# OSCILLATION OF ERROR TERMS ; LITTLEWOOD'S RESULT.

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ABSTRACT. This is a note for MATH 613 E (Topics in Number theory) class on Littlewood's result on error terms of prime number theorem where he proves that $\pi(x) - \mathrm{li}(x)$ can take both positive and negative values for infinitely many times. Here we follow closely chapter 15 of [14] .

## 1. INTRODUCTION

The problem of counting the number of primes has been fascinated people from the ancient time. In 1791, Gauss conjectured that the number of primes $\leq x$ (denoted by $\pi(x)$) is approximated by $x/\log x$. In a letter to Encke in 1849, Gauss gave a better guess given by the logarithmic integral.

$$\pi(x) \approx \mathrm{li}(x) = \int_2^x \frac{1}{\log t}dt$$

This statement were proved precisely in 1896 (The Prime Number Theorem). After Gauss, Riemann (see [3, Chapter 1]) gave an even better estimate. First, we begin with

$$\Pi(x) := \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + ...$$

Using Mӧbius Inversion Formula [14, Section 10.9], we can solve for $\pi(x)$.

$$\pi(x) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m}\Pi(x^{1/m}).$$

Now Riemann replaced $\Pi(x)$ in the formula above by the function $\mathrm{Li}(x)$. Here

$$\mathrm{Li}(x) = \lim_{\epsilon \searrow 0}\left(\int_0^{1-\epsilon} \frac{dt}{\log t} + \int_{1+\epsilon}^x \frac{dt}{\log t}\right).$$

To do this, Riemann gave a heuristic proof (the formal proofs are given later by many people) of the following analytic formula.

$$\Pi(x) = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}, x > 1. \tag{1}$$

Here the sum over $\rho$ is the sum over nontrivial zero of Riemann zeta function. Note that $\mathrm{Li}(x)$ is the main term, the sum over $\rho$ is an oscillation term, the remaining are neglectable error terms that does not grow with $x$. We see that, for a suitable $N$,

$$\pi(x) - \sum_{n=1}^{N} \frac{\mu(n)}{n} \mathrm{Li}(x^{1/n}) = \sum_{n=1}^{N}\sum_{\rho} \mathrm{Li}(x^{\rho/n}) + \text{ lesser terms.} \tag{2}$$

If we ignore the oscillation and the constant terms, we get an approximation

$$\pi(x) \approx \mathrm{Li}(x) - \frac{1}{2}\mathrm{Li}(x^{1/2}) - \frac{1}{3}\mathrm{Li}(x^{1/3}) - \frac{1}{5}\mathrm{Li}(x^{1/5}) + \frac{1}{6}\mathrm{Li}(x^{1/6}) - ... \tag{3}$$

Notice that the first term is Gauss's approximation. Emperically, (3) turns out to be a very good approximation to $\pi(x)$. This means the oscillation error terms are small. This may be surprising to the fact that $\sum_\rho[\mathrm{Li}(x^\rho)+\mathrm{Li}(x^{1-\rho})]$ is only conditionally convergent and hence the smallness of this sum depends on cancellation. Also, each term $\mathrm{Li}(x^\rho)$ grows in magnitude like $x^\rho/\log(x^\rho) = x^{\mathrm{Re}(\rho)}/\rho\log x$ so many of them grow at least as fast as $x^{1/2}/\log x \sim 2\,\mathrm{Li}(x^{1/2}) > \mathrm{Li}(x^{1/3})$. Thus this is expected to be as significant as $-(1/2)\,\mathrm{Li}(x^{1/2})$ term and more significant than other terms.

Gauss observed that $\mathrm{Li}(x)$ always exceeds the actual number of primes where he calculated up to $x = 3,000,000$. People began to believe that this may be always the case as we may see from (3) that $\mathrm{Li}(x) - (1/2)\,\mathrm{Li}(x^{1/2})$ should be a better approximation (see also Table 6 in [5] that $\pi(x) - [\mathrm{Li}(x) - (1/2)\,\mathrm{Li}(x^{1/2})]$ has no bias in sign.) and the reason that $\pi(x) < \mathrm{Li}(x)$ comes from the term $-\mathrm{Li}(x^{1/2})$. However it turns out that it is actually not true that $\pi(x) < \mathrm{Li}(x)$. E.Schmidt [12] proved in 1903 by elementary mean that $\psi(x) - x$ changes sign infinitely often. A hard part would be to deduce results on $\pi(x) - \mathrm{Li}(x)$ from this. E.Schmidt could do this under the assumption that the Riemann Hypothesis (RH) is false. Littlewood [12] proved in 1914 that this is also the case that RH is true:

$$\mathrm{Li}(x) - \pi(x) = \Omega_\pm(x^{1/2}\log\log\log x) \tag{4}$$

Here the $\Omega_\pm$ means the error achieves the given order of magnitude infinitely often in both positive and negative signs. We see from (4) that actually the formula (3) is not much better approximation for $\pi(x)$ than $\mathrm{Li}(x)$ since the term $-(1/2)\,\mathrm{Li}(x^{1/2}) - (1/3)\,\mathrm{Li}(x^{1/3}) - \ldots = O(x^{1/2}/\log x)$ have no influence on (4); each approximation will deviate as widely as the other for some arbitrarily large $x$.(However, $\mathrm{Li}(x) - \frac{1}{2}\,\mathrm{Li}(x^{1/2})$ is better than $\mathrm{Li}(x)$ on average, see discusion in [9, p.106]).

In this note we will present the proof of this $\Omega$ result of Schmidt and Littlewood following [14]. In section we briefly review some backgrounds and briefly mention some theorems needed in the proof. We present the result of Schmidt in section 2 and of Littlewood in section 3. We briefly discuss the problem of finding explicit $x$ such that $\pi(x) > \mathrm{Li}(x)$ and some related (similar) problems in the final section.

### 1.1. **Preliminaries and Some Tools.**

**Definition 1.1.** *For a real-valued functions $f$ and positive function $g$*

$$f(x) \asymp g(x) \textit{ if there are constants } c, C \textit{ such that } cg(x) \le |f(x)| \le Cg(x).$$
$$f(x) = \Omega(g(x)) \textit{ if } \limsup \frac{|f(x)|}{g(x)} > 0.$$
$$f(x) = \Omega_+(g(x)) \textit{ if } \limsup \frac{f(x)}{g(x)} > 0.$$
$$f(x) = \Omega_-(g(x)) \textit{ if }, \liminf \frac{f(x)}{g(x)} < 0.$$
$$f(x) = \Omega_\pm(g(x)) \textit{ if } \limsup \frac{f(x)}{g(x)} > 0, \liminf \frac{f(x)}{g(x)} < 0.$$

**Definition 1.2** (Logarithmic Integral)**.**

$$\mathrm{Li}(x) = \lim_{\epsilon \searrow 0}\left(\int_0^{1-\epsilon}\frac{dt}{\log t} + \int_{1+\epsilon}^x \frac{dt}{\log t}\right).$$
$$\mathrm{li}(x) = \int_2^x \frac{1}{\log t}dt = \mathrm{Li}(x) - \mathrm{Li}(2). \ (\mathrm{Li}(2) = 1.04...)$$

**Definition 1.3** (Sum Over Primes)**.** *Note that $p$ will always denote primes.*

$$\psi(x) = \sum_{n\le x}\Lambda(n),\ \vartheta(x) = \sum_{p\le x}\log p.$$

$$\Pi(x) := \sum_{n\le x} \frac{\Lambda(n)}{\log n} = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + ...$$

*Note that the following are equivalent form of Prime Number Theorem.*

$$\psi(x) = x + O(\frac{x}{exp(c\sqrt{\log x})})$$
$$\vartheta(x) = x + O(\frac{x}{exp(c\sqrt{\log x})})$$
$$\pi(x) = \mathrm{li}(x) + O(\frac{x}{exp(c\sqrt{\log x})})$$

**Theorem 1.4.** [14, Section 10.2]*We will denote the nontrivial zeros of $\zeta$ by $\rho = \sigma + i\gamma$. $N(T)$ is the number of zeros of $\zeta$ with $0 < \gamma \le T$.*

$$T \ge 4, N(T) = \frac{T}{2\pi}(\log\frac{T}{2\pi}) - \frac{T}{2\pi} + O(\log T)$$
$$N(T+h) - N(T) \asymp h \log T$$

*From the last inequality with $h = 1$ ,we have*

$$\sum_{0<\gamma\le T} 1/\gamma \ll \sum_{0<N\le T} \frac{\log N}{N} \ll (\log T)^2$$
$$\sum_{\gamma>T} 1/\gamma^2 \ll \sum_{T}^{\infty} \frac{\log N}{N^2} \ll \frac{\log T}{T}$$
$$\textit{For } \alpha > 1 , \sum_{\gamma} \frac{1}{\gamma^\alpha} \ll \sum_{N>1} \frac{\log N}{N^\alpha} < \infty.$$

**Theorem 1.5** (Explicit Formula For $\psi$ )**.** *Let $\psi_0(x) = \frac{\psi(x^+)+\psi(x^-)}{2}$,then*

$$\psi_0(x) = \frac{1}{1\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^s}{s}\frac{\zeta'}{\zeta}(s)ds, x > 0, c > 0. \tag{5}$$

$$\psi_0(x) = x - \lim_{T\to\infty} \sum_{|\rho|\le T} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0) + \sum_{k\ge1} \frac{x^{-2k}}{2k}. \tag{6}$$

*Note that $\sum_{k\ge1} x^{-2k}/2k = -(1/2)\log(1-1/x^2)$. Define $\psi_1(x) = \int_0^x \psi(u)du = \int_0^x \sum_{n\le u}\Lambda(n)du = \frac{1}{(k-1)!}\sum_{n\le x}\int_n^x \Lambda(n)du = \sum_{n\le x}(x-n)\Lambda(n)$. In general for $k \ge 1$, we define*

$$\psi_k(x) = \frac{1}{(k-1)!}\int_0^x \sum_{n\le u}(u-n)^{k-1}\Lambda(n)du = \frac{1}{(k-1)!}\sum_{n\le x}\int_n^x (u-n)^{k-1}\Lambda(n)du = \frac{1}{k!}\sum_{n\le x}(x-n)^k\Lambda(n). \tag{7}$$

*We have for $c > 0, x > 0$*

$$\psi_k(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)...(s+k)}\left(\frac{\zeta'(s)}{\zeta(s)}\right)ds. \tag{8}$$

*We have*

$$\psi_k(x) = \frac{x^2}{(k+1)!} - \lim_{T\to\infty}\sum_{|\gamma|\le T}\frac{x^{\rho+1}}{\rho(\rho+1)...(\rho+k)} - \frac{x}{k!}\frac{\zeta'(0)}{\zeta} + O(1) - \sum_{r=1}^{\infty}\frac{x^{-2r+1}}{(-2r)(-2r+1)...(-2r+m)} \tag{9}$$

*Here, the $O(1)$ term is the terms arising from the residue of (7) at $-1, ..., -k$.*

The idea of the proof is to integrate the integrand (4) or (7) over large rectangles. The terms in explicit formula arise from the residue of the integrand and we need some estimate of zeta function to estimate the integral. See Chapter 4 of [9] for details.

The next theorem is useful for converting convergent dirichlet series to absolutely convergent integral.

**Theorem 1.6.** [14, Theorem 1.3] *Suppose that $\alpha(s) = \sum \alpha_n n^{-s}$ has abcissa of convergence $\sigma_c$ ($\alpha(s)$ converges for all $s, \sigma > \sigma_c$ and for no $s, \sigma < \sigma_c$). Let $A(x) = \sum_{n\le x} a_n$ and suppose that $\sigma_c \ge 0$, then for $\sigma > \sigma_c$,*

$$\sum_{n=1}^{\infty} a_n n^{-s} = s\int_1^{\infty} A(x)x^{-s-1}dx.$$

The next result will be useful when converting $\psi(x) - x$ to $\pi(x) - \mathrm{li}(x)$.

**Theorem 1.7.** [14, Theorem13.2 ] *Assuming RH , then*

$$\nu(x) = \psi(x) - x^{\frac{1}{2}} + O(x^{\frac{1}{3}}).$$

$$\pi(x) - \mathrm{li}(x) = \frac{\vartheta(x) - x}{\log x} + O(\frac{x^{\frac{1}{2}}}{\log^2 x}).$$

## 2. First $\Omega$-Result.

The first oscillation of error term result comes from the work of E.Schmidt in 1903 assuming RH is false. Our main tool is the following theorem of Landau.

**Lemma 2.1** (Landau)**.** *Suppose $A(x)$ is bounded and Riemann integrable function on any finite interval $[1, x]$ and $A(x) \ge 0$ for all sufficiently large $x$. Let $\sigma_c$ be the infimum of $\sigma$ for which $\int_1^{\infty} A(x)x^{-\sigma}dx < \infty$. Then the function $F(s) = \int_1^{\infty} A(x)x^{-s}dx$ is analytic on $Re(s) > \sigma_c$ but is not analytic at $s = \sigma_c$.*

*Proof.* It follows from the definition of $\sigma_c$ that $\int_1^{\infty} A(x)x^{-s}dx$ is absolutely convergent for $\mathrm{Re}(s) > \sigma_c$. Hence $\int_1^{N} A(x)x^{-s}dx \to \int_1^{\infty} A(x)x^{-s}dx$ uniformly in $\{s : Re(s) \ge \sigma_c + \epsilon\}$ for any $\epsilon > 0$ and hence $\int_1^{\infty} A(x)x^{-s}dx$ is analytic on $\{s : Re(s) > \sigma_c\}$.

Now assume on the contrary that $F(s)$ is analytic at $s = \sigma_c$. Since the integral on finite interval $[1, x]$ is entire we may assume $A(x) \ge 0$ for $x \ge 1$ and by replacing $A(x)$ by $A(x)x^{\sigma}, \sigma \ge 0$, we may asume $\sigma_c = 0$. Now $A(x)$ would be analytic on a neighborhood of 0 ,say $\{z \in \mathbb{C} : |z| < \delta\}$. Let $\Omega = \{s : \sigma > 0\} \cup \{s : |s| < \delta\}$. Now $F$ is analytic in $\Omega$. Write $F(S) = \sum_{k\ge 0} c_k(s-1)^k$. Since the nearest points to 1 that are not in $\Omega$ are $\pm i\delta$, radius of convergence $\ge \sqrt{1+\delta^2} = 1 + \delta'$ ,say. For $s$ near 1 ,$|\int_1^{\infty} A(x)x^{-s}dx| \le \int_1^{\infty} |A(x)|dx < \infty$, we differentiate under the integral sign,

$$c_k = \frac{1}{k!}F^{(k)}(1) = \frac{1}{k!}\int_1^{\infty} A(x)\frac{d^k}{ds^k}x^{-s}\Big|_{s=1} dx = \frac{1}{k!}\int_1^{\infty} A(x)(-\log(x)^{-k})dx.$$

So,

$$F(s)=\sum_{k\geq 0}\frac{1}{k!}\int_1^\infty A(x)(-\log(x)^{-k})x^{-1}(s-1)^k dx$$

Now if $-\delta' < s < 1$, the integrand is nonnegative, we may switch the order of summation and integration:

$$\begin{aligned}F(s)&=\int_1^\infty\sum_{k\geq 0}\frac{1}{k!}A(x)(-\log(x)^{-k})x^{-1}(s-1)^k dx\\&=\int_1^\infty \exp(\log(x)(1-s))A(x)x^{-1}dx\\&=\int_1^\infty A(x)x^{-s}dx.\end{aligned}$$

In particular $\int_1^\infty A(x)x^{-s}dx$ converges at $s=\frac{-\delta'}{2}$ which contradicts the definition of $\sigma_c$. □

**Theorem 2.2.** *Let $\Theta$ denotes the supremum of real parts of the zeros of zeta functions. Then for any $\epsilon > 0$ ,*

$$\psi(x)-x=\Omega_\pm(x^{\Theta-\epsilon}), x\to\infty.$$

*Proof.* By theorem 1.6 , we have

$$-\frac{\zeta'}{\zeta}(s)=s\int_1^\infty \psi(x)x^{-s-1}dx, \sigma>1$$

Hence for $\sigma > 1$,

$$\frac{1}{s-\Theta+\epsilon}+\frac{\zeta'(s)}{s\zeta(s)}-\frac{1}{s-1}=\int_1^\infty (x^{\Theta-\epsilon}+\psi(x)-x)x^{-s-1}dx, \sigma>1 \tag{10}$$

Now assume that $\psi(x)-x \geq -x^{\Theta-\epsilon}$ for $x$ large enough. LHS of (10) has a pole at $\Theta-\epsilon$ and $\zeta(s)$ is nonzero for real $s\in(0,1)$, we see that LHS of (10) is analytic for real $s>\Theta-\epsilon$ i.e. none o such point is the abcissa $\sigma_c$. Applying the Landau's theorem, $\int_1^\infty (x^{\Theta-\epsilon}+\psi(x)-x)x^{-s-1}dx$ is analytic for $\mathrm{Re}(s)>\Theta-\epsilon$ and hence the equation (10) holds for $\mathrm{Re}(s)>\Theta-\epsilon$. This is a contradiction since $\frac{\zeta'}{\zeta}$ has a pole with real part $>\Theta-\epsilon$ by definition of $\Theta$. It follows that

$$\psi(x)-x=\Omega_-(x^{\Theta-\epsilon})$$

Similarly , if $\psi(x)-x<x^{\Theta-\epsilon}$ for large enough $x$, we consider the following function.

$$\frac{1}{s-\Theta+\epsilon}+\frac{\zeta'(s)}{s\zeta(s)}+\frac{1}{s-1}=\int_1^\infty (x^{\Theta-\epsilon}-\psi(x)+x)x^{-s-1}dx, \sigma>1. \tag{11}$$

Then the same argument as in previous case gives $\psi(x)-x=\Omega_+(x^{\Theta-\epsilon})$. □

Next we will prove the $\Omega$ result of $\Pi$ to obtain $\Omega$ result of $\pi(x)$.

**Theorem 2.3.** *Let $\Theta$ denote the supremum of real parts of the zeros of zeta functions. Then for any $\epsilon>0$ ,*

$$\Pi(x)-\mathrm{li}(x)=\Omega_\pm(x^{\Theta-\epsilon})$$

*and assuming RH is false, then*

$$\pi(x)-\mathrm{li}(x)=\Omega_\pm(x^{\Theta-\epsilon})$$

*Proof.* We consider the Mellin's transform of $\mathrm{li}(x)$. Using integration by part and a change of variable, we have

$$\begin{aligned} s\int_2^\infty \mathrm{li}(x)x^{-s-1}dx &= -\frac{\mathrm{li}(x)}{x^s}\bigg|_2^\infty + \int_2^\infty \frac{dx}{x^s\log x} \\ &= \int_2^\infty \frac{dx}{x^s\log x}. \\ &= \int_{(s-1)\log 2}^\infty \frac{e^{-u}}{u}du \text{ (substitute } x = e^{\frac{u}{s-1}}). \\ &= \int_1^\infty \frac{e^{-u}}{u}du + \int_{(s-1)\log 2}^1 \frac{e^{-u}-1}{u}du + \int_{(s-1)\log 2}^1 \frac{1}{u}du. \\ &= -\int_0^{(s-1)\log 2} \frac{e^{-u}-1}{u}du - C_0 - \log(s-1) - \log\log 2. \end{aligned}$$

Here $C_0 = \int_0^1 (e^{-t}-1)/tdt + \int_1^\infty (e^{-t}/t)dt$ is a finite constant. Then we can write

$$s\int_2^\infty \mathrm{li}(x)x^{-s-1}dx = -\log(s-1) + r(s).$$

Where $r(s) = \int_0^{(s-1)\log 2}(e^{-z}-1)/z\ dz - C_0 - \log\log 2$ is an entire function (as $(e^{-z}-1)/z$ is continuous on $\mathbb{C}$).
Now for $\sigma > 1$,

$$\begin{aligned} \log\zeta(s) = \log\prod_p (1-\frac{1}{p^{-s}}) = -\sum_p \log(1-p^{-s}) &= \sum_p (p^{-s} - \frac{p^{-2s}}{2} + \frac{p^{-3s}}{3} + ...) \\ &= \sum_n (\frac{\Lambda(n)}{\log n}n^{-s}). \end{aligned}$$

Hence, in the view of theorem 1.6, we have for $\sigma > 1$

$$s\int_2^\infty \Pi(x)x^{-s-1}dx = \log\zeta(s)$$

So for $\sigma > 1$,

$$\frac{1}{s-\Theta+\epsilon} - \frac{1}{s}\log(\zeta(s)(s-1)) + \frac{r(s)}{s} = \int_2^\infty (x^{\Theta-\epsilon} - \Pi(x) + \mathrm{li}(x))x^{-s-1}dx. \tag{12}$$

LHS of (12) is analytic for real $s > \Theta - \epsilon$ so if we assume that $x^{\Theta-\epsilon} - \Pi(x) + \mathrm{li}(x)$ is eventually nonnegative then by Landau's Theorem, both sides are analytic for $\mathrm{Re}(s) > \Theta - \epsilon$. This is a contradiction since $\zeta$ has a zero $\rho = \sigma + i\gamma$ with $\rho > \Theta - \epsilon$. So there is arbitrarily large $x, x^{\Theta-\epsilon} - \Pi(x) + \mathrm{li}(x) < 0$ That is

$$\Pi(x) - \mathrm{li}(x) = \Omega_-(x^{\Theta-\epsilon})$$

Now,

$$\Pi(x) - \pi(x) = \sum_{k=2}^{O(\log x)} \frac{1}{k}\pi(x^{1/k}) = \frac{1}{2}\pi(x^{1/2}) + O(x^{1/3}\log x) = O(\frac{x^{1/2}}{\log x}).$$

so if we assume RH is false i.e. $\Theta > 1/2$.Suppose $\Theta - \epsilon > 1/2$, then

$$\pi(x) - \mathrm{li}(x) = O(\frac{x^{1/2}}{\log x}) + \Omega_{\pm}(x^{\Theta-\epsilon}) = \Omega_{\pm}(x^{\Theta-\epsilon}).$$

□

*Remark.* In case that RH is true, i.e. $\Theta = \frac{1}{2}$. Since $\Pi(x) > \pi(x)$ we have $\pi(x) - \mathrm{li}(x) = \Omega_{-}(x^{\frac{1}{2}-\epsilon})$but the $\Omega_{+}$ result are not obtainable by this method.

Next we show that if we assume that there is a zero of $\zeta$ on the line $\sigma = \Theta$ then we can have a stronger conclusion.

**Theorem 2.4.** *Let $\Theta$ denotes the supremum of real parts of the zeros of zeta functions and there is a zero $\rho$ with $Re(\rho) = \Theta$. Then,*

$$\limsup_{x\to\infty} \frac{\psi(x) - x}{x^{\Theta}} \geq \frac{1}{|\rho|} \tag{13}$$

$$\liminf_{x\to\infty} \frac{\psi(x) - x}{x^{\Theta}} \leq -\frac{1}{|\rho|} \tag{14}$$

*In particular ,* $\psi(x) - x = \Omega_{\pm}(x^{\Theta})$.

*Proof.* Let $\rho = \Theta + i\gamma$ be a zero of $\zeta$. Assume $\psi(x) \leq x + cx^{\Theta}$ for $x \geq X_0$ i.e. $c \geq \limsup_{x\to\infty}(\psi(x) - x)/x^{\Theta}$, then by Landau's theorem, for $\sigma > \Theta$

$$\frac{c}{s-\Theta} + \frac{\zeta'(s)}{s\zeta(s)} + \frac{1}{s-1} = \int_{1}^{\infty} (x^{\Theta-\epsilon} - \psi(x) + x)x^{-s-1}dx. \tag{15}$$

and both sides are analytic. Call this function $F(s)$. Then for $\sigma > \Theta$,

$$F(s) + \frac{1}{2}e^{i\phi}F(s+i\gamma) + \frac{1}{2}e^{-i\phi}F(s-i\gamma) = \int_{1}^{\infty} (cx^{\Theta-\epsilon} - \psi(x) + x)(1 + \cos(\phi - \gamma \log x))x^{-s-1}dx \tag{16}$$

Now integral of RHS of (16) on the finite interval $[1, X_0]$ is uniformly bounded while the integrand on $[X_0, \infty)$ is nonnegative. We have that $\liminf_{s\to\Theta^+}$ of RHS of (16) is $> -\infty$. On the other hand, recall that if $f$ is analytic with zero of order $m$ at $z_0 \neq 0$ then residue of $\frac{f'(z)}{zf(z)}$ at $z_0$ is given by $m/z_0$. Let $m$ be the multiplicity of $\rho$ then by LHS of (15),

- $F(s)$ has a pole at $s = \Theta$ with residue $c$.
- $\frac{e^{i\phi}}{2}F(s + i\gamma)$ has a pole at $s = \Theta$ with residue $\frac{me^{i\phi}}{2\rho}$.
- $\frac{e^{i\phi}}{2}F(s - i\gamma)$ has a pole at $s = \Theta$ with residue $\frac{me^{-i\phi}}{2\rho}$.

Choose $\psi$ so that$\frac{e^{i\phi}}{\rho} = -\frac{1}{|\rho|}$. Then the residue of RHS of (5) at $\Theta$ is $c - \frac{m}{|\rho|}$. Now as $\liminf_{s\to\Theta^+}$ of RHS of (16) is $> -\infty$, $c - \frac{m}{|\rho|} \geq 0$ i.e. $c \geq \frac{1}{|\rho|}$. We have (13). (14) is proved similarly. □

**Corollary 2.5.** *As $x \to \infty$,*
$$\psi(x) - x = \Omega_{\pm}(x^{1/2})$$
$$\vartheta(x) - x = \Omega_{-}(x^{1/2})$$
$$\pi(x) - \mathrm{li}(x) = \Omega_{-}(\frac{x^{1/2}}{\log x}).$$

*Proof.* If RH is false then we have theorem 2.3 as a stronger result so we may assume $\Theta = \frac{1}{2}$ Now by Theorem 2.4, we have
$$\psi(x) - x = \Omega_{\pm}(x^{1/2}).$$
So by Theorem 1.7 we have $\vartheta(x) - x = \psi(x) - x - x^{1/2} + O(x^{1/3}) = \Omega_{-}(x^{1/2})$. Hence by Theorem 1.7,
$$\pi(x) - \mathrm{li}(x) = \frac{\vartheta(x) - x}{\log x} + O(\frac{x^{1/2}}{\log^2 x}) = \Omega_{-}(\frac{x^{1/2}}{\log x}).$$
□

The problem of proving $\Omega_{+}$ results for $\pi(x) - \mathrm{li}(x)$ is more difficult and this is first done by Littlewood as we discribe in the next section.

## 3. Littlewood's $\Omega-$ Result.

Our goal in this section is to prove the following theorem of Littlewood. In the view of Theorem 2.3 , we may assume RH throughout this section.

**Theorem 3.1** (Littlweeod1914)**.**
$$\psi(x) - x = \Omega_{\pm}(x^{1/2} \log\log\log(x)).$$

$$\pi(x) - x = \Omega_{\pm}(\frac{x^{1/2}}{\log(x)} \log\log\log(x)).$$

For the rough idea of our aproach, substituting $e^y$ for $x$ in the explicit formula for $\psi(x)$ (with RH), we get
$$\frac{\psi(e^y) - e^y}{e^{y/2}} = -\sum_{\rho} \frac{e^{i\rho y}}{\rho e^{y/2}} - \frac{C}{e^{y/2}} + \frac{1}{e^{y/2}} \sum_{k \geq 1} \frac{e^{-2ky}}{2k} = -\sum_{\rho} \frac{e^{i\gamma y}}{\rho} + O(e^{-y/2})$$
uniformly for $y \geq 1$.
Now since we assume RH then $1/\rho = 1/i\gamma + O(1/\gamma^2)$. Since $\sum_{\rho} 1/\gamma^2 < \infty$. the above expression becomes
$$-2 \sum_{\gamma > 0} \frac{\sin \gamma y}{\gamma} + O(1)$$
Now suppose that we could approximate this by the truncated sum
$$-2 \sum_{0 < \gamma < T} \frac{\sin \gamma y}{\gamma}. \tag{17}$$
The sum of the absolute value of coefficients in (17) is $\asymp (\log T)^2$ and the sum will be of this order of magnitude (of both signs) if we can find $y$ so that the fractional part $\frac{\gamma y}{2\pi}$ is $\approx 1/4$ for all $0 < \gamma \leq T$. This problem is a kind of *inhomogeneous problem of diophantine approximation*. In general,

this kind of problem has a solution only if $\gamma$ above are linearly independent over $\mathbb{Q}$ which we don't have information about this. Instead , we look at *homogeneous diophantine approximation.*

Dirichlet's theorem tells us that there exist large $y$ such that $\frac{\gamma y}{2\pi}$ are near integers for all $0 < \gamma \leq T$. Now observe that the sum (17) is small when $\frac{\gamma y}{2\pi}$ are near integers.However if we take $y = \pi/T$ then $\sin(\gamma y) \asymp \gamma/T$ and the total sum becomes $\asymp N(T)/T \asymp \log T$ which is large enough. Now choose $y_0$ so that $\frac{\gamma y_0}{2\pi}$ are small. Then take $y = y_0 \pm \pi/T$ so that the sum in (17) is large in both signs. Now in the next lemma we can form a weighted sum that is similar to (17).

**Lemma 3.2.** *Assume RH then*

$$\frac{1}{x(e^{-\delta} - e^{\delta})} \int_{e^{-\delta}x}^{e^{\delta}x} (\psi(u) - u)du = -2x^{1/2} \sum_{\gamma > 0} \frac{\sin(\gamma\delta)}{\gamma\delta} \frac{\sin(\gamma \log x)}{\gamma} + O(x^{1/2}). \tag{18}$$

*Here* $\frac{1}{2x} \leq \delta \leq \frac{1}{2}$ *and* $O$ *is uniform for* $x \geq 4$.

*Proof.* By the explicit formula, we have

$$\int_0^x (\psi(u) - u)du = -\sum_\rho \frac{x^\rho}{\rho(\rho+1)} - cx + O(1)$$

Take the average, we have

$$\frac{1}{(e^\delta - e^{-\delta})x} \int_{e^{-\delta}x}^{e^\delta x} (\psi(u) - u)du = \frac{-\delta}{\sinh(\delta)} \sum_\rho \frac{e^{\delta(\rho+1)} - e^{\delta(\rho+1)}}{2\delta\rho(\rho+1)} x^\rho + O(1) \tag{19}$$

Now observe that $e^{\pm\delta(\rho+1)} = (e^{\pm\delta(\mathrm{Re}(\rho)+1+i\gamma)}) = e^{\pm\delta i\gamma}(1 + O(\delta)) = e^{\pm\delta i\gamma} + O(\delta)$.
We have $\sum_\rho \frac{1}{|\rho(\rho+1)|} \leq \sum \gamma^{-2} < \infty$, in particular $\sum_\rho \frac{1}{\rho(\rho+1)} < \infty$. Note also that $\frac{\delta}{\sinh(\delta)} = \frac{\delta}{\delta + O(\delta^3)} = 1 + O(\delta^2) \ll 1,\ \delta \to 0$ .
Thus assuming RH then $|x^\rho| = x^{1/2}$. Replacing $e^{\delta(\rho+1)}$ by $e^{i\delta\gamma}$ in (19) gives

$$\begin{aligned}\frac{-\delta}{\sinh(\delta)} \sum_\rho \frac{e^{\delta(\rho+1)} - e^{\delta(\rho+1)}}{2\delta\rho(\rho+1)} x^\rho &= -ix^{1/2} \frac{\delta}{\sinh(\delta)} \sum_\rho \frac{\sin(\gamma\delta)}{\delta} \frac{x^{i\gamma}}{\rho(\rho+1)} + O(x^{1/2} \frac{\delta}{\sinh(\delta)} \sum_\rho \frac{1}{\rho(\rho+1)}) \\ &= -ix^{1/2} \frac{\delta}{\sinh(\delta)} \sum_\rho \frac{\sin(\gamma\delta)}{\delta} \frac{x^{i\gamma}}{\rho(\rho+1)} + O(x^{1/2}).\end{aligned}$$

Now

$$\begin{aligned}\sum_\rho \frac{\sin(\gamma\delta)}{\delta} \frac{x^{i\gamma}}{\rho(\rho+1)} \ll \sum_\rho \frac{\gamma}{\rho(\rho+1)} &\ll \sum_{0<\gamma<\delta^{-1}} \frac{1}{\gamma} + \delta^{-1} \sum_{\gamma > \delta^{-1}} \frac{1}{\gamma^2} \\ &\ll \log(\delta^{-1})^2 + \delta^{-1}\delta \log \delta^{-1} \\ &\ll \log^2(\delta^{-1}).\end{aligned}$$

Substitute $\frac{\delta}{\sinh(\delta)} = 1 + O(\delta^2)$, RHS of (19) becomes

$$-ix^{1/2} \sum_\rho \frac{\sin \gamma\delta}{\delta} \frac{x^{i\gamma}}{\rho(\rho+1)} + O(x^{1/2}).$$

Now if we assume RH, we have $1/\rho = 1/(2\gamma) + O(1/\gamma^2)$ and notice that $\frac{\sin \gamma\delta}{\delta} \leq |\gamma|$. Hence if we replace $1/\rho$ by $1/i\gamma$ then we have the error term $\ll x^{1/2} \sum \gamma^{-2} \ll x^{1/2}$. Similarly, we replace $1/(\rho + 1)$ by $1/i\gamma$. Our expression becomes

$$
\begin{aligned}
-x^{1/2} \sum_{\rho} \frac{\sin \gamma\delta}{\gamma\delta} \frac{x^{i\gamma}}{i\gamma} + O(x^{1/2}) &= -2x^{1/2} \sum_{\gamma>0} \frac{\sin \gamma\delta}{\gamma\delta} \frac{e^{i \log x\gamma} - e^{i \log x\gamma}}{i\gamma} + O(x^{1/2}) \\
&= -2x^{1/2} \sum_{\gamma>0} \frac{\sin(\gamma\delta)}{\gamma\delta} \frac{\sin(\gamma \log x)}{\gamma} + O(x^{1/2}).
\end{aligned}
$$

□

**Lemma 3.3** (Dirichlet)**.** *For $x \in \mathbb{R}$, define $\|x\|$ to be the distance to the nearest integer.Let $\theta_1, \ldots\theta_K$ be real numbers and $N$ be a positive integer. Then there is a positive integer $n, 1 \leq n \leq N^K$ with $\|\theta_i n\| < \frac{1}{N}$ for all $i$.*

*Proof.* Partition $[0,1)^K$ into $N^K$ equal subcubes then there exist $0 \leq n_1 < n_2 \leq N^K$ such that $(\theta_1 n_1, \theta_2 n_1 \ldots., \theta_K n_1), (\theta_1 n_2, \theta_2 n_2, \ldots, \theta_k n_2)$ are in the same subcube. Let $n = n_2 - n_1 \in [1, N^K]$. Then for $1 \leq i \leq k$

$$
\|n\theta_i\| = \|n_2\theta_i - n_1\theta_i\| \leq |n_2\theta_i - n_1\theta_i| < \frac{1}{N}.
$$

□

Now we are ready to give the proof of the result of Littlewood.

*Proof of theorem 3.1.* Note first that if RH is false then theorem 2.3 gives a stronger result so we may assume RH. Note also that it we can prove that

$$
\psi(x) - x = \Omega_{\pm}(x^{1/2} \log\log\log(x)) \tag{20}
$$

Then since $\psi(x) - \vartheta(x) = O(x^{1/2})$, we have

$$
\nu(x) - x = \Omega_{\pm}(x^{1/2} \log\log\log x)
$$

Now under RH, using theorem , we have $\pi(x) - \text{li}(x) = \frac{\vartheta(x) - x}{\log x} + O(\frac{x^{1/2}}{\log^2 x})$ Then we have,

$$
\pi(x) - x = \Omega_{\pm}(\frac{x^{1/2}}{\log(x)} \log\log\log(x)).
$$

We now prove (20) , let $N$ be a large integer then apply Lemma 3.3 to $\gamma\frac{\log N}{2\pi}, 0 \leq \gamma \leq T = N \log N$ Here ,$K$ , the number of elements in that set, is $N(T) \asymp T \log T$ and there exists $n, 1 \leq n \leq N^k$ such that

$$
\left\| \frac{\gamma n}{2\pi} \log N \right\| < \frac{1}{N}, 0 < \gamma \leq T \tag{21}
$$

Note the inequality

$$
|\sin(\pi x)| \leq \pi \|x\|. \tag{22}
$$

By periodiity, it is enough to verify this for $x \in [0,1]$ which can be done directly. We have from (22) that $|\sin(2\pi\alpha) - \sin(2\pi\beta)| = |2\sin(\pi(\alpha - \beta))\cos(\pi(\alpha + \beta))| \leq 2\pi \|\alpha - \beta\|$.

Take $x = N^n e^{\pm 1/n}, \delta = \frac{1}{N}$,

$$\begin{aligned}
|\sin(\gamma \log x) \mp \sin(\frac{\gamma}{N})| &\leq 2\pi \left\| \frac{\gamma(\log x \mp \frac{1}{N})}{2\pi} \right\| \\
&= 2\pi \left\| \frac{n\gamma \log N}{2\pi} \right\| \\
&\leq \frac{2\pi}{N} \text{ (by the lemma).}
\end{aligned}$$

Next we have,

$$\begin{aligned}
\sum_{\gamma > N\log N} \frac{\sin(\gamma/N)}{\gamma/N} \frac{\sin(\gamma \log x)}{\gamma} &\ll N \sum_{\gamma > N \log N} \frac{1}{\gamma^2} \\
&\ll N(N\log N)^{-1} \log(N \log N) \\
&\ll 1.
\end{aligned}$$

The RHS of (18) is

$$\begin{aligned}
-2x^{1/2} \sum_{\gamma>0} \frac{\sin(\gamma/N)}{\gamma/N} \frac{\sin(\gamma \log x)}{\gamma} &= \mp 2x^{1/2} \sum_{\gamma>0} \frac{\sin(\gamma/N)}{\gamma/N} \frac{\sin(\gamma/N)}{\gamma} + O(\frac{1}{N} x^{1/2} \sum_{\gamma>0} |\frac{\sin(\gamma/N)}{\gamma/N} \frac{1}{\gamma}|). \\
&= \mp \frac{2x^{1/2}}{N} \sum_{\gamma>0} \left(\frac{\sin(\gamma/N)}{\gamma/N}\right)^2 \mp 2x^{1/2} N \sum_{\gamma > N\log N} \left(\frac{\sin(\gamma/N)}{\gamma}\right)^2 + O(x^{1/2}). \\
&\asymp \mp \frac{2x^{1/2}}{N} N \log N \mp N \frac{2x^{1/2} \log(N \log N)}{N \log N} + O(x^{1/2}). \\
&= \mp 2x^{1/2} \log(N) + O(x^{1/2}).
\end{aligned}$$

Since $x \leq N^{N^K} e^{1/N}, K = N(T) \asymp T \log T \asymp N(\log^2 N)$, we have

$$\log\log x \leq K \log N + \log\log N \ll N(\log N)^3.$$

Then for some constants $C$,

$$\log N \geq \log\log\log x - \log C - 3 \log\log N$$

Since $x \geq N e^{\pm(1/N)} \gg N$, we have $\log\log N = o(\log\log\log x)$ so

$$\log N \geq (1 + o(1)) \log\log\log x.$$

Now by lemma 3.2 , the quantity (11) is an average of $\psi(u) - u$ over a neighborhood of $x$, where $x \gg N$ and $N$ can be arbitrarily large. It follows that

$$\psi(x) - x = \Omega_{\pm}(x^{1/2} \log\log\log(x))$$

$\square$

**Theorem 3.4.** *Let $\Theta$ denote the supremum of the real parts of zeros of $\zeta(s)$. Of $\zeta$ has a zero with real part $\Theta$ then there exists a constant $C > 0$ such that $\psi(x) - x$ changes sign in every interval $[x, Cx]$ for $x \geq 2$.*

*Proof.* For each integer $k \geq 0$, we define

$$R_k(y) := \frac{1}{k!} \sum_{n \leq e^y} (y - \log n)^k \Lambda(n) - e^y. \tag{23}$$

Note, for example,

$$R_1(y) = \int_0^y \psi(e^u) du = \int_0^x \sum_{n \leq e^u} \Lambda(n) du = \sum_{\log n \leq x} \int_{\log n}^x \Lambda(n) du = \sum_{n \leq e^x} (x - \log n) \Lambda(n).$$

We have that $R_1$ is continuous. For $k > 1$, $R_k$ is differentiable with $R_k' = R_{k-1}$. By the explicit formula, we have

$$R_k(y) = -\sum_{\rho} \frac{e^{\rho y}}{\rho^{k+1}} + O(y^{k+1}). \tag{24}$$

Suppose that zeros of $\zeta(s)$ on the line $\sigma = \Theta$ are given by $\rho_j = \Theta + i\gamma_j$ where $0 < \gamma_1 < \gamma_2 < \ldots$. Let $m_j$ denote the multiplicity of $\rho_j$. Since for $\alpha > 1$, $\sum_\rho \rho^{-\alpha} < \infty$ we can rearrange the terms in the summation. We have that, for $k \geq 1$,$y \to \infty$,

$$R_k(y) = -\sum_{\rho, \operatorname{Re}\rho = \Theta} \frac{e^{\rho y}}{\rho^{k+1}} - \sum_{\rho, \operatorname{Re}\rho < \Theta} \frac{e^{\rho y}}{\rho^{k+1}} + O(y^{k+1}) \tag{25}$$

$$= -\sum_{\rho, \operatorname{Re}\rho = \Theta, \gamma > 0} \left( \frac{e^{\rho y}}{\rho^{k+1}} + \overline{\frac{e^{\rho y}}{\rho^{k+1}}} \right) + o(e^{\Theta y}) \tag{26}$$

$$= -2e^{\Theta y} \operatorname{Re} \sum_j \frac{m_j e^{i\rho_j y}}{\rho_j^{k+1}} + o(e^{\Theta y}). \tag{27}$$

Let $K$ be the least positive integer such that $m_1 > \sum_{j>1} m_j |\rho_1/\rho_j|^k$, i.e.

$$\frac{m_1}{|\rho_1|^K} > \sum_{j>1} \frac{m_j}{|\rho_1|^K}. \tag{28}$$

Choose $\phi$ such that $e^{i\gamma_1 \phi}/\rho_1^k > 0$, take $k = K$ in (27) then

$$R_K(\phi + \frac{\pi n}{\gamma_1}) = -2e^{\Theta y} \operatorname{Re} \frac{m_1 e^{i\gamma_1(\phi + \frac{\pi n}{\gamma_1})}}{\rho_1^{K+1}} - 2e^{\Theta y} \sum_{j>1} \operatorname{Re} \frac{m_j e^{i\gamma_j(\phi + \frac{\pi n}{\gamma_j})}}{\rho_j^{K+1}} \tag{29}$$

Then, in view of (28) we have that for $n$ large enough, $n$ is even ,$R_K(\phi + (\pi n/\gamma_1))$ is negative while if $n$ is odd then $R_K(\phi + (\pi n/\gamma_1))$ is positive.

Take $C = \exp(\pi(K+2)/\gamma_1)$. Then any intervals $[y_0, y_0 + \log C]$ contains at least $K+2$ points of the form $\phi + (\pi n/\gamma_1)$. Hence if $y_0$ is large enough then the interval contains $K+2$ points for which $R_K(y)$ alternates in signs. By the Mean Value Theorem, we know that if $f$ is differentiable on $[\alpha, \beta]$ with $f(\alpha) > 0, f(\beta) < 0$ then $\exists \xi, \alpha < \xi < \beta, f'(\xi) > 0$. (Similarly, if $f(\alpha) < 0, f(\beta) > 0$ then $\exists \xi, \alpha < \xi < \beta, f'(\xi) < 0$.) Hence as $R_{k-1} = R_k'$,we can find $K+1$ points in $[y_0, y_0 + \log C]$ such that $R_{k-1}$ alternates signs. Continue this process until we get 3 points in ihe interval such that $R_1(y)$ alternats signs. Now since $R_1$ is an indefinit integral of $R_0$, if $R_0$ is monotone in any interval then so is $R_1$ but $R_1$ is not monotone in$[y_0, y_0 + \log C]$ so $R_0$ changes signs in the interval as required.

□

The same result of this theorem is also true for $\pi(x) - \mathrm{li}(x)$ under the same hypothesis, see [13]. See also Exercise 15.2.4 in [14].

## 4. Conclusion

Note first that the proof of Littlewood's result in [13] (see also Exercise 15.2.5 in [14]) gives us more quantitative information :

$$\limsup_{x\to\infty} \frac{\psi(x) - x}{x^{1/2}\log\log\log x} \geq \frac{1}{2}, \ \liminf_{x\to\infty} \frac{\psi(x) - x}{x^{1/2}\log\log\log x} \leq -\frac{1}{2}.$$

With the same result for $\pi(x) - \mathrm{li}(x)$ in place of $\psi(x) - x$ above.

The proof of Littlewood is ineffective in the sense that it does not give us any information about the value of $c$ for which $\pi(x) > \mathrm{li}(x)$ for some $x < c$ (which is stil unknown). Now it has been calculated that $\pi(x) < \mathrm{li}(x)$ to $x = 10^{14}$(see[7]). Skewes [19], using Littlewood's proof assume RH, gave the first $c = 10^{10^{10^{34}}}$ and later[20] he gave $c = 10^{10^{10^{963}}}$ unconditionally. This is connected with the problem of constructing a function $f(x)$ for which in any interval $[x_0, f(x_0)]$ such that the sum of over non trivial zeros is not much cancel.Turan study these power sums and his power sum method has many applications to oscillation of error terms that srise in analytic number theory, see [22] .In 1961, Knaposkwi [6],without extensive numerical calculations, used Turan's method to gave the upper bound $c$. Lehmer(1966)[8],using extensive numerical calculation about zeros $\rho$, $\mathrm{li}(x) < \pi(x)$ for $10^{500}$ consecutive integers between $1.53 \times 10^{1165}, 1.65 \times 10^{1165}$. Lehmer also gave a useful theorem that enables us to obtain a lower bound of $c$ unconditionally (depending on how far you can verify RH). In 1987, H.J.Riele [23] gave $c = 7 \times 10^{370}$. In 2000, Bays,Hudson [1] gave $c = 1.398 \times 10^{316}$ and they believe this is close to the first $c$. Note also recent slight improvements [2], [17]. Note that [23], [1], [2], [17] also use Lehman's Theorem. In 1941, Winter [24] showed that the proportion that $\pi(x) > \mathrm{Li}(x)$ is positive in logarithmic scale: $\limsup \frac{1}{\log(x)} \sum_{x\leq X, \pi(x)>\mathrm{Li}(x)} 1/x > 0$. Rubinstein,Sarnak (1994)[16] showed under Generalized Riemann Hypotheis and Grand Simplicity Hypothesis (about linearly independence of zeros of L-functions on the critical line) that the proportion is about 0.00000026. Montogomery and Monach [10] has the following conjecture concerning linear forms of the zeros:

**Conjecture 4.1.** *For every $\epsilon > 0, K > 0$, there exists $T_0(K, \epsilon)$ such that*

$$|\sum_{0<\gamma\leq T} k_\gamma \gamma| > exp(-T^{1+\epsilon})$$

*For $T \geq T_0$. Here $k_\gamma$ are integers, not all zeros, $|k_\gamma| < K$.*

From this conjecture they can show

$$\limsup_{x\to\infty} \frac{\psi(x) - x}{x^{1/2}(\log\log\log x)^2} \geq \frac{1}{2\pi}. \ \liminf_{x\to\infty} \frac{\psi(x) - x}{x^{1/2}(\log\log\log x)^2} \leq -\frac{1}{2\pi}.$$

and they believe that they are actually equal in the view of results in [24] (see also equation 13.48 in [14]).

A similar method can be used to obtain a similar result for the function $M(x) = \sum_{n\leq x} \mu(n)$ It is conjectured by Merten that $M(x) \leq x^{1/2}$ (this would imply RH but not vice versa). However, this is disproved by Odlyzko and te Riele [15] who showed that the size of the sum can be slightly larger than expected. In fact it has been shown using a similar method to ours (see [14] theorem

15.7)that if the zeros of $\zeta$ are linearly independent over $\mathbb{Q}$ and they are all simple (so that $\zeta'(\rho) \neq 0$ )then

$$\limsup_{x\to\infty} \frac{M(x)}{x^{1/2}} = \infty, \ \liminf_{x\to\infty} \frac{M(x)}{x^{1/2}} = -\infty.$$

Another function that we can obtain $\Omega$ result (under some hypothesis,see [4] and Exercise 15.1.8 in [14]) is $L(x) = \sum_{n\le x} \lambda(n)$ where $\lambda(n) = (-1)^{\Omega(n)}$ is the Louville Function. Here $\Omega(n)$ is the number of distinct prime factors of $n$. Pólya conjectured that $L(x) \le 0$ for $x \ge 2$ and this has been verified up to $10^6$. This conjecture is disproved by Haselgrove[4] in 1958. In 1960 Lehmer [11] found that $L(906, 180, 359) = 1$. It can be shown [4] that if RH is true and ordinates $\gamma > 0$ are linearly independent over $\mathbb{Q}$ then

$$\limsup_{x\to\infty} \frac{L(x)}{x^{1/2}} = \infty, \ \liminf_{x\to\infty} \frac{L(x)}{x^{1/2}} = -\infty.$$

Also, if $\sum_{n\le x}(\lambda(n)/n) > 0, \ x \ge 1$ then RH is true.

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