

## Essays on topological vector spaces

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### Quasi-complete topological vector spaces

In the theory of representations of a locally compact group  $G$ , an indispensable role is played by an action of the convolution algebra  $C_c(G)$  on the space  $V$  of a continuous representation  $\pi$  of  $G$ . In order to define this, one must know how to make sense of integrals

$$\int_G f(x)\pi(x)v \, dx,$$

in which  $f$  is a continuous function of compact support on  $G$ ,  $v$  in  $V$ , and  $dx$  a Haar measure on  $G$ . This is not quite a trivial matter. If  $V$  is assumed to be complete, a definition by Riemann sums is adequate, but this is an unnecessarily strong requirement. In defining  $V$ -valued integrals one does not need the whole space to be complete. The natural condition on  $V$  seems to be that it be **quasi-complete**, a condition somewhat weaker than completeness, which requires only that bounded closed subsets of  $V$  be complete. Continuous representations of a locally compact group are almost always assumed to be on a quasi-complete topological vector space.

The problem can be put in quite general terms. Suppose  $dx$  to be a Radon measure on the locally compact space  $X$ ,  $f$  a continuous function of compact support on  $X$  with values in the TVS  $V$ . How can the integral

$$I = \int_X f(x) \, dx$$

be defined? What requirements do we place on a suitable definition? There is one natural condition—if  $\hat{v}$  is in the continuous dual of  $V$ , then we might expect an equation of scalar integrals

$$\langle \hat{v}, I \rangle = \int_X \langle \hat{v}, f(x) \rangle \, dx.$$

So we adopt this as a necessary requirement for the integral, and it must be asked, given  $dx$  and the continuous function  $f(x)$  of compact support from  $X$  to  $V$ , does there exist  $I$  in  $V$  satisfying this equation? One satisfying feature of this requirement is that it guarantees uniqueness of  $I$ , because the Hahn-Banach theorem assures us that there are enough candidates for  $\hat{v}$  to separate points of  $V$ .

This equation does at least define an element of some vector space, that is to say the linear dual of the continuous linear dual  $\hat{V}$ . There is a canonical embedding of  $V$  into this double dual, and we want to know that the vector defined by the equation above lies in its image.

A second question is rather similar, but is about differentiation. Suppose  $f(x)$  to be a continuous function on an open subset  $\Omega$  of  $\mathbb{R}^n$  with values in a TVS. It is said to be **weakly smooth** if the function  $\langle \hat{v}, f(x) \rangle$  is smooth for all  $\hat{v}$  in  $\hat{V}$ . Is  $f(x)$  itself then smooth? The answer is again affirmative, as long as  $V$  is quasi-complete.

In this essay, a TVS will always be a locally convex, Hausdorff, topological vector space. Such spaces possess a topology defined by a collection of semi-norms  $\|v\|_{\rho}$ , or equivalently by a basis of neighbourhoods of 0 that are convex, balanced, and absorbing. I assume some familiarity with these spaces, but the most important thing required will be the Hahn-Banach theorem, which guarantees the existence of sufficiently many linear functions on  $V$ .

I have used many references useful in writing up this material, but the two papers by Garrett stand out among these. I should also point out that the introduction to [Schwartz:1954] contains a useful if brief introduction to quasi-complete spaces.

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**1. Introduction**

A subset of a TVS is **bounded** if every (continuous) semi-norm is bounded on it. A subset is **complete** if every Cauchy filter in it converges. In finite dimensions the whole vector space is complete, but in general it is not even true that bounded closed sets are always complete. If this is true, the vector space is said to be **quasi-complete**. An equivalent condition is that every bounded Cauchy filter converge. If the vector space is complete, then of course it is also quasi-complete, but the converse is not necessarily true. The really good news is that almost all TVS encountered in practice are quasi-complete. They are ubiquitous, and make up the natural class of TVS to use in representation theory.

For our purposes, the basic result relating integrals and quasi-complete spaces is this:

**1.1. Proposition.** *In a quasi-complete TVS, the closure of the convex hull of a compact subset is compact.*

**Remark.** It can happen in an arbitrary TVS that the closure of the convex hull of a compact set is not compact. For example, let  $V$  be the space of all sequences  $(x_i)$  of finite support, assigned the  $L^2$  norm. For each  $n$  let  $e_n$  be the sequence with  $e_{n,i} = 0$  unless  $i = n$ , and  $e_{n,n} = 1/n$ . Thus

$$\begin{aligned} e_1 &= (1, 0, 0, 0, 0, \dots) \\ e_2 &= (0, 1/2, 0, 0, \dots) \\ e_3 &= (0, 0, 1/3, 0, \dots) \\ &\dots \end{aligned}$$

The  $e_n$  converge to 0 in  $V$ , and the set

$$E = \{e_i \mid i \in \mathbb{N}\} \cup \{0\}$$

is closed in this space. Its convex hull is made up of all finite sequences  $(x_i)$  with

$$0 \leq x_i \leq 1, \quad \sum x_i \leq 1,$$

and its closure in  $L^2$  amounts to all such infinite sequences. The convex hull of  $E$  is hence not compact. Closure in Proposition 1.1 is a necessary requirement. It can happen even in a complete space that the convex hull of a compact set is not compact. The set  $\Omega$  is compact in the Hilbert space of all square-summable sequences, but its convex hull is not even closed.



The proof of Proposition 1.1 will be in several steps.

**Step 1.** A subset  $C$  of  $V$  is **totally bounded** if for every neighbourhood  $X$  of 0 in  $V$  the set  $C$  is covered by a finite number of translates of  $X$ . Every totally bounded subset of a TVS is clearly bounded.

**1.2. Lemma.** *The following conditions on the subset  $C$  of a TVS are equivalent:*

- (a) every filter to which  $C$  belongs possesses a Cauchy refinement;
- (b) the set  $C$  is totally bounded.

A set satisfying these conditions is often called **precompact**, for reasons that will appear in a moment. The equivalence of the two criteria is extremely useful.

*Proof.* This is a standard result in the theory of uniform spaces—see, for example, Theorem 6.32 of [Kelley:1955]—and the proof in the general case involves a few notions I don't want to introduce. But I'll prove it in a special case, when  $V$  possesses a countable basis of semi-norms, and is hence metrizable. In this case, we can restrict consideration to sequences instead of filters. Let  $|u - v|$  be the metric.

Suppose  $C$  to satisfy (b), and that  $v_i$  is a sequence in  $C$ . For every  $n \geq 0$  the set  $C$  can be covered by a finite number  $U_n$  of open sets of the form  $|c - c_k| < 1/2^n$ . Since  $U_0$  is finite, there exist an infinite number of the  $v_i$  in some single set in it, hence a subsequence  $v_{i_k}$  at most a unit distance apart from each other. But then this in turn must contain a subsequence with all of its elements at distance at most  $1/2$  from each other. And so on. Therefore  $(c_i)$  contains a Cauchy subsequence.

Conversely, suppose that  $C$  does not satisfy (b). Then for some  $\varepsilon > 0$  it cannot be covered by a finite number of disks of radius  $\varepsilon$ . Choose an arbitrary element  $c_1$  of  $C$ . The disk  $|x - c| < \varepsilon$  doesn't cover  $C$ , so we can choose  $c_2$  not in it. But then we can choose  $c_3$  with  $|c - c_i| \geq \varepsilon$  for  $i = 1, 2$ . And so on. The sequence you get has no Cauchy subsequence. ▣

**Step 2.** This notion is part of a well known criterion for compactness.

**1.3. Proposition.** *A subset  $C$  of a TVS is compact if and only if it is complete and totally bounded.*

*Proof.* I again restrict myself to a metrizable space. A subset  $C$  is compact if and only if every sequence contains a convergent subsequence, or (by Heine-Borel) if every covering of  $C$  by open sets contains a finite sub-covering. That every compact subset is totally bounded is thus a matter of definition.

Suppose, conversely, that  $C$  is totally bounded. Since  $C$  is bounded, its closure is complete and every Cauchy sequence in it converges. Combining this with (a) of Lemma 1.2, we see that  $C$  is compact. ▣

**Step 3.** This leads immediately to:

**1.4. Corollary.** *A subset  $C$  of a quasi-complete TVS is compact if and only if it is closed and totally bounded.*

**Step 4.** Finally, the basic relation between convex hulls and totally bounded sets:

**1.5. Lemma.** *The convex hull of any totally bounded subset of an arbitrary TVS is totally bounded.*

*Proof.* Let  $U$  be a neighbourhood of 0 in  $V$ , and  $W$  a convex neighbourhood such that  $W + W \subseteq U$ . Suppose  $X$  totally bounded. We can find a finite set of  $x_i$  in  $X$  with  $X$  contained in the union of the  $x_i + W$ . Let  $C$  be the convex hull of the  $x_i$ , which is contained in the finite-dimensional space they span. As I mentioned a moment ago, since it is closed and bounded in that space it is compact, and hence we can find a finite set of  $y_j$  such that  $C$  is contained in the union of the  $y_j + W$ . Since  $W$  is convex, the convex hull of  $X$  is contained in the union of the  $y_j + U$ . ▣

**Step 5.** Combine Lemma 1.5 and Lemma 1.2 to finish the proof of Proposition 1.1. ▣

As a final remark about quasi-complete spaces, here is what [Reed-Simon:1972] call the BLT (for **B**ounded **L**inear **T**ransformation) theorem in a narrower context:

**1.6. Proposition.** *Suppose  $V$  and  $W$  to be TVS,  $U$  a subspace of  $V$  with the property that every point of  $V$  is in the closure of a bounded subset of  $U$ . If  $F: U \rightarrow W$  is a linear map from  $U$  to  $W$ , continuous in the induced topology, and  $W$  is quasi-complete, then  $F$  extends uniquely to a continuous linear map from all of  $V$  to  $W$ .*

*Proof.* The space  $U$  is dense in  $V$ , so that by the standard extension theorem for complete spaces the map  $f$  may be extended to a continuous map  $\bar{f}$  into the completion  $\overline{W}$  of  $W$ . But since every point of  $V$  is in the closure of a bounded subset  $B$  of  $U$ ,  $\bar{f}(v)$  is in the closure of  $f(B)$ . But  $f(B)$  is closed in  $W$ , so its closure is complete, and  $\bar{f}(v)$  lies in  $W$ . ▣

## 2. Examples

Of course a complete space is quasi-complete, so in particular Fréchet spaces are quasi-complete. From these other quasi-complete spaces may be constructed.

**SUBSPACES.** The following is immediate from the definition.

**2.1. Proposition.** *If  $V$  is quasi-complete and  $U$  is a closed subspace of  $V$  then  $U$  is also quasi-complete.*

**PRODUCTS.** Suppose  $A$  to be any set and  $V_\alpha$  a TVS for each  $\alpha$  in  $A$ . Let  $V = \prod_A V_\alpha$ , and let  $p_\alpha$  be the projection onto  $V_\alpha$ . Assign  $V$  the product topology—a subset  $X$  is a neighbourhood of 0 in  $V$  if and only if each  $p_\alpha(X)$  is a neighbourhood of 0 in  $V_\alpha$  and  $p_\alpha(X) = V_\alpha$  for all but a finite number of  $\alpha$ . Defining semi-norms are the finite products of semi-norms on some finite set of the  $V_\alpha$ .

A product  $\prod X_\alpha$  is bounded in  $V$  if and only if each  $X_\alpha$  is bounded in  $V_\alpha$ . From this observation and the fact that the product of complete spaces is complete:

**2.2. Proposition.** *The product of quasi-complete TVS is quasi-complete.*

**PROJECTIVE LIMITS.** Suppose  $A$  to be any partially ordered set. Suppose given for each  $\alpha$  in  $A$  a TVS  $V_\alpha$ , and for each  $\alpha \leq \beta$  a continuous linear map  $p_{\alpha,\beta}$  from  $V_\beta$  to  $V_\alpha$ . The **projective limit** of these data is the subset  $(v_\alpha)$  in  $\prod V_\alpha$  such that  $p_{\alpha,\beta}(v_\beta) = v_\alpha$  whenever  $\alpha \leq \beta$ . It is a closed subspace of the product, hence:

**2.3. Proposition.** *A projective limit of quasi-complete TVS is quasi-complete.*

Incidentally, the product  $\prod_A V_\alpha$  is a projective limit, if one takes as the indexing sets the finite subsets of  $A$ .

Now fix a TVS  $V$ . Suppose  $X$  to be a locally compact set, which we assume to be the countable union of compact subsets  $X_i$  with  $X_i \subseteq X_{i+1}$ . Let  $U_i = C(X_i, V)$ , and for  $i \leq j$  let  $p_{i,j}$  be the restriction map from  $U_j$  to  $U_i$ . There is an obvious map from  $C(X, V)$  to the projective limit of the  $U_i$ .

**2.4. Lemma.** *This map is an isomorphism of  $C(X, V)$  with the projective limit of the  $U_i$ .*

**2.5. Lemma.** *If  $X$  is compact and  $V$  is quasi-complete, so is  $C(X, V)$ .*

*Proof.* Every bounded set in  $C(X, V)$  is made up of functions with values in a fixed bounded subset of  $V$ . ▢

From Proposition 2.3, Lemma 2.4, and Lemma 2.5:

**2.6. Proposition.** *In these circumstances, if  $V$  is quasi-complete, so is  $C(X, V)$ .*

**INDUCTIVE LIMITS.** Suppose given an increasing chain of TVS

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq \dots$$

with each  $V_i$  closed in  $V_{i+1}$ . Assume that the topology on each  $V_i$  is the same as that induced from  $V_{i+1}$ . Let  $V = \bigcup V_i$ . A neighbourhood of 0 in  $V$  is a subset  $U$  of  $V$  such that each  $U \cap V_i$  is one in  $V_i$ . With this topology, the TVS  $V$  is called the **strict inductive limit** of the  $V_i$ .

**2.7. Proposition.** *The strict inductive limit of quasi-complete TVS is quasi-complete.*

*Proof.* A subset of the limit is bounded if and only if it is contained in one of the  $V_i$  and bounded in it. ▢

**DUALS.** An **adequate cover** of the TVS  $V$  is a collection  $\mathfrak{S}$  of subsets such that:

- every set in  $\mathfrak{S}$  is weakly bounded;
- any two sets in  $\mathfrak{S}$  are contained in a third;
- if  $X$  is in  $\mathfrak{S}$  then every  $cX$  with  $c \neq 0$  is in  $\mathfrak{S}$ ;
- $\mathfrak{S}$  covers  $V$ .

Adequate covers are used to define topologies on spaces of continuous maps from one TVS to another. I'll look at just one class—the dual space  $\widehat{V}$  of continuous linear functions on  $V$ .

A set  $X$  in  $\mathfrak{S}$  is weakly bounded, so for any  $\tilde{v}$  in  $\tilde{V}$

$$\|f\|_X = \sup_{v \in X} |\langle \tilde{v}, v \rangle| < \infty.$$

As  $X$  ranges over  $\mathfrak{S}$  these semi-norms define a topology on  $\widehat{V}$ , hence the TVS  $\widehat{V}_{\mathfrak{S}}$ . When is it quasi-complete?

A **barrel** in a TVS  $V$  is a subset that is convex, absorbing, balanced, and closed. Every TVS has a basis of the origin made up of barrels. A TVS is barreled if every barrel is a neighbourhood of 0. Fréchet spaces and  $LF$  spaces are barreled.

**2.8. Proposition.** *If  $V$  is barreled and  $\mathfrak{S}$  an adequate cover, then  $\widehat{V}_{\mathfrak{S}}$  is quasi-complete.*

This is Corollary 2 to Theorem 34.2 in [Treves:1967]. The proof would require a long digression I do not want to go into.

**REFLEXIVITY.** The dual of the strong dual  $\widehat{V}_b$  is called the bidual  $\widehat{\widehat{V}}$ . The space is **reflexive** if the canonical map from  $V$  to its bidual (with the strong dual topology) is an isomorphism.

**2.9. Proposition.** *Every reflexive TVS is quasi-complete.*

This is a consequence of Corollary 1 of Theorem 5.5 in [Schaefer-Wolff:1999].

### 3. Integration of vector-valued functions

From now on, assume  $V$  to be a quasi-complete TVS.

One of the principal reasons for interest in quasi-complete spaces is that they furnish the right context for integration. The problem is how to make sense of integrals

$$I(f) = \int_X f(x) dx$$

where  $X$  is a locally compact space,  $dx$  a Radon measure on  $X$ , and  $f$  a function in  $C_c(X, V)$ . To tell the truth, in just about all cases we'll deal with the integral will have a direct interpretation by-passing the machinery I'll introduce here, but it is worth something not to have to deal with each case separately.

Suppose  $X$  to be a locally compact space that's the union of a countable number of compact subsets. There are two different ways to define the notion of a positive measure on  $X$  compatible with its topology. The space  $C(X)$  is that of all continuous functions on  $X$ , with semi-norms

$$\|f\|_{\Omega} = \sup_{x \in \Omega} |f(x)|$$

where  $\Omega$  is a compact subset of  $X$ . For each compact subset  $\Omega$  of  $X$  let  $C_c(\Omega)$  be the space of functions  $f$  in  $C(X)$  with support in  $\Omega$  assigned the semi-norm  $\|f\|_{\infty, \Omega}$ . The TVS  $C_c(X)$  is the inductive limit of the  $C_c(\Omega)$ . The simplest definition of a positive measure  $\mu$  on  $X$  compatible with its topology is that it is a continuous linear functional on  $C_c(X)$  such that  $\langle \mu, f \rangle \geq 0$  if  $f \geq 0$ . By the **Riesz-Markov-Kakutani Theorem** (see page 111 of [Reed & Simon:1972]), this gives rise also to a measure in the usual sense—called a **Baire measure**—that assigns a measure to various subsets of  $X$ , including the open ones.

**3.1. Theorem.** *Suppose  $X$  to be a locally compact space with a positive Baire measure  $dx$ , and suppose  $f(x)$  to be a continuous function with compact support  $\Omega$  on  $X$  with values in the quasi-complete TVS  $V$ . Then*

(a) *there exists a vector  $I_f$  in  $V$  such that*

$$\langle \widehat{v}, I_f \rangle = \int_X \langle \widehat{v}, f(x) \rangle dx$$

*for all  $\widehat{v}$  in  $\widehat{V}$ ;*

- (b) it is unique;
- (c) it lies in the closure of the convex hull of  $f(\Omega)$  scaled by  $\text{meas}(\Omega)$ ;
- (d) if  $\rho$  is a (continuous) semi-norm of  $V$  then

$$\|I_f\|_\rho \leq \int_X \|f(x)\|_\rho dx.$$

*Proof.* Replacing  $X$  if necessary by the support of  $f$ , and then perhaps scaling, I may assume that  $\text{meas}(X) = 1$ . I may also assume that  $V$  is a vector space over  $\mathbb{R}$ .

Condition **(b)** is an immediate consequence of the Hahn-Banach theorem.

Condition **(c)** is also straightforward, given the validity of **(a)**. The Hahn-Banach theorem asserts that the closure of the convex hull of any set is the intersection of all the closed half-spaces containing the set, so what must be shown is that if  $\hat{v}$  is any linear function on  $V$  such that  $\langle \hat{v}, f(X) \rangle \leq C$  then

$$\left\langle \hat{v}, \int_X f(x) dx \right\rangle = \int_X \langle \hat{v}, f(x) \rangle dx \leq C \cdot \text{meas}(X).$$

This is a basic fact about integration of Radon measures.

It is condition **(a)** that is the difficult point. There seem to be two approaches to a proof of existence in the literature, rather different from each other. One is that of [Bourbaki:1959–65]. It applies a technique that occurs frequently in French expositions. The integral exists always, without further hypothesis, in a very weak sense. Let  $\tilde{V}$  be the simple linear dual of  $\hat{V}$ . Each  $v$  in  $V$  determines an element of  $\tilde{V}$ , but more generally an element of  $\tilde{V}$  is specified by any linear function on  $\hat{V}$ , with no topological restriction imposed. The formula above tells us that we can interpret the integral without trouble as an element of  $\tilde{V}$ . The question is whether it lies in the image of the canonical embedding  $\iota: V \hookrightarrow \tilde{V}$ . That it does takes several steps of rather delicate reasoning about double duals.

**IN FINITE DIMENSIONS.** I prefer to follow here the note [Garrett:2011]. The first step is to prove the theorem when  $V$  has finite dimension, which I now assume.

In this case, defining the integral is trivial. Choose a basis  $e_i$  of  $V$ , express

$$f(x) = \sum f_i(x) e_i,$$

and then set

$$\int_X f(x) dx = \sum_i \left( \int_X f_i(x) dx \right) e_i.$$

At this point I'll prove a stronger version of **(c)**:

**3.2. Lemma.** *If  $V$  is finite-dimensional, the integral*

$$\int_X f(X) dx$$

*lies in the convex hull of  $f(X)$  scaled by  $\text{meas}(X)$ .*

That is to say, the convex hull in finite dimensions is already closed.

*Proof of the Lemma.* I recall that a **polyhedron** is the convex hull of a finite number of points in some  $\mathbb{R}^n$ .

**3.3. Lemma.** *A polyhedron in  $\mathbb{R}^n$  may be partitioned into a finite number of simplices of dimension  $n$ .*

The simplices may be degenerate.

*Proof.* The result is trivially true in dimension one, since a polyhedron is just an interval. One may assume the given points to be extremal for the convex hull. Fix one of these vertices, say  $P$ , and look at

the maximal faces of the polyhedron not containing it. By induction, each of these may be partitioned into simplices of dimension  $n - 1$ , and then the original polyhedron is the union of cones with these as bases and  $P$  as vertex. ▣

In an arbitrary quasi-complete TVS it is only the *closure* of the convex hull of a compact set that is necessarily compact, but in a finite-dimensional space things are simpler:

**3.4. Lemma.** *In a finite-dimensional vector space the convex hull of a compact set is compact.*

*Proof.* Say  $V$  has dimension  $n$ . Any point in the convex hull of the compact set  $\Omega$  will be in the convex hull of a finite number of points of  $\Omega$ . By the Lemma just proved, it will be in the convex hull of  $n + 1$  of them. If

$$\Sigma_n = \left\{ (x_i) \mid \sum x_i = 1, x_i \geq 0 \right\}$$

is the standard  $n$ -dimensional simplex then the map

$$\Sigma_n \times \Omega^{n+1} \rightarrow V$$

therefore has image  $\Omega$ , but it also has compact image. ▣

**CONCLUSION OF THE PROOF OF EXISTENCE.** The motivation of the argument is that if  $\varphi$  is a continuous linear map from  $V$  to  $W$ , then **(a)** implies immediately that

$$\varphi \left( \int_X f(x) dx \right) = \left( \int_X \varphi(f(x)) dx \right).$$

For  $W$  we shall take various finite-dimensional spaces that in some sense, in the limit, approximate  $V$ .

We are looking for a vector  $I_f$  in  $V$  such that

$$\langle \hat{v}, I_f \rangle = \int_X \langle \hat{v}, f(x) \rangle dx$$

for all  $\hat{v}$  in  $\hat{V}$ . We impose only a finite number of such conditions, and increase the number. If  $\mathcal{F}$  is a finite set of continuous linear functions on  $V$ , we can define the map

$$F_{\mathcal{F}}: V \rightarrow \mathbb{R}^{\mathcal{F}}, \quad v \mapsto (\langle \hat{v}, v \rangle)_{\hat{v} \in \mathcal{F}}.$$

We obtain a map from  $X$  to  $\mathbb{R}^{\mathcal{F}}$  by composing  $F_{\mathcal{F}}$  with  $f$ . We know that the integral of this composite over  $X$  lies in the convex hull of its image. But the convex hull of an image is the image of a convex hull, so that the set  $\mathcal{C}_{\mathcal{F}}$  of all  $v$  in the convex hull of  $f(X)$  such that

$$\langle \hat{v}, v \rangle = \int_X \langle \hat{v}, f(x) \rangle dx$$

for all  $\hat{v}$  in  $\mathcal{F}$  is not empty. Since  $f$  has compact support,  $f(X)$  is compact. The closure of its convex hull is therefore compact, by Proposition 1.1. Hence the closure  $\Gamma_{\mathcal{F}}$  of  $\mathcal{C}_{\mathcal{F}}$  is also compact.

If  $\Phi$  is a finite set of sets  $\mathcal{F}$ , then

$$\bigcap_{\mathcal{F} \in \Phi} \Gamma_{\mathcal{F}} = \Gamma_{\cup \Phi}$$

so we may invoke the well known finite intersection property to deduce that there exists some  $v$  in the intersection of all the  $\Gamma_{\mathcal{F}}$ , which is exactly what we are looking for.

**POLARS.** Now for **(d)**. This is [Bourbaki:Intégration], Chapitre III.3, Proposition 6, and I follow the argument sketched there. It requires a few basic facts about TVS. Suppose  $\rho$  to be a semi-norm on  $V$ . Let  $D$  be the set where  $\|v\|_{\rho} \leq 1$ . It is convex and balanced. Its **polar** is the set

$$D^{\circ} = \{ \hat{v} \in \hat{V} \mid |\langle \hat{v}, v \rangle| \leq 1 \text{ for all } v \in D \}.$$

It is a basic fact that  $D^{\circ\circ}$  is the canonical copy of  $D$  in  $\widehat{V}$ .

From the definition of the polar, we obtain

$$|\langle v, \widehat{v} \rangle| \leq \|v\|_\rho$$

for every  $v$  in  $V$ ,  $\widehat{v}$  in  $D^\circ$ . From this we deduce

$$(3.5) \quad \left| \left\langle \int_X f(x) dx, \widehat{v} \right\rangle \right| = \left| \int_X \langle f(x), \widehat{v} \rangle \right| \leq \int_X \|f(x)\|_\rho dx$$

for all  $\widehat{v}$  in  $D^\circ$ .

On the other hand, the Banach-Alaoglu theorem tells us that if  $U$  is a neighbourhood of 0 in  $V$ , then  $U^\circ$  is weakly compact. In particular,  $D^\circ$  is compact. If  $v$  is  $V$  then  $|\langle v, \widehat{v} \rangle|$  is a continuous non-negative function on  $D^\circ$ . We know it is bounded from above by  $\|v\|_\rho$ . Let  $\widehat{v}_0$  be a point in the compact set  $D^\circ$  where it takes its maximum value, which is at most  $\|v\|_\rho$ . I claim that this maximum value is exactly  $\|v\|_\rho$ . Well, say it is  $c\|v\|_\rho$  with  $c \leq 1$ . Replace  $v$  by  $v/\|v\|_\rho$ , so now  $\|v\|_\rho = 1$ . Then  $|\langle v, \widehat{v} \rangle|$ . But then  $v/c$  lies in  $D^{\circ\circ}$ , which is equal to  $D$ , but this happens only if  $c = 1$ .

Thus for every  $v$  in  $V$  there exists some  $\widehat{v}_0$  in  $D^\circ$  such that

$$(3.6) \quad \|v\|_\rho \leq |\langle v, \widehat{v}_0 \rangle|.$$

Hence

$$\left\| \int_X f(x) dx \right\|_\rho \leq \left| \left\langle \int_X f(x) dx, \widehat{v}_0 \right\rangle \right|.$$

Combining (3.5) and (3.6) :

$$\left\| \int_X f(x) dx \right\|_\rho \leq \int_X \|f(x)\|_\rho dx.$$

#### 4. Differentiation of vector-valued functions

Continue to assume  $V$  to be a quasi-complete TVS.

Suppose  $\Omega$  to be an open set in  $\mathbb{R}^n$ . A continuous function in  $C(\Omega, V)$  is said to be **differentiable** at  $x \in \Omega$  if the dirrectional derivative

$$[\partial_v f](x) = \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

exists for every  $v$  in  $\mathbb{R}^n$ . If it is differentiable at all points of  $\Omega$ , we get in this way a function in  $C(\Omega \times \mathbb{R}^n, V)$ . It is said to be **continuously differentiable** in  $\Omega$  if this in turn is differentiable in all of  $\Omega \times \mathbb{R}^n$ . Let  $C^1(\Omega, V)$  be the space of continuously differentiable functions in  $C(\Omega, V)$ .

Thus  $f$  is in  $C^1(\Omega, V)$  if and only if

$$\frac{f(y) - f(x)}{y - x},$$

which is defined and continuous on the complement of the diagonal in  $\Omega \times \Omega$ , extend to a continuous function on all of the product.

Suppose for the moment that  $\Omega$  is an open interval in  $\mathbb{R}$ . If  $f$  is in  $C^1(\Omega, V)$  then according to Theorem 3.1 the function  $f'$  is integrable. It then follows easily that

$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t) dt \\ &= (y - x) \int_0^1 f'(x + t(y - x)) dt. \end{aligned}$$

This leads to:

**4.1. Proposition.** *The function  $f$  in  $C(\Omega, V)$  lies in  $C^1(\Omega, V)$  if and only if there exists  $\varphi$  in  $C(\Omega, V)$  such that*

$$f(y) - f(x) = (y - x) \int_0^1 \varphi(t) dt$$

for all  $x, y$  in  $\Omega$ . In this case  $\varphi = f'$ .

As a further consequence:

**4.2. Proposition.** *If  $f$  is in  $C^1(\Omega, V)$  then  $f(y) - f(x) = (y - x)f'(\theta)$  for some  $\theta$  in  $[x, y]$ .*

Now return to the general case, in which  $\Omega$  is open in  $\mathbb{R}^n$ . If  $(\varepsilon_i)$  is the standard basis of  $\mathbb{R}^n$  and  $v = \varepsilon_i$ , then  $\partial_v f$  is the  $i$ -th partial derivative of  $f$ .

**4.3. Lemma.** *If the partial derivatives of  $f$  are continuous, then  $f$  lies in  $C^1(\Omega, V)$ . For any  $x$  in  $\Omega$  and small  $v = \sum v_i \varepsilon_i$*

$$f(x + v) - f(x) = \sum v_i \cdot [\partial f / \partial x_i](x) + o(|v|).$$

*Proof.* Given Proposition 4.2, the classic argument (pp. 60–62 in [Courant:1936]) remains valid. ▣

Now suppose  $m = (m_i)$  with  $|m| \geq 2$ . The function  $f$  is in  $C^m(\Omega, V)$  if all  $\partial_v f$  are in  $C^k(\Omega, V)$  for  $|k| \leq m - 1$ . Furthermore

$$C^\infty(\Omega, V) = \bigcap_m C^m(\Omega, V).$$

These can all be made into TVS in an obvious way.

The classical argument shows that the  $\partial/\partial x_i$  all commute, but this is also a consequence of Hahn-Banach and this:

**4.4. Lemma.** *If  $f$  in  $C^m(\Omega, V)$  then for every function in  $\widehat{V}$  the function  $\langle \widehat{v}, f \rangle$  is in  $C^m(\Omega, \mathbb{R})$ .*

This, together with Hahn-Banach, can be used to extend classical results about scalar-valued differential operators—those in, say, §§3–4 of Chapter II in [Courant:1936]—to  $V$ -valued ones.

Although the converse to Lemma 4.4 isn't true, it almost is.

**4.5. Theorem.** *Suppose  $f$  to be any function on  $\Omega$  with values in  $V$ ,  $m \geq 1$ . If  $V$  is quasi-complete and the function*

$$x \mapsto \langle \widehat{v}, f(x) \rangle$$

*is in  $C^m(\Omega)$  for every  $\widehat{v}$  in  $\widehat{V}$ , then  $f$  is in  $C^{m-1}(\Omega, V)$ .*

The hypothesis means that  $f$  is **weakly  $m$ -smooth** in  $C^m(\Omega, V)$ . This is apparently due to Grothendieck. A sketch of a proof is contained in [Grothendieck:1953], with more details [Grothendieck:1958]. A full proof is also contained in [Schwartz:1954].

*Proof.* The proofs of Grothendieck and Schwartz involve a delicate analysis of double duals. I incorporate a simplifying suggestion of [Garrett:2008].

The case  $m = 1$  is very different from the cases  $m \geq 2$ .

**4.6. Lemma.** *If  $f$  is any function on  $\Omega$  with values in  $V$  and  $\langle \widehat{v}, f(x) \rangle$  is in  $C^1(\Omega, V)$  for all  $\widehat{v}$  in  $\widehat{V}$ , then  $f$  is in  $C^0(\Omega, V)$ .*

*Proof.* Suppose  $\langle \widehat{v}, f(x) \rangle$  to be in  $C^1(\Omega, V)$ . Set

$$g(x, y) = \frac{f(x) - f(y)}{x - y}.$$

Then every  $\langle \widehat{v}, g(x, y) \rangle$  is a continuous function on  $\Omega \times \Omega$ . It is in fact bounded in the weak topology on  $V$ . But according to the Banach-Steinhaus theorem, it is then also bounded in the initial topology of  $V$ . That is to say, there exists a bounded subset  $B$  of  $V$  such that

$$|f(x) - f(y)| \in |x - y| \cdot B$$

on  $\Omega \times \Omega$ . This guarantees that  $f$  is continuous. ▣

Now for the general case. The proof easily reduces to the case  $m = 2$ .

We assume that  $\langle \widehat{v}, f(x) \rangle$  is in  $C^2(\Omega)$  for every  $\widehat{v}$  in  $\widehat{V}$ , and want to prove that  $f$  is in  $C^1(\Omega, V)$ . The Lemma tells us at least that  $f$  is continuous, but we do not even know at first that  $f'$  is defined. In fact, that is how we begin—it suffices to show that  $f$  is differentiable at every point of  $\Omega$ , for then according to the Lemma, since  $f$  is weakly  $C^2$ , its directional derivatives will then be continuous.

To deal with directional derivatives, it suffices to assume  $\Omega$  is an open interval in  $\mathbb{R}^n$ . We shall work locally, and may assume that  $I$  is a compact interval in  $\Omega$  containing the point  $\omega$ . Let

$$F(x, y) = \frac{f(x) - f(y)}{x - y}$$

for every  $x, y$  in  $\Omega$ . By assumption, every

$$\langle \widehat{v}, F(x, y) \rangle$$

is in  $C^2(\Omega)$ . Therefore the corresponding image of  $I \times I$  in  $\mathbb{R}$  is compact, hence bounded. That is to say, the set  $B$  of  $F(x, y)$  for  $x, y$  in  $V$  is weakly bounded. Applying Banach-Steinhaus again, it is therefore also strongly bounded in  $V$ . Hence for all  $x, y$  in  $I$

$$f(x) - f(y) \in (x - y) \cdot B.$$

This implies that the filter  $\{f(x) \mid x \in U\}$  as  $U$  traverses the filter of neighbourhoods of  $\omega$  is a bounded Cauchy filter. Since  $V$  is quasi-complete, it converges, necessarily to the derivative of  $f$  at  $\omega$ . ▣

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