vec v\,dS = \iiint_V \vec\nabla\times\vec v\ dV$$

where $\hat{\mathbf n}$ is the outward unit normal of ∂V.

Memory Aid. All three formulae can be combined into

$$\iint_{\partial V} \hat{\mathbf n}*\tilde v\,dS = \iiint_V \vec\nabla*\tilde v\ dV$$

where $*$ can be either $\cdot$, $\times$ or nothing. When $*=\cdot$ or $*=\times$, then $\tilde v=\vec v$. When $*$ is nothing, $\tilde v = v$.

Proof: The first formula is the divergence theorem and was proven in class.

To prove the second, assuming the first, apply the first with $\vec v=\psi\vec a$, where $\vec a$ is any constant vector.

$$\begin{aligned}\iint_{\partial V} \psi\vec a\cdot\hat{\mathbf n}\,dS &= \iiint_V \vec\nabla\cdot(\psi\vec a)\ dV\\ &= \iiint_V \big[(\vec\nabla\psi)\cdot\vec a+\psi\vec\nabla\cdot\vec a\big]\ dV\\ &= \iiint_V (\vec\nabla\psi)\cdot\vec a\ dV\end{aligned}$$

To get the second line, we used vector identity # 8. To get the third line, we just used that $\vec a$ is a constant, so that it is annihilated by all derivatives. Since $\vec a$ is a constant, we can factor it out of both integrals, so

$$\vec a\cdot\iint_{\partial V} \psi\hat{\mathbf n}\,dS = \vec a\cdot\iiint_V \vec\nabla\psi\ dV$$
$$\Longrightarrow \vec a\cdot\left\{\iint_{\partial V} \psi\hat{\mathbf n}\,dS - \iiint_V \vec\nabla\psi\ dV\right\} = 0$$

In particular, choosing $\vec{a} = \hat{\imath}, \hat{\jmath}, \hat{\mathbf{k}}$, we see that all three components of the vector $\iint_{\partial V} \psi \hat{\mathbf{n}}\, dS - \iiint_V \vec{\nabla}\psi\ dV$ are zero. So

$$\iint\limits_{\partial V} \psi \hat{\mathbf{n}}\, dS - \iiint_V \vec{\nabla}\psi\ dV = 0$$

which is what we wanted show.

To prove the third, assuming the first, apply the first with $\vec{v}$ replaced by $\vec{v} \times \vec{a}$, where $\vec{a}$ is any constant vector.

$$\begin{aligned}
\iint\limits_{\partial V} \vec{v} \times \vec{a} \cdot \hat{\mathbf{n}}\ dS &= \iiint_V \vec{\nabla} \cdot (\vec{v} \times \vec{a})\ dV \\
&= \iiint_V \big[(\vec{\nabla} \times \vec{v}) \cdot \vec{a} - \vec{v} \cdot \vec{\nabla} \times \vec{a}\big]\ dV \\
&= \iiint_V (\vec{\nabla} \times \vec{v}) \cdot \vec{a}\ dV
\end{aligned}$$

To get the second line, we used vector identity # 9. To get the third line, we just used that $\vec{a}$ is a constant, so that it is annihilated by all derivatives. For all vectors

$$\vec{v} \times \vec{a} \cdot \hat{\mathbf{n}} = \hat{\mathbf{n}} \cdot \vec{v} \times \vec{a} = \hat{\mathbf{n}} \times \vec{v} \cdot \vec{a}$$

so

$$\begin{aligned}
\vec{a} \cdot \iint\limits_{\partial V} \hat{\mathbf{n}} \times \vec{v}\ dS &= \vec{a} \cdot \iiint_V \vec{\nabla} \times \vec{v}\ dV \\
\Longrightarrow \vec{a} \cdot \Big\{ \iint\limits_{\partial V} \hat{\mathbf{n}} \times \vec{v}\ dS &- \iiint_V \vec{\nabla} \times \vec{v}\ dV \Big\} = 0
\end{aligned}$$

In particular, choosing $\vec{a} = \hat{\imath}, \hat{\jmath}, \hat{\mathbf{k}}$, we see that all three components of the vector $\iint_{\partial V} \hat{\mathbf{n}} \times \vec{v}\ dS - \iiint_V \vec{\nabla} \times \vec{v}\ dV$ are zero. So

$$\iint\limits_{\partial V} \hat{\mathbf{n}} \times \vec{v}\ dS - \iiint_V \vec{\nabla} \times \vec{v}\ dV = 0$$

which is what we wanted show. ■