Essays on mathematical astronomy

The physics of orbits

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How do initial position and velocity determine an orbit?

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1. Newton's laws

If two spherical bodies interact due to gravity, then according to Newton's laws the force exerted by one on the other is attracting and of magnitude

$$F = \frac{G\, m_1 m_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|^2}\,,$$

where G is the gravitational constant, m_1 and m_2 are the masses, $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ the positions. Since the units of force are mlt^{-2}, those of G are $m^{-1}\ell^3 t^{-2}$.

The force acts along the line between them. As a vector, the force drawing 1 to 2 is thus

$$-G\, m_1 m_2 \left(\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|^3} \right)$$

and that drawing 2 to 1 is of the same magnitude in the opposite direction. By Newton's second law, under the assumption that no other forces act on the bodies—i.e. that the system is isolated—

$$\begin{aligned} m_1 \boldsymbol{r}_1'' &= -G\, m_1 m_2 \left(\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|^3} \right) \\ m_2 \boldsymbol{r}_2'' &= -G\, m_1 m_2 \left(\frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|^3} \right) . \end{aligned}$$

The sum cancels, so

$$m_1 \boldsymbol{r}_1'' + m_2 \boldsymbol{r}_2'' = (m_1 \boldsymbol{r}_1' + m_2 \boldsymbol{r}_2')' = 0\,.$$

If

$$m = m_1 + m_2\,, \tag{1.1}$$

is the sum of masses, the location of the center of mass is

$$\boldsymbol{c_m} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m}\,.$$

Under our assumptions, since $\boldsymbol{r}'' = 0$, it can be expressed as a linear function of time

$$\boldsymbol{c_m} = \boldsymbol{a} + \boldsymbol{b}\, t$$

for some point $\boldsymbol{a}$ and vector $\boldsymbol{b}$. *The center of mass moves with constant velocity.* Choose dynamical coordinates so as to make the center of mass the origin and set $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$. Since $m_2 r_2 = -m_1 r_1$, Newton's equations can be rewritten

$$\begin{aligned} \boldsymbol{r}_1'' &= -Gm_2 \, \frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|^3} \\ &= -G \, \frac{m_2\boldsymbol{r}_1 - m_2\boldsymbol{r}_2}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|^3} \\ &= -G \, \frac{m_2\boldsymbol{r}_1 + m_1\boldsymbol{r}_1}{\|\boldsymbol{r}_1 - \boldsymbol{r}_2\|^3} \\ &= -Gm \, \frac{\boldsymbol{r}_1}{r^3} \\ \boldsymbol{r}_2'' &= -Gm \, \frac{\boldsymbol{r}_2}{r^3} \end{aligned}$$

where $r = \|\boldsymbol{r}\|$.

Set

(1.2)
$$\mu = Gm\,.$$

Its units are $\ell^3 t^{-2}$. The basic physical parameter of the system is the constant μ. The basic fact is that the potential energy per unit of mass at distance r is $-\mu/r$. With this convention concerning an arbitrary integration constant, the total energy of an object with unit mass and speed v is

$$\mathcal{E} = \frac{v^2}{2} - \frac{\mu}{r}\,.$$

The point of this convention is that for a body escaping arbitrarily far, with limiting zero velocity, the energy is zero. It is negative for bodies in a bounded orbit.

From $m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 = 0$ and $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ we solve to get

$$\begin{aligned} \boldsymbol{r}_1 &= \frac{m_2}{m}\boldsymbol{r} \\ \boldsymbol{r}_2 &= -\frac{m_1}{m}\boldsymbol{r}\,, \end{aligned}$$

so we can recover $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ from $\boldsymbol{r}$. Subtracting one equation from the other gives us

(1.3)
$$\boldsymbol{r}'' = -\mu \, \frac{\boldsymbol{r}}{r^3}\,.$$

If we take the cross-product of it with $\boldsymbol{r}$ we get

$$\boldsymbol{r} \times \boldsymbol{r}'' = \boldsymbol{r} \times \boldsymbol{r}'' + \boldsymbol{r}' \times \boldsymbol{r}' = (\boldsymbol{r} \times \boldsymbol{r}')' = 0$$

from which we deduce that

(1.4)
$$\boldsymbol{r} \times \boldsymbol{r}' = \boldsymbol{h}$$

for some fixed vector $\boldsymbol{h}$. This is by definition the angular momentum of the system, up to a mass constant. Since $\boldsymbol{r}' \bullet \boldsymbol{h} = 0$, *the motion is restricted to a plane perpendicular to* $\boldsymbol{h}$.

Recall the vector product identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \bullet \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \bullet \boldsymbol{b})\, \boldsymbol{c}$$

and calculate

$$\boldsymbol{r} \bullet \boldsymbol{r}' = \sum x_i x_i' = \sum (1/2)(x_i^2)' = (1/2)(r^2)' = r r' \, .$$

Here r' is the time derivative of r, not the length of $\boldsymbol{r}'$. We now have

$$\begin{aligned}
(\boldsymbol{h} \times \boldsymbol{r})' &= \boldsymbol{h} \times \boldsymbol{r}'' \\
&= -\mu \, \frac{\boldsymbol{h} \times \boldsymbol{r}}{r^3} \\
&= -\mu \, \frac{\boldsymbol{r} \times \boldsymbol{r}' \times \boldsymbol{r}}{r^3} \\
&= -\mu \, \frac{r^2 \boldsymbol{r}' - (\boldsymbol{r} \bullet \boldsymbol{r}')\, \boldsymbol{r}}{r^3} \\
&= -\mu \, \frac{r^2 \boldsymbol{r}' - (r r')\boldsymbol{r}}{r^3} \\
&= -\mu \, \frac{r \boldsymbol{r}' - r' \boldsymbol{r}}{r^2} \\
&= -\mu \, \frac{d}{dt} \left(\frac{\boldsymbol{r}}{r} \right) \\
&= -\mu \, \frac{d\widehat{\boldsymbol{r}}}{dt}
\end{aligned}$$

Integrating, we see that

$$\boldsymbol{h} \times \boldsymbol{r}' = -\mu \widehat{\boldsymbol{r}} - \boldsymbol{P} \qquad (\widehat{\boldsymbol{r}} = \boldsymbol{r}/\|\boldsymbol{r}\|)$$

for some fixed vector $\boldsymbol{P}$. This is in effect the definition of $\boldsymbol{P}$ as

(1.5) .

$$\boldsymbol{P} = -\boldsymbol{h} \times \boldsymbol{r}' - \mu \, \widehat{\boldsymbol{r}}$$

It is perpendicular to $\boldsymbol{h}$, and points from the focus to 'perihelion'. From this we deduce that

$$\begin{aligned}
\boldsymbol{r} \bullet (\boldsymbol{h} \times \boldsymbol{r}') &= \mu r - \boldsymbol{P} \bullet \boldsymbol{r} \\
-\boldsymbol{h} \bullet (\boldsymbol{r} \times \boldsymbol{r}') &= \mu r - \boldsymbol{P} \bullet \boldsymbol{r} \\
h^2 &= \mu r + \boldsymbol{P} \bullet \boldsymbol{r} \\
\frac{h^2/\mu}{r} &= 1 + \frac{\boldsymbol{P}}{\mu} \bullet \widehat{r} \\
\frac{h^2/\mu}{r} &= 1 + \frac{P}{\mu} \cos\theta
\end{aligned}$$

if θ is the angle between $\boldsymbol{r}$ and $\boldsymbol{P}$, and $P = \|\boldsymbol{P}\|$. Set

$$p = h^2/\mu, \quad e = P/\mu \, .$$

Then in polar coordinates the equation of the orbit becomes

(1.6)

$$\frac{p}{r} = 1 + e \cos\theta$$

or

$$r = \frac{p}{1 + e\cos\theta}\,.$$

This is the polar equation of a conic section with **focus** at the origin, **eccentricity** e, and **semi-latus rectum** p. All these constants are determined from (1.2) , (1.4) , (1.5) , and (1.6) if we are given initial values of $\boldsymbol{r}$ and $\boldsymbol{r}'$.

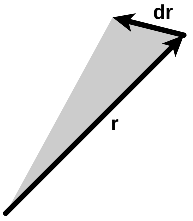
It can happen that $e < 0$, in which case 'perihelion' will lie to the left of teh given focus. We can change this into the usual form, replacing θ by $\pi - \theta$.

If $0 \le e < 1$ the conic section is an ellipse, if $e = 1$ it is a parabola, and if $e > 1$ an hyperbola.

Kepler's Second Law says that equal areas are swept out around the focus in equal times. As the following picture shows,

$$dA/dt = (1/2)\|\,\boldsymbol{r} \times \boldsymbol{r}'\| = h/2\,.$$

so Kepler's law is equivalent to conservation of angular momentum.



At any rate, we now know the geometry of the orbit, but almost nothing about motion along it, which is determined by Kepler's second law. That is another story.

2. Summary

Here is a summary of the path from given data $\boldsymbol{r}$, $\boldsymbol{r}'$ to geometric properties of the orbit.

$$\begin{aligned}
m &= m_1 + m_2 \\
\mu &= Gm \\
\boldsymbol{h} &= \boldsymbol{r} \times \boldsymbol{r}' \\
\boldsymbol{P} &= -(\boldsymbol{h}| \times \boldsymbol{r}' + \mu\widehat{\boldsymbol{r}}) \\
p &= h^2/\mu \\
e &= P/\mu \\
a &= \frac{p}{1 - e^2} \\
t_{\text{period}} &= 2\pi \cdot \sqrt{\frac{a^3}{\mu}}\,.
\end{aligned}$$

Here a is the semi-major axis. The semi-latus rectum is $p = a(1 - e^2)$. In case $e < 1$ and the orbit is an ellipse, the semi-minor axis is

$$b = a\sqrt{1 - e^2}\,.$$

There are some important things to note in case the orbit is an ellipse. Its total area $A = \pi ab$ is finite and is the product of $h/2$ and its period. The last formula is the basic relation between physics and geometry. To derive it:

$$\begin{aligned} \pi ab &= (h/2)t_{\mathrm{period}} \\ 2\pi a^2\sqrt{1-e^2} &= t_{\mathrm{period}}\sqrt{\mu p} \\ &= t_{\mathrm{period}}\sqrt{\mu}\sqrt{a}\sqrt{1-e^2} \\ t_{\mathrm{period}} &= 2\pi\cdot\sqrt{\frac{a^3}{\mu}}\,. \end{aligned}$$

This last equation amounts to a precise form of Kepler's third law.

The eccentricity e determines the shape of the conic section, p determines its size, and the period determines the rate at which things happen. All of these depend only on the product $\mu = Gm$, not the individual factors. This agrees with the fact that μ has dimensions $\ell^3 t^{-2}$.

The length a is that from the x-intersection nearest to the principal focus to the other x-intersection (assuming the foci are on the x-axis). For parabolas $a = \infty$, and for hyperbolas $a < 0$, which agrees with the fact that the direction of this x-intersection is in the direction opposite to what it is for ellipses.

In the solar system, the basic unit of length is the A.U., which is (officially) $149,597,870,700$ metres. The basic unit of time is, accordingly, one yaer, which is 365.2422 days. In thsi system, $t_{\mathrm{period}} = 1$, $a = 1$, and $\mu = 4\pi^2$.

Example. Let's look at a frequently occurring case. Take

$$\begin{aligned} \boldsymbol{r} &= (x,0,0) \\ \boldsymbol{r}' &= (0,v,0)\,. \end{aligned}$$

Then

$$\begin{aligned} \boldsymbol{h} &= (0,0,xv) \\ \widehat{\boldsymbol{r}} &= (1,0,0) \\ \boldsymbol{h}\times\boldsymbol{r}' &= (-xv^2,0,0) \\ \boldsymbol{P} &= (xv^2-\mu,0,0) \\ e &= (xv^2/\mu)-1 \\ p &= xv/\mu \end{aligned}$$

If $xv^2/2/\mu < 1$ (i.e. v is small), the perihelion will be to the left; if $xv^2/\mu = 1$ the orbit is a circle; and if $xv^2/\mu = 2$ the orbit is a parabola. As a special case, to check things, I take the numerical example mentioned in the Feynman lectures, with $\mu = 1$.

$$\begin{aligned} \boldsymbol{r} &= (0.5,0,0) \\ \boldsymbol{r}' &= (0,1.63,0) \\ \boldsymbol{h} &= [0,0,0.815] \\ e &= 0.328 \\ a &= 0.745 \\ p &= 0.664 \\ \boldsymbol{P} &= [0.0328,0,0] \\ t_{\mathrm{period}} &= 4.037\,. \end{aligned}$$

This agrees with Feynman's computation of about 2.0 units of time to get half-way around.

3. References

1. Roger R. Bate, Donald D. Mueller, and Jerry E. White, **Fundamentals of astrodynamics**, Dover, 1971.
2. J. M. A. Danby, **Fundamentals of celestial mechanics**, Willmann-Bell, 1988.