The Cayley Transform and Self–adjoint Extensions

Throughout these notes $\mathcal{H}$ is a Hilbert space. We shall use $D(B)$ and $R(B)$ to denote the domain and range, respectively, of the linear operator B.

Theorem 1 (Cayley Transform)

(a) If $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is a densely defined symmetric linear operator, then

$$C_+(T) = (T - i\mathbb{1})(T + i\mathbb{1})^{-1}$$

is a well–defined isometric operator with domain $R(T + i\mathbb{1})$ and range $R(T - i\mathbb{1})$. Furthermore 1 is not an eigenvalue of $C_+(T)$ and

$$T = i(\mathbb{1} + C_+(T))\big(\mathbb{1} - C_+(T)\big)^{-1}$$

WARNING: It is possible for the domain $D\big(C_+(T)\big) = R(T + i\mathbb{1})$ and/or the range $R\big(C_+(T)\big) = R(T - i\mathbb{1})$ to not even be dense in $\mathcal{H}$.

(b) If $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is a densely defined symmetric linear operator and is closed, then the domain $D\big(C_+(T)\big) = R(T + i\mathbb{1})$ and the range $R\big(C_+(T)\big) = R(T - i\mathbb{1})$ are closed.

(c) If $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is a densely defined self–adjoint linear operator, then

$$C_+(A) = (A - i\mathbb{1})(A + i\mathbb{1})^{-1}$$

is a well–defined unitary operator on $\mathcal{H}$. Furthermore 1 is not an eigenvalue of $C_+(A)$.

(d) If U is a unitary operator on $\mathcal{H}$ which does not have 1 as an eigenvalue, then

$$C_-(U) = i(\mathbb{1} + U)(\mathbb{1} - U)^{-1}$$

is a well–defined, densely defined self–adjoint operator with domain $R(\mathbb{1} - U)$ and range $R(\mathbb{1} + U)$.

(e) If $A : D(A) \to \mathcal{H}$ is a self–adjoint linear operator, then $C_-\big(C_+(A)\big) = A$.
If $U : \mathcal{H} \to \mathcal{H}$ is a unitary operator which does not have 1 as an eigenvalue, then $C_+\big(C_-(U)\big) = U$.

(f) Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is a densely defined symmetric operator.
If T' is a symmetric extension of T, then $C_+(T')$ is isometric extension of $C_+(T)$.
If $A : D(A) \to \mathcal{H}$ is a self–adjoint extension of T, then $C_+(A)$ is a unitary extension of $C_+(T)$.
Conversely, if U is a unitary extension of $C_+(T)$, then $C_-(U)$ is a self–adjoint extension of T.

Proof: (a) *Well–defined:*

If $\mathbf{v} \in D(T)$, then, as T is symmetric,

$$\|(T + i\mathbb{1})\mathbf{v}\|^2 = \langle (T + i\mathbb{1})\mathbf{v}\,,\, (T + i\mathbb{1})\mathbf{v}\rangle = \|T\mathbf{v}\|^2 + \|\mathbf{v}\|^2 \ge \|\mathbf{v}\|^2$$

Consequently $(T+i\mathbb{1})$ is injective, so that $(T+i\mathbb{1})^{-1}$ is well–defined with domain $R(T+i\mathbb{1})$ and range $D(T+i\mathbb{1}) = D(T)$. So $C_+(T)$ is well–defined with domain $R(T+i\mathbb{1})$ and range $R(T - i\mathbb{1})$.

Isometric:

Let $\psi \in D\big(C_+(T)\big) = R(T + i\mathbb{1})$, and write $\varphi = (T + i\mathbb{1})^{-1}\psi$. Then

$$\begin{aligned} \|C_+(T)\,\psi\|^2 &= \|(T - i\mathbb{1})(T + i\mathbb{1})^{-1}\psi\|^2 = \|(T - i\mathbb{1})\varphi\|^2 = \|T\varphi\|^2 + \|\varphi\|^2 \\ \|\psi\|^2 &= \|(T + i\mathbb{1})(T + i\mathbb{1})^{-1}\psi\|^2 = \|(T + i\mathbb{1})\varphi\|^2 = \|T\varphi\|^2 + \|\varphi\|^2 \end{aligned}$$

So $\|C_+(T)\,\psi\| = \|\psi\|$.

$1 \notin \sigma_p\big(C_+(T)\big)$*:*

Let $\psi \in D\big(C_+(T)\big) = R(T + i\mathbb{1})$, and write $\varphi = (T + i\mathbb{1})^{-1}\psi$. Then

$$C_+(T)\,\psi = \psi \implies (T - i\mathbb{1})\varphi = (T + i\mathbb{1})\varphi \implies \varphi = 0 \implies \psi = (T + i\mathbb{1})\varphi = 0$$

$T = i(\mathbb{1} + C_+(T))\big(\mathbb{1} - C_+(T)\big)^{-1}$*:*

Since 1 is not an eigenvalue of $C_+(T)$, the inverse $\big(\mathbb{1} - C_+(T)\big)^{-1}$ is well defined and has range $D\big(\mathbb{1} - C_+(T)\big) = D\big(C_+(T)\big)$. So at least $i(\mathbb{1} + C_+(T))\big(\mathbb{1} - C_+(T)\big)^{-1}$ is well–defined. Since

$$\begin{aligned} \mathbb{1} - C_+(T) &= (T + i\mathbb{1})(T + i\mathbb{1})^{-1} - (T - i\mathbb{1})(T + i\mathbb{1})^{-1} = 2i(T + i\mathbb{1})^{-1} \\ \mathbb{1} + C_+(T) &= (T + i\mathbb{1})(T + i\mathbb{1})^{-1} + (T - i\mathbb{1})(T + i\mathbb{1})^{-1} = 2T(T + i\mathbb{1})^{-1} \end{aligned}$$

the operator $i(\mathbb{1} + C_+(T))\big(\mathbb{1} - C_+(T)\big)^{-1}$ has domain

$$R(\mathbb{1} - C_+(T)) = R\big(2i(T + i\mathbb{1})^{-1}\big) = D(T + i\mathbb{1}) = D(T)$$

and

$$i\big[\mathbb{1} + C_+(T)\big]\big[\mathbb{1} - C_+(T)\big]^{-1} = i\big[2T(T + i\mathbb{1})^{-1}\big]\big[2i(T + i\mathbb{1})^{-1}\big]^{-1} = T$$

(b) Let $\big\{\psi_n = (T \pm i\mathbb{1})\varphi_n\big\}_{n\in\mathbb{N}} \subset R(T \pm i\mathbb{1})$ converge to $\psi \in \mathcal{H}$. Then $\big\{(T \pm i\mathbb{1})\varphi_n\big\}_{n\in\mathbb{N}}$ is Cauchy. As $(T \pm i\mathbb{1})^{-1}$ is bounded, $\big\{\varphi_n\big\}_{n\in\mathbb{N}}$ is also Cauchy and so converges to some $\varphi \in \mathcal{H}$. As T (and hence $T \pm i\mathbb{1}$) is closed, $\psi = (T \pm i\mathbb{1})\varphi \in R\big(T \pm i\mathbb{1}\big)$.

(c) By part (a), $C_+(A)$ is a well–defined isometric operator with domain $R(A + i\mathbb{1})$ and range $R(A - i\mathbb{1})$ and furthermore does not have 1 as an eigenvalue. As A is self–adjoint, $R(A + i\mathbb{1}) = R(A - i\mathbb{1}) = \mathcal{H}$, so that $C_+(A)$ is unitary.

(d) *Densely defined:*

We first observe that 1 cannot be an eigenvalue of U^* because

$$U^*\psi = \psi \implies UU^*\psi = U\psi \implies U\psi = \psi \implies \psi = 0$$

Consequently

$$\big[D\big(C_-(U)\big)\big]^\perp = \big[R(\mathbb{1} - U)\big]^\perp = \ker(\mathbb{1} - U^*) = \{0\}$$

so that $C_-(U)$ is well–defined and densely defined.

Symmetric:

We now verify that $C_-(U)$ is symmetric. If $\varphi, \psi \in D\big(C_-(U)\big) = R(\mathbb{1} - U)$, then, writing $\varphi = (\mathbb{1} - U)\Phi$ and $\psi = (\mathbb{1} - U)\Psi$,

$$\begin{aligned}
\langle \varphi\,,\, C_-(U)\psi\rangle &= \big\langle \varphi\,,\, i(\mathbb{1}+U)(\mathbb{1}-U)^{-1}\psi\big\rangle = \langle (\mathbb{1}-U)\Phi\,,\, i(\mathbb{1}+U)\Psi\rangle \\
&= \langle -i(\mathbb{1}+U^*)(\mathbb{1}-U)\Phi\,,\, \Psi\rangle \\
&= \langle -i[U^* - U]\Phi\,,\Psi\rangle \\
\langle C_-(U)\varphi\,,\, \psi\rangle &= \big\langle i(\mathbb{1}+U)(\mathbb{1}-U)^{-1}\varphi\,,\, \psi\big\rangle = \langle i(\mathbb{1}+U)\Phi\,,\, (\mathbb{1}-U)\Psi\rangle \\
&= \langle i(\mathbb{1}-U^*)(\mathbb{1}+U)\Phi\,,\, \Psi\rangle \\
&= \langle i[U - U^*]\Phi\,,\Psi\rangle
\end{aligned}$$

So $\langle \varphi\,,\, C_-(U)\psi\rangle = \langle C_-(U)\varphi\,,\, \psi\rangle$ for all $\varphi, \psi \in D\big(C_-(U)\big)$ and $C_-(U)$ is symmetric.

Self–adjoint:

Finally, we verify that $R\big(C_-(U) \pm i\mathbb{1}\big) = \mathcal{H}$. As

$$\begin{aligned}
C_-(U) + i\mathbb{1} &= i(\mathbb{1}+U)(\mathbb{1}-U)^{-1} + i(\mathbb{1}-U)(\mathbb{1}-U)^{-1} = 2i(\mathbb{1}-U)^{-1} \\
C_-(U) - i\mathbb{1} &= i(\mathbb{1}+U)(\mathbb{1}-U)^{-1} - i(\mathbb{1}-U)(\mathbb{1}-U)^{-1} = 2iU(\mathbb{1}-U)^{-1}
\end{aligned}$$

the range of $C_-(U) + i\mathbb{1}$ is $D(\mathbb{1} - U) = \mathcal{H}$. As U is unitary, it has range $\mathcal{H}$, so that the range of $C_-(U) - i\mathbb{1} = 2iU(\mathbb{1} - U)^{-1}$ is also $\mathcal{H}$.

(e) We already proved that $C_-\big(C_+(A)\big) = A$ (just with A renamed to T) in part (a). To prove that $C_+\big(C_-(U)\big) = U$, we just recall that

$$C_-(U) + i\mathbb{1} = 2i(\mathbb{1}-U)^{-1} \qquad \text{and} \qquad C_-(U) - i\mathbb{1} = 2iU(\mathbb{1}-U)^{-1}$$

so that

$$\begin{aligned}
C_+\big(C_-(U)\big) &= \big[C_-(U) - i\mathbb{1}\big]\big[C_-(U) + i\mathbb{1}\big]^{-1} = \big[2iU(\mathbb{1}-U)^{-1}\big]\big[2i(\mathbb{1}-U)^{-1}\big]^{-1} \\
&= U
\end{aligned}$$

(f) This is (mostly) obvious from the following (also obvious) observations. Let V, $\tilde V$, W, $\tilde W$ be linear operators with

$$V \subset \tilde V \qquad W \subset \tilde W$$

- For all $\alpha, \beta \in \mathbb{C}$,

  $$\alpha V + \beta W \subset \alpha \tilde V + \beta \tilde W$$

  Here $D(\alpha V + \beta W) = D(V) \cap D(W)$, $D(\alpha \tilde V + \beta \tilde W) = D(\tilde V) \cap D(\tilde W)$
- If $R(W) \subset D(V)$ and $R(\tilde W) \subset D(\tilde V)$, then

  $$VW \subset \tilde V \tilde W$$

- If $\tilde V$ is injective, then so is V and

  $$V^{-1} \subset \tilde V^{-1}$$

All of the claims follow from those observations, combined with the fact that
- if U is a unitary extension of $C_+(T)$ then it is automatically true that $1 \notin \sigma_p(U)$ (see Problem Set 6 #4)

■

Suppose that T is a densely defined symmetric linear operator and that A is a self–adjoint extension of T. By part (f) of the last theorem, $C_+(A)$ is a unitary extension of $C_+(T)$, which has domain $R(T + i\mathbb{1})$ and range $R(T - i\mathbb{1})$. In particular, $C_+(A)$ is an isometric bijection from $R(T + i\mathbb{1})$ to $R(T - i\mathbb{1})$ and that part of $C_+(A)$ is completely determined by $C_+(T)$. For $C_+(A)$ to be unitary, it has to also be an isometric bijection from $R(T + i\mathbb{1})^\perp$ to $R(T - i\mathbb{1})^\perp$. That is, using $\restriction$ to denote "restricted to", $C_+(A)$ is determined by
- $C_+(A) \restriction R(T + i\mathbb{1}) = C_+(T)$
- $C_+(A) \restriction R(T + i\mathbb{1})^\perp$ which can be any isometric bijection to $R(T - i\mathbb{1})^\perp$.

So we have

Corollary 2 (Self–adjoint Extensions) *Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined symmetric linear operator.*

(a) If $\dim R(T + i\mathbb{1})^\perp = \dim R(T - i\mathbb{1})^\perp = 0$ (i.e. if $R(T + i\mathbb{1})$ and $R(T - i\mathbb{1})$ are dense), then T has a unique self–adjoint extension.

(b) If $\dim R(T + i\mathbb{1})^\perp = \dim R(T - i\mathbb{1})^\perp \geq 1$, then T has infinitely many distinct self–adjoint extensions.

(c) If $\dim R(T + i\mathbb{1})^\perp \neq \dim R(T - i\mathbb{1})^\perp$, then T has no self–adjoint extensions.

Definition 3 If $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is a densely defined symmetric linear operator, then the integers $n_+(T) = \dim R(T + i\mathbb{1})^\perp$ and $n_-(T) = \dim R(T - i\mathbb{1})^\perp$ are called the *deficiency indices* of T.

Remark 4 Suppose that $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is a densely defined symmetric linear operator and $\dim R(T + i\mathbb{1})^\perp = \dim R(T - i\mathbb{1})^\perp$. Fix any orthonormal basis $\{\mathbf{e}_i\}_{i\in\mathcal{I}}$ for $R(T + i\mathbb{1})^\perp$. Then, for each orthonormal basis $\{\mathbf{f}_i\}_{i\in\mathcal{I}}$ for $R(T - i\mathbb{1})^\perp$,

$$\begin{aligned} C_+(A_\mathbf{f})\,\mathbf{v} &= C_+(T)\,\mathbf{v} && \text{for all } \mathbf{v} \in R(T + i\mathbb{1}) \\ C_+(A_\mathbf{f})\,\mathbf{e}_i &= \mathbf{f}_i && \text{for all } i \in \mathcal{I} \end{aligned}$$

determines a self–adjoint extension of T.

Remark 5 Suppose that $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is a densely defined symmetric linear operator and $\dim R(T+i\mathbb{1})^\perp = \dim R(T-i\mathbb{1})^\perp = 1$. Let $\mathbf{e}$ be a unit vector in $R(T+i\mathbb{1})^\perp$ and $\mathbf{f}$ be a unit vector in $R(T - i\mathbb{1})^\perp$. Then there is a 1–1 correspondence between self–adjoint extensions of T and $\{\ \alpha \in \mathbb{C}\ \big|\ |\alpha| = 1\ \}$. If A_α is the α^{th} extension, then

$$\begin{aligned} C_+(A_\alpha)\,\mathbf{v} &= C_+(T)\,\mathbf{v} && \text{for all } \mathbf{v} \in R(T + i\mathbb{1}) \\ C_+(A_\alpha)\,\mathbf{e} &= \alpha\,\mathbf{f} \end{aligned}$$

Definition 6 (Essentially Self–adjoint)

(a) A densely defined symmetric operator T is said to be *essentially self–adjoint* if its closure $\bar{T}$ is self–adjoint.

(b) If A is a self–adjoint operator and D is a dense subspace of $D(A) \subset \mathcal{H}$ and the restriction, $A \restriction D$, of A to D obeys $\overline{A \restriction D} = A$, then D is called a *core* for A.

Theorem 7 *Let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a densely defined symmetric linear operator. Then*

T is essentially self–adjoint $\iff$ T has a unique self–adjoint extension

Proof: $\implies$ Let T be essentially self–adjoint and A be any self–adjoint extension of A. Then

$$T \subset A \implies A = A^* \subset T^* \implies \bar{T} = T^{**} \subset A^* = A$$

and

$$\bar{T} = T^{**} \subset A \implies A = A^* \subset (\bar{T})^* = \bar{T}$$

since T is essentially self–adjoint. So $A = \bar{T}$.
$\Longleftarrow$ Let T have a unique self–adjoint extension. Then by Corollary 2, $R(T + i\mathbb{1})$ and $R(T - i\mathbb{1})$ are dense. As $\bar{T}$ is a symmetric extension of T, $C_+(\bar{T})$ is an isometric extension of $C_+(T)$, which by part (b) of Theorem 1 has domain and range $\mathcal{H}$. That is, $C_+(\bar{T})$ is unitary and $\bar{T}$ is the hypothesized unique self–adjoint extension.

■

Example 8 In this example, we consider the operator $i\frac{d}{dx}$ acting on a number of different domains, all of which are subspaces of

$$AC[0,1] = \Big\{ \varphi : [0,1] \to \mathbb{C} \ \Big|\ \varphi(x) = C + \textstyle\int_0^x f(t)\ dt \text{ for some } C \in \mathbb{C}, f \in L^2([0,1]) \Big\}$$

When $\varphi(x) = C + \int_0^x f(t)\ dt \in AC[0,1]$, we define $\frac{d}{dx}\varphi = f$. Note that, by Cauchy–Schwarz, every $\varphi \in AC[0,1]$ is continuous.

All of the operators in this example will be extensions of

$$T = i\tfrac{d}{dx} \quad \text{with } D(T) = AC[0,1] \cap \big\{ \ \varphi \in AC[0,1] \ \big|\ \varphi(0) = \varphi(1) = 0 \ \big\}$$

By Problem Set 6, #1,

$$T^* = i\tfrac{d}{dx} \quad \text{with } D(T^*) = AC[0,1]$$

In particular, $T \subset T^*$, so that T is symmetric. If A is any self–adjoint extension of T, then, as $T \subset A$, we have $A = A^* \subset T^*$. Thus A must be $i\frac{d}{dx}$ with

$$AC[0,1] \cap \big\{ \ \varphi \in AC[0,1] \ \big|\ \varphi(0) = \varphi(1) = 0 \ \big\} \subset D(A) \subset AC[0,1] \tag{$*$}$$

Now

$$\begin{aligned}
\varphi \in D(A^*) \iff & \quad \big\langle i\tfrac{d}{dx}\varphi\,,\,\psi\big\rangle = \big\langle \varphi\,,\, i\tfrac{d}{dx}\psi\big\rangle && \text{for all } \psi \in D(A) \\
\iff & \quad -\int_0^1 \overline{\varphi'(x)}\,\psi(x)\,dx = \int_0^1 \overline{\varphi(x)}\,\psi'(x)\,dx && \text{for all } \psi \in D(A) \\
\iff & \quad \int_0^1 \big[\overline{\varphi'(x)}\,\psi(x) + \overline{\varphi(x)}\,\psi'(x)\big]\,dx = 0 && \text{for all } \psi \in D(A) \\
\iff & \quad \overline{\varphi(1)}\,\psi(1) - \overline{\varphi(0)}\,\psi(0) = 0 && \text{for all } \psi \in D(A)
\end{aligned}$$

So A is self–adjoint if and only if $D(A)$ obeys $(*)$ and

$$\varphi \in D(A) \iff \overline{\varphi(1)}\,\psi(1) - \overline{\varphi(0)}\,\psi(0) = 0 \quad \text{for all } \psi \in D(A) \tag{$**$}$$

Now

- in particular, choosing $\varphi = \psi$, we must have $|\psi(1)|^2 = |\psi(0)|^2$ for all $\psi \in D(A)$.
- There must be at least one $\psi \in D(A)$ with $\psi(0) \neq 0$, because otherwise we would have $D(A) = D(T)$ and we know that T is not self–adjoint.
- If $\psi, \varphi \in D(A)$ and $\psi(0) \neq 0$ and $\varphi(0) \neq 0$, then $(**)$ gives

$$\frac{\psi(1)}{\psi(0)} = \overline{\left(\frac{\varphi(0)}{\varphi(1)}\right)} = \frac{\varphi(1)}{\varphi(0)}$$

since $\frac{\varphi(1)}{\varphi(0)}$ has modulus one. Thus every $\varphi \in D(A)$ with $\varphi(0) \neq 0$ must have the same value of $\frac{\varphi(1)}{\varphi(0)}$. Call that common value, which must have modulus one, α.
- Conversely, for each $\alpha \in \mathbb{C}$ with $|\alpha| = 1$,

$$D(A_\alpha) = AC[0,1] \cap \left\{ \varphi \in AC[0,1] \,\middle|\, \varphi(1) = \alpha\,\varphi(0) \right\}$$

obeys both $(*)$ and $(**)$.

We have found all self–adjoint extensions of T, namely

$$A_\alpha = i\tfrac{d}{dx} \quad \text{with } D(A_\alpha) = AC[0,1] \cap \left\{ \varphi \in AC[0,1] \,\middle|\, \varphi(1) = \alpha\,\varphi(0) \right\}$$

with $\alpha \in \mathbb{C}$, $|\alpha| = 1$.

This conclusion implies that the deficiency indices of T must both be one. Let's compute the deficiency indices directly, as a consistency check. Using "Ker" to denote the kernel,

$$\begin{aligned} R(T \pm i\mathbb{1})^\perp &= \operatorname{Ker}(T^* \mp i\mathbb{1}) \\ &= \left\{ \varphi \in AC[0,1] \,\middle|\, \varphi' = \pm\varphi \right\} \\ &= \left\{ Ce^{\pm x} \,\middle|\, C \in \mathbb{C} \right\} \end{aligned}$$

Thus

$$n_\pm(T) = \dim R(T \pm i\mathbb{1})^\perp = 1$$

as expected.

Example 9 In this example we construct a symmetric operator T that has no self–adjoint extensions. Motivated by Corollary 2, we do so by first constructing an isometry J with $\dim D(J)^\perp \neq \dim R(J)^\perp$ and with $1 \notin \sigma_p(J)$, and then defining $T = i(\mathbb{1} + J)\big(\mathbb{1} - J\big)^{-1}$.

The right shift operator $J : \ell^2 \to \ell^2$, defined by

$$J\big(a_1, a_2, a_3, \cdots\big) = \big(0, a_1, a_2, a_3, \cdots\big)$$

does the job. It
- obeys $\|J\mathbf{a}\| = \|\mathbf{a}\|$ and so is an isometry, and

- has domain $D(J) = \ell^2$, so that $\dim D(J)^\perp = 0$, and
- has range $R(J) = \left\{ \ (b_1, b_2, b_3, \cdots) \in \ell^2 \ \middle|\ b_1 = 0 \ \right\}$ so that the orthogonal complement $R(J)^\perp = \left\{ \ (b_1, 0, 0, \cdots) \in \ell^2 \ \middle|\ b_1 \in \mathbb{C} \ \right\}$ has $\dim R(J)^\perp = 1$, and
- does not have 1 as an eigenvalue, since

$$\begin{aligned} & J\big(a_1, a_2, a_3, \cdots\big) = \big(a_1, a_2, a_3, \cdots\big) \\ \iff & \big(0, a_1, a_2, a_3, \cdots\big) = \big(a_1, a_2, a_3, \cdots\big) \\ \iff & 0 = a_1 = a_2 = a_3 = \cdots \end{aligned}$$

As J does not have 1 as an eigenvalue,

$$T = i(\mathbb{1} + J)\big(\mathbb{1} - J\big)^{-1}$$

is well–defined and has domain $D(T) = R(\mathbb{1} - J)$. We can check directly that T has the desired properties. We shall use that J has domain ℓ^2, that $J^* = L$, the left shift operator, which also has domain ℓ^2, and that $LJ = \mathbb{1}$. (But beware that $JL \neq \mathbb{1}$.)

- T is symmetric:

  Let $\mathbf{v} = (\mathbb{1} - J)\mathbf{a}, \ \mathbf{w} = (\mathbb{1} - J)\mathbf{b} \in D(T)$. Then

$$\begin{aligned} \langle \mathbf{v}, T\mathbf{w} \rangle &= \langle (\mathbb{1} - J)\mathbf{a}, i(\mathbb{1} + J)\mathbf{b} \rangle = \langle -i(\mathbb{1} + L)(\mathbb{1} - J)\mathbf{a}, \mathbf{b} \rangle \\ &= \langle -i[L - J]\mathbf{a}, \mathbf{b} \rangle \\ \langle T\mathbf{v}, \mathbf{w} \rangle &= \langle i(\mathbb{1} + J)\mathbf{a}, (\mathbb{1} - J)\mathbf{b} \rangle = \langle i(\mathbb{1} - L)(\mathbb{1} + J)\mathbf{a}, \mathbf{b} \rangle \\ &= \langle i[J - L]\mathbf{a}, \mathbf{b} \rangle \end{aligned}$$

  so that $\langle \mathbf{v}, T\mathbf{w} \rangle = \langle T\mathbf{v}, \mathbf{w} \rangle$.
- $\dim R(T + i\mathbb{1})^\perp = 0$ but $\dim R(T - i\mathbb{1})^\perp = 1$:

  Since

$$\begin{aligned} T + i\mathbb{1} &= i(\mathbb{1} + J)(\mathbb{1} - J)^{-1} + i(\mathbb{1} - J)(\mathbb{1} - J)^{-1} = 2i(\mathbb{1} - J)^{-1} \\ T - i\mathbb{1} &= i(\mathbb{1} + J)(\mathbb{1} - J)^{-1} - i(\mathbb{1} - J)(\mathbb{1} - J)^{-1} = 2iJ(\mathbb{1} - J)^{-1} \end{aligned}$$

  we have that $R(T + i\mathbb{1}) = D(\mathbb{1} - J) = \ell^2$ and $R(T - i\mathbb{1}) = R(J)$ and hence $\dim R(T + i\mathbb{1})^\perp = 0$ and $\dim R(T - i\mathbb{1})^\perp = \dim R(J)^\perp = 1$.