Notes on Kazhdan-Lusztig polynomials

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This represents my attempts to digest [Kazhdan-Lusztig:1979]. My exposition will follow that paper closely, with a few ideas taken from [Soergel:1997] and [Shi:1986].

In the final section, I include an argument essentially due to Susumu Ariki to show that KL cells in $\mathfrak{S}_n$ coincide with those described by the Robinson-Schensted process. In a future version, I'll look at the analogous result for affine groups of type A_n found in [Shi:1986].

Also in a later version of this essay, I'll discuss simple ways to compute in Coxeter groups. Fokko du Cloux has written the only practical suite of programs for calculating KL structures, which he calls `Coxeter`. Version 3.0 was the last version he finished before his untimely death, succeeded by the suite of programs called `Atlas` that computed related structures related to representations of real reductive groups.

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If $w = xy$ with $\ell(w) = \ell(x) + \ell(y)$ I shall write $w = x{\diamond}y$.

1. The Hecke algebra

Let $\mathcal{A}$ be the ring $\mathbb{Z}[v^{\pm 1}]$ of Laurent polynomials, $q = v^{-2}$, and let (W, S) be a Coxeter system. The Hecke algebra $\mathcal{H} = \mathcal{H}(W, S)$ is a free module over $\mathcal{A}$ with basis T_w parametrized by W and multiplicative identity $I = T_1$, satisfying relations

$$\begin{aligned} T_s^2 &= (q-1)T_s + q \\ T_w T_s &= T_{ws} \quad \text{if } ws > w \\ T_s T_w &= T_{sw} \quad \text{if } sw > w\,. \end{aligned}$$

It is associative. Since $w = ws{\diamond}s$ if $ws < w$, we deduce equations

$$\begin{aligned} T_w T_s &= (q-1)T_w + qT_{ws} \quad \text{if } ws < w \\ T_s T_w &= (q-1)T_w + qT_{sw} \quad \text{if } sw < w \end{aligned}$$

and then

$$T_{x\diamond y} = T_x T_y\,.$$

Consequently, if $w = s_{i_1}{\diamond}\cdots{\diamond}s_{i_\ell}$ then

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}\,.$$

It is not quite trivial that an associative algebra satisfying these conditions exists. As far as I am aware, there are three different approaches to constructing it. Refer to [Bourbaki:1968] or [Humphreys:1992] for variations on the original approach due to Tits; to [Eriksson:1994] for an elegant graph-theoretical proof; or to [Casselman:2011] for what I believe to be the most direct.

Since

$$T_s(T_s - (q-1)) = q$$

the element T_s is invertible, and

(1.1) $$T_s^{-1} = T_s/q - (1 - 1/q)\,.$$

Therefore every T_w is invertible.

The inclusion of $v^{\pm 1}$ in the coefficient ring is natural. If W is finite then with this choice the complex Hecke algebra is isomorphic to $\mathbb{C}[W]$, and it is known that for groups of type E_7 and E_8 this is a necessary assumption. As we shall see, it allows much simplification, too. Let ι be the linear map from $\mathcal{H}$ to itself taking

$$v \longmapsto v^{-1}, \quad T_w \longmapsto T_{w^{-1}}^{-1}\,.$$

1.2. Lemma. *The involution ι is an automorphism.*

Proof. It must be shown that

$$\begin{aligned} T_w^\iota T_s^\iota &= (T_w T_s)^\iota \ (ws > w) \\ T_s^\iota T_w^\iota &= (T_s T_w)^\iota \ (sw > w) \\ T_s^\iota T_s^\iota &= (T_s T_s)^\iota\,. \end{aligned}$$

The first two are immediate. The last is equivalent to

$$T_s^{-2} - (q^{-1} - 1)T_s^{-1} - q^{-1} = (T_s^{-1} - q^{-1})(T_s^{-1} + I) = 0$$

which follows from

$$(T_s - q)(T_s + I) = 0$$

upon multiplication by $q^{-1}T_s^{-2}$. ∎

For many purposes, it is convenient to work with normalized elements $\tau_w = v^{\ell(x)}T_x$ of $\mathcal{H}$, rather than the T_w. Then $(\tau_s + v)(\tau_s - v^{-1}) = 0$ and the defining relations of the T_w become

(1.3) $$\begin{aligned} \tau_s^2 &= (v^{-1} - v)\tau_s + I \\ \tau_{x\diamond y} &= \tau_x \tau_y\,. \end{aligned}$$

Since $\tau_w^\iota = \tau_{w^{-1}}^{-1}$, Equation (1.1) becomes

$$\tau_s^\iota = \tau_s^{-1} = I + (v - v^{-1})\tau_s\,.$$

If I set

$$\gamma_s = \tau_s + v$$

this can be rewritten as

(1.4) $$\gamma_s^\iota = (\tau_s + v)^\iota = \tau_s + v = \gamma_s\,.$$

From (1.3) we deduce

(1.5) $$\gamma_s \tau_x = \begin{cases} \tau_{sx} \ + \ v\tau_x & \text{if } sx > x \\ v^{-1}\tau_x + \tau_{sx} & \text{if } sx < x\,. \end{cases}$$

2. Kazhdan-Lusztig polynomials

The main result is:

2.1. Theorem. *For every x in W there exists a unique element γ_x in $\mathcal{H}$ that's invariant under the involution ι and of the form*

$$\gamma_x = \tau_x + \sum_{y<x} v\pi_{y,x}(v)\tau_y \,,$$

where, for every $y < x$, $\pi_{y,x}(v)$ is a polynomial in $\mathbb{Z}[v]$ of degree at most $\ell(x) - \ell(y) - 1$.

The formula in the theorem can be written as

$$\gamma_x = \sum_{y} v\pi_{y,x}(v)\tau_y$$

if we set $\pi_{x,x} = 1/v$ and $\pi_{y,x} = 0$ for $y \not\leq x$.

For $y \leq x$ define

$$P_{y,x} = v^{-(\ell(x)-\ell(y)-1)}\,\pi_{y,x}(v), \quad \Gamma_x = v^{-\ell(x)}\gamma_x \,.$$

Thus $P_{y,x}$ is a polynomial in v^{-1} of degree at most $\ell(x) - \ell(y) - 1$. If we multiply the equation for γ by $v^{-\ell(x)}$ we get

$$\Gamma_x = T_x + \sum_{y<x} P_{y,x}\,T_y \,.$$

What I call Γ_x is the same as C'_x in [Kazhdan-Lusztig:1979]. For example,

$$\Gamma_s = T_s + 1 \,.$$

The previous theorem will be supplemented by:

2.2. Theorem. *The polynomial $P_{y,x}$ is a polynomial in $\mathbb{Z}[q]$ of degree at most $\big(\ell(x)-\ell(y)-1\big)/2$. Its constant term is 1.*

Equivalently, *if $\pi_{y,x}(v) = \sum p_n v^n$ then $p_n = 0$ unless $n \equiv \ell(x) - \ell(y) - 1$ modulo 2.*

There are many reasons for introducing these polynomials. One important one is that if W is the Weyl group of a Kac-Moody group, they are the Poincaré polynomials of local intersection cohomology on the subvarieties $B\backslash ByB \subseteq B\backslash\overline{BxB}$. The ι-invariance of γ_x in this case is related to the self-duality of the perverse sheaves giving rise to intersection cohomology. This interpretation of $P_{y,x}$ implies that all its coefficients are non-negative. This was found to be true for all finite Coxeter groups, and conjectured many years ago to be true for all Coxeter groups. This was proved recently by Elias and Williamson, using an idea of Wolfgang Soergel.

For the proof of these theorems, which I give in a later section, I'll follow closely the treatment in [Soergel:1997], which is somewhat more concise than others in the literature, although not substantially different from the original one of [Kazhdan-Lusztig:1979].

Now for some simple examples.

2.3. Proposition. *We have*

$$\begin{aligned}
\gamma_1 &= \tau_1 = I \\
\gamma_s &= \tau_s + v \\
\gamma_{st} &= \tau_{st} + v\tau_s + v\tau_t + v^2 \quad (s \neq t)\,.
\end{aligned}$$

Proof. (a) It is trivial that $\gamma_1 = \tau_1$.

(b) We have already defined γ_s and seen that $\gamma_s^\iota = \gamma_s$, but now I'll show that no other definition of γ_s is possible. We have

$$\begin{aligned}\gamma_s &= \tau_s + v\pi_{1,s}(v) \\ \gamma_s^\iota &= \tau_s + (v - v^{-1}) + v^{-1}\pi_{1,s}(v^{-1}) = \tau_s + v + \big(v^{-1}\pi_{1,s}(v^{-1}) - 1\big)\end{aligned}$$

which implies that $\pi_{1,s} - 1 = 0$, so γ_s has to be $\tau_s + v$.

(c) Suppose $s \neq t$. What is γ_{st}? As we'll see in a while, the key to the construction of all the γ_w is the observation that *if γ_x is invariant under ι then so is $\gamma_s\gamma_x$*. In this case

$$\begin{aligned}\gamma_s\gamma_t &= (\tau_s + v)(\tau_t + v) \\ &= \tau_s\tau_t + v\tau_s + v\tau_t + v^2 \\ &= \tau_{st} + v\tau_s + v\tau_t + v^2 \,,\end{aligned}$$

so that we can choose $\gamma_{st} = \gamma_s\gamma_t$, with $\pi_{s,st} = \pi_{t,st} = 1$, $\pi_{1,st} = v$. Uniqueness in this case is simple—if γ is another element of $\mathcal{H}$ satisfying the conditions, then $\gamma_{st} - \gamma$ will be ι-invariant and of the form

$$\Pi = vp_s(v)\tau_s + vp_t(v)\tau_t + vp_1(v) \,,$$

hence also of the form

$$vp_s(v)\gamma_s + vp_t(v)\gamma_t + vp_1(v) \,.$$

But then

$$\Pi^\iota = v^{-1}p_s(v^{-1})\gamma_s^\iota + v^{-1}p_t(v^{-1})\gamma_t^\iota + v^{-1}p_1(v^{-1})$$

which implies that all of $vp_s(v)$, $vp_t(v)$, and $vp_1(v)$ vanish.

3. The Bruhat closure

In this section I review what we'll need to know eventually about Bruhat closures in Coxeter groups. The standard reference is Chapter 2 of [Bjorner-Brenti:2005], but it is has far more material than we'll need. More succinct, and adequate for my purposes here, is [Casselman:2012].

Given x and y in W, I write $x \Leftarrow y$ if $x = ry$ for some reflection r with $\ell(x) < \ell(y)$. Since the conjugation of a reflection is also a reflection and $ry = y \cdot y^{-1}ry$, whether r appears on left or right does not matter. Then $x < y$ if there exists a chain

$$x = x_0 \Leftarrow x_1 \Leftarrow \ldots \Leftarrow x_n = y \,.$$

In particular, $x < y$ if $y = sx$ or sx with $\ell(y) = \ell(x) + 1$. The closure $\overline{y}$ is the set of all $x \leq y$.

As far as I can see, all we need in addition about this partial order is:

3.1. Proposition. (s-stability) *If $sy < y$ then the closure of y is stable under left multiplication by s.*

This is equivalent to the 'lifting theorem' (Proposition 2.2.7) of [Bjorner-Brenti:2005]. It is proved directly in §4.2 of [Casselman:2012].

3.2. Corollary. *If $y = sx > x$ then $\overline{y} = s\overline{x} \cup \overline{x}$, and similarly if $y = xs > x$.*

4. Proofs

In a moment, we are going to need this elementary observation:

4.1. Lemma. *Suppose that for every $y \le x$ we have defined polynomials Π_y such that*

$$\Pi_y = \tau_y + \sum_{z<y} v p_{z,y}(v)\tau_z .$$

with $p_{z,y} \in \mathbb{Z}[v]$. Then we also have

$$\tau_y = \Pi_y + \sum_{z<y} v q_{z,y}(v)\Pi_z$$

for suitable $q_{z,y}(v)$ in $\mathbb{Z}[v]$.

Proof. By an easy induction. This amounts to solving a system of equations whose coefficient matrix is unipotent.

Now I begin the proof of Theorem 2.1, which will come in two parts.

Existence. I am going to prove by induction on $\ell(x)$ that *there exists a self-dual γ_x such that*

$$\gamma_x = \tau_x + \sum_{y<x} v\pi_{y,x}(v)\tau_y .$$

with $\pi_{y,x}(v) \in \mathbb{Z}[v]$.

We have already found $\gamma_1 = 1$, $\gamma_s = \tau_s + v$. Now assume $\ell(x) > 1$ and assume the Proposition true for $y < x$. Choose s with $y = sx < x$. We know the claim to be true for y so that

$$\gamma_y = \tau_y + \sum_{z<y} v\pi_{z,y}(v)\tau_z .$$

Multiplying on the left by γ_s and applying (1.5), we get the ι-invariant product:

$$\begin{aligned}
\gamma_s\gamma_y &= \gamma_s\tau_y + \sum_{z<y} v\pi_{z,y}(v)\gamma_s\tau_z \\
&= \tau_x + v\tau_y + \sum_{\substack{z<y\\ sz>z}} v\pi_{z,y}(v)(\tau_{sz} + v\tau_z) + \sum_{\substack{z<y\\ sz<z}} v\pi_{z,y}(v)(v^{-1}\tau_z + \tau_{sz}) \\
&= \tau_x + v\tau_y + \sum_{\substack{z<y\\ sz>z}} v\pi_{z,y}(v)(\tau_{sz} + v\tau_z) + \sum_{\substack{z<y\\ sz<z}} v\pi_{z,y}(v)\tau_{sz} + \sum_{\substack{z<y\\ sz<z}} \pi_{z,y}(v)\tau_z .
\end{aligned}$$

This is of the right form, except for the last sum. Its easy to fix this problem by defining

$$\gamma_x = \gamma_s\gamma_y - \sum_{\substack{z<y\\ sz<z}} \pi_{z,y}(0)\,\gamma_z , \tag{4.2}$$

which gives us

$$\begin{aligned}
\gamma_x = \tau_x + v\tau_y &+ \sum_{\substack{z<y\\ sz>z}} v^2\pi_{z,y}(v)\tau_z + \sum_{\substack{z<y\\ sz>z}} v\pi_{z,y}(v)\tau_{sz} \\
&+ \sum_{\substack{z<y\\ sz<z}} v\pi_{z,y}(v)\tau_{sz} + \sum_{\substack{z<y\\ sz<z}} \big(\pi_{z,y}(v) - \pi_{z,y}(0)\big)\tau_z + \sum_{\substack{z<y\\ sz<z}} \pi_{z,y}(0)\big(\tau_z - \gamma_z\big) .
\end{aligned} \tag{4.3}$$

and therefore corrects the last sum. This concludes the first step.

Uniqueness. We now have for each x in W an element γ_x in $\mathcal{H}$ which is fixed by ι and of the form

$$\gamma_x = \tau_x + \sum_{y<x} v p_{y,x}(v)\tau_y \,.$$

It remains to see that it is unique with these properties. This will follow from the claim: *If* $\sum_y v\pi_y(v)\tau_v$ *is fixed by the involution* ι*, it is* 0.

Proof of the claim. By Lemma 4.1 it is equivalent to the claim that if

$$\sum_x v p_x(v)\gamma_x \quad \big(p_x(v) \in \mathbb{Z}[v]\big)$$

is invariant under ι then it vanishes. This is immediate. The proof of Theorem 2.1 is concluded.

It remains to prove .

We can deduce from this argument a recursive formula for the $\pi_{y,x}(v)$. From now on let

$$\mu(w,z) = \begin{cases} \pi_{w,z}(0) & \text{if } w < z \\ \mu(z,w) & \text{if } z < w \\ 0 & \text{otherwise.} \end{cases}$$

From (4.3) and the formula

$$\gamma_z = \tau_z + \sum_{w<z} v\pi_{w,z}(v)\tau_w$$

the following is immediate:

$$\begin{aligned} \gamma_x = \tau_x + v\tau_y + \sum_{\substack{z<y \\ sz>z}} v\pi_{z,y}(v)\,\tau_{sz} + \sum_{\substack{z<y \\ sz>z}} v^2\pi_{z,y}(v)\,\tau_z \\ + \sum_{\substack{z<y \\ sz<z}} \big(\pi_{z,y}(v) - \mu(z,y)\big)\,\tau_z + \sum_{\substack{z<y \\ sz<z}} v\pi_{z,y}(v)\,\tau_{sz} \\ - \sum_{\substack{w,z \\ sz<z<y \\ w<z}} \mu(z,y)\,v\,\pi_{w,z}(v)\tau_w \,. \end{aligned}$$

Because of s-stability of Bruhat closures, the map $z \mapsto sz$ is a bijection of $\{z < y \,|\, sz > z\}$ with $\{w < x \,|\, sw < w\}$. Since $\pi_{y,z} = 0$ if $y \not\leq z$ the expression above becomes

$$\begin{aligned} \gamma_x = \tau_x + v\tau_y + \sum_{\substack{z<x \\ sz<z}} v\pi_{sz,y}(v)\,\tau_z + \sum_{\substack{z<y \\ sz>z}} v^2\pi_{z,y}(v)\,\tau_z \\ + \sum_{\substack{z<y \\ sz<z}} \big(\pi_{z,y}(v) - \mu(y,z)\big)\,\tau_z + \sum_{\substack{z<y \\ sz>z}} v\pi_{sz,y}(v)\,\tau_z \\ - \sum_{\substack{w,z \\ sz<z<y \\ w<z}} \mu(z,y)\,v\,\pi_{w,z}(v)\tau_w \,, \end{aligned}$$

leading to this recursive formula for $\pi_{y,x}$:

4.4. Proposition. *Suppose $sx < x$, $y \le x$. Then*

$$\pi_{y,x} = \begin{cases} 1/v & y = x \\ 1 & y = sx \\ \dfrac{\pi_{y,sx}(v) - \mu(y,sx)}{v} + \pi_{sy,sx}(v) - \displaystyle\sum_{\substack{w \\ y<w<sx \\ sw<w}} \mu(w,sx)\pi_{y,w}(v) & sy < y \\ \pi_{sy,sx}(v) + v\pi_{y,sx}(v) - \displaystyle\sum_{\substack{w \\ y<w<sx \\ sw<w}} \mu(w,sx)\pi_{y,w}(v) & sy \ne x, sy > y. \end{cases}$$

From the third formula:

4.5. Corollary. *Suppose $sx < x$, $sy < y < x$ but $y \not\le sx$. Then $\pi_{y,x} = \pi_{sy,sx}$.*

Recall that $T_x = v^{\tau\ell(x)}T_x$, $\Gamma_x = v^{-\ell(x)}\gamma_x$. In particular, since $\gamma_s = \tau_s + v$,

$$\Gamma_s = T_s + 1\,.$$

If we multiply the equation

$$\gamma_x = \tau_x + \sum_{y<x} v\pi_{y,x}(v)\,\tau_y$$

by $v^{-\ell(x)} = q^{\ell(x)/2}$ we get

$$\begin{aligned} \Gamma_x &= T_x + \sum_{y<x} q^{-1/2}q^{(\ell(x)-\ell(y))/2}\pi_{y,x}(v)\,T_y \\ &= T_x + \sum_{y<x} P_{y,x}\,T_y \qquad \big(P_{y,x} = q^{(\ell(x)-\ell(y)-1)/2}\pi_{y,x}(q^{-1/2})\big)\,. \end{aligned}$$

We know that $\pi_{y,x}(v)$ is a polynomial in v, hence $P_{y,x}$ is $v^{-(\ell(x)-\ell(y)-1)}$ times a polynomial in v. In particular, $\mu(y,x) = \pi_{y,x}(0)$ is the coefficient of $q^{(\ell(x)-\ell(y)-1)/2}$ in $P_{y,x}$. The recursion formula for γ_x leads to:

4.6. Corollary. *Suppose $x = sz > z$. Then for $y < x$*

$$\begin{aligned} P_{y,x} &= P_{sy,z} + qP_{y,z} - \sum_{\substack{w \\ y\le w<z \\ sw<w}} \mu(w,z)q^{(\ell(x)-\ell(w))/2}\,P_{y,w} \quad (sy < y) \\ &= qP_{sy,z} + P_{y,z} - \sum_{\substack{w \\ y\le w<z \\ sw<w}} \mu(w,z)q^{(\ell(x)-\ell(w))/2}\,P_{y,w} \quad (sy > y)\,. \end{aligned}$$

Here we set $P_{y,x} = 0$ if $y \not\le x$. The formula can also be written more succinctly, as it is in [Kazhdan-Lusztig:1979]:

$$P_{y,x} = q^{1-c}P_{sy,z} + q^cP_{y,z} - \sum_{\substack{w \\ y\le w<z \\ sw<w}} \mu(w,z)q^{(\ell(x)-\ell(y))/2}\,P_{y,w} \quad c = \begin{cases} 1 & sy < y \\ 0 & sy > y \end{cases}$$

Finally, to prove that $P_{y,x}$ is a polynomial in $\mathbb{Z}[q]$. It is equivalent to the claim that if $\pi_{y,x}(v) = \sum p_nv^n$ then $p_n = 0$ unless $nequiv\ell(x) - \ell(y) - 1$ modulo 2. It implies that $\mu(y,x) = 0$ unless $\ell(x)$ and $\ell(y)$ have different parity. Proceed by induction on $\ell(x)$. We know from earlier examples that it is true for $\ell(x) \le 2$. But then induction and Proposition 4.4 allow us to conclude.

5. Elementary properties

Define $w \prec z$ to mean $w < z$ and $\mu(w,z) \neq 0$.

5.1. Proposition. *We have*

$$\tau_s \gamma_x = \begin{cases} -v\gamma_x + \gamma_{sx} + \displaystyle\sum_{\substack{y \\ y \prec x \\ sy < y}} \mu(y,x)\gamma_y & \textit{if } sx > x \\ v^{-1}\gamma_x & \textit{otherwise.} \end{cases}$$

These play a role in the representation theory of $\mathcal{H}$.

Proof. A simple analogue of (4.2) tells us that if $w = sx > x$ then

$$\gamma_w = \gamma_s\gamma_x - \sum_{\substack{y<x \\ sy<y}} \mu(y,x)\gamma_y \,,$$

which translates immediately to the first equation, since $\gamma_s = \tau_s + v$. For the second, I start with the simplest case $\tau_s\gamma_s = v^{-1}\gamma_s$, which is an easy calculation. If $sx < x$ we can rewrite the definition of γ_x as

$$\gamma_x = \gamma_s\gamma_w - \sum_{\substack{y<w \\ sy<y}} \mu(y,w)\gamma_y \,,$$

where $sx = w$. But then I apply induction:

$$\begin{aligned} \tau_s\gamma_x &= \tau_s\gamma_s\gamma_w - \sum_{\substack{y<w \\ sy<y}} \mu(y,w)\tau_s\gamma_y \\ &= v^{-1}\gamma_x \,. \ \square \end{aligned}$$

5.2. Proposition. *If $sx < x$ but $sy > y$ then*

$$\pi_{y,x} = v\pi_{sy,x} \,.$$

Proof. If $sx < x$ then from the second formula above we obtain

$$\begin{aligned} \tau_s\gamma_x = v^{-1}\gamma_x = v^{-1}\sum_{y \le x} v\pi_{y,x}(v)\tau_y \\ = \sum_{\substack{y<x \\ sy>y}} \pi_{y,x}(v)\tau_y + \sum_{\substack{y\le x \\ sy<y}} \pi_{y,x}(v)\tau_y \,. \end{aligned}$$

But also

$$\begin{aligned} \Big(\sum_{y\le x} v\pi_{y,x}(v)\tau_s\tau_y\Big) &= \sum_{\substack{y\le x \\ sy<y}} v\pi_{y,x}(v)\tau_s\tau_y + \sum_{\substack{y<x \\ sy>y}} v\pi_{y,x}(v)\tau_s\tau_y \\ &= \sum_{\substack{y\le x \\ sy<y}} v\pi_{y,x}(v)\big((v^{-1}-v)\tau_y + \tau_{sy}\big) + \sum_{\substack{y\le x \\ sy>y}} v\pi_{y,x}(v)\tau_{sy} \,. \end{aligned}$$

Cancelling, we have

$$\sum_{\substack{y<x\\ sy>y}} \pi_{y,x}(v)\tau_y = \sum_{\substack{y\le x\\ sy<y}} v\pi_{y,x}(v)(\tau_{sy} - v\tau_y) + \sum_{\substack{y<x\\ sy>y}} v\pi_{y,x}(v)\tau_{sy}$$

to which I again apply s-stability.

5.3. Corollary. *if $x < y$, $x > x$, $sy < y$, then either $y = sx$, in which case $\mu(x, y) = 1$, or $\mu(x, y) = 0$.*

All these results have analogues for the right action of s as well.

For w in W, let

$$\begin{aligned} L_w &= \{s \in S \mid sw < w\} \\ R_w &= \{s \in S \mid ws < w\} \\ \mathcal{L}_w &= \{r \text{ a reflection} \mid rw < w\} \\ \mathcal{R}_w &= \{r \text{ a reflection} \mid wr < w\} \end{aligned}$$

If r is a reflection then there exists exactly one geometric root $\lambda > 0$ such that $r = s_\lambda$, and $rw < w$ if and only if $w^{-1}\lambda < 0$. Each of the sets defined above could have similarly been specified in terms of positive roots as well. The following is well known:

5.4. Lemma. *Suppose $w = xy$. The following are equivalent:*

(a) $\ell(w) = \ell(x) + \ell(y)$;
(b) $\mathcal{L}_{xy}$ is the disjoint union of $\mathcal{L}_x$ and $x\mathcal{L}_y x^{-1}$;
(c) $\mathcal{R}_{xy}$ is the disjoint union of $y^{-1}\mathcal{R}_x y$ and $\mathcal{R}_y$.

Define $x \rightarrow_L y$ if $\mu(x, y) \ne 0$ and $L_y \not\subseteq L_x$. These oriented links define the **left W-graph** of W. To each link is associated also the label $L_y - L_x$, so *the right W-graph is an oriented, labelled graph*. Similarly define $x \rightarrow_R y$ and the right W-graph. For example, if $x < y = sx$ then $s \in L_y - L_x$ and $\mu(x, y) = 1$ according to extremal, so $x \rightarrow_L sx$. These turn out to be the only edges in the left W-graph going up:

5.5. Proposition. *Suppose $x \rightarrow_L y$ with $x < y$. Then $y = sx$ for some s in S.*

Proof. Choose s in $L_y - L_x$, so $sy < y$ but $sx > x$. Then by Corollary 5.3 $\pi_{x,y} = v\pi_{sx,y}$. If $y \ne sx$ then both π are polynomials, so $\mu(x, y) = 0$.

If $xs > s$ then there is certainly an edge $x \rightarrow_R xs$ in the W-graph. Under what circumstances is there an edge backwards? Well, $\mu(x, y)$ is symmetric, so the answer is, precisely when there exists t in S with $xt < x$ but $xst > xs$. In this case, I shall say that x and y are R-equivalent: $x \equiv_R y$. More generally, $x \equiv_R y$ if there exists a sequence of relations $x_i \rightarrow_R x_{n+1}$ with $x = x_0 = x_n$ with y equal to one of the x_i. Similarly $x \equiv_L y$.

This criterion covers all such equivalence relations for the dihedral Coxeter groups, say generated by s and t. The identity and the longest element w_ℓ each make up an equivalence class. So do all the w such that $ws < w$ except for w_ℓ, and all the w such that $wt < w$ except for w_ℓ.

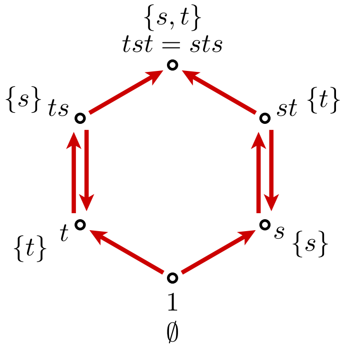
Example. Let W be the Coxeter system generated by s, t with $sts = tst$. (The group is $\mathfrak{S}_3$.) We have already seen

$$\begin{aligned} \gamma_1 &= \tau_1 \\ \gamma_s &= \tau_s + v\tau_1 \\ \gamma_t &= \tau_1 + v\tau_1 \\ \gamma_{st} &= \tau_{st} + v\tau_s + v\tau_t + v^2\tau_1 \\ \gamma_{ts} &= \tau_{ts} + v\tau_s + v\tau_t + v^2\tau_1 \end{aligned}$$

and now we calculate

$$
\begin{aligned}
\gamma_s\gamma_t\gamma_s &= \tau_{sts} + v\tau_{st} + v\tau_{ts} + v^2\tau_t + v^2\tau_s + \tau_s + (v + v^3)\tau_1 \\
\gamma_{sts} &= \gamma_s\gamma_t\gamma_s - \gamma_s \\
&= \gamma_s\gamma_t\gamma_s - (\tau_s + v\tau_1) \\
&= \tau_{sts} + v\tau_{st} + v\tau_{ts} + v^2\tau_t + v^2\tau_s + v^3\tau_1 \,.
\end{aligned}
$$

All links in the right W-graph are of length 1. The graph, with its nodes labelled by right descent sets, looks like this:



In $\mathfrak{S}_3$ as in every dihedral group and as the figure shows, there are four right-equivalence classes.

5.6. Proposition. *If $x \to_L y$ then $R_x \subseteq R_y$. and if $x \equiv_L y$ then $R_x = R_y$. Similarly for links $x \to_R y$.*

Proof. First of all, suppose $x \to_L y$ and $x < y$. Say $y = sx > x$. Then

$$
\begin{aligned}
\mathcal{R}_y &= \mathcal{R}_{sx} = \{x^{-1}sx\} \sqcup \mathcal{R}_x \supset \mathcal{R}_x \\
R_y &= \mathcal{R}_y \cap S \supseteq \mathcal{R}_x \cap S = R_x \,.
\end{aligned}
$$

That takes care of the links in the W-graph going up.

Now suppose $x \to_L y$ and $y < x$. This means that $\mu(y, x) \neq 0$ and $L_y \not\subseteq L_x$. Suppose s lies in R_x but $s \notin R_y$. Then $xs < x$, but $ys > y$. According to Corollary 5.3, $\pi_{y,x} = v\pi_{ys,x}$. If $ys \neq x$ then $\mu(y, x) = 0$, a contradiction.

So we are reduced to the case $ys = x$. But if $ys > y$ then $L_y \not\subseteq L_x$ and $y \to_L x$. According to Proposition 5.5, $x = ty$ for some t in S. However, if $x = ty > y$ then

$$
\mathcal{R}_x = \mathcal{R}_{ty} = \mathcal{R}_y \sqcup \{y^{-1}ty\}, \ \text{ hence } R_y \subseteq R_x \,,
$$

which contradicts the assumption on s.

The example of $\mathfrak{S}_3$ exhibits a special case of a very useful observation. Let $\mathcal{U}$ be the set of all $w \neq 1$ in an arbitrary W possessing a unique reduced expression. For each w in $\mathcal{U}$ the set L_w is a singleton. Hence $\mathcal{U}$ is the disjoint union of its subsets $\mathcal{U}_s$ made up of w such that $L_w = \{s\}$.

5.7. Proposition. *The set $\mathcal{U}_s$ is a single left cell in W. The union $\mathcal{U}$ is a single two-sided cell in W.*

Proof. Left as exercise.

6. The star operations

As we have already seen, if $xs > s$ and there exists t in S with $xt < x$ but $xst > xs$, then $x \equiv_R xs$.

I shall reformulate this criterion in special circumstances. Suppose s, t to be in S with $W_{s,t}$ isomorphic to $\mathfrak{S}_3$. Following [Kazhdan-Lusztig:1979], define $\mathcal{D}_R(s,t)$ to be the set of w such that $R_w \cap \{s,t\}$ consists of a single element. Another way to see this: we can write $w = xy$ with y in $W_{s,t}$ and x a distinguished coset element satisfying the condition $xs > x$, $xt > x$. Then w is in $\mathcal{D}_R(s,t)$ if and only if $y \in \{s, t, st, ts\}$. The set $\mathcal{D}_R(s,t)$ is the domain of an operation to be defined here. First of all, on the set $W_{s,t} \cap \mathcal{D}_R(s,t)$, define the involution $x \mapsto x^{|s,t}$:

$$
\begin{aligned}
s &\longleftrightarrow st \\
t &\longleftrightarrow ts\,.
\end{aligned}
$$

Next, if $w = xy$ as above then set $w^{|s,t} = x\,y^{|s,t}$. These are called the 'star' operations because of the original notation $x^* = x^{|s,t}$ in [Kazhdan-Lusztig:1979]. The following is my reformulation:

6.1. Proposition. *If x lies in $\mathcal{D}_R(s,t)$ then $x \equiv_R x^{|s,t}$.*

Similarly, one can define left operations ${}^{s,t|}x$ and similar results hold for it.

So far, the discussion is elementary, but the following result is more subtle. It looks at first sight rather technical, but upon a second look seems quite remarkable. It is most important in analyzing the equivalence relation $\equiv_R$ for the symmetric groups and the affine Coxeter groups of type $\widetilde{A}_n$ reported on in [Shi:1976]. However, I wonder whether it might, in the long run, serve as a model for similar results about more general Coxeter groups. Although W-graphs (defined in the next section) are very complicated, there is some reason to think that the equivalence classes defined by them are determined by simpler, more local relations.

6.2. Theorem. *Suppose x and y both to lie in $\mathcal{D}_R(s,t)$, and assume that $y \notin x\,W_{s,t}$. Then $y \prec x$ if and only if $y^{|s,t} \prec x^{|s,t}$, and in this case $\mu(x,y) = \mu(x^{|s,t}, y^{|s,t})$.*

Similarly for ${}^{s,t|}x$, ${}^{s,t|}y$. What is remarkable about this is that it will not usually be the case that $\pi_{y,x} = \pi_{y^{|s,t},x^{|s,t}}$.

Proof. Let $x^* = x^{|s,t}$, same for y^*. Define $\pi(v) \sim \rho(v)$ to mean that $\pi(v) - \rho(v)$ is divisible by v. In particular $\pi_{y,x} \sim \mu(y,x)$ (its constant term). The proof in all cases will show that $\pi_{y,x} \sim \pi_{y^*,x^*}$. There are two basic cases. The elements $x^{-1}x^*$ and $y^{-1}y^*$ are both in $\{s,t\}$. Are they the same or not?

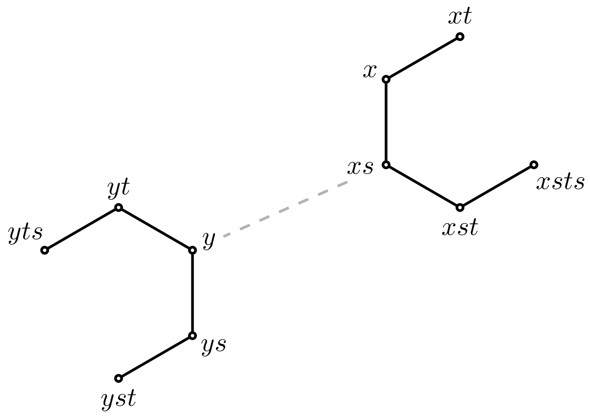
Case 1. Suppose $x^* = xu$, $y^* = yu$ for $u \in \{s,t\}$. Renaming s and t if necessary, we may assume $u = s$. This case is completely symmetric, so it suffices to show that if $y \prec x$ then $y^* \prec x^*$. There are four cases to be considered, depending on how x, y relate respectively to xs, ys.

Case (i) $ys > y$, $xs < x$. This is impossible by Corollary 5.3, since by assumption we cannot have $ys = x$.

Case (ii) $ys < y$, $xs > x$. This is also impossible, again by Corollary 5.3, because now $xt < x$, $yt > y$.

Cases (iii) $ys < y$, $xs < x$ and **(iv)** $ys > y$, $xs > x$. By symmetry, we only have to deal with one of them, say (iii). So we assume now that $x^* = xs < x$, $y^* = ys < y$.

◇ If $y \not\leq xs$ then, by Corollary 4.5, $\pi_{y^*,x^*} = \pi_{ys,xs} = \pi_{y,x}$ and we are done.



$\diamond$ Otherwise, since $y \neq xs$ we now have $ys < y$, $xs < x$, $y < xs$, as indicated in the figure above. Since $y \prec x$, $\ell(x) - \ell(y)$ is odd, $\ell(xs) - \ell(y)$ is even and necessarily $\mu(y, xs) = 0$. Therefore by Proposition 4.4

$$\begin{aligned} \pi_{y,x} &= \pi_{ys,xs} + \frac{\pi_{y,xs} - \mu(y,xs)}{v} - \sum_{\substack{z \\ y<z<xs \\ sz<z}} \mu(z,xs)\pi_{y,z} \\ &\sim \pi_{ys,xs} + \frac{\pi_{y,xs}}{v} - \sum_{\substack{z \\ y<z<xs \\ sz<z}} \mu(y,z)\mu(z,xs) \\ &\sim \pi_{ys,xs} + \pi_{yt,xs} - \sum_{\substack{z \\ y<z<xs \\ sz<z}} \mu(y,z)\mu(z,xs)\,. \end{aligned}$$

The transition from line two to line three comes from Corollary 5.3, because $xst < xs$, $yt > y$.

$\diamond$ I claim that $z = yt$ occurs in the sum. The conditions on z are $y < z$, $z < xs$, $sz < z$.

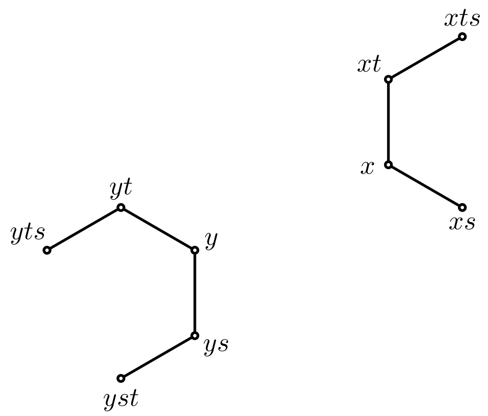
(a) $y < yt$ is immediate from the assumption $y \in \mathcal{D}_R(s,t)$. So is (b) $yts < yt$ (here is where we are using the assumption that $W_{s,t} = \mathfrak{S}_3$). It is also true that (c) $yt < xst < xs$, by S-stability of $\overline{xs}$ applied to t.

$\diamond$ It is the only term in the sum. If z occurs there, then $\mu(z, xs) \neq 0$, hence either $z = xst$ or $zt < z$, by Corollary 5.3. But if $z = xst$ then $zs > z$ and does not occur in the sum. Hence $zt < t$. But then since $\mu(y, z) \neq 0$, again by Corollary 5.3, $yt = z$ or $yt < y$. Since in fact $yt > y$, we must have $z = yt$.

$\diamond$ Since $\mu(y, yt) = 1$, we can now write

$$\pi_{y,x} \sim \pi_{ys,xs} + \pi_{yt,xs} - \mu(yt, xs) \sim \pi_{ys,xs}\,.$$

Case 2. We have $x^{|s,t} = xu$, $y^{|s,t} = yv$, $u \neq v$. Renaming if necessary, we may as well take $u = s$, $v = t$. There are two cases: $x^{|s,t} = xt < x$ and $x^{|s,t} = xt > x$. Again, because the final result is symmetric with respect to x, y and $x^{|s,t}$, $y^{|s,t}$, we may suppose $x^{|s,t} = xt > x$. Then $xs < x < xt < xts$, as in the right of this figure:



$\diamond$ We want to show that $y \prec x$ if and only if $ys \prec xt$. If $y \prec x$ then Corollary 5.3 tells us that since $xs < x$ and $ys \neq x$ so too is $ys < y$. If $ys \prec xt$ then (again by Corollary 5.3) since $xt \cdot t = x < xt$ so too is $yst < ys$. In either case we must have $yst < ys < y < yt$ as on the left in the figure above.

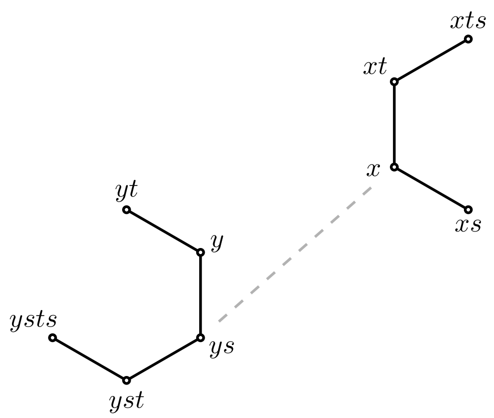
$\diamond$ I claim that if either $y \prec x$ or $ys \prec xt$, then $ys \leq x$. One case is trivial—if $y \prec x$ then $ys < y < x$.

For the other, suppose $ys \not\leq x$. Since $xt \cdot t = x < xt$, by Corollary 4.5 we have

$$\pi_{ys,xt} = \pi_{yst,x}$$

and $\mu(ys, xt) = \mu(yst, x)$. But since $xs < x$, $yst \cdot s > yst$ Corollary 5.3 tells us that $\mu(yst, x) = \mu(ys, xt) = 0$. Hence $ys \not\prec xt$.

$\diamond$ So we may now assume that $ys \leq x$, indicated in the following figure:



Under either assumption $y \prec x$ or $ys \prec xt$, $\ell(ys, x) = 0$, so by Proposition 4.4

$$\pi_{ys,xt} \sim \pi_{yst,x} + \pi_{ys,x}/v - \sum_{\substack{z \\ ys<z<x \\ zT<z}} \mu(ys, z)\mu(z, x)\,.$$

$\diamond$ I claim that the sum is empty. If $\mu(ys, z) \neq 0)$ then by Corollary 5.3 since $zs < xz$ and $ys \cdot s > ys$ we must have $z = y$. But then if $\mu(z, x) \neq 0$, since $y \neq zt$ byt $zt < z$ we must have $yt < y$, a contradiction.

$\diamond$ Therefore $\pi_{ys,xt} \sim \pi_{yst,x} + \pi_{ys,x}/v$. But (as I have observed before) since $ys \leq x$ and $s \in R_x$, $y \leq x$, $\pi_{ys,x}/v = \pi_{y,x}$. Furthermore, $\mu(yst, x) = 0$ since $xs < x$ while $ysts > yst$ (we are using again the fact that $W_{s,t} = \mathfrak{S}_3$). Therefore $\pi_{ys,xt} \sim \pi_{y,x}$.

6.3. Corollary. *Suppose $x \equiv_R y$. Then x is in $\mathcal{D}_L(s,t)$ if and only if y is in $\mathcal{D}_L(s,t)$, and then ${}^{s,t|}x \equiv_R {}^{s,t|}y$.*

Applied to x^{-1} and y^{-1}, this implies something similar when $x \equiv_L y$.

Proof. It suffices to prove just one half.

Again let ${}^*x = {}^{s,t|}x$, etc. Suppose $x \rightarrow_R y$. Recall that if $x \in \mathcal{D}_L(s,t)$ then $x \equiv_L {}^*x$, hence by Proposition 5.6 $R_x = R_{^*x}$. Because $x \rightarrow_R y$, $R_{^*y} = R_y \not\subseteq R_{^*x} = R_x$. Since $x \rightarrow_R y$ we have $x \prec y$, but then by the Proposition ${}^*x \prec {}^*y$. Therefore ${}^*x \rightarrow_R {}^*y$.

If $x \equiv_R y$, there exists a chain $x_i \rightarrow_R x_{i+1}$ from x through y and back to x. All the x_i are equivalent. The argument in the previous paragraph then tells us that ${}^*x_i \rightarrow_R {}^*x_{i+1}$, and we have a chain in the right W-graph from ${}^*x$ to ${}^*y$ and back again, so ${}^*x \equiv_R {}^*y$. ▯

7. W-graphs

Recall that $x \rightarrow_L y$ if $\mu(x,y) \neq 0$ and $L_y \not\subseteq L_x$. According to Proposition 5.5, this happens if and only if $y = sx > x$ or $y \prec x$ and $L_y \not\subseteq L_x$. The formula in Proposition 5.1 may therefore be written

$$\tau_s \gamma_x = \begin{cases} -v\gamma_x + \displaystyle\sum_{\substack{x \rightarrow_L y \\ sy < y}} \mu(y,x)\gamma_y & \text{if } sx > x \\ v^{-1}\gamma_x & \text{otherwise.} \end{cases}$$

More generally, I modify the definition in [Kazhdan-Lusztig:1979] to define a **W-graph** to be generalization of this. Let X be a set of nodes of the graph. For each node x one is given a set $I_x \subseteq S$ for each node x, a constant $\mu(x,y)$ in a ring F for each pair of nodes, and for each pair of nodes x, y with $\mu(x,y) \neq 0$ and $I_y \not\subseteq I_x$ is given an edge $x \rightarrow y$. For each s in S one defines an endomorphism of $F[X]$:

$$t_s x = \begin{cases} v^{-1}x & \text{if } s \in I_x \\ -vx + \displaystyle\sum_{\substack{x \rightarrow y \\ s \in I_y}} \mu(y,x)y & \text{otherwise.} \end{cases}$$

The important point is that all targets except x have opposite s-parity. It is immediate from this definition that $t_s^2 = (v^{-1} - v)t_s + I$, and these data are said to define a W-graph if in addition the braid relations are satisfied. In these circumstances $F[X]$ becomes a module over $\mathcal{H}(W)$.

Alternatively, one is given for each s in S a partition of $X = X_s^+ \sqcup X_s^-$ and a set of edges $x \rightarrow_s y$ from nodes x in X_s^+ to nodes y in X_s^-, and for each such edge a function $\mu(x,y)$, with corresponding assumptions.

The nodes of a W-graph may be partitioned into equivalence classes—I write $x \equiv y$ if there exists a chain of links in the graph from x to y and back again. The equivalence classes are called the cells of the graph. If C and D are two cells, then I write $C \prec D$ if there exists a link from a node in D to one in C, and $C < D$ if there exists a chain of such links.

If C is a cell of a W-graph, then the sum of the $F[D]$ for $D < C$ is stable under the Hecke algebra, and $F[C]/\sum_{D<C} F[D]$ is therefore a module over the Hecke algebra.

For example, in the left W-graph of W itself the sum of nodes $x \neq I$ span a submodule, and $F[W]/\sum_{x \neq I} F{\cdot}x$ is the one-dimensional module on which τ_s acts as $-vI$. Similarly, if W is finite with longest element w_ℓ, then $F[w_\ell]$ spans a one-dimensional module on which τ_s acts by v^{-1}.

8. KL-cells and the Robinson-Schensted correspondence

Let (W, S) for the moment be an arbitrary Coxeter group. Suppose c to be a chain of pairs (s_i, t_i) with each s_i, t_i in S and generating a copy of $\mathfrak{S}_3$. I define the domain $\mathcal{D}_L(c)$ as well as the operator $x \mapsto {}^{c|}x$ from $\mathcal{D}_L(c)$ to W by induction.

(a) If c is the single pair (s, t) then $\mathcal{D}_L(c) = \mathcal{D}_L(s, t)$ and ${}^{c|}x = {}^{s,t|}x$;

(b) if d is the concatenation of (s, t) and c then $\mathcal{D}(d)$ the set of x in $\mathcal{D}_L(c)$ such that ${}^{c|}x$ lies in $\mathcal{D}_L(s, t)$ and then ${}^{d|}x = {}^{s,t|c|}x$.

Similarly for a right operator $x \mapsto x^{|c}$ with domian $\mathcal{D}_R(c)$. From Corollary 9.2 we deduce immediately:

8.1. Lemma. *Suppose $x \equiv_R y$. Then x is in $\mathcal{D}_L(c)$ if and only if y is in $\mathcal{D}_L(c)$, and then ${}^{c|}x \equiv_R {}^{c|}y$.*

I'll say that $x \equiv_\ell y$ (they are said to be **left Knuth-equivalent**) if $y = {}^{c|}x$ for some c. Similarly, $x \equiv_r y$ is right equivalence.

From Proposition 6.1 we deduce immediately that if $x \equiv_r y$ then $x \equiv_R y$. This is true for any Coxeter group. It turns out that for two classes of Coxeter groups the converse is true. I'll explain one in detail.

Let $W = \mathfrak{S}_n$. For each i in $[1, n)$ let s_i be the swap of i and $i + 1$, and let S be the set of all s_i. This makes W into a Coxeter group.

8.2. Theorem. *If $W = \mathfrak{S}_n$ then $x \equiv_R y$ if and only if $x \equiv_r y$.*

This result originated in [Kazhdan-Lusztig:1979], but the argument there depended on some non-elementary results about Lie algebras due to David Vogan. As far as I can tell the first purely combinatorial proof was published by Ariki, but his original paper was in Japanese and escaped notice. He published the English version [Ariki:2000] several years later. The argument I'll follow is only a mild expansion of his. I'd like to thank Peter Trapa for some assistance in working this out.

The consequence is that the Kazhdan-Lusztig cells of $\mathfrak{S}_n$ agree with structures constructed a long time ago with very different aims in mind.

Proof. In several steps.

Step 1. I'll begin by recalling the Robinson-Schensted correspondence. The principal reference for this is [Knuth:1970].

To each x in $\mathfrak{S}_n$ the Robinson-Schensted correspondence associates to each permutation x a pair (P_x, Q_x) of Young tableaux of the same shape, size n, entries in $[1, n]$. This induces a bijection between the two sets, permutations and pairs.

∘ Two permutations x, y are right Knuth-equivalent if and only if $P_x = P_y$, and they are left Knuth-equivalent if and only if $Q_x = Q_y$.

∘ If $x \equiv_r y$ then x lies in $\mathcal{D}_L(c)$ if and only if y does, and in this case ${}^{c|}x \equiv_r {}^{c|}y$. (This is the analogue of Lemma 8.1. I am not aware of a clear formulation of this other than in §11 of [Casselman:2015].)

Step 2. The dual of a partition π is the partition $\overline{\pi}$ associated to the shape obtained by reflecting that of π around its NW-SE axis—replacing rows by columns. For example, the dual of the partition in the figure below is $(4, 3, 2)$.

1	5	8
2	6	9
3	7	
4		

Duality inverts order.

Suppose π to be a partition, $\overline{\pi}$ its dual. Define

$$\gamma_k = \overline{\pi}_1 + \cdots + \overline{\pi}_k \, .$$

Here γ_0 is taken to be 0. Define the tableau T_π to be that in which the $i+1$-st column is

$$\gamma_i + 1, \ldots, \gamma_{i+1} \, .$$

For example, the tableau pictured just above is the standard tableau T_π for $\pi = (3,3,2,1)$.

Step 3. Suppose T to be a tableau. A **simple inversion** in T is a pair $i, i+1$ occurring in T with the occurrence of i weakly to the right of that of $i+1$. Let $\mathcal{L}(T)$, the **descent set** of T, be the set of simple inversions of T. For example, if T is the tableau in the figure above, its simple inversions are $(1,2)$, $(2,3)$, $(3,4)$, $(5,6)$, $(6,7)$, $(8,9)$.

Let $\pi(T)$ be the partition determined by a tableau T. There is a natural partial order on partitions of n: $\pi \le \rho$ if

$$\sum_{i \le m} \pi_i \le \sum_{i \le m} \rho_i$$

for all $1 \le m \le n$.

8.3. Lemma. *Suppose T to be a tableau, π a partition of the same size. If $\mathcal{L}(T) = \mathcal{L}(T_\pi)$, then $\pi(T) \le \pi$. If $\pi(T) = \pi$ as well, then $T = T_\pi$.*

In other words, T_π has the largest shape of all tableau with a given descent set, and is unique with that property.

Proof. Let T be given with $\mathcal{L}(T) = \mathcal{L}(T_\pi)$.

The dual of a partition π is that associated to the shape obtained by reflecting that of π around its NW-SE axis—replacing rows by columns. For example, the dual of the partition in the figure above is $(4,3,2)$. Duality inverts order. So it must be shown that the partition associated to the columns of T weakly dominates that of T_π. This follows from the assertion that *all of the items in the first k columns of T_π must lie in the first k columns of T, for all k.* This also implies the last claim of the Proposition.

Why is this assertion true for $k = 1$? The descent set of T_π contains all pairs $(k, k+1)$ in its first column, for $k < \overline{\pi}_1$. Suppose $(1,2)$ to be one of these pairs. Then by definition 2 must lie in the first column of Q, and the only possible position is just below 1 since there are no integers between 1 and 2. By induction on $\overline{\pi}_1$ we see that the first column of T starts out exactly as the first column of T_π, and furthermore that if $\pi(T) = \pi$ then these columns coincide.

Continue on by induction on k. Suppose some item in column $k+1$ of T_π does not lie in the first $k+1$ columns of T. Suppose it to be $\gamma_k + \ell$ with $\ell \ge 1$, smallest with this property. If $\ell > 1$ then $\gamma_k + \ell - 1$, $\gamma_{\ell+2}$ is not an inversion for T_π, a contradiction. So we then have $\gamma_k + 1$ in some column larger then k. But then all of the other items in the first row must be the $\gamma_i + 1$ for $i \le k-1$. But $k-1$ items cannot fill k places, so this is a contradiction.

Step 4. Suppose a permutation x given, and let $\pi = \pi(x)$ be the partition associated to its diagram P_x. Because the Robinson-Schensted correspondence is a bijection, it makes sense to define $\overline{x}$ to be the permutation with $P_{\overline{x}} = P_x$, and $Q_{\overline{x}} = T_\pi$. Then $\overline{x} \equiv_r x$.

Step 5. Suppose that $x \equiv_R y$. Choose c such that ${}^{c|}x = \overline{x}$. By Corollary 9.2 we know y lies in $\mathcal{D}_c$, which means that ${}^{c|}y$ is well defined. From Corollary 9.2 we also know that $\overline{x} \equiv_R {}^{c|}y$, and hence that $\mathcal{L}(\overline{x}) = \mathcal{L}({}^{c|}y)$. By the Lemma we know that $\pi(y) = \pi({}^{c|}y) \le \pi(\overline{x}) = \pi(x)$.

By symmetry, we see also that $\pi(x) \le \pi({}^{c|}y) = \pi(y)$ and then by the Lemma $Q_{{}^{c|}y} = T_\pi$. Therefore $Q_{{}^{c|}x} = Q_{{}^{c|}y}$, and hence ${}^{c|}x \equiv_r {}^{c|}y$. But then by the remark at the end of Step 1 we see that $x \equiv_r y$.

Remark. This result is well known, but I don't think it is well understood, at least in so far what it suggests for other Coxeter groups. What it does *not* do is say what the structure of the W-graphs attached to cells looks like. Indeed, there is every reason to think these are quite complicated. For example, for a long time all the coefficients $\mu(x, y)$ computed for symmetric groups turned out to be 1, and many wondered—without any real insight—if this were always the case. Counter-examples were exhibited for $\mathfrak{S}_{10}$ and $\mathfrak{S}_{16}$ in [McLarnan-Warrington:2003]. In spite of this complexity, the Theorem above says that the structure of cells for $\mathfrak{S}_n$ is relatively simple. I have conjectured that this is true, in some sense, for all Coxeter groups—that the KL-cells in any Coxeter group possess a **regular** structure, and more particularly that their structure is determined by local information in W-graphs, as it is for $\mathfrak{S}_n$ in terms of Knuth relations. For finite groups, regularity is not a problem, but the degree to which structures are determined locally is not known in general. The regular structure has been verified for affine Weyl groups in [Gunnells:2010], but not locality of structure. Nor has an algorithm been proposed to compute finite automata that implement regularity.

The Corollary is also true for affine Weyl groups of type A_n. This is proved in [Shi:1986]).

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