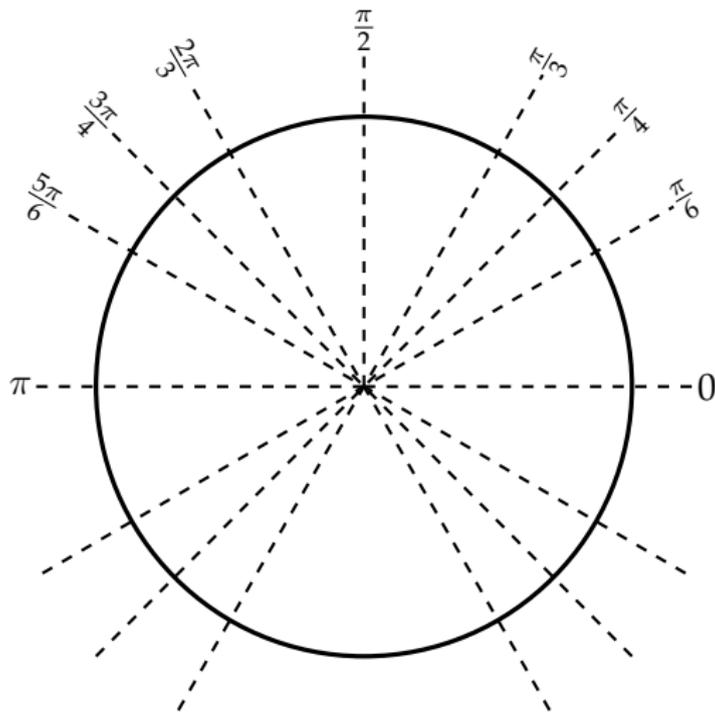


COMMONLY USED FACTS

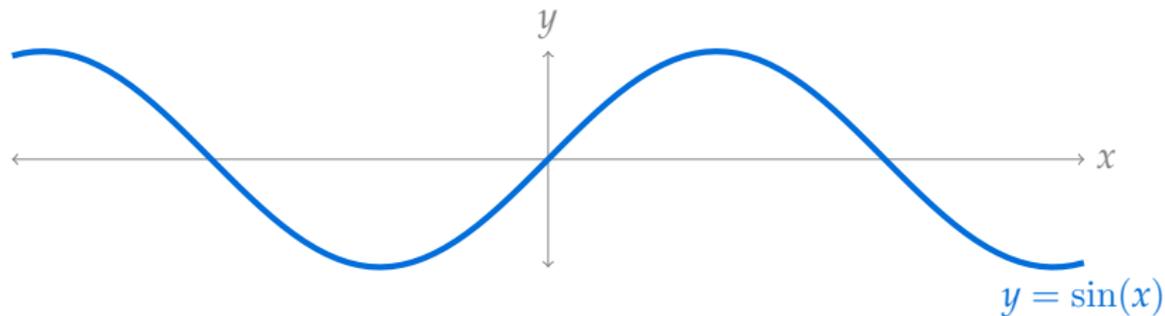
- ▶ Graphs of sine, cosine, tangent
- ▶ Sine, cosine, and tangent of reference angles: $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$
- ▶ How to use reference angles to find sine, cosine and tangent of other angles
- ▶ Identities: $\sin^2 x + \cos^2 x = 1;$ $\tan^2 x + 1 = \sec^2 x;$
 $\sin^2 x = \frac{1 - \cos(2x)}{2};$ $\cos^2 x = \frac{1 + \cos 2x}{2}$
- ▶ Conversion between radians and degrees

CLP-1 has an appendix on high school trigonometry that you should be familiar with.

REFERENCE ANGLES



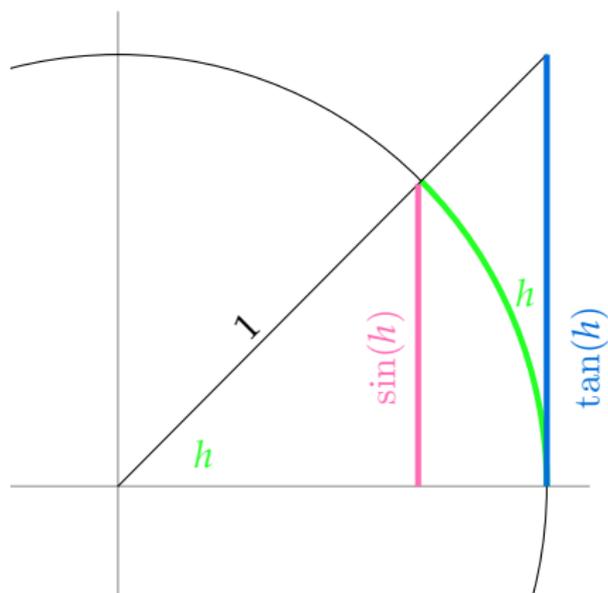
DERIVATIVE OF SINE



Consider the derivative of $f(x) = \sin(x)$.

$$\begin{aligned}
\frac{d}{dx} \{\sin x\} &= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sin(x)(\cos(h) - 1)}{h} + \lim_{h \rightarrow 0} \frac{\cos(x)\sin(h)}{h} \\
&= \sin(x) \lim_{h \rightarrow 0} \frac{\cos(0+h) - \cos(0)}{h} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} \\
&= \sin(x) \frac{d}{dx} \{\cos(x)\} \Big|_{x=0} + \cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = \boxed{\cos(x) \lim_{h \rightarrow 0} \frac{\sin(h)}{h}}
\end{aligned}$$

since $\cos(x)$ has a horizontal tangent, and hence has derivative zero, at $x = 0$.



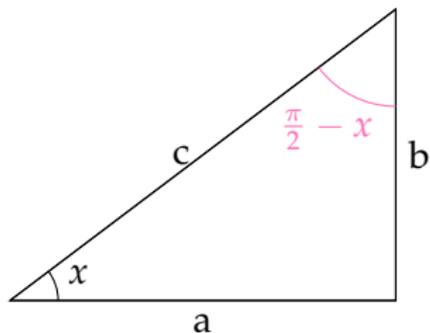
DERIVATIVES OF SINE AND COSINE

From before,

$$\frac{d}{dx}\{\sin(x)\} = \cos(x) \cdot \lim_{h \rightarrow 0} \frac{\sin(h)}{h} = \cos(x)$$

DERIVATIVE OF COSINE

Now for the derivative of \cos . We already know the derivative of \sin , and it is easy to convert between \sin and \cos using trig identities.



$$\sin x = \frac{b}{c} = \cos\left(\frac{\pi}{2} - x\right)$$

$$\cos x = \frac{a}{c} = \sin\left(\frac{\pi}{2} - x\right)$$

When we use radians:

Derivatives of Trig Functions

$$\frac{d}{dx} \{\sin(x)\} = \cos(x)$$

$$\frac{d}{dx} \{\cos(x)\} = -\sin(x)$$

$$\frac{d}{dx} \{\tan(x)\} =$$

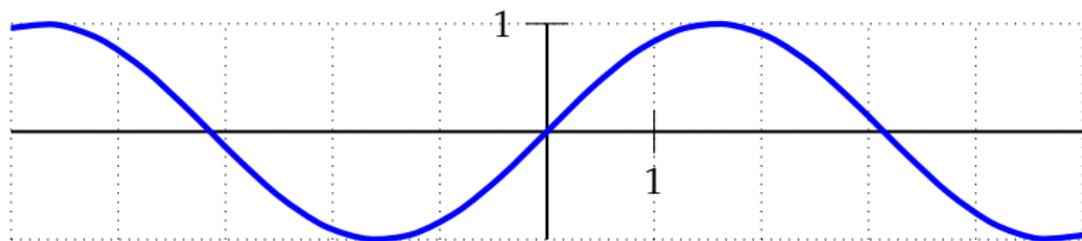
$$\frac{d}{dx} \{\sec(x)\} =$$

$$\frac{d}{dx} \{\csc(x)\} =$$

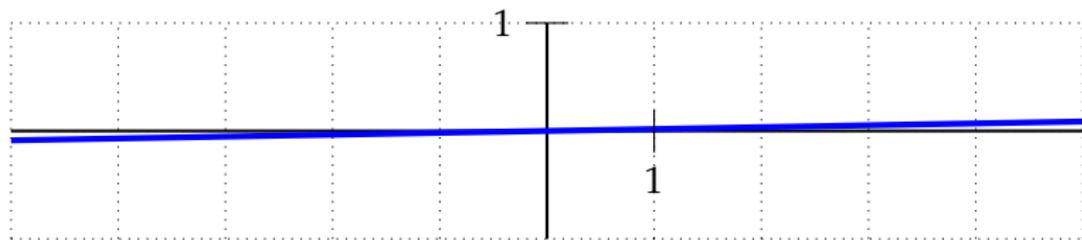
$$\frac{d}{dx} \{\cot(x)\} =$$

Honorable Mention

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$



$$y = \sin x, \text{ radians}$$



$$y = \sin x, \text{ degrees}$$

OTHER TRIG FUNCTIONS

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

OTHER TRIG FUNCTIONS

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\begin{aligned}\frac{d}{dx}[\sec(x)] &= \frac{d}{dx} \left[\frac{1}{\cos(x)} \right] \\ &= \frac{\cos(x)(0) - (1)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\sin(x)}{\cos^2(x)} \\ &= \frac{1}{\cos(x)} \frac{\sin(x)}{\cos(x)} \\ &= \sec(x) \tan(x)\end{aligned}$$

OTHER TRIG FUNCTIONS

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\begin{aligned}\frac{d}{dx}[\csc(x)] &= \frac{d}{dx} \left[\frac{1}{\sin(x)} \right] \\ &= \frac{\sin(x)(0) - (1)\cos(x)}{\sin^2(x)} \\ &= \frac{-\cos(x)}{\sin^2(x)} \\ &= \frac{-1}{\sin(x)} \frac{\cos(x)}{\sin(x)} \\ &= -\csc(x) \cot(x)\end{aligned}$$

OTHER TRIG FUNCTIONS

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\begin{aligned}\frac{d}{dx}[\cot(x)] &= \frac{d}{dx} \left[\frac{\cos(x)}{\sin(x)} \right] \\ &= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\ &= \frac{-1}{\sin^2(x)} \\ &= -\csc^2(x)\end{aligned}$$

MEMORIZE

$$\frac{d}{dx} \{\sin(x)\} = \cos(x)$$

$$\frac{d}{dx} \{\cos(x)\} = -\sin(x)$$

$$\frac{d}{dx} \{\tan(x)\} = \sec^2(x)$$

$$\frac{d}{dx} \{\sec(x)\} = \sec(x) \tan(x)$$

$$\frac{d}{dx} \{\csc(x)\} = -\csc(x) \cot(x)$$

$$\frac{d}{dx} \{\cot(x)\} = -\csc^2(x)$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Let $f(x) = \frac{x \tan(x^2 + 7)}{15e^x}$. Use the definition of the derivative to find $f'(0)$.

Differentiate $(e^x + \cot x)(5x^6 - \csc x)$.



$$\text{Let } h(x) = \begin{cases} \frac{\sin x}{x} & , \quad x < 0 \\ \frac{ax+b}{\cos x} & , \quad x \geq 0 \end{cases}$$

Which values of a and b make $h(x)$ continuous at $x = 0$?

Practice and Review

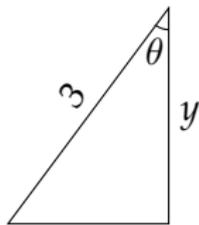
$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x = 0$?

$$g(x) = \begin{cases} e^{\frac{\sin x}{x}} & , \quad x < 0 \\ (x - a)^2 & , \quad x \geq 0 \end{cases}$$

What value(s) of a makes $g(x)$ continuous at $x = 0$?

A ladder 3 meters long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, measured in radians, and let y be the height of the top of the ladder. If the ladder slides away from the wall, how fast does y change with respect to θ ? When is the top of the ladder sinking the fastest? The slowest?



Suppose a point in the plane that is r centimetres from the origin, at an angle of θ ($0 \leq \theta \leq \frac{\pi}{2}$), is rotated $\pi/2$ radians. What is its new coordinate (x, y) ? If the point rotates at a constant rate of a radians per second, when is the x coordinate changing fastest and slowest with respect to θ ?

