Asymptotic Dynamics of Nonlinear Schrödinger Equations: Resonance-Dominated and Dispersion-Dominated Solutions

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Abstract

We consider a linear Schrödinger equation with a nonlinear perturbation in $\mathbb{R}^3$. Assume that the linear Hamiltonian has exactly two bound states and its eigenvalues satisfy some resonance condition. We prove that if the initial data is sufficiently small and is near a nonlinear ground state, then the solution approaches to certain nonlinear ground state as the time tends to infinity. Furthermore, the difference between the wave function solving the nonlinear Schrödinger equation and its asymptotic profile can have two different types of decay: The resonance-dominated solutions decay as $t^{-1/2}$ or the dispersion-dominated solutions decay at least like $t^{-3/2}$. © 2002 John Wiley & Sons, Inc.

Contents

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1 Introduction

Consider the nonlinear Schrödinger equation

$$i\partial_t \psi = (-\Delta + V)\psi + \lambda|\psi|^2\psi\,, \quad \psi(t=0) = \psi_0\,, \tag{1.1}$$

where V is a smooth localized potential, λ is an order-1 parameter, and $\psi = \psi(t,x) : \mathbb{R}\times\mathbb{R}^3 \longrightarrow \mathbb{C}$ is a wave function. Let $e_0 < 0$ be the ground state energy to $-\Delta + V$, and denote $H_1 = -\Delta + V - e_0$. The nonlinear bound states to the Schrödinger equation (1.1) are solutions to the equation

$$(-\Delta + V)Q + \lambda|Q|^2 Q = EQ\,. \tag{1.2}$$

They are critical points to the energy functional

$$\mathcal{H}[\phi] = \int \frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}V|\phi|^2 + \frac{1}{4}\lambda|\phi|^4\,dx$$

subject to the constraint of fixed L^2 norm. For each bound state $Q = Q_E$, $\psi(t) = Qe^{-iEt}$, is a solution to the nonlinear Schrödinger equation. We may obtain a family of such bound states by standard bifurcation theory: For each E sufficiently close to e_0 so that $E - e_0$ and λ share the same sign, there is a unique small positive solution $Q = Q_E$ to equation (1.2) that decays exponentially as $x \to \infty$; see Lemma 2.1. We call this family the *nonlinear ground states* and shall refer to it as $\{Q_E\}_E$.

Let

$$H_E = -\Delta + V - E + \lambda Q_E^2\,. \tag{1.3}$$

We have $H_E Q_E = 0$. Since Q_E is small and E is close to e_0, the spectral properties of H_E are similar to those of H_1.

Suppose the initial data of the nonlinear Schrödinger equation ψ_0 is near some Q_E. Under rather general conditions, the family of nonlinear ground states is stable in the sense that if

$$\inf_{\Theta,E} \left\| \psi(t) - Q_E e^{i\Theta} \right\|_{L^2}$$

is small for $t = 0$, it remains so for all t; see, for example, [13] for the case $\lambda < 0$. See also [18, 19]. Let $\|\cdot\|_{L^2_{\mathrm{loc}}}$ denote a local L^2 norm; a precise choice will be made later on. One expects that this difference actually approaches zero when measured by a local L^2 norm, i.e.,

$$\lim_{t\to\infty} \inf_{\Theta,E} \left\| \psi(t) - Q_E e^{i\Theta} \right\|_{L^2_{\mathrm{loc}}} = 0\,. \tag{1.4}$$

If $-\Delta + V$ has only one bound state, it is proven in [15] that the evolution will eventually settle down to some ground state Q_{E_∞} with E_∞ close to E. (See also [9] for another proof using techniques from dynamical systems.)

Suppose now that $-\Delta + V$ has multiple bound states, say, two bound states: a ground state ϕ_0 with eigenvalue e_0 and an excited state ϕ_1 with eigenvalue e_1, i.e., $H_1\phi_1 = e_{01}\phi_1$ where $e_{01} = e_1 - e_0 > 0$. The question is whether the evolution with initial data ψ_0 near some Q_E will eventually settle down to some ground state Q_{E_∞}

with E_∞ close to E. Furthermore, can we characterize the asymptotic evolution? In this paper, we shall answer this question positively in the case of two bound states and estimate precisely the rate of relaxation for a certain class of initial data.

We now state the main assumptions of this paper.

ASSUMPTION A0: $-\Delta + V$ acting on $L^2(\mathbb{R}^3)$ has two simple eigenvalues $e_0 < e_1 < 0$ with normalized eigenvectors ϕ_0 and ϕ_1.

ASSUMPTION A1: *Resonance condition.* Let $e_{01} = e_1 - e_0$ be the spectral gap of the ground state. We assume that $2e_{01} > |e_0|$ so that $2e_{01}$ is in the continuum spectrum of H_1. Furthermore, for some constant $\gamma_0 > 0$ and all real s sufficiently small,

$$\lim_{\sigma\to 0+} \left(\phi_0\phi_1^2, \operatorname{Im} \frac{1}{H_1 - \sigma i - 2e_{01} - s} \mathbf{P}_c^{H_1} \phi_0\phi_1^2 \right) \geq \gamma_0 > 0\,. \tag{1.5}$$

We shall use $0i$ to replace σi and the limit $\lim_{\sigma\to 0+}$ later on.

ASSUMPTION A2: For λQ_E^2 sufficiently small, the bottom of the continuous spectrum to $-\Delta + V + \lambda Q_E^2$, 0, is not a generalized eigenvalue; i.e., it is not a resonance. Also, we assume that V satisfies the assumption in Yajima [20] so that the $W^{k,p}$ estimates $k \leq 2$ for the wave operator $W_H = \lim_{t\to\infty} e^{itH} e^{it(\Delta+E)}$ hold for $k \leq 2$; i.e., there is a small $\sigma > 0$ such that

$$|\nabla^\alpha V(x)| \leq C\langle x\rangle^{-5-\sigma} \quad \text{for } |\alpha| \leq 2\,.$$

Also, the functions $(x\cdot\nabla)^k V$ for $k = 0, 1, 2, 3$, are $-\Delta$ bounded with a $-\Delta$ bound less than 1:

$$\left\| (x\cdot\nabla)^k V\phi \right\|_2 \leq \sigma_0 \left\| -\Delta\phi \right\|_2 + C\left\|\phi\right\|_2\,, \quad \sigma_0 < 1\,,\ k = 0, 1, 2, 3.$$

Assumption A2 contains some standard conditions to assure that most tools in linear Schrödinger operators apply. These conditions are certainly not optimal. The main assumption above is the condition $2e_{01} > |e_0|$ in assumption A1. The rest of assumption A1 just consists of generic assumptions. This condition states that the excited state energy is closer to the continuum spectrum than to the ground state energy. It guarantees that twice the excited state energy of H_1 (which one obtains from taking the square of the excited state component) becomes a resonance in the continuum spectrum (of H_1). This resonance produces the main relaxation mechanism. If this condition fails, the resonance occurs in higher-order terms and a proof of relaxation will be much more complicated. Also, the rate of decay will be different.

Define the notation

$$\langle x\rangle = \sqrt{1+x^2}\,, \quad \{t\}_\varepsilon = \varepsilon^{-2}n^{-2} + 2\Gamma t\,, \quad \{t\}_\varepsilon^{-1/2} \sim \min\left\{\varepsilon n, n^{-1}t^{-1/2}\right\}, \tag{1.6}$$

where $n = \|\psi_0\|_{L^2}$ and $\Gamma = O(n^2)$ is a positive constant to be specified later in (6.3) of Lemma 6.1. For the moment, we remark that Γ is of order $2\lambda^2 n^2$ times the quantity in (1.5). The subscript ε is a small parameter to be specified in Theorems 1.3 and 1.4. We shall often drop it in the proofs of Theorems 1.3 and 1.4. We denote by L^2_r the weighted L^2 spaces (r may be positive or negative),

$$L^2_r(\mathbb{R}^3) \equiv \left\{\phi \in L^2(\mathbb{R}^3) : \langle x\rangle^r \phi \in L^2(\mathbb{R}^3)\right\} . \tag{1.7}$$

Our space for initial data is

$$Y \equiv H^1(\mathbb{R}^3) \cap L^2_{r_0}(\mathbb{R}^3) , \quad r_0 > 3 . \tag{1.8}$$

We shall use L^2_{loc} to denote $L^2_{-r_0}$. The parameter $r_0 > 3$ is fixed, and we can choose, say, $r_0 = 4$ for the rest of this paper.

Our first theorem states that if the initial data ψ_0 is small in Y and the distance between ψ_0 and a nonlinear ground state Q_* is small, then the solution $\psi(t)$ has to settle down to some asymptotic nonlinear ground state as $t \to \infty$. Furthermore, the difference between ψ_0 and the asymptotic nonlinear ground state at $t = \infty$ is bounded *above* by the order $O(t^{-1/2})$. Recall that $\lambda = O(1)$ is fixed.

THEOREM 1.1 *Assume that assumptions* A0, A1, *and* A2 *on V hold. Then there are small universal positive constants ε_0 and $n_0 > 0$ such that, for any nonlinear ground state Q_* with mass $n = \|Q_*\|_{L^2} < n_0$ and any initial data ψ_0 satisfying $\|\psi_0 - e^{i\Theta_0} Q_*\|_Y \le \varepsilon_0^2 n^2$ for some $\Theta_0 \in \mathbb{R}$, there exist an energy E_∞ and a function $\Theta(t)$ such that $\|Q_{E_\infty}\|_{L^2} - n = O(\varepsilon_0^2 n)$, $\Theta(t) = -E_\infty t + O(\log t)$, and*

$$\left\|\psi(t) - Q_{E_\infty} e^{i\Theta(t)}\right\|_{L^2_{\text{loc}}} \le C(1+t)^{-1/2} . \tag{1.9}$$

To describe more detailed behavior of the solution $\psi(t)$, we need various spectral properties of the linearized operator. All statements we make here will be proven in Section 2. Let $\mathcal{L}$ be the operator obtained from linearizing the Schrödinger equation (1.1) around the trivial solution Qe^{-iEt} (see Section 2), i.e.,

$$\mathcal{L}h = i^{-1}\left\{(-\Delta + V - E + \lambda Q^2)h + \lambda Q^2(h + \overline{h})\right\} . \tag{1.10}$$

Notice that $\mathcal{L}$ is not self-adjoint due to the conjugation. If we can decompose h into its real and imaginary parts, the operator $\mathcal{L}$ can be written in the matrix forms

$$\mathcal{L} \longleftrightarrow \begin{bmatrix} 0 & L_- \\ -L_+ & 0 \end{bmatrix}$$

where

$$L_- := -\Delta + V - E + \lambda Q_E^2 = H_E , \quad L_+ = L_- + 2\lambda Q_E^2 . \tag{1.11}$$

Explicitly, we have

$$\mathcal{L}(f + ig) = L_- g - iL_+ f .$$

Notice that $\mathcal{L}$ is a small perturbation of the Hamiltonian H_1. Hence the spectral properties of $\mathcal{L}$ are closely related to those of H_1. Since H_1 has two eigenvalues, 0 and e_{01}, we expect $\mathcal{L}$ to have two eigenvalues as well.

From (1.2) we have $L_- Q_E = 0$. If we differentiate (1.2) with respect to E, we have $L_+ R_E = Q_E$, where $R_E = \partial_E Q_E$. Let S_E be the space spanned by iQ_E and its tangent R_E, i.e.,

$$S = S_E = \text{span}_{\mathbb{R}}\{R_E, iQ_E\} . \tag{1.12}$$

Clearly, S is the generalized eigenspace of $\mathcal{L}$ with eigenvalue 0.

$\mathcal{L}$ has a pair of eigenvalues $\pm i\kappa$, where κ is a perturbation of $e_{01} = e_1 - e_0$ of H_1. The corresponding generalized eigenfunctions, u and v, are perturbations of the linear excited state ϕ_1. They are real-valued functions characterized by

$$L_+ u = \kappa v\,, \quad L_- v = \kappa u\,, \quad (u, v) = 1\,. \tag{1.13}$$

In particular, for all real numbers α and β, we have $\mathcal{L}^2(\alpha u + \beta i v) = -\kappa^2(\alpha u + \beta i v)$. Thus

$$\mathbf{E}_\kappa(\mathcal{L}) = \operatorname{span}_{\mathbb{R}} \{u, iv\} \tag{1.14}$$

is the generalized eigenspace with eigenvalues $\pm i\kappa$.

Finally, the space of the continuous spectrum, $\mathbf{H}_c(\mathcal{L})$, is characterized by the relation

$$\mathbf{H}_c(\mathcal{L}) = \left\{ f + ig \in L^2 :\ f, g \text{ real},\ \ f \perp Q_E, v;\ \ g \perp R_E, u \right\}\,. \tag{1.15}$$

We have the following spectral decomposition result, to be proven in Section 2.2.

LEMMA 1.2 (Spectral Decomposition) *The space of complex-valued L^2 functions in $\mathbb{R}^3$ can be decomposed as the direct sum of $S, \mathbf{E}_\kappa(\mathcal{L})$ and $\mathbf{H}_c(\mathcal{L})$, i.e.,*

$$L^2(\mathbb{R}^3) = S \oplus M = S \oplus \mathbf{E}_\kappa(\mathcal{L}) \oplus \mathbf{H}_c(\mathcal{L})\,, \quad M := \mathbf{E}_\kappa(\mathcal{L}) \oplus \mathbf{H}_c(\mathcal{L})\,. \tag{1.16}$$

The decomposition is not orthogonal; the three real subspaces S, $\mathbf{E}_\kappa(\mathcal{L})$, and $\mathbf{H}_c(\mathcal{L})$ are invariant under $\mathcal{L}$ but not under multiplication by i. The space M can also be characterized by

$$M = M_E = \left\{ f + ig \in L^2 :\ f, g \text{ real},\ \ f \perp Q_E;\ \ g \perp R_E \right\}\,. \tag{1.17}$$

From this lemma, for a fixed energy E we can decompose a wave function via (1.16). It is more convenient for our analysis to decompose the wave function into the following form:

$$\psi(t, x) = [Q_E(x) + a_E(t) R_E(x) + h_E(t, x)]\, e^{i\Theta(t)} \tag{1.18}$$

with $h_E(t, x) \in M_E$. We shall prove in Lemma 2.2 the existence and uniqueness of such a decomposition for $\psi(t, x)$ near a ground state. If we allow the energy E to vary, it is possible to choose E such that the component along the R_E direction vanishes; i.e., there exists $E(t)$ such that

$$\psi(t, x) = [Q_{E(t)}(x) + h(t, x)] e^{i\Theta(t)} \tag{1.19}$$

with $h_E(t, x) \in M_E$; see Lemma 2.3. We can view this ground state $Q_{E(t)}$ as the best approximation to the wave function $\psi(t)$. This choice is different from choosing the best approximation by minimizing the difference in the L^2 norm $\inf_{E,\Theta} \|\psi(t, \cdot) - Q_E e^{i\Theta}\|_{L^2}$.

Although (1.19) can be viewed as the best decomposition of a wave function, it introduces a time-dependent linearized operator. So for analytical purposes, it is more convenient to work with (1.18). Notice that $Q_E + a_E(t) R_E$ is the first-order approximation of $Q_{(E+a_E(t))}$. So the decomposition (1.18) can be viewed as the best time-independent decomposition. Since $h_E(t,x) \in M_E$, we can write it as

$$h_E(t) = \zeta_E(t) + \eta_E(t)\,, \quad \zeta_E \in \mathbf{E}_\kappa(\mathcal{L})\,, \quad \eta_E \in \mathbf{H}_{\rm c}(\mathcal{L})\,. \tag{1.20}$$

Theorem 1.1 gives an upper bound to the difference between the wave function and the asymptotic ground state. In fact, we can estimate the component along the excited states and the continuum spectrum more precisely in the following theorem:

THEOREM 1.3 *Suppose the assumptions of Theorem* 1.1 *hold. Let*

$$\varepsilon \equiv n^{-1}\big(\|\zeta_{E_0,0}\| + \|\eta_{E_0,0}\|_Y^{1/2}\big) \le \varepsilon_0\,. \tag{1.21}$$

If we decompose $\psi(t)$ as in (1.18) *with $Q = Q_{E_\infty}$ being the profile at time $t=\infty$ and $h(t) = \zeta(t) + \eta(t)$ as in* (1.20), *then we have*

$$\begin{aligned} &\|\zeta(t)\|_{L^2} \le C\,\{t\}^{-1/2}\,, &&|a(t)| \le C\,\{t\}^{-1}\,, \\ &\|\eta(t)\|_{L^4} \le C\,\{t\}^{-3/4+\sigma}\,, &&\|\eta(t)\|_{L^2_{\rm loc}} \le C\,\{t\}^{-1}\,, \end{aligned} \tag{1.22}$$

where $\{t\} = \{t\}_\varepsilon$ and $\sigma = 0.01$.

Theorem 1.3 provides rather precise upper bounds on the asymptotic evolution. These bounds are optimal for a large class of initial data described by the next theorem. In the following, for two functions f and g, we denote

$$f \approx g \quad \text{if } C_1\,\|g\| \le \|f\| \le C_2\,\|g\| \tag{1.23}$$

for some constants $C_1, C_2 > 0$.

THEOREM 1.4 (Resonance-Dominated Solutions) *Assume that assumptions* A0 *through* A2 *on V hold. Suppose that the initial data ψ_0 is decomposed as in* (1.19) *with respect to the unique E_0, i.e.,*

$$\psi_0 = [Q_{E_0} + \zeta_{E_0,0} + \eta_{E_0,0}]e^{i\Theta_{E_0,0}}\,.$$

Write $\zeta_{E_0,0} = z_0 u + i z_1 v$ and denote $z_{E_0,0} = z_0 + i z_1$. Suppose that these components satisfy

$$\|\psi_0\| = n < n_0\,, \quad 0 < \varepsilon := \frac{\|z_{E_0,0}\|}{n} \le \varepsilon_0\,, \quad \|\eta_{E_0,0}\|_Y \le \varepsilon^2 n^2\,, \tag{1.24}$$

where n_0 and ε_0 are the same constants as in Theorem 1.1. *Then the conclusions of Theorem* 1.1 *and Theorem* 1.3 *hold. In particular, there is a limit frequency E_∞ such that the estimates* (1.22) *hold for $\{t\} = \{t\}_\varepsilon$. If $\zeta(t) = \zeta_{E_\infty}(t)$ denotes the excited state component with respect to the linearized operator with energy E_∞, the lower bound holds as well, and we have*

$$\zeta_{E_\infty}(t) \approx \{t\}_\varepsilon^{-1/2}\,. \tag{1.25}$$

Furthermore, for all $\lambda > 0$ or $\lambda < 0$, we have $\|Q_{E_\infty}\|_2 > \|Q_{E_0}\|_2$ and

$$\|Q_{E_\infty}\|_{L^2}^2 = \|Q_{E_0}\|_{L^2}^2 + \frac{1}{2}\|\zeta_{E_0,0}\|_{L^2}^2 + o(\varepsilon^2 n^2)\,. \tag{1.26}$$

Solutions satisfying (1.25) *are called* resonance-dominated solutions.

Roughly speaking, Theorem 1.4 states that, if the excited state component is much bigger than the dispersive component, then the decay mechanism is dominated by resonance. Furthermore, (1.26) states that approximately half of the probability density of the excited state, $|\zeta_{E_0}(x)|^2/2$, is transferred to the ground state independently of the sign of λ. Since the total probability is conserved, the other half is transferred to the dispersive part.

We have decomposed the wave function according to the optimal energy E_0 (1.19). Alternatively, we can first fix an energy E_* and decompose the initial data ψ_0 as

$$\psi_0 = [Q_{E_*} + a_* R_{E_*} + \zeta_* + \eta_*] e^{i\Theta_*} . \tag{1.27}$$

If the components satisfy

$$\begin{gathered} \|\psi_0\| = n < n_0 , \quad \|\zeta_*\| = \varepsilon n , \quad 0 < \varepsilon \le \varepsilon_0 , \\ \|\eta_*\|_Y \le C\varepsilon^2 n^2 , \quad |a_*| \le \varepsilon^2 n^2 , \end{gathered} \tag{1.28}$$

then we can re-decompose ψ_0 as in Theorem 1.4 with respect to E_0 satisfying the estimate (1.24); see Lemma 2.3. This implies that the set of all such ψ_0 contains an *open set* with nonlinear ground states in its boundary. Therefore, the class of resonance-dominated solutions is in a sense large.

The condition (1.24) is very subtle. It states that the excited-state component is much bigger than the dispersive component with respect to a decomposition according to $\mathcal{L}$. Since $\mathcal{L}$ differs from H_E by order n^2, the condition (1.24) does not hold for a decomposition of ψ_0 with respect to the linear Hamiltonian H_E.

Finally, the following existence result shows that the resonance-dominated solutions are not all solutions. We expect that the dispersion-dominated solutions constructed by the following theorem are rare:

THEOREM 1.5 (Dispersion-Dominated Solutions) *Let assumptions* A0 *through* A2 *apply on* V. *Let* $Z = H^2 \cap W^{2,1}(\mathbb{R}^3)$. *For a given nonlinear ground state* Q_{E_∞} *with* $\|Q_{E_\infty}\| = n \le n_0$, *let* $E = E_\infty$ *and* $\mathcal{L} = \mathcal{L}_{E_\infty}$. *For any given* $\xi_\infty \in \mathbf{H}_c(\mathcal{L}) \cap Z$ *with sufficiently small* Z *norm, there exists a solution* $\psi(t)$ *of* (1.1) *and a real function* $\theta(t) = O(t^{-1})$ *for* $t > 0$ *so that*

$$\|\psi(t) - \psi_{\mathrm{as}}(t)\|_{H^2(\mathbb{R}^3)} \le C t^{-2} ,$$

where

$$\psi_{\mathrm{as}}(t) = Q_E e^{-iEt + i\theta(t)} + e^{-iEt} e^{t\mathcal{L}} \xi_\infty .$$

In particular, $\psi(t) - Q_{E_\infty} e^{-iEt+i\theta(t)} = O(t^{-3/2})$ *in* L^2_{loc}.

Theorem 1.5 constructs dispersion-dominated solutions based on the operator $e^{t\mathcal{L}}$. The scattering property of this operator is very similar to the standard operator $e^{i\Delta t}$. In particular, for $\chi_\infty \in \widetilde{Z} = H^2(\mathbb{R}^3) \cap W^{2,1}(\mathbb{R}^3, (1 + |x|^4)dx)$ sufficiently small in $\widetilde{Z}$ with $\hat{\chi}_\infty(0) = 0$ and $\nabla\hat{\chi}_\infty(0) = 0$, the same statement holds if we replace ψ_{as} by

$$\widetilde{\psi}_{\text{as}}(t) = Q_E e^{-iEt} + e^{i\Delta t}\chi_\infty\,.$$

This will be shown in the proof of Theorem 1.5.

The resonance solutions were first observed by Buslaev and Perel′man [3] for some one-dimensional Schrödinger equation with two bound states and a higher nonlinear term (we thank the referee for supplying us with this reference). In the physical dimension $d = 3$, this type of solution was proven by Soffer and Weinstein in an important paper [16], but for real solutions to the nonlinear Klein-Gordon equation

$$\partial_t^2 u + B^2 u = \lambda u^3\,, \quad B^2 := (-\Delta + V + m^2)\,, \tag{1.29}$$

where λ is a small nonzero number. Assume that B^2 has only *one* eigenvector (the ground state) ϕ, $B^2\phi = \Omega^2\phi$, with the resonance condition $\Omega < m < 3\Omega$ (and some positivity assumption similar to that appearing in assumption A1). Rewrite real solutions to equation (1.29) as $u = a\phi + \eta$ with

$$a(t) = \operatorname{Re} A(t)e^{i\Omega t}\,, \quad \operatorname{Re} A'(t)e^{i\Omega t} = 0\,.$$

Then A and η satisfy the equations

$$\dot{A} = \frac{1}{2i\Omega} e^{-i\Omega t}\left(\phi, \lambda(a\phi + \eta)^3\right) \tag{1.30}$$

$$(\partial_t^2 + B^2)\eta = P_c\lambda(a\phi + \eta)^3\,. \tag{1.31}$$

Theorem 1.1 in [16] states that all solutions decay as

$$A(t) \sim \langle t\rangle^{-1/4}\,, \quad \|\eta(t)\|_{L^\infty} \sim \langle t\rangle^{-3/4}\,.$$

In particular, the ground state is unstable and will decay as a resonance with rate $t^{-1/4}$.

We first remark that the proof in [16] has only established the upper bound $t^{-1/4}$. Furthermore, a universal lower bound of the form $t^{-1/4}$ is in fact incorrect. From the previous work of [1, 5], it is clear that dispersion-dominated solutions decaying much faster than $t^{-1/4}$ exist. Similar to Theorems 1.4 and 1.5, we have the following two cases:

(1) $\eta(0) \ll A(0)$: The dominant term on the right side of (1.30) is $\lambda a^3\phi^3$.
(2) $\eta(0) \gg A(0)$: The dominant term is $\lambda\eta^3$.

In case 2 another type of solutions arises, namely, those with decay rate

$$A(t) \sim \langle t\rangle^{-2}\,, \quad \|\eta\|_{L^\infty} \sim \langle t\rangle^{-3/2}\,.$$

We shall sketch a construction of such solutions at the end of Section 8.

Notice that all solutions in [16] decay as a function of t. Therefore, we can view [16] as a study of asymptotic dynamics around a vacuum. Although most works concerning asymptotic dynamics of nonlinear evolution equations have been concentrated on cases with vacuum as the unique profile at $t = \infty$, more interesting and relevant cases are asymptotic dynamics around solitons (such as the Hartree equations [5]; see also [2, 3]). The soliton dynamics have extra complications involving translational invariance. The current setting of nonlinear Schrödinger equations with local potentials eliminates the translational invariance and constitutes a useful intermediate step. This greatly simplifies the analysis but preserves a key difficulty that we now explain.

Recall that we need to approximate the wave function $\psi(t)$ by nonlinear ground states for all t. Since we aim to show that the error between them decay like $t^{-1/2}$, we have to track the nonlinear ground states with accuracy at least like $t^{-1/2}$. Although the nonlinear ground states approximating the wave function $\psi(t)$ can in principle be defined, say, via equation (1.4) or (1.6), neither characterization is useful unless we know the wave functions $\psi(t)$ precisely. Furthermore, even assuming we can track the approximate ground states reasonably well, the linearized evolution around these approximate ground states will be based on *time-dependent, non-self-adjoint* operators $\mathcal{L}_t$. At this point we would like to mention the approach of [15] based on perturbation around the unitary evolution e^{itH_1}, where H_1 is the original self-adjoint Hamiltonian. While we do not know whether this approach can be extended to the current setting by adding the ideas of [16] (it was announced in [16] that its method can be extended to (1.1) as well), such an approach can be difficult to extend to the Hartree or other equations with nonvanishing solitons. The main reason is that these dynamics are not perturbations of linear dynamics.

We believe that perturbation around the profile at $t = \infty$ is a more natural setup. In this approach, at least we do not have to worry about the time dependence of approximate ground states in the beginning. But the linearized operator $\mathcal{L} = \mathcal{L}_\infty$ is still non-self-adjoint, and it does not commute with the multiplication by i. So calculations and estimates based on $\mathcal{L}$ are rather complicated. Our idea is to map this operator to a self-adjoint operator by a bounded transformation in Sobolev spaces. This map in a sense brings the problem back to the self-adjoint case for various calculations and estimates. Another important input we used is the existence and boundedness of the wave operator for $\mathcal{L}$. The boundedness of the wave operator in Sobolev spaces for the standard one-body Schrödinger operator is a classical theorem of Yajima [20]. In the current setting it was recently proved by Cuccagna [4]. See Section 2 for more details.

The next step is to identify and calculate the leading oscillatory terms of the nonlinear systems involving the bound-states components and the continuum-spectrum components. The leading-order terms, however, depend on the relative sizes of these components, and thus we have two different asymptotic behaviors: The resonance-dominated solutions and the dispersion-dominated solutions. Finally, we represent the continuum spectrum component in terms of the bound-states components, and this leads to a system of ordinary differential-integral equations for the bound-states components. This system can be put into a normal form, and the size of the excited-state component can be seen to decay as $t^{-1/2}$. Notice that the phase and the size of the excited-state component decay differently. It is thus important to isolate the contribution of the phase in the system. Finally, we estimate the error terms using estimates of the linearized operators. These estimates are based on standard methods from scattering theory [6, 14, 16] and estimates on the wave operators [4, 20].

2 Preliminaries

We first fix our notation. Let H^k denote the Sobolev spaces $W^{k,2}(\mathbb{R}^3)$. The weighted Sobolev space $L^2_r(\mathbb{R}^3)$ is defined in (1.7). The inner product $(\,,\,)$ is

$$(f, g) = \int_{\mathbb{R}^3} \bar{f} g \, d^3x \,. \tag{2.1}$$

For two functions f and g, we denote

$$f = O(g) \quad \text{if } \|f\| \le C\,\|g\| \text{ for some constant } C > 0\,.$$

We will denote, as in (1.23),

$$f \approx g \qquad \text{if } C_1\,\|g\| \le \|f\| \le C_2\,\|g\| \text{ for some constants } C_1, C_2 > 0\,.$$

For a double index $\alpha = (\alpha_0, \alpha_1)$, α_0 and α_1 nonnegative integers, we denote

$$z^\alpha = z^{\alpha_0}\bar{z}^{\alpha_1}\,, \quad |\alpha| = \alpha_0 + \alpha_1\,, \quad [\alpha] = -\alpha_0 + \alpha_1\,. \tag{2.2}$$

For example, $z^{(32)} = z^3\bar{z}^2$. We will define $z = \mu^{-1}p$ where $\mu = e^{i\kappa t}$. Hence $z^\alpha = \mu^{[\alpha]}p^\alpha$. In what follows $|\alpha| = 2$ and $|\beta| = 3$.

2.1 Nonlinear Ground States Family

We establish here the existence and basic properties of the nonlinear ground-states family mentioned in Section 1. Our statement and proof are also valid for nonlinear excited states.

LEMMA 2.1 *Suppose that* $-\Delta + V$ *satisfies assumptions* A0 *and* A2. *Then there is an* n_0 *sufficiently small such that for* E *between* e_0 *and* $e_0 + \lambda n_0^2$ *there is a nonlinear ground state* $\{Q_E\}_E$ *solving* (1.2). *The nonlinear ground state* Q_E *is real, local, and smooth,* $\lambda^{-1}(E - e_0) > 0$, *and*

$$Q_E = n\phi_0 + O(n^3)\,, \quad n \approx C[\lambda^{-1}(E - e_0)]^{1/2}\,, \quad C = \left(\int \phi_0^4\,dx\right)^{-1/2}\,.$$

Moreover, we have $R_E = \partial_E Q_E = O(n^{-2})Q_E + O(n) = O(n^{-1})$ *and* $\partial_E^2 Q_E = O(n^{-3})$. *If we define* $c_1 \equiv (Q, R)^{-1}$, *then* $c_1 = O(1)$ *and* $\lambda c_1 > 0$.

PROOF: Suppose that $Q = Q_E$ is a nonlinear ground state satisfying

$$(-\Delta + V - e_0)Q + \lambda Q^3 = E'Q$$

where $E' = E - e_0$ is small. Write $Q = n\phi_0 + h$ with real $h \perp \phi_0$. Then h satisfies

$$(-\Delta + V - e_0 - E')h + \lambda(n\phi_0 + h)^3 = E'n\phi_0\,. \tag{2.3}$$

Taking the inner product with ϕ_0, we see that it is necessary that λ and E' be of the same sign. For E' sufficiently small, since the spectral gap of the operator $-\Delta + V - e_0$ is of order 1, the same conclusion holds for the operator $-\Delta + V - e_0 - E'$. Thus we can check directly from (2.3) that

$$\|Q_E\| := n^2 \sim \lambda^{-1} E', \quad \|h\|_2 = O(\lambda n^3). \tag{2.4}$$

The existence of solutions h and E' can be established from the implicit function theorem, or the solutions can simply be obtained by a contraction mapping argument.

By differentiating the equation of Q_E with respect to E, we have

$$L_+ R_E = Q_E, \quad L_+ := -\Delta + V - E + 3\lambda Q^2. \tag{2.5}$$

Denote by $|0\rangle$ the ground state to L_+ with L^2 norm 1 and eigenvalue $\approx \lambda n^2$. We have

$$Q_E = O(n)|0\rangle + O(\lambda n^3).$$

Hence (using the spectral gap)

$$\begin{aligned} R = (L_+)^{-1}[O(n)|0\rangle + O(\lambda n^3)] &= O\left(\frac{n}{\lambda n^2}\right)|0\rangle + O(\lambda n^3) \\ &= O(\lambda^{-1} n^{-2}) Q + O(n). \end{aligned} \tag{2.6}$$

Hence $c_1 = O(1)$. Since the sign of the ground-state energy to L_+ is the same as λ, we have $\lambda(Q, R) > 0$. By differentiating the equation, we can prove $\partial_E^2 Q_E = O(n^{-3})$ in a similar way. □

Suppose $Q_E = O(n)$ and a is a small parameter with $|a| \ll n$. By Taylor expansion and $\partial_E^2 Q_E = O(n^{-3})$, we have

$$Q_{E+a} = Q_E + a R_E + O(n^{-3} a^2) = Q_E + O(n^{-1} a). \tag{2.7}$$

Since $(a R_E)^2 = O(n^{-2} a^2)$, from (2.7) we also have

$$\|Q_{E+a}\|^2 = \|Q_E\|^2 + 2a(Q_E, R_E) + O(n^{-2} a^2). \tag{2.8}$$

We will need the following renormalization lemmas. We will formulate them in terms of nonlinear ground states, although they also hold for nonlinear excited states. Let Y_1 be either Y, defined in (1.8), or $L^2_{-r_0}$, defined in (1.7) with $r_0 > 3$. Recall from Lemma 1.2 that $h \in M$ if and only if $\operatorname{Re} h \perp Q$ and $\operatorname{Im} h \perp R$.

LEMMA 2.2 *Let $\psi \in Y_1$ be given with $\|\psi\|_{Y_1} = n$, $0 < n \le n_0$. Suppose that for some nonlinear ground state $Q_E e^{i\Theta}$ we have $\|\psi - Q_E e^{i\Theta}\|_{Y_1} \le C\tau n$, $0 \le \tau \le \varepsilon_0$. Then there are unique small a, θ, and h such that*

$$\psi = [Q_E + a R_E + h]\, e^{i(\Theta+\theta)}, \quad h \in M_E. \tag{2.9}$$

Furthermore, $\|Q_E\|_{Y_1} > \frac{9}{10} n$, $a = O(\tau n^2)$, $\theta = O(\tau)$, and $h = O(\tau n)$.

PROOF: By considering $\psi e^{-i\Theta}$ instead of ψ, we may assume $\Theta = 0$. Write $\psi = Q + k$ with a small $k = O(\tau n)$. We want to solve small a and θ such that $h = \psi e^{-i\theta} - Q_E - aR_E \in M_E$, i.e.,

$$\begin{bmatrix} (Q, \operatorname{Re}(Q+k)e^{-i\theta} - Q - aR) \\ (R, \operatorname{Im}[(Q+k)e^{-i\theta}]) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

with the left side of order $[\begin{smallmatrix} \tau n^2 \\ \tau \end{smallmatrix}]$ if $(a, \theta) = (0, 0)$. We compute the derivative of this vector with respect to a and θ at $(a, \theta) = (0, 0)$ and obtain the matrix

$$\begin{bmatrix} -(Q, R) & (Q, \operatorname{Im} k) \\ 0 & (R, -\operatorname{Re}(Q+k)) \end{bmatrix} = \begin{bmatrix} -(Q, R) & O(\tau n^2) \\ 0 & -(R, Q) + O(\tau) \end{bmatrix},$$

which is invertible. Here we have used $Q = O(n)$ and $R = O(n^{-1})$. Since $(Q, R) = O(1)$, by the implicit function theorem we can solve $a = O(\tau n^2)$ and $\theta = O(\tau)$ satisfying the equation. Hence $h = (Q_E + k)e^{-i\theta} - Q_E - aR_E = O(\tau n)$. $\square$

The next lemma shows that the best decomposition as defined by (1.19) can always be achieved for a small vector near a nonlinear ground state.

LEMMA 2.3 *Let $\psi \in Y_1$ with $\|\psi\|_{Y_1} = n$ bounded by a small number n_0, i.e., $0 < n \le n_0$. If ψ is near a nonlinear ground state $Q_{E_1} e^{i\Theta_1}$ in the sense that $\|\psi - Q_{E_1} e^{i\Theta_1}\|_{Y_1} \le C\tau n$ for some τ with $0 \le \tau \le \varepsilon_0$ and ϵ_0 small. Then there is a unique E near E_1 such that the component along the R_E direction as defined by* (2.9) *vanishes; i.e., there is unique small θ and h such that*

$$\psi = [Q_E + h]\, e^{i(\Theta_1 + \theta)}, \quad h \in M_E\,.$$

Moreover, $E - E_1 = O(\tau n^2)$, $\theta = O(\tau^2)$, and $h = O(\tau n)$.

If ψ is given explicitly as $\psi = [Q_{E_1} + a_1 R_{E_1} + h_1]e^{i\Theta_1}$ with $h_1 \in M_{E_1}$ and $a_1 = O(\tau^2 n^2)$, we have

$$|E - E_1| \le \frac{5}{4}|a_1| = O(\tau^2 n^2)\,, \quad \theta = O(\tau^3)\,, \quad h - h_1 = O(\tau^2 n)\,. \tag{2.10}$$

PROOF: As in the proof of the previous lemma, we may assume $\Theta_1 = 0$. Write $\psi = Q_{E_1} + k_1$ with $h_1 \in M_{E_1}$ and $k_1 = a_1 R_{E_1} + h_1 = O(\tau n)$ small. Let $E = E_1 + \gamma$. We want to find small γ and θ such that $h = \psi e^{-i\theta} - Q_E \in M_E$, that is,

$$\begin{bmatrix} (Q_{E_1+\gamma}, \operatorname{Re}(Q_{E_1} + k_1)e^{-i\theta} - Q_{E_1+\gamma}) \\ (R_{E_1+\gamma}, \operatorname{Im}[(Q_{E_1} + k_1)e^{-i\theta}]) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{2.11}$$

When $(\gamma, \theta) = (0, 0)$, the left side equals

$$\begin{bmatrix} (Q_{E_1}, a_1 R_{E_1}) \\ 0 \end{bmatrix}, \quad \text{which is of order } \begin{bmatrix} \tau n^2 \\ 0 \end{bmatrix}.$$

The derivative of this vector with respect to γ and θ at $(\gamma, \theta) = (0, 0)$ is given by

$$\begin{bmatrix} (R_{E_1}, -Q_{E_1} + \operatorname{Re} k_1) & (Q_{E_1}, \operatorname{Im} k_1) \\ (\partial_E R_E|_{E=E_1}, \operatorname{Im} k_1) & (R_{E_1}, -\operatorname{Re}(Q_{E_1} + k_1)) \end{bmatrix} = \begin{bmatrix} -(Q, R) & O(\tau n^2) \\ O(\tau n^{-2}) & -(R, Q) + O(\tau) \end{bmatrix},$$

which is invertible. Here we have used $Q = O(n)$, $R = O(n^{-1})$, and $\partial_E R_E = O(n^{-3})$. By the implicit function theorem we can solve $\gamma = O(\tau n^2)$ and $\theta = O(\tau^2)$ satisfying the equation. Hence $h = (Q_{E_1} + k_1)e^{-i\theta} - Q_{E_1+\gamma} = O(\tau n)$.

Finally, when $a_1 = O(\tau^2 n^2)$, the vector on the left side of (2.11) at $(\gamma, \theta) = (0, 0)$ is of order

$$\begin{bmatrix} \tau^2 n^2 \\ 0 \end{bmatrix}.$$

Hence we can solve $\gamma = O(\tau^2 n^2)$ and $\theta = O(\tau^3)$. We also have $h - h_1 = (Q_{E_1} + a_1 R_{E_1} + h_1)e^{-i\theta} - h_1 - Q_{E_1+\gamma} = O(n\theta) + O(n^{-1}a_1) = O(\tau^2 n)$. The last statement is proven. □

The next lemma provides estimates on the components in (1.18) when the reference nonlinear ground state varies.

LEMMA 2.4 *Let $\psi = [Q_{E_1} + a_1 R_{E_1} + h_1]e^{i\Theta_1}$ with $\|Q_{E_1}\|_{Y_1} = n \le n_1$, $\|h_1\|_{Y_1} = \rho \le \varepsilon_1 n$, and $|a_1| \le C\rho^2$. Suppose $E_2 = E_1 + \gamma$ with $|\gamma| \le C\rho^2$. By Lemma 2.2 we can rewrite ψ uniquely with respect to E_2 as*

$$\psi = \left[Q_{E_2} + a_2 R_{E_2} + h_2\right] e^{i\Theta_2}, \tag{2.12}$$

where $h_2 \in M_{E_2}$, a_2, and $\theta = \Theta_2 - \Theta_1$ are small. We have the estimates

$$\begin{aligned} \theta &= O(n^{-3}\rho\gamma), \\ E_1 + a_1 - E_2 - a_2 &= O(n^{-1}\rho\gamma), \\ h_1 - h_2 &= O(n^{-2}\rho\gamma). \end{aligned} \tag{2.13}$$

Notice that $n^{-3}\rho\gamma \le C\varepsilon_1^3$ is small.

PROOF: Note $|\gamma| \le C\rho^2 \le \varepsilon_1^2 n^2$. Using (2.7) we have

$$\begin{aligned} Q_{E_1} - Q_{E_2} = O(n^{-1}\gamma), \quad R_{E_1} - R_{E_2} = O(n^{-3}\gamma), \\ \lambda Q_{E_1}^2 - \lambda Q_{E_2}^2 = O(\gamma). \end{aligned} \tag{2.14}$$

Since

$$\begin{aligned} e^{-i\Theta_1}\psi - Q_{E_2} &= [Q_{E_1} - Q_{E_2}] + a_1 R_{E_1} + h_1 \\ &= O(n^{-1}\gamma) + O(\rho^2 n^{-1}) + O(\rho) = O(\rho) \le C\varepsilon_1 n, \end{aligned}$$

Lemma 2.2 is applicable with $\tau = \rho/n$, and we have (2.12) with

$$a_2 = O(\tau n^2) = O(n\rho), \quad h_2 = O(\tau n) = O(\rho), \quad \theta = O(\tau^2) = O(n^{-2}\rho^2).$$

We have

$$[Q_{E_1} + a_1 R_{E_1} + h_1]e^{-i\theta} = Q_{E_2} + a_2 R_{E_2} + h_2. \tag{2.15}$$

Denote by (RS) the right side and by (LS) the left side. Taking the imaginary part and then taking the inner product with R_{E_2}, we have

$$(R_{E_2}, \mathrm{Im(RS)}) = (R_{E_2}, \mathrm{Im}\, h_2) = 0\,,$$

since $h \in M_{E_2}$, and

$$(R_{E_2}, \mathrm{Im(LS)}) = (R_{E_2}, Q_{E_1} + a_1 R_{E_1} + \mathrm{Re}\, h_1)(-\sin\theta) + (R_{E_2}, \mathrm{Im}\, h_1)\cos\theta\,.$$

Since $(R_{E_2}, Q_{E_1} + a_1 R_{E_1} + \mathrm{Re}\, h_1) \approx 1$ and

$$(R_{E_2}, \mathrm{Im}\, h_1) = (R_{E_1} + O(n^{-3}\gamma), \mathrm{Im}\, h_1) = 0 + O(n^{-3}\rho\gamma)\,,$$

we conclude that $\sin\theta + O(n^{-3}\rho\gamma) = 0$. Since θ is small, we have $\theta = O(n^{-3}\rho\gamma)$.

Write $e^{i\theta} = 1 + O(\theta)$ in (2.15). Since

$$[Q_{E_1} + aR_{E_1} + h_1]O(\theta) = O(n)O(n^{-3}\rho\gamma)\,,$$

equation (2.15) gives

$$Q_{E_1} + a_1 R_{E_1} - Q_{E_2} - a_2 R_{E_2} = O(n^{-2}\rho\gamma) + h_2 - h_1\,. \tag{2.16}$$

Taking the real part and then taking the inner product with Q_{E_2}, we get from the right side $O(n \cdot n^{-2}\rho\gamma) + 0 + (Q_{E_1} + O(n^{-1}\gamma), h_1) = O(n^{-1}\rho\gamma)$. On the other hand, by (2.7)

$$\begin{aligned} &Q_{E_1} + a_1 R_{E_1} - Q_{E_2} - a_2 R_{E_2} \\ &\quad = \big[Q_{E_2} - \gamma R_{E_2} + O(\gamma^2 n^{-3})\big] + a_1\big[R_{E_2} + O(\gamma n^{-3})\big] - Q_{E_2} - a_2 R_{E_2} \\ &\quad = (-\gamma + a_1 - a_2)R_{E_2} + O(n^{-3}\rho^2\gamma)\,, \end{aligned}$$

where we have used $E_1 = E_2 - \gamma$. Hence we conclude

$$(E_1 - E_2 + a_1 - a_2)(Q_{E_2}, R_{E_2}) + O(n^{-3}\rho^2\gamma) = O(n^{-1}\rho\gamma)\,.$$

Since $(Q_{E_2}, R_{E_2}) = O(1)$, we have $E_1 - E_2 + a_1 - a_2 = O(n^{-1}\rho\gamma)$. From (2.16), we thus have

$$h_2 - h_1 = O(n^{-1}\rho\gamma)R_{E_2} + O(n^{-3}\rho^2\gamma) - O(n^{-2}\rho\gamma) = O(n^{-2}\rho\gamma)\,.$$

□

2.2 Linearized Operator: Spectral Analysis

We now study the spectral properties of $\mathcal{L}$. In this section, we identify $\mathbb{C}$ with $\mathbb{R}^2$ and the multiplication by i becomes

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Recall the definition of M and the decomposition $L^2(\mathbb{R}^3) = S \oplus M$. This decomposition is nonorthogonal and presents some problems in analysis. Let X be the space orthogonal to Q,

$$X = \Pi(L^2) = \big[\phi \in L^2(\mathbb{R}^3) : \phi \perp Q\big]; \quad X \longleftrightarrow \begin{bmatrix} Q^\perp \\ Q^\perp \end{bmatrix},$$

where Π is the orthogonal projection that eliminates the Q direction:

$$\Pi h = h - \frac{(Q, h)}{(Q, Q)} Q .$$

We claim that there is a "nice" operator U from M to X and a self-adjoint operator A that is a perturbation of H_1 such that

$$\mathcal{L}\big|_M = -U^{-1} J A U . \tag{2.17}$$

Let P_M be the projection from L^2 onto M according to the decomposition $L^2(\mathbb{R}^3) = S \oplus M$. It has the matrix form

$$P_M : \begin{bmatrix} L^2 \\ L^2 \end{bmatrix} \longrightarrow \begin{bmatrix} Q^\perp \\ R^\perp \end{bmatrix}, \quad P_M = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix},$$

where the projections are given by

$$\begin{aligned} &P_1 : L^2 \longrightarrow Q^\perp , \quad P_1 = \mathrm{Id} - c_1 |R\rangle\langle Q| , \quad c_1 = (Q, R)^{-1} , \\ &P_2 : L^2 \longrightarrow R^\perp , \quad P_2 = \mathrm{Id} - c_1 |Q\rangle\langle R| . \end{aligned} \tag{2.18}$$

Clearly $P_1 R = 0$ and $P_2 Q = 0$. One can easily check that

$$R_{MX} \equiv \begin{bmatrix} I & 0 \\ 0 & \Pi \end{bmatrix} : M \longrightarrow X , \quad R_{XM} \equiv \begin{bmatrix} I & 0 \\ 0 & P_2 \end{bmatrix} : X \longrightarrow M . \tag{2.19}$$

We now define

$$A := \left[(H^2 + H^{1/2} 2\lambda Q^2 H^{1/2}) \right]^{1/2} = \left[H^{1/2} L_+ H^{1/2} \right]^{1/2} , \quad H = L_- . \tag{2.20}$$

A is a self-adjoint operator acting on $L^2(\mathbb{R}^3)$, with Q as an eigenvector with eigenvalue 0. We shall often view A as an operator restricted to its invariant subspace X. Define

$$U_0 : X \longrightarrow X , \quad U_0 \equiv \begin{bmatrix} A^{1/2} H^{-1/2} & 0 \\ 0 & A^{-1/2} H^{1/2} \end{bmatrix}, \tag{2.21}$$

and let

$$U = U_0 R_{MX} : M \longrightarrow X , \quad U^{-1} = R_{XM} U_0^{-1} : X \longrightarrow M . \tag{2.22}$$

Notice that $H^{-1/2}$ is defined only on $Q^\perp$. We can easily check that $U^{-1} U|_M = \mathrm{Id}_M$ and $U U^{-1}|_X = \mathrm{Id}_X$. Moreover, we have

$$\begin{aligned} -(U^{-1} J)(A U)_M &= \begin{bmatrix} 0 & H^{1/2} A^{-1/2} \\ -P_2 H^{-1/2} A^{1/2} & 0 \end{bmatrix} \begin{bmatrix} A^{3/2} H^{-1/2} & 0 \\ 0 & A^{1/2} H^{1/2} \end{bmatrix} \\ &= \begin{bmatrix} 0 & H \\ -P_2 \Pi L_+ & 0 \end{bmatrix}. \end{aligned}$$

Since $P_2 \Pi L_+ = L_+$ when acting on $R^\perp$, we have proven (2.17).

If we define

$$U_\pm = \frac{1}{2} \left(\Pi A^{1/2} H^{-1/2} P_1 \pm \Pi A^{-1/2} H^{1/2} \Pi \right) , \tag{2.23}$$

then $(U_\pm)^* = \frac{1}{2}(P_2 H^{-1/2} A^{1/2} \Pi \pm \Pi H^{1/2} A^{-1/2} \Pi)$ and

$$U = U_+ + \mathbf{C} U_- , \quad U^{-1} = (U_+)^* - \mathbf{C}(U_-)^* , \tag{2.24}$$

where $\mathbf{C} = [\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}]$ is the conjugation operator in $L^2(\mathbb{R}^3)$. Although U_+ and U_- are not self-adjoint operators, they commute with the multiplication operator i. Hence, as operators in $L^2(\mathbb{R}^3)$,

$$U i = (U_+ + \mathbf{C} U_-) i = i (U_+ - \mathbf{C} U_-) . \tag{2.25}$$

Since the mass of Q_E is small, A is a self-adjoint perturbation of $H_1 = -\Delta + V - e_0$. As H_1 has two simple eigenvalues, so does A. The ground state of A is just Q (which with the normalized ground state equals $\phi_0^A = Q / \|Q\|_2$) with energy 0. Standard perturbation theory implies that A has an excited state $A\phi_1^A = \kappa \phi_1^A$ such that $\kappa = e_{10} + O(n^2)$ and $\phi_1^A = \phi_1 + O(n^2)$. The spectral decomposition Lemma 1.1 is a corollary of (2.17) and the spectral properties of A. For example, define

$$\begin{bmatrix} u \\ v \end{bmatrix} = U^{-1} \begin{bmatrix} \phi_1^A \\ \phi_1^A \end{bmatrix} .$$

We can check easily that u and v are the generalized eigenvectors of (1.13). We summarize the results here in the following lemma:

LEMMA 2.5 *Let X be the orthogonal complement of Q in L^2. Let A be defined by* (2.20). *Let U and U^{-1} be defined by* (2.22). *Then* (2.17) *holds, i.e., $\mathcal{L}|_M = -U^{-1} J A U$. If we denote by P_κ^A and $\mathbf{P}_c^A$ the orthogonal projections onto the eigenspace and continuous spectrum space of A, we have*

$$U P_M = U , \quad U P_\kappa^{\mathcal{L}} = P_\kappa^A U , \quad U \mathbf{P}_c^{\mathcal{L}} = \mathbf{P}_c^A U . \tag{2.26}$$

Furthermore, we have $U = U_+ + U_- \mathbf{C}$ and $U^{-1} = U_+^* - U_-^* \mathbf{C}$, with U_+ and U_- commuting with i.

2.3 The Equations

In this subsection we derive the equations for the components of the solution $\psi(t)$ of the Schrödiger equation (1.1) according to the decomposition (1.18).

Fix an energy E, the corresponding ground state $Q = Q_E$, and its tangent $R = R_E = \partial_E Q_E$. Recall the decomposition (1.18) of the wave function $\psi(t,x)$. We let

$$\Theta(t) = \theta(t) - Et . \tag{2.27}$$

Denote by $h_a = a(t) R_E(x) + h(t,x)$. Substituting the above ansatz into (1.1), we have the equations

$$\begin{aligned} \partial_t h_a &= \mathcal{L} h_a + i^{-1}(F + \dot\theta (Q + h_a)) , \\ F &= \lambda Q (2|h_a|^2 + h_a^2) + \lambda |h_a|^2 h_a \\ &= \lambda Q (2|h|^2 + h^2) + 2\lambda Q R a (2h + \bar h) + 3\lambda Q R^2 a^2 + \lambda (aR + h)^2 (aR + \bar h) , \end{aligned}$$

where $\mathcal{L}$ is already defined in (1.10). By (1.2) and (2.5), we have $(\mathcal{L} - \partial_t)aR = -iaQ - \dot{a}R$. Hence

$$\partial_t h = \mathcal{L}h + i^{-1}(F + \dot{\theta}(Q + aR + h)) - iaQ - \dot{a}R\,.$$

Since $h(t) \in M$ and M is an invariant subspace of $\mathcal{L}$, we have $i^{-1}(F + \dot{\theta}(Q + aR + h)) - iaQ - \dot{a}R \in M$. From the characterization $M = [\,f + ig : f \perp Q, g \perp R]$, the functions $a(t)$ and $\theta(t)$ satisfy

$$(Q, \operatorname{Im}(F + \dot{\theta}h) - \dot{a}R) = 0\,,$$
$$(R, \operatorname{Re}(F + \dot{\theta}(Q + aR + h)) + aQ) = 0\,.$$

The equation on M is

$$\partial_t h = \mathcal{L}h + P_M F_{\text{all}}\,, \quad F_{\text{all}} = i^{-1}(F + \dot{\theta}(aR + h))\,. \tag{2.28}$$

Recall the representation of $\mathcal{L}$ in terms of A. The previous equation can be rewritten as

$$\partial_t(Uh) = -iA(Uh) + UP_M F_{\text{all}}\,. \tag{2.29}$$

We can write $Uh = z\phi_1^A + w$ where w denotes the part on the continuum spectrum of A. Then we have

$$\dot{z} = -i\kappa z + (\phi_1^A, UP_M F_{\text{all}})\,, \quad \dot{w} = -iAw + \mathbf{P}_{\text{c}}^A U P_M F_{\text{all}}\,.$$

Since $U\mathbf{P}_{\text{c}}^{\mathcal{L}} = \mathbf{P}_{\text{c}}^A U$, we have $\mathbf{P}_{\text{c}}^A U P_M = \mathbf{P}_{\text{c}}^A U$. Using the definitions of u and v, we have

$$(\phi_1^A, UP_M F_{\text{all}}) = (v, \operatorname{Re} F_{\text{all}}) + i(u, \operatorname{Im} F_{\text{all}}) = (u_+, F_{\text{all}}) - (u_-, \overline{F}_{\text{all}})$$

where $u_\pm = \frac{1}{2}(u \pm v)$. If we define $p(t) = e^{i\kappa t}z(t)$, we have

$$e^{-i\kappa t}\dot{p}(t) = i^{-1}\{(u_+, F) + (u_-, \overline{F}) + [(u_+, h) + (u_-, \overline{h}) + (u, R)a]\dot{\theta}\}\,.$$

Summarizing, we have the decomposition

$$\psi = (Q + aR + h)e^{-iEt + i\theta}\,, \quad Uh = z\phi_1^A + w\,, \quad z = e^{-i\kappa t}p\,,\ \mu = e^{i\kappa t}\,,$$

and the equation

$$\begin{cases} \dot{a} = (c_1 Q, \operatorname{Im}(F + \dot{\theta}h))\,, \quad c_1 = (Q, R)^{-1}\,, \\ \partial_t w = -iAw + \mathbf{P}_{\text{c}}^A U i^{-1}(F + \dot{\theta}(aR + h))\,, \\ ie^{-i\kappa t}\dot{p} = (u_+, F) + (u_-, \overline{F}) + [(u_+, h) + (u_-, \overline{h}) + (u, R)a]\dot{\theta}\,, \end{cases} \tag{2.30}$$

where

$$\begin{aligned} F = {}& \lambda Q(2|h|^2 + h^2) + 2\lambda QRa(2h + \bar{h}) + 3\lambda QR^2a^2 \\ & + \lambda(aR + h)^2(aR + \bar{h})\,, \end{aligned} \tag{2.31}$$

$$\dot{\theta} = -[a + (c_1 R, \operatorname{Re} F)] \cdot [1 + a(c_1 R, R) + (c_1 R, \operatorname{Re} h)]^{-1}\,. \tag{2.32}$$

This is a system of equations involving a, z, and η only. It is the system that we shall solve in the rest of this paper. Note that θ does not appear explicitly in the system (2.30). We see $\dot{\theta}$ only on the right side. Although θ will appear when we integrate the main term of η, it appears only in the form $e^{i\theta}$; hence we do not need estimates for θ.

We also have the following decomposition of h according to (1.16):

$$h = \zeta + \eta\,, \quad \eta = U^{-1}w \in \mathbf{H}_{\mathrm{c}}(\mathcal{L})\,, \tag{2.33}$$

$$\zeta = U^{-1}z\phi_1^A = \mathrm{Re}(z)u + \mathrm{Im}(z)iv = zu_+ + \bar{z}u_- \in \mathbf{E}_\kappa(\mathcal{L})\,. \tag{2.34}$$

The equation for η can be obtained from that of w or more directly from projecting the original equation of h onto the continuum spectrum of $\mathcal{L}$:

$$\partial_t \eta = \mathcal{L}\eta + \mathbf{P}_{\mathrm{c}}^{\mathcal{L}} i^{-1}(F + \dot{\theta}(aR + h))\,. \tag{2.35}$$

2.4 Linear Estimates

Here we collect a few estimates on the operators U, $\mathcal{L}$, and A. They will be proven at the end of this paper.

We denote the wave operators for $\mathcal{L}$ (respectively, A and H) by $W_{\mathcal{L}}$ (respectively, W_A and W_H). They are defined by

$$\begin{gathered} W_{\mathcal{L}} = \lim_{t\to\infty} e^{-t\mathcal{L}}e^{-itH_*}\,, \quad W_A = \lim_{t\to\infty} e^{itA}e^{-itH_*}\,, \\ W_H = \lim_{t\to\infty} e^{itH}e^{-itH_*}\,, \end{gathered} \tag{2.36}$$

where

$$H_* = -\Delta - E\,. \tag{2.37}$$

Note that if W_A exists, we have the intertwining property that $f(A)P_c(A) = W_A f(H_*)W_A^*$ for suitable functions f. We also have a similar property for $\mathcal{L}$.

LEMMA 2.6 *For $k = 0, 1, 2$, there is a positive constant C_1 so that*

$$C_1^{-1}\,\|\phi\|_{H^k} \le \|e^{-itA}\phi\|_{H^k} \le C_1\,\|\phi\|_{H^k} \tag{2.38}$$

for all $\phi \in X \cap H^k$ and all $t \in \mathbb{R}$.

LEMMA 2.7 (Decay Estimates for e^{-itA}) *For $q \in [2, \infty]$ and $q' = q/(q-1)$,*

$$\left\| e^{-itA}\mathbf{P}_{\mathrm{c}}^A \Pi\phi \right\|_{L^q} \le C|t|^{-3\left(\frac{1}{2}-\frac{1}{q}\right)}\,\|\phi\|_{L^{q'}}\,. \tag{2.39}$$

For smooth local functions ϕ and sufficiently large r_1, we have

$$\left\| \langle x\rangle^{-r_1} e^{-itA}\frac{1}{(A - 0i - 2\kappa)^l}\mathbf{P}_{\mathrm{c}}^A \Pi \langle x\rangle^{-r_1}\phi \right\|_{L^2} \le C\langle t\rangle^{-9/8} \tag{2.40}$$

where $l = 1, 2$ and $0i$ means σi with $\lim_{\sigma\to 0+}$ outside of the bracket.

Estimate (2.39) for $A = -\Delta + V$ was proven in [8] using estimates from [7] and [10]. Estimate (2.40) for $A = \sqrt{H}$ was proven by Soffer and Weinstein [16].

LEMMA 2.8

$$\left(\phi_0\phi_1^2, \operatorname{Im}\frac{1}{A-0i-2\kappa}\mathbf{P}_c^A\Pi\phi_0\phi_1^2\right) = \left(\phi_0\phi_1^2, \operatorname{Im}\frac{1}{H_1-0i-2\kappa}\mathbf{P}_c^{H_1}\phi_0\phi_1^2\right) + O(n^2) > 0\,. \tag{2.41}$$

This lemma is a perturbation result. Notice that $\kappa = e_{01} + O(n^2)$; hence the second term is greater than γ_0 by assumption A1.

LEMMA 2.9 (Operator U) (i) *The operators U_0 and U_0^{-1} are bounded operators in $W^{k,p} \cap X$ for $k \le 2$, $1 \le p < \infty$, and in $L_r^2 \cap X$ for $|r| \le r_0$. (L_r^2 is the weighted L^2 space defined in* (1.7).) *Hence $U : M \to X$ and $U^{-1} : X \to M$ are bounded in $W^{k,p}$ and L_r^2 norms.*

(ii) *The commutator $[U, i]$ is a local operator in the sense that*

$$\|[U, i]\phi\|_{L^{8/7}\cap L^{4/3}} \le C\,\|\phi\|_{L^4}\ . \tag{2.42}$$

LEMMA 2.10 (Wave Operators) *The wave operators $W_{\mathcal{L}}$ and W_A defined by* (2.36) *exist and satisfy $W^{k,p}$ estimates for $k \le 2$, $1 \le p < \infty$* (*similar estimates hold for their adjoints*):

$$\left\|W_{\mathcal{L}}\mathbf{P}_c^{\mathcal{L}}\right\|_{(W^{k,p},W^{k,p})} \le C\,, \quad \left\|W_A\mathbf{P}_c^A\Pi\right\|_{(W^{k,p},W^{k,p})} \le C\,.$$

The statement on $W_{\mathcal{L}}$ was proven in [4], following the proof of [20]. Hence we need only to prove the statement on W_A.

3 Main Oscillation Terms

We now identify the main oscillation terms in equation (2.30). Most quantities treated in this paper, such as F, a, z, and η, are strongly oscillatory. Hence it is necessary to identify their oscillatory parts before we can estimate. We shall use the complex amplitude of the excited state, z, as the reference. Recall $z(t) = e^{-i\kappa t}p(t)$. We will show that $p(t) \sim t^{-1/2}$ and its oscillation (phase speed) is much smaller than κ. The change of mass on the direction of the nonlinear ground state is given by a. We will also show that $a = O(z^2)$ and the order of the dispersive wave $\eta(t)$ is also of order $O(z^2)$. Assuming these orders, the second-order term in F is given explicitly by

$$F^{(2)} = \lambda Q(2|\zeta|^2 + \zeta^2) = z^2\phi_{(20)} + z\bar{z}\phi_{(11)} + \bar{z}^2\phi_{(02)} \tag{3.1}$$

where $\zeta = zu_+ + \bar{z}u_-$ and

$$\begin{aligned}
\phi_{(20)} &= \lambda Q(u_+^2 + 2u_+u_-) = \lambda Q\phi_1^2 + O(n^3)\,,\\
\phi_{(11)} &= 2\lambda Q(u_+^2 + u_-^2 + u_+u_-) = 2\lambda Q\phi_1^2 + O(n^3)\,,\\
\phi_{(02)} &= \lambda Q(u_-^2 + 2u_+u_-) = O(n^3)\,.
\end{aligned} \tag{3.2}$$

We shall write $F^{(2)} = z^\alpha \phi_\alpha$, where α is a double indices (ij) with $i + j = 2$ and $i, j \geq 0$. The repeated indices mean summation. We shall use β later on to denote double indices summing to 3 and γ summing to 4.

3.1 Leading Oscillatory Terms in *a*

We first identify the main oscillation terms of $a(t)$. Recall from (2.30) that $\dot{a} = (c_1 Q, \operatorname{Im} F + \dot{\theta} h)$ and $c_1 = (Q, R)^{-1}$. We shall impose the boundary condition of a at $t = T$. This is in fact a condition imposed on the choice of E_T. Hence we use the following equivalent integral equation:

$$a(t) = a(T) + \int_T^t (c_1 Q, \operatorname{Im} F + \dot{\theta} h)(s) ds \,. \tag{3.3}$$

The main term of $\operatorname{Im} F + \dot{\theta} h$ is $\operatorname{Im} F^{(2)} = \operatorname{Im} \lambda Q \zeta^2$, and thus the leading oscillation term of $a(t)$ is from the integral $\int_T^t A^{(2)}\, ds$ with

$$A^{(2)} = (c_1 Q, \lambda Q \operatorname{Im} \zeta^2) \,.$$

Since $\zeta = z u_+ + \bar{z} u_-$, we have $\operatorname{Im} \zeta = \operatorname{Im} z (u_+ - u_-)$ and $\operatorname{Im} \zeta^2 = (\operatorname{Im} z^2)(u_+^2 - u_-^2)$; therefore, we have

$$A^{(2)} = C_1 \operatorname{Im} z^2 \,, \quad C_1 = (c_1 Q, \lambda Q (u_+^2 - u_-^2)) \,.$$

Write

$$z^2(s) = e^{-2i\kappa s} p^2 = \frac{1}{-2i\kappa} \frac{d}{ds} (e^{-2i\kappa s}) p^2 \,. \tag{3.4}$$

We can integrate $A^{(2)}$ by parts to get

$$\int_T^t A^{(2)}\, ds = \operatorname{Im} C_1 \int_T^t z^2\, ds = 2a_{20} \operatorname{Re} \left\{ [z^2]_T^t - \int_T^t e^{-2i\kappa s} 2 p \dot{p}\, ds \right\} \,, \tag{3.5}$$

where

$$a_{20} = \frac{C_1}{4\kappa} = \frac{1}{4\kappa} \left(c_1 Q, \lambda Q (u_+^2 - u_-^2) \right) = O(n^2) \tag{3.6}$$

and the last integral is a higher-order term. Let us denote

$$a^{(2)}(t) = a_{20} (z^2 + \bar{z}^2)(t) \,, \tag{3.7}$$

which is the main oscillatory term in a. We denote the rest of a by b, i.e.,

$$a(t) = a^{(2)}(t) + b(t) \,, \tag{3.8}$$

and we have

$$\begin{aligned} b(t) &= a(T) - a^{(2)}(T) \\ &\quad + \int_T^t \left\{ (c_1 Q, \operatorname{Im}[F - F^{(2)} + \dot{\theta} h]) - 4 a_{20} \operatorname{Re} e^{-2i\kappa s} p \dot{p} \right\} (s) ds \,. \end{aligned} \tag{3.9}$$

It is easy to see that $a, b, \dot{a} = O(z^2)$, but $\dot{b} = O(z^3)$. In other words, $a^{(2)}$ is the part of a with strong oscillation and b is the part of a that has slower oscillation. The use of b is convenient when we work on normal forms of p and a later. (We will integrate by parts terms involving b, and in some sense replace terms involving b by terms involving $\dot{b}$, which is smaller. If we work with a instead, then we replace a by $\dot{a}$, which is of the same order and we need one more step.) In fact, in principle one should treat the normal forms of p and a together, since they correspond to the excited states and the ground state. See [17] for more elaboration on this point.

It should be noted that, although $b(t)$ is not oscillatory, it is in fact larger than $a^{(2)}(t)$, and hence is the main term of $a(t)$. Another point to make here is that the introduction of $b(t)$ is for computational convenience. The true variable we work with is still $a(t)$. In particular, the induction assumption we make later in Section 5 is on a, not b.

Notice that when integrating $\dot{\theta}$, we take $\operatorname{Re} F$ instead of $\operatorname{Im} F$. Hence the term $C|z|^2$ survives and cannot be integrated; thus $\theta(t) = O(\log t)$, although $a(t) = O(t^{-1})$.

3.2 Leading Oscillatory Terms in η

We now identify the main oscillation term in η. From the basic equation (2.30), we rewrite the w equation as

$$\partial_t w = -iAw - i\dot{\theta} w + \mathbf{P}_c^A U i^{-1}[F + \dot{\theta}(aR + \zeta)] - \mathbf{P}_c^A [U, i]\dot{\theta}\eta\,.$$

where we have used the commutator $[U, i]$ to interchange U and i so as to produce the term $i\dot{\theta} w$. This term is a global linear term in η and cannot be treated as an error term (however, $[U, i]\dot{\theta}\eta$ is an error term). We can eliminate it by rewriting the last equation in terms of $\tilde{\eta} = e^{i\theta} w = e^{i\theta} U\eta$, i.e.,

$$\partial_t \tilde{\eta} = -iA\tilde{\eta} + e^{i\theta}\mathbf{P}_c^A U i^{-1}[F + \dot{\theta}(aR + \zeta)] - e^{i\theta}\mathbf{P}_c^A [U, i]\dot{\theta}\eta\,,$$

or in integral form,

$$\tilde{\eta}(t) = e^{-iAt}\tilde{\eta}_0 + \int_0^t e^{-iA(t-s)}\mathbf{P}_c^A F_\eta(s)\,ds\,, \tag{3.10}$$

where

$$F_\eta = e^{i\theta} U i^{-1}[F + \dot{\theta}(aR + \zeta)] - e^{i\theta}[U, i]\dot{\theta}\eta\,. \tag{3.11}$$

Since U and U^{-1} are bounded in Sobolev spaces by Lemma 2.9 and

$$\eta(t) = U^{-1} e^{-i\theta(t)}\tilde{\eta}(t)\,, \tag{3.12}$$

for the purpose of estimation we can treat η and $\tilde{\eta}$ as the same.

Let $F_{\eta,2}$ be the second-order term of F_η, and let $F_{\eta,3}$ denote the rest, i.e.,

$$\begin{aligned} F_\eta &= F_{\eta,2} + F_{\eta,3}\,,\\ F_{\eta,2} &= e^{i\theta} U i^{-1} F^{(2)} = e^{i\theta} U i^{-1} z^\alpha \phi_\alpha\,,\\ F_{\eta,3} &= e^{i\theta} U i^{-1}\big[(F - F^{(2)}) + \dot{\theta}(aR + \zeta)\big] - e^{i\theta}[U, i]\dot{\theta}\eta\,. \end{aligned} \tag{3.13}$$

Since $U = U_+ + \mathbf{C}U_-$ with U_+ and U_- commuting with i, see (2.24), we can write $F_{\eta,2}$ as

$$F_{\eta,2} = e^{i\theta}(U_+ + \mathbf{C}U_-)i^{-1}z^\alpha\phi_\alpha = i^{-1}e^{i\theta}(U_+ - \mathbf{C}U_-)z^\alpha\phi_\alpha = i^{-1}e^{i\theta}z^\alpha\Phi_\alpha\,, \tag{3.14}$$

$$\Phi_{(20)} = U_+\phi_{(20)} - U_-\phi_{(02)}\,, \quad \Phi_{(02)} = U_+\phi_{(02)} - U_-\phi_{(20)}\,, \quad \Phi_{(11)} = (U_+ - U_-)\phi_{(11)}\,. \tag{3.15}$$

Replacing F_η in (3.10) by $F_{\eta,2}$ and integrating by parts, we have

$$\int_0^t e^{-iA(t-s)}\mathbf{P}_c^A F_{\eta,2}\,ds = \int_0^t e^{-itA}e^{is(A-0i+[\alpha]\kappa)}\left((e^{i\theta}p^\alpha)(s)\right)i^{-1}\mathbf{P}_c^A\Phi_\alpha\,ds = \tilde{\eta}^{(2)}(t) - e^{-itA}(e^{i\theta}z^\alpha)(0)\widetilde{\eta}_\alpha + (*)\,, \tag{3.16}$$

where

$$\tilde{\eta}^{(2)}(t) = e^{i\theta(t)}z^\alpha(t)\widetilde{\eta}_\alpha\,, \quad \widetilde{\eta}_\alpha = \frac{-1}{A - 0i + [\alpha]\kappa}\mathbf{P}_c^A\Phi_\alpha\,, \tag{3.17}$$

and $(*)$ denotes the remainder term from integration by parts,

$$(*) = -\int_0^t e^{-i(t-s)A}\left\{e^{is\kappa[\alpha]}\frac{d}{ds}(e^{i\theta}p^\alpha)\widetilde{\eta}_\alpha\right\}ds\,.$$

In the integration we have inserted the regularizing factor $e^{is(-0i)}$ with the sign of $0i$ chosen so that $e^{-itA}\widetilde{\eta}_\alpha$ decays as $t \to +\infty$. See Lemma 2.7.

We shall see that $\tilde{\eta}^{(2)}(t)$ is the main oscillation term in $\tilde{\eta}(t)$. We denote the rest by $\tilde{\eta}^{(3)}(t)$:

$$\tilde{\eta} = \tilde{\eta}^{(2)} + \tilde{\eta}^{(3)}\,. \tag{3.18}$$

Notice that this is not a decomposition in L^2: Both $\tilde{\eta}^{(2)}$ and $\tilde{\eta}^{(2)}$ are not in L^2. Nevertheless, this decomposition is useful for studying the local behavior of $\tilde{\eta}$. Also note that, although $\tilde{\eta}_\alpha \notin L^2$ but is still "orthogonal" to the eigenvector of A, $(\phi_1^A, \tilde{\eta}_\alpha) = 0$.

From (3.10) we obtain the equation for $\tilde{\eta}^{(3)}$:

$$\begin{aligned}\tilde{\eta}^{(3)}(t) &= e^{-iAt}\tilde{\eta}_0 - e^{-iAt}(e^{i\theta}z^\alpha)(0)\widetilde{\eta}_\alpha \\ &\quad - \int_0^t e^{-iA(t-s)}\left\{e^{is\kappa[\alpha]}\frac{d}{ds}(e^{i\theta}p^\alpha)\widetilde{\eta}_\alpha\right\}ds \\ &\quad + \int_0^t e^{-iA(t-s)}\mathbf{P}_c^A\left\{F_{\eta,3} - e^{i\theta}Ui^{-1}\eta^2\bar{\eta}\right\}ds \\ &\quad + \int_0^t e^{-iA(t-s)}\mathbf{P}_c^A e^{i\theta}Ui^{-1}\eta^2\bar{\eta}\,ds \\ &\equiv \tilde{\eta}_1^{(3)} + \tilde{\eta}_2^{(3)} + \tilde{\eta}_3^{(3)} + \tilde{\eta}_4^{(3)} + \tilde{\eta}_5^{(3)}\,.\end{aligned} \tag{3.19}$$

We treat $\tilde{\eta}_5^{(3)}$ separately because $\eta^2\bar{\eta}$ is a nonlocal term. Recall that $\eta = U^{-1}e^{-i\theta}\tilde{\eta}$. We have the similar decomposition for η,

$$\begin{aligned} \eta(t) &= \eta^{(2)}(t) + \eta^{(3)}(t)\,, \quad & \eta^{(2)} &= U^{-1}e^{-i\theta}\tilde{\eta}^{(2)} = U^{-1}z^\alpha \widetilde{\eta}_\alpha\,, \\ \eta^{(3)} &= U^{-1}e^{-i\theta}\tilde{\eta}^{(3)}\,, & \eta_j^{(3)} &= U^{-1}e^{-i\theta}\tilde{\eta}_j^{(3)}\,. \end{aligned} \tag{3.20}$$

If we view b and $\eta^{(2)}$ as order z^2, and $\eta^{(3)}$ as order z^3, we can now decompose F into

$$F = F^{(2)} + F^{(3)} + \widetilde{F}^{(3)} + F^{(4)} \tag{3.21}$$

where

$$\begin{aligned} F^{(2)} &= \lambda Q(2|\zeta|^2 + \zeta^2)\,, \\ F^{(3)} &= 2\lambda Q\big[(\zeta+\bar{\zeta})\eta^{(2)} + \zeta\bar{\eta}^{(2)}\big] + \lambda|\zeta|^2\zeta + 2\lambda QRa^{(2)}(2\zeta+\bar{\zeta})\,, \\ \widetilde{F}^{(3)} &= 2\lambda QRb(2\zeta+\bar{\zeta}) = 2\lambda QR(a-a^{(2)})(2\zeta+\bar{\zeta})\,, \\ F^{(4)} &= 2\lambda Q\big[(\zeta+\bar{\zeta})\eta^{(3)} + \zeta\bar{\eta}^{(3)}\big] + \lambda Q\big[2|\eta|^2+\eta^2\big] + 2\lambda a QR(2\eta+\bar{\eta}) \\ &\quad + 3\lambda a^2 QR^2 + \lambda\big[|k|^2k - |\zeta|^2\zeta\big]\,. \end{aligned} \tag{3.22}$$

In view of (4.5), $F^{(2)}$ consists of terms of order z^2; $F^{(3)}$ and $\widetilde{F}^{(3)}$ of terms of order z^3; and $F^{(4)}$ of higher-order terms. We separate the $\widetilde{F}^{(3)}$ term since it depends on b and requires different methods to estimate.

From (2.32), we have

$$\begin{aligned} \dot{\theta} &= -[a + (c_1R, \operatorname{Re}F)]\cdot[1 + a(c_1R, R) + (c_1R, \operatorname{Re}h)]^{-1} \\ &= -[a + (c_1R, \operatorname{Re}F)]\cdot[1 - (c_1R, \operatorname{Re}h)] + O(z^4 + a^2 + \eta^2) \\ &= -[a + (c_1R, \operatorname{Re}F^{(2)})] + F_{\theta,3} + F_{\theta,4} \end{aligned} \tag{3.23}$$

where

$$F_{\theta,3} = -\big(c_1R,\ \operatorname{Re}F^{(3)} + \widetilde{F}^{(3)}\big) + \big[a + (c_1R, \operatorname{Re}F^{(2)})\big]\,(c_1R, \operatorname{Re}\zeta)$$

and $F_{\theta,4} = O(z^4 + a^2 + \eta^2)$.

4 Outline of Proofs and Basic Estimates

We are now ready to outline the proofs for Theorems 1.1 through 1.4.

4.1 Initial Data and Basic Quantities

Under the assumptions of either Theorem 1.1 or Theorem 1.3, we can rewrite the initial data ψ_0 using Lemma 2.3 as

$$\psi_0 = [Q_{E_0} + h_{E_0,0}]e^{i\Theta_{E_0,0}}\,, \quad h_{E_0,0} \in M_{E_0}\,,$$

for some E_0, $h_{E_0,0}$, and $\Theta_{E_0,0}$, with

$$n = \|\psi_0\|_Y\,, \quad \|Q_{E_0}\| \geq \tfrac{9}{10}n\,, \quad \|h_{E_0,0}\|_Y \leq C\varepsilon_0 n\,.$$

We can assume that $\Theta_{E_0,0} = 0$ without loss of generality. We now write $h_{E_0,0} = \zeta_{E_0,0} + \eta_{E_0,0}$ according to the spectral decomposition (1.16) in M_{E_0}. Recall the definition of ε in (1.21). We have

$$\|\zeta_{E_0,0}\| \le C\varepsilon n\,, \quad \|\eta_{E_0,0}\|_Y \le C\varepsilon^2 n^2\,, \quad \varepsilon \le \varepsilon_0\,. \tag{4.1}$$

For the rest of this paper, we shall take ϵ as either given by (1.21) or (1.24). The choice will not be important, and, whenever a specific choice is needed, we shall remark upon it.

For all $t \ge 0$, we have $\|\psi(t)\|_{L^2} = n$. If we write $\psi(t) = x\phi_0 + y\phi_1 + \xi$, $\xi \in \mathbf{H}_c(-\Delta + V)$, then we have $|x|^2 + |y|^2 + \|\xi\|_{L^2}^2 = n^2$. Since the spectral projections of $-\Delta + V$ and $\mathcal{L}$ differ by an order of $O(\lambda Q^2) = O(n^2)$, we have $\|Q_E\| = |x| + O(n^3)$, $\|\zeta\|_{L^2} = |y| + O(n^3)$, and $\|\eta\|_{L^2} = \|\xi\|_{L^2} + O(n^3)$. In particular,

$$\|Q_E\| \le \tfrac{9}{8}n\,, \quad \|\zeta(t)\|_{L^2} \le \tfrac{9}{8}n\,, \quad \|\eta(t)\|_{L^2} \le \tfrac{9}{8}n\,. \tag{4.2}$$

We now give a list of the expected sizes of frequently used quantities. Let

$$\Gamma \equiv \operatorname{Im}\left(2\Phi_{20}, \frac{1}{A - 0i - 2\kappa}\mathbf{P}_c^A\Phi_{20}\right) \ge \lambda^2 n^2 \gamma_0\,, \tag{4.3}$$

where $\Phi_{20} = O(n)$ is defined as in (3.15), and γ_0 is the constant from assumption A1. The last statement $\Gamma \ge \lambda^2 n^2 \gamma_0$ will be proven in Lemma 6.1. Recall that

$$\{t\} \equiv \varepsilon^{-2} n^{-2} + \Gamma t, \quad \{t\}^{-1/2} \sim \min\left\{\varepsilon n, n^{-1} t^{-1/2}\right\}\,. \tag{4.4}$$

$\{t\}^{-1/2}$ will be the typical size of $|z(t)|$. The following is a table of order in t for functions:

$$\begin{aligned} |z(t)| &= O(t^{-1/2})\,, \quad |a(t)| = O(t^{-1})\,, \\ \|\eta(t)\|_{\text{loc}} &= O(t^{-1})\,, \quad |\theta(t)| = O(\log t)\,. \end{aligned} \tag{4.5}$$

The following is a table of order in n for constants:

$$\begin{aligned} Q &= O(n)\,, \quad R = O(n^{-1})\,, \\ c_1 &= O(1)\,, \quad u_+ = O(1)\,, \quad u_- = O(n^2)\,. \end{aligned} \tag{4.6}$$

The first three estimates in the last equation are due to Lemma 2.1. The last one is because the differences between L_+, L_-, and $H_1 = -\Delta + V - e_0$ are of order $O(\lambda n^2)$, and hence so are the differences between ϕ_1, u, and v.

4.2 Outlines of the Proofs for Theorems 1.1 and 1.3

We now estimate solutions to the equations (2.30) with the decompositions into main oscillatory and higher-order terms in Section 3. We first fix a time T and let $E = E(T)$ be the best approximation at the time T so that (1.19) holds at time T, i.e.,

$$\psi(T, x) = [Q_E(x) + h_E(T, x)]\, e^{i\Theta(T)}\,. \tag{4.7}$$

We can now decompose the wave function for all time with respect to this ground state as in (1.18), namely,

$$\psi(t,x) = \left[Q_{E(T)}(x) + a_{E(T)}(t)R_{E(T)}(x) + h_{E(T)}(t,x)\right]e^{i\Theta(t)} ,$$

where $h_{E(T)}(t,x) = \zeta_{E(T)}(t) + \eta_{E(T)}(t) \in M_{E(T)}$. To simplify the notation, we write $a_{E(T)} = a_T$, etc. By assumption, we have $a_T(T) = 0$. We now wish to estimate a_T, ζ_T, and η_T for all time $t \le T$. The following propositions are stated with respect to a decomposition of a fixed nonlinear ground state profile E.

We first choose a suitable norm. We need to control the excited-state component z and a local norm of η. We also need a global norm of η to control the nonlocal term η^3. Recall that we can decompose η into a sum of $\eta^{(2)} + \eta^{(3)}$ with $\eta^{(3)}$ decomposed further into a sum of five terms; see (3.20). Since $\eta^{(2)}$, $\eta_1^{(3)}$, and $\eta_2^{(3)}$ are explicit, we need only to control

$$\eta_{3-5}^{(3)} \equiv \sum_{j=3}^{5} \eta_j^{(3)} = \eta_3^{(3)} + \eta_4^{(3)} + \eta_5^{(3)} , \tag{4.8}$$

and we define

$$\begin{aligned} M(T) := \sup_{0\le t\le T} \Big\{ &\{t\}^{1/2} |z(t)| + \{t\}^{3/4-\sigma} \|\eta(t)\|_{L^4} + n^\sigma \{t\} \|\eta(t)\|_{L^2_{\mathrm{loc}}} \\ &+ \varepsilon^\sigma (\varepsilon n)^{-3/4} \{t\}^{9/8} \left\|\eta_{3-5}^{(3)}(t)\right\|_{L^2_{\mathrm{loc}}} \Big\} \end{aligned} \tag{4.9}$$

where $\Gamma = O(n^2)$ is defined in (4.3) and $B_{22} = \frac{1}{2}c_1\Gamma + O(n^3)$ will be defined explicitly later in (7.22). For the time being, we need only know that B is a constant with the size given above; hence $D = c_1 + O(n)$.

If we assume that $a(t)$ is bounded by t^{-1} up to some time T, we have the following control of $M(t)$.

PROPOSITION 4.1 *Suppose $z(t)$ and $\eta(t)$ are solutions to the equations* (2.30) *for* $0 \le t \le T$. (*T can be finite or infinite.*) *Assume*

$$|a(t)| \le D\{t\}^{-1} , \quad 0 \le t \le T . \tag{4.10}$$

Then we have

$$M(t) \le 2 \quad \textit{for all } t \le T .$$

Moreover, if we further assume $|z_0| = \varepsilon n > 0$ and $\|\eta_0\|_Y \le \varepsilon^2 n^2$, then $|z(t)| \ge c\{t\}^{-1/2}$.

Recall $b(t) := a(t) - a^{(2)}(t)$. Let $S_T(t) = b(t) - b(T)$ so that

$$a(t) = a(T) + \left[a_{20}(z^2 + \bar z^2)\right]_T^t + S_T(t) . \tag{4.11}$$

Assuming that $a(t)$ is bounded by t^{-1}, we can control $S_T(t)$ by the following proposition:

PROPOSITION 4.2 *Suppose that $M(t) \le 2$ and $|a(t)| \le D\{t\}^{-1}$ for $0 \le t \le T$. Recall $D = 2B_{22}/\Gamma = c_1 + O(n)$. Then we have*

$$|S_T(t)| \le \frac{D}{2}\{t\}^{-1}, \quad |a(t)| \le |a(T)| + \frac{D}{2}\{t\}^{-1}. \tag{4.12}$$

PROOF OF THEOREMS 1.1 AND 1.3: Assuming the previous two propositions, we now prove that

$$\sup_{t \le T} |a_T(t)|\{t\} < D \tag{4.13}$$

for all time, where a_T is the component with respect to the best approximation at the time T defined in (1.19). From (4.1), the relation (4.13) holds for $T = 0$. Suppose it holds for time $T = T_1$. From the continuity of the Schrödinger equation, there exists an $\delta > 0$ (depending on T_1) such that $|E(T) - E(T_1)| \le \exp[-T_1](D - \sup_{t \le T_1} |a_{T_1}(t)|\{t\})$ for all $|T - T_1| < \delta$. Thus the bound (4.13) holds for all $|T - T_1| < \delta$. This proves that the set of such T is open. Now suppose that T_1 is the first time that (4.13) is violated. Clearly, we have $\sup_{t \le T_1} |a_{T_1}(t)|\{t\} \le D$. From Propositions 4.1 and 4.2, formula (4.13) holds for T_1.

Once we have proven (4.13), the conclusions of Propositions 4.1 and 4.2 hold, and they imply both Theorem 1.1 and Theorem 1.3. □

We remark that the main reason that the previous continuity argument works is due to the explicit identification of the leading terms in a, η, and z. Since these leading terms are essentially computed without using assumptions on the decay of a, the bound on a, i.e., (4.13), is used only to control error terms. We shall prove Theorem 1.4 in Section 7, where we obtain more detailed information about the wave function.

The proof of Proposition 4.1 will be given in the next two sections and is followed by a proof of Proposition 4.2. Once again, our strategy of proof is to show that $M(0) \le \frac{3}{2}$ and that $M(t) \le \frac{3}{2}$ if $M(t) \le 2$. By continuity of $M(t)$, this would imply that $M(t) \le \frac{3}{2}$ for all $t \le T$. So for the rest of this and the next sections, we shall freely use that $M(t) \le 2$ and $|a(t)| \le D\{t\}^{-1}$. It will follow from Lemmas 5.2 and 5.3 that we can bound

$$\{t\}^{3/4-\sigma}\|\eta(t)\|_{L^4} + \varepsilon^\sigma(\varepsilon n)^{-3/4}\{t\}^{9/8}\left\|\eta^{(3)}_{3-5}(t)\right\|_{L^2_{\text{loc}}}$$

by $\frac{1}{4}$. It will follow from Lemma 6.2 that

$$\{t\}^{1/2}|z(t)| \le (1 + 2\sigma)$$

for some small σ. We have thus $M(T) \le \frac{3}{2}$ and conclude Proposition 4.1.

We note that, under the assumptions (4.10) and $M(T) \le 2$, we have, from the equations (2.31) for F, (2.32) for $\dot\theta$, (2.30) for $\dot p$, and (3.9) for b,

$$\begin{aligned} &\|F\|_{L^2_{\text{loc}}} \le Cn\{t\}^{-1}, \quad \left\|F - F^{(2)}\right\|_{L^2_{\text{loc}}} \le C\{t\}^{-3/2}, \\ &|\dot\theta| \le C\{t\}^{-1}, \quad |\dot p| \le Cn\{t\}^{-1}, \quad |\dot b| \le Cn\{t\}^{-3/2}. \end{aligned} \tag{4.14}$$

It follows from (4.14) that

$$|\theta(t)| \le C \log \{t\} \ . \tag{4.15}$$

Note that, although $\dot\theta$ and $\dot a$ are of the same order, $\dot\theta$ is not oscillatory and we cannot expect an estimate better than (4.15). This estimate will not be used in the rigorous proof, but it provides an idea about its size.

5 Estimates of the Dispersive Wave

For the dispersive part η, our main interest is its local decay estimate. Due to the presence of the nonlocal term η^3 in F, we will also need a global estimate on η. Our goal is to prove, for $\sigma = 1/100$,

$$\|\eta(t)\|_{L^4} \le \tfrac{1}{8} \{t\}^{-3/4+\sigma} \ , \quad \left\|\eta^{(3)}_{3-5}(t)\right\|_{L^2_{\rm loc}} \le C\varepsilon^{-\sigma}(\varepsilon n)^{3/4} \{t\}^{-9/8} \ .$$

We will need the following calculus lemma:

LEMMA 5.1 *Let* $0 < d < 1 < m$, *and* $\{t\} \equiv \varepsilon^{-2} + 2\Gamma t$.

$$\int_0^t |t-s|^{-d}\{s\}^{-m}\, ds \le C\varepsilon^{2m-2}n^{2m+2d-4}\{t\}^{-d} \ , \tag{5.1}$$

$$\int_0^t |t-s|^{-d}\{s\}^{-1}\, ds \le Cn^{2d-2}\{t\}^{-d}\log(1+\varepsilon^2 n^4 t) \, . \tag{5.2}$$

If, instead, $d \ge m > 1$,

$$\int_0^t \langle t-s\rangle^{-d}\{s\}^{-m}\, ds \le C\{t\}^{-m} \ . \tag{5.3}$$

PROOF: Denote the first integral by (I). If $t \le \varepsilon^{-2}n^{-4}$, then $\{t\} \sim \varepsilon^{-2}n^{-2}$ and

$$\begin{aligned}(\mathrm{I}) \sim \int_0^t |t-s|^{-d}(\varepsilon n)^{2m}\, ds &\lesssim (\varepsilon n)^{2m}(\varepsilon^{-2}n^{-4})^{1-d} \\ &= \varepsilon^{2m-2}n^{2m+2d-4}(\varepsilon n)^{2d} \sim \varepsilon^{2m-2}n^{2m+2d-4}\{t\}^{-d} \ .\end{aligned}$$

If $t \ge \varepsilon^{-2}n^{-4}$, then $\{t\} \sim 2\Gamma t$ and

$$\begin{aligned}(\mathrm{I}) &\le \int_0^{t/2} Ct^{-d}\{s\}^{-m}\, ds + \int_{t/2}^t C|t-s|^{-d}C\{t\}^{-m}\ ds \\ &\le \frac{Ct^{-d}(\varepsilon^2 n^4)^{m-1}}{\Gamma} + Ct^{1-d}\{t\}^{-m} \le C\varepsilon^{2m-2}n^{2m+2d-4}\{t\}^{-d} \ .\end{aligned}$$

Estimate (5.2) is an obvious modification of (5.1). For the last estimate (5.3), denote the second integral by (II). If $t \le \varepsilon^{-2}n^{-4}$, then $\{t\} \sim \varepsilon^{-2}n^{-2}$ and

$$(\mathrm{II}) \le C\int_0^t \langle t-s\rangle^{-d}(\varepsilon n)^{2m}\, ds \le (\varepsilon n)^{2m}\cdot C \sim C\{t\}^{-m} \ .$$

If $t \geq \varepsilon^{-2} n^{-4}$, then $\{t\} \sim \Gamma \langle t \rangle$ and

$$(\mathrm{II}) \sim \int_0^t \langle t-s \rangle^{-d}\, \Gamma^{-m} \langle s \rangle^{-m}\, ds \leq C \Gamma^{-m} \langle t \rangle^{-m} \sim C\, \{t\}^{-m} \ .$$

We conclude the lemma. □

5.1 Estimates in L^4

LEMMA 5.2 *Suppose that $\tilde{\eta}$ is given by equation* (3.10) *and recall $\eta = U^{-1} e^{-i\theta} \tilde{\eta}$. Assuming the estimate* (4.10) *on $a(t)$ and $M(T) \leq 2$, we have*

$$\|\eta(t)\|_{L^4} \leq \tfrac{1}{8}\, \{t\}^{-3/4+\sigma} \ ,$$

where $\sigma = 1/100$ if ε is sufficiently small.

PROOF: From the defining equation (3.10) of $\tilde{\eta}$, we have

$$\|\widetilde{\eta}(t)\|_{L^4} \leq C\, \|\widetilde{\eta}_0\|_Y\, \langle t \rangle^{-3/4} + \int_0^t C |t-s|^{-3/4} \|F_\eta(s)\|_{4/3}\, ds \ ,$$

where $F_\eta = e^{i\theta} U i^{-1} [F + \dot{\theta}(aR + \zeta)] - e^{i\theta} [U, i] \dot{\theta} \eta$. By Lemma 2.9,

$$\left\| e^{i\theta} [U, i] \dot{\theta} \eta(s) \right\|_{L^{4/3}} \leq C |\dot{\theta}|\, \|\eta(s)\|_4 \ .$$

Therefore we have

$$\|F_\eta(s)\|_{4/3} \leq C |\dot{\theta}| \left(|a| + \|\zeta(s)\|_{4/3} + \|\eta(s)\|_4 \right) + \|F(s)\|_{L^{4/3}} \ .$$

From the Hölder inequality and the definition of $\|F(s)\|_{L^{4/3}}$, we can bound $\|F(s)\|_{L^{4/3}}$ by

$$\|F(s)\|_{L^{4/3}} \leq C \left(\|h\|_{L^4}^2 + |a|\, \|h\|_{L^4} + |a|^2 + \|h^3\|_{L^{4/3}} + |a|^3 \right) \ .$$

Since $\|h^3\|_{L^{4/3}} = \|h\|_{L^4}^3$ and $\|h(s)\|_{L^4} \leq \|\zeta(s)\|_{L^4} + \|\eta(s)\|_{L^4}$, we have from the assumption $M_T(t) \leq 2$ that

$$\|h(s)\|_{L^4} \leq C\{s\}^{-1/2} + C\{s\}^{-3/4} \log\{s\} \leq C\{s\}^{-1/2} \, .$$

Therefore we have $\|F(s)\|_{L^{4/3}} \leq C n \{s\}^{-1}$. Similarly, we have $|\dot{\theta}(s)| \leq C\{s\}^{-1}$.

Since $C\, \|\tilde{\eta}_0\|_Y\, \langle t \rangle^{-3/4+\sigma} \leq C\, \|\tilde{\eta}_0\|_Y\, \varepsilon^{-3/2+2\sigma}\, \{t\}^{-3/4+\sigma} \leq \frac{1}{16}\, \{t\}^{-3/4+\sigma}$, we conclude

$$\begin{aligned} \|\eta(t)\|_{L^4} &\leq \tfrac{1}{16}\, \{t\}^{-3/4+\sigma} + \int_0^t |t-s|^{-3/4} C n \{s\}^{-1}\, ds \\ &\leq \tfrac{1}{16}\, \{t\}^{-3/4+\sigma} + C n n^{-1/2}\, \{t\}^{-3/4} \log[(\varepsilon n)^2\, \{t\}] \leq \tfrac{1}{8}\, \{t\}^{-3/4+\sigma} \ . \end{aligned}$$

Here we have used Lemma 5.1 for the integration. □

5.2 Local Decay

Recall that $\eta^{(3)} = U^{-1}e^{-i\theta}\tilde{\eta}^{(3)}$ and $\tilde{\eta}^{(3)}$ satisfies the equation (3.19). We want to show that $\tilde{\eta}^{(3)}$ is smaller than $\tilde{\eta}^{(2)}$ locally. Recall that the local L^2 norm is defined in (1.7) with r_0 the exponent in Lemma 2.7.

LEMMA 5.3 *Assuming the estimate* (4.10) *on* $a(t)$ *and* $M(T) \le 2$, *we have*

$$\|\eta^{(3)}_{1-2}\|_{L^2_{\text{loc}}} \le C\varepsilon^2 n^2 \langle t\rangle^{-9/8}, \quad \|\eta^{(3)}_{3-5}\|_{L^2_{\text{loc}}} \le C(\varepsilon n)^{3/4}\{t\}^{-9/8}. \tag{5.4}$$

Hence we have $\|\eta\|_{L^2_{\text{loc}}} \le C\{t\}^{-1}$, *and for a local function* ϕ *we have*

$$\begin{aligned} \left|(\phi, |\eta|^2 + |\eta|^3)\right| &\le C\left\|\langle x\rangle^{2r_0}\phi\right\|_{L^\infty}\left(\|\eta\|^2_{L^2_{\text{loc}}} + \|\eta\|_{L^2_{\text{loc}}}\|\eta\|^2_{L^4}\right) \\ &\le C\left\|\langle x\rangle^{2r_0}\phi\right\|_{L^\infty}\{t\}^{-2}. \end{aligned} \tag{5.5}$$

PROOF: Since U is bounded, it is sufficient to prove the corresponding estimates for each $\tilde{\eta}^{(3)}_j$, $j = 1, 2, \dots, 5$, defined in the decomposition (3.19) for $\tilde{\eta}^{(3)}$. We first estimate $\tilde{\eta}^{(3)}_1$,

$$\|\tilde{\eta}^{(3)}_1\|_{L^2_{\text{loc}}} \le C\|\eta_0\|_Y\langle t\rangle^{-9/8} \le C\varepsilon^2 n^2\langle t\rangle^{-9/8}.$$

For $\tilde{\eta}^{(3)}_2$ and $\tilde{\eta}^{(3)}_3$, the two terms involving η_α ($\eta_\alpha \notin L^2$) from the definition and the linear estimates in Lemma 2.7, we have

$$\begin{aligned} \|\tilde{\eta}^{(3)}_2\|_{L^2_{\text{loc}}} &\le (\varepsilon n)^2 n\langle t\rangle^{-9/8}, \\ \|\tilde{\eta}^{(3)}_3\|_{L^2_{\text{loc}}} &\le C\int_0^t \langle t-s\rangle^{-9/8} n^2\{s\}^{-3/2}\,ds \le n^2(\varepsilon n)^{3/4}\{t\}^{-9/8}. \end{aligned}$$

Here we have used $\widetilde{\eta}_\alpha = O(n)$, $|\dot{p}| \le Cn\{t\}^{-1}$, and $|\dot{\theta}| \le C\{t\}^{-1}$ in (4.14) and (5.3). Note that, when t is of order 1, $\tilde{\eta}^{(3)}_1$ and $\tilde{\eta}^{(3)}_2$ are of order η_0 and $\varepsilon^2 n^3$, which are larger than $C(\varepsilon n)^{3/4}\{1\}^{-9/8} = O((\varepsilon n)^3)$.

We treat $\tilde{\eta}^{(3)}_4$ and $\tilde{\eta}^{(3)}_5$ together. For $\tilde{\eta}^{(3)}_4 = \int_0^t e^{-iA(t-s)}\mathbf{P}^A_c\{F_{\eta,3} - e^{i\theta}Ui^{-1}\eta^2\bar{\eta}\}ds$, we can rewrite the integrand using the definition of $F_{\eta,3}$ from (3.13),

$$F_{\eta,3} - e^{i\theta}Ui^{-1}\eta^2\bar{\eta} = e^{i\theta}Ui^{-1}\left[(F - F^{(2)} - \eta^2\bar{\eta}) + \dot{\theta}(aR + \zeta)\right] - e^{i\theta}[U, i]\dot{\theta}\eta,$$

which consists of only local terms. Since $M_T(t) \le 2$, we have

$$\left\|F_{\eta,3} - e^{i\theta}Ui^{-1}\eta^2\bar{\eta}\right\|_{L^{4/3}\cap L^{8/7}} \le C\{s\}^{-3/2}.$$

Here we have used Lemma 2.9 to estimate $e^{i\theta}[U, i]\dot{\theta}\eta$. Note that, in terms like $Q\zeta\eta$, η is estimated by $\|\eta\|_{L^2_{\text{loc}}} \le Cn^{-\sigma}\{s\}^{-1}$.

From the Hölder inequality, the L^2 bound (4.2), and the global estimate of η in Lemma 5.2, we have

$$\begin{aligned} \|\eta^3(s)\|_{L^{8/7}} &\le \|\eta\|_2^{1/2}\|\eta\|_4^{5/2} \le Cn^{1/2}\left(\{s\}^{-3/4+\sigma}\right)^{5/2} \le C\{s\}^{-7/4}, \\ \|\eta^3(s)\|_{L^{4/3}} &\le \|\eta\|_4^3 \le C\left(\{s\}^{-3/4+\sigma}\right)^3 \le C\{s\}^{-7/4}. \end{aligned}$$

Thus we have $\tilde\eta_4^{(3)} + \tilde\eta_5^{(3)} = \int_0^t e^{-iA(t-s)}\mathbf{P}_c^A F_{\eta,3}(s)ds$, with

$$\left\| F_{\eta,3}(s) \right\|_{L^{4/3}\cap L^{8/7}} \le C\{s\}^{-3/2}\,.$$

We can bound the L^2_{loc} norm by either the L^4 or L^8 norm. If $t > 1$, we bound the L^2_{loc} norm by L^8 in $0 \le s \le t-1$, and by L^4 in $t-1 \le s \le t$. By Lemma 2.7,

$$\begin{aligned}\left\| \tilde\eta_4^{(3)} + \tilde\eta_5^{(3)} \right\|_{L^2_{\text{loc}}} &\le C\int_0^{t-1} \frac{1}{|t-s|^{9/8}} \|F_{\eta,3}(s)\|_{L^{8/7}}\,ds \\ &\quad + C\int_{t-1}^{t} \frac{1}{|t-s|^{3/4}} \|F_{\eta,3}(s)\|_{L^{4/3}}\,ds \\ &\le C\int_0^t C\,\langle t-s\rangle^{-9/8}\,\{s\}^{-3/2}\,ds + C\,\{t\}^{-3/2} \\ &\le C(\varepsilon n)^{3/4}\,\{t\}^{-9/8} + C(\varepsilon n)^{3/4}\,\{t\}^{-9/8}\ .\end{aligned}$$

In the last line we have used (5.3). If $t \le 1$, we can use the second integral to estimate; hence we get the desired estimate for $\|\tilde\eta_4^{(3)} + \tilde\eta_5^{(3)}\|_{L^2_{\text{loc}}}$. □

6 Excited-State Equation and Normal Form

6.1 Excited-State Equation

Let us return to the equation for $\dot p$ from (2.30),

$$\dot p = -ie^{i\kappa t}\left\{(u_+, F) + (u_-, \overline{F}) + \left[(u_+, h) + (u_-, \overline{h}) + (u, R)a\right]\dot\theta\right\}. \tag{6.1}$$

In view of the decompositions (3.22) of F and (3.23) of $\dot\theta$, we can expand the right-hand side of equation (6.1) into terms in order of z:

$$\dot p = e^{i\kappa t}\left\{c_\alpha z^\alpha + d_\beta z^\beta + d_1 bz + d_2 b\bar z + P^{(4)}\right\}. \tag{6.2}$$

Here α and β are summing over all $|\alpha| = 2$ and $|\beta| = 3$. The coefficients are computed and their properties summarized in the following lemma:

LEMMA 6.1 *We can rewrite equation* (6.1) *of* $\dot p$ *into the form* (6.2). *The coefficients* d_1 *and* d_2 *and all* c_α *are purely imaginary.* $P^{(4)}$ *denotes higher-order terms and will be defined in* (6.6). *Moreover,* $\operatorname{Re} d_{21} = -\Gamma$, *with*

$$\Gamma = \left(2\Phi_{20}, \operatorname{Im}\frac{1}{A - 0i - 2\kappa}\mathbf{P}_c^A\Phi_{20}\right) \ge \lambda^2 n^2 \gamma_0 > 0\,, \tag{6.3}$$

where γ_0 *is the constant in assumption* A1, *and* $\Phi_{20} = \Pi\lambda Q\phi_1^2 + O(n^3)$ *is defined in* (3.15). *We also have* $c_\alpha = O(n)$ *and* $d_\beta, d_1, d_2 = O(1)$.

PROOF: There are two parts in equation (6.1): The part with F and the part with $\dot\theta$. We first consider the second part. Recall from (3.23),

$$\dot\theta = -a - (c_1 R, \operatorname{Re} F^{(2)}) + O(z^3)\,,$$

where $O(z^3)$ denotes terms of the order $z^3+az+z\eta$ plus higher-order terms. Hence

$$\begin{aligned}
&-ie^{i\kappa t}\{(u_+,h)+(u_-,\bar h)+(u,R)a\}\dot\theta\\
&\quad=-ie^{i\kappa t}\big\{(u_+,\zeta)+(u_-,\bar\zeta)+O(z^2)\big\}\cdot\big\{-a-(c_1R,\operatorname{Re}F^{(2)})+O(z^3)\big\}\\
&\quad=-ie^{i\kappa t}\big\{(u_+,\zeta)+(u_-,\bar\zeta)\big\}\\
&\qquad\cdot\big\{-a_{20}(z^2+\bar z^2)-b-(c_1R,\operatorname{Re}F^{(2)})\big\}+O(z^4)\,.
\end{aligned}$$

The leading terms are of the form z^β, bz, and $b\bar z$. These terms are of order $O(z^3)$. Since the coefficients in each bracket are real, the coefficients of the leading terms, that is, their contributions to d_β, d_1, and d_2, are all purely imaginary due to the i factor in front.

Now we look at the first part of the equation (6.1). The contribution to $c_\alpha z^\alpha$ is from $F^{(2)}=z^\alpha\phi_\alpha$,

$$-i\,\{(u_+,z^\alpha\phi_\alpha)+(u_-,\bar z^\alpha\phi_\alpha)\}\,.$$

Clearly all coefficients of z^α are purely imaginary. Since there is no contribution from the second part to $c_\alpha z^\alpha$, we know that the c_α are purely imaginary. We also have $c_\alpha=O(\phi_\alpha)=O(n)$.

The contribution from the first part to $d_1bz+d_2b\bar z$ is from $\widetilde F^{(3)}$:

$$-i\,\big\{(u_+,2\lambda QRb(2\zeta+\bar\zeta)+(u_-,2\lambda QRb(\zeta+2\bar\zeta)\big\}$$

with $\zeta=zu_++\bar zu_-$. Hence all coefficients of bz and $b\bar z$ are purely imaginary. Together with the analysis of the second part of (6.1), we know d_1 and d_2 are purely imaginary.

The contribution from the first part to $d_\beta z^\beta$ is from $F^{(3)}$:

$$-i\,\big\{(u_+,F^{(3)})+(u_-,\bar F^{(3)})\big\}\,.$$

Among all d_β, we are only interested in the real part of d_{21}, which does not come from the second part as we already showed. A coefficient in $F^{(3)}$ has to have an imaginary part in order to have a contribution to $\operatorname{Re}d_{21}$. The only such source is $z^2\tilde\eta_{20}$ in $\eta^{(2)}$, which lies in the first group of terms in $F^{(3)}$. Let us call these terms $F_1^{(3)}$:

$$F_1^{(3)}=2\lambda Q\big[(\zeta+\bar\zeta)\eta^{(2)}+\zeta\bar\eta^{(2)}\big]\,.$$

Recall that $\eta^{(2)}=U^{-1}z^\alpha\tilde\eta_\alpha$. Denote $\eta'=z^\alpha\tilde\eta_\alpha$ and recall (2.24) that $U^{-1}=(U_+)^*-(U_-)^*\mathbf{C}$. Hence $\eta^{(2)}=[(U_+)^*-(U_-)^*\mathbf{C}]\eta'=(U_+)^*\eta'-(U_-)^*\bar\eta'$, and

$$\begin{aligned}
&\big(u_+,F_1^{(3)}\big)+\big(u_-,\bar F_1^{(3)}\big)\\
&\quad=\big(2\lambda Q,(u_+\zeta+u\bar\zeta)\eta^{(2)}\big)+\big(2\lambda Q,(u\zeta+u_-\bar\zeta)\bar\eta^{(2)}\big)\\
&\quad=\big(2\lambda Q,(u_+\zeta+u\bar\zeta)[(U_+)^*\eta'-(U_-)^*\bar\eta']\big)\\
&\qquad+\big(2\lambda Q,(u\zeta+u_-\bar\zeta)[(U_+)^*\bar\eta'-(U_-)^*\eta']\big)
\end{aligned}$$

$$= \int \left[U_+\big(2\lambda Q(u_+\zeta + u\bar\zeta)\big) - U_-\big(2\lambda Q(u\zeta + u_-\bar\zeta)\big)\right] \eta'\, dx$$
$$+ \int \left[-U_-\big(2\lambda Q(u_+\zeta + u\bar\zeta)\big) + U_+\big(2\lambda Q(u\zeta + u_-\bar\zeta)\big)\right] \bar\eta'\, dx\,.$$

We want to collect terms of the form $Cz^2\bar z$ with $\operatorname{Re} C \neq 0$. However, the only term from $\bar\eta'$ with a resonance coefficient is $\overline{z^2\tilde\eta_{20}}$, which is of the form $\bar z^2$, with two bars. Hence the last integral does not contain $z^2\bar z$ and is irrelevant. From the first integral, we want to choose z^2 from η', i.e., $z^2\tilde\eta_{20}$, and choose $\bar z$ from ζ or $\bar\zeta$. The terms with $\bar z$ are

$$U_+2\lambda Q(u_+\bar z u_- + u\bar z u_+) - U_-2\lambda Q(u\bar z u_- + u_-\bar z u_+) = 2(U_+\phi_{20} - U_-\phi_{02})\bar z$$
$$= 2\Phi_{20}\bar z\,.$$

Here we have used (3.2) and (3.15); hence

$$\operatorname{Re} d_{21} = \operatorname{Re}(-i)\,(2\Phi_{20}, \eta_{20}) = \operatorname{Im}\left(2\Phi_{20}, \frac{-1}{A - 0i - 2\kappa}\mathbf{P}_c^A\Phi_{20}\right) = -\Gamma\,. \tag{6.4}$$

To show $\Gamma \geq \lambda^2 n^2\gamma_0$, it suffices to show that

$$\Phi_{20} = \Pi\lambda Q\phi_1^2 + O(n^3)\,. \tag{6.5}$$

Note that $\Phi_{20} = U_+\phi_{20} - U_-\phi_{02} = U_+\lambda Q\phi_1^2 + O(n^3)$ by (3.15) and (3.2). Now $U_+ = \frac{1}{2}(\Pi A^{1/2}H^{-1/2}P_1 + \Pi A^{-1/2}H^{1/2}\Pi)$ is defined in (2.23). Since $A^{1/2}H^{-1/2} = 1 + O(n^2)$ and $A^{-1/2}H^{1/2} = 1 + O(n^2)$ by (9.7) and (9.8), we have $U_+ = \frac{1}{2}(\Pi P_1 + \Pi) + O(n^2)$. By (2.18), $P_1 = \Pi + O(n^2)$. Therefore we have (6.5). It will follow from the induction assumption that $Q = m\phi_0 + O(n^3)$ with $m \geq \frac{4}{5}n$. Then we have, by Lemma 2.8,

$$\Gamma = 2\lambda^2 m^2 \operatorname{Im}\left(\Pi\phi_0\phi_1^2, \frac{1}{A - 0i - 2\kappa}\mathbf{P}_c^A\Pi\phi_0\phi_1^2\right) + O(n^4)$$
$$\geq 2\lambda^2\big(\tfrac{4}{5}n\big)^2\gamma_0 + O(n^4)\,.$$

Collecting terms of order $O(z^4 + a^2 + \eta^2)$, we have

$$P^{(4)} = -i\big\{(u_+, F^{(4)}) + (u_-, \overline{F}^{(4)}) + [(u_+, \eta) + (u_-, \overline{\eta}) + (u, R)a]\dot\theta\big\}$$
$$- i\left[(u_+, \zeta) + (u_-, \overline{\zeta})\right](F_{\theta,3} + F_{\theta,4})\,. \tag{6.6}$$

□

6.2 Normal Form

LEMMA 6.2 *We can rewrite equation* (6.2) *of $\dot p$ into a* normal form:

$$\dot q = \delta_{21}|q|^2 q + d_1 b q + g\,, \tag{6.7}$$

where q is a perturbation of p given in the proof. The coefficient δ_{21} satisfies

$$\operatorname{Re}\delta_{21} = \operatorname{Re} d_{21} = -\Gamma\,. \tag{6.8}$$

If we assume estimate (4.10) *on* $a(t)$ *and* $M(T) \leq 2$*, then the error term* $g(t)$*, to be given by* (6.22)*, satisfies the bound*

$$|g(t)| \leq C\varepsilon^{3/4-\sigma} n^{7/4} \{t\}^{-13/8} \ . \tag{6.9}$$

Furthermore, there is a small positive constant σ *such that* $|q(t)|$ *and* $|z(t)|$ *are bounded by*

$$|q(t)| \leq (1+\sigma)\{t\}^{-1/2} \ , \quad |z(t)| \leq (1+2\sigma)\{t\}^{-1/2} \ . \tag{6.10}$$

If we have $|z(0)| = \varepsilon n + o(\varepsilon n)$*, we also have lower bound*

$$|q(t)| \geq (1-\sigma)\{t\}^{-1/2} \ , \quad |z(t)| \geq (1-2\sigma)\{t\}^{-1/2} \ .$$

PROOF: From (6.2) we have

$$\dot{p} = \mu\left[c_\alpha z^\alpha + d_\beta z^\beta + d_1 b z + d_2 b\bar{z} + P^{(4)}\right] , \quad \mu = e^{i\kappa t} ,$$

and we want to obtain the normal form (6.7). We will repeatedly use the following formula:

$$\mu^m p^\alpha = \frac{d}{dt}\left(\frac{\mu^m p^\alpha}{i\kappa m}\right) - \frac{\mu^m p^\alpha}{i\kappa m} f_\alpha(z) \tag{6.11}$$

where, if $\alpha = (\alpha_0 \alpha_1)$, $|\alpha| = \alpha_0 + \alpha_1 = 2, 3, 4, \ldots$,

$$\begin{aligned} f_\alpha(z) &= (\alpha_0 + \alpha_1 \mathbf{C})(p^{-1}\dot{p}) \\ &= (\alpha_0 + \alpha_1 \mathbf{C}) z^{-1}\left[c_\alpha z^\alpha + d_\beta z^\beta + d_1 b z + d_2 b\bar{z} + P^{(4)}\right] , \end{aligned} \tag{6.12}$$

and $\mathbf{C}$ denotes the conjugation operator. The formula is equivalent to integration by parts.

We first remove $c_\alpha z^\alpha$. Let

$$p_1 = p - \frac{c_\alpha}{i\kappa(1+[\alpha])}\mu z^\alpha \ .$$

Since $[\alpha]$ is even, $1 + [\alpha] \neq 0$. By (6.11)

$$\dot{p}_1 = \dot{p} - c_\alpha \mu z^\alpha - \frac{c_\alpha}{i\kappa(1+[\alpha])}\mu z^\alpha f_\alpha(z) \ .$$

Decomposing $f_\alpha(z)$ into two parts, we can write

$$\begin{aligned} d_\beta^+ z^\beta &= -\frac{c_\alpha}{i\kappa(1+[\alpha])}\mu z^\alpha(\alpha_0 + \alpha_1 \mathbf{C}) z^{-1} c_{\tilde{\alpha}} z^{\tilde{\alpha}} , \\ g_1 &= -\frac{c_\alpha}{i\kappa(1+[\alpha])}\mu z^\alpha(\alpha_0 + \alpha_1 \mathbf{C}) z^{-1}\left[d_\beta z^\beta + d_1 b z + d_2 b\bar{z} + P^{(4)}\right] , \end{aligned}$$

and we get

$$\dot{p}_1 = \delta_\beta \mu z^\beta + d_1 \mu b z + d_2 \mu b\bar{z} + \mu P^{(4)} + g_1$$

with $\delta_\beta = d_\beta + d_\beta^+$. Since c_α are purely imaginary, hence so are d_β^+, and we have the relation

$$\operatorname{Re}\delta_\beta = \operatorname{Re} d_\beta \ . \tag{6.13}$$

Next we remove $d_2\mu b\bar z$. Let

$$p_2 = p_1 - \frac{\mu d_2 b\bar z}{2i\kappa}\,.$$

We have

$$\dot p_2 = \dot p_1 - \mu d_2 b\bar z - \frac{\mu^2 d_2}{2i\kappa}(\dot b\bar p + b\dot{\bar p}) = \mu\delta_\beta z^\beta + d_1\mu bz + (\mu P^{(4)} + g_1 + g_2)\,,$$

where

$$g_2 = -\frac{\mu^2 d_2}{2i\kappa}(\dot b\bar p + b\dot{\bar p})\,.$$

Now we deal with $\delta_\beta z^\beta$ terms. Let

$$p_3 = p_2 - \sum_{\beta\neq(21)} \frac{\delta_\beta \mu z^\beta}{i\kappa(1+[\beta])}\,.$$

Note $1+[\beta]\neq 0$ for $\beta\neq(21)$. We have

$$\dot p_3 = \dot p_2 - \mu\delta_\beta z^\beta + g_3 = \delta_{21}\mu z^2\bar z + \mu d_1 bz + (\mu P^{(4)} + g_1 + g_2 + g_3)$$

with

$$g_3 = -\sum_{\beta\neq(21)} \frac{\delta_\beta \mu^{1+[\beta]}}{i\kappa(1+[\beta])}\frac{d}{dt}(p^\beta)\,.$$

Notice that the equation for p_3 has the desired form (6.7), and that the error term $(\mu P^{(4)} + g_1 + g_2 + g_3)$ has the desired decay in t when t is large. However, certain terms in the error term are not small compared to Γz^3 when t is of order 1, and hence need to be treated. (Recall $\Gamma = -\operatorname{Re}\delta_{21}$.) We will also integrate these terms and include them in the perturbation of p.

One such term is of the form $Q\zeta\eta^{(3)}_{1,2}$ from $P^{(4)}$, where $\eta^{(3)}_{1,2} = \eta^{(3)}_1 + \eta^{(3)}_2$. Since the size of $\eta^{(3)}_1$ and $\eta^{(3)}_2$ are comparable to $\eta^{(2)}$ when the time t is of order 1, and Γz^3 comes from terms of the form $Q\zeta\eta^{(2)}$, we need to treat terms of the form $Q\zeta\eta^{(3)}_{1,2}$. We will integrate these terms and include them in the perturbation of p. Recall $\eta^{(3)}_j = U^{-1}e^{-i\theta}\widetilde\eta^{(3)}_j$, $j=1,2$, and

$$\widetilde\eta^{(3)}_1 + \widetilde\eta^{(3)}_2 = e^{-iAt}\left[\widetilde\eta_0 - (e^{i\theta}z^\alpha)(0)\widetilde\eta_\alpha\right]$$

where $\widetilde\eta_0 = e^{i\theta(0)}U\eta_0$, and $\widetilde\eta_\alpha$, $|\alpha| = 2$, are defined as in (3.17). Denote $\chi = \eta^{(3)}_{1,2} = \eta^{(3)}_1 + \eta^{(3)}_2$ and $\widetilde\chi_0 = e^{i\theta(0)}[U\eta_0 - z^\alpha(0)\widetilde\eta_\alpha] \in \mathbf{H}_c(A)$ for the computation below. We have

$$\chi(t) = \left(\eta^{(3)}_1 + \eta^{(3)}_2\right)(t) = U^{-1}e^{-i\theta(t)}e^{-iAt}\widetilde\chi_0\,. \tag{6.14}$$

Recall (6.1). The only source of $Q\zeta\chi$ is from $F^{(4)}$, where we have a term $2\lambda Q((\zeta+\bar\zeta)\chi + \zeta\overline{\chi})$. This is the same source for the resonance term $Q\zeta\eta^{(2)}$. Hence terms of the form $Q\zeta\chi$ in $\mu P^{(4)}$ are exactly

$$P_{z\eta^{(3)}_{1,2}} \equiv -i\mu\big(u_+, 2\lambda Q((\zeta+\bar\zeta)\chi + \zeta\bar\chi)\big) - i\mu\mathbf{C}\big(u_-, 2\lambda Q((\zeta+\bar\zeta)\chi + \zeta\bar\chi)\big)\,.$$

Clearly these terms can be summed in the form

$$P_{z\eta_{1,2}^{(3)}} = \mu(z\phi + \bar{z}\phi, \chi) + \mathbf{C}\mu^{-1}(z\phi + \bar{z}\phi, \chi)\,. \tag{6.15}$$

Here each $\phi = O(n)$ stands for a different local smooth function. Recall $U^{-1} = U_+^+ - U_-^*\mathbf{C}$ from (2.24); hence

$$U^{-1}(zf + \bar{z}g) = z(U_+^* f - U_-^*\bar{g}) + \bar{z}(U_+^* g - U_-^* \bar{f})$$

(cf. (7.14)). Together with (6.14) we have

$$\begin{aligned} P_{z\eta_{1,2}^{(3)}} &= \int \left(p\phi_1 + \mu^2 \bar{p}\phi_2\right) e^{-i\theta} e^{-iAt} \widetilde{\chi}_0\, dx \\ &\quad + \mathbf{C} \int \left(\mu^{-2} p\phi_3 + \bar{p}\phi_4\right) e^{-i\theta} e^{-iAt} \widetilde{\chi}_0\, dx \end{aligned} \tag{6.16}$$

for some local functions $\phi_1, \phi_2, \phi_3, \phi_4 = O(n)$. Recall $\mu(s) = e^{i\kappa s}$. Define

$$\begin{aligned} f_1(t) &= \int_\infty^t \int \phi_1 e^{-iAs} \widetilde{\chi}_0\, dx\, ds\,, \\ f_2(t) &= \int_\infty^t \int \phi_2 e^{i2\kappa s} e^{-i(A-0i)s} \widetilde{\chi}_0\, dx\, ds\,, \\ f_3(t) &= \int_\infty^t \int \phi_3 e^{-i2\kappa s} e^{-iAs} \widetilde{\chi}_0\, dx\, ds\,, \\ f_4(t) &= \int_\infty^t \int \phi_4 e^{-iAs} \widetilde{\chi}_0\, dx\, ds\,. \end{aligned}$$

These functions in t depend on the initial data only. Recall $\widetilde{\chi}_0$ contains $\widetilde{\eta}_{20} = -(A - 0i - 2\kappa)^{-1}\mathbf{P}_c^A \Phi_{20}$, which is not a local term. It is clear by Lemma 2.7 that

$$|f_j(t)| \le \int_\infty^t Cn\langle s\rangle^{-9/8}\, \|\widetilde{\chi}_0\|\, ds \le C\varepsilon^{3/2} n^2 \langle t\rangle^{-1/8}\,.$$

However, we claim we have

$$|f_j(t)| \le C\varepsilon^{3/2} n^2 \langle t\rangle^{-9/8}\,, \quad j = 1, 2, 3, 4\,. \tag{6.17}$$

In fact, (6.17) is clear for $f_j(t)$, $j \ne 2$, since we can integrate them explicitly. For example,

$$f_1(t) = \int (-iA)^{-1}(\mathbf{P}_c^A \phi_1) e^{-iAt} \widetilde{\chi}_0\, dx = O(\langle t\rangle^{-9/8})\,,$$

by Lemma 2.7. Here we have used that A^{-1} is bounded in $L_r^2 \cap \mathbf{H}_c(A)$. The problem for the term $f_2(t)$ is that the factor $e^{i2\kappa s} e^{-iAs}$ gives resonance while $\widetilde{\chi}_0$ is not a local term. However, the main term in f_2 that concerns us is

$$\int_\infty^t \int \phi_2 e^{i2\kappa s} e^{-i(A-0i)s} \widetilde{\eta}_{20}\, dx\, ds = -i \int \phi_2 (A - 2\kappa - 0i)^{-2} \mathbf{P}_c^A \Phi_{20}\, dx\,,$$

which is still of order $O(\langle t\rangle^{-9/8})$ by Lemma 2.7. Thus we also have (6.17) for f_2.

We can now integrate $P_{z\eta^{(3)}_{1,2}}$:

$$P_{z\eta^{(3)}_{1,2}} = \frac{d}{dt}(p_4) + g_4 \,, \tag{6.18}$$

where p_4 is of similar form to $P_{z\eta^{(3)}_{1,2}}$,

$$\begin{aligned} p_4 &= pe^{-i\theta} f_1 + \bar{p}e^{-i\theta} f_2 + \mathbf{C}\left(pe^{-i\theta} f_3 + \bar{p}e^{-i\theta} f_4\right) , \\ g_4 &= \frac{d}{dt}(pe^{-i\theta})\, f_1 + \frac{d}{dt}(\bar{p}e^{-i\theta})\, f_2 + \mathbf{C}\left(\frac{d}{dt}(pe^{-i\theta})\, f_3 + \frac{d}{dt}(\bar{p}e^{-i\theta})\, f_4\right). \end{aligned} \tag{6.19}$$

By (6.17),

$$|p_4| \le C\varepsilon^{3/2} n^2 \langle t\rangle^{-9/8} |z| \,, \quad |g_4| \le C\varepsilon^{3/2} n^2 \langle t\rangle^{-9/8} (|z\dot{\theta}| + |\dot{p}|) \,. \tag{6.20}$$

We have integrated terms of the order $Qz\eta^{(3)}_{1-2}$.

Other terms of concern for $t = O(1)$ are related to aR and η from $P^{(4)}$. (We are not worried about terms from g_1, g_2, and g_3, since the coefficients of these remainder terms of integration by parts are multiplied by $O(n)$; see (4.14).) These terms are of the form

$$P_{a^2,a\eta} = \mu \left\{(\phi, z^2 aR) + (\phi, Q(aR)^2) + (\phi, QaR\eta) + (\phi, zaR\eta)\right\} \,,$$

where ϕ denotes some local functions. Since $a = O(\{t\}^{-1})$, $R = O(n^{-1})$, and $\eta = O(\{t\}^{-1})$ locally, the largest terms here are of order $n^{-1}\{t\}^{-2}$, which is larger than Γz^3 when t is small. However, terms linear in η can be integrated as $A^{(z\eta)}$ in the next section. The first two terms, $\mu z^2 aR$ and $\mu Q(aR)^2$, are multiplied by $\mu = e^{i\kappa t}$; hence they are oscillatory and can be integrated as $c_\alpha z^\alpha$ terms and as $A^{(zb)}$ in the next section. (Note $a = a_{20}(z^2 + \bar{z}^2) + b$ and all terms of the form $\mu z^\gamma b^m$ with $|\gamma| + 2m = 4$ are oscillatory; i.e., they have a nonzero phase.) Since the computation is the same as for $A^{(z\eta)}$, $c_\alpha z^\alpha$, and $A^{(zb)}$, we shall skip the computation and just give the result:

$$P_{a^2,a\eta} = \frac{d}{dt}(p_5) + g_5 \,,$$

where p_5 is of similar form to $P_{a^2,a\eta}$ and

$$|p_5| \le Cn^{-1}\{t\}^{-2} \,, \quad |g_5| \le C\{t\}^{-5/2} \,. \tag{6.21}$$

Now we let $q = p_3 - p_4 - p_5$. We have

$$\begin{aligned} \dot{q} &= \delta_{21}\mu z^2\bar{z} + \mu d_1 bz + \left(\mu P^{(4)} + g_1 + g_2 + g_3\right) - P_{z\eta^{(3)}_{1,2}} + g_4 - P_{a^2,a\eta} + g_5 \\ &= \delta_{21}|p|^2 p + d_1 bp + \left[\left(\mu P^{(4)} - P_{z\eta^{(3)}_{1,2}} - P_{a^2,a\eta}\right) + g_1 + g_2 + g_3 + g_4 + g_5\right] \\ &= \delta_{21}|q|^2 q + d_1 bq + g \,, \end{aligned}$$

where

$$\begin{aligned} g = {} & \left[\left(\mu P^{(4)} - P_{z\eta^{(3)}_{1,2}} - P_{a^2,a\eta}\right) + g_1 + \cdots + g_5\right] \\ & + \delta_{21}(|p|^2 p - |q|^2 q) + d_1 b(p - q) \,. \end{aligned} \tag{6.22}$$

Hence we have arrived at the normal form. (Note that, although we kept $\dot{p}$ and $\dot{b}$ in the definition of g_2, it should be replaced by their corresponding equations (6.2) and (7.2)). Also, note we have

$$q = p - \frac{c_\alpha}{i\kappa(1+[\alpha])}\mu z^\alpha - \frac{\mu d_2 b\bar{z}}{2i\kappa} - \sum_{\beta \neq (21)} \frac{\delta_\beta \mu z^\beta}{i\kappa(1+[\beta])} - p_4 \,. \tag{6.23}$$

Finally, from the explicit form of $g(t)$, it is bounded by terms of the form

$$nz^4 + nb^2 + (n\phi, \eta^2) + nz\eta^{(3)}_{3-5} \,.$$

Assuming the bounds on $a(t)$ and $M(t)$, we get the bound (6.9) for $g(t)$:

$$|g(t)| \le C\varepsilon^{3/4-\sigma} n^{7/4} \{t\}^{-13/8} \,.$$

To conclude this lemma, it remains to prove the estimate on q, which follows from the next lemma. □

Our main restriction on the size of η_0 comes from the term $(n\phi, (\eta_1^{(3)})^2)$ in $g(t)$. Since we want it to be smaller than $\Gamma|z|^3$, we need $\eta_0 \ll \varepsilon^{3/2} n^2$ for $t = O(1)$. We assume $\eta_0 \le \varepsilon^2 n^2$ for simplicity.

6.3 Decay Estimates

In this subsection we present a calculus lemma that deals with the decay of $q(t)$. We will write

$$q(t) = \rho(t) e^{i\omega(t)} \,, \tag{6.24}$$

where $\rho = |q|$ and ω is the phase of q. Recall

$$\{t\} = \varepsilon^{-2} n^{-2} + 2\Gamma t \,, \quad \{t\} \sim \max\{\varepsilon^{-2} n^{-2}, t\} \,.$$

Before we proceed with the proof, we give some simple facts of an ordinary differential equation.

Example. Consider real functions $r(t) > 0$ that solve

$$\dot{r}(t) = -r(t)^3 - \varepsilon(1+t)^{-3} \tag{6.25}$$

for $t > 0$, where $\varepsilon > 0$ is small. We have the following facts:

(1) All solutions $r(t)$ satisfy

$$r(t) \le (C + 2t)^{-1/2} \quad \text{with } C = r(0)^{-2} \,.$$

(2) There is a number $r_1 > 0$ such that if $r(0) > r_1$, then $r(t) \sim (C + 2t)^{-1/2}$.

(3) There is a unique global solution $r_0(t)$ of (6.25) such that

$$r_0(t) \sim t^{-2} \quad \text{as } t \to \infty \,.$$

(4) If $r(0) < r_0(0)$, then $r(t) = 0$ in finite time.

(5) If $r(0) > r_0(0)$, then

$$\int_0^\infty r(s)^2 ds = \infty \,.$$

Thus the behavior of $r(t)$ depends on which term on the right side of (6.25) dominates.

LEMMA 6.3 *Recall that n_0, ε_0, n, and ε given in Theorems* 1.1 *through* 1.4 *satisfy* $0 < n \le n_0$ *and* $0 < \varepsilon \le \varepsilon_0$. *Let* $\Gamma \approx n^2$ *and* $\sigma = 0.01$. *Suppose a positive function* $\rho(t)$ *satisfies*

$$\dot{\rho} = -\Gamma\rho^3 + \widetilde{g}(t)\,, \tag{6.26}$$

where the error term

$$|\widetilde{g}(t)| \le C\varepsilon^{3/4-\sigma} n^{7/4} \{t\}^{-13/8}\,, \quad \{t\} = \{t\}_\varepsilon = \varepsilon^{-2}n^{-2} + 2\Gamma t\,.$$

(i) *Suppose* $\rho 0 = \varepsilon n$. *Then there is a constant* $m = m(\varepsilon_0) > 1$ *such that*

$$m^{-1}\{t\}^{-1/2} \le \rho(t) \le m\,\{t\}^{-1/2}\,.$$

Moreover, $m(\varepsilon_0) \to 1^+$ *as* $\varepsilon_0 \to 0^+$.

(ii) *Suppose that* $\rho(0) \le \varepsilon n$. *Then we have*

$$\rho(t) \le m\,\{t\}^{-1/2}\,.$$

The above example shows that we cannot expect a lower bound for $\rho(t)$ under the assumption of part (ii), when $\rho(0)$ is too small compared with the error term.

PROOF: We first prove part (i). Let $\rho_+ = m\,\{t\}^{-1/2}$ and $\rho_- = m^{-1}\{t\}^{-1/2}$, with $m > 1$ to be determined. We have $\rho_+(0) > \rho(0) > \rho_-(0)$. Moreover,

$$\dot{\rho}_+ = -\Gamma m^{-2}\rho_+^3 = -\Gamma\rho_+^3 + \Gamma(1-m^{-2})\rho_+^3 \ge -\Gamma\rho_+^3 + \widetilde{g}$$

if

$$\Gamma(1-m^{-2})m^3\{t\}^{-3/2} \ge C\varepsilon^{3/4-\sigma}n^{7/4}\{t\}^{-13/8}\,.$$

Also,

$$\dot{\rho}_- = -\Gamma m^2\rho_-^3 = -\Gamma\rho_-^3 - \Gamma(m^2-1)\rho_-^3 \le -\Gamma\rho_-^3 + \widetilde{g}$$

if

$$\Gamma(m^2-1)m^{-3}\{t\}^{-3/2} \ge C\varepsilon^{3/4-\sigma}n^{7/4}\{t\}^{-13/8}\,.$$

Since $\{t\}^{-3/2} \ge (\varepsilon n)^{1/4}\{t\}^{-13/8}$, both inequalities hold if

$$\Gamma(1-m^{-2})m^3,\ \Gamma(m^2-1)m^{-3} \ge C\varepsilon^{1-\sigma}n^2\,.$$

Since $\Gamma \approx n^2$ and $\varepsilon \le \varepsilon_0$, the above is true for $m > 1$ arbitrarily close to 1 by choosing ε_0 sufficiently small. By comparison, we have $\rho_-(t) < \rho(t) < \rho_+(t)$ for all t.

The upper bound in part (ii) follows from the same proof by comparing $\rho(t)$ with $\rho_+(t)$. □

7 Change of the Mass of the Ground State

Recall that $a(t)$ satisfies the integral equation (3.3), and we have derived the main oscillatory terms of $a(t)$ in (3.8): $a(t) = a^{(2)}(t)+b(t)$ and $a^{(2)} = a_{20}(z^2+\bar{z}^2)$, with $b(t)$ given by (3.9); that is,

$$a(t) = a(T) + \int_T^t (c_1 Q, \operatorname{Im}(F + \dot{\theta}h))ds \tag{7.1}$$
$$= a(T) + a^{(2)}(t) - a^{(2)}(T) + \int_T^t \dot{b}(s)ds\,,$$

where $c_1 = (Q, R)^{-1}$ and

$$\dot{b} = \big(c_1 Q, \operatorname{Im}[F - F^{(2)} + \dot{\theta}h]\big) - 4a_{20} \operatorname{Re} e^{-2i\kappa s} p\dot{p}\,. \tag{7.2}$$

Note that, after the substitution $\eta = \eta^{(2)} + \eta^{(3)}$ and $a = a_{20}(z^2 + \bar{z}^2) + b$, $\dot{b}$ is of the form $\operatorname{Re}\sum\{Cz^3 + Czb + (z\phi, \eta) + Cz^4 + Cz^2 b\}$ plus error. We will perform several integrations by parts to arrive at the form

$$a(t) = O(t^{-1}) + \int_T^t O(t^{-2})ds = O(t^{-1})$$

and obtain the estimate $|a(t)| \le O(t^{-1})$.

We first decompose the integrand of $a(t)$ in (7.1) according to order in z; see table (4.5). Recall $\eta = \eta^{(2)} + \eta^{(3)}$ and $a = a_{20}(z^2 + \bar{z}^2) + b$. Also, recall from (3.23) that

$$\dot{\theta} = -\big[a^{(2)} + b + (c_1 R, \operatorname{Re} F^{(2)})\big] + F_{\theta,3} + F_{\theta,4} \tag{7.3}$$
$$F_{\theta,3} = -\big(c_1 R,\ \operatorname{Re} F^{(3)} + \widetilde{F}^{(3)}\big) + \big[a^{(2)} + b + (c_1 R, \operatorname{Re} F^{(2)})\big]\,(c_1 R, \operatorname{Re}\zeta)\,,$$

and $F_{\theta,4} = O(z^4 + a^2 + \eta^2)$. Thus we have

$$(c_1 Q, \operatorname{Im}\dot{\theta}h) = \dot{\theta}\,\big(c_1 Q, \operatorname{Im}\zeta + \eta^{(2)} + \eta^{(3)}\big)$$
$$= -\big[a^{(2)} + b + (c_1 R, \operatorname{Re} F^{(2)})\big](c_1 Q, v) \operatorname{Im} z \tag{7.4}$$
$$+ F_{\theta,3}(c_1 Q, v) \operatorname{Im} z - \big[a^{(2)} + b + (c_1 R, \operatorname{Re} F^{(2)})\big]\big(c_1 Q, \operatorname{Im}\eta^{(2)}\big) \tag{7.5}$$
$$\begin{aligned}&+ F_{\theta,4}(c_1 Q, v) \operatorname{Im} z + \big(F_{\theta,3} + F_{\theta,4}\big)\big(c_1 Q, \operatorname{Im}\eta^{(3)}\big)\\ &+ \dot{\theta}\big(c_1 Q, \operatorname{Im}\eta^{(3)}\big)\,.\end{aligned} \tag{7.6}$$

Here the second line (7.4) is of order $O(z^2)$ and the third line (7.5) of order $O(z^3)$, and the last line (7.6) contains higher-order terms. Together with the decompositions (3.22) of F we can decompose the integrand of $a(t)$ in (7.1) according to order in z,

$$(c_1 Q, \operatorname{Im}(F + \dot{\theta}h)) = A^{(2)} + A^{(3)} + A^{(4)} + A^{(5)}\,,$$
$$A^{(3)} = [A^{(zb)} + A^{(z\eta)} + A^{(z^3)}]\,,$$

where $A^{(2)}$ consists of $O(z^2)$ terms, $A^{(3)}$ of $O(z^3)$ terms, $A^{(4)}$ of $O(z^4)$ terms, and $A^{(5)}$ of higher-order terms. They are given explicitly by

$$
\begin{aligned}
A^{(2)} &= (c_1 Q, \operatorname{Im} \lambda Q \zeta^2)\,,\\
A^{(zb)} &= (c_1 Q, \operatorname{Im} 2\lambda Q R b \zeta) - b(c_1 Q, v) \operatorname{Im} z\,,\\
A^{(z\eta)} &= (c_1 Q, \operatorname{Im} 2\lambda Q \zeta \eta)\,,\\
A^{(z^3)} &= \big(c_1 Q, \operatorname{Im} \lambda |\zeta|^2 \zeta + 2\lambda Q R a^{(2)} \zeta\big)\\
&\quad - \big[a^{(2)} + (c_1 R, \operatorname{Re} F^{(2)})\big](c_1 Q, v) \operatorname{Im} z\,,\\
A^{(4)} + A^{(5)} &= \big(c_1 Q, \operatorname{Im} \big\{\lambda Q \eta^2 + 2\lambda Q R a \eta + \lambda \big[|aR + \zeta|^2 (aR + \zeta) - |\zeta|^2 \zeta\big]\big\}\big)\\
&\quad + (7.5) + (7.6)\,.
\end{aligned}
$$

$$
\begin{aligned}
A^{(4)} &= (7.5) + \big(c_1 Q, \operatorname{Im} \big\{\lambda Q (\eta^{(2)})^2 + 2\lambda Q R a \eta^{(2)}\\
&\qquad + \lambda\big[2|\zeta|^2 (aR + \eta^{(2)}) + \zeta^2 (aR + \overline{\eta^{(2)}})\big]\big\}\big)\,, \qquad (7.7)\\
A^{(5)} &= \big(c_1 Q, \operatorname{Im} \big\{\lambda Q [2\eta^{(2)} \eta^{(3)} + (\eta^{(3)})^2] + 2\lambda Q R a \eta^{(3)}\big\}\big)\\
&\quad + \big(c_1 Q, \operatorname{Im} \big\{\lambda\big[2|\zeta|^2 \eta^{(3)} + \zeta^2 \overline{\eta^{(3)}} + \ell^2 \bar\zeta + 2|\ell|^2 \zeta + \ell^2 \bar\ell\big]\big\}\big) + (7.6)\,,\\
\ell &= aR + \eta\,.
\end{aligned}
$$

Since $\zeta = z u_+ + \bar z u_-$, we have

$$\operatorname{Im} \zeta = \operatorname{Im} z (u_+ - u_-) \quad \text{and} \quad \operatorname{Im} \zeta^2 = (\operatorname{Im} z^2)(u_+^2 - u_-^2)\,.$$

We now proceed to integrate them term by term. We have already integrated $A^{(2)}$ in Section 3.1. Write

$$e^{-i\kappa t}\, \dot p = c_\alpha z^\alpha + d_\beta z^\beta + d_1 b z + d_2 b \bar z + P^{(4)} := c_\alpha z^\alpha + P^{(3,4)}\,, \qquad (7.8)$$

where $P^{(3,4)}$ is defined by the last equality. We can rewrite our result in Section 3.1 as

$$\int_T^t A^{(2)}\, ds = a^{(2)}(t) - a^{(2)}(T) + \int_T^t A_{2,3} + A_{2,4} + A_{2,5}\, ds\,, \qquad (7.9)$$

$$a^{(2)} = a_{20}(z^2 + \bar z^2)\,, \quad a_{20} = (4\kappa)^{-1} \big(c_1 Q,\ \lambda Q (u_+^2 - u_-^2)\big) = O(n^2)\,,$$

$$
\begin{aligned}
A_{2,3} &= -4 a_{20} \operatorname{Re} z c_\alpha z^\alpha\,, \quad A_{2,4} = -4 a_{20} \operatorname{Re} \big[d_\beta z z^\beta + d_1 b z^2\big]\,, \qquad (7.10)\\
A_{2,5} &= -4 a_{20} \operatorname{Re} z P^{(4)}\,.
\end{aligned}
$$

Note here we have used $\operatorname{Re} z d_2 b \bar z = 0$ from Lemma 6.1.

We can also rewrite (7.2), the equation for $b = a - a^{(2)}$, as

$$\dot b = \big[A^{(zb)} + A^{(z\eta)} + A^{(z^3)} + A_{2,3}\big] + A^{(4)} + A_{2,4} + A^{(5)} + A_{2,5}\,. \qquad (7.11)$$

7.1 Integration of $O(z^3)$ Terms

There are three terms of order $O(z^3)$: $A^{(zb)}$, $A^{(z\eta)}$, and $A^{(z^3)}$ (we will absorb $A_{2,3}$ into $A^{(z^3)}$). We first integrate $A^{(zb)}$. From its explicit form we have $A^{(zb)} = C_2\, b \operatorname{Im} z$ for a real constant $C_2 = O(n)$. Hence, using the integration-by-parts formula (6.11) and the decompositions (7.8) and (7.11),

$$\begin{aligned}\int_T^t A^{(zb)}\, ds = \int_T^t C_2 b \operatorname{Im} z\, ds &= C_2 \operatorname{Im} \frac{1}{-i\kappa}\left\{bz - \int_T^t e^{-i\kappa s}\frac{d}{dt}(bp)\, ds\right\} \\ &= \frac{C_2}{2\kappa} 2 \operatorname{Re}\left\{zb - \int_T^t z\dot{b} + b(c_\alpha z^\alpha + P^{(3,4)}) ds\right\} \\ &= c_{zb} b(z+\bar{z}) + \int_T^t A_{zb,4} + A_{zb,5}\, ds\,,\end{aligned}$$

where $P^{(3,4)}$ is defined in (7.8) and

$$c_{zb} = \frac{C_2}{2\kappa} = O(n)\,,$$

$$\begin{aligned}A_{zb,4} &= -c_{zb}(z+\bar{z})\big[A^{(zb)} + A^{(z\eta)} + A^{(z^3)} + A_{2,3}\big] - 2c_{zb} b \operatorname{Re} c_\alpha z^\alpha\,, \\ A_{zb,5} &= -c_{zb}(z+\bar{z})\big[A^{(4)} + A_{2,4} + A^{(5)} + A_{2,5}\big] - 2c_{zb} b \operatorname{Re} P^{(3,4)}\,.\end{aligned} \tag{7.12}$$

We now integrate $A^{(z\eta)}$. Recall from (2.24) that $U = U_+ + U_-\mathbf{C}$ and $U^{-1} = U_+^* - U_-^*\mathbf{C}$. We will also use the following formulae:

$$\begin{aligned}\operatorname{Re}\int dx f(\mathbf{C}g) &= \operatorname{Re}\int dx(\mathbf{C}f)g\,, \\ \operatorname{Im}\int dx f(\mathbf{C}g) &= -\operatorname{Im}\int dx(\mathbf{C}f)g\,,\end{aligned} \tag{7.13}$$

$$U(zf + \bar{z}g) = z(U_+ f + U_-\bar{g}) + \bar{z}(U_+ g + U_- \bar{f})\,. \tag{7.14}$$

Recall $\eta = U^{-1}e^{-i\theta}\tilde{\eta}$. Denote $\eta' = e^{-i\theta}\tilde{\eta}$ and hence $\eta = U^{-1}\eta'$. We have from (7.13) and (7.14) the identity

$$\begin{aligned}A^{(z\eta)} &= (c_1 Q, \operatorname{Im} 2\lambda Q\zeta\eta) \\ &= \operatorname{Im}\int dx 2c_1\lambda Q^2\zeta\eta \\ &= \operatorname{Im}\int dx 2c_1\lambda Q^2(zu_+ + \bar{z}u_-)(U_+^* - U_-^*\mathbf{C})\eta' \\ &= \operatorname{Im}\int dx \left\{(U_+ + U_-\mathbf{C})[z(2c_1\lambda Q^2 u_+) + \bar{z}(2c_1\lambda Q^2 u_-)]\right\}\eta' \\ &= \operatorname{Im}\int dx(z\phi_3 + \bar{z}\phi_4)\eta'\,,\end{aligned}$$

where

$$\phi_3 = U_+(2c_1\lambda Q^2 u_+) + U_-(2c_1\lambda Q^2 u_-)\,,$$
$$\phi_4 = U_+(2c_1\lambda Q^2 u_-) + U_-(2c_1\lambda Q^2 u_+)\,.$$

We now rewrite

$$\eta'(s) = e^{-i\theta}\tilde{\eta} = e^{-i\theta}e^{-iAs} f(s)\,,$$
$$f(s) = e^{iAs}\tilde{\eta} = \tilde{\eta}_0 + \int_0^s e^{i\tau A}\mathbf{P}_c^A F_\eta(\tau)d\tau\,. \tag{7.15}$$

The reason we work with f instead of $\tilde{\eta}$ is that those terms of the same order (z^2) in $\partial_s f(s) = e^{isA}\mathbf{P}_c^A F_\eta(s)$ are explicit. We now have

$$\begin{aligned}\int_T^t A^{(z\eta)}\,ds &= \operatorname{Im}\int_T^t (\phi_3, z\eta') + (\phi_4, \bar{z}\eta')ds\\ &= \operatorname{Im}\int_T^t \left(\phi_3, e^{-is(A+\kappa)}\left(pe^{-i\theta} f\right)\right) ds\\ &\quad + \operatorname{Im}\int_T^t \left(\phi_4, e^{-is(A-\kappa)}\left(\bar{p}e^{-i\theta} f\right)\right) ds\,.\end{aligned}$$

Note $f(s) \in \mathbf{H}_c(A)$. We first compute the first integral, which is equal to

$$\begin{aligned}&= \left[\operatorname{Im}\left(\phi_3, \frac{1}{-i(A+\kappa)}e^{-is(A+\kappa)}\left(pe^{-i\theta} f\right)\right)\right]_T^t\\ &\quad - \operatorname{Im}\int_T^t \left(\phi_3, \frac{1}{-i(A+\kappa)}e^{-is(A+\kappa)}\frac{d}{ds}\left(pe^{-i\theta} f\right)\right) ds\\ &= \left[\operatorname{Re}\left(\frac{1}{A+\kappa}\mathbf{P}_c^A\Pi\phi_3, ze^{-i\theta}\tilde{\eta}\right)\right]_T^t\\ &\quad - \operatorname{Re}\int_T^t \left(\frac{1}{A+\kappa}\mathbf{P}_c^A\Pi\phi_3, \left[e^{-i\kappa s}\dot{p}\eta' - i\dot{\theta}z\eta' + ze^{-i\theta}\mathbf{P}_c^A F_\eta\right]\right) ds\\ &= \left[\operatorname{Re}\left(\phi_5, z\eta'\right)\right]_T^t - \operatorname{Re}\int_T^t \left(\phi_5, \left[e^{-i\kappa s}\dot{p}\eta' - i\dot{\theta}z\eta' + ze^{-i\theta}\mathbf{P}_c^A F_\eta\right]\right) ds\end{aligned}$$

where

$$\phi_5 = \frac{1}{A+\kappa}\mathbf{P}_c^A\Pi\phi_3\,.$$

We are careful in adding $\mathbf{P}_c^A\Pi$ so that ϕ_5 makes sense. We can do so since $f(s) \in \mathbf{H}_c(A)$. Similarly, the second integral is equal to

$$= \left[\operatorname{Re}\left(\phi_6, \bar{z}\eta'\right)\right]_T^t - \operatorname{Re}\int_T^t \left(\phi_6, \left[e^{i\kappa s}\overline{\dot{p}}\eta' - i\dot{\theta}\bar{z}\eta' + \bar{z}e^{-i\theta}\mathbf{P}_c^A F_\eta\right]\right) ds$$

with

$$\phi_6 = \frac{1}{A-\kappa}\mathbf{P}_c^A\Pi\phi_4\,.$$

Using $\eta' = U\eta$, we can rewrite the leading terms of $\int_T^t A^{(z\eta)}\,ds$ in the form

$$\begin{aligned}\mathrm{Re}(\phi_5, z\eta') + \mathrm{Re}(\phi_6, \bar z\eta') &= \mathrm{Re}\int dx(z\phi_5 + \bar z\phi_6)(U_+ + U_-\mathbf{C})\eta\\ &= \mathrm{Re}\int dx[(U_+^* + U_-^*\mathbf{C})(z\phi_5 + \bar z\phi_6)]\eta\\ &= \mathrm{Re}\int dx(z\phi_8 + \bar z\phi_7)\eta\\ &= \mathrm{Re}(z\phi_7 + \bar z\phi_8, \eta)\,,\end{aligned}$$

where we have used (7.13) and (7.14) again, with the convention (2.1) and

$$\phi_8 = U_+^*\phi_5 + U_-^*\phi_6\,, \quad \phi_7 = U_+^*\phi_6 + U_-^*\phi_5\,.$$

Tracking our definition, we have $\phi_i = O(n^2)$, $i = 3, 4, \ldots, 8$.

The remaining integral has the integrand

$$\begin{aligned}&= -\mathrm{Re}\left(\phi_5, \left[e^{-i\kappa s}\dot p\eta' - i\dot\theta\eta' + ze^{-i\theta}\mathbf{P}_{\mathrm{c}}^A F_\eta\right]\right)\\ &\quad - \mathrm{Re}\left(\phi_6, \left[e^{i\kappa s}\overline{\dot p}\eta' - i\dot\theta\bar z\eta' + \bar z e^{-i\theta}\mathbf{P}_{\mathrm{c}}^A F_\eta\right]\right)\\ &= -\mathrm{Re}\int dx\left(e^{-i\kappa s}\dot p\phi_5 + e^{i\kappa s}\overline{\dot p}\phi_6\right)U\eta\\ &\quad + (z\phi_5 + \bar z\phi_6)\left[-i\dot\theta U\eta + e^{-i\theta}\mathbf{P}_{\mathrm{c}}^A F_\eta\right]\,.\end{aligned}$$

Using (3.11) we have

$$\left[-i\dot\theta U\eta + e^{-i\theta}\mathbf{P}_{\mathrm{c}}^A F_\eta\right] = \mathbf{P}_{\mathrm{c}}^A U i^{-1}[F + \dot\theta(aR + \zeta + \eta)]\,.$$

Thus the integrand of the remaining integral can be written as $A_{z\eta,3} + A_{z\eta,4} + A_{z\eta,5}$, and we have

$$\int_T^t A^{(z\eta)}\,ds = [\mathrm{Re}(z\phi_7 + \bar z\phi_8, \eta)]_T^t + \int_T^t A_{z\eta,3} + A_{z\eta,4} + A_{z\eta,5}\,ds\,, \tag{7.16}$$

$$\begin{aligned}A_{z\eta,3} &= -\mathrm{Re}\int dx(z\phi_5 + \bar z\phi_6)\mathbf{P}_{\mathrm{c}}^A U i^{-1} z^\alpha\phi_\alpha\\ A_{z\eta,4} &= -\mathrm{Re}\int dx\left(c_\alpha z^\alpha\phi_5 + \overline{c_\alpha z^\alpha}\phi_6\right)U\eta^{(2)}\\ &\quad + (z\phi_5 + \bar z\phi_6)\mathbf{P}_{\mathrm{c}}^A U i^{-1}\big\{F^{(3)} + \widetilde F^{(3)}\\ &\qquad\qquad - \left[a^{(2)} + b + \left(c_1 R, \mathrm{Re}\,F^{(2)}\right)\right]\zeta\big\}\\ A_{z\eta,5} &= -\mathrm{Re}\int dx\left(c_\alpha z^\alpha\phi_5 + \overline{c_\alpha z^\alpha}\phi_6\right)U\eta^{(3)} + \left(P^{(3,4)}\phi_5 + \overline{P^{(3,4)}}\phi_6\right)U\eta\\ &\quad + (z\phi_5 + \bar z\phi_6)\mathbf{P}_{\mathrm{c}}^A U i^{-1}\left\{F^{(4)} + \dot\theta(aR + \eta) + (F_{\theta,3} + F_{\theta,4})\zeta\right\}\,,\end{aligned} \tag{7.17}$$

where $P^{(3,4)}$ stands for the higher-order terms in $e^{-i\alpha s}\dot p$ (7.8), and we have used the definitions of F (3.22) and θ (7.3). We observe that the only appearance of θ

is as the exponent of $e^{i\theta}$. Also, $A_{z\eta,3}$ is a sum of monomials of the form cz^β with $|\beta| = 3$. There is no a or η in $A_{z\eta,3}$.

Summarizing our effort, we have obtained

$$\begin{aligned} a(t) = a(T) &+ \left[a_{20}\left(z^2+\bar z^2\right) + c_{zb} b(z+\bar z) + \operatorname{Re}(z\phi_7 + \bar z \phi_8, \eta)\right]_T^t \\ &+ \int_T^t \left[A^{(z^3)} + A_{2,3} + A_{(z\eta,3)}\right] + \left[A^{(4)} + A_{2,4} + A_{(zb,4)} + A_{(z\eta,4)}\right] \\ &+ \left[A^{(5)} + A_{2,5} + A_{(zb,5)} + A_{(z\eta,5)}\right] ds\,. \end{aligned} \tag{7.18}$$

All terms in $[A^{(z^3)} + A_{2,3} + A_{(z\eta,3)}]$ are of the form cz^β, $|\beta| = 3$. We define A_β so that

$$\begin{aligned} A_\beta z^\beta &= A^{(z^3)} + A_{2,3} + A_{(z\eta,3)} \\ &= \left(c_1 Q, \operatorname{Im}\lambda|\zeta|^2\zeta + 2\lambda Q R a^{(2)}\zeta\right) - \left[a^{(2)} + \left(c_1 R, \operatorname{Re} F^{(2)}\right)\right](c_1 Q, v)\operatorname{Im} z \\ &\quad - 4a_{20}\operatorname{Re} z c_\alpha z^\alpha - \operatorname{Re}\int dx\,(z\phi_5 + \bar z\phi_6)\mathbf{P}_c^A U i^{-1} z^\alpha \phi_\alpha\,. \end{aligned}$$

From this explicit form, we have $A_\beta = O(n)$. By integration by parts, we have

$$\begin{aligned} \int_T^t A_\beta z^\beta\,ds &= a_\beta z^\beta - \int_T^t a_\beta z^\beta f_\beta(z)\,ds\,, \quad a_\beta := \frac{A_\beta}{i[\beta]\kappa} = O(n)\,, \\ &= a_\beta z^\beta + \int_T^t A_{3,4} + A_{3,5}\,ds \end{aligned} \tag{7.19}$$

where f_β is defined in (6.12) and

$$\begin{aligned} A_{3,4} &= \sum_{\beta=(\beta_0,\beta_1),\beta_0+\beta_1=3} -a_\beta z^\beta(\beta_0 + \beta_1\mathbf{C})z^{-1}c_\alpha z^\alpha \\ A_{3,5} &= \sum_{\beta=(\beta_0,\beta_1),\beta_0+\beta_1=3} -a_\beta z^\beta(\beta_0 + \beta_1\mathbf{C})z^{-1}P^{(3,4)}\,. \end{aligned} \tag{7.20}$$

Notice that $[\beta] \neq 0$ and $\overline{a_\beta} = a_{\bar\beta}$.

Substituting (7.19) into (7.18), we obtain

$$\begin{aligned} a(t) = a(T) &+ \left[a_{20}\left(z^2+\bar z^2\right) + c_{zb} b(z+\bar z) + \operatorname{Re}(z\phi_7 + \bar z\phi_8, \eta) + a_\beta z^\beta\right]_T^t \\ &+ \int_T^t \left[A^{(4)} + A_{2,4} + A_{(zb,4)} + A_{(z\eta,4)} + A_{3,4}\right] \\ &\quad + \left[A^{(5)} + A_{2,5} + A_{(zb,5)} + A_{(z\eta,5)} + A_{3,5}\right] ds\,. \end{aligned} \tag{7.21}$$

We have finished the integration of all terms of order $O(z^3)$ in equation (7.1) for $a(t)$. Thus there are no order-$O(z^3)$ terms in the integrand in (7.21). The order z^4 terms are collected in first group of integrands in (7.21), and the higher-order terms in the second group.

In principle, order-z^4 terms have the following forms: z^4, b^2, η^2, $z^2 b$, $z^2\eta$, $b\eta$, and $z\eta^{(3)}$. Closer examination shows that b^2, $b|z|^2$, or $z\eta^{(3)}$ never occur. (Recall that $z\eta^{(3)}$ is part of $z\eta$, which we already treated when we integrated $A^{(z\eta)}$.) Furthermore, the terms involving η can be removed by making the substitution $\eta = \eta^{(2)} + \eta^{(3)}$. A precise statement on these integrands is given in the following lemma. It is crucial that no term involving b^2 appear in the integrand. Otherwise, it will result in an inequality of the form $b(t) \le C/t + \int^t b(s)^2 ds$, which does not guarantee $b(t) \le C/t$.

LEMMA 7.1 *The integrands of order $O(z^4)$ in* (7.21) *can be summed into the form*

$$\left[A^{(4)} + A_{2,4} + A_{(zb,4)} + A_{(z\eta,4)} + A_{3,4}\right] = B_{22}|z|^4 + \operatorname{Re}\left\{A_{40}z^4 + A_{31}z^3\bar z + A_{b2}bz^2\right\} .$$

There are no terms of the form b^2, $b|z|^2$, or $z\eta^{(3)}$. Moreover, we have

$$B_{22} = \tfrac{1}{2}c_1\Gamma + O(n^3)\,, \quad A_{40}, A_{31} = O(n)\,, \quad A_{b2} = O(n)\,. \tag{7.22}$$

PROOF: Recall the definitions (7.7), (7.10), (7.12), (7.17), and (7.20):

$$\begin{aligned}
A^{(4)} &= \Big(c_1 Q, \operatorname{Im}\big\{\lambda Q(\eta^{(2)})^2 + 2\lambda QRa\eta^{(2)} \\
&\qquad + \lambda\big[2|\zeta|^2(aR + \eta^{(2)}) + \zeta^2(aR + \overline{\eta^{(2)}})\big]\big\}\Big) \\
&\quad + F_{\theta,3}(c_1 Q, v)\operatorname{Im} z - \big[a^{(2)} + b + \big(c_1 R, \operatorname{Re} F^{(2)}\big)\big](c_1 Q, \operatorname{Im}\eta^{(2)}) \\
A_{2,4} &= -4a_{20}\operatorname{Re}\big[d_\beta z z^\beta + d_1 bz^2\big] \\
A_{(zb,4)} &= -c_{zb}(z+\bar z)\big[A^{(zb)} + A^{(z\eta)} + A^{(z^3)} + A_{2,3}\big] - 2c_{zb} b \operatorname{Re} c_\alpha z^\alpha \\
A_{(z\eta,4)} &= -\operatorname{Re}\int dx\,\big(c_\alpha z^\alpha \phi_5 + \overline{c_\alpha z^\alpha}\phi_6\big)\, U\eta^{(2)} \\
&\quad + (z\phi_5 + \bar z\phi_6)\mathbf{P}_c^A U i^{-1}\big\{F^{(3)} + \widetilde{F}^{(3)} - \big[a^{(2)} + b + \big(c_1 R, \operatorname{Re} F^{(2)}\big)\big]\zeta\big\} \\
A_{3,4} &= \sum_{\beta=(\beta_0,\beta_1),\beta_0+\beta_1=3} -a_\beta z^\beta(\beta_0 + \beta_1 \mathbf{C})z^{-1}c_\alpha z^\alpha\,.
\end{aligned}$$

The first part of the lemma is obtained by direct inspection. Note we can deal with $A_{z\eta,4}$ in the same manner as we dealt with $A^{(z\eta)}$. The orders of the coefficients are also obtained by direct check with the following table in mind:

$$\begin{gathered}
Q = n\,, \quad R = n^{-1}\,, \quad c_1 = 1\,, \quad c_\alpha = n\,, \quad d_\beta, d_1, d_2 = 1\,, \\
a_{20} = n^2\,, \quad a_\beta = n\,, \quad c_{zb} = n\,, \quad \phi_7, \phi_8 = n^2 \\
\zeta \sim z + n^2\bar z\,, \quad aR \sim n^{-1}\rho^2\,, \quad \eta \sim n\rho^2\,.
\end{gathered}$$

We note that all $C|z|^4$ terms with $C = O(n^2)$ but one are killed by the Im operator. The only surviving term is the following resonance term of $A^{(4)}$:

$$\big(c_1 Q, \operatorname{Im} \lambda\zeta^2\overline{\eta^{(2)}}\big) = \big(c_1 Q, \operatorname{Im} \lambda z^2 u_+^2 \overline{z^2\eta_{20}}\big) + O(n^3)|z|^4 + \sum_{|\gamma|=4,\gamma\neq(22)} O(n^2) z^\gamma .$$

The first term on the right side is equal to $\frac{1}{2}c_1\Gamma|z|^4 + O(n^3)$, and this is the main term in B_{22}. This proves Lemma 7.1. □

7.2 Integration of Higher-Order Terms

Next we proceed to integrate out those oscillatory terms in the integrand of (7.21) for $a(t)$. The first group consists of terms of the form z^γ, $|\gamma| = 4$.

$$\begin{aligned}&\operatorname{Re} \int_T^t A_{40}z^4 + A_{31}z^3\bar{z} + A_{b2}bz^2\, ds \\ &\quad = \operatorname{Re}\left[\frac{A_{40}z^4}{-4i\kappa} + \frac{A_{31}z^3\bar{z}}{-2i\kappa} + \frac{A_{b2}bz^2}{-2i\kappa}\right]_T^t \\ &\qquad - \operatorname{Re}\int_T^t \frac{A_{40}z^4}{-4i\kappa} f_{40}(z) + \frac{A_{31}z^3\bar{z}}{-2i\kappa} f_{31}(z) + \frac{A_{b2}\mu^{-2}}{-2i\kappa}\frac{d}{ds}(bp^2)ds\end{aligned}$$

where $\frac{d}{ds}(bp^2) = \dot{b}p^2 + 2bp\dot{p} = O(z^5)$. Let

$$a_{40} = \frac{A_{40}}{-4i\kappa} = O(n)\,, \quad a_{31} = \frac{A_{31}}{-2i\kappa} = O(n)\,, \quad a_{b2} = \frac{A_{b2}}{-2i\kappa} = O(n)\,,$$

$$A_{4,5} = -\operatorname{Re}\left\{a_{40}z^4 f_{40}(z) + a_{31}z^3\bar{z}\, f_{31}(z) + a_{b2}\frac{d}{ds}(bp^2)\right\}.$$

Then we have the identity

$$\operatorname{Re}\int_T^t A_{40}z^4 + A_{31}z^3\bar{z} + A_{b2}bz^2\, ds = \operatorname{Re}\big[a_{40}z^4 + a_{31}z^3\bar{z} + a_{b2}bz^2\big]_T^t + \int_T^t A_{4,5}\, ds\,.$$

After this integration, the integrand in (7.21) becomes

$$A^{(5)} + A_{2,5} + A_{(za,5)} + A_{(z\eta,5)} + A_{3,4} + A_{4,5}\,.$$

Since $|A_{4,5}|$ is of order z^5, the integrands are of order z^5 or higher. More precisely, they are of the orders

$$n\big(z^5 + z(aR)^2 + z(aR)\eta + (aR)^2\eta + z\eta^2 + z^2\eta^{(3)}_{1-2} + z^2\eta^{(3)}_{3-5} + \cdots\big)\,.$$

(The first n is from the $c_1 Q$ in equation (2.30) for $\dot{a}$.) These terms are of order $t^{-2-1/8}$ and thus can be considered as error terms when t large. Unfortunately, we need to control their behavior even for t of order 1. In this region, their orders in terms of n are crucial.

We first note that nz^5 is bounded by $\varepsilon n^2|z|^4 \ll |B_{22}||z|^4$. Since $\|\eta\|_{L^2_{\rm loc}} \le C\,\{t\}^{-1}$, the term $nz\eta^2$ is also much smaller than $|B_{22}||z|^4$. The term $nz^2\eta^{(3)}_{3-5}$ is bounded by $\varepsilon^{-\sigma} n(\varepsilon n)^{3/4}\,\{t\}^{-2-1/8} \ll |B_{22}||z|^4$. Thus the main trouble lies in terms of order

$$nz(aR)^2 + nz(aR)\eta + n(aR)^2\eta + nz^2\eta^{(3)}_{1-2}\,.$$

These terms have faster time-decay order than $|B_{22}||z|^4$, namely, $t^{-2-1/8}$, but they are larger than $|B_{22}||z|^4$ when time $t = O(1)$. They are oscillatory terms of order z^5 that we will integrate. (In contrast, terms of order z^{2k} may be nonoscillatory.) Since the integration procedures for these terms are the same as those for $O(z^3)$ terms, we only sketch the main steps:

(1) Replace all a in the above group by $a_{20}(z^2+\bar z^2)+b$.

(2) All terms z^γ with $|\gamma| = 5$ are oscillatory and can be integrated as $A^{(z^3)}$.

(3) nz^3bR^2 and nzb^2R^2 are also oscillatory and can be integrated as $A^{(zb)}$.

(4) All terms linear in η—$z^3\eta$, $zb\eta$, and $b^2\eta$—can be integrated as $A^{(z\eta)}$.

(5) $z^2\eta^{(3)}_{1-2}$ can be integrated as $P_{z\eta^{(3)}_{1-2}}$, defined in (6.15). Specifically, we have terms of the form

$$A_{z^2\eta^{(3)}_{1-2}} = \sum_{|\alpha|=2} \operatorname{Re}\left(n\phi\,,\ z^\alpha \eta^{(3)}_{1-2}\right) = \frac{d}{dt}\left(a_{z^2\eta^{(3)}_{1-2}}\right) + A_{z^2\eta^{(3)}_{1-2},5}$$

where $\phi = O(1)$ are some complex local functions and $a_{z^2\eta^{(3)}_{1-2}}$ and $A_{z^2\eta^{(3)}_{1-2},5}$ are defined similarly to p_4 and g_4 in (6.19) with the decay

$$\left|a_{z^2\eta^{(3)}_{1-2}}\right| \le C\,\{t\}^{-1}\,\varepsilon^2 n^3 \langle t\rangle^{-9/8}\,,$$
$$\left|A_{z^2\eta^{(3)}_{1-2},5}\right| \le C\,\{t\}^{-3/2}\,\varepsilon^2 n^3 \langle t\rangle^{-9/8} \le C\varepsilon |B_{22}|\,\{t\}^{-2}\ .$$

Note that, as in p_4 and g_4, we have terms of the form $(\phi, (A-2\kappa-0i)^{-2}\mathbf{P}^A_{\rm c}\Phi)nz^2$ in $a_{z^2\eta^{(3)}_{1-2}}$, which still has the claimed decay by Lemma 2.7.

To summarize, we have obtained

$$\begin{aligned} a(t) = a(T) &+ \left[a_{20}\left(z^2+\bar z^2\right) + c_{zb}b(z+\bar z) + \operatorname{Re}(z\phi_7 + \bar z\phi_8, \eta) + a_\beta z^\beta\right]_T^t \\ &+ \operatorname{Re}\left[a_{40}z^4 + a_{31}z^3\bar z + a_{b2}bz^2 + a_{z^2\eta_0}\right]_T^t \\ &+ \operatorname{Re}\left[Cz^5 + Cz^3b + Czb^2 + Cz^3\eta + Czb\eta + Cb^2\eta + Cz^2\eta^{(3)}_{1-2}\right]_T^t \\ &+ \int_T^t B_{22}|z|^4 + B_5\,ds\,, \end{aligned} \tag{7.23}$$

where $B_{22} = \frac{1}{2}c_1\Gamma + O(n^3)$, the terms in the third line denote various terms of similar form (e.g., Cz^5 means Cz^γ with $|\gamma| = 5$), and

$$|B_5(t)| \le C(\varepsilon+n)^\sigma n^{7/4}\,\{t\}^{-2-1/8} \ll \Gamma\,\{t\}^{-2}\ , \tag{7.24}$$

assuming the bound (4.10) on $a(t)$ and $M(T) \leq 2$.

Recall $b(t) := a(t) - a^{(2)}(t)$ and S_T (4.11) is defined by

$$a(t) = a(T) + \left[a_{20}\left(z^2 + \bar{z}^2\right)\right]_T^t + S_T(a, z, \eta, \theta)(t)\,.$$

From (7.23), S_T can be viewed as a function of a, z, η, and θ and explicitly given by

$$\begin{aligned} &S_T(a, z, \eta, \theta)(t) \\ &\quad := \left[c_{zb} b(z + \bar{z}) + \mathrm{Re}(z\phi_7 + \bar{z}\phi_8, \eta) + a_\beta z^\beta\right]_T^t \\ &\qquad + \mathrm{Re}\left[a_{40} z^4 + a_{31} z^3 \bar{z} + a_{b2} b z^2 + a_{z^2\eta_0}\right]_T^t \\ &\qquad + \mathrm{Re}\left[C z^5 + C z^3 b + C z b^2 + C z^3 \eta + C z b \eta + C b^2 \eta + C z^2 \eta_{1-2}^{(3)}\right]_T^t \\ &\qquad + \int_T^t B_{22} |z|^4 + B_5\, ds\,. \end{aligned} \tag{7.25}$$

The right-hand side depends on b, $\eta^{(3)}$, etc., which are not explicitly given as variables in S_T. But all these variables can be traced back to the basic variables a, z, η, and θ, e.g., $b = a - a_{20}(z^2 + \bar{z}^2)$.

Making the same assumptions as in Proposition 4.1, we have

$$|S_T(a, z, \eta, \theta)(t)| \leq C_1(D)\,\{t\}^{-3/2} + \frac{B_{22}}{2\Gamma}\,\{t\}^{-1} \leq \left[o(1) + \frac{D}{4}\right]\{t\}^{-1}\,. \tag{7.26}$$

We also have $a(t) = a(T) + [a_{20}(z^2 + \bar{z}^2)]_T^t + S(a)(t) = a(T) + [o(1) + D/4]\,\{t\}^{-1}$. We conclude with the following lemma:

LEMMA 7.2 *Suppose that* $M(t) \leq 2$ *and* $|a(t)| \leq D\,\{t\}^{-1}$ *for* $0 \leq t \leq T$. *Recall* $D = 2B_{22}/\Gamma = c_1 + O(n)$. *Then we have*

$$|S_T(a, z, \eta, \theta)(t)| \leq \frac{D}{2}\,\{t\}^{-1}\,, \quad |a(t)| \leq |a(T)| + \frac{D}{2}\,\{t\}^{-1}\,. \tag{7.27}$$

This lemma gives Proposition 4.2 and concludes the proof of Theorems 1.1 and 1.3.

Since $|E_0 - E_{j+1}| \leq |E_0 - E_j| + |\gamma| \leq \frac{9}{8} D\varepsilon^2 n^2 + |\gamma| \leq 2D\varepsilon^2 n^2$, by Lemma 2.4 again with E_1 standing for E_0 and E_2 standing for E_{j+1},

$$|E_0 - E_{j+1} - a_{j+1}(0)| \leq C n^{-1}(\varepsilon n)(2D\varepsilon^2 n^2) = C\varepsilon^3 n^2\,. \tag{7.28}$$

The constant C here is independent of j since we use a bound for $|E_0 - E_{j+1}|$ that is independent of j. Since $|a_{j+1}(0)| \leq \frac{1}{2} D\varepsilon^2 n^2$, we have

$$|E_0 - E_{j+1}| \leq D\varepsilon^2 n^2\,. \tag{7.29}$$

From this, we also get $\|\psi_0 - Q_{j+1} e^{i\Theta_{j+1}(0)}\|_Y \leq 2\varepsilon n$. We have thus proven the induction hypothesis (I_{j+1}) for $t = T_{j+1}$. The induction proof is complete.

PROOF OF THEOREM 1.4: Suppose the assumption of Theorem 1.4 holds. The existence of E_∞ and the upper bound (1.22) follows from the same proof for Theorem 1.3. Recall from (1.24) that

$$|z_{E_0,0}| = \varepsilon n\,, \quad \|\eta_{E_0,0}\|_Y \le C\varepsilon^2 n^2\,, \quad a_{E_0,0} = 0\,.$$

Let $\gamma = E_0 - E_\infty$ be the energy shift. The energy shift can be bounded by $|\gamma| \le D\varepsilon^2 n^2$ (2.10). Thus with respect to E_∞ Lemma 2.4 guarantees

$$|z_{E_\infty}(0)| = \varepsilon n + O(\varepsilon^2 n)\,, \quad \|\eta_{E_\infty}(0)\|_Y \le C\varepsilon^2 n^2\,, \quad a_{E_\infty}(0) \le C\varepsilon^2 n^2\,.$$

The second part of Proposition 4.1 thus provides the lower bound $|z_\infty(t)| \ge (1 - 2\sigma)\,\{t\}^{-1/2}$ for all times t.

We now prove (1.26), the last statement of Theorem 1.4. Since $|\gamma| \le D\varepsilon^2 n^2$, by (2.8) we have

$$\|Q_{E_0}\|^2 = \|Q_{E_\infty}\|^2 + 2\gamma(Q_\infty, R_\infty) + O(n^{-2}\gamma^2)\,. \tag{7.30}$$

We can also estimate the energy shift $\gamma = a_\infty(0) + O(\varepsilon^3 n^2)$ by Lemma 2.4 with E_1 standing for E_0 and E_2 standing for E_∞. Thus we have

$$a_\infty(0) = \int_\infty^0 B_{22}|z|^4\,ds + O\big((\varepsilon n)^{9/4}\big) = \int_\infty^0 B_{22}|q|^4\,ds + O\big((\varepsilon n)^{9/4}\big)$$

where $q = q_\infty(t)$. Recall $B_{22} = \frac{1}{2}c_1\Gamma + O(n^3)$ and $c_1 = (Q_\infty,\, R_\infty)^{-1}$. On the other hand, we have $\frac{d}{dt}|q|^2 = -2\Gamma|q|^4 + \{t\}^{-17/8}$; hence

$$|q(0)|^2 = \int_0^\infty -2\Gamma|q|^4\,ds + O((\varepsilon n)^{9/4})\,.$$

From these formulae for $a_\infty(0)$ and $q(0)$ we have

$$4a_\infty(0)(Q_\infty, R_\infty) + |q(0)|^2 = O\big((\varepsilon n)^{9/4}\big)\,.$$

Recall that the eigenfunctions for $\mathcal{L}$ satisfy $u, v = \phi_1 + O(n^2)$. From the definition of z, we have $|z(0)|^2 = \|\zeta(0)\|_{L^2}^2 \cdot [1 + O(n^2)]$. Since $|q(0)|^2 = |z(0)|^2 + O((\varepsilon n)^{9/4})$, we conclude $2\gamma(Q_\infty, R_\infty) = -\frac{1}{2}\,\|\zeta(0)\|_{L^2}^2 + O(\varepsilon^2 n^2(\varepsilon^{1/4} + n^{1/4}))$ and

$$\|Q_{E_0}\|^2 = \|Q_{E_\infty}\|^2 - \tfrac{1}{2}\,\|\zeta(0)\|_{L^2}^2 + O\big(\varepsilon^2 n^2(\varepsilon^{1/4} + n^{1/4})\big)\,. \tag{7.31}$$

Therefore (1.26) and thus Theorem 1.4 are proven. □

8 Existence of Dispersion-Dominated Solutions

In this section we prove Theorem 1.5. For a given profile ξ_∞, we will construct a solution of the form (1.18),

$$\psi = [Q(x) + a(t)R(x) + h(t,x)]e^{i[-Et+\theta(t)]}\,,$$

so that $\psi(t) - \psi_{\text{as}}(t) = O(t^{-2})$. Here $a(t), \theta(t) \in \mathbb{R}$ and $h(t) \in M$.

By (2.29) we have

$$\partial_t U h = -iAUh - i\dot\theta U h + U i^{-1}(F + \dot\theta a R) - [U, i]\dot\theta h\,.$$

(Recall $UP_M = U$.) We single out $-i\dot{\theta}Uh$ since it is a global linear term in h. Let $\widetilde{h} = e^{i\theta}Uh$; we have

$$\partial_t \widetilde{h} = -iA\widetilde{h} + e^{i\theta}\left\{Ui^{-1}(F + \dot{\theta}aR) - [U, i]\dot{\theta}h\right\}. \tag{8.1}$$

Let

$$\widetilde{h}(t) = e^{-iAt}U\xi_\infty + g(t)\,, \quad h = U^{-1}e^{-i\theta}\widetilde{h}\,. \tag{8.2}$$

Here $g(t) \in X$ is the error term. Note that $g(t)$ consists of both excited state and subspace of continuity components. From (8.1) we have

$$g(t) = \int_\infty^t e^{-iA(t-s)}e^{i\theta}\left\{Ui^{-1}(F + \dot{\theta}aR) - [U, i]\dot{\theta}h\right\}ds\,. \tag{8.3}$$

We will solve $\{a, \theta, g\}$ satisfying (2.30), (2.32), and (8.3), respectively, with $\widetilde{h}$ and h defined in (8.2).

Note $F = F(aR + h)$. The main term in F is

$$F_0 = \lambda Q\left(2|\xi|^2 + \xi^2\right) + \lambda|\xi|^2\xi\,, \qquad \xi(t) = U^{-1}e^{-i\theta(t)}e^{-iAt}U\xi_\infty\,.$$

By assumption ξ_∞ is small in the space $Z = H^2 \cap W^{2,1}(\mathbb{R}^3)$. Since U is bounded in $W^{k,p}$, $\|U\xi_\infty\|_{H^2\cap W^{2,1}(\mathbb{R}^3)} \le \varepsilon$ for some ε sufficiently small. From Lemmas 2.6 and 2.7, we have the bounds $\|\xi(t)\|_{H^2} \le C_1\varepsilon$ and $\|\xi(t)\|_{W^{2,\infty}} \le C_1\varepsilon|t|^{-3/2}$. Hence the term $\lambda|\xi|^2\xi$ in F_0 is bounded by $\||\xi|^2\xi(t)\|_{H^2} \le C\|\xi(t)\|_{H^2}^3 \le C\varepsilon^3$. Moreover, if $t > 1$,

$$\left\||\xi|^2\xi(t)\right\|_{L^2} \le C\left\|\xi(t)\right\|_\infty^2\left\|\xi(t)\right\|_2 \le C\varepsilon^3 t^{-3}\,,$$
$$\left\|\nabla^2(|\xi|^2\xi)(t)\right\|_{L^2} \le C\left\|\xi(t)\right\|_\infty^2\left\|\nabla^2\xi(t)\right\|_2 + C\left\|\xi(t)\right\|_\infty\left\|\nabla\xi(t)\right\|_4^2 \le C\varepsilon^3 t^{-3}\,.$$

We conclude that

$$\left\||\xi|^2\xi(t)\right\|_{H^2} \le C\varepsilon^3\langle t\rangle^{-3}\,. \tag{8.4}$$

We also have the following estimate: For a function $\phi \in L^1 \cap L^2$, we have

$$\begin{aligned}\left|(\phi, |\xi|^2\xi)\right| &\le \min\left\{\|\phi\|_{L^1}\cdot\||\xi|^2\xi\|_{L^\infty}, \|\phi\|_{L^2}\cdot\||\xi|^2\xi\|_{L^2}\right\}\\ &\le C\,\|\phi\|_{L^1\cap L^2}\,\varepsilon^3\langle t\rangle^{-9/2}\,.\end{aligned} \tag{8.5}$$

The other term $\lambda Q(2|\xi|^2 + \xi^2)$ in F_0 is actually larger than $\lambda|\xi|^2\xi$ locally and can be estimated similarly. We conclude

$$\|F_0(t)\|_{H^2} \le C\varepsilon^2\langle t\rangle^{-3}\,.$$

If $\|g(t)\|_{H^2} \le C\varepsilon^2\langle t\rangle^{-2}$, one can prove, for example,

$$\left\||\xi + g|^2(\xi + g)(t)\right\|_{H^2} \le C\varepsilon^3\langle t\rangle^{-3}, \quad \|Q\xi g(t)\|_{H^2} \le C\varepsilon^3\langle t\rangle^{-7/2}\,.$$

From these estimates on F_0, the main term of F, the following bounds follow from definitions (8.3), (2.30), and (2.32) of g, a, and $\dot{\theta}$, respectively,

$$g(t) \lesssim t^{-2}\,, \quad a \lesssim t^{-2}\,, \quad \dot{\theta} \lesssim t^{-2}\,, \quad \theta \lesssim t^{-1}\,. \tag{8.6}$$

One verifies with these orders that the main term of F is indeed F_0. Although $\theta(t) = O(t^{-1})$, θ only appears in the form $e^{i\theta}$ and does not change the estimates.

We now proceed to construct a solution. For convenience, we introduce a new variable $\omega = \dot\theta$. Let $\mathcal{A}$ be the space

$$\mathcal{A} = \Big\{(\omega,\theta,a,g) : [0,\infty) \to \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times H^2,\ |\omega(t)| \le \varepsilon^{2-2\sigma}\langle t\rangle^{-2},\\ |\theta(t)| \le \varepsilon^{2-3\sigma}\langle t\rangle^{-1},\ |a(t)| \le \varepsilon^{2-\sigma}\langle t\rangle^{-2},\ \|g(t)\|_{H^2} \le \varepsilon^{2-\sigma}\langle t\rangle^{-2}\Big\},$$

where $\sigma = 1/100$ is small. We define a Cauchy sequence on the space $\mathcal{A}$ by iterating the following map: (cf. (2.30), (2.32), and (8.3))

$$\omega^{\triangle}(t) := -[a + (c_1 R, \operatorname{Re} F)]\cdot[1 + a(c_1R, R) + (c_1 R, \operatorname{Re} h)]^{-1}\ ,$$

$$\theta^{\triangle}(t) := \int_\infty^t \omega\, ds\ ,$$

$$a^{\triangle}(t) := \int_\infty^t (c_1 Q, \operatorname{Im}(F + \omega h))\, ds\ ,$$

$$g^{\triangle}(t) := \int_\infty^t e^{-iA(t-s)}e^{-iA(t-s)}e^{i\theta}\left\{Ui^{-1}(F + \omega a R) - [U,i]\omega h\right\} ds\ ,$$

where $F = F(aR + h)$ and $h(t) = U^{-1}e^{-i\theta}(e^{-iAt}U\xi_\infty + g(t))$. Our initial data are

$$\omega(t) \equiv 0,\quad \theta(t) \equiv 0\,,\quad a(t) \equiv 0\,,\quad g(t) \equiv 0\,.$$

Given $(\omega,\theta,a,g) \in \mathcal{A}$, using this assumption and (8.4) and (8.5), we have

$$\||g|^2 g\|_{H^2} \le \|g\|_{H^2}^3 \le C\varepsilon^{6-3\sigma}\langle t\rangle^{-6}\,,$$

$$\|F\|_{H^2} \le C\varepsilon^2\langle t\rangle^{-3}\,,$$

$$|(\phi, F)| \le C\,\|\phi\|_{L^1\cap L^2}\, C\varepsilon^2\langle t\rangle^{-3}\,,$$

$$|\omega^{\triangle}(t)| \le C\varepsilon^{2-\sigma}\langle t\rangle^{-2} \le \varepsilon^{2-2\sigma}\langle t\rangle^{-2}\,,$$

$$|\theta^{\triangle}(t)| \le \int_\infty^t \varepsilon^{2-2\sigma}\langle s\rangle^{-2} ds \le \varepsilon^{2-3\sigma}\langle t\rangle^{-1}\,,$$

$$|a^{\triangle}(t)| \le \int_\infty^t C\varepsilon^2\langle s\rangle^{-3} ds \le \varepsilon^{2-\sigma}\langle t\rangle^{-2}\,,$$

$$\|g^{\triangle}(t)\|_{H^2} \le \int_\infty^t C\varepsilon^2\langle s\rangle^{-3} ds \le \varepsilon^{2-\sigma}\langle t\rangle^{-2}\,,$$

provided that ε is small enough. In the last line we have used Lemma 2.6 on the boundedness of e^{-itA} on $X \cap H^2$. We have also used the estimate

$$\|[U,i]h\|_{H^2} \le C\,\|\xi\|_{W^{2,\infty}} + C\,\|g\|_{H^2}\ .$$

The estimate of ξ is similar to (2.42) in Lemma 2.9, which in some sense says $[U, i]$ is a local operator. Although it is not stated in Lemma 2.9, the proof of Lemma 2.9 in Section 9 gives the estimate. We have shown that $(\omega^\triangle, \theta^\triangle, a^\triangle, g^\triangle) \in \mathcal{A}$, that is, our mapping maps $\mathcal{A}$ into itself.

Next we show that the mapping is a contraction. Given $(\omega_1, \theta_1, a_1, g_1)$ and $(\omega_2, \theta_2, a_2, g_2) \in \mathcal{A}$, we denote

$$\delta_0 = \sup_{0\le t<\infty} \left\{8\langle t\rangle^2|\delta\omega(t)| + \langle t\rangle|\delta\theta(t)| + 2^7\langle t\rangle^2|\delta a(t)| + \langle t\rangle^2 \|\delta g(t)\|_{H^2}\right\};$$

we know $\delta_0 \le C\varepsilon^{2-3\sigma}$. Note that F_0 is cancelled in δF and that $\delta(e^{i\theta}) = O(\delta\theta)$. We have (the norms of F, g, and h are taken in H^2)

$$\begin{aligned}
\|\delta(|g|^2 g)\|_{H^2} &\le C\|g\|^2\|\delta g\| \le C\varepsilon^{4-2\sigma}\delta_0\langle t\rangle^{-6},\\
|(\phi, \delta h)| &\le C_\phi\varepsilon\langle t\rangle^{-3/2}|\delta\theta| + C_\phi\|\delta g\| \le C_\phi\delta_0\langle t\rangle^{-2},\\
\|\delta F\|_{H^2} &\le C\|aR + h\|\cdot(\varepsilon\langle t\rangle^{-3/2}|\delta\theta| + \|\delta g\| + |\delta a|) \le C\varepsilon\delta_0\langle t\rangle^{-7/2},\\
|\delta\omega^\triangle(t)| &\le \frac{5}{4}|\delta a| + C\|\delta F\| + C\varepsilon\langle t\rangle^{-3/2}|(R, \delta h)| \le \frac{1}{64}\delta_0\langle t\rangle^{-2},\\
|\delta\theta^\triangle(t)| &\le \int_\infty^t |\delta\omega| ds \le \frac{1}{8}\delta_0\langle t\rangle^{-1},\\
|\delta a^\triangle(t)| &\le C\int_\infty^t \|\delta F\| + \varepsilon^{2-2\sigma}\langle s\rangle^{-2}(|\delta\omega| + |(Q, \delta h)|)ds \le 2^{-10}\delta_0\langle t\rangle^{-2},\\
\|\delta g^\triangle(t)\|_{H^2} &\le C\int_\infty^t \varepsilon^2\langle s\rangle^{-3}|\delta\theta| + \|\delta F\|,\\
&\quad + \varepsilon^{2-2\sigma}\langle s\rangle^{-2}(|\delta\omega| + |\delta a| + \delta_0\langle s\rangle^{-2})ds \le \frac{1}{8}\delta_0\langle t\rangle^{-2}.
\end{aligned}$$

Here we have used Lemma 2.6 and the localness of $[U, i]$ again. Therefore we have

$$\sup_{0\le t<\infty}\left\{8\langle t\rangle^2|\delta\omega^\triangle(t)| + \langle t\rangle|\delta\theta^\triangle(t)| + 2^7\langle t\rangle^2|\delta a^\triangle(t)| + \langle t\rangle^2\|\delta g^\triangle(t)\|_{H^2}\right\} \le \frac{1}{2}\delta_0,$$

and thus our map is a contraction. We conclude that we do have solutions $\widetilde{h}$ with the main profile $e^{-iAt}U\xi_\infty$.

Recall $\psi_{\rm as}(t) = Q_E e^{-iEt+i\theta(t)} + e^{-iEt}e^{t\mathcal{L}}\xi_\infty$. By (8.2) we have

$$\begin{aligned}
\psi(t) &= [Q + aR + U^{-1}e^{-i\theta}(e^{-iAt}U\xi_\infty + g)]e^{-iEt+i\theta(t)}\\
&= \psi_{\rm as}(t) + [aR + U^{-1}e^{-i\theta}g]e^{-iEt+i\theta(t)} + e^{-iEt+i\theta}[U^{-1}, e^{-i\theta}]e^{-iAt}U\xi_\infty.
\end{aligned}$$

Since $a(t)$, $\|g(t)\|_{H^2} = O(t^{-2})$, and by the localness of $[U^{-1}, i]$

$$[U^{-1}, e^{-i\theta}]e^{-iAt}U\xi_\infty = -\sin\theta[U, i]e^{-iAt}U\xi_\infty = O(t^{-1}) \cdot O(t^{-3/2}) \,,$$

we have $\psi(t) - \psi_{\rm as}(t) = O(t^{-2})$ in H^2. Hence Theorem 1.5 is proved.

Finally, we address the remark after Theorem 1.5. Rewrite $\mathcal{L} = i(\Delta + E) + V_1$ where $V_1 = i^{-1}(V + 2\lambda Q^2 + \lambda Q^2 \mathbf{C})$ is a local operator. One may replace the definition of $\xi(t)$ by $\xi(t) = U^{-1}e^{-i\theta(t)}U\widetilde{\xi}(t)$ with

$$\widetilde{\xi}(t) = \mathbf{P}_{\rm c}^{\mathcal{L}} e^{i(\Delta+E)t}\chi_\infty + \int_\infty^t e^{(t-s)\mathcal{L}}\,\mathbf{P}_{\rm c}^{\mathcal{L}} V_1\, e^{i(\Delta+E)s}\chi_\infty\, ds \,, \tag{8.7}$$

and proceed as in the previous proof to construct a solution $h = U^{-1}e^{-i\theta}(U\widetilde{\xi} + g)$. The assumption $\hat{\chi}_\infty(0) = 0$ and $\nabla\hat{\chi}_\infty(0) = 0$ ensures that $e^{i(\Delta+E)t}\chi_\infty$ has a local decay of order $O(t^{-7/2})$; see [5, lemma 5.2]. Hence the difference between $\widetilde{\xi}(t)$ and $e^{i(\Delta+E)t}\chi_\infty$ is of order $O(t^{-5/2})$ in H^2. Thus globally $F_0(t) = O(t^{-3})$ and locally $F_0(t) = O(t^{-5})$. Hence $a, \dot{\theta} = O(t^{-4})$ and $\theta(t) = O(t^{-3})$. Therefore in H^2

$$\begin{aligned}\psi(t) &= [Q + U^{-1}e^{-i\theta(t)}U\widetilde{\xi}(t)]e^{-iEt+i\theta(t)} + O(t^{-2}) \\ &= Qe^{-iEt} + e^{i\Delta t}\chi_\infty + O(t^{-2}) \,.\end{aligned}$$

Here we use again that $e^{i\Delta t}\chi_\infty$ has a fast local decay and that $[U^{-1}, i]$ is a local operator. Note that (8.7) is in fact an expansion of $e^{t\mathcal{L}}W_{\mathcal{L}}\chi_\infty$ by the Duhamel's formula, where $W_{\mathcal{L}}$ is the wave operator of $\mathcal{L}$ defined in (2.36). See [5, section 5] for a similar argument.

Remark (Remark on Dispersion-Dominated Solutions to Klein-Gordon Equations). We now sketch a construction for dispersion-dominated solutions to Klein-Gordon equations. We follow the notation in the introduction. For a specified profile $\eta_\pm$, let $u = \xi + g$ where $\xi(t) = e^{iBt}\eta_+ + e^{-iBt}\eta_-$ and g denotes the rest. Then we have

$$(\partial_t^2 + B^2)\xi = 0 \,, \quad (\partial_t^2 + B^2)g = \lambda(\xi + g)^3 \,.$$

Hence $g(t)$ satisfies

$$g(t) = \int_\infty^t \left\{e^{iB(t-s)} - e^{-iB(t-s)}\right\} \frac{1}{2iB}\lambda(\xi + g)^3 ds \,.$$

Since the main source term is bounded by $\|\xi^3\|_2 \le \|\xi\|_\infty^2 \|\xi\|_2 \le Ct^{-3}$, we have $\|g\|_2 \le \int_\infty^t Cs^{-3}\, ds \le Ct^{-2}$. Then we proceed as in Section 8 to construct one such solution by a contraction mapping argument.

9 Proof of Linear Estimates

We now proceed to prove the lemmas in Section 2.4. To simplify the presentation, we will assume $\lambda > 0$. The proof for the case $\lambda < 0$ is exactly the same.

Recall that X is the space of all functions in $L^2(\mathbb{R}^3)$ that are orthogonal to Q, and Π is the orthogonal projection from $L^2(\mathbb{R}^3)$ onto X. In what follows we will only consider the restrictions of H and A on X. Hence we often omit the projection

Π in the definition of A. (It should be noted, however, H_* acts on $L^2(\mathbb{R}^3)$.) In the rest of this section, when we write L^2, $W^{k,p}$, or L^2_r, we often mean their intersection with X: $L^2 \cap X$, $W^{k,p} \cap X$, or $L^2_r \cap X$.

We recall assumption A2 on V. We assume that 0 is neither an eigenvalue nor a resonance for $-\Delta + V$. We also assume that V satisfies the assumption in Yajima [20] so that the $W^{k,p}$ estimates for $k \le 2$ for the wave operator W_H holds: For a small $\sigma > 0$,

$$|\nabla^\alpha V(x)| \le C\langle x\rangle^{-5-\sigma} \quad \text{for } |\alpha| \le 2\,.$$

Also, the functions $(x \cdot \nabla)^k V$, for $k = 0, 1, 2, 3$, are $-\Delta$ bounded with an $-\Delta$ bound less than 1:

$$\|(x\cdot\nabla)^k V\phi\|_2 \le \sigma_0\,\|-\Delta\phi\|_2 + C\,\|\phi\|_2\,, \quad \sigma_0 < 1\,,\ k = 0, 1, 2, 3. \tag{9.1}$$

By the assumption, the following operators are bounded in L^2:

$$H_*^{-1/2}(x\cdot\nabla)^k V H_*^{-1/2}\,, \quad (x\cdot\nabla)^k V H_*^{-1}\,, \quad H_*^{-1}(x\cdot\nabla)^k V\,, \tag{9.2}$$

for $k = 0, 1, 2, 3$.

Since Q is the ground state of H with V satisfying the previous assumptions, Q is a smooth function with exponential decay at infinity. Hence the above statements on V also hold for Q and Q^2. Since $V + \lambda Q^2$ and V have the same properties, in what follows we will replace $V + \lambda Q^2$ in H by V and write $H = H_* + V$ to make the presentation simpler. So it should be kept in mind that the potential V in this section is in fact $V + \lambda Q^2$.

For two operators S and T with bounded inverses, S is said to be *T-bounded* if ST^{-1} is a bounded operator. If both S and T are self-adjoint, this implies $T^{-1}S$ is also bounded. A deeper result says $S^{1/2}$ is $T^{1/2}$-bounded; see [11, theorem X.18]. We say S and T are *mutually bounded* if both ST^{-1} and TS^{-1} are bounded operators. This is the case if $\|(S-T)T^{-1}\|_{(L^2,L^2)} = \theta < 1$ for some θ. (It implies immediately that $\|ST^{-1}\| < 2$. Since $\|T\phi\| \le \|S\phi\| + \|(T-S)\phi\| \le \|S\phi\| + \theta\,\|T\phi\|$, we have $\|T\phi\| \le C\,\|S\phi\|$, which implies T is S-bounded.)

LEMMA 9.1 *For each $k = \frac{1}{2}, 1, \frac{3}{2}, 2, 3$, the operators H_*^k, H^k, and A^k are mutually bounded.*

PROOF: That H_*^k and H^k are mutually bounded follows from our assumption on V by standard argument. To show H^k and A^k are mutually bounded, it suffices to prove the cases $k = 2$ and $k = 3$ by the previous remark. We first show $\|(A^2 - H^2)H^{-2}\| < 1$, which implies the case $k = 2$.

$$\begin{aligned}\left\|(A^2 - H^2)H^{-2}\right\| &= \left\|H^{1/2}\lambda Q^2 H^{1/2} H^{-2}\right\| \\ &\le \left\|H^{1/2}\lambda Q^2 H^{-1}\right\| \le \left\|H_*^{1/2}\lambda Q^2 H_*^{-1}\right\| \le 1/2\,.\end{aligned}$$

The last inequality can be obtained by writing

$$\begin{aligned}H_*^{1/2} Q^2 H_*^{-1} &= H_*^{-1/2} Q^2 + H_*^{1/2}[Q^2, H_*^{-1}] \\ &= H_*^{-1/2} Q^2 + H_*^{-1/2}[Q^2, H_*] H_*^{-1}\,,\end{aligned}$$

and noting $[Q^2, H_*] = \Delta Q^2 + 2\nabla Q^2 \cdot \nabla$.

To prove the case $k = 3$, it suffices to prove $A^6 \le CH^6$ and $H^6 \le CA^6$. Note that

$$(f A^6 f) = (f A^2 A^2 A^2 f) \le (f A^2 H^2 A^2 f).$$

Since $A^2 = H^2 + H^{1/2}\lambda Q^2 H^{1/2}$, we have

$$(f A^2 H^2 A^2 f) \le \\ C(f H^2 H^2 H^2 f) + C(f (H^{1/2}\lambda Q^2 H^{1/2}) H^2 (H^{1/2}\lambda Q^2 H^{1/2}) f)$$

where the cross terms are estimated by the Schwarz inequality. To show that the last term is bounded by $C(f H^6 f)$, we shall show that $H^{3/2} Q^2 H^{-5/2}$ is bounded in X. Rewrite

$$\begin{aligned} H^{3/2} Q^2 H^{-5/2} &= H^{3/2} H^{-2} Q^2 H^{-1/2} + H^{3/2} [Q^2, H^{-2}] H^{-1/2} \\ &= H^{-1/2} Q^2 H^{-1/2} + H^{-1/2} [Q^2, H^2] H^{-5/2}. \end{aligned}$$

Since $[Q^2, H^2]$ is of the form $\sum_{|\alpha| \le 3} G_\alpha(x) \nabla^\alpha$, the operators on the right side of the equation are bounded in X. This shows $A^6 \le CH^6$. That $H^6 \le CA^6$ is proven similarly. □

Recall the standard formula

$$T^{-\sigma} = \int_0^\infty \frac{1}{s+T} \frac{ds}{s^\sigma}, \qquad 0 < \sigma < 1. \tag{9.3}$$

The operator T in the above formula will be A^2 or H. Hence we also need to estimate operators of the form $\frac{H^m}{s+H^2}$. Clearly, for $s \ge 0$,

$$\left\| \frac{H^2}{s+H^2} \right\|_{(W^{k,p}, W^{k,p})} \le 1, \quad \left\| H^{-1/2} \right\|_{(W^{k,p}, W^{k,p})} \le C. \tag{9.4}$$

LEMMA 9.2 *Let $s \ge 0$. The operator $H/(s+H^2)$ is bounded in $W^{k,p} \cap X$ with*

$$\left\| \frac{H}{s+H^2} \right\|_{(W^{k,p}, W^{k,p})} \le C\langle s\rangle^{-1/2}. \tag{9.5}$$

Also, for $k = \pm 1, \pm 2, \pm 3$,

$$\begin{aligned} \left\| \langle x\rangle^k \frac{1}{s+H} \langle x\rangle^{-k} \right\|_{(L^2, L^2)} &\le C\langle s\rangle^{-1}, \\ \left\| \langle x\rangle^k \frac{H}{s+H^2} \langle x\rangle^{-k} \right\|_{(L^2, L^2)} &\le C\langle s\rangle^{-1/2}. \end{aligned} \tag{9.6}$$

PROOF: We can rewrite

$$\frac{H}{s+H^2} = \frac{1}{H+\sqrt{s}i} + \frac{1}{H-\sqrt{s}i}.$$

Therefore, to prove statements for $H/(s+H^2)$, it suffices to prove the corresponding statements for $1/(H \pm \sqrt{s}i)$. We first prove (9.5) for $k=0$. Let κ_1 denote the eigenvalue of the excited state of H, and let P_1 denote the projection onto the corresponding eigenspace. We can write

$$\left. \frac{1}{H+\sqrt{s}i} \right|_X = \frac{1}{\kappa_1+\sqrt{s}i} P_1 + W_H \frac{1}{p^2 - E + \sqrt{s}i} W_H^* \mathbf{P}_c^H ,$$

where $p = -i\nabla$ and W_H is the wave operator of H. Note $E < 0$. Since W_H and W_H^* are bounded in $W^{k,p}$ for sufficiently nice V [20], it is sufficient to prove that $1/(p^2 - E \pm \sqrt{s}i)$ are bounded in $W^{k,p}$. However, $1/(p^2 - E \pm \sqrt{s}i)$ are convolution operators with explicit Green functions:

$$\frac{C}{|x|} e^{-|x|(-E\pm\sqrt{s}i)^{1/2}} .$$

Since $|e^{-|x|(-E\pm\sqrt{s}i)^{1/2}}| \le e^{-c|x|\langle s\rangle^{1/4}}$, the L^1 norms of the Green functions are bounded by $\langle s\rangle^{-1/2}$. By Young's inequality we have

$$\left\| \frac{1}{p^2 - E \pm \sqrt{s}i} \right\|_{(L^p, L^p)} \le C\langle s\rangle^{-1/2} ,$$

which proves (9.5) for $k=0$. For $k \ge 1$ and for $\phi \in W^{k,p}$, we have

$$\begin{aligned} \left\| H^{k/2} \frac{H}{s+H^2} \phi \right\|_{W^{k,p}} &\sim \left\| H^{k/2} \frac{H}{s+H^2} \phi \right\|_{L^p} = \left\| \frac{H}{s+H^2} H^{k/2} \phi \right\|_{L^p} \\ &\le C\langle s\rangle^{-1/2} \left\| H^{k/2}\phi \right\|_{L^p} \sim \langle s\rangle^{-1/2} \|\phi\|_{W^{k,p}} . \end{aligned}$$

This proves (9.5) for $k \ge 1$.

For (9.6), we prove the second part. The proof for the first part is similar. For $k > 0$, since

$$\left[\langle x\rangle^k, \frac{1}{H+\sqrt{s}i} \right] = \frac{1}{H+\sqrt{s}i} [\langle x\rangle^k, H+\sqrt{s}i] \frac{1}{H+\sqrt{s}i}$$

and

$$[\langle x\rangle^k, H+\sqrt{s}i] = 2\nabla^*(\nabla\langle x\rangle^k) - (\Delta\langle x\rangle^k) ,$$

we have

$$\begin{aligned} &\left\| \left[\langle x\rangle^k, \frac{1}{H+\sqrt{s}i} \right] \langle x\rangle^{-k} \right\| \\ &\quad \le C \left\| \frac{1}{H+\sqrt{s}i} (\nabla^* + 1) \right\| \cdot \left\| \langle x\rangle^{k-1} \frac{1}{H+\sqrt{s}i} \langle x\rangle^{-k+1} \right\| \\ &\quad \le C\langle s\rangle^{-1/2} \end{aligned}$$

by induction in k. We have the same estimate for $[\langle x\rangle^k, \frac{1}{H-\sqrt{s}i}]$, and hence (9.6) holds for positive k. The proof for the case $k < 0$ is similar. □

Recall L^2_r is the weighted L^2 space with norm $\|f\|_{L^2_r} = \|\langle x\rangle^r f\|_{L^2}$.

LEMMA 9.3 *The operators $H^{1/2}A^{-1/2}$, $A^{-1/2}H^{1/2}$, $H^{-1/2}A^{1/2}$, and $A^{1/2}H^{-1/2}$ are bounded operators in $W^{k,p}\cap X$ and $L^2_r \cap X$.*

PROOF: By (9.3) we can write

$$\begin{aligned}
H^{1/2}A^{-1/2} &= H^{1/2}\int_0^\infty \frac{1}{s+H^2+H^{1/2}\lambda Q^2 H^{1/2}}\frac{ds}{s^{1/4}}\\
&= H^{1/2}\int_0^\infty \Bigg[\frac{1}{s+H^2}+\frac{1}{s+H^2}H^{1/2}\lambda Q\\
&\qquad \sum_{j=0}^{\infty}\left(\lambda Q\frac{H}{s+H^2}Q\right)^j QH^{1/2}\frac{1}{s+H^2}\Bigg]\frac{ds}{s^{1/4}}\\
&= 1+\int_0^\infty\left[\frac{H}{s+H^2}\lambda Q\sum_{j=0}^{\infty}\left(\lambda Q\frac{H}{s+H^2}Q\right)^j Q\frac{H}{s+H^2}H^{-1/2}\right]\frac{ds}{s^{1/4}}\,.
\end{aligned}$$

Since $\|\frac{H}{s+H^2}\| \le \langle s\rangle^{-1/2}$ by Lemma 9.2, we have

$$\begin{aligned}
\left\|H^{1/2}A^{-1/2}\right\|_{(W^{k,p},W^{k,p})} &\le 1 + C\int_0^\infty \langle s\rangle^{-1/2} n\sum_{j=0}^{\infty}(n^2\langle s\rangle^{-1/2})^j n\langle s\rangle^{-1/2}\frac{ds}{s^{1/4}}\\
&\le 1+Cn^2\,.
\end{aligned}$$

Similarly,

$$\left\|A^{-1/2}H^{1/2}\right\|_{(W^{k,p},W^{k,p})} \le 1+Cn^2\,.$$

Also, using (9.6), for $r\le 3$ we have

$$\left\|H^{1/2}A^{-1/2}\right\|_{(L^2_r,L^2_r)} + \left\|A^{-1/2}H^{1/2}\right\|_{(L^2_r,L^2_r)} \le 1+Cn^2\,.$$

The above proves that $H^{1/2}A^{-1/2}$ and $A^{-1/2}H^{1/2}$ are bounded in $W^{k,p}$ and L^2_r. Indeed, we have proven

$$\begin{aligned}
&\left\|\langle x\rangle^3(H^{1/2}A^{-1/2}-1)\langle x\rangle^3\right\|_{(L^2,L^2)}\\
&\qquad + \left\|\langle x\rangle^3(A^{-1/2}H^{1/2}-1)\langle x\rangle^3\right\|_{(L^2,L^2)} \le Cn^2\,.
\end{aligned}\tag{9.7}$$

We now consider $H^{-1/2}A^{1/2}$ and $A^{1/2}H^{-1/2}$. Since

$$A^{1/2} = A^2A^{-3/2} = A^2\int_0^\infty\frac{1}{s+A^2}\frac{ds}{s^{3/4}}\,,$$

we have

$$\begin{aligned} H^{-1/2}A^{1/2} &= H^{-1/2}(H^2 + H^{1/2}\lambda Q^2 H^{1/2}) \int_0^\infty \frac{1}{s + H^2 + H^{1/2}\lambda Q^2 H^{1/2}} \frac{ds}{s^{3/4}} \\ &= (H^{3/2} + \lambda Q^2 H^{1/2}) \int_0^\infty \frac{1}{s + H^2 + H^{1/2}\lambda Q^2 H^{1/2}} \frac{ds}{s^{3/4}} \\ &= I_1 + \lambda Q^2 I_2 \,. \end{aligned}$$

The main term is I_1. The term I_2 is similar to $H^{1/2}A^{-1/2}$, and its integrand has a better decay in s for large s. Hence

$$\left\| \lambda Q^2 I_2 \right\| \le C\lambda \left\| I_2 \right\| \le Cn^2 \,.$$

For the main term I_1,

$$I_1 = 1 + \int_0^\infty \frac{H^2}{s+H^2} Q \sum_{j=0}^\infty \left(Q \frac{H}{s+H^2} Q \right)^j Q \frac{H}{s+H^2} H^{-1/2} \frac{ds}{s^{3/4}} \,.$$

Hence

$$\begin{aligned} \|I_1\| &\le 1 + C \int_0^\infty n \sum_{j=0}^\infty (n^2 \langle s \rangle^{-1/2})^j n \langle s \rangle^{-1/2} \frac{ds}{s^{3/4}} \\ &\le 1 + C \int_0^\infty n^2 \langle s \rangle^{-1/2} \frac{ds}{s^{3/4}} \le 1 + Cn^2 \,. \end{aligned}$$

Here the norms are taken in $(W^{k,p}, W^{k,p})$ and (L^2_r, L^2_r). Hence we have proven Lemma 9.3. □

In fact, the last part of the above proof also shows

$$\left\| \langle x \rangle^3 (H^{-1/2}A^{1/2} - 1) \langle x \rangle^3 \right\|_{(L^2,L^2)} + \left\| \langle x \rangle^3 (A^{1/2}H^{-1/2} - 1) \langle x \rangle^3 \right\|_{(L^2,L^2)} \le Cn^2 \,. \tag{9.8}$$

PROOF OF LEMMA 2.9: The above lemma proves that U_0 and U_0^{-1} are bounded in $W^{k,p}$ and L^2_r. Moreover, (9.7) and (9.8) mean that $U_0 - 1$ and $U_0^{-1} - 1$ are "local" operators. In particular, they imply $\|[U_0, i]\,\phi\|_{L^{8/7} \cap L^{4/3}} \le C\,\|\phi\|_{L^4}$. Since

$$U = U\widetilde{\Pi} \quad \text{where } \widetilde{\Pi} = \begin{bmatrix} P_1 & 0 \\ 0 & \Pi \end{bmatrix}$$

is a bounded projection, U is also bounded. Similarly U^{-1} is also bounded. Moreover, we have

$$[U, i] = U_0[\widetilde{\Pi}, i] + [U_0, i]\widetilde{\Pi} = U_0(P_1 - \Pi) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + [U_0, i]\widetilde{\Pi} \,,$$

where by (2.18) $P_1 - \Pi = -c_1 |\Pi R\rangle \langle Q|$ is a local operator; hence $[U, i]$ satisfies the last estimate in Lemma 2.9. Lemma 2.9 is thus proven. □

PROOF OF LEMMA 2.10: We need only to prove the statement on W_A. Note that the estimates (9.7) and (9.8) also show, for any $\phi \in L^2$,

$$(U^{\pm 1} - 1)e^{-itH_*}\phi \to 0 \quad \text{in } L^2 \qquad \text{as } t \to \infty\,. \tag{9.9}$$

Notice

$$\begin{aligned} W_A = \lim_{t\to\infty} e^{itA}e^{-itH_*} &= \lim_{t\to\infty} Ue^{t\mathcal{L}}U^{-1}e^{-itH_*} \\ &= \lim_{t\to\infty} Ue^{t\mathcal{L}}e^{-itH_*} + \lim_{t\to\infty} Ue^{t\mathcal{L}}(U^{-1}-1)e^{-itH_*}\,. \end{aligned}$$

By (9.9) we have

$$W_A = \lim_{t\to\infty} Ue^{t\mathcal{L}}e^{-itH_*} = UW_{\mathcal{L}}\,.$$

The boundedness of W_A follows from that of U and $W_{\mathcal{L}}$. This proves Lemma 2.10. □

PROOF OF LEMMA 2.7: Since

$$e^{-itA}\mathbf{P}_c^A\phi = W_A e^{-itH_*}W_A^*\mathbf{P}_c^A\phi\,,$$

the estimate (2.39) follows from the usual (L^p, L^q) estimate for e^{-itH_*} and the boundedness of W_A and $\mathbf{P}_c^A$ in L^p spaces. To prove (2.40), either we prove the boundedness of W_A in weighted spaces L_r^2, or we use the Mourre estimate. We will follow the second approach and the argument in [16].

Let $a = 2\kappa$. We consider intervals $\Delta = (a-r, a+r)$. Let $g_\Delta(t) = g_0((t-a)/r)$, where g_0 is a fixed smooth function with support in $(-2,2)$ and $g_0(t) = 1$ for $|t| < 1$. We will consider $g_\Delta(A)$ with r small enough. Let $D = xp + px$, $p = -i\nabla$, and the commutators

$$\mathrm{ad}_D^0(A) = A\,, \quad \mathrm{ad}_D^{k+1}(A) = [\mathrm{ad}_D^k(A), D]\,.$$

We need to prove the following lemma:

LEMMA 9.4 *For Δ small enough, the Mourre estimate*

$$g_\Delta(A)[iA, D]g_\Delta(A) \geq \theta g_\Delta(A)^2$$

holds for some $\theta > 0$. Also, $g_\Delta(A)\,\mathrm{ad}_D^k(A)g_\Delta(A)$ are bounded operators in L^2 for $k = 0, 1, 2, 3$.

We will use the following lemma:

LEMMA 9.5 *The operators*

$$H^{-3}D^kH^{m/2}\langle x\rangle^{-3} \quad \text{and} \quad \langle x\rangle^{-3}H^{m/2}D^kH^{-3}$$

are bounded in L^2 for $k, m = 0, 1, 2, 3$.

PROOF: This is standard and we only sketch the proof. If m is even, we can compute the commutator $[D^k, H^{m/2}]$ explicitly and estimate

$$H^{-3}D^kH^{m/2}\langle x\rangle^{-3} = H^{-3}H^{m/2}D^k\langle x\rangle^{-3} + H^{-3}[D^k, H^{m/2}]\langle x\rangle^{-3}\,.$$

If m is odd, we write

$$H^{-3}D^k H^{m/2}\langle x\rangle^{-3} = \int_0^\infty H^{-3}D^k H^{(m+1)/2}\frac{1}{s+H}\langle x\rangle^{-3}\frac{ds}{\sqrt{s}}$$

and proceed as in the case when m is even, by using (9.6). Here we have used formula (9.3). ☐

PROOF OF LEMMA 9.4: Let $G = A - H$ and write $A = H + G$. Since

$$\begin{aligned}[iA, D] = [H_* + V + G, iD] &= -\Delta + [V + G, iD]\\ &= A - V - G + [V + G, iD]\end{aligned}$$

and $g_\Delta(A)Ag_\Delta(A) \ge 2\theta g_\Delta(A)^2$ for some $2\theta > 0$, it suffices to show that, for $M = -V + [V, iD]$, $-G$, and $[G, D]$, the operators

$$g_\Delta(A)Mg_\Delta(A) = (g_\Delta(A)H_*^2)(H_*^{-2}MH_*^{-2})\,(H_*^2 g_\Delta(A))$$

are bounded by $g_\Delta(A)^2$, and the bound goes to zero when the interval Δ shrinks to zero. Since both

$$g_\Delta(A)H_*^2 = (g_\Delta(A)A^2)(A^{-2}H_*^2) \quad \text{and} \quad H_*^2 g_\Delta(A) = (H_*^2A^{-2})(A^2 g_\Delta(A))$$

are bounded and converge to zero weakly when Δ shrinks to zero, this will be true if one can show that $H_*^{-2}MH_*^{-2}$ is compact. The case $M = -V + [V, iD]$ is standard and follows from our assumption, so we only consider $H_*^{-2}GH_*^{-2}$ and $H_*^{-2}[G, D]H_*^{-2}$.

We proceed to find an explicit form of G. By (9.3) with $T = A^2$, $\sigma = \frac{1}{2}$, we write

$$\begin{aligned}A^{-1} &= \int_0^\infty \frac{1}{s + H^2 + H^{1/2}\lambda Q^2 H^{1/2}}\frac{ds}{\sqrt{s}}\\ &= \int_0^\infty \frac{1}{s+H^2} + \frac{1}{s+H^2}H^{1/2}\lambda Q\sum_{j=0}^\infty\left(\lambda Q\frac{H}{s+H^2}Q\right)^j QH^{1/2}\frac{1}{s+H^2}\frac{ds}{\sqrt{s}}\\ &= H^{-1} + H^{-1/2}\langle x\rangle^{-3}J_0\langle x\rangle^{-3}H^{-1/2}\end{aligned}$$

where

$$J_0 = \int_0^\infty \langle x\rangle^3\frac{H}{s+H^2}\lambda Q\sum_{j=0}^\infty\left(\lambda Q\frac{H}{s+H^2}Q\right)^j Q\frac{H}{s+H^2}\langle x\rangle^3\frac{ds}{\sqrt{s}}\,.$$

By Lemma 9.2,

$$\|J_0\|_{(L^2,L^2)} \le \int_0^\infty \langle s\rangle^{-1/2}\cdot n^2\cdot\langle s\rangle^{-1/2}s^{-1/2}\,ds \le Cn^2\,.$$

Hence

$$
\begin{aligned}
A &= A^2 A^{-1} \\
&= (H^2 + H^{1/2}\lambda Q^2 H^{1/2}) \left(H^{-1} + H^{-1/2}\langle x\rangle^{-3} J_0 \langle x\rangle^{-3} H^{-1/2}\right) \\
&= H + G\,, \qquad (9.10)\\
G &= H^{1/2}\lambda Q^2 H^{-1/2} + H^{3/2}\langle x\rangle^{-3} J_0 \langle x\rangle^{-3} H^{-1/2} \\
&\quad + H^{1/2}\lambda Q^2 \langle x\rangle^{-3} J_0 \langle x\rangle^{-3} H^{-1/2}\,.
\end{aligned}
$$

Since $H^{-1/2}\langle x\rangle^{-1}$ and $\langle x\rangle^{-1}H^{-1/2}$ are compact, from (9.10) $H_*^{-2}GH_*^{-2}$ is compact. We can also write

$$
H_*^{-2}GDH_*^{-2} = \left\{H_*^{-2}GH^{1/2}\langle x\rangle\right\}\cdot\left\{\langle x\rangle^{-1}H^{-1/2}DH_*^{-2}\right\}.
$$

The second operator is bounded by Lemma 9.5. The first is compact since its terms are of the form $H^{-m}\cdot\langle x\rangle^{-k}\cdot$ (bounded operator). Similarly, $H_*^{-2}DGH_*^{-2}$ is also compact. Hence we conclude the Mourre estimate.

To show that $g_\Delta(A)\,\mathrm{ad}_D^k(A) g_\Delta(A)$ are bounded for $k = 0, 1, 2, 3,$ we rewrite

$$
\begin{aligned}
&g_\Delta(A)\,\mathrm{ad}_D^k(A) g_\Delta(A) = \\
&\qquad (g_\Delta(A)A^3)(A^{-3}H^3)(H^{-3}\,\mathrm{ad}_D^k(A)H^{-3})(H^3A^{-3})(A^3 g_\Delta(A))\,.
\end{aligned}
$$

We need only to show that $H^{-3}\,\mathrm{ad}_D^k(A)H^{-3}$ are bounded since the other terms are bounded by Lemma 9.1. Recall $A = H + G$. It is standard to prove that $H^{-3}\,\mathrm{ad}_D^k(H)H^{-3}$ is bounded. For $H^{-3}\,\mathrm{ad}_D^k(G)H^{-3}$, since it is a sum of terms of the form

$$
H^{-3}D^kGD^mH^{-3}\,, \quad k+m \le 3\,,
$$

it suffices to show that these terms are bounded. By the explicit form (9.10) of G and Lemma 9.5, they are indeed bounded. For example,

$$
\begin{aligned}
&H^{-3}D^2\left\{H^{3/2}\langle x\rangle^{-3}J_0\langle x\rangle^{-3}H^{-1/2}\right\}D^1H^{-3} = \\
&\qquad \left\{H^{-3}D^2H^{3/2}\langle x\rangle^{-3}\right\} J_0 \left\{\langle x\rangle^{-3}H^{-1/2}D^1H^{-3}\right\}\,,
\end{aligned}
$$

a product of three bounded operators. We conclude that $g_\Delta(A)\,\mathrm{ad}_D^k(A)g_\Delta(A)$ are bounded for $k = 0, 1, 2, 3$. □

With Lemma 9.4 (cf. the remark in [16, p. 27]), the minimal velocity estimate in [6] and theorem 2.4 of [14] implies

$$
\left\| F(D \le \theta t/2)e^{-iAt}g_\Delta(A)\,\langle D\rangle^{-3/2}\right\|_{(L^2,L^2)} \le C\langle t\rangle^{-5/4}\,.
$$

The same argument in [16] then gives the desired decay estimate (2.40). □

PROOF OF LEMMA 2.8: Let $\psi_n = \mathbf{P}_c^A \Pi\phi_0\phi_1^2$ and $\psi_0 = \mathbf{P}_c^{H_1}\phi_0\phi_1^2$. Recall $H_1 = -\Delta + V - e_0$. We have $\psi_n = \psi_0 + O(n^2)$. We write $\psi_n = \psi_0 + b\phi_1 + \eta$, where

$\eta \in \mathbf{H}_c(H_1)$ and $b, \eta = O(n^2)$. Rewrite

$$\left(\psi_n, \operatorname{Im} \frac{1}{A - 0i - 2\kappa} \psi_n\right)$$

$$= \operatorname{Im} i \int_0^\infty \left(\psi_n, e^{-it(A-0i-2\kappa)} \psi_n\right) dt$$

(9.11)

$$= \operatorname{Im} i \int_0^\infty \left(\psi_n, e^{-it(H_1-0i-2\kappa)} \psi_n\right) dt$$

(9.12)

$$+ \operatorname{Im} i \int_0^\infty \int_0^t \left(\psi_n, e^{-i(t-s)(A-0i-2\kappa)} (\lambda Q^2 + G) e^{-is(H_1-0i-2\kappa)} \psi_n\right) ds\, dt\,.$$

The main term lies in (9.11). It is

$$\operatorname{Im} i \int_0^\infty \left(\psi_0, e^{-it(H_1-0i-2\kappa)} \psi_0\right) dt = \left(\psi_0, \operatorname{Im} \frac{1}{H_1 - 0i - 2\kappa} \psi_0\right),$$

which is the desired main term in Lemma 2.8.

We want to show that the rest of (9.11) and (9.12) are integrable and of order $O(n^2)$. Recall that we write $\psi_n = \psi_0 + b\phi_1 + \eta$. For the term η in ψ_n, by the decay estimate we have

$$\text{(9.13)} \quad \left| \left(\psi_n, e^{-it(H_1-0i-2\kappa)} \eta\right) \right| \le C\langle t\rangle^{-3/2} \|\psi_n\|_{L^1\cap L^2} \|\eta\|_{L^1\cap L^2} \le C\langle t\rangle^{-3/2} n^2\,;$$

hence this term is integrable. Also, since $H_1\phi_1 = e_{01}\phi_1$,

$$\left(\psi_n, e^{-it(H_1-0i-2\kappa)} b\phi_1\right) = \left(\psi_n, e^{-it(e_{01}-2\kappa-0i)} b\phi_1\right),$$

so we can integrate this oscillation term explicitly. (The boundary term at $t = \infty$ vanishes due to the decay of $e^{-it(-0i)}$.) We conclude that the rest of (9.11) is integrable and of order $O(n^2)$.

For (9.12), it suffices to show its integrability since $\lambda Q^2 + G$ gives the order $O(n^2)$. Rewrite the last ψ_n in (9.12) as $b\phi_1 + \mathbf{P}_c^{H_1}\psi_n$. For the part containing $b\phi_1$, we have

$$\left(\psi_n, e^{-i(t-s)(A-0i-2\kappa)} (\lambda Q^2 + G) e^{-is(H_1-0i-2\kappa)} b\phi_1\right) = \left(\psi_n, e^{-it(A-0i-2\kappa)}\, e^{is(A-e_{01})} (\lambda Q^2 + G) b\phi_1\right).$$

Integration in s gives

$$\left((A - e_{01})^{-1}\psi_n, e^{-it(A-0i-2\kappa)} (\lambda Q^2 + G) b\phi_1\right).$$

Since e_{01} lies outside the continuous spectrum of A, the last expression is integrable in t following the same argument as (9.13). For the part containing $\mathbf{P}_c^{H_1}\psi_n$, since $(\lambda Q^2 + G)$ is a "local" operator in the sense that it sends L^∞ functions to L^1, we have

$$\left| \left(\psi_n, e^{-i(t-s)(A-0i-2\kappa)}(\lambda Q^2+G)e^{-is(H_1-0i-2\kappa)}\mathbf{P}_{\mathrm{c}}^{H_1}\psi_n\right) \right| \le \\ Cn^2 \langle t-s\rangle^{-3/2} \langle s\rangle^{-3/2} \|\psi_n\|^2_{L^1\cap L^2} \ ,$$

which can be integrated in s and t. Hence we have proven Lemma 2.8. □

PROOF OF LEMMA 2.6: Given $\phi \in H^k \cap X$, $k = 0, 1, 2$, let $u(t) = e^{-itA}\phi \in X$. We have $\frac{d}{dt}(u(t),\ A^m u(t)) = 0$; hence $(u(t),\ A^m u(t))$ is a conserved quantity. When $k = 0$, this implies that the L^2 norm is conserved. For $k = 2$, we have

$$\begin{aligned}(u, A^2 u) &= \left(u, (H^2 + H^{1/2}\lambda Q^2 H^{1/2})u\right) \\ &= (u, H^2 u) + O(n^2 \|u\|^2_{H^1}) \le C\|u\|^2_{H^2}\ .\end{aligned}$$

On the other hand, since $\|u\|^2_{H^1} \le C\|u\|^2_{H^2} + C\|u\|^2_{L^2}$ and $(u,u) \le C(u, A^2 u)$ due to the spectral gap of A, we have

$$\begin{aligned}C^{-1}\|u\|^2_{H^2} \le (u, H^2 u) &= (1 - O(n^2))^{-1}\left[(u, A^2 u) + O(n^2)\|u\|^2_{L^2}\right] \\ &\le C(u, A^2 u)\,.\end{aligned}$$

Thus we have $(u(t), A^2 u(t)) \approx \|u(t)\|^2_{H^2}$ in the sense of (1.23), and we have

$$\|u(t)\|^2_{H^2} \approx (u(t),\ A^2 u(t)) = (u(0),\ A^2 u(0)) \approx \|u(0)\|^2_{H^2}\ .$$

The case $k = 2$ is thus proven. The case $k = 1$ can be obtained by interpolation. □

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