Last revised 3:18 p.m. March 10, 2016

Essays in analysis

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Fourier series and elliptic regularity

This essay is in two parts. Part I is a brief introduction to 'soft' analysis on a compact torus, including Fourier series, distributions, and Sobolev spaces. My purpose is to make the logic of Sobolev spaces as clear as possible, and I give no examples.

As an application, in Part II it will be shown that solutions of a smooth elliptic differential equation on any open subset of $\mathbb{R}^n$ are smooth. In this latter part, I follow the sketch at the beginning of [Duistermaat:1996] combined with a more classical part taken from Appendix 4 of [Lang:1973], but as far as I can tell my exposition is significantly different from others. This seems to me the simplest approach to elliptic regularity. The appeal to analysis on the torus in order to prove results on Euclidean space seems a bit inelegant, but I know no method more efficient. . Another, in some ways more interesting, technique can be found in the lecture notes [Nelson:1960]. I wish to thank Ed Nelson for making a copy available to me.

I should perhaps remark that 'smooth' in this essay means C^∞.

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Part I. Fourier series

1. Smooth functions on a torus

Let $\mathbb{S}$ be $\mathbb{R}/\mathbb{Z}$. It may be identified with the unit circle

$$\bigl\{ z \in \mathbb{C} \bigm| |z| = 1 \bigr\}$$

via the exponential map $x \mapsto e^{2\pi i x}$ (but only upon making a choice of sign for $\sqrt{-1}$).

Let $\mathbb{T} = \mathbb{S}^n = \mathbb{R}^n/\mathbb{Z}^n$. For each integer $m \ge 0$ the space $C^m(\mathbb{T})$ of all functions on $\mathbb{T}$ with continuous derivatives of order up to m becomes a Banach space with norm

$$|||f|||_m = \sup_{\mathbb{T},\, |k| \le m} \bigl| \partial^k f / \partial x^k \bigr| .$$

Here $k = (k_j)$ is a multi-index and $|k| = \sum k_j$. The space $C^m(\mathbb{T})$ embeds continuously into $C^k(\mathbb{T})$ for $k < m$. The space C^∞ is the intersection of all these spaces with the norms $|||f|||_m$ as semi-norms, making it into a Fréchet space.

For two continuous functions f, g on $\mathbb{T}$ define the bilinear pairing

$$\langle f, g\rangle = \int_{\mathbb{T}} f(x)g(x)\,dx\,.$$

The pairing $f \bullet g = \langle f, \overline{g}\,\rangle$ is a positive definite Hermitian inner product.

Let

$$\mathcal{Z} = 2\pi i\mathbb{Z}\,.$$

For κ in $\mathcal{Z}^n$ let ε^κ define the character of $\mathbb{T}$

$$\varepsilon^\kappa\colon\ x = (x_j) \longmapsto e^{\langle \kappa, x\rangle} = \prod e^{\kappa_j x_j} \quad \Big(x_j \in \mathbb{R}/\mathbb{Z},\ \langle \kappa, x\rangle = \sum \kappa_i x_i\Big)\,.$$

Assign $\mathbb{S}$ the invariant measure with total measure 1, so that

$$\int_{\mathbb{S}} f(s)\,ds = \int_0^1 f(x)\,dx\,,$$

and assign $\mathbb{T}$ the associated product measure. Since

$$\int_{\mathbb{T}} \varepsilon^\kappa(x)\,dx = \begin{cases} 1 & \text{if } \kappa = 0 \\ 0 & \text{otherwise} \end{cases}$$

and $\overline{\varepsilon}^\lambda = \varepsilon^{-\lambda}$ we have the orthogonality relations

$$\varepsilon^\kappa \bullet \varepsilon^\lambda = \begin{cases} 1 & \text{if } \kappa = \lambda \\ 0 & \text{otherwise.} \end{cases}$$

The characters ε^κ form a topological basis of $C^\infty(\mathbb{T})$. Formally, if

$$f(x) = \sum_\kappa c_\kappa\, \varepsilon^\kappa(x)$$

then

$$f \bullet \varepsilon^\lambda = \sum c_\kappa (\varepsilon^\kappa \bullet \varepsilon^\lambda) = c_\lambda\,.$$

This motivates the following definition. If f is a smooth function on $\mathbb{T}$, its **Fourier series** is

$$\sum_{\lambda \in \mathcal{Z}^n} c_\lambda\, \varepsilon^\lambda \quad \text{where} \quad c_\lambda = \widehat{f}_\lambda = f \bullet \varepsilon^\lambda = \int_{\mathbb{T}} f(x) e^{-\langle \lambda, x\rangle}\,dx\,.$$

Since

$$\big|\widehat{f}_\lambda\big| \le \sup_{\mathbb{T}} |f(x)|\,,$$

the terms in the Fourier series are bounded. We can do better.

1.1. Proposition. *If f in $C^k(\mathbb{T})$ has Fourier coefficients c_λ then $\partial^k f/\partial x^k$ has Fourier coefficients $\lambda^k c_\lambda$.*

Here k is a multi-index, and

$$\lambda^k = \prod \lambda_j^{k_i}\,.$$

Proof. Let $F = \partial f/\partial x_k$. Integration by parts gives us

$$\widehat{F}_\lambda = \int_{\mathbb{T}} \big(\partial f/\partial x_j\big)\, e^{-\langle\lambda,x\rangle}\, dx = \lambda_j \int_{\mathbb{T}} f(x) e^{-\langle\lambda,x\rangle}\, dx = \lambda_j \widehat{f}_\lambda\,.$$

Repeat as needed. ▮

Combining this and an earlier observation, we deduce that the Fourier coefficients of f in $C^\infty(\mathbb{T})$ are rapidly decreasing.

More precisely, define on $\mathcal{Z}^n$ the norms

$$|\lambda| = \sup_j |\lambda_j/2\pi i|, \quad |\!|\!|\lambda|\!|\!|_n = \sup_{m\le n} |\lambda|^m\,.$$

The norm $|\lambda|$ is equivalent to the usual Euclidean norm $\|\lambda\|$, in the sense that each is bounded by a multiple of the other. In subsequent doiscussions, the equivalence of various norms will be implicit.

Let $\mathcal{S}(\mathcal{Z}^n)$ be the space of all functions $c = (c_\lambda)$ on $\mathcal{Z}^n$ that are rapidly decreasing, with the topology defined by semi-norms

$$|c|_N = \sup_{\mathcal{Z}^n} |\!|\!|\lambda|\!|\!|_N\, |c_\lambda|\,.$$

1.2. Proposition. *The Fourier coefficients of f define a function in $\mathcal{S}(\mathcal{Z}^n)$.*

This follows from:

1.3. Corollary. *For f in $C^\infty(\mathbb{T})$ and $N \ge 0$ we have*

$$\widehat{f}_\lambda \underset{f,N}{\ll} 1/|\!|\!|\lambda|\!|\!|_N$$

for all λ in $\mathcal{Z}$.

Here I have used Serge Lang's notation:

$$A \underset{B}{\ll} C$$

means that $|A| \le \mathrm{const}\cdot C$ where the constant depends only on B.

Proof. The Lemma gives us

$$\big(1 + |\lambda|^N\big)\big|\widehat{f}_\lambda\big| \le |\!|\!|f|\!|\!|_0\,.$$ ▮

The map from $C^\infty(\mathbb{T})$ to $\mathcal{S}(\mathcal{Z}^n)$ is continuous. We now want to define an inverse map. For $n \ge 0$ let

$$\begin{aligned}\sigma_d(n) &= \mathrm{card}\big\{\lambda \in \mathcal{Z}^d \,\big|\, |\lambda| = n\big\}\\ \Sigma_d(n) &= \mathrm{card}\big\{\lambda \in \mathcal{Z}^d \,\big|\, |\lambda| \le n\big\}\,.\end{aligned}$$

Then $\Sigma_d(n) = (2n+1)^d$ and

$$\sigma_d(n) = \begin{cases} 1 & \text{if } n = 0\\ (2n+1)^d - (2n-1)^d & \text{otherwise.}\end{cases}$$

In particular, for $n > 0$ the function $\sigma_d(n)$ is a polynomial of degree $d-1$. These functions occur in the formula

$$\sum_{\lambda\in\mathcal{Z}^n} f\big(|\lambda|\big) = \sum_{m=0}^{\infty} \sigma_d(m) f(m)\,.$$

Using these functions, an easy calculation gives us:

1.4. Lemma. *The sum*

$$\sum_{\lambda \in \mathcal{Z}^d} \frac{1}{|\lambda|^N}$$

converges as long as $N > d$.

Proof. Since then the sum is comparable to $\sum 1/n^2$. ∎

1.5. Proposition. *The trigonometrical series*

$$\sum c_\lambda \, \varepsilon^\lambda$$

associated to any sequence in $\mathcal{S}(\mathcal{Z}^n)$ *converges absolutely on* $\mathbb{T}$ *to a smooth function on* $\mathbb{T}$ *whose Fourier series is* (c_λ). *The map taking a sequence in* $\mathcal{S}(\mathcal{Z}^n)$ *to the function that the series converges to defines a continuous map from* $\mathcal{S}(\mathcal{Z}^n)$ *to* $C^\infty(\mathbb{T})$.

Proof. The first assertion is valid because

$$\begin{aligned} \big|f(x)\big| &\le \sum |c_\lambda| \\ &= \sum |c_\lambda| \cdot |\lambda|^N \cdot \frac{1}{|\lambda|^N} \\ &\le \|\!|c|\!\|_N \cdot \sum \frac{1}{|\lambda|^N} \, . \end{aligned}$$

I leave the second and third as an exercise. ∎

If f lies in $C^\infty(\mathbb{T})$ then its Fourier series satisfies the hypotheses of this Proposition. The following is natural to expect:

1.6. Proposition. *The Fourier series of a function* f *in* $C^\infty(\mathbb{T})$ *converges uniformly to it.*

Proof. We want to show that the Fourier series

$$\sum_\lambda \widehat{f}_\lambda e^{\langle \lambda, x \rangle}$$

converges to $f(x)$. If $\varphi(x) = f(x+a)$ then $\varphi_n = e^{2\pi i n a} \widehat{f}_n$, so we may assume that $x = 0$. The Proposition is certainly true for $f(x) = 1$, so we may subtract it, and are then reduced to showing that if $f(0) = 0$ then

$$\sum \widehat{f}_\lambda \, e^{\langle \lambda, x \rangle} = 0 \, .$$

1.7. Lemma. *If* f *lies in* $C^\infty(\mathbb{T})$ *and* $f(0) = 0$ *then there exist functions* $f_k(x)$ *in* $C^\infty(\mathbb{T})$ *such that*

$$f(x) = \sum_k (e^{2\pi i x_k} - 1) f_k(x) \, .$$

This certainly implies the Proposition, since the Fourier transform of $(e^{2\pi i x_k} - 1) f_k(x)$ is $\widehat{f}_{\lambda - e_k} - \widehat{f}_\lambda$, and ithe sum over all λ certainly vanishes.

Proof. For each $k < n$ let $\mathbb{T}_k$ be the copy $\mathbf{S}^k$ in $\mathbb{T}$ where $x_i = 0$ for $i > k$. The first step in the proof is the elementary observation that if f vanishes on $\mathbb{T}_{n-1}$ then $f/(e^{2\pi i x_n} - 1)$ may be identified with a smooth function on $\mathbb{T}$.

Now suppose $f(0) = 0$. This observation tells us that

$$f_{(1)}(x) = f(x_1, x_2, \ldots, x_n) - (e^{2\pi i x_1} - 1) f_1(x)$$

with

$$f_1(x) = \frac{f(x_1, 0, \ldots, 0)}{e^{2\pi i x_1} - 1}$$

is smooth on $\mathbb{T}$. The function $f_{(1)}$ vanishes on all of $\mathbb{T}_1$, so we may apply the observation again to get

$$f_{(2)}(x) = f_{(1)}(x) - (e^{2\pi i x_2} - 1) f_2(x)$$

smooth on $\mathbb{T}$ and vanishing on $\mathbb{T}_2$. Repeat as needed.

Combining the previous Proposition and Theorem:

1.8. Theorem. *The Fourier transform is a topological isomorphism of* $C^\infty(\mathbb{T})$ *with* $\mathcal{S}(\mathcal{Z}^n)$.

2. The Plancherel formula

For f and g in $C^\infty(\mathbb{T})$ with Fourier series (c_κ) and (d_κ)

$$\begin{aligned} \langle f, g \rangle &= \int_{\mathbb{T}} f(x) g(x) \, dx \\ &= \int_{\mathbb{T}} \Big(\sum_\kappa c_\kappa \varepsilon^\kappa \Big) \Big(\sum_\lambda d_\lambda \varepsilon^\lambda \Big) \, dx \\ &= \sum_{\kappa,\lambda} c_\kappa d_\lambda \int_{\mathbb{T}} \varepsilon^{\kappa+\lambda} \, dx \\ &= \sum_\kappa c_\kappa d_{-\kappa} \, . \end{aligned}$$

If

$$f(x) = \sum c_\lambda \varepsilon^\lambda$$

then

$$\overline{f}(x) = \sum \overline{c}_\lambda \varepsilon^{-\lambda} = \sum \overline{c}_{-\lambda} \varepsilon^\lambda$$

so that

$$\langle f, \overline{g} \rangle = \sum c_\lambda \overline{d}_\lambda \, .$$

Lemma 1.4 implies that $\mathcal{S}(\mathcal{Z}^n)$ embeds into $\mathrm{L}^2(\mathcal{Z}^n)$. So this leads to:

2.1. Theorem. *The isomorphism of* $C^\infty(\mathbb{T})$ *with* $\mathcal{S}(\mathcal{Z}^n)$ *extends to one of* $\mathrm{L}^2(\mathbb{T})$ *with* $\mathrm{L}^2(\mathcal{Z}^n)$.

One consequence is that the Hilbert space $\mathrm{L}^2(\mathbb{T})$ is the completion of $C^\infty(\mathbb{T})$ with respect to the norm $\|f\|^2 = f \bullet f$.

3. Products and convolutions

The group $\mathcal{Z}^n$ acts on $\mathcal{S}(\mathcal{Z}^n)$ by left translation:

$$[L_\lambda c]_\mu = c_{\mu-\lambda} .$$

3.1. Proposition. *The left action of the group $\mathcal{Z}^n$ on $\mathcal{S}(\mathcal{Z}^n)$ is continuous, and more precisely*

$$|||L_\lambda c|||_n \underset{n}{\ll} |\lambda|^n \, |||c|||_n .$$

Proof. We have

$$\begin{aligned} |||L_\lambda c|||_n &= \sup_\mu |\mu|^n |c_{\mu-\lambda}| \\ &= \sup_\mu |\lambda+\mu|^n |c_\mu| \\ &\le \sup_\mu \sum_{k=0}^{n} \binom{n}{k} |\lambda|^{n-k} \, |\mu|^k \, |c_\mu| \\ &= (n+1) \sup_k \binom{n}{k} \, |\lambda|^n \, |||c|||_n . \end{aligned}$$ ▯

If $c = (c_\lambda)$ and $d = (d_\lambda)$ both lie in $\mathrm{L}^2(\mathcal{Z}^n)$, their convolution $e = c * d$ is the array

$$e_\lambda = \sum c_\mu d_{\lambda-\mu} .$$

The series converges by the Cauchy-Schwartz inequality, and the result is a bounded function on $\mathcal{Z}^n$. The left action of $\mathcal{Z}^n$ induces one of the group algebra $C_c(\mathcal{Z}^n)$, which acts on $\mathcal{S}(\mathcal{Z}^n)$ by convolution:

$$[L_a c]_\lambda = \sum_{\mu\in\mathcal{Z}^n} a_\mu [L_\mu c]_\lambda = \sum_\mu a_\mu c_{\lambda-\mu} .$$

3.2. Proposition. *If (c_λ) and (d_λ) are both in $\mathcal{S}(\mathcal{Z}^n)$, their convolution $c * d$ is also in $\mathcal{S}(\mathcal{Z}^n)$.*

Proof of Proposition 3.2. An immediate consequence of Proposition 3.1. ▯

Another way to put this:

3.3. Corollary. *The left action of $C_c(\mathcal{Z}^n)$ on $\mathcal{S}(\mathcal{Z}^n)$ extends to one of $\mathcal{S}(\mathcal{Z}^n)$.*

Suppose f and g to be in $C^\infty(\mathbb{T})$. Their convolution is

$$[f * g](x) = \int_{\mathbb{T}} f(y) g(x-y) \, dy .$$

3.4. Proposition. *For f and g in $C^\infty(\mathbb{T})$, the has Fourier series of $f * g$ is*

$$\sum_\lambda \widehat{f}_\lambda \widehat{g}_\lambda \, \varepsilon^\lambda .$$

This is straightforward.

4. Distributions

A **distribution** on $\mathbb{T}$ is a continuous linear functional on $C^\infty(\mathbb{T})$. This means it is a linear functional Φ on $C^\infty(\mathbb{T})$ such that

$$\langle \Phi, f\rangle \underset{\Phi}{\ll} |||f|||_m$$

for some m, loosely called the **order** of Φ. The space of distributions $C^{-\infty}(\mathbb{T})$ is the direct limit of subspaces of finite order, each with norm $\sup_f \langle T, f\rangle / |||f|||_m$ defining the Banach space $C^{-m}(\mathbb{T})$. In effect, the space $C^{-\infty}$ of distributions is the union of all the space $C^{-m}(\mathbb{T})$.

Since every ε^μ is in $C^\infty(\mathbb{T})$, every distribution has Fourier coefficients

$$\widehat{\Phi}_\mu = \langle \Phi, \varepsilon^{-\mu}\rangle \,.$$

4.1. Lemma. *If Φ is a distribution on $\mathbb{T}$ then*

$$|\widehat{\Phi}_\mu| \underset{\Phi,m}{\ll} \|\mu\|^m$$

for some m.

Proof. Because

$$|||\varepsilon^{-\mu}|||_m \underset{m}{\ll} (1 + \|\mu\|)^m \,.$$

This result may be completed to a converse, which I leave as an exercise:

4.2. Proposition. *The space $C^{-\infty}(\mathbb{T})$ may be identified with the space of Fourier series with the sequences (c_μ) of moderate growth in the sense that*

$$c_\mu \ll \|\mu\|^m$$

for some m.

The distribution may be evaluated explicitly in terms of its Fourier series by the formula

$$\langle \Phi, f\rangle = \sum \widehat{\Phi}_\mu \widehat{f}_{-\mu} \,.$$

The series $\sum \widehat{\Phi}_\mu \, \varepsilon^\mu$ converges to Φ in the sense that

$$\lim_{R\to\infty} \sum_{\|\mu\|\le R} \widehat{\Phi}_\mu \langle \, \varepsilon^\mu, f\rangle$$

converges to $\langle \Phi, f\rangle$ for all f in $C^\infty(\mathbb{T})$.

Integration by parts applied to two smooth functions on $\mathbb{T}$ gives

$$\langle \partial f/\partial x_i, F\rangle = -\langle f, \partial F/\partial x_i\rangle \,.$$

Defining the derivatives of a distribution defined by the formula

$$\langle \partial^k \Phi/\partial x^k, f\rangle = (-1)^{|k|}\langle \Phi, \partial^k f/\partial x^k\rangle$$

is consistent with this.

Distributions are really a miraculous tool. It's like living in some kind of paradise where all functions are smooth and all Fourier series converge. Of course the 'functions' are not really functions, but as someone has pointed out, it is fruitful to think of them as functions of 'fuzzy' points.

The convolution of a distribution and a function in $\mathcal{S}(\mathbb{T})$ is well defined. Let $f^{\#}(x) = f(-x)$. Then

$$F * f(y) = \int_{\mathbb{T}} F(x) f(y-x)\, dx = \langle F, R_y f^{\#} \rangle \,.$$

In terms of Fourier series, it remains true that the Fourier series of a convolution is the product of Fourier transforms.

Among the distributions are the derivatives of the Dirac delta:

$$\langle \delta^{(k)}, f \rangle = (-1)^k f^{(k)}(0) \,.$$

Its Fourier transform is the function λ^k. The derivative $f^{(k)}$ is $\delta^{(k)} * f$.

In terms of Fourier series, partial differential operators are represented as multiplication by polynomials with constant coefficients. Convolution with an arbitrary distribution is a natural generalization of this. It is a **pseudo-differential operator** with constant coefficients. We shall see other pseudo-differential operators later on.

How are properties of a distribution mirrored in properties of its Fourier series? The simplest result is that if f is a distribution such that $\sum |\widehat{f}_\mu| < \infty$ then f is a continuous function and

$$\begin{aligned} |\widehat{f}_\mu| &\le \sup |f(x)| \\ \sup |f(x)| &\le \sum |\widehat{f}_\mu| \,. \end{aligned}$$

But beyond this and its immediate corollaries it is notoriously difficult to say much. The condition in , for example, is sufficient but not necessary for continuity. When a simple question is hard to answer in mathematics, it is perhaps the wrong question to ask. For this and other reasons, it is reasonable to speculate that the filtration

$$C^\infty(\mathbb{T}) \subset \ldots \subset C^m(\mathbb{T}) \subset \ldots \subset C^0(\mathbb{T}) \subset C^{-1}(\mathbb{T}) \subset \ldots \subset C^{-\infty}(\mathbb{T})$$

is not, in some sense, natural. We shall see in the next section the **Sobolev filtration**, one that's perhaps a little less intuitive, but that is in many ways much more useful.

5. Sobolev spaces

The space $\mathrm{L}^2(\mathbb{T})$ is naturally embedded in the space of distributions. For any function F in $\mathrm{L}^2(\mathbb{T})$, its derivatives are also distributions. This makes sense of the following definition of the Soboloev spaces:

$$\boldsymbol{H}^m = \left\{ F \in \mathrm{L}^2(\mathbb{T}) \;\middle|\; \text{all distributional derivatives } \partial^k F/\partial x^k \text{ with } \textstyle\sum k_i \le m \text{ lie in } \mathrm{L}^2(\mathbb{T}) \right\} .$$

In particular, $\boldsymbol{H}^0 = \mathrm{L}^2(\mathbb{T})$. This definition translates directly to a simple condition on Fourier series. If F is a distribution then the Fourier coefficients of $\partial^k F/\partial x^k$ are the $\mu^k \widehat{F}_\mu$. Its L^2 norm is then

$$\sum_\mu |\mu_1|^{2k_1} \ldots |\mu_n|^{2k_n} |F_\mu|^2 \,.$$

Since

$$\left(1 + \textstyle\sum_1^n c_i^2\right)^m = \textstyle\sum_{|k| \le m} C_{m,n,k} \prod c_i^{2k_i}$$

with all $C_{m,n,k} \geq 1$ we deduce:

5.1. Proposition. *For all $m \geq 0$ the map taking F to $(\widehat{F}_\mu)$ is an isomorphism of $\boldsymbol{H}^m$ with the space of all sequences $C = (c_\kappa)$ with finite m-th Sobolev norm*

$$\|C\|_m^2 = \sum_\kappa \big(1 + \|\kappa\|^2\big)^m |c_\kappa|^2 < \infty\,.$$

The factor $1 + \|\kappa\|^2$ may be of course replaced by any equivalent one.

This result suggests immediately the right definition of $\boldsymbol{H}^m(\mathbb{T})$ for $m < 0$, as the space of distributions F on $\mathbb{T}$ whose Fourier transforms satisfy

$$\|\widehat{F}\|_m = \sum_k \big(1 + \|\kappa\|^2\big)^m |\widehat{F}_\kappa|^2 < \infty\,.$$

Thus we have a single condition on functions in $\boldsymbol{H}^m$ for all m, one that agrees with the L^2-condition for $m \geq 0$, in which case conditions defining $\boldsymbol{H}^m$ can be formulated relatively simply directly in terms of either F or of its Fourier coefficients.

Another simple observation:

5.2. Proposition. *For any m the space $\boldsymbol{H}^m$ is the completion of $C^\infty(\mathbb{T})$ with respect to the m-th Sobolev norm.*

There is a natural pairing between $\boldsymbol{H}^m$ and $\boldsymbol{H}^{-m}$, since the sum

$$\langle F, f\rangle = \sum \widehat{F}_\mu \widehat{f}_{-\mu}$$

converges.

5.3. Proposition. *This identifies $\boldsymbol{H}^{-m}$ with the dual of $\boldsymbol{H}^m$.*

5.4. Proposition. *The differential operator $\partial^k/\partial x^k$ is a bounded operator from $\boldsymbol{H}^m$ to $\boldsymbol{H}^{m-|k|}$.*

A little more interesting:

5.5. Proposition. *The inclusion of $\boldsymbol{H}^m$ in $\boldsymbol{H}^k$ for $m > k$ is compact.*

Proof. We must show that the embedding of $\boldsymbol{H}^m$ into $\boldsymbol{H}^k$ is well approximated in norm by maps of finite rank. I'll just do $\boldsymbol{H}^1 \subseteq \boldsymbol{H}^0$ to show the basic idea.

It suffices to show that for every $\varepsilon > 0$ there exists a finite subspace U of $\boldsymbol{H}^0$ such that $\|f - \Pi_U f\|^2 < \varepsilon$ for every f in $\boldsymbol{H}^1$ with $\|f\|_1 \leq 1$, where Π_U is orthogonal projection onto U. Given f, let $c_\lambda = \widehat{f}_\lambda$. We have

$$\sum (1 + \|\lambda\|^2)|c_\lambda|^2 < \infty$$

Choose R such that $1/(1 + R^2) < \varepsilon$, and let U be the subspace spanned by the ε^λ with $\|\lambda\| \leq R$. Then

$$\|f - \Pi_U f\|^2 < \sum_{\|\lambda\| > R} |c_\lambda|^2 < \frac{1}{1 + R^2} \sum (1 + \|\lambda\|^2)|c_\lambda|^2 < \varepsilon\,.$$

5.6. Lemma. *For $c = (c_\kappa)$ in $\mathcal{S}(\mathcal{Z}^n)$ and all $m > n/2$*

$$\sum_{\mathcal{Z}^n} |c_\kappa| \underset{m}{\ll} \|c\|_m\,.$$

Proof. Let $f = \sum c_\mu \varepsilon^\mu$. We have

$$\begin{aligned} \sum |c_\kappa| &= \sum |c_\kappa| \big(1 + \|\kappa\|^2\big)^{m/2} \cdot \frac{1}{\big(1 + \|\kappa\|^2\big)^{m/2}} \\ &\le \left[\sum \frac{1}{\big(1 + \|\kappa\|^2\big)^m} \right]^{1/2} \left[\sum |c_\kappa|^2 \big(1 + \|\kappa\|^2\big)^m \right]^{1/2} \qquad \text{(Cauchy-Schwarz)} \\ &= C_m \, \|f\|_m \, . \end{aligned}$$

∎

5.7. Proposition. *Any function f in $\boldsymbol{H}^m$ with $m > n/2$ is continuous, and*

$$\sup_{\mathbb{T}} |f(x)| \underset{m}{\ll} \|f\|_m \, .$$

Proof. Because the embedding of $C^\infty(\mathbb{T})$ into $C(\mathbb{T})$ extends continuously to an embedding of $\boldsymbol{H}^m$. ∎

5.8. Corollary. *The intersection of all $\boldsymbol{H}^m$ is $C^\infty(\mathbb{T})$.*

5.9. Proposition. *Every distribution on $\mathbb{T}$ lies in some $\boldsymbol{H}^m$.*

How does multiplication by a function act on the Sobolev spaces? *Suppose φ to be a function on $\mathbb{T}$. Under what circumstances does multiplication by φ take $\boldsymbol{H}^m$ to itself?*

5.10. Lemma. *If φ is in $C^m(\mathbb{T})$, multiplication by φ is takes both $\boldsymbol{H}^m$ and $\boldsymbol{H}^{-m}$ to themselves.*

Proof. If f is in $\boldsymbol{H}^m$ and φ in $C^m(\mathbb{T})$ then Leibniz' rule tells us that $D^k(\varphi f)$ is square integrable for $|k| \le m$, and asserts an explicit bound in terms of the derivatives of φ. Thus $f \mapsto \varphi f$ is a continuous linear map of $\boldsymbol{H}^m$ to itself. If $m < 0$, multiplication by φ on $\boldsymbol{H}^m$ is the adjoint of multiplication by φ on its dual $\boldsymbol{H}^{-m}$. ∎

Since a diffeomorphism T changes volumes locally by a factor $\det \operatorname{Jac}(T)$, it is an easy exercise to show:

5.11. Lemma. *If T is a diffeomorphism of $\mathbb{T}$ and F lies in $\boldsymbol{H}^s$ then so does T^*F.*

The implication of these two results is that the notion of Sobolev spaces is really a notion defined locally in Euclidean space, and then on any manifold, and in a coordinate-free manner. Let U be an open subset of $\mathbb{R}^n$, x a point of U, Φ a distribution on U. Let x be a point of U, $B_r \subset B_R$ two disks centred at x. If χ is a smooth function with support in B_R, identically 1 on B_r, then $\chi\Phi$ may be identified with a distribution on the quotient $\mathbb{T}_{2R}$ of $\mathbb{R}^n$ by $4R\mathbb{Z}^n$, agreeing with Φ locally at x. The distribution Φ is said to be in $\boldsymbol{H}^m(U)$ if this periodic distribution lies in some $\boldsymbol{H}^m(\mathbb{T}_R)$ for all v. This specification turns out to be independent of all choices.

The filtration

$$C^\infty(\mathbb{T}) = \boldsymbol{H}^\infty(\mathbb{T}) \subset \ldots \subset \boldsymbol{H}^1 \subset \boldsymbol{H}^0(\mathbb{T}) \subset \boldsymbol{H}^{-1}(\mathbb{T}) \subset \ldots \subset \boldsymbol{H}^{-\infty}(\mathbb{T}) = C^{-\infty}(\mathbb{T})$$

is called the **Sobolev filtration**. It has much to be said for it.

Part II. Differential operators

6. Operators and adjoints

In this essay, a differential operator on an open subset of $\mathbb{R}^n$ is a linear combination of partial derivatives with smooth functions as coefficients:

$$L = \sum_{|k| \le p} a_k(x)\, \partial^k / \partial x^k \,.$$

If there exists a coefficient $a_p(x)$ that does not vanish identically, it is said to be of order p. This allows us also to interpret differential operators on a torus $\mathbb{T}$, taking $C^\infty(\mathbb{T})$ to itself. Of course, this suggests defining a dual map on distributions. Making this explicit involves defining the **adjoint** of a differential operator.

Suppose F to be a smooth function on U, φ to be in $C_c^\infty(U)$. Then

$$\begin{aligned}
\langle LF, \varphi\rangle &= \sum \langle a_k(x)\, \partial^k F/\partial x^k, \varphi\rangle \\
&= \sum \langle \partial^k F/\partial x^k, a_k(x)\varphi\rangle \\
&= \sum (-1)^{|k|} \langle F, \partial^k\big(a_k(x)\varphi\big)/\partial x^k\rangle \\
&= \langle F, L^*\varphi\rangle \ \text{ where } \ L^*\varphi = \sum (-1)^{|k|} \partial^k\big(a_k(x)\varphi\big)/\partial x^k \,.
\end{aligned}$$

which suggests immediately that we should define LF for any distribution by the same formula. The operator L^* is the **formal adjoint** of L. The adjoint can be calculated more explicitly in the standard form $\sum a_k^*(x)\, \partial^k/\partial x^k$ by applying the formula

$$\partial(af)/\partial x_j = a(\partial f/\partial x_j) + (\partial a/\partial x_j) f$$

and recursion. If M_a is multiplication by the function $a(x)$ and $\partial_k = \partial/\partial_k$, this can be written as

$$\partial_k M_a = M_a\, \partial_k + M_{\partial_k a} \,.$$

This in turn, for example, leads to

$$\partial_j \partial_k M(a) = M_a\, \partial_j \partial_k + M_{\partial_j a}\, \partial_k + M_{\partial_k a}\, \partial_j + M_{\partial_j \partial_k a}$$

and leads to a variant of Leibniz' formula. If X is a multi-set (set with multiplicities) of indices i_1, i_2, … , i_m, let $\partial_X = \partial_{i_1} \partial_{i_2} \ldots\ \partial_{i_m}$. Since the ∂_k commute, it doesn't matter what order the indices are arrayed, but their multiplicity certainly counts.

6.1. Proposition. *For $a(x)$ a smooth function on an open subset of $\mathbb{R}^n$ and X a multi-subset of $[1, n]$*

$$\partial_X M_a = \sum_{\Theta \subseteq X} M_{\partial_\Theta a} \partial_{X - \Theta} \,.$$

6.2. Corollary. *If $a(x)$ is a smooth function and L a differential operator of order p, then $La(x) - a(x)L$ is an operator of order $\le p - 1$.*

6.3. Lemma. *If $L^* = \sum (-1)^k a_k^*(x) \partial^k/\partial x^k$, then*

$$\widehat{LF}_\mu = \sum_k \widehat{(a_k^* F)}_\mu \mu^k \,.$$

From this and Lemma 5.10, it is easy to see:

6.4. Proposition. *A smooth differential operator of order k from $C^\infty(\mathbb{T})$ to itself extends to a continuous operator from each $\boldsymbol{H}^m$ to $\boldsymbol{H}^{m-k}$.*

If D is a differential operator

$$D = \sum_{|k| \le m} a_k(x)\, \partial^k / \partial x^k \,,$$

of order m defined on an open connected subset U on $\mathbb{R}^n$, then its **symbol** at x is the polynomial valued function

$$\sigma_D(x, \xi) = \sum_{|k| = m} a_k(x)\, \xi^k \,.$$

It is a homogeneous polynomial of degree m. It is canonically defined as a section of the m-th symmetric power of the tangent bundle.

6.5. Lemma. *If σ is the symbol of the operator L of order m, the symbol of L^* is $(-1)^m \sigma$.*

7. Elliptic differential operators

A (smooth) differential operator L on U is called **elliptic** if its symbol $\sigma_L(x, \xi) \ne 0$ for all x in U, $\xi \ne 0$ in $\mathbb{R}^n$. One well known example is the Laplacian Δ with symbol $\|\xi\|^2$. Any ordinary differential operator (on $\mathbb{R}$) with a non-vanishing coefficient of highest order is elliptic, but if $n > 1$ it is necessary that the order p of the operator be even.

Elliptic operators possess a very special property. Given a Lebesgue measure dx, integration embeds the space $C^\infty(U)$ into that of distributions on U.

7.1. Theorem. (Smooth elliptic regularity) *If E is a smooth elliptic differential operator defined on an open subset U of $\mathbb{R}^n$, and φ is a smooth function on U, any distribution solution Φ of $E\Phi = \varphi$ is also in $C^\infty(U)$.*

Proving this will take some time. I'll follow an idea attributed to Peter Lax, which first looks at elliptic operators on a torus, then reduces the general case to a property of such operators concerning the way they act on Sobolev spaces. The idea is plausible since $\mathbb{R}^n$ and $\mathbb{T}$ are locally indistinguishable, but a few technicalities intrude. The proof will also introduce some very simple pseudo-differential operators, at least on the torus.

It is also true that if the coefficients $a_k(x)$ and the function φ are real analytic, then Φ also is real analytic, but that is a different—more difficult and perhaps more interesting—story. Furthermore, proofs of this that I am aware of start off by calling on the regularity theorem above.

In the rest of this section I'll motivate the eventual argument by looking at elliptic operators with constant coefficients on a torus. In the next I'll conclude the proof in the general case.

Suppose

$$P(\xi) = \sum a_k \xi^k$$

to be a polynomial of order p with constant coefficients, $L = P(\partial/\partial x)$, with symbol

$$\sigma(\xi) = \sum_{|k| = p} a_k \xi^k$$

Assume L to be elliptic, so that $\sigma(\xi) \ne 0$ for $\xi \ne 0$. In terms of Fourier coefficients,

$$\widehat{Lf}_\mu = P(\mu) \widehat{f}_\mu \,.$$

If $P(\xi)$ itself has no zeroes in $\mathcal{Z}^n$, then we can simply calculate an inverse for L in terms of Fourier series:

$$\widehat{L^{-1} f}_\mu = P(\mu)^{-1} \widehat{f}_\mu \,.$$

This happens, for example, when $P(\xi) = 1 - \xi \bullet \xi$, $L = I - \Delta$, and $P(\mu) = 1 + \|\mu\|^2$ for μ in $\mathcal{Z}^n$. But this is exceptional. What we know in general is that

$$P(\xi) = \sigma(\xi) + \text{ terms of degree at most } p-1$$

so that for large μ in $\mathcal{Z}^n$ we have asymptotically $P(\mu) \sim \sigma(\mu)$, which by assumption does not vanish. Thus we can find R such that $P(\mu) \neq 0$ for $\|\mu\| > R$, and the number of μ with $P(\mu) = 0$ is finite. We can then define an *approximation* A to L^{-1} by the formula

$$\widehat{Af}_\mu = \begin{cases} \widehat{f}_\mu / P(\mu) & \text{if } P(\mu) \neq 0 \\ 0 & \text{if } P(\mu) = 0. \end{cases}$$

If f is smooth, the $\widehat{f}_\mu$ are rapidly decreasing and so are the $(Af)_\mu$, so that Af is also smooth. We now have

$$\begin{aligned} (I - AL)f &= \sum_{P(\mu)=0} \widehat{f}_\mu \varepsilon^\mu \\ f &= ALf + \sum_{P(\mu)=0} \widehat{f}_\mu \varepsilon^\mu \,. \end{aligned}$$

If $Lf = \varphi$ is smooth then $A\varphi = ALf$ is also smooth, but so is the finite sum of characters in this formula, and hence so is f.

Thus when L has constant coefficients, the operator A defined above is an approximation to an inverse for L, in the sense that $I - AL$ has finite rank and its image is in $C^\infty(\mathbb{T})$. It is the simplest example of what is called a **parametrix** for L. If L has variable coefficients, there is no simple formula for such an approximation to L^{-1}, at least not in terms of Fourier series, but there is a way to find successive approximations to it in terms of the Sobolev filtration. This involves introducing a generalization of differential operators called **pseudo-differential operators**. I'll give a preview of what is to come in the next section by looking again here at the case of an elliptic differential operator with constant coefficients.

Instead of dividing by $P(\mu)$ where possible, divide by $\sigma(\mu)$ for $\mu \neq 0$:

$$\widehat{Bf}_\mu = \begin{cases} 0 & \text{if } \mu = 0 \\ \widehat{f}_\mu / \sigma(\mu) & \text{otherwise.} \end{cases}$$

It is not true that $(I - BL)f$ is a finite exponential sum, and we have to make a more subtle argument. Suppose that

$$P(\mu) = \sigma(\mu) + \sum_{k=0}^{p-1} P_k(\mu) \,,$$

in which each $P_k(\mu)$ is a polynomial of degree k. If $F = (I - BL)f$ then

$$F_\mu = \begin{cases} f_\mu & \text{if } \mu = 0 \\ -f_\mu \Big(\sum_{k=0}^{p-1} P_k(\mu)/\sigma(\mu) \Big) & \text{otherwise.} \end{cases}$$

Thus $I - BL$ is not an operator of finite rank, but it is convolution with a certain distribution—what I have called earlier a pseudo-differential operator with coinsnat coefficients. We shall see in the next section what this does for us.

8. Proving regularity

Suppose

$$L = \sum a_k(x)\, \partial^k / \partial x^k$$

to be a smooth differential operator on $\mathbb{T}$ with adjoint

$$L^* = \sum (-1)^k a_k^*(x)\, \partial^k / \partial x^k \, .$$

There are two ways to express Lf in terms of Fourier series. On the one hand we can write

$$\begin{aligned} f &= \sum_\lambda \widehat{f}_\lambda \, e^{\langle \lambda, x \rangle} \\ Lf &= \sum_\lambda \Big(\sum_k a_k(x) \lambda^k \Big) \widehat{f}_\lambda \, e^{\langle \lambda, x \rangle} \, , \end{aligned}$$

while on the other we can also write

$$Lf = \sum_\lambda \widehat{Lf}_\lambda e^{\langle \lambda, x \rangle} \, ,$$

where

$$[Lf]_\lambda = \langle Lf, e^{-\langle \lambda, x \rangle} \rangle = \langle f, L^* e^{-\langle \lambda, x \rangle} \rangle = \sum_k (a_k^* f)_\lambda \lambda^k \, .$$

The first expression tells us in particular that

$$L\varepsilon^\lambda = \Big(\sum_k a_k(x) \lambda^k \Big) \varepsilon^\lambda \, .$$

If the $a_k(x)$ are smooth functions, L takes $C^\infty(\mathbb{T})$ to itself.

A **pseudo-differential operator** is one defined by a formula that generalizes this construction. Any smooth function $a(x, \lambda)$ on $\mathbb{T} \times \mathcal{Z}$ defines a linear map from the subspace $\mathbb{C}[\varepsilon^{\pm 1}] = \mathbb{C}[\varepsilon_1^{\pm 1}, \ldots , \varepsilon_n^{\pm 1}]$ of finite linear combinations of characters of $\mathbb{T}$ to $C^\infty(\mathbb{T})$, assigning

$$[P_a \varepsilon^\lambda](x) = a(x, \lambda) \varepsilon^\lambda(x) \, .$$

If

$$a(x, \lambda) = \sum_\mu a_{\lambda,\mu} \varepsilon^\mu$$

we can also express this as

$$P_a \varepsilon^\lambda = \sum_\mu a_{\lambda,\mu} \varepsilon^{\mu+\lambda} = \sum_\mu a_{\lambda,\mu-\lambda} \varepsilon^\mu \, ,$$

so that the Fourier coefficient of $P_a \varepsilon^\lambda$ is a convolution. *In what circumstances does this operator extend to be defined on all of* $C^\infty(\mathbb{T})$*?*

PSEUDO-DIFFERENTIAL OPERATORS. A pseudo-differential operator on $\mathbb{T}$ is one defined in this way on $\mathbb{C}[\varepsilon^{\pm 1}]$ that extends to $C^\infty(\mathbb{T})$. In order that each $P_a \varepsilon^\lambda$ be defined on $\mathbb{C}[\varepsilon^{\pm 1}]$, it is only necessary that $a(x, \lambda)$ be smooth for every λ, but if P_a is to be extended to an operator from $C^\infty(\mathbb{T})$ to itself some condition on the way the various $a(x, \lambda)$ grow as $\|\lambda\|$ passes off to infinity. The simplest is that

$$\frac{\partial^k a(x, \lambda)}{\partial x^k} \ll_k \big(1 + |\lambda|\big)^N \text{ for some } N \text{ independent of } k. \tag{Ψ}$$

Integral operators with various degrees of differentiability imposed on the kernel are examples with $N < 0$.

This condition on $a(x,\lambda)$ has one immediate consequence. Suppose that

$$a(x,\lambda) = \sum_\mu a_{\lambda,\mu}\varepsilon^\mu \,.$$

Then for all $n \ge 0$

$$|a_{\lambda,\mu}| \ll_{n,\lambda} \frac{1}{\big(1+\|\mu\|^2\big)^n}\,\big(1+|\lambda|\big)^N \,. \tag{8.1}$$

8.2. Proposition. *If $a(x,\lambda)$ on $\mathbb{T}\times\mathcal{Z}$ satisfies (Ψ), then for each m the map from $\mathbb{C}[\varepsilon^{\pm1}]$ to $C^\infty(\mathbb{T})$ extends to a continuous linear map from $\boldsymbol{H}^m$ to $\boldsymbol{H}^{m-N}$.*

In particular, since $C^\infty(\mathbb{T}) = \bigcap \boldsymbol{H}^m$, it takes $C^\infty(\mathbb{T})$ to itself.

Proof. Suppose f in $\mathbb{C}[\varepsilon^{\pm1}]$, which means that $c_\lambda = \widehat{f}_\lambda$ vanishes except for a finite number of λ. Then

$$\begin{aligned}
[P_a f](x) &= \sum_\lambda c_\lambda\, a(x,\lambda)\,\varepsilon^\lambda(x) \\
&= \sum_{\lambda,\mu} c_\lambda\, a_{\lambda,\mu}\,\varepsilon^{\mu+\lambda}(x) \\
&= \sum_{\lambda,\nu} c_\lambda\, a_{\lambda,\nu-\lambda}\,\varepsilon^{\nu}(x)\,. \\
\|P_a f\|^2_{m-N} &= \sum_\nu (1+\|\nu\|^2)^{m-N}\cdot\Big|\sum_\lambda a_{\lambda,\nu-\lambda}c_\lambda\Big|^2 .
\end{aligned}$$

By (8.1) we have for all n

$$\|P_a f\|^2_{m-N} \le C_n \sum_\nu (1+\|\nu\|^2)^{m-N}\Big|\sum_\lambda \frac{1}{(1+\|\nu-\lambda\|^2)^n}\,(1+|\lambda|)^N c_\lambda\Big|^2 .$$

The sum over λ is the convolution of the two sequences

$$\frac{1}{(1+\|\lambda\|^2)^n}\,, \qquad \text{and } (1+|\lambda|)^N c_\lambda \,.$$

Define distributions $\varphi(x)$ and $\gamma(x)$ by the formulas

$$\widehat{\varphi}_\lambda = \frac{1}{(1+|\lambda|^2)^n}\,, \quad \widehat{\gamma}_\lambda = (1+\|\lambda\|)^N c_\lambda \,.$$

The distribution γ belongs to $\boldsymbol{H}^{m-N}$. According to a mild extension of , the convolution is the Fourier series of the product $\varphi\gamma$. But according to Lemma 5.10, multiplication by φ is continuous on $\boldsymbol{H}^{m-N}$ if we take n large enough. This concludes the proof. ▯

THE BASIC LEMMAS. Suppose L to be a smooth elliptic differential operator of order p and symbol $\sigma_L(x,\xi)$. The proof of the following is almost excatly the same as what we have seen in the previous section.

8.3. Proposition. *The function $1/\sigma(x,\xi)$ satisfies the condition Ψ). The associated pseudo-differential operator E has order $-p$. The difference $I-EL$ is a pseudo-differential operator of order -1.*

And now comes the conclusion:

8.4. Corollary. *If L is an elliptic differential operator of order p then for each k there exists a pseudo-differential operator A_k of order $-p$ such that $I - A_k L$ has order $-k$.*

Proof. Let E be the pseudo-differential operator $1/\sigma(x, \xi)$. It has order $-p$. Define A_k by the condition that

$$I - A_k L = (I - EL)^k \,.$$

Explicitly, apply induction, setting

$$\begin{aligned} A_1 &= E \\ A_{k+1} &= A_k + (I - A_k L)E \,. \end{aligned}$$ ∎

8.5. Proposition. *Suppose L to be an elliptic differential operator of order p with smooth coefficients. If Φ is a distribution with $L\Phi$ in $\mathbf{H}^m$ then Φ lies in $\mathbf{H}^{m+p}$.*

Proof. Since $L\Phi$ lies in $\mathbf{H}^m$, each $A_k L\Phi$ lies in $\mathbf{H}^{m+p}$, since A_k has order $-p$. But $(I - A_k L)\Phi = \Phi - A_k L\Phi$ lies in $\mathbf{H}^{m+k}$, so that if k is large enough Φ lies in $\mathbf{H}^{m+p}$. ∎

Now we can prove the Theorem on the torus. Suppose that Φ is a distribution on $\mathbb{T}$ with $L\Phi = \varphi$ smooth. Say Φ lies in $\mathbf{H}^m$. Then $A_k L\Phi$ is smooth, and $(A_k L - I)\Phi$ lies in $\mathbf{H}^{m+k}$, so Φ lies in $\mathbf{H}^{m+k}$ for all k, and Φ itself is smooth. ∎

Unfortunately this cannot be applied directly to the proof of the Theorem in $\mathbb{R}^n$.

MOVING OFF THE TORUS. Let's now prove the Theorem itself. We are given an elliptic operator L and a distribution Φ on the open subset U of $\mathbb{R}^n$, such that $L\Phi = \varphi$ is smooth. We want to know that Φ is smooth in the neighbourhood of any point in U, which we can take to be 0.

We want to transfer the problem to the torus. Let L_0 be the constant coefficient operator agreeing with L at 0. Let $\sigma_0(\xi)$ be its symbol. Choose R small enough so B_{2R} is contained in U. Suppose $|\sigma(x, \xi)| > \varepsilon$ for $|x| \le 2R$. Choose R small enough so that $|\sigma(x, \xi) - \sigma(\xi)| < \varepsilon/2$ for $|x| \le 2R$. Choose a smooth function χ identically 1 on B_R and with support B_{2R}. The operator $L_\# = \chi L + (1 - \chi)L_0$ will be elliptic with symbol $\chi\sigma_L(x, \xi) + (1 - \chi)\sigma_0(\xi)$. The operator can be defined on $\mathbb{T}$, and agrees with L on B_R.

The differential operator $K = L\chi - \chi L$ has order $p - 1$. Then

$$K\Phi = L\chi\Phi - \chi L\Phi \,.$$

Since $L\Phi$ is smooth, so is $\chi L\Phi$, The first term $\chi\Phi$ lies in $\mathbf{H}^m$ for some m. Thus $K\Phi$ lies in $\mathbf{H}^{m-(p-1)}$. But $L\chi\Phi$ agrees with $L_\#\chi\Phi$, and may be considered a function on $\mathbb{T}$. Therefore according to Proposition 8.5, $\chi\Phi$ lies in $\mathbf{H}^{m+1}$. Loop. ∎

Part III. References

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