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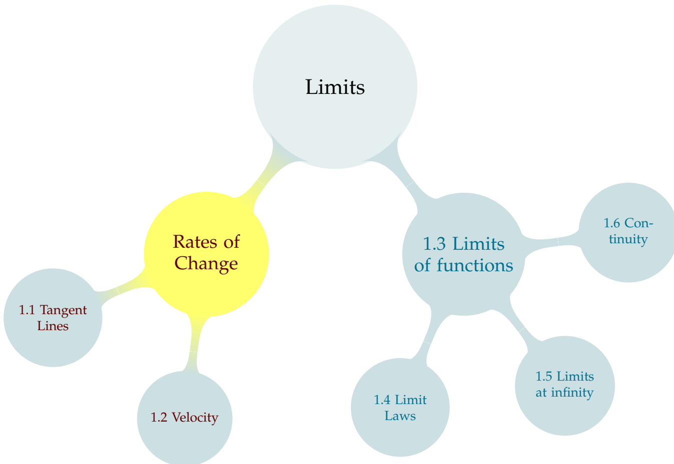
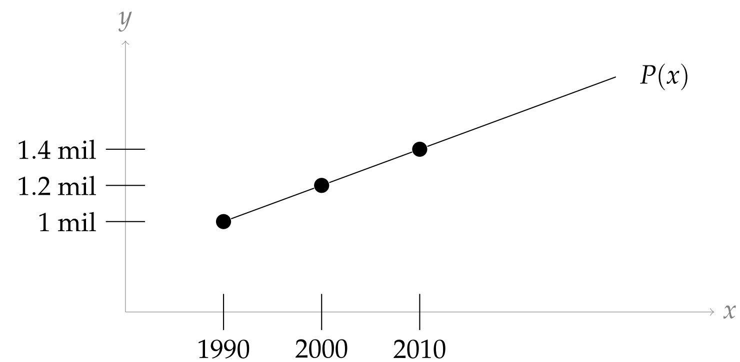


Rates of Change

Suppose the population of a small country was 1 million individuals in 1990, and is growing at a steady rate of 20,000 individuals per year.

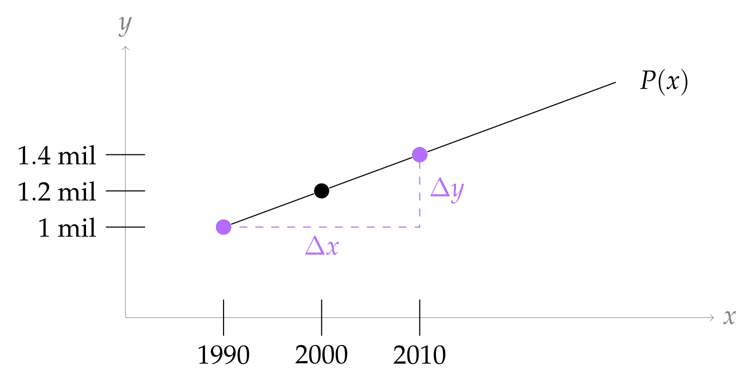


Definition

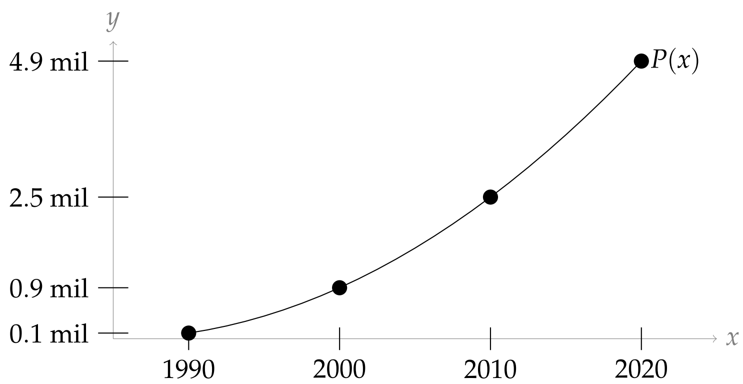
The **slope** of a line that passes through the points (x_1, y_1) and (x_2, y_2) is "rise over run"

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

This is also called the **rate of change** of the function.
If a line has equation $y = mx + b$, its slope is m.



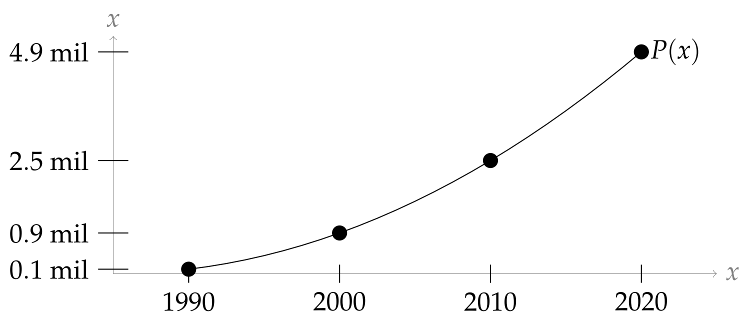
Suppose the population of a small country is given in the chart below.



Definition

Let $y = f(x)$ be a curve that passes through (x_1, y_1) and (x_2, y_2). Then the **average rate of change** of $f(x)$ when $x_1 \leq x \leq x_2$ is

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



Average Rate of Change and Slope

The average rate of change of a function $f(x)$ on the interval $[a, b]$ (where $a \neq b$) is "change in output" divided by "change in input:"

$$\frac{f(b) - f(a)}{b - a}$$

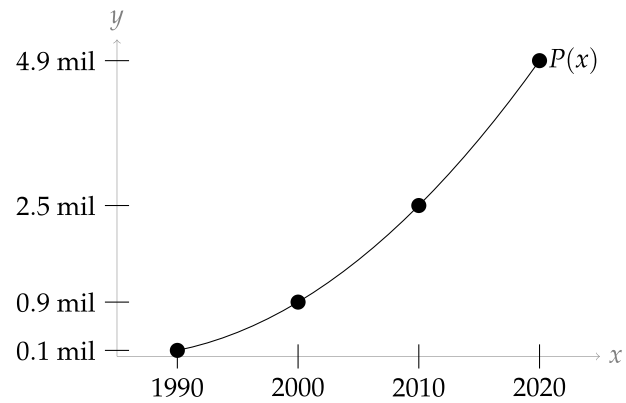
If the function $f(x)$ is a line, then the slope of the line is "rise over run,"

$$\frac{f(b) - f(a)}{b - a}$$

If a function is a line, its slope is the same as its average rate of change, which is the same for every interval.

If a function is not a line, its average rate of change might be different for different intervals, and we don't have a definition (yet) for its "slope."

How fast was this population growing in the year 2010? (What was its instantaneous rate of change?)



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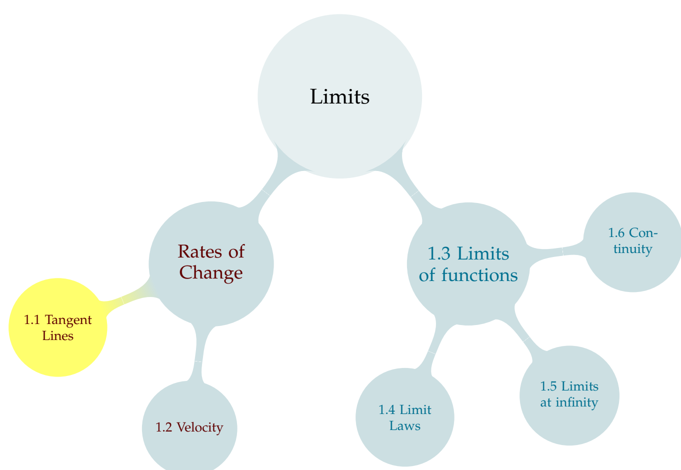


Definition

The **secant line** to the curve $y = f(x)$ through points R and Q is a line that passes through R and Q.

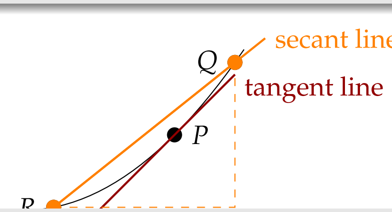
We call the slope of the secant line the **average rate of change of** $f(x)$ **from** R **to** Q.

Definition

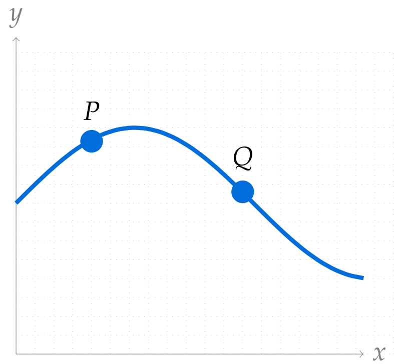
The **tangent line** to the curve $y = f(x)$ at point P is a line that

- passes through P and
- has the same slope as $f(x)$ at P.

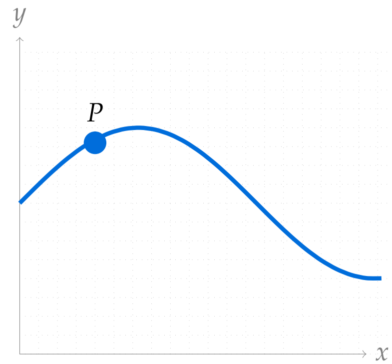
We call the slope of the tangent line the **instantaneous rate of change of** $f(x)$ **at** P.



On the graph below, draw the secant line to the curve through points P and Q.



On the graph below, draw the tangent line to the curve at point P.



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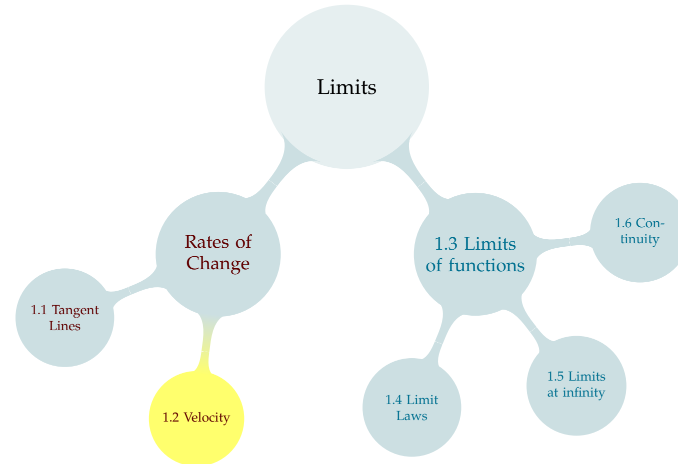
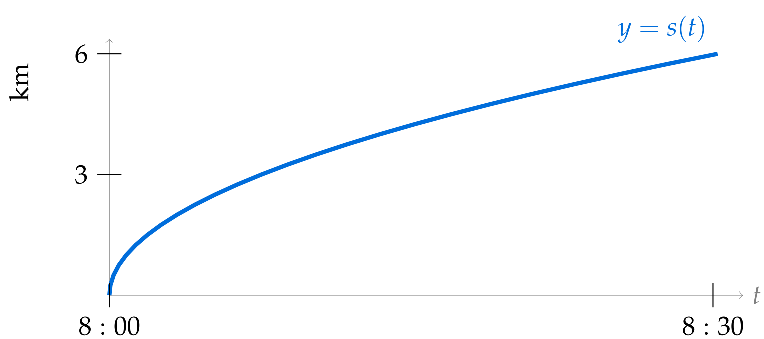
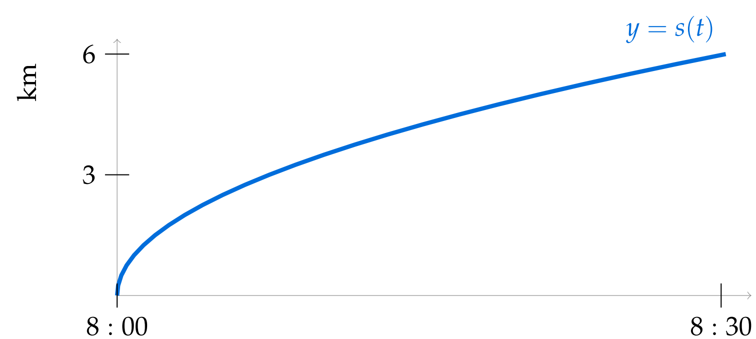






It took $\frac{1}{2}$ hour to bike 6 km. 12 kph represents the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:30$

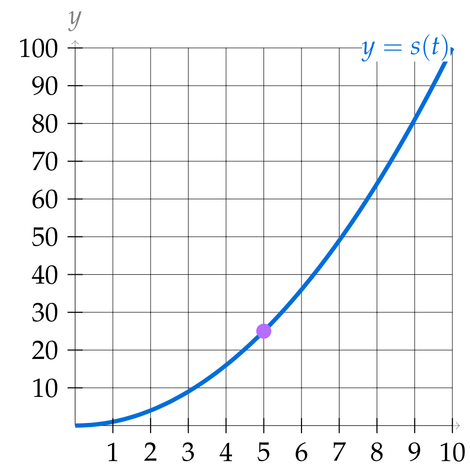
C. tangent line to $y = s(t)$ at $t = 8:30$

D. slope of the tangent line to $y = s(t)$ at $t = 8:30$

At 8:25, the speedometer on my bike reads 5 kph. 5 kph represents the:

A. secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$
B. slope of the secant line to $y = s(t)$ from $t = 8:00$ to $t = 8:25$
C. tangent line to $y = s(t)$ at $t = 8:25$
D. slope of the tangent line to $y = s(t)$ at $t = 8:25$

Suppose the distance from the ground s (in meters) of a helium-filled balloon at time t over a 10-second interval is given by $s(t) = t^2$. Try to estimate how fast the balloon is rising when $t = 5$.



Let's look for an algebraic way of determining the velocity of the balloon when $t = 5$.

$$\text{vel} = \frac{\Delta \text{ height}}{\Delta \text{ time}} = \frac{s(5+h) - s(5)}{(5+h) - 5} = \frac{(5+h)^2 - 5^2}{h}$$

Our First Limit

Average Velocity, $t = 5$ to $t = 5 + h$:

$$\begin{aligned}\frac{\Delta s}{\Delta t} &= \frac{s(5+h) - s(5)}{h} \\ &= \frac{(5+h)^2 - 5^2}{h} \\ &= 10 + h \qquad \text{when } h \neq 0\end{aligned}$$

When h is very small,

$$\text{Vel} \approx 10$$

Limit Notation

We write:

$$\lim_{h \to 0}(10+h) = 10$$

We say: "The limit as h goes to 0 of $(10+h)$ is 10."

It means: As h gets extremely close to 0, $(10+h)$ gets extremely close to 10.

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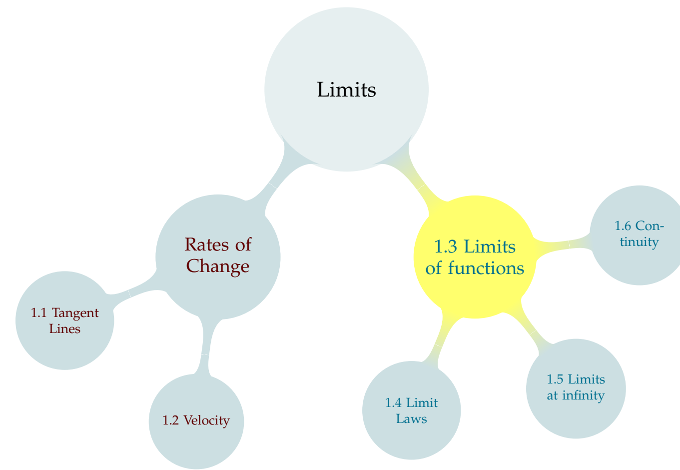


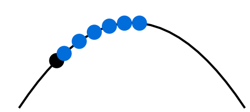
Notation 1.3.1 and Definition 1.3.3

$$\lim_{x \to a} f(x) = L$$

where a and L are real numbers
We read the above as "the limit as x goes to a of $f(x)$ is L."
Its meaning is: as x gets very close to (but not equal to) a, $f(x)$ gets very close to L.

Finding Slopes of Tangent Lines

We NEED limits to find slopes of tangent lines.



Slope of secant line: $\dfrac{\Delta y}{\Delta x}$, $\Delta x \neq 0$.
Slope of tangent line: can't do the same way.

If the position of an object at time t is given by $s(t)$, then its instantaneous velocity is given by

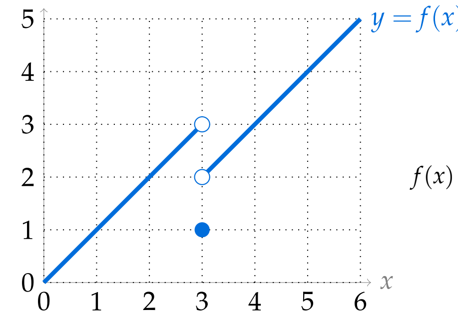
$$\lim_{h \to 0} \frac{s(t+h)-s(t)}{h}$$

Evaluating Limits

$$\text{Let } f(x) = \frac{x^3 + x^2 - x - 1}{x - 1}.$$

We want to evaluate $\lim\limits_{x \to 1} f(x)$.

One-Sided Limits



$$f(x) = \begin{cases} x & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ x - 1 & \text{if } x > 3 \end{cases}$$

What do you think $\lim\limits_{x \to 3} f(x)$ should be?

Definition 1.3.7

The limit as x goes to a **from the left** of $f(x)$ is written

$$\lim_{x \to a^-} f(x)$$

We only consider values of x that are less than a.

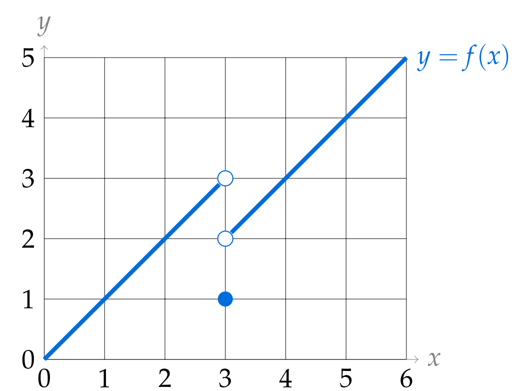
The limit as x goes to a **from the right** of $f(x)$ is written

$$\lim_{x \to a^+} f(x)$$

We only consider values of x greater than a.

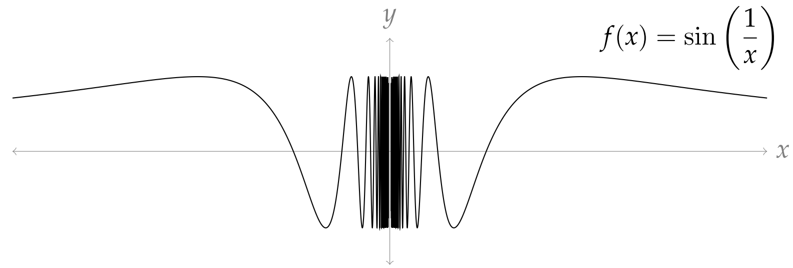
Theorem 1.3.8

In order for $\lim\limits_{x \to a} f(x)$ to exist, both one-sided limits must exist and be equal.



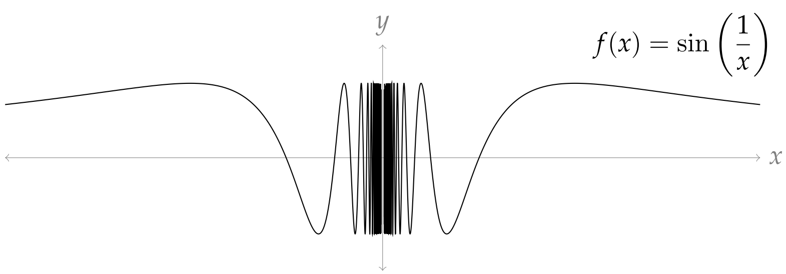
Consider the function $f(x) = \dfrac{1}{(x-1)^2}$. For what value(s) of x is $f(x)$ not defined?

A Stranger Limit Example



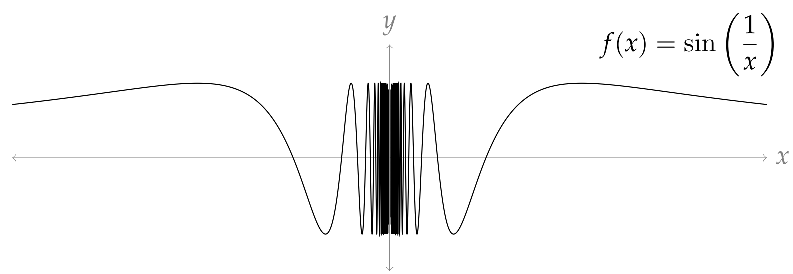
What is $\lim\limits_{x\to\infty} f(x)$?

A Stranger Limit Example



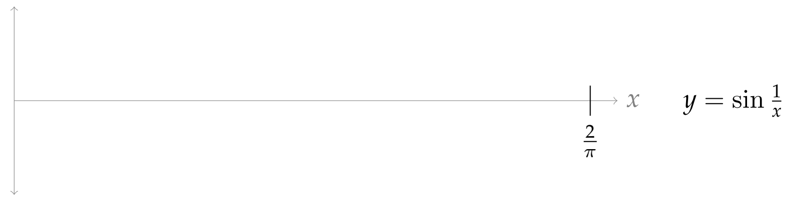
What is $\lim\limits_{x\to 0} f(x)$?

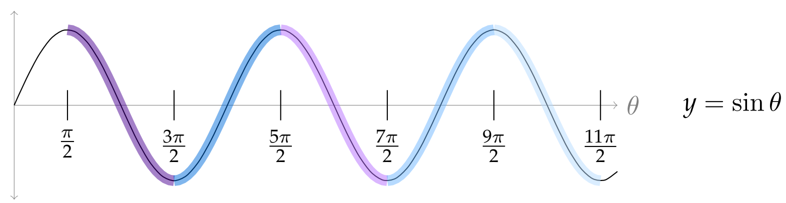
A Stranger Limit Example



What is $\lim\limits_{x\to \pi} f(x)$?

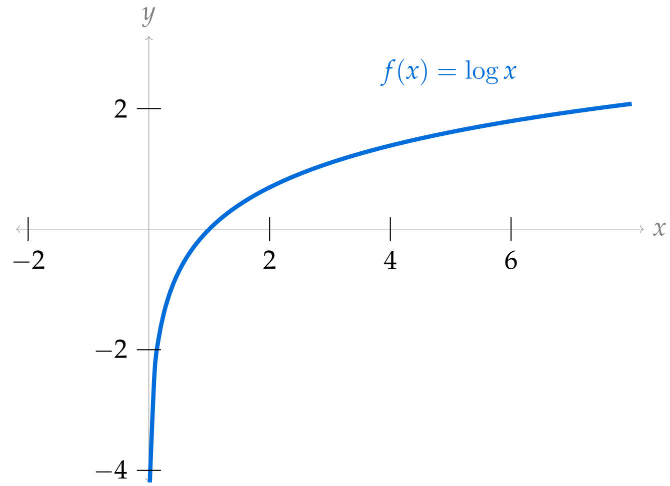
Optional: Sketching $f(x) = \sin\left(\frac{1}{x}\right)$





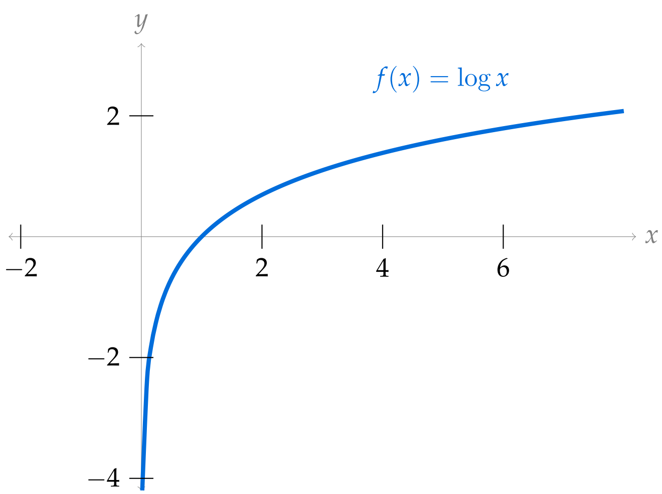
Limits and the Natural Logarithm

Where is $f(x)$ defined, and where is it not defined?

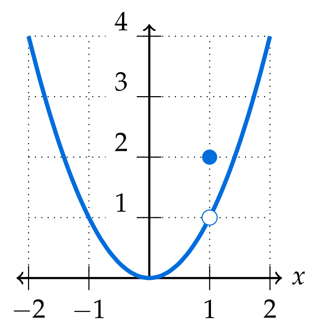


Limits and the Natural Logarithm

What can you say about the limit of $f(x)$ near 0?



$$f(x) = \begin{cases} x^2 & x \neq 1 \\ 2 & x = 1 \end{cases}$$



What is $\lim_{x\to 1} f(x)$?

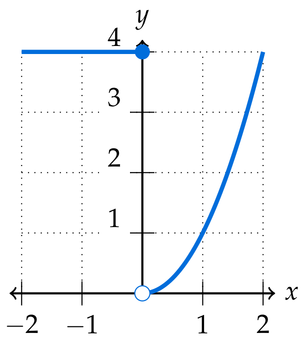
A. $\lim_{x\to 1} f(x) = 2$

B. $\lim_{x\to 1} f(x) = 1$

C. $\lim_{x\to 1} f(x)$ DNE

D. none of the above

$$f(x) = \begin{cases} 4 & x \le 0 \\ x^2 & x > 0 \end{cases}$$



What is $\lim_{x\to 0} f(x)$? What is $\lim_{x\to 0^+} f(x)$? What is $f(0)$?

A. $\lim_{x\to 0^+} f(x) = 4$

B. $\lim_{x\to 0^+} f(x) = 0$

C. $\lim_{x\to 0^+} f(x) = \begin{cases} 4 & x \le 0 \\ 0 & x > 0 \end{cases}$

D. none of the above

Suppose $\lim_{x\to 3^-} f(x) = 1$ and $\lim_{x\to 3^+} f(x) = 1.5$.

Does $\lim_{x\to 3} f(x)$ exist?

A. Yes, certainly, because the limits from both sides exist.

B. No, never, because the limit from the left is not the same as the limit from the right.

C. Can't tell. For some functions is might exist, for others not.

Suppose $\lim_{x\to 3^-} f(x) = 22 = \lim_{x\to 3^+} f(x)$.

Does $\lim_{x\to 3} f(x)$ exist?

A. Yes, certainly, because the limits from both sides exist and are equal to each other.

B. No, never, because we only talk about one-sided limits when the actual limit doesn't exist.

C. Can't tell. We need to know the value of the function at $x = 3$.

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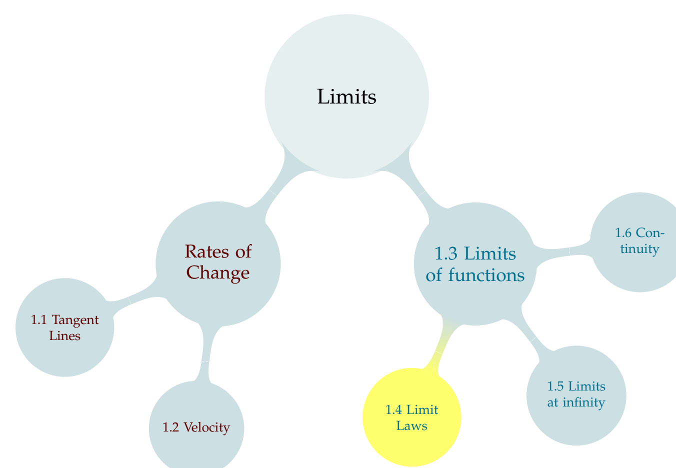


Calculating Limits in Simple Situations

Direct Substitution – Theorem 1.4.10

If $f(x)$ is a polynomial or rational function, and a is in the domain of f, then:

$$\lim_{x\to a} f(x) = f(a).$$

Calculate: $\displaystyle\lim_{x\to 3}\left(\frac{x^2-9}{x+3}\right)$

Calculate: $\displaystyle\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$

Algebra with Limits: Theorem 1.4.2

Suppose $\lim_{x\to a} f(x) = F$ and $\lim_{x\to a} g(x) = G$, where F and G are both real numbers. Then:

- $\lim_{x\to a}(f(x)+g(x)) = F+G$
- $\lim_{x\to a}(f(x)-g(x)) = F-G$
- $\lim_{x\to a}(f(x)g(x)) = FG$
- $\lim_{x\to a}(f(x)/g(x)) = F/G$ provided $G \neq 0$

Calculate: $\displaystyle\lim_{x\to 1}\left[\frac{2x+4}{x+2}+13\left(\frac{x+5}{3x}\right)\left(\frac{x^2}{2x-1}\right)\right]$

Limits involving Powers and Roots

Which of the following gives a real number?

A. $4^{\frac{1}{2}}$ B. $(-4)^{\frac{1}{2}}$ C. $4^{-\frac{1}{2}}$ D. $(-4)^{-\frac{1}{2}}$

E. $8^{1/3}$ F. $(-8)^{1/3}$ G. $8^{-1/3}$ H. $(-8)^{-1/3}$

Powers of Limits – Theorem 1.4.8

If n is a positive integer, and $\lim_{x\to a} f(x) = F$ (where F is a real number), then:

$$\lim_{x\to a}(f(x))^n = F^n.$$

Furthermore, unless n is even and F is negative,

$$\lim_{x\to a}(f(x))^{1/n} = F^{1/n}$$

$$\lim_{x\to 4}(x+5)^{1/2}$$

Cautionary Tales

- $\lim_{x\to 0} \dfrac{(5+x)^2-25}{x}$
- $\lim_{x\to 3} \left(\dfrac{x-6}{3}\right)^{1/8}$
- $\lim_{x\to 0} \dfrac{32}{x}$
- $\lim_{x\to 5} \left(x^2+2\right)^{1/3}$

Suppose you want to evaluate $\lim_{x\to 1} f(x)$, but $f(1)$ doesn't exist. What does that tell you?

A $\lim_{x\to 1} f(x)$ may exist, and it may not exist.

B We can find $\lim_{x\to 1} f(x)$ by plugging in 1 to $f(x)$.

C Since $f(1)$ doesn't exist, it is not meaningful to talk about $\lim_{x\to 1} f(x)$.

D Since $f(1)$ doesn't exist, automatically we know $\lim_{x\to 1} f(x)$ does not exist.

E $\lim_{x\to 1} f(x)$ does not exist if we are "dividing by zero," but may exist otherwise.

Which of the following statements is true about $\lim_{x\to 0} \dfrac{\sin x}{x^3-x^2+x}$?

A $\lim_{x\to 0} \dfrac{\sin x}{x^3-x^2+x} = \dfrac{\sin 0}{0^3-0^2+0} = \dfrac{0}{0}$

B Since the function $\dfrac{\sin x}{x^3-x^2+x}$ is not rational, its limit at 0 does not exist.

C Since the numerator and denominator of $\dfrac{\sin x}{x^3-x^2+x}$ are both 0 when $x=0$, the limit exists.

D Since the function $\dfrac{\sin x}{x^3-x^2+x}$ is not defined at 0, plugging in $x=0$ will not tell us the limit.

E Since the function $\dfrac{\sin x}{x^3-x^2+x}$ consists of the quotient of polynomials and trigonometric functions, its limit exists everywhere.

Which of the following statements is true about $\lim_{x\to 1} \dfrac{\sin x}{x^3-x^2+x}$?

A $\lim_{x\to 1} \dfrac{\sin x}{x^3-x^2+x} = \dfrac{\sin 1}{1^3-1^2+1} = \sin 1$

B Since the function $\dfrac{\sin x}{x^3-x^2+x}$ is not rational, its limit at 1 does not exist.

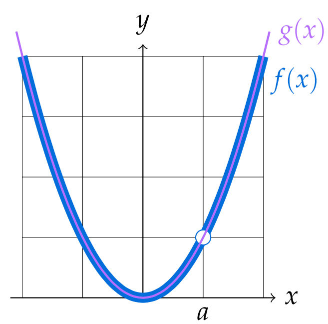
C Since the function $\dfrac{\sin x}{x^3-x^2+x}$ is not defined at 1, plugging in $x=1$ will not tell us the limit.

D Since the numerator and denominator of $\dfrac{\sin x}{x^3-x^2+x}$ are both 0 when $x=1$, the limit exists.

Functions that Differ at a Single Point – Theorem 1.4.12

Suppose $\lim\limits_{x \to a} g(x)$ exists, and $f(x) = g(x)$ when x is close to a (but not necessarily equal to a).

Then $\lim\limits_{x \to a} f(x) = \lim\limits_{x \to a} g(x)$.



Evaluate $\lim\limits_{x \to 1} \dfrac{x^3 + x^2 - x - 1}{x - 1}$.

Evaluate $\lim\limits_{x \to 5} \dfrac{\sqrt{x+20} - \sqrt{4x+5}}{x - 5}$

A Few Strategies for Calculating Limits

First, hope that you can directly substitute (plug in). If your function is made up of the sum, difference, product, quotient, or power of polynomials, you can do this provided the function exists where you're taking the limit.

$$\lim_{x \to 1} \left(\sqrt{35 + x^5} + \frac{x-3}{x^2} \right)^3 =$$

To take a limit outside the domain of a function (that is made up of the sum, difference, product, quotient, or power of polynomials) try to simplify and cancel.

$$\lim_{x\to 0} \frac{x+7}{\frac{1}{x} - \frac{1}{2x}}$$

Otherwise, you can try graphing the function, or making a table of values, to get a better picture of what is going on.

Denominators Approaching Zero

$$\lim_{x\to 1} \frac{1}{(x-1)^2}$$

$$\lim_{x\to 1} \frac{-1}{(x-1)^2}$$

$$\lim_{x\to 1^-} \frac{1}{x-1}$$

$$\lim_{x\to 1^+} \frac{1}{x-1}$$

Denominators Approaching Zero

Now You

$$\lim_{x\to 2^+} \frac{x}{x^2-4}$$

$$\lim_{x\to 2^-} \frac{x}{4-x^2}$$

$$\lim_{x\to 2} \frac{x-2}{x^2-4}$$

Squeeze Theorem – Theorem 1.4.17

Suppose, when x is near (but not necessarily equal to) a, we have functions $f(x)$, $g(x)$, and $h(x)$ so that

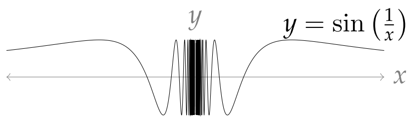
$$f(x) \leq g(x) \leq h(x)$$

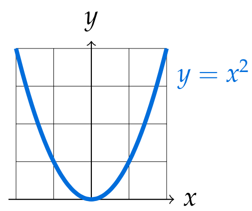
and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x)$. Then $\lim_{x\to a} g(x) = \lim_{x\to a} f(x)$.

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Evaluate:

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$





$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right)$$

$$-1 \quad \leq \quad \sin\left(\frac{1}{x}\right) \quad \leq \quad 1$$

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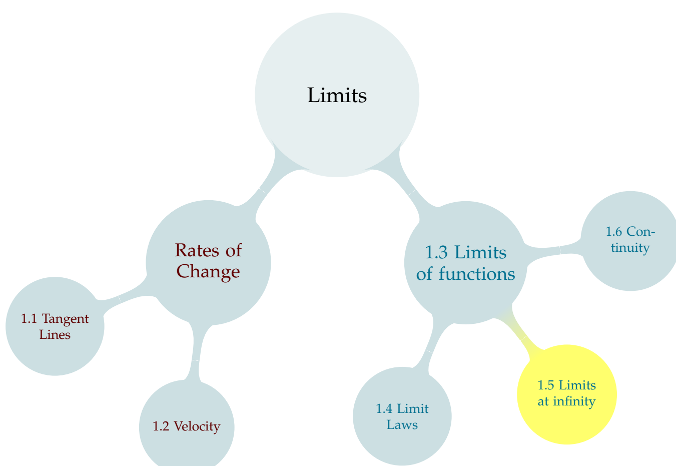


End Behavior

We write:

$$\lim_{x\to\infty} f(x) = L$$

to express that, as x grows larger and larger, $f(x)$ approaches L.

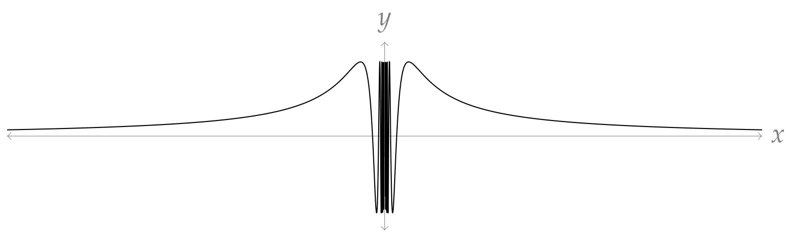
Similarly, we write:

$$\lim_{x\to-\infty} f(x) = L$$

to express that, as x grows more and more strongly negative, $f(x)$ approaches L.

If L is a number, we call $y = L$ a horizontal asymptote of $f(x)$.

Horizontal Asymptotes



$y = 0$ is a horizontal asymptote for $y = \sin\left(\frac{1}{x}\right)$

Common Limits at Infinity

$$\lim_{x\to\infty} 13 = \qquad \lim_{x\to-\infty} 13 =$$

$$\lim_{x\to\infty} \frac{1}{x} = \qquad \lim_{x\to-\infty} \frac{1}{x} =$$

$$\lim_{x\to\infty} x^2 = \qquad \lim_{x\to-\infty} x^2 =$$

$$\lim_{x\to\infty} x^3 = \qquad \lim_{x\to-\infty} x^3 =$$

$$\lim_{x\to-\infty} x^{5/3} = \qquad \lim_{x\to-\infty} x^{2/3} =$$

Arithmetic with Limits at Infinity

Now You

$$\lim_{x\to\infty} \left(x + \frac{x^2}{10}\right) =$$

$$\lim_{x\to\infty} \left(x - \frac{x^2}{10}\right) =$$

$$\lim_{x\to-\infty} \left(x^2 + x^3 + x^4\right) =$$

$$\lim_{x\to-\infty} (x+13)\left(x^2+13\right)^{1/3} =$$

Calculating Limits at Infinity

$$\lim_{x\to\infty} \frac{x^2+2x+1}{x^3}$$

Calculating Limits at Infinity

$$\lim_{x\to-\infty}\left(x^{7/3}-x^{5/3}\right)$$

Again: factor out largest power of x.

Calculating Limits at Infinity

Suppose the height of a bouncing ball is given by $h(t)=\frac{\sin(t)+1}{t}$, for $t\geq 1$. What happens to the height over a long period of time?

Calculating Limits at Infinity

Now You

$$\lim_{x\to\infty}\sqrt{x^4+x^2+1}-\sqrt{x^4+3x^2}$$

Now You

Evaluate $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{3+x^2}}{3x}$

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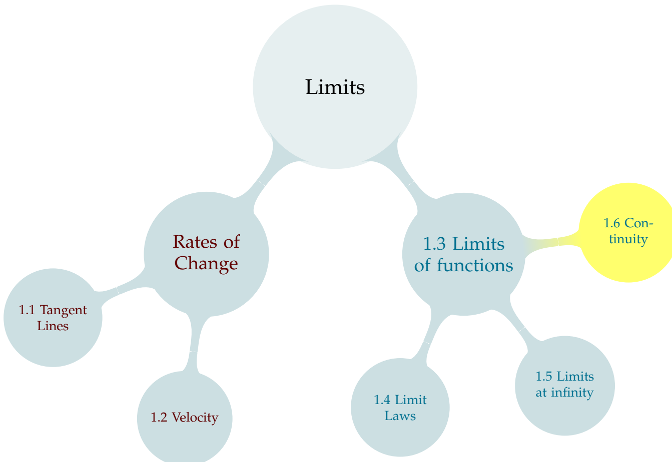
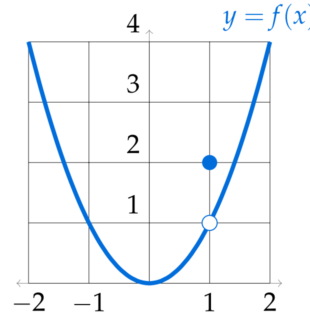


Continuity

Definition 1.6.1

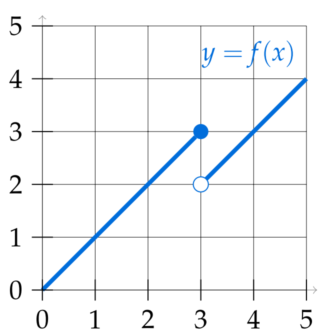
A function $f(x)$ is continuous at a point a if $\lim_{x \to a} f(x)$ exists AND is equal to $f(a)$.



Does $f(x)$ exist at $x = 1$?
Is $f(x)$ continuous at $x = 1$?

Definitions 1.6.1 and 1.6.2

A function $f(x)$ is continuous from the left at a point a if $\lim_{x \to a^-} f(x)$ exists AND is equal to $f(a)$.



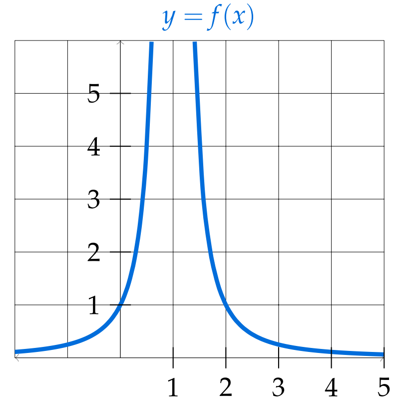
Is $f(x)$ continuous at $x = 3$?

Is $f(x)$ continuous from the left at $x = 3$?

Is $f(x)$ continuous from the right at $x = 3$?

Definition

A function $f(x)$ is continuous at a point a if $\lim_{x \to a} f(x)$ exists AND is equal to $f(a)$.



Definition

A function $f(x)$ is continuous at a point a if $\lim\limits_{x \to a} f(x)$ exists AND is equal to $f(a)$.

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is $f(x)$ continuous at 0?

CONTINUOUS FUNCTIONS

Functions made by adding, subtracting, multiplying, dividing, and taking appropriate powers of polynomials are continuous for every point in their domain.

$$f(x) = \frac{x^2}{2x-10} - \left(\frac{x^2+2x-1}{x-1} + \frac{\sqrt[5]{25-x} - \frac{1}{x}}{x+2} \right)^{1/3}$$

A continuous function is continuous for every point in $\mathbb{R}$.

We say $f(x)$ is continuous over (a, b) if it is continuous at every point in (a, b).

Common Functions – Theorem 1.6.8

Functions of the following types are continuous over their domains:

- polynomials and rationals
- roots and powers
- trig functions and their inverses
- exponential and logarithm
- The products, sums, differences, quotients, powers, and compositions of continuous functions

Where is the following function continuous?

$$f(x) = \left(\frac{\sin x}{(x-2)(x+3)} + e^{\sqrt{x}} \right)^3$$

A Technical Point

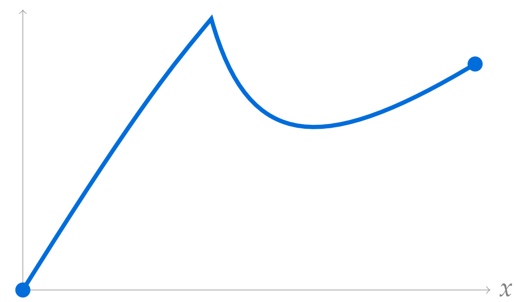
Definition 1.6.3

A function $f(x)$ is continuous on the closed interval $[a, b]$ if:

- $f(x)$ is continuous over (a, b), and
- $f(x)$ is continuous from the left at b, and
- $f(x)$ is continuous from the right at a

Intermediate Value Theorem (IVT) – Theorem 1.6.12

Let $a < b$ and let $f(x)$ be continuous over $[a, b]$. If y is any number between $f(a)$ and $f(b)$, then there exists c in (a, b) such that $f(c) = y$.



Intermediate Value Theorem (IVT) – Theorem 1.6.12

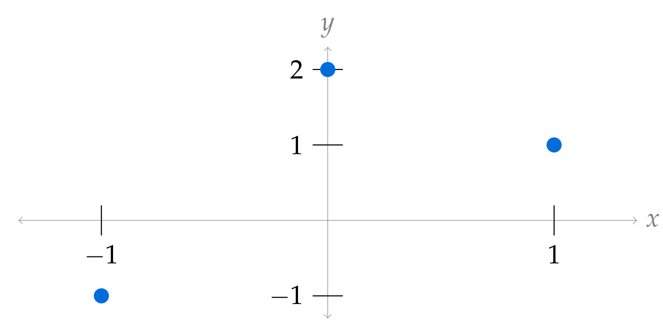
Let $a < b$ and let $f(x)$ be continuous over $[a, b]$. If y is any number between $f(a)$ and $f(b)$, then there exists c in (a, b) such that $f(c) = y$.

Suppose your favourite number is 45.54. At noon, your car is parked, and at 1pm you're driving 100kph.

Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value x for which $f(x) = 0$. Let's find some points:

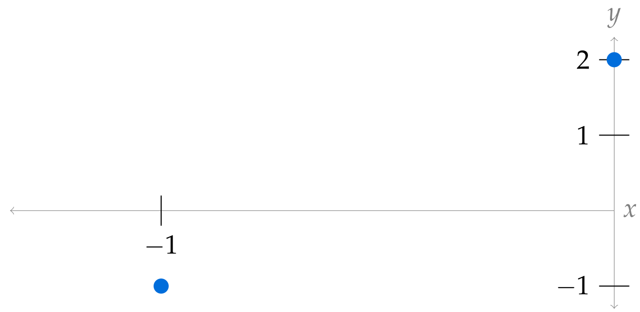
$f(0) = 2$ $\qquad$ $f(1) = 1$ $\qquad$ $f(-1) = -1$



Using IVT to Find Roots: "Bisection Method"

Let $f(x) = x^5 - 2x^4 + 2$. Find any value x for which $f(x) = 0$.

$f(0) = 2, f(-1) = -1$



Use the Intermediate Value Theorem to show that there exists some solution to the equation $\ln x \cdot e^x = 4$, and give a reasonable interval where that solution might occur.

Now You

Use the Intermediate Value Theorem to give a reasonable interval where the following is true: $e^x = \sin(x)$. (Don't use a calculator – use numbers you can easily evaluate.)

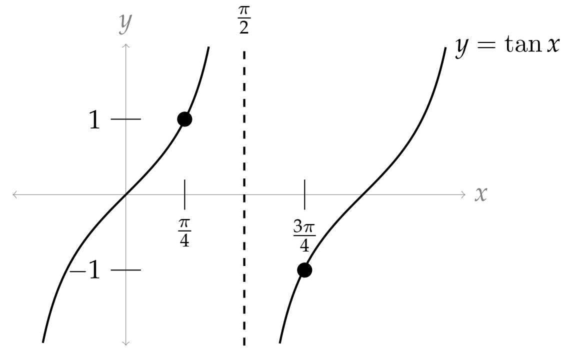
Now You

Is there any value of x so that $\sin x = \cos(2x) + \frac{1}{4}$?

Now You

Is the following reasoning correct?

- $f(x) = \tan x$ is continuous over its domain, because it is a trigonometric function.
- In particular, $f(x)$ is continuous over the interval $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.
- $f\left(\frac{\pi}{4}\right) = 1$, and $f\left(\frac{3\pi}{4}\right) = -1$.
- Since $f\left(\frac{3\pi}{4}\right) < 0 < f\left(\frac{\pi}{4}\right)$, by the Intermediate Value Theorem, there exists some number c in the interval $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ such that $f(c) = 0$.



Continuity

Section 1.6 Review

Suppose $f(x)$ is continuous at $x = 1$. Does $f(x)$ have to be defined at $x = 1$?

Suppose $f(x)$ is continuous at $x=1$ and $\lim_{x\to 1^-} f(x) = 30$.

True or false: $\lim_{x\to 1^+} f(x) = 30$.

Suppose $f(x)$ is continuous at $x=1$ and $f(1)=22$. What is $\lim_{x\to 1} f(x)$?

Suppose $\lim_{x\to 1} f(x) = 2$. Must it be true that $f(1)=2$?

$$f(x) = \begin{cases} ax^2 & x \geq 1 \\ 3x & x < 1 \end{cases}$$

For which value(s) of a is $f(x)$ continuous?

$$f(x) = \begin{cases} \frac{\sqrt{3}x+3}{x^2-3} & x \neq \pm\sqrt{3} \\ a & x = \pm\sqrt{3} \end{cases}$$

For which value(s) of a is $f(x)$ continuous at $x = -\sqrt{3}$?

$$f(x) = \begin{cases} \frac{\sqrt{3}x+3}{x^2-3} & x \neq \pm\sqrt{3} \\ a & x = \pm\sqrt{3} \end{cases}$$

For which value(s) of a is $f(x)$ continuous at $x = \sqrt{3}$?

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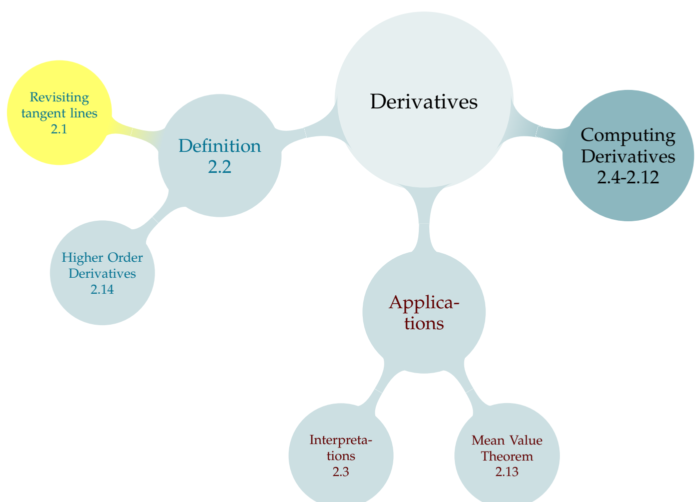
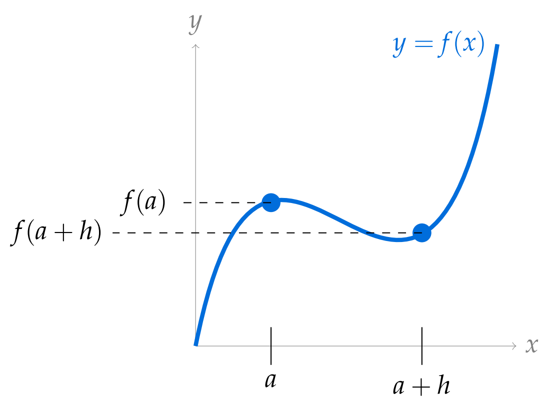


SLOPE OF SECANT AND TANGENT LINE

Slope

Recall, the slope of a line is given by any of the following:

$$\frac{\text{rise}}{\text{run}} \qquad \frac{\Delta y}{\Delta x} \qquad \frac{y_2 - y_1}{x_2 - x_1}$$



Slope of secant line: $\frac{f(a+h)-f(a)}{h}$

Slope of tangent line: $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$

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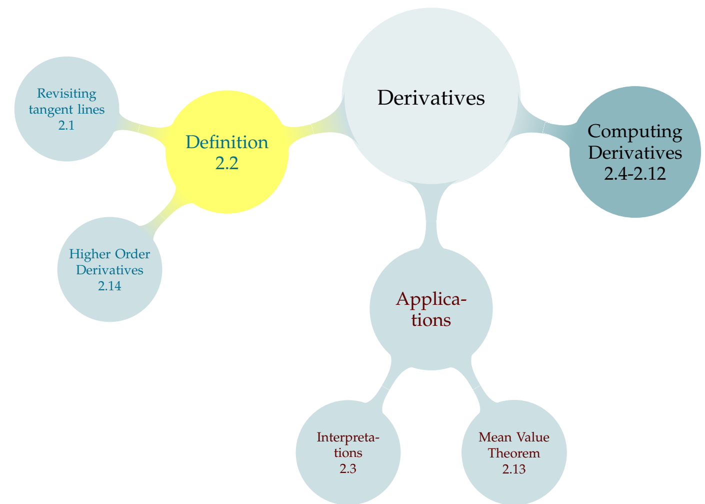


Derivative at a Point

Definition 2.2.1

Given a function $f(x)$ and a point a, the slope of the tangent line to $f(x)$ at a is the derivative of f at a, written $f'(a)$.

So, $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

$f'(a)$ is also the instantaneous rate of change of f at a.

Derivative

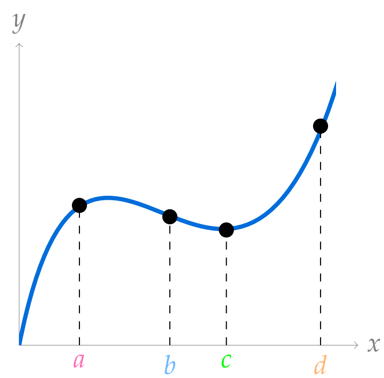
$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

If $f'(a) > 0$, then f is increasing at a. Its graph “points up.”

If $f'(a) < 0$, then f is decreasing at a. Its graph “points down.”

If $f'(a) = 0$, then f looks constant or flat at a.

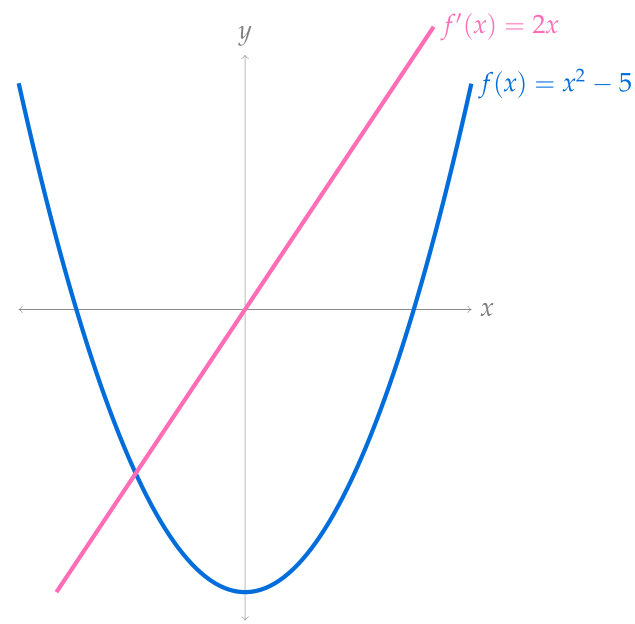
PRACTICE: INCREASING AND DECREASING



Where is $f'(x) < 0$? Where is $f'(x) > 0$? Where is $f'(x) \approx 0$?

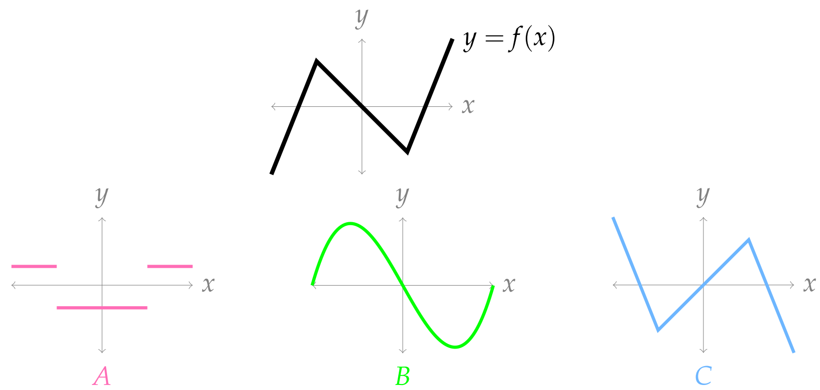
Use the definition of the derivative to find the slope of the tangent line to $f(x) = x^2 - 5$ at the point $x = 3$.

Let's keep the function $f(x) = x^2 - 5$. We just showed $f'(3) = 6$. We can also find its derivative at an arbitrary point x:



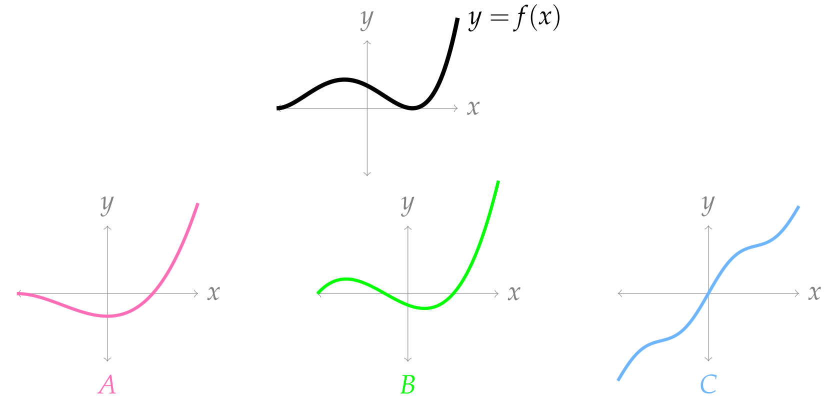
Increasing and Decreasing

In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?



Increasing and Decreasing

In black is the curve $y = f(x)$. Which of the coloured curves corresponds to $y = f'(x)$?



Derivative as a Function – Definition 2.2.6

Let $f(x)$ be a function.
The derivative of $f(x)$ with respect to x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists. Notice that x will be a part of your final expression: this is a function.

If $f'(x)$ exists for all x in an interval (a, b), we say that f is differentiable on (a, b).

Notation 2.2.8

The "prime" notation $f'(x)$ and $f'(a)$ is sometimes called Newtonian notation. We will also use Leibniz notation:

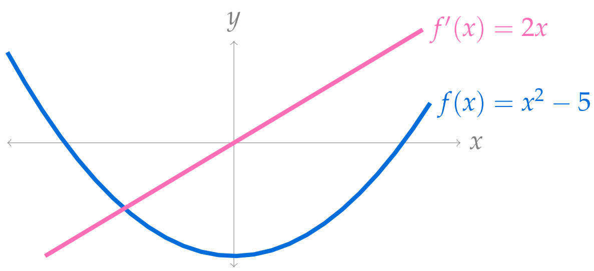
$\dfrac{\mathrm{d}f}{\mathrm{d}x}$	$\dfrac{\mathrm{d}f}{\mathrm{d}x}(a)$	$\dfrac{\mathrm{d}}{\mathrm{d}x}f(x)$	$\left.\dfrac{\mathrm{d}}{\mathrm{d}x}f(x)\right\vert_{x=a}$
function	number	function	number

Newtonian Notation:

$$f(x) = x^2 + 5 \qquad f'(x) = 2x \qquad f'(3) = 6$$

Leibniz Notation:

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(3) = \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x) = \qquad \frac{\mathrm{d}}{\mathrm{d}x}f(x)\bigg|_{x=3} =$$



Alternate Definition – Definition 2.2.1

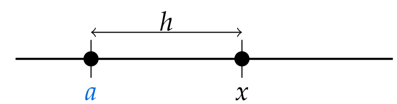
Calculating

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

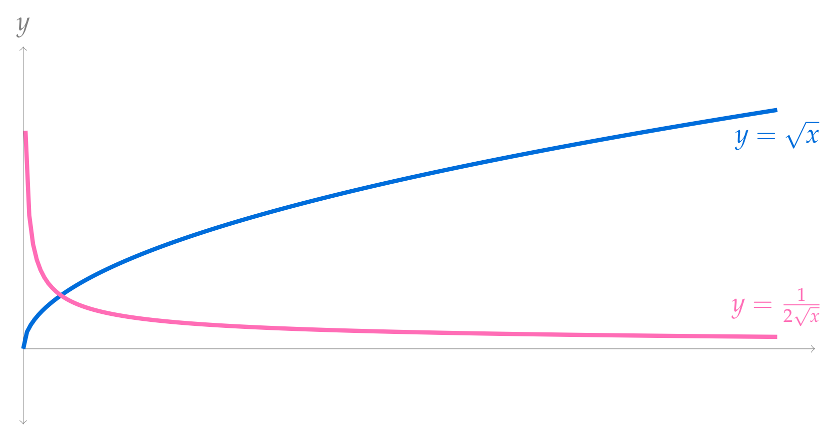
is the same as calculating

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios, $h = x - a$.



Let $f(x) = \sqrt{x}$. Using the definition of a derivative, calculate $f'(x)$.



$\boxed{\text{Review:}}$ $\lim_{x \to \infty} \sqrt{x} =$ $\qquad \lim_{x \to \infty} \frac{1}{2\sqrt{x}} =$

$\lim_{x \to 0^+} \sqrt{x} =$ $\qquad \lim_{x \to 0^+} \frac{1}{2\sqrt{x}} =$

Now You

Using the definition of the derivative, calculate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{x}\right\}$.

Using the definition of the derivative, calculate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{2x}{x+1}\right\}$.

Using the definition of the derivative, calculate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{\sqrt{x^2+x}}\right\}$.

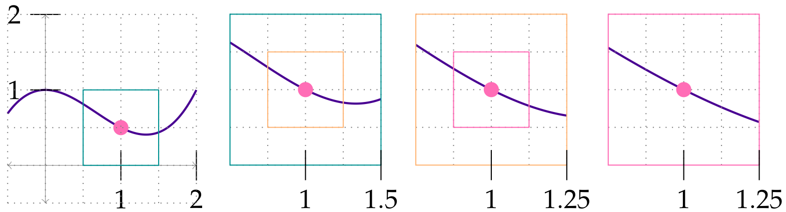
Memorize

The derivative of a function f at a point a is given by the following limit, if it exists:

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Zooming In

For a smooth function, if we zoom in at a point, we see a line:



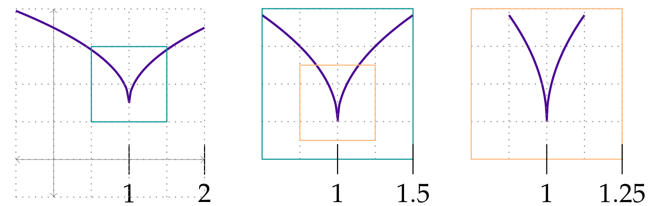
In this example, the slope of our zoomed-in line looks to be about:

$$\frac{\Delta y}{\Delta x} \approx -\frac{1}{2}$$

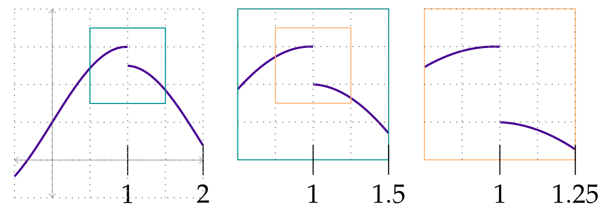
Zooming in on functions that aren't smooth

For a function with a cusp or a discontinuity, even though we zoom in very closely, we don't see simply a single straight line.

Cusp:



Discontinuity:



Alternate Definition – Definition 2.2.1

Calculating

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is the same as calculating

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Notice in these scenarios, $h = x - a$.

The derivative of $f(x)$ does not exist at $x = a$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

does not exist.
Note this is the slope of the tangent line to $y = f(x)$ at $x = a$, $\frac{\Delta y}{\Delta x}$.

When Derivatives Don't Exist

What happens if we try to calculate a derivative where none exists?

Find the derivative of $f(x) = x^{1/3}$ at $x = 0$.

Theorem 2.2.14

If the function $f(x)$ is differentiable at $x = a$, then $f(x)$ is also continuous at $x = a$.

Proof:

Let $f(x)$ be a function and let a be a constant in its domain. Draw a picture of each scenario, or say that it is impossible.

$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$	$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$
$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$	$f(x)$ continuous at $x = a$ $f(x)$ differentiable at $x = a$

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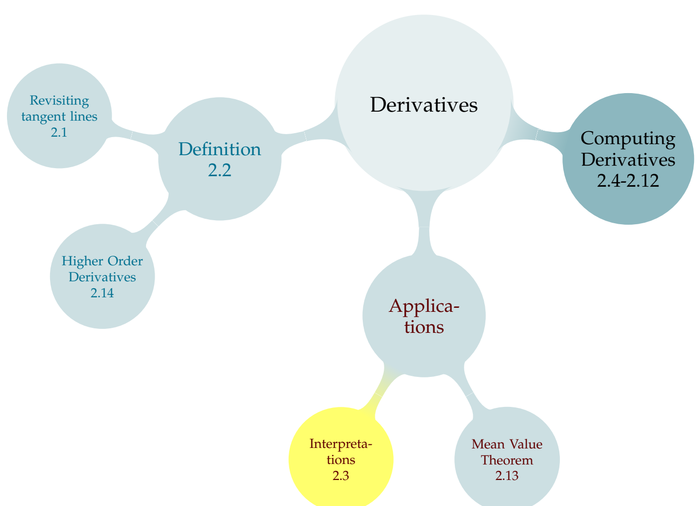

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Suppose $P(t)$ gives the number of people in the world at t minutes past midnight, January 1, 2012. Suppose further that $P'(0) = 156$. How do you interpret $P'(0) = 156$?

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Suppose $P(n)$ gives the total profit, in dollars, earned by selling n widgets. How do you interpret $P'(100)$?

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Suppose $h(t)$ gives the height of a rocket t seconds after liftoff. What is the interpretation of $h'(t)$?

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Suppose $M(t)$ is the number of molecules of a chemical in a test tube t seconds after a reaction starts. Interpret $M'(t)$.

Interpreting the Derivative

The derivative of $f(x)$ at a, written $f'(a)$, is the instantaneous rate of change of $f(x)$ when $x = a$.

Suppose $G(w)$ gives the diameter in millimetres of steel wire needed to safely support a load of w kg. Suppose further that $G'(100) = 0.01$. How do you interpret $G'(100) = 0.01$?

A paper[1] on the impacts of various factors in average life expectancy contains the following:

> *The only statistically significant variable in the model is physician density. The coefficient for this variable 20.67 indicating that a one unit increase in physician density leads to a 20.67 unit increase in life expectancy. This variable is also statistically significant at the 1% level demonstrating that this variable is very strongly and positively correlated with quality of healthcare received. This denotes that access to healthcare is very impactful in terms of increasing the quality of health in the country.*

[1] Natasha Deshpande, Anoosha Kumar, Rohini Ramaswami, *The Effect of National Healthcare Expenditure on Life Expectancy*, page 12.
Remark: physician density is measured as number of doctors per 1000 members of the population.

If $L(p)$ is the average life expectancy in an area with a density p of physicians, write the statement as a derivative: "a one unit increase in physician density leads to a 20.67 unit increase in life expectancy."

Equation of the Tangent Line

The tangent line to $f(x)$ at a has slope $f'(a)$ and passes through the point $(a, f(a))$.

Tangent Line Equation – Theorem 2.3.2

The tangent line to the function $f(x)$ at point a is:

$$(y - f(a)) = f'(a)(x - a)$$

Point-Slope Formula

In general, a line with slope m passing through point (x_1, y_1) has the equation:

$$(y - y_1) = m(x - x_1)$$

Find the equation of the tangent line to the curve $f(x) = \sqrt{x}$ at $x = 9$. (Recall $\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{x}\right] = \frac{1}{2\sqrt{x}}$).

Memorize

The tangent line to the function $f(x)$ at point a is:

$$(y - f(a)) = f'(a)(x - a)$$

Now You Let $s(t) = 3 - 0.8t^2$. Then $s'(t) = -1.6t$. Find the equation for the tangent line to the function $s(t)$ when $t = 1$.

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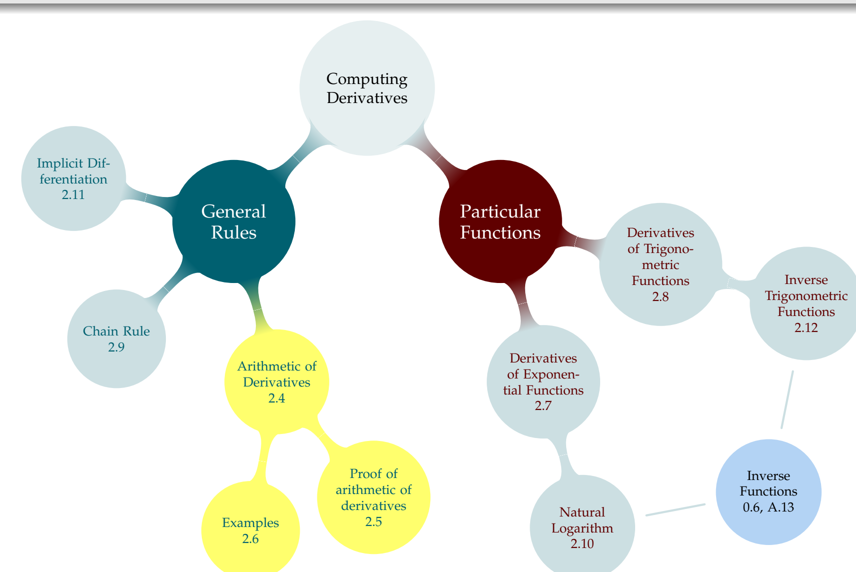


Derivatives of Lines

$$f(x) = 2x - 15$$

The equation of the tangent line to $f(x)$ at $x = 100$ is:

$f'(1) =$

A. 0 B. 1 C. 2 D. -15 E. -13

$f'(5) =$

$f'(-13) =$

$$g(x) = 13$$

$g'(1) =$

A. 0 B. 1 C. 2 D. 13

Adding a Constant

Adding or subtracting a constant to a function does not change its derivative.

We saw

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(3 - 0.8t^2\right)\Big|_{t=1} = -1.6$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(10 - 0.8t^2\right)\Big|_{t=1} =$$

Differentiating Sums

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(x) + g(x)\} =$$

Constant Multiple of a Function

Let a be a constant.

$$\frac{\mathrm{d}}{\mathrm{d}x}\{a \cdot f(x)\} =$$

Rules – Lemma 2.4.1

Suppose $f(x)$ and $g(x)$ are differentiable, and let c be a constant number. Then:

- $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)+g(x)\} = f'(x)+g'(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)-g(x)\} = f'(x)-g'(x)$
- $\frac{\mathrm{d}}{\mathrm{d}x}\{cf(x)\} = cf'(x)$

For instance: let $f(x) = 10\left((2x-15)+13-\sqrt{x}\right)$. Then $f'(x) =$

Now You

Suppose $f'(x) = 3x$, $g'(x) = -x^2$, and $h'(x) = 5$.

Calculate:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)+5g(x)-h(x)+22\}$$

A. $3x-5x^2$
B. $3x-5x^2-5$
C. $3x-5x^2-5+22$
D. none of the above

Derivatives of Products

$$\frac{\mathrm{d}}{\mathrm{d}x}\{x\} = 1$$

True or False:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\{2x\} &= \frac{\mathrm{d}}{\mathrm{d}x}\{x+x\}\\ &= [1]+[1]\\ &= 2\end{aligned}$$

True or False:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\{x^2\} &= \frac{\mathrm{d}}{\mathrm{d}x}\{x\cdot x\}\\ &= [1]\cdot[1]\\ &= 1\end{aligned}$$

What to do with Products?

Suppose $f(x)$ and $g(x)$ are differentiable functions of x. What about $f(x)g(x)$?

Product Rule – Theorem 2.4.3

For differentiable functions $f(x)$ and $g(x)$:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Example:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] =$$

Example: suppose $f(x) = 3x^2, f'(x) = 6x$, $g(x) = \sin(x), g'(x) = \cos(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[3x^2 \sin(x)\right] =$$

Given $\quad \frac{\mathrm{d}}{\mathrm{d}x}[2x+5] = 2, \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(x^2)\right] = 2x\cos(x^2), \quad \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] = 2x$

Now You

$$f(x) = (2x+5)\sin(x^2)$$

A. $f'(x) = (2)\left(2x\cos(x^2)\right)(2x)$
B. $f'(x) = (2)\left(2x\cos(x^2)\right)$
C. $f'(x) = (2x+5)(2) + \sin(x^2)\left(2x\cos(x^2)\right)$
D. $f'(x) = (2x+5)\left(2x\cos(x^2)\right) + (2)\sin(x^2)$
E. none of the above

Now You

$$f(x) = a(x) \cdot b(x) \cdot c(x)$$

What is $f'(x)$?

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{2x+5}{3x-6}\right\} =$$

Quotient Rule – Theorem 2.4.5

Let $f(x)$ and $g(x)$ be differentiable and $g(x) \neq 0$. Then:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

Mnemonic: Low d'high minus high d'low over lowlow.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{5x}{\sqrt{x}-1}\right\} =$$

Now You

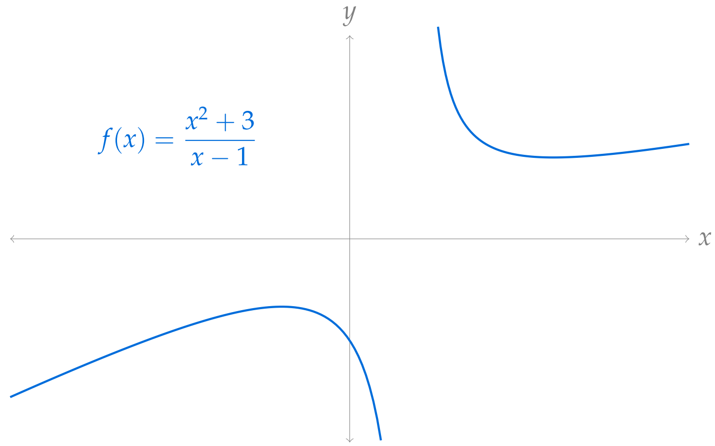
Differentiate the following.

$$f(x) = 2x+5$$
$$g(x) = (2x+5)(3x-7)+25$$
$$h(x) = \frac{2x+5}{8x-2}$$
$$j(x) = \left(\frac{2x+5}{8x-2}\right)^2$$

Rules

Product: $\frac{\mathrm{d}}{\mathrm{d}x}\{f(x)g(x)\} = f(x)g'(x) + g(x)f'(x)$

Quotient: $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{f(x)}{g(x)}\right\} = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$

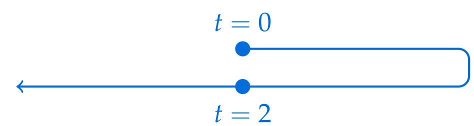


For which values of x is the tangent line to the curve horizontal?

The position of an object moving left and right at time t, $t \geq 0$, is given by

$$s(t) = -t^2(t-2)$$

where a positive position means it is to the right of its starting position, and a negative position means it is to the left. First it moves to the right, then it moves left forever.



What is the farthest point to the right that the object reaches?

More About the Product Rule

$\frac{d}{dx}\{x^2\} = \frac{d}{dx}\{x \cdot x\} = x(1) + x(1)$
$= 2x$

$\frac{d}{dx}\{x^3\} = \frac{d}{dx}\{x \cdot x^2\}$
$= (x)(2x) + (x^2)(1) = 3x^2$

$\frac{d}{dx}\ \{x^4\} = \frac{d}{dx}\{x \cdot x^3\}$
$= x(3x^2) + x^3(1) = 4x^3$

Where are these functions defined?

function	derivative
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^{30}	$30x^{29}$
x^n	nx^{n-1}

Cautionary Tale

With *functions* raised to a power, it's more complicated.

Differentiate $(2x+1)^2$

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

$$\frac{\mathrm{d}}{\mathrm{d}x}\{3x^5 + 7x^2 - x + 15\} =$$

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

Differentiate $\dfrac{(x^4+1)(\sqrt[3]{x}+\sqrt[4]{x})}{2x+5}$

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

Suppose a motorist is driving their car, and their position is given by $s(t) = 10t^3 - 90t^2 + 180t$ kilometres. At $t = 1$ (t measured in hours), a police officer notices they are driving erratically. The motorist claims to have simply suffered a lack of attention: they were in the act of pressing the brakes even as the officer noticed their speed.

At $t = 1$, how fast was the motorist going, and were they pressing the gas or the brake?

Challenge: What about $t = 2$?

Power Rule – Corollary 2.6.17

$\frac{\mathrm{d}}{\mathrm{d}x}\{x^a\} = ax^{a-1}$ (where defined)

Recall that a sphere of radius r has volume $V = \frac{4}{3}\pi r^3$.
Suppose you are winding twine into a gigantic twine ball, filming the process, and trying to make a viral video. You can wrap one cubic meter of twine per hour. (In other words, when we have V cubic meters of twine, we're at time V hours.) How fast is the radius of your spherical twine ball increasing?

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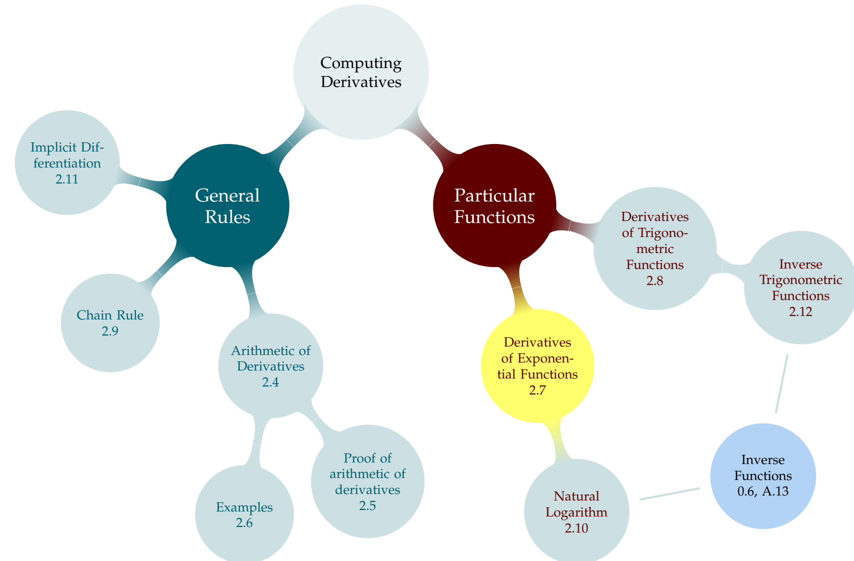
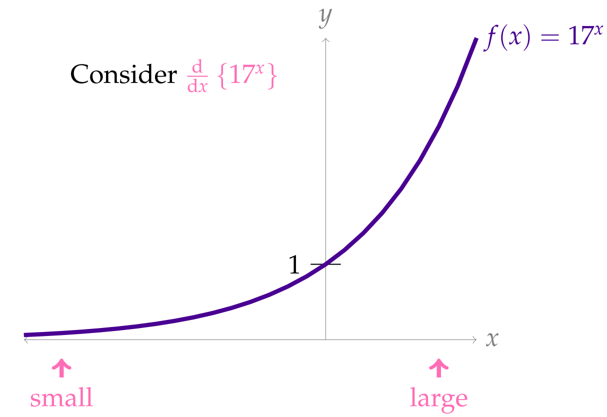


Exponential Functions



$f(x)$ is always increasing, so $f'(x)$ is always positive.
$f'(x)$ might look similar to $f(x)$.

Exponential Functions

$$\frac{d}{dx}\{17^x\} =$$

$$\frac{d}{dx}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0}\frac{(17^h - 1)}{h}}_{\text{constant}}$$

Given what you know about $\frac{d}{dx}\{17^x\}$, **is it possible** that

$$\lim_{h\to 0}\frac{17^h - 1}{h} = 0?$$

A. Sure, there's no reason we've seen that would make it impossible.
B. No, it couldn't be 0, that wouldn't make sense.
C. I do not feel equipped to answer this question.

$$\frac{d}{dx}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0}\frac{(17^h-1)}{h}}_{\text{constant}}$$

Given what you know about $\frac{d}{dx}\{17^x\}$, **is it possible** that $\displaystyle\lim_{h\to 0}\frac{17^h-1}{h} = \infty$?

A. Sure, there's no reason we've seen that would make it impossible.
B. No, it couldn't be ∞, that wouldn't make sense.
C. I do not feel equipped to answer this question.

$$\frac{d}{dx}\{17^x\} = 17^x \cdot \underbrace{\lim_{h\to 0}\frac{(17^h-1)}{h}}_{\text{constant}}$$

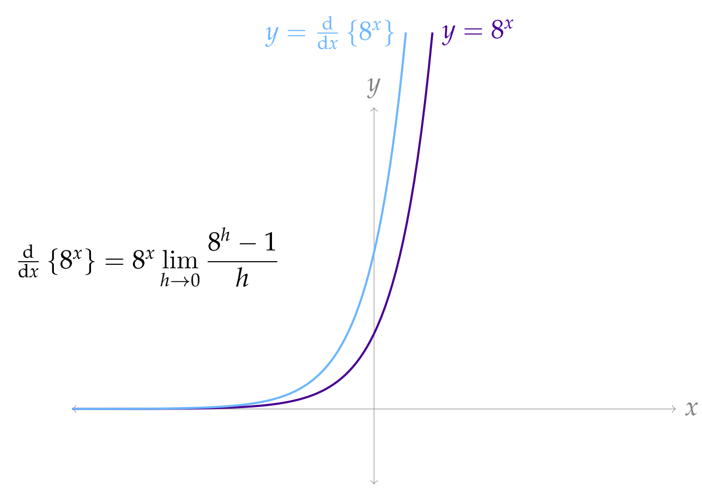
h	$\frac{17^h-1}{h}$
0.001	2.83723068608
0.00001	2.83325347992
0.0000001	2.83321374583
0.000000001	2.83321344163

$$\begin{aligned}\frac{d}{dx}\{17^x\} &= \lim_{h\to 0}\frac{17^{x+h}-17^x}{h}\\ &= \lim_{h\to 0}\frac{17^x 17^h-17^x}{h}\\ &= \lim_{h\to 0}\frac{17^x(17^h-1)}{h}\\ &= 17^x\lim_{h\to 0}\frac{(17^h-1)}{h}\end{aligned}$$

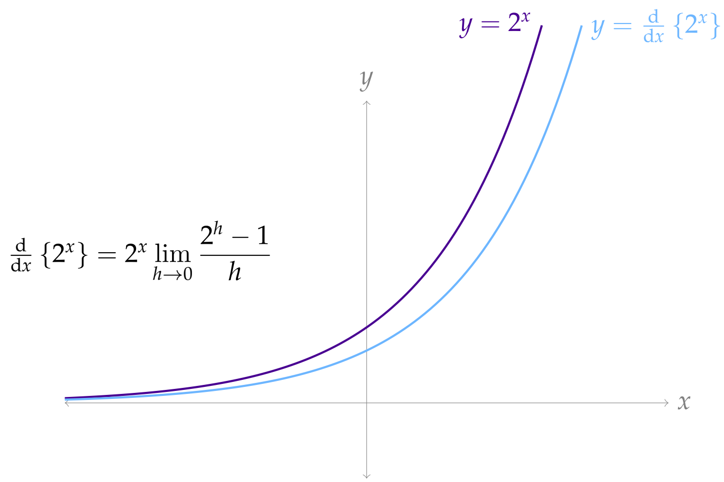
In general, for any positive number a,

$$\frac{d}{dx}\{a^x\} = a^x\lim_{h\to 0}\frac{a^h-1}{h}$$

Exponential Functions



Exponential Functions



In general, for any positive number a, $\frac{d}{dx}\{a^x\} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$

Euler's Number – Theorem 2.7.4

We define e to be the unique number satisfying

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

$e \approx 2.71828182845904523536028747135266624\ldots$ (Wikipedia)

Theorem 2.7.4 and Corollary 2.10.6

Using this definition of e,

$$\frac{d}{dx}\{e^x\} = e^x \underbrace{\lim_{h\to 0} \frac{e^h - 1}{h}}_{1} = e^x$$

In general, $\lim_{h\to 0} \frac{a^h - 1}{h} = \log_e(a)$, so $\frac{d}{dx}\{a^x\} = a^x \log_e(a)$

That $\lim_{h\to 0} \frac{a^h - 1}{h} = \log_e(a)$ and $\frac{d}{dx}\{a^x\} = a^x \log_e(a)$ are consequences of

$$a^x = \left(e^{\log_e(a)}\right)^x = e^{x\log_e(a)}$$

For the details, see the end of Section 2.7.

Things to Have Memorized

$$\frac{d}{dx}\{e^x\} = e^x$$

When a is any constant,

$$\frac{d}{dx}\{a^x\} = a^x \log_e(a)$$

Let $f(x) = \frac{e^x}{3x^5}$. When is the tangent line to $f(x)$ horizontal?

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{e^{3x}\right\}$

Suppose the deficit, in millions, of a fictitious country is given by

$$f(x) = e^x(4x^3 - 12x^2 + 14x - 4)$$

where x is the number of years since the current leader took office. Suppose the leader has been in power for exactly two years.

1. Is the deficit increasing or decreasing?

2. Is the rate at which the deficit is growing increasing or decreasing?

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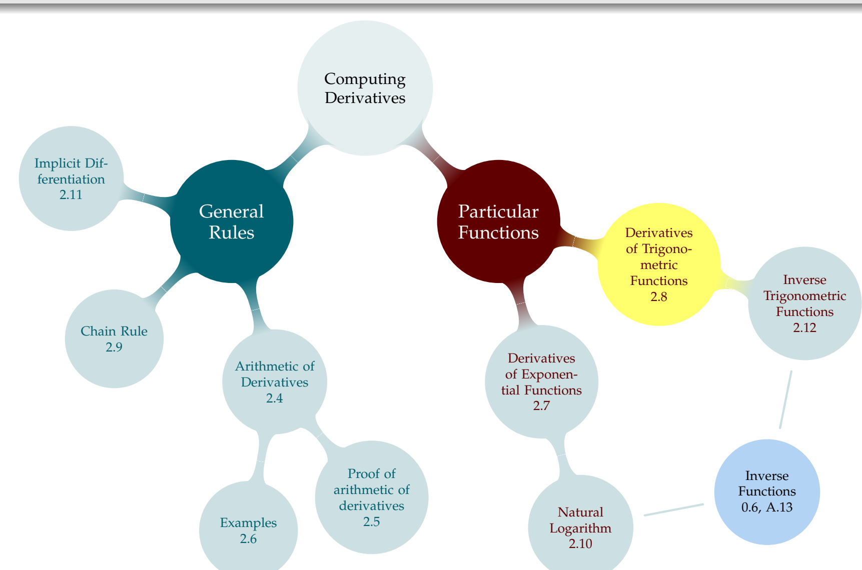
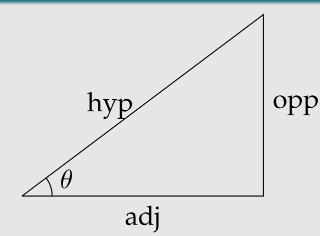


Basic Trig Functions



$$\sin(\theta) = \frac{\text{opp}}{\text{hyp}} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} \qquad \sec(\theta) = \frac{1}{\cos(\theta)}$$

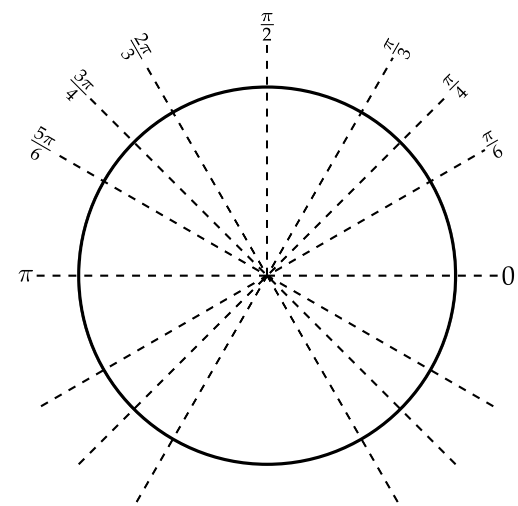
$$\tan(\theta) = \frac{\text{opp}}{\text{adj}} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

Commonly used facts

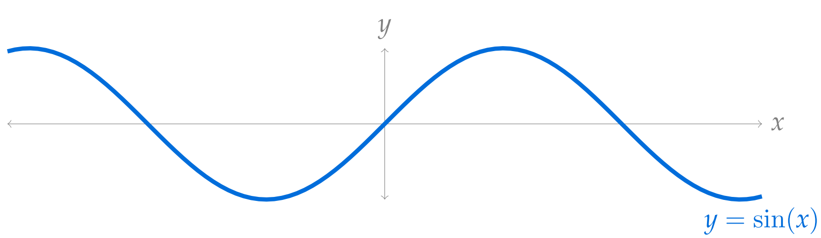
- Graphs of sine, cosine, tangent
- Sine, cosine, and tangent of reference angles: $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$
- How to use reference angles to find sine, cosine and tangent of other angles
- Identities: $\sin^2 x + \cos^2 x = 1;\qquad \tan^2 x + 1 = \sec^2 x;$
  $\sin^2 x = \dfrac{1-\cos(2x)}{2};\qquad \cos^2 x = \dfrac{1+\cos 2x}{2}$
- Conversion between radians and degrees

CLP-1 has an appendix on high school trigonometry that you should be familiar with.

Reference Angles



Derivative of Sine



Consider the derivative of $f(x) = \sin(x)$.

$$\frac{d}{dx}\{\sin x\} = \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

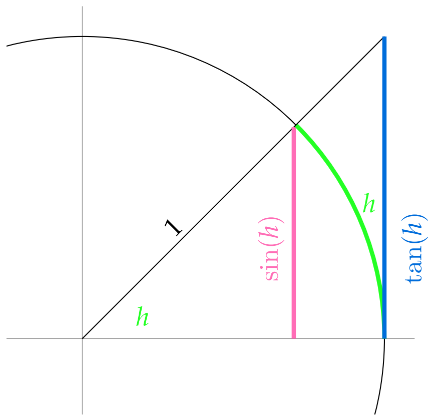
$$= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{\sin(x)(\cos(h)-1)}{h} + \lim_{h\to 0}\frac{\cos(x)\sin(h)}{h}$$

$$= \sin(x)\lim_{h\to 0}\frac{\cos(0+h)-\cos(0)}{h} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \sin(x)\,\frac{d}{dx}\{\cos(x)\}\Big|_{x=0} + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h} \qquad = \qquad \boxed{\cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}}$$

since $\cos(x)$ has a horizontal tangent, and hence has derivative zero, at $x=0$.



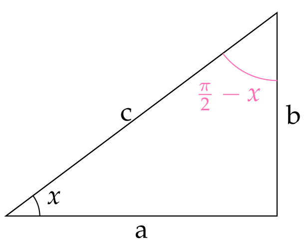
Derivatives of Sine and Cosine

¿From before,

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x) \cdot \lim_{h \to 0} \frac{\sin(h)}{h} = \cos(x)$$

Derivative of Cosine

Now for the derivative of $\cos$. We already know the derivative of $\sin$, and it is easy to convert between $\sin$ and $\cos$ using trig identities.



$$\sin x = \frac{b}{c} = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \frac{a}{c} = \sin\left(\frac{\pi}{2} - x\right)$$

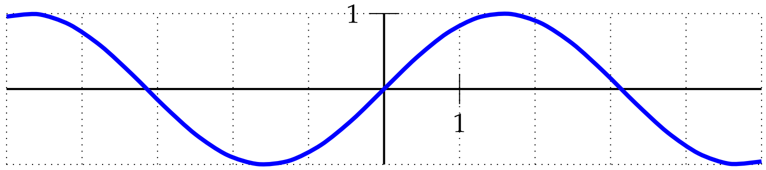
When we use radians:

Derivatives of Trig Functions

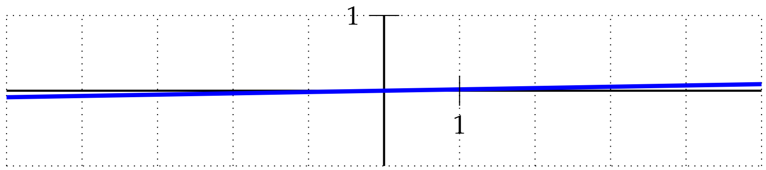
$\frac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} = \cos(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}\{\sec(x)\} =$
$\frac{\mathrm{d}}{\mathrm{d}x}\{\cos(x)\} = -\sin(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}\{\csc(x)\} =$
$\frac{\mathrm{d}}{\mathrm{d}x}\{\tan(x)\} =$	$\frac{\mathrm{d}}{\mathrm{d}x}\{\cot(x)\} =$

Honorable Mention

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$



$y = \sin x$, radians



$y = \sin x$, degrees

Other Trig Functions

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

Other Trig Functions

$$\sec(x) = \frac{1}{\cos(x)}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[\sec(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\cos(x)}\right]\\ &= \frac{\cos(x)(0) - (1)(-\sin(x))}{\cos^2(x)}\\ &= \frac{\sin(x)}{\cos^2(x)}\\ &= \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)}\\ &= \sec(x)\tan(x)\end{aligned}$$

Other Trig Functions

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[\csc(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\sin(x)}\right]\\ &= \frac{\sin(x)(0) - (1)\cos(x)}{\sin^2(x)}\\ &= \frac{-\cos(x)}{\sin^2(x)}\\ &= \frac{-1}{\sin(x)}\frac{\cos(x)}{\sin(x)}\\ &= -\csc(x)\cot(x)\end{aligned}$$

Other Trig Functions

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}[\cot(x)] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\cos(x)}{\sin(x)}\right] \\ &= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\ &= \frac{-1}{\sin^2(x)} \\ &= -\csc^2(x)\end{aligned}$$

Memorize

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}x}\{\sin(x)\} &= \cos(x) & \tfrac{\mathrm{d}}{\mathrm{d}x}\{\sec(x)\} &= \sec(x)\tan(x) \\ \tfrac{\mathrm{d}}{\mathrm{d}x}\{\cos(x)\} &= -\sin(x) & \tfrac{\mathrm{d}}{\mathrm{d}x}\{\csc(x)\} &= -\csc(x)\cot(x) \\ \tfrac{\mathrm{d}}{\mathrm{d}x}\{\tan(x)\} &= \sec^2(x) & \tfrac{\mathrm{d}}{\mathrm{d}x}\{\cot(x)\} &= -\csc^2(x)\end{aligned}$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

Let $f(x) = \dfrac{x\tan(x^2+7)}{15e^x}$. Use the definition of the derivative to find $f'(0)$.

Differentiate $(e^x + \cot x)\left(5x^6 - \csc x\right)$.

Let $h(x) = \begin{cases} \frac{\sin x}{x} & , \quad x < 0 \\ \frac{ax+b}{\cos x} & , \quad x \geq 0 \end{cases}$

Which values of a and b make $h(x)$ continuous at $x = 0$?

Practice and Review

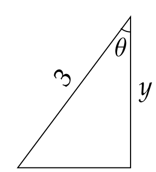
$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases}$$

Is $f(x)$ differentiable at $x = 0$?

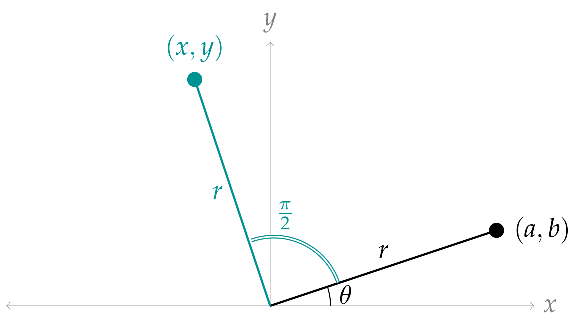
$$g(x) = \begin{cases} e^{\frac{\sin x}{x}} & , \quad x < 0 \\ (x-a)^2 & , \quad x \geq 0 \end{cases}$$

What value(s) of a makes $g(x)$ continuous at $x = 0$?

A ladder 3 meters long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall, measured in radians, and let y be the height of the top of the ladder. If the ladder slides away from the wall, how fast does y change with respect to θ?
When is the top of the ladder sinking the fastest? The slowest?



Suppose a point in the plane that is r centimetres from the origin, at an angle of θ ($0 \le \theta \le \frac{\pi}{2}$), is rotated $\pi/2$ radians. What is its new coordinate (x, y)? If the point rotates at a constant rate of a radians per second, when is the x coordinate changing fastest and slowest with respect to θ?



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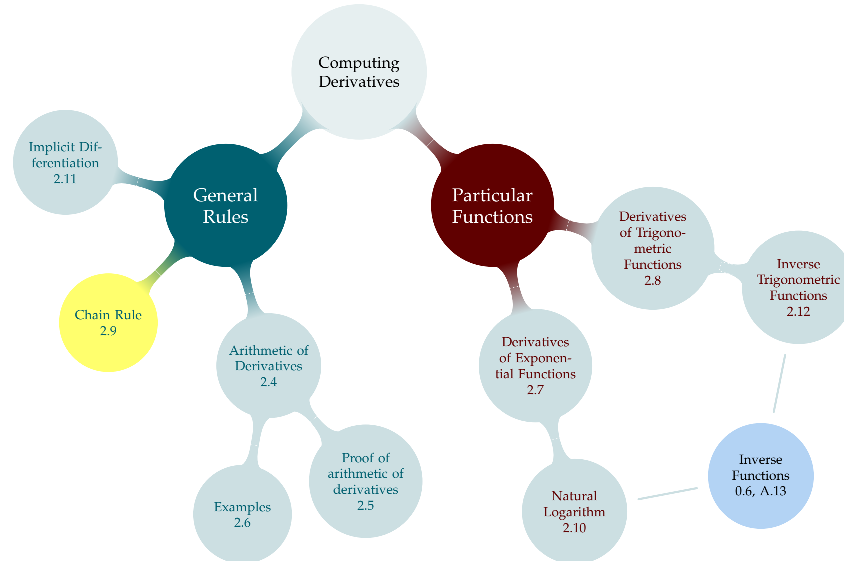
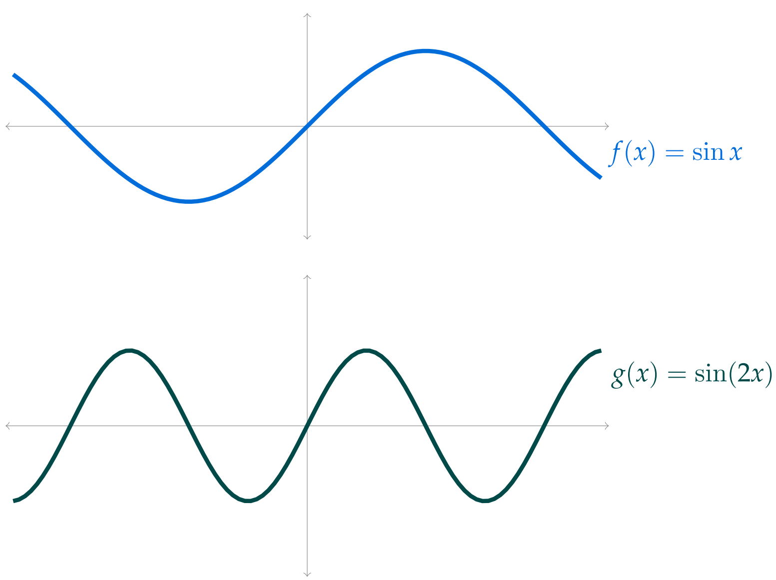


Intuition: $\sin x$ versus $\sin(2x)$



Compound Functions

Video: 2:27-3:50

Morton, Jennifer. (2014). *Balancing Act: Otters, Urchins and Kelp*. Available from `https://www.kqed.org/quest/67124/balancing-act-otters-urchins-and-kelp`

Kelp Population

k kelp population
u urchin population
o otter population
p public policy

$k(u)$ $k(u(o))$ $k(u(o(p)))$

These are examples of compound functions.

Should $\frac{\mathrm{d}}{\mathrm{d}o}k(u(o))$ be positive or negative?
A. positive B. negative C. I'm not sure

Should $k'(u)$ be positive or negative?
A. positive B. negative C. I'm not sure

Differentiating Compound Functions

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\} &= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\left(\frac{g(x+h)-g(x)}{g(x+h)-g(x)}\right)\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f\left(\boxed{g(x+h)}\right)-f\left(\boxed{g(x)}\right)}{\boxed{g(x+h)}-\boxed{g(x)}}\cdot g'(x)
\end{aligned}
$$

Set $H=g(x+h)-g(x)$. As $h\to 0$, we also have $H\to 0$. So

$$
\begin{aligned}
&= \lim_{H\to 0}\frac{f(g(x)+H)-f(g(x))}{H}\cdot g'(x)\\
&= f'(g(x))\cdot g'(x)
\end{aligned}
$$

Chain Rule

Chain Rule – Theorem 2.9.3

Suppose f and g are differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\}=f'(g(x))\,g'(x)=\frac{\mathrm{d}f}{\mathrm{d}g}(g(x))\,\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

In the case of kelp, $\frac{\mathrm{d}}{\mathrm{d}o}k(u(o))=\frac{\mathrm{d}k}{\mathrm{d}u}(u(o))\,\frac{\mathrm{d}u}{\mathrm{d}o}(o)$

Chain Rule

Suppose f and g are differentiable functions. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{f(g(x))\}=f'(g(x))\,g'(x)=\frac{\mathrm{d}f}{\mathrm{d}g}(g(x))\,\frac{\mathrm{d}g}{\mathrm{d}x}(x)$$

Example: suppose $F(x)=\sin(e^x+x^2)$.

$$F(v) = \left(\frac{v}{v^3+1}\right)^6$$

Now You

Let $f(x) = (10^x + \csc x)^{1/2}$. Find $f'(x)$.

Now You

Suppose $o(t) = e^t$, $u(o) = \frac{1}{o+\sin(o)}$, and $t \geq 10$ (so all these functions are defined). Using the chain rule, find $\frac{\mathrm{d}}{\mathrm{d}t} u(o(t))$.
Note: your answer should depend only on t: not o.

Evaluate $\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^2 + \sec\left(x^2 + \frac{1}{x}\right)\right\}$

Evaluate $\dfrac{\mathrm{d}}{\mathrm{d}x}\left\{\dfrac{1}{x+\dfrac{1}{x+\frac{1}{x}}}\right\}$

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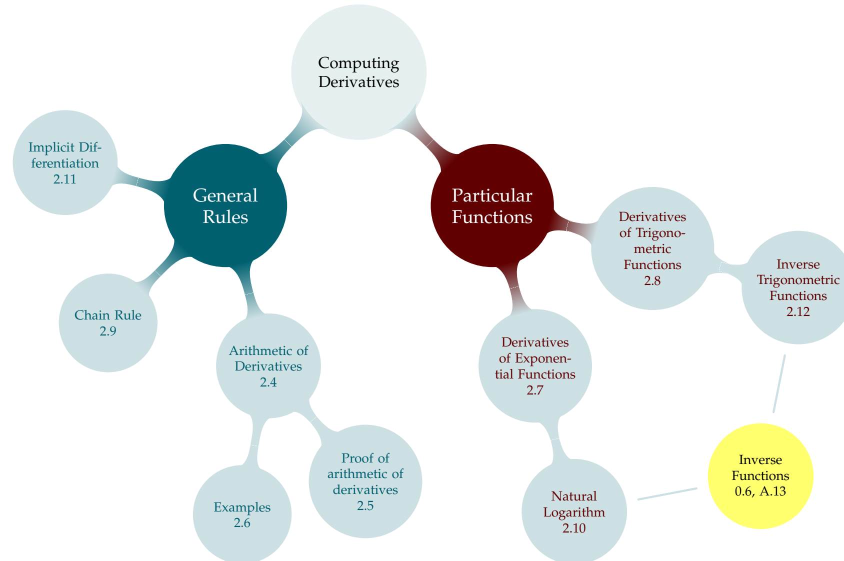


Invertibility Game

- A function $y = f(x)$ is known to both players
- Player A chooses a secret value x in the domain of $f(x)$
- Player A tells Player B what $f(x)$ is
- Player B tries to guess Player A's x-value.

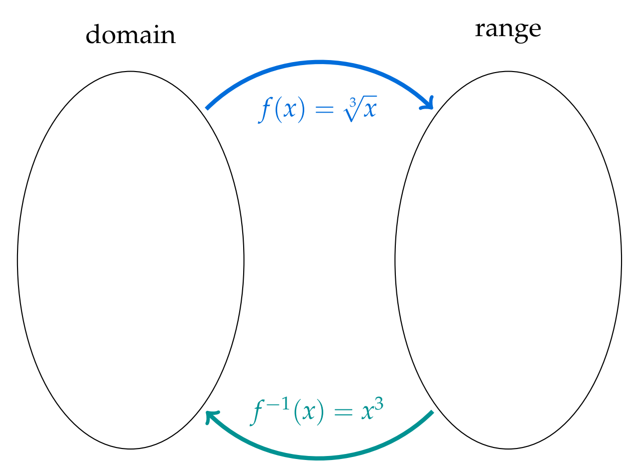
Round 1: $f(x) = 2x$

Round 2: $f(x) = \sqrt[3]{x}$

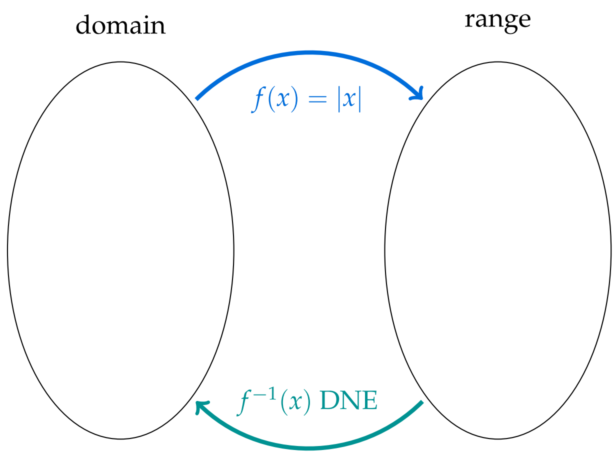
Round 3: $f(x) = |x|$

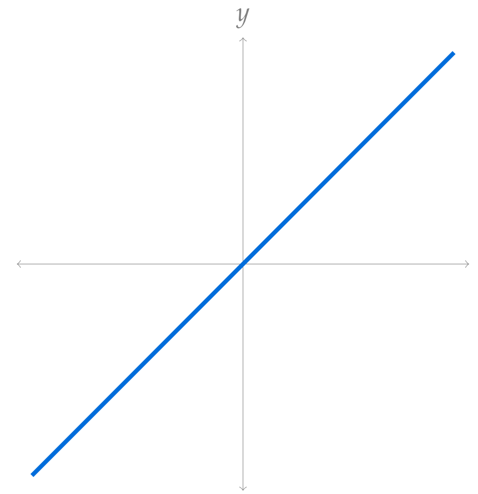
Round 4: $f(x) = \sin x$

Functions are Maps



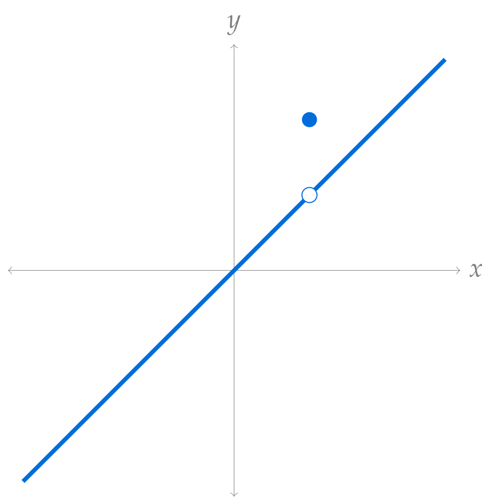
Functions are Maps





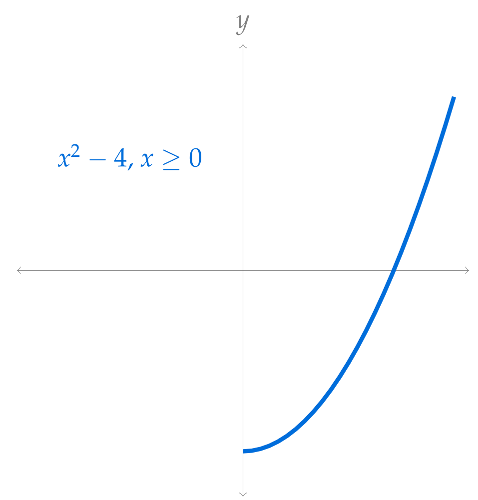
A. invertible

B. not invertible



A. invertible

B. not invertible



A. invertible

B. not invertible

Relationship between $f(x)$ and $f^{-1}(x)$

Let f be an invertible function.
What is $f^{-1}(f(x))$?

A. x
B. 1
C. 0
D. not sure

Invertibility

In order for a function to be invertible , different x values cannot map to the same y value.
We call such a function **one-to-one**, or **injective**.

Suppose $f(x) = \sqrt[3]{19+x^3}$. What is $f^{-1}(3)$? (simplify your answer)

What is $f^{-1}(10)$? (do not simplify)

What is $f^{-1}(x)$?

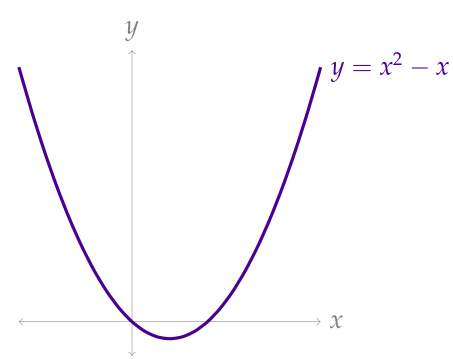
Let $f(x) = x^2 - x$.

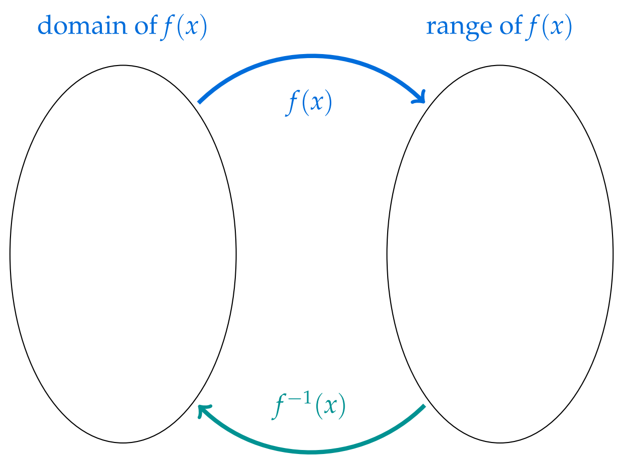
1. Sketch a graph of $f(x)$, and choose a (large) domain over which it is invertible.

2. For the domain you chose, evaluate $f^{-1}(20)$.

3. For the domain you chose, evaluate $f^{-1}(x)$.

4. What are the domain and range of $f^{-1}(x)$? What are the (restricted) domain and range of $f(x)$?





Invertibility Game: $f(x) = e^x$ $\quad f^{-1}(x) = \log_e x$

- I'm thinking of an x. Your clue: $f(x) = e$. What is my x?
- I'm thinking of an x. Your clue: $f(x) = 1$. What is my x?
- I'm thinking of an x. Your clue: $f(x) = \frac{1}{e}$. What is my x?
- I'm thinking of an x. Your clue: $f(x) = e^3$. What is my x?
- I'm thinking of an x. Your clue: $f(x) = 0$. What is my x?

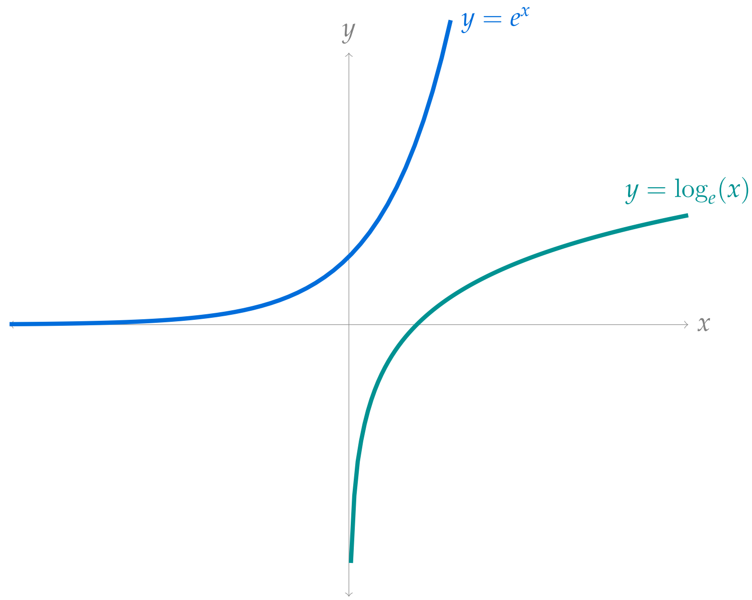
1. Suppose $0 < x < 1$. Then $\log_e(x)$ is...
2. Suppose $-1 < x < 0$. Then $\log_e(x)$ is...
3. Suppose $e < x$. Then $\log_e(x)$ is...

A. positive
B. negative
C. greater than one
D. less than one
E. undefined

Exponents and Logarithms

$f(x) = e^x \qquad f^{-1}(x) = \log_e(x) = \ln(x) = \log(x)$

x	e^x	e fact $\leftrightarrow$ $\log_e$ fact	x	$\log_e(x)$
0	1			
1	e			
-1	$\frac{1}{e}$			
n	e^n			



Logs of Other Bases: $\log_n(x)$ is the inverse of n^x

$\log_{10} 10^8 =$

A. 0
B. 8
C. 10
D. other

$\log_2 16 =$

A. 1
B. 2
C. 3
D. other

Logarithm Rules

Let A and B be positive, and let n be any real number.

$\log(A \cdot B) = \log(A) + \log(B)$

Proof: $\log(A \cdot B) = \log(e^{\log A} e^{\log B}) = \log(e^{\log A + \log B}) = \log(A) + \log(B)$

$\log(A/B) = \log(A) - \log(B)$

Proof: $\log(A/B) = \log\left(\frac{e^{\log A}}{e^{\log B}}\right) = \log(e^{\log A - \log B}) = \log A - \log B$

$\log(A^n) = n\log(A)$

Proof: $\log(A^n) = \log\left(\left(e^{\log A}\right)^n\right) = \log\left(e^{n \log A}\right) = n \log A$

Logarithm Rules

Let A and B be positive, and let n be any real number.

$\log(A \cdot B) = \log(A) + \log(B)$

$\log(A/B) = \log(A) - \log(B)$

$\log(A^n) = n\log(A)$

Write as a single logarithm:

$$f(x) = \log\left(\frac{10}{x^2}\right) + 2\log x + \log(10 + x)$$

Base Change

Fact: $b^{\log_b(a)} = a$

$$\Rightarrow \log\left(b^{\log_b(a)}\right) = \log(a)$$
$$\Rightarrow \log_b(a)\log(b) = \log(a)$$
$$\Rightarrow \log_b(a) = \frac{\log(a)}{\log(b)}$$

In general, for positive a, b, and c:

$$\boxed{\log_b(a) = \frac{\log_c(a)}{\log_c(b)}}$$

In general, for positive a, b, and c:

$$\boxed{\log_b(a) = \frac{\log_c(a)}{\log_c(b)}}$$

Suppose your calculator can only compute logarithms base 10. What would you enter to calculate $\log(17)$?

Suppose your calculator can only compute natural logarithms. What would you enter to calculate $\log_2(57)$?

Suppose your calculator can only compute logarithms base 2. What would you enter to calculate $\log(2)$?

Decibels: For a particular measure of the power P of a sound wave, the decibels of that sound is:

$$10\log_{10}(P)$$

So, every ten decibels corresponds to a sound being ten **times** louder.

A lawnmower emits a 100dB sound. How much sound will two lawnmowers make?

A. 100 dB
B. 110 dB
C. 200 dB
D. other

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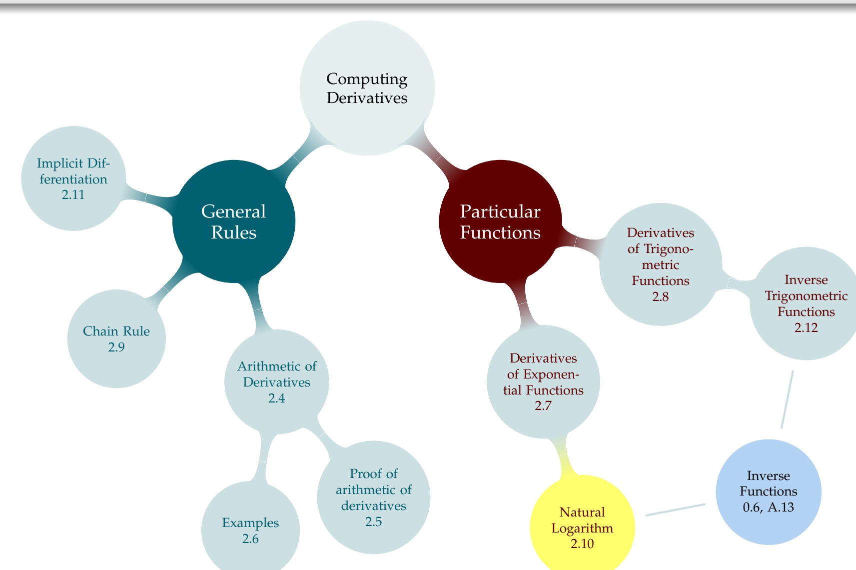

Differentiating the Natural Logarithm

Calculate $\frac{\mathrm{d}}{\mathrm{d}x}\{\log_e x\}$.
One Weird Trick:

$$\begin{aligned} x &= e^{\log_e x} \\ \frac{\mathrm{d}}{\mathrm{d}x}\{x\} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{e^{\log_e x}\right\} \\ 1 &= e^{\log_e x} \cdot \frac{\mathrm{d}}{\mathrm{d}x}\{\log_e x\} = x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\{\log_e x\} \\ \frac{1}{x} &= \frac{\mathrm{d}}{\mathrm{d}x}\{\log_e x\} \end{aligned}$$

Derivative of Natural Logarithm

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\log_e |x|\} = \frac{1}{x} \qquad (x \neq 0)$$

Differentiate: $f(x) = \log_e |x^2 + 1|$

Derivatives of Logarithms – Corollary 2.10.6

For $a > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log_a |x|] = \frac{1}{x \log a}$$

In particular:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\log |x|] = \frac{1}{x}$$

Differentiate: $f(x) = \log_e |\cot x|$

Logarithmic Differentiation - A Fancy Trick

- $\log(f \cdot g) = \log f + \log g$

  multiplication turns into addition

- $\log\left(\frac{f}{g}\right) = \log f - \log g$

  division turns into subtraction

- $\log\left(f^g\right) = g \log f$

  exponentiation turns into multiplication

  We can exploit these properties to differentiate!

Logarithmic Differentiation

In general, if $f(x) \neq 0$, $\frac{d}{dx}[\log|f(x)|] = \frac{f'(x)}{f(x)}$.

$$f(x) = \left(\frac{(2x+5)^4(x^2+1)}{x+3}\right)^5$$

Find $f'(x)$.

Logarithmic Differentiation - A Fancy Trick

$$f(x) = \left(\frac{(2x+5)^4(x^2+1)}{x+3}\right)^5$$

Logarithmic Differentiation - A Fancy Trick

Differentiate:

$$f(x) = x^x$$

Logarithmic Differentiation - A Fancy Trick

Differentiate:

$$f(x) = \left(\frac{(x^{15}-9x^2)^{10}(x+x^2+1)}{(x^7+7)(x+1)(x+2)(x+3)}\right)^5$$

$$f(x) = \frac{(x^8 - e^x)(\sqrt{x} + 5)}{\csc^5 x}$$

$$f(x) = (x^2 + 17)(32x^5 - 8)(x^{98} - x^{57} + 32x^2)^4(32x^{10} - 10x^{32})$$

Find $f'(x)$.

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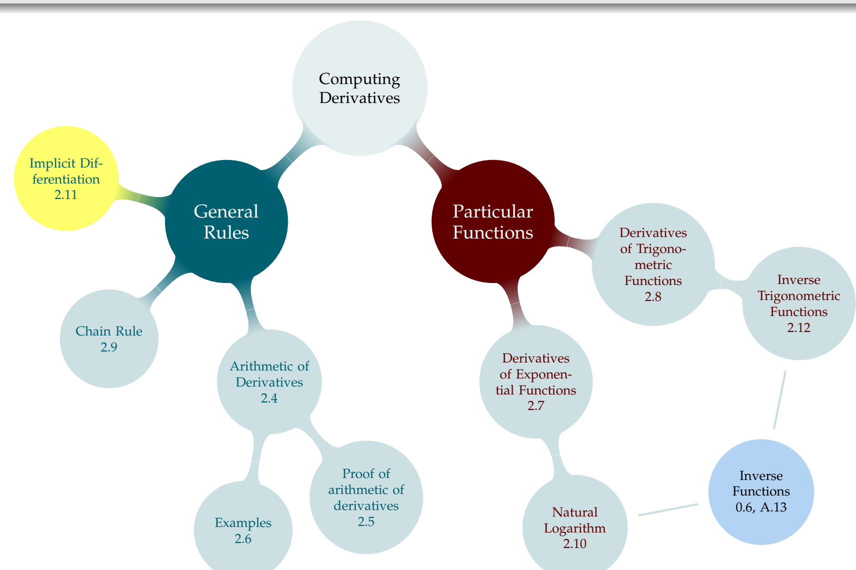


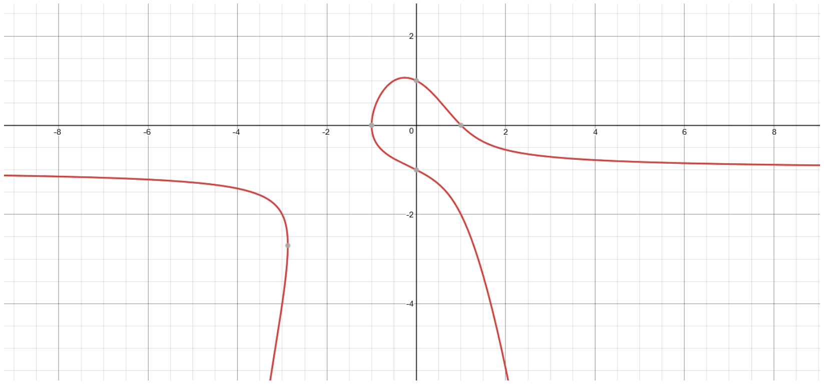
Implicitly Defined Functions

$$y^2 + x^2 + xy + x^2y = 1$$

Which of the following points are on the curve?
$(0,1)$, $(0,-1)$, $(0,0)$, $(1,1)$

If $x = -3$, what is y?

$$y^2 + x^2 + xy + x^2y = 1$$



Still has a slope: $\frac{\Delta y}{\Delta x}$
Locally, y is still a function of x.

$$y^2 + x^2 + xy + x^2y = 1$$

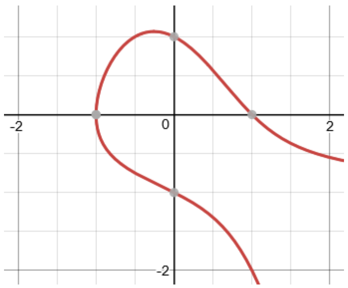
Consider y as a function of x. Can we find $\frac{dy}{dx}$?

$\frac{d}{dx}[y] =$ $\qquad$ $\frac{d}{dx}[x] =$ $\qquad$ $\frac{d}{dx}[1] =$

$$y^2 + x^2 + xy + x^2y = 1$$

$$\frac{dy}{dx} = -\frac{2x + y + 2xy}{2y + x + x^2}$$

Necessarily, $\frac{dy}{dx}$ depends on **both** y and x. Why?



Now You

Suppose $x^4y + y^4x = 2$. Find $\frac{dy}{dx}$ at the point $(1, 1)$.

Now You Suppose $\dfrac{3y^2+2y+y^3}{x^2+1} = x$. Find $\dfrac{dy}{dx}$ when $x=0$, and the equations of the associated tangent line(s).

Use implicit differentiation to differentiate $\log(x)$, $x>0$.

$$\begin{aligned}\log x &= y(x)\\ x &= e^{y(x)}\end{aligned}$$

Use implicit differentiation to differentiate $\log|x|$, $x<0$.

Use implicit differentiation to differentiate $\log_a(x)$, where $a>0$ is a constant and $x>0$.

Use implicit differentiation to differentiate $\log_a|x|$, $a>0$.

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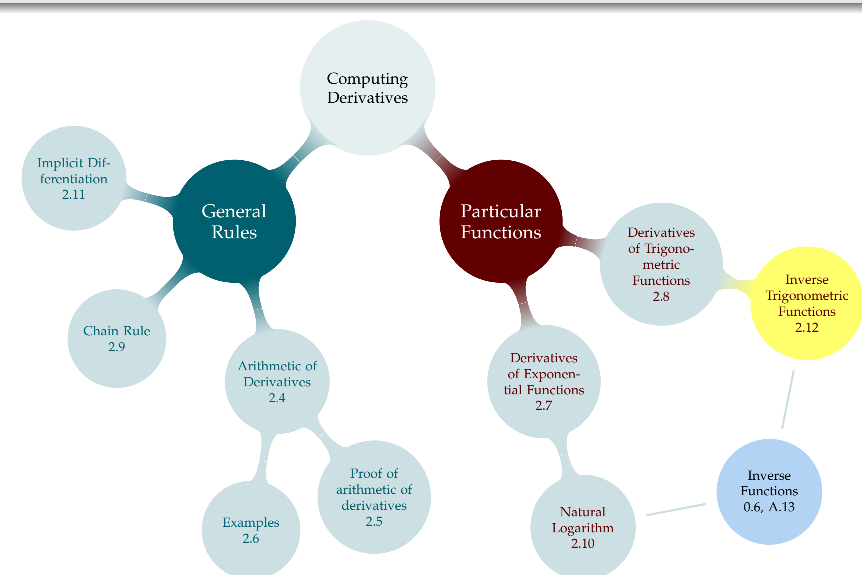
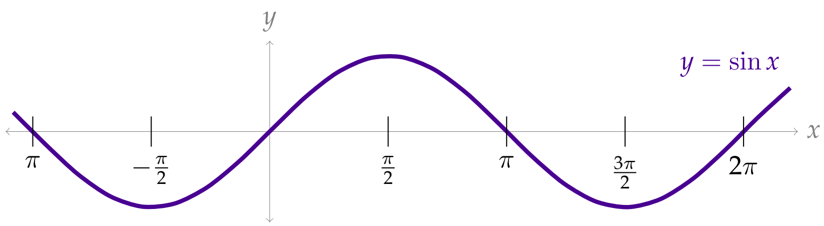


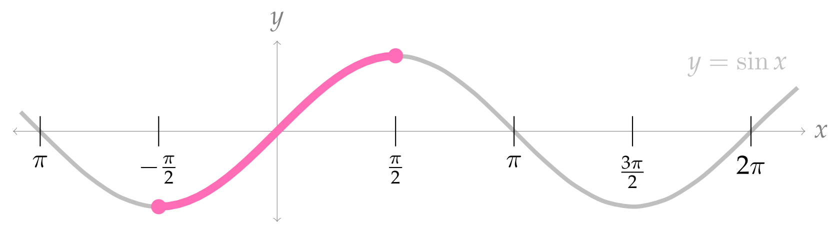
Invertibility Game



I'm thinking of a number x. Your hint: $\sin(x) = 0$. What number am I thinking of?

I'm thinking of a number x, and x is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Your hint: $\sin(x) = 0$. What number am I thinking of?

Arcsine



$\arcsin(x)$ is the inverse of $\sin x$ restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\arcsin x$ is the (unique) number θ such that:

- $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, and
- $\sin\theta = x$

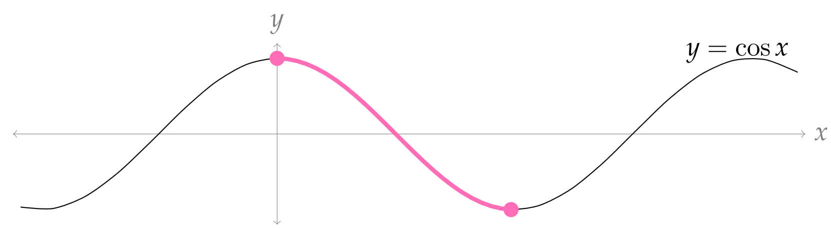
Arcsine

Reference Angles:

θ	$\sin\theta$
0	0
$\frac{\pi}{6}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

- $\arcsin(0)$
- $\arcsin\left(\frac{1}{\sqrt{2}}\right)$
- $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$
- $\arcsin\left(\frac{\pi}{2}\right)$
- $\arcsin\left(\frac{\pi}{4}\right)$

Arccosine

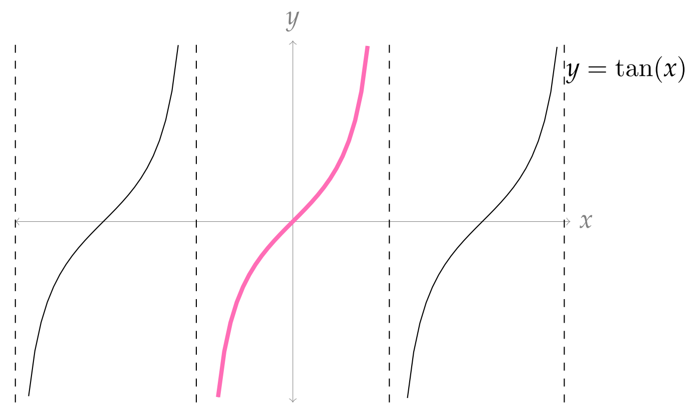


$\arccos(x)$ is the inverse of $\cos x$ restricted to $[0, \pi]$.

$\arccos(x)$ is the (unique) number θ such that:

- $\cos(\theta) = x$ and
- $0 \leq \theta \leq \pi$

Arctangent



$\arctan(x) = \theta$ means:
(1) $\tan(\theta) = x$ and
(2) $-\pi/2 < \theta < \pi/2$

Arcsecant, arcsine, and arccotangent

$\text{arcsec}(x) =$

Arcsecant, arcsine, and arccotangent

$\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$

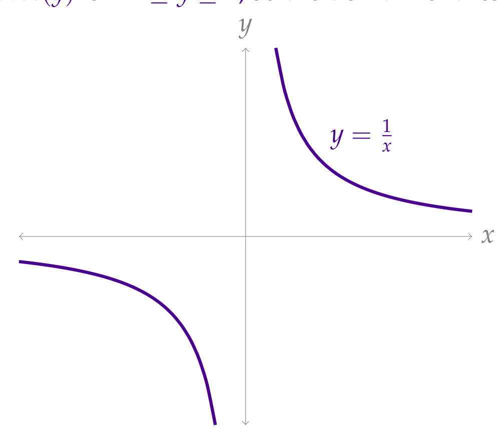
$$\begin{aligned} \text{arccsc}(x) &= y \\ \csc y &= x \\ \frac{1}{\sin y} &= x \\ \sin y &= \frac{1}{x} \\ y &= \arcsin\left(\frac{1}{x}\right) \\ \text{arccsc}(x) &= \arcsin\left(\frac{1}{x}\right) \end{aligned}$$

$\text{arccot(x)} = \arctan\left(\frac{1}{x}\right)$

$$\begin{aligned} \text{arccot(x)} &= y \\ \cot y &= x \\ \frac{1}{\tan y} &= x \\ \tan y &= \frac{1}{x} \\ y &= \arctan\left(\frac{1}{x}\right) \\ \text{arccot(x)} &= \arctan\left(\frac{1}{x}\right) \end{aligned}$$

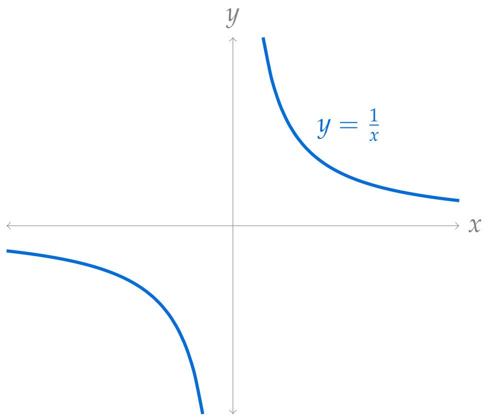
$\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$

The domain of $\arccos(y)$ is $-1 \le y \le 1$, so the domain of $\text{arcsec}(y)$ is



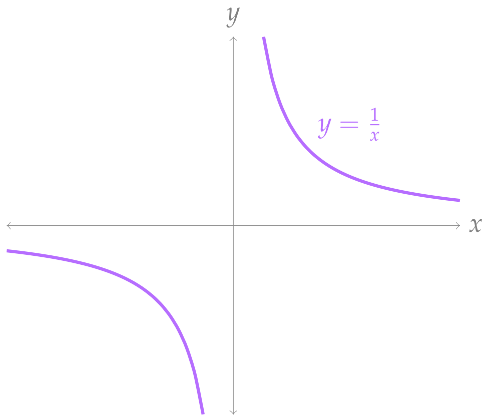
$$\operatorname{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$$

Domain of $\arcsin(y)$ is $-1 \leq y \leq 1$, so the domain of $\operatorname{arccsc}(x)$ is



$$\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$$

Domain of $\arctan(x)$ is all real numbers, so the domain of $\operatorname{arccot}(x)$ is



$$y = \arcsin x$$

Find $\frac{dy}{dx}$.

$$y = \arctan x$$

Find $\frac{dy}{dx}$.

$$y = \arccos x$$

Find $\frac{dy}{dx}$.

To differentiate arcsecant, arccosecant, and arccotangent, you can use the chain rule!

$$\frac{d}{dx}\left[\operatorname{arccsc}(x)\right] = \frac{d}{dx}\left[\arcsin\left(\frac{1}{x}\right)\right] = \frac{d}{dx}\left[\arcsin\left(x^{-1}\right)\right]$$

Derivatives of Inverse Trigonometric Functions – Theorem 2.12.7

Memorize:

$$\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arccos x] = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}[\arcsin x] = \frac{1}{1+x^2}$$

Be able to derive:

$$\frac{d}{dx}[\operatorname{arccsc} x] = -\frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\operatorname{arcsec} x] = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}[\operatorname{arccot} x] = -\frac{1}{1+x^2}$$

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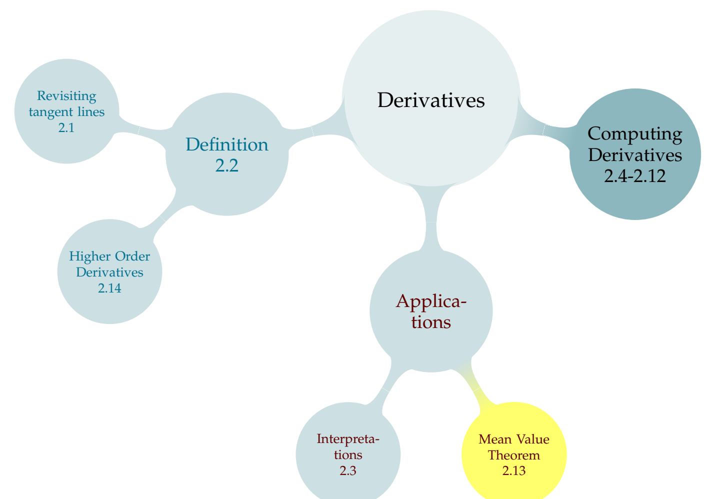
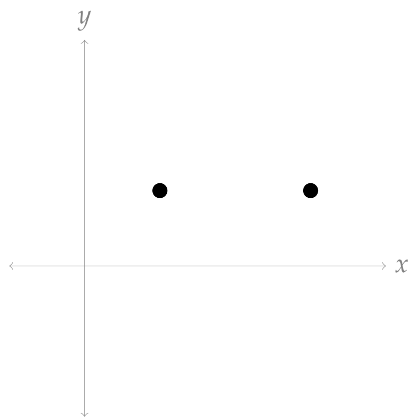


Rolle's Theorem



Rolle's Theorem – Theorem 2.13.1

Let a and b be real numbers, with $a < b$. And let f be a function with the properties:

- $f(x)$ is continuous for every x with $a \leq x \leq b$;
- $f(x)$ is differentiable when $a < x < b$;
- and $f(a) = f(b)$.

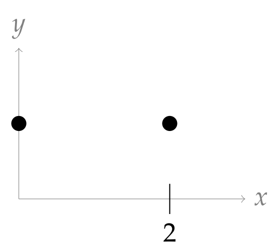
Then there exists a number c with $a < c < b$ such that

$$f'(c) = 0.$$

Rolle's Theorem – Theorem 2.13.1

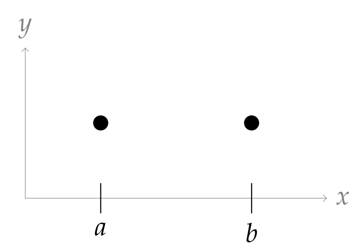
Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.

Example: Let $f(x) = x^3 - 2x^2 + 1$, and observe $f(2) = f(0) = 1$. Since $f(x)$ is a polynomial, it is continuous and differentiable everywhere.



Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.



Suppose $a < b$ and $f(a) = f(b)$, $f(x)$ is continuous over $[a, b]$, and $f(x)$ is differentiable over (a, b).

How many different values of x between a and b have $f'(x) = 0$?

A. 0 or 1
B. 1
C. 0, 1, or more
D. 1 or more
E. I'm not sure

Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f(x)$ has precisely seven roots, all different. How many roots does $f'(x)$ have?

A. precisely six
B. precisely seven
C. at most seven
D. at least six

Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f'(x)$ is also continuous and differentiable for all real numbers, and $f(x)$ has precisely seven roots, all different. How many roots does $f''(x)$ have?

A. precisely six
B. precisely five
C. at most five
D. at least five

Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and there are precisely three places where $f'(x) = 0$. How many distinct roots does $f(x)$ have?

A. at most three
B. at most four
C. at least three
D. at least four

Rolle's Theorem – Theorem 2.13.1

Let $f(x)$ be continuous on the interval $[a, b]$, differentiable on (a, b), and let $f(a) = f(b)$. Then there is a number c strictly between a and b such that $f'(c) = 0$.

Suppose $f(x)$ is continuous and differentiable for all real numbers, and $f'(x) = 0$ for precisely three values of x. How many distinct values x exist with $f(x) = 17$?

A. at most three
B. at most four
C. at least three
D. at least four

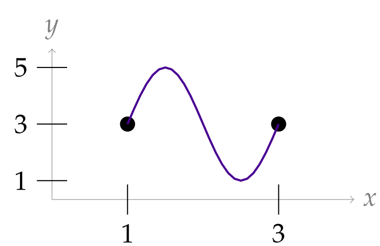
Applications of Rolle's Theorem

Prove that the function $f(x) = x^3 + x - 1$ has at most one real root.

How would you show that $f(x)$ has precisely one real root?

Use Rolle's Theorem to show that the function $f(x) = \frac{1}{3}x^3 + 3x^2 + 9x - 3$ has at most two distinct real roots.

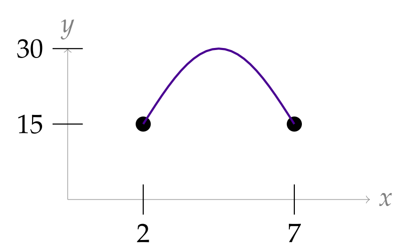
Average Rate of Change



What is the average rate of change of $f(x)$ from $x = 1$ to $x = 3$?

A. 0
B. 1
C. 2
D. 4
E. I'm not sure

Average Rate of Change



What is the average rate of change of $f(x)$ from $x = 2$ to $x = 7$?

A. 0
B. 3
C. 5
D. 15
E. I'm not sure

Rolle's Theorem and Average Rate of Change

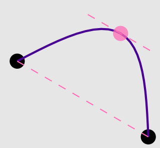
Suppose $f(x)$ is continuous on the interval $[a, b]$, differentiable on the interval (a, b), and $f(a) = f(b)$. Then there exists a number c strictly between a and b such that

$$f'(c) = 0 = \frac{f(b) - f(a)}{b - a}.$$

So there exists a point where the derivative is the same as the average rate of change.

Mean Value Theorem – Theorem 2.13.4

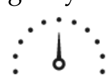
Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on (a, b). Then there is a number c strictly between a and b such that:



$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

That is: there is some point c in (a, b) where the instantaneous rate of change of the function is equal to the average rate of change of the function on the interval $[a, b]$.

Suppose you are driving along a long, straight highway with no shortcuts. The speed limit is 100 kph. A police officer notices your car going 90 kph, and uploads your plate and the time they saw you to their database. 150 km down this same straight road, 75 minutes later, another police officer notices your car going 85kph, and uploads your plates to the database. Then they pull you over, and give you a speeding ticket. Why were they justified?



According to this website, Canada geese may fly 1500 miles in a single day under favorable conditions. It also says their top speed is around 70mph. Does this seem like a typo? (If it contradicts the Mean Value Theorem, it's probably a typo.)

The record for fastest wheel-driven land speed is around 700 kph. [2] However, non-wheel driven cars (such as those powered by jet engines) have achieved higher speeds. [3]
Suppose a driver of a jet-powered car starts a 10km race at 12:00, and finishes at 12:01. Did they beat 700kph?

[2] (at time of writing) George Poteet, https://en.wikipedia.org/wiki/Wheel-driven_land_speed_record

[3] https://en.wikipedia.org/wiki/Land_speed_record

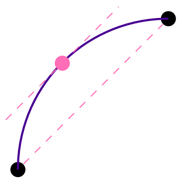
Suppose you want to download a file that is 3000 MB (slightly under 3GB). Your internet provider guarantees you that your download speeds will always be between 1 MBPS (MB per second) and 5 MBPS (because you bought the cheap plan). Using the Mean Value Theorem, give an upper and lower bound for how long the download can take (assuming your providers aren't lying, and your device is performing adequately).

Suppose $1 \leq f'(t) \leq 5$ for all values of t, and $f(0) = 0$. What are the possible solutions to $f(t) = 3000$?
Notice: since the derivative exists for all real numbers, $f(x)$ is differentiable and continuous for all real numbers!

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) = 0$ for all x in (a, b), then

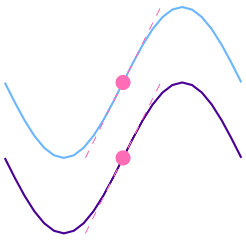


If $f(c) \neq f(d)$, then $\frac{f(d)-f(c)}{d-c} \neq 0$, so $f'(e) \neq 0$ for some e.

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) = g'(x)$ for all x in (a, b), then

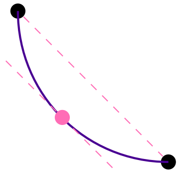


Define a new function $k(x) = f(x) - g(x)$. Then $k'(x) = 0$ everywhere, so (by the last corollary) $k(x) = A$ for some constant A.

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) > 0$ for all x in (a, b), then

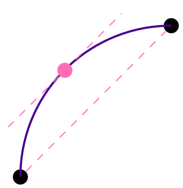


If $f(c) > f(d)$ and $c < d$, then $\frac{f(d)-f(c)}{d-c} = \frac{\text{(negative)}}{\text{(positive)}} < 0$. Then $f'(e) < 0$ for some e between c and d.

Corollary to the MVT

Let $a < b$ be numbers in the domain of $f(x)$ and $g(x)$, which are continuous over $[a, b]$ and differentiable over (a, b).

If $f'(x) < 0$ for all x in (a, b), then



If $f(c) < f(d)$ and $c < d$, then $\frac{f(d)-f(c)}{d-c} = \frac{\text{(positive)}}{\text{(positive)}} > 0$. Then $f'(e) > 0$ for some e between c and d.

Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on (a, b). Then there is a number c strictly between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

WARNING: The MVT has two hypotheses.

- $f(x)$ has to be continuous on $[a, b]$.
- $f(x)$ has to be differentiable on (a, b).

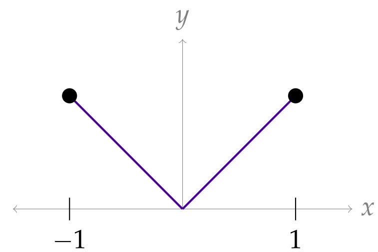
If either of these hypotheses are violated, the conclusion of the MVT can fail. Here are two examples.

Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on (a, b). Then there is a number c strictly between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example: Let $a = -1$, $b = 1$ and $f(x) = |x|$.

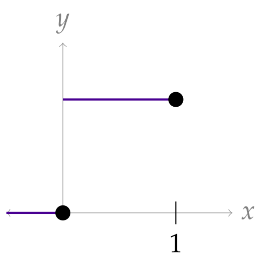


Mean Value Theorem – Theorem 2.13.4

Let $f(x)$ be continuous on the interval $[a, b]$ and differentiable on (a, b). Then there is a number c strictly between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Example: Let $a = 0$, $b = 1$ and $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0 \end{cases}$.



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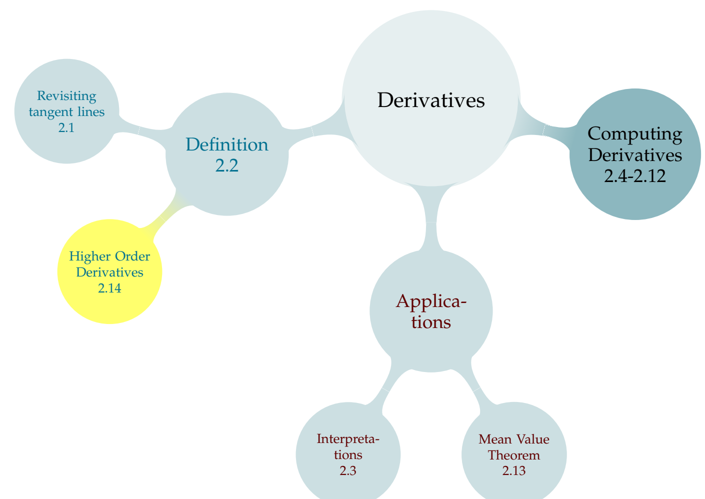

Higher Order Derivatives

Evaluate $\frac{d}{dx}\left[\frac{d}{dx}[x^5 - 2x^2 + 3]\right]$

$$\frac{d}{dx}[x^5 - 2x^2 + 3] =$$

Notation 2.14.1

The derivative of a derivative is called the **second derivative**, written

$$f''(x) \qquad \text{or} \qquad \frac{d^2y}{dx^2}(x)$$

Similarly, the derivative of a second derivative is a third derivative, etc.

Notation 2.14.1

- $f''(x)$ and $f^{(2)}(x)$ and $\frac{d^2f}{dx^2}(x)$ all mean $\frac{d}{dx}\left(\frac{d}{dx}f(x)\right)$
- $f'''(x)$ and $f^{(3)}(x)$ and $\frac{d^3f}{dx^3}(x)$ all mean $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}f(x)\right)\right)$
- $f^{(4)}(x)$ and $\frac{d^4f}{dx^4}(x)$ both mean $\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}\left(\frac{d}{dx}f(x)\right)\right)\right)$
- and so on.

Typical Example: Acceleration

- Velocity: rate of change of position
- Acceleration: rate of change of velocity.

The position of an object at time t is given by $s(t) = t(5-t)$. *Time is measured in seconds, and position is measured in metres.*

1. Sketch the graph giving the position of the object.
2. What is the velocity of the object when $t = 1$? Include units.
3. What is the acceleration of the object when $t = 1$? Include units.

Concept Check

True or False: If $f'(1) = 18$, then $f''(1) = 0$, since the $\frac{d}{dx}\{18\} = 0$.

Which of the following is always true of a QUADRATIC polynomial $f(x)$?

A. $f(0) = 0$
B. $f'(0) = 0$
C. $f''(0) = 0$
D. $f'''(0) = 0$
E. $f^{(4)}(0) = 0$

Which of the following is always true of a CUBIC polynomial $f(x)$?

A. $f(0) = 0$
B. $f'(0) = 0$
C. $f''(0) = 0$
D. $f'''(0) = 0$
E. $f^{(4)}(0) = 0$

Implicit Differentiation

Suppose $y(x)$ is a function such that

$$y(x) = y^3x + x^2 - 1$$

Find $y''(x)$ at the point $(-2, 1)$.

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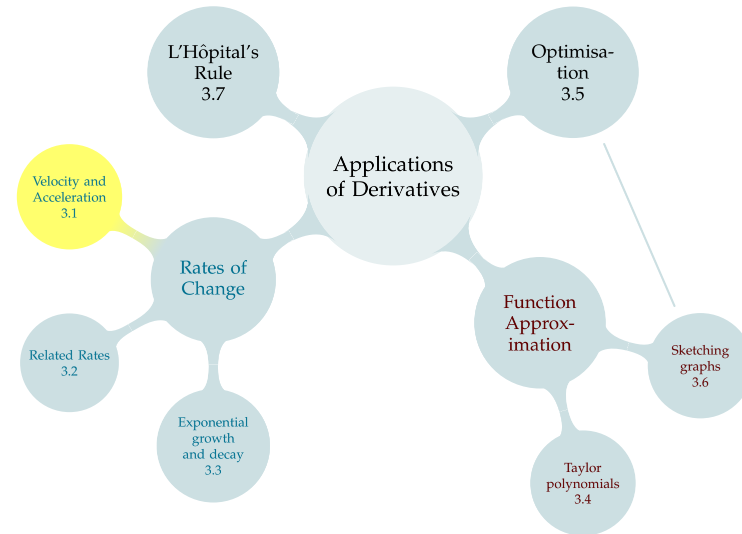

The position of a unicyclist along a tightrope is given by

$$s(t) = t^3 - 3t^2 - 9t + 10$$

where $s(t)$ gives the distance in meters to the right of the middle of the tightrope, and t is measured in seconds, $-2 \leq t \leq 4$.

Describe the unicyclist's motion: when they are moving right or left; when they are moving fastest and slowest; and how far to the right or left of centre they travel.

A solution in a beaker is undergoing a chemical reaction, and its temperature (in degrees Celsius) at t seconds from noon is given by

$$T(t) = t^3 + 3t^2 + 4t - 273$$

1. When is the reaction increasing the temperature, and when is it decreasing the temperature?
2. What is the slowest rate of change of the temperature?

You roll a magnetic marble across the floor towards a metal fridge, giving it an initial velocity of 50 centimetres per second. The magnet imparts an acceleration on the magnet of 1 meter per second per second. If the magnet hits the fridge after 2 seconds, how far away was it when you rolled it?

The deceleration of a particular car while braking is 9 m/s^2.
1. Suppose the car needs to stop in 30m. How fast can it be going? (Give your answer in kph.)

2. Suppose the car needs to stop in 50m. How fast can it be going? (Give your answer in kph.)

Suppose your brakes decelerate your car at a constant rate. That is, d meters per second per second, for some constant d.
Is it true that if you double your speed, you double your stopping time?

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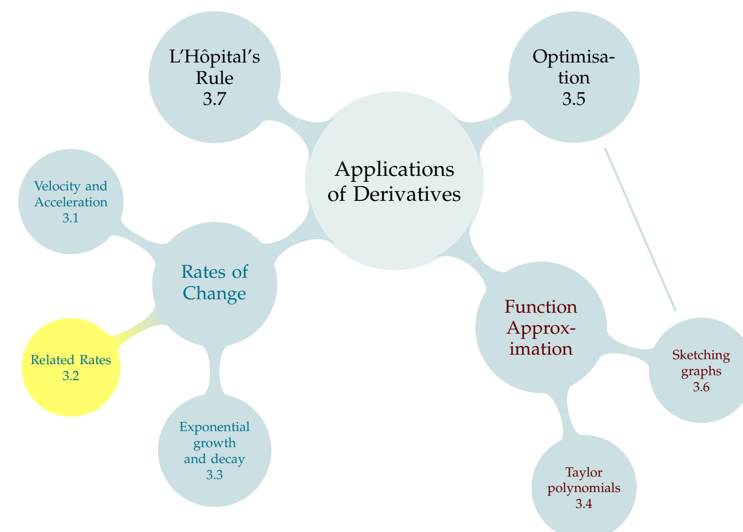


Related Rates - Introduction

"Related rates" problems involve finding the rate of change of one quantity, based on the rate of change of a related quantity.

Suppose P and Q are quantities that are changing over time, t. Suppose they are related by the equation

$$3P^2 = 2Q^2 + Q + 3.$$

If $\frac{dP}{dt}(t) = 5$ when $P(t) = 1$ and $Q(t) = 0$, then what is $\frac{dQ}{dt}$ at that time?

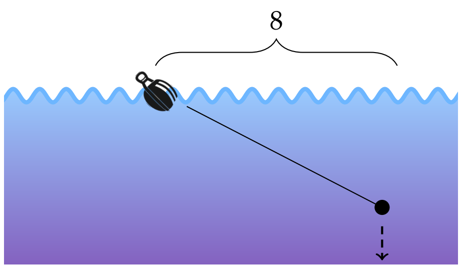
Related rates problems often involve some kind of geometric or trigonometric modeling

A garden hose can pump out a cubic meter of water in about 20 minutes. Suppose you're filling up a rectangular backyard pool, 3 meters wide and 6 meters long, with a garden hose. How fast is the water rising?

Solving Related Rates

1. Draw a Picture
2. Write what you know, and what you want to know. Note units.
3. Relate all your relevant variables in one equation.
4. Differentiate both sides (with respect to the appropriate variable!)
5. Solve for what you want.

A weight is attached to a rope, which is attached to a pulley on a boat, at water level. The weight is taken 8 (horizontal) metres from its attachment point on the boat, then dropped in the water.
The weight sinks straight down. The rope stays taught as it is let out at a constant rate of one metre per second, and two seconds have passed. How fast is the weight descending?

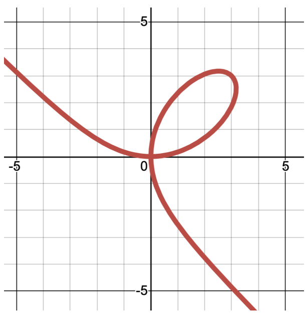


You are pouring water through a funnel with an extremely small hole. The funnel lets water out at 100mL per second, and you are pouring water into the funnel at 300mL per second. The funnel is shaped like a cone with height 20 cm and with the diameter at the top also 20 cm. (Ignore the hole in the bottom.) How fast is the height of the water in the funnel rising when it is 10 cm high?

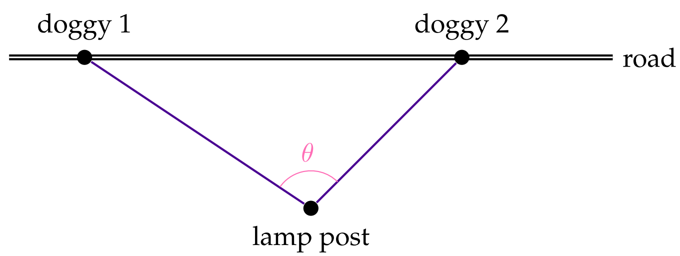
A cone with radius r and height h has volume $\frac{\pi}{3}r^2h$.

A sprinkler is 3m from a long, straight wall. The sprinkler sprays water in a circle, making three revolutions per minute. Let P be the point on the wall closest to the sprinkler. The water hits the wall at some spot, and that spot moves as the sprinkler rotates. When the spot where the water hits the wall is 1m away from P, how fast is the spot moving horizontally?
(You may assume the water travels from the sprinkler to the wall instantaneously.)

A roller coaster has a track shaped in part like the folium of Descartes: $x^3 + y^3 = 6xy$. When it is at the position $(3, 3)$, its horizontal position is changing at 2 units per second in the negative direction. How fast is its vertical position changing?



Two dogs are tied with elastic leashes to a lamp post that is 2 metres from a straight road. At first, both dogs are on the road, at the closest part of the road to the lamp post. Then, they start running in opposite directions: one dog runs 3 metres per second, and the other runs 2 metres per second. After one second of running, how fast is the angle made by the two leashes increasing?



A crow is one kilometre due east of the math building, heading east at 5 kph. An eagle is two kilometres due north of the math building, heading north at 7kph. How fast is the distance between the two birds increasing at this instant?

A triangle has one side that is 1cm long, and another side that is 2cm, and the third side is formed by an elastic band that can shrink and stretch. The two fixed sides are rotated so that the angle they form, θ, grows by 1.5 radians each second. Find the rate of change of the area inside the triangle when $\theta = \pi/4$.

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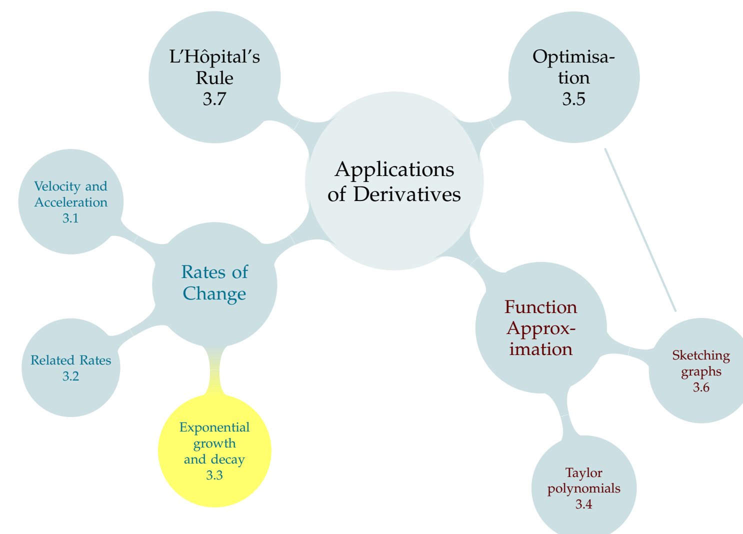


Radioactive Decay

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.

Differential Equation

Let $Q = Q(t)$ be the amount of a radioactive substance at time t. Then for some positive constant k:

$$\frac{dQ}{dt} = -kQ$$

Solution – Theorem 3.3.2

Let $\boxed{Q(t) = Ce^{-kt}}$, where k and C are constants. Then:

Radioactive Decay

Quantity of a Radioactive Isotope

$$Q(t) = Ce^{-kt}$$

$Q(t)$: quantity at time t

What is the sign of $Q(t)$?

A. positive or zero
B. negative or zero
C. could be either
D. I don't know

What is the sign of C?

A. positive or zero
B. negative or zero
C. could be either
D. I don't know

Seaborgium Decay

The amount of ^{266}Sg (Seaborgium-266) in a sample at time t (measured in seconds) is given by

$$Q(t) = Ce^{-kt}$$

Let's approximate the half life of ^{266}Sg as 30 seconds. That is, every 30 seconds, the size of the sample halves.

What are C and k?

A sample of radioactive matter is stored in a lab in 2000. In the year 2002, it is tested and found to contain 10 units of a particular radioactive isotope. In the year 2005, it is tested and found to contain only 2 units of that same isotope. How many units of the isotope were present in the year 2000?

$Q'(t) = kQ(t)$

The number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample.

The rate of growth of a population in a given time interval is propotional to the number of individuals in the population, when the population has ample resources.

The amount of interest a bank account accrues in a given time interval is proportional to the balance in that bank account.

Exponential Growth – Theorem 3.3.2

Let $Q = Q(t)$ satisfy:

$$\frac{dQ}{dt} = kQ$$

for some constant k. Then for some constant $C = Q(0)$,

$$Q(t) = Ce^{kt}$$

Suppose $y(t)$ is a function with the properties that

$$\frac{dy}{dt} + 3y = 0 \quad \text{and} \quad y(1) = 2.$$

What is $y(t)$?

Population Growth

Suppose a petri dish starts with a culture of 100 bacteria cells and a limited amount of food and space. The population of the culture at different times is given in the table below. At approximately what time did the culture start to show signs of limited resources?

time	population
0	100
1	1000
3	100000
5	1000000

Flu Season

The CDC keeps records (link) on the number of flu cases in the US by week. At the start of the flu season, the 40th week of 2014, there are 100 cases of a particular strain. Five weeks later (at week 45), there are 506 cases. What do you think was the first week to have 5,000 cases? What about 10,000 cases?

Newton's Law of Cooling – Equation 3.3.7

The rate of change of temperature of an object is proportional to the difference in temperature between that object and its surroundings.

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

where $T(t)$ is the temperature of the object at time t, A is the (constant) ambient temperature of the surroundings, and K is some constant depending on the object.

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

$T(t)$ is the temperature of the object, A is the ambient temperature, K is some constant.

What is true of K?

A. $K \geq 0$
B. $K \leq 0$
C. $K = 0$
D. K could be positive, negative, or zero, depending on the object
E. I don't know

Newton's Law of Cooling – Equation 3.3.7

$$\frac{dT}{dt}(t) = K[T(t) - A]$$

$T(t)$ is the temperature of the object, A is the ambient temperature, and K is some constant.

$$T(t) = [T(0) - A]e^{Kt} + A$$

is the only function satisfying Newton's Law of Cooling

If $T(10) < A$, then:

A. $K > 0$
B. $T(0) > 0$
C. $T(0) > A$
D. $T(0) < A$

Evaluate $\lim_{t\to\infty} T(t)$.

A. A
B. 0
C. ∞
D. $T(0)$

What assumptions are we making that might not square with the real world?

Newton's Law of Cooling – Equation 3.3.7

$$\frac{dT}{dt} = K[T(t) - A]$$

$T(t)$ is the temperature of the object, A is the ambient temperature, and K is some constant.

Temperature of a Cooling Body – Corollary 3.3.8

$$T(t) = [T(0) - A]e^{Kt} + A$$

A farrier forms a horseshoe heated to 400° C, then dunks it in a river at room-temperature (25° C). The water boils for 30 seconds. The horseshoe is safe for the horse when it's 40° C. When can the farrier put on the horseshoe?

$$T(t) = [T(0) - A]e^{Kt} + A$$

A glass of just-boiled tea is put on a porch outside. After ten minutes, the tea is 40°, and after 20 minutes, the tea is 25°. What is the temperature outside?

In 1963, the US Fish and Wildlife Service recorded a bald eagle population of 487 breeding pairs. In 1993, that number was 4015. How many breeding pairs would you expect there were in 2006? What about 2015?

link: Wood Bison Restoration in Alaska, Alaska Department of Fish and Game

Excerpt:

> *Based on experience with reintroduced populations elsewhere, wood bison would be expected to increase at a rate of 15%-25% annually after becoming established.... With an average annual growth rate of 20%, an initial precalving population of 50 bison would increase to 500 in approximately 13 years.*

Now You Are they using our same model?

Compound Interest

Suppose you invest \$10,000 in an account that accrues interest each month. After one month, your balance (with interest) is \$10,100. How much money will be in your account after a year?

Compound interest is calculated according to the formula Pe^{rt}, where r is the interest rate and t is time.

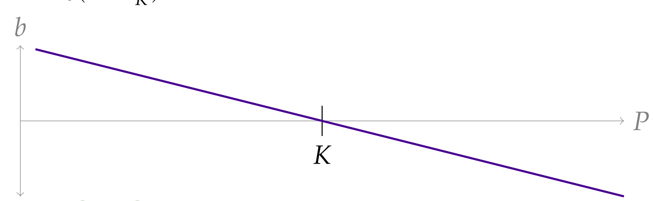
Carrying Capacity

For a population of size P with unrestricted access to resources, let β be the average number of offspring each breeding pair produces per generation, where a generation has length t_g. Then $b = \frac{\beta-2}{2t_g}$ is the net birthrate (births minus deaths) per member per unit time. This yields $\frac{dP}{dt}(t) = bP(t)$, hence:

But as resources grow scarce, b might change.

Carrying Capacity

b is the net birthrate (births minus deaths) per member per unit time. If K is the carrying capacity of an ecosystem, we can model $b = b_0(1 - \frac{P}{K})$.



Now You Describe to your neighbour what the following mean in terms of the model:

- $b > 0, b = 0, b < 0$
- $P = 0, P > 0, P < 0$

Carrying Capacity

Then:

$$\frac{dP}{dt}(t) = \underbrace{b_0\left(1 - \frac{P(t)}{K}\right)}_{\text{per capita birthrate}} P(t)$$

This is an example of a differential equation that we don't have the tools to solve. (If you take more calculus, though, you'll learn how!) It's also an example of a way you might tweak a model so its assumptions better fit what you observe.

Radiocarbon Dating

Researchers at Charlie Lake in BC have found evidence[2] of habitation dating back to around 8500 BCE. For instance, a butchered bison bone was radiocarbon dated to about 10,500 years ago.

Suppose a comparable bone of a bison alive today contains 1μg of ^{14}C. If the half-life of ^{14}C is about 5730 years, roughly how much ^{14}C do you think the researchers found in the sample?

A. About $\frac{1}{10{,}500}$ μg
B. About $\frac{1}{4}$ μg
C. About $\frac{1}{2}$ μg
D. About 1 μg
E. I'm not sure how to estimate this

[2] http://pubs.aina.ucalgary.ca/arctic/Arctic49-3-265.pdf

Suppose a body is discovered at 3:45 pm, in a room held at 20°, and the body's temperature is 27°, not the normal 37°. At 5:45 pm, the temperature of the body has dropped to 25.3°. When did the inhabitant of the body die?

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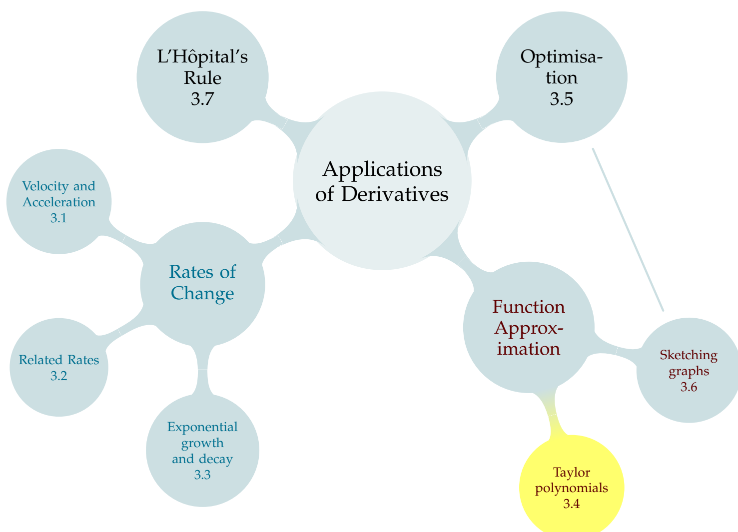
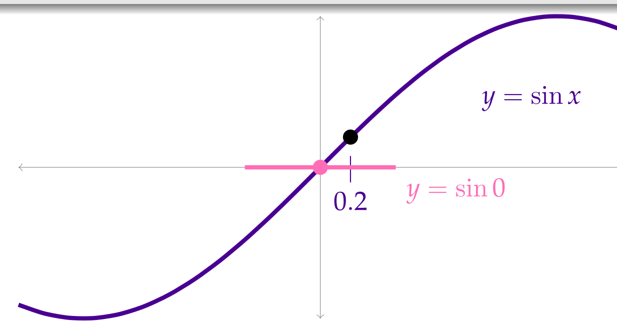


Approximating a Function

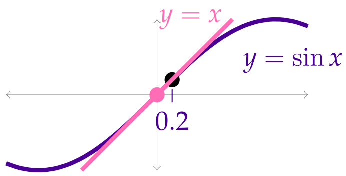


Constant Approximation – Equation 3.4.1

We can approximate $f(x)$ near a point a by

$$f(x) \approx f(a)$$

Approximating a Function



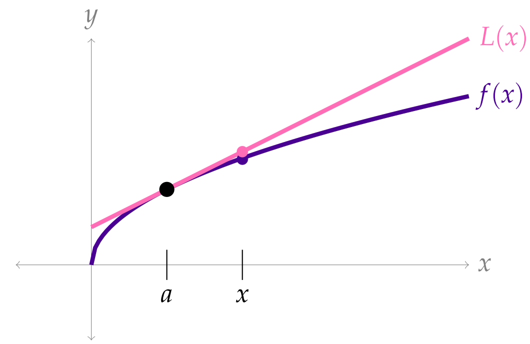
Linear Approximation (Linearization) – Equation 3.4.3

We can approximate $f(x)$ near a point a by the tangent line to $f(x)$ at a, namely

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

To find a linear approximation of $f(x)$ at a particular point x, pick a point a near to x, such that $f(a)$ and $f'(a)$ are easy to calculate.

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$



To find a linear approximation of $f(x)$ at a particular point x, pick a point a near to x, such that $f(a)$ and $f'(a)$ are easy to calculate.

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

Let $f(x) = \sqrt{x}$. Approximate $f(8.9)$.

Can we compute?

Suppose we want to approximate the value of $\cos(1.5)$. Which of the following linear approximations could we calculate by hand? (You can leave things in terms of π.)

A. tangent line to $f(x) = \cos x$ when $x = \pi/2$

B. tangent line to $f(x) = \cos x$ when $x = 3/2$

C. both

D. neither

Can we compute?

Which of the following tangent lines is probably the most accurate in approximating $\cos(1.5)$?

A. tangent line to $f(x) = \cos x$ when $x = \pi/2$
B. tangent line to $f(x) = \cos x$ when $x = \pi/4$
C. constant approximation: $\cos 1.5 \approx \cos(\pi/2) = 0$
D. the linear approximations should be better than the constant approximation, but both linear approximations should have the same accuracy

Linear Approximation

Approximate $\sin(3)$ using a linear approximation. You may leave your answer in terms of π.

Linear Approximation

Approximate $e^{1/10}$ using a linear approximation.

If $f(x) = e^x$ and $a = 0$:

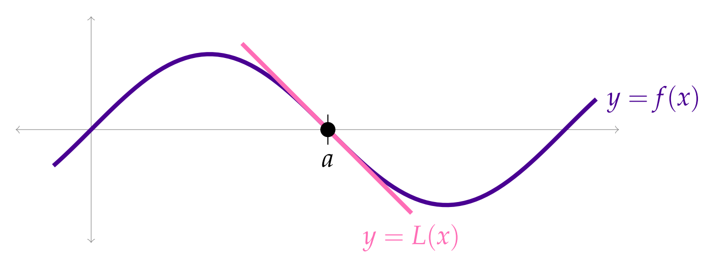
Linear Approximation Wrap-Up

Let $L(x) = f(a) + f'(a)(x - a)$, so $L(x)$ is the linear approximation (linearization) of $f(x)$ at a.

What is $L(a)$?

What is $L'(a)$?

What is $L''(a)$? (Recall $L''(x)$ is the derivative of $L'(x)$.)



LINEAR APPROXIMATION WRAP-UP

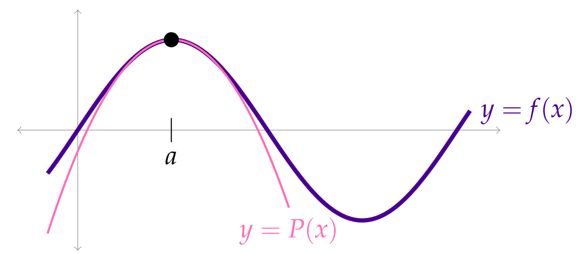
Let $L(x)$ be a linear approximation of $f(x)$.

$f(a)$	$L(a)$	same
$f'(a)$	$L'(a)$	same
$f''(a)$	$L''(a)$	different[3]

[3] unless $f''(a) = 0$

QUADRATIC APPROXIMATION

Imagine we approximate $f(x)$ at $x = a$ with a parabola, $P(x)$.



	Constant	Linear	Quadratic
Function value matches at $x = a$	✓	✓	✓
First derivative matches at $x = a$	×	✓	✓
Second derivative matches at $x = a$	×	×	✓

Constant: $f(x) \approx f(a)$

Linear: $f(x) \approx f(a) + f'(a)(x-a)$

Quadratic: $f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

Quadratic Approximation

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\log(1.1)$ using a quadratic approximation.

Quadratic Approximation

$$P(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

Approximate $\sqrt[3]{28}$ using a quadratic approximation.
You may leave your answer unsimplified, as long as it is an expression you could figure out from integers using only plus, minus, times, and divide.

Determine what $f(x)$ and a should be so that you can approximate the following using a quadratic approximation.

$\log(.9)$

$e^{-1/30}$

$\sqrt[5]{30}$

$(2.01)^6$

	Constant	Linear	Quadratic	degree n
match $f(a)$	✓	✓	✓	✓
match $f'(a)$	×	✓	✓	✓
match $f''(a)$	×	×	✓	✓
...				
match $f^{(n)}(a)$	×	×	×	✓
match $f^{(n+1)}(a)$	×	×	×	×

Constant:

$f(x) \approx f(a)$

Linear:

$f(x) \approx f(a) + f'(a)(x-a)$

Quadratic:

$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$

Degree-n:

$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots ?$

Brief Detour: Sigma (Summation) Notation

$$\sum_{i=a}^{b} f(i)$$

- a, b (integers) "bounds"
- i "index": runs over integers from a to b
- $f(i)$ "summand": compute for every i, add

Sigma Notation

$$\sum_{i=2}^{4}(2i+5)$$

Sigma Notation

$$\sum_{i=1}^{4}(i+(i-1)^2)$$

Write the following expressions in sigma notation:

1. $3 + 4 + 5 + 6 + 7$
2. $8 + 8 + 8 + 8 + 8$
3. $1 + (-2) + 4 + (-8) + 16$

Factorial – Definition 3.4.9

We read "$n!$" as "n factorial."
For a natural number n, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.
By convention, $0! = 1$.

We write $f^{(n)}(x)$ to mean the n^{th} derivative of $f(x)$. By convention, $f^{(0)}(x) = f(x)$.

Taylor Polynomial – Definition 3.4.11

Given a function $f(x)$ that is differentiable n times at a point a, the n-th degree **Taylor polynomial** for $f(x)$ about a is

$$T_n(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

If $a = 0$, we also call it a **Maclaurin polynomial**.

$$T_n(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$=$$

Find the 7th degree Maclaurin[4] polynomial for e^x.

[4] A Maclaurin polynomial is a Taylor polynomial with $a = 0$.

Find the 8th degree Maclaurin polynomial for $f(x) = \sin x$.

Now You

Find the 7th degree Taylor polynomial for $f(x) = \log x$, centered at $a = 1$.

skip Δx notation

Notation 3.4.18

Let x, y be variables related such that $y = f(x)$. Then we denote a small change in the variable x by Δx (read as "delta x"). The corresponding small change in the variable y is denoted Δy (read as "delta y").

$$\Delta y = f(x + \Delta x) - f(x)$$

Thinking about change in this way can lead to convenient approximations.

Let $y = f(x)$ be the amount of water needed to produce x apples in an orchard.
A farmer wants to know how a much water is needed to increase their crop yield. Δx is shorthand for some change in the number of apples, and Δy is shorthand for some change in the amount of water.

- Consider changing the number of apples grown from a to $a + \Delta x$
- Then the change in water requirements goes from $y = f(a)$ to $y = f(a + \Delta x)$

$$\Delta y = f(a + \Delta x) - f(a)$$

Linear Approximation of Δy

- Using a linear approximation, setting $x = a + \Delta x$:

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) && \text{linear approximation} \\ f(a+\Delta x) &\approx f(a) + f'(a)(\Delta x) && \text{set } x = a + \Delta x \\ \Delta y = f(a+\Delta x) - f(a) &\approx f'(a)\Delta x && \text{subtract } f(a) \text{ both sides} \end{aligned}$$

Linear Approximation of Δy (Equation 3.4.20)

$$\Delta y \approx f'(a)\Delta x$$

If we set $\Delta x = 1$, then $\Delta y \approx f'(a)$. So, if we want to produce $a+1$ apples instead of a apples, the extra water needed for that one extra apple is about $f'(a)$. We call this the *marginal* water cost of the apple.

Quadratic Approximation of Δy

If we wanted a more accurate approximation, we can use other Taylor polynomials. For example, let's try the quadratic approximation.

Quadratic Approximation of Δy (Equation 3.4.21)

$$\Delta y \approx f'(a)\Delta x + \frac{1}{2}f''(a)(\Delta x)^2$$

skip further examples

Approximate $\tan(65°)$ three ways: using constant, linear, and quadratic approximation.
Your answer may consist of the sum, difference, product, and quotient of integers, roots of integers, and π.

You measure an angle $x \approx \frac{\pi}{2}$, and use it to calculate $y = \sin x \approx 1$. However, you suspect the angle was not *exactly* equal to $\frac{\pi}{2}$, which means the actual value y is slightly *less than* 1. In order for your value of y to have an error of no more than $\frac{1}{200}$, how accurate does your measurement of θ have to be?

Definition 3.4.25

Let Q_0 be the exact value of a quantity and let $Q_0 + \Delta Q$ be the measured value. We call

$$|\Delta Q|$$

the absolute error of the measurement, and

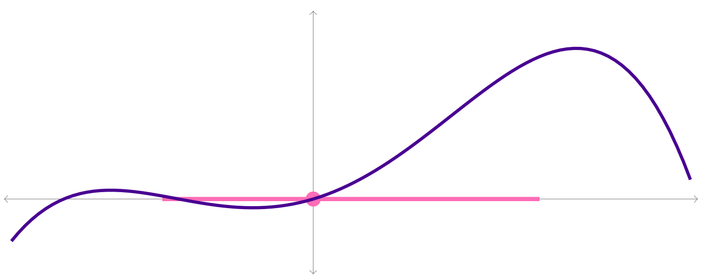
$$100\frac{|\Delta Q|}{Q_0}$$

the percentage error of the measurement.

Suppose a bottle of water is labelled as having 500 mL of water, but in fact contains 502.

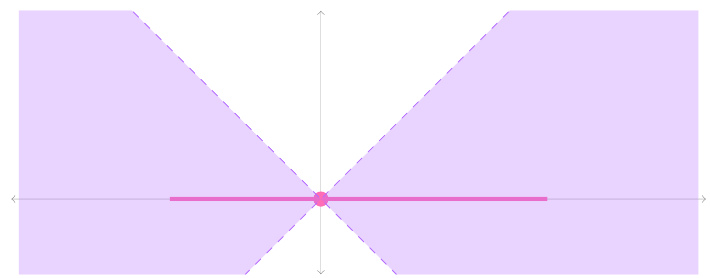
Once again, you find yourself in the position of measuring an angle x, which you use to compute $y = \sin x$. Let's say both x and y are positive. If your percentage error in measuring x is at most 1%, what is the corresponding maximum percentage error in y?
Use a linear approximation.

Error: what "causes" error in an estimation?



Constant approximation: We assume the function doesn't change, but in fact the function does change (its derivative is not always zero).

Controlling the "cause" of the error



Constant approximation: We assume the function doesn't change, but in fact the function does change (its derivative is not always zero).
BUT: suppose we know the max and min values of the function's slope.

Error

The error in an estimation $f(x) \approx T_n(x)$ is $f(x) - T_n(x)$. We often use $|f(x) - T_n(x)|$ if we don't care whether the approximation is too big or too little, but only that it is not too egregious.

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

The trick is bounding $f^{(n+1)}(c)$. It's usually OK to be sloppy here! Also, usually what we care about is the magnitude of the error: $|f(x) - T_n(x)|$.

Third degree Maclaurin polynomial for $f(x) = e^x$:

$$\begin{aligned} T_3(x) &= f(0) + f'(0)(x-0) + \tfrac{1}{2!}f''(0)(x-0)^2 + \tfrac{1}{3!}f'''(0)(x-0)^3 \\ &= e^0 + e^0 x + \frac{1}{2!}e^0 x^2 + \frac{1}{3!}e^0 x^3 \\ &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \end{aligned}$$

Bound the error associated with using $T_3(x)$ to approximate $e^{1/10}$.

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Bound the error associated with using $T_3(x)$ to approximate $e^{1/10}$.

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose we use the 5th degree Taylor polynomial centered at $a = \pi/2$ to approximate $f(x) = \cos x$. What could the magnitude of the error be if we approximate $\cos(2)$?

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose we use a third degree Taylor polynomial centred at 4 to approximate $f(x) = \sqrt{x}$. If we use this Taylor polynomial to approximate $\sqrt{4.1}$, give a bound for our error.

Taylor's Theorem – Equation 3.4.33

For some c strictly between x and a,

$$f(x) - T_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c)(x-a)^{n+1}$$

Suppose you want to approximate the value of e, knowing only that it is somewhere between 2 and 3. You use a 4th degree Maclaurin polynomial for $f(x) = e^x$ to approximate $f(1) = e^1 = e$. Bound your error.

Computing approximations uses resources. We might want to use as few resources as possible while ensuring sufficient accuracy.

A reasonable question to ask is: which approximation will be good enough to keep our error within some fixed error tolerance?

Which Degree?

Suppose you want to approximate $\sin 3$ using a Taylor polynomial of $f(x) = \sin x$ centered at $a = \pi$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

Which Degree?

Suppose you want to approximate e^5 using a Maclaurin polynomial of $f(x) = e^x$. If the magnitude of your error must be less than 0.001, what degree Maclaurin polynomial should you use?

Which Degree?

Suppose you want to approximate $\log \frac{4}{3}$ using a Taylor polynomial of $f(x) = \log x$ centred at $a = 1$. If the magnitude of your error must be less than 0.001, what degree Taylor polynomial should you use?

Which Degree?

Let $f(x) = \sqrt[4]{x}$. Suppose you use a second-degree Taylor polynomial of $f(x)$ centered at $a = 81$ to approximate $\sqrt[4]{81.2}$. Bound your error, and tell whether $T_2(10)$ is an overestimate or underestimate.

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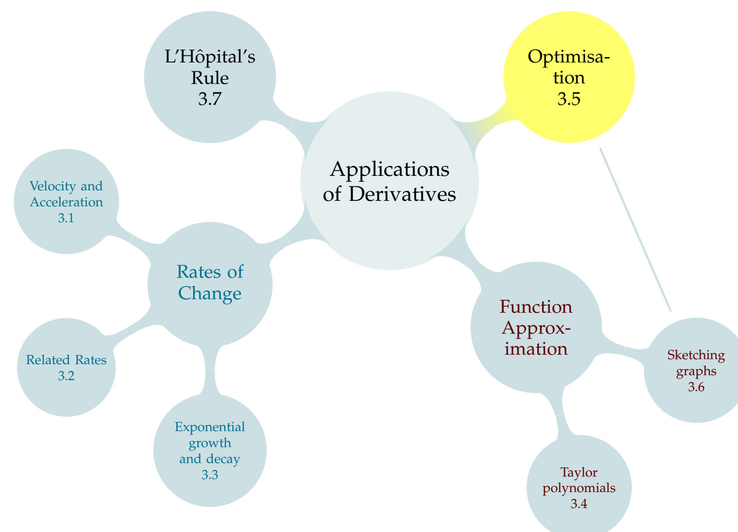
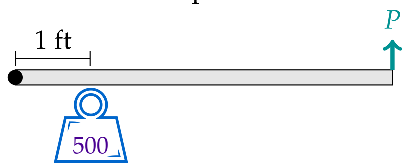


Optimisation:
finding the biggest/smallest/highest/lowest, etc.

Lots of non-standard problems! Opportunities to work on your problem-solving skills.

Engineering Design Example

A lever of density 3 lbs/ft is being used to lift a 500-pound weight, attached one foot from the fixed point.



For an L-foot-long lever, the force P required to lift the system satisfies

$$500(1) + 3L\left(\frac{L}{2}\right) - PL = 0$$

What length of lever will require the least amount of force to lift?

Source: Drexel (2006)

Medical Dosing Example

Let D be the size of a dose, α be the absorption rate, and β the elimination rate of a drug.
Caffeine is absorbed and eliminated by first-order kinetics. Its blood concentration over time is modelled as

$$c(t) = \frac{D}{1 - \beta/\alpha}\left(e^{-\beta t} - e^{-\alpha t}\right)$$

Will the blood concentration reach a toxic level?

Source (including links to a study): Vectornaut (2015)

Circuit Example

When a critically damped RLC circuit is connected to a voltage source, the current I in the circuit varies with time according to the equation

$$I(t) = \left(\frac{V}{L}\right) te^{-\frac{Rt}{2L}}$$

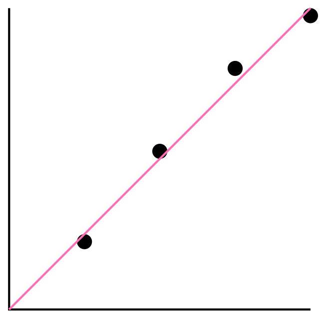
where V is the applied voltage, L is the inductance, and R is the resistance (all of which are constant).

We need to choose wires that will be able to safely carry the current at all times.

Source: Belk (2014)

Least Squares Example

You have a lot of data that more-or-less resembles a line.
Which line does it most resemble?



Extrema – Definition 3.5.3

Let I be an interval, and let the function $f(x)$ be defined for all $x \in I$. Now let $c \in I$.

- We say that $f(x)$ has a global (or absolute) minimum on the interval I at the point $x = c$ if $f(x) \geq f(c)$ for all $x \in I$.
- We say that $f(x)$ has a global (or absolute) maximum on I at $x = c$ if $f(x) \leq f(c)$ for all $x \in I$.
- We say that $f(x)$ has a local minimum at $x = c$ if $f(x) \geq f(c)$ for all $x \in I$ that are near c.
- We say that $f(x)$ has a local maximum at $x = c$ if $f(x) \leq f(c)$ for all $x \in I$ that are near c.

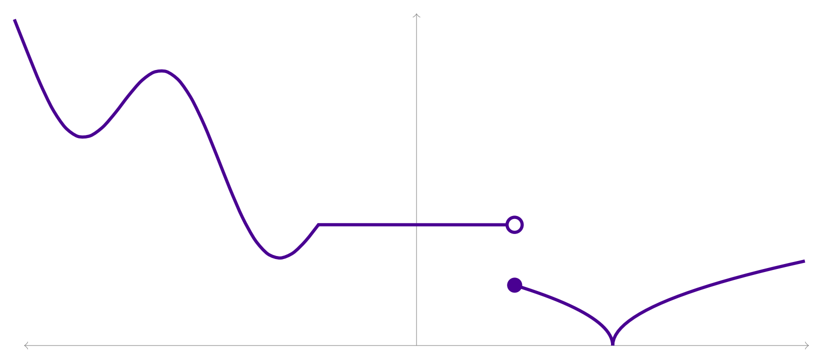
The maxima and minima of a function are called the extrema of that function.

Critical and Singular Points – Definition 3.5.6

Let $f(x)$ be a function and let c be a point in its domain. Then

- If $f'(c)$ exists and is zero we call $x = c$ a critical point of the function, and
- If $f'(c)$ does not exist then we call $x = c$ a singular point of the function.

Anatomy of a Function



c is a critical point if $f'(c) = 0$.
c is a singular point if $f'(c)$ does not exist.

Theorem 3.5.4

If a function $f(x)$ has a local maximum or local minimum at $x = c$ and if $f'(c)$ exists, then $f'(c) = 0$.

MULTIPLE CHOICE

Suppose $f(x)$ has domain $(-\infty, \infty)$.
If $f'(5) = 0$, then:

A. $f'(5)$ DNE
B. f has a local maximum at 5
C. f has a local minimum at 5
D. f has a local extremum (maximum or minimum) at 5
E. f may or may not have a local extremum (max or min) at 5

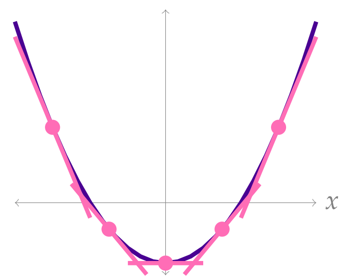
SKETCH

Draw a continuous function $f(x)$ with a local maximum at $x = 3$ and a local minimum at $x = -1$.

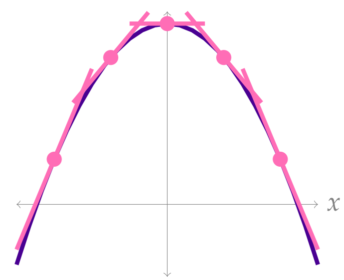
Draw a continuous function $f(x)$ with a local maximum at $x = 3$ and a local minimum at $x = -1$, but $f(3) < f(-1)$.

Draw a function $f(x)$ with a singular point at $x = 2$ that is NOT a local maximum, or a local minimum.

SECOND DERIVATIVES



- Is slope increasing, decreasing, or constant?
- Is second derivative positive, negative, or zero?
- Is critical point a local max, local min, or neither?



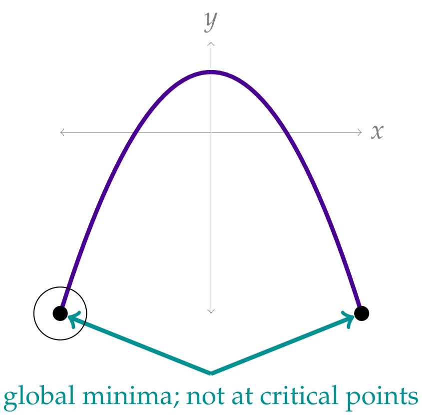
- Is slope increasing, decreasing, or constant?
- Is second derivative positive, negative, or zero?
- Is critical point a local max, local min, or neither?

Suppose $f'(x) = (x+5)^2(x-5)$. Then f has no singular points, and its critical points are ± 5. Identify whether the critical points are local maxima, local minima, or neither.

Second Derivative Test:
Suppose $f'(a) = 0$ and $f''(a) > 0$.
Then $x = a$ is a local

Suppose $f'(a) = 0$ and $f''(a) < 0$.
Then $x = a$ is a local

ENDPOINTS



Theorems 3.5.11 and 3.5.12

A function that is continuous on the interval $[a, b]$ (where a and b are real numbers–not infinite) has a global max and min, and they occur at endpoints, critical points, or singular points.

DETERMINING EXTREMA

To find local extrema:

- Could be at
- Could be at
- Could be at
- At these points, check whether there is some interval around x where $f(x)$ is no larger than the other numbers, or no smaller. (A sketch helps. The signs of the derivatives on either side of x are also a clue.)

To find global extrema:

- Could be at
- Could be at
- Could be at

- Check the value of the function at all of these, and compare.

Find All Extrema[4]:

$$f(x) = x^3 - 3x$$

[4]Extrema: local and global maxima and minima

Find All Extrema

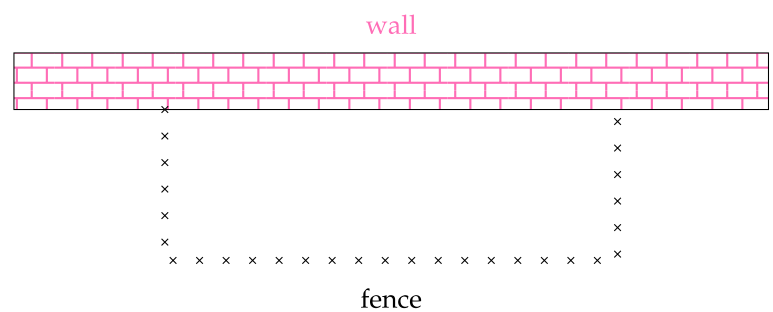
$$f(x) = \sqrt[3]{x^2 - 64}, \quad x \text{ in } [-1, 10]$$

Find the largest and smallest value of $f(x) = x^4 - 18x^2$.

Find the largest and smallest values of $f(x) = \sin^2 x - \cos x$.

MAX/MIN WORD PROBLEMS

A rancher wants to build a rectangular pen, using an existing wall for one side of the pen, and using 100m of fencing for the other three sides. What are the dimensions of the pen built this way that has the largest area?



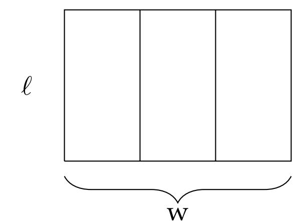
General Idea

We know how to find the global extrema of a function over an interval.

Problems often involve multiple variables, but we can only deal with functions of one variable.

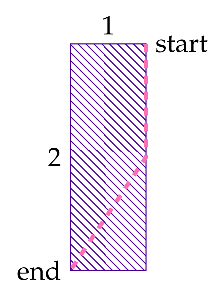
Find all the variables in terms of ONE variable, so we can find extrema.

You want to build a pen, as shown below, in the shape of a rectangle with two interior divisions. If you have 1000m of fencing, what is the greatest area you can enclose?



Suppose you want to make a rectangle with perimeter 400. What dimensions give you the maximum area?

You are standing on the bank of a river that is 1km wide, and you want to reach the opposite side, two km down the river. You can paddle 3 kilometres per hour, and walk 6 kph while carrying your boat. What route takes you to your desired destination in the least amount of time?



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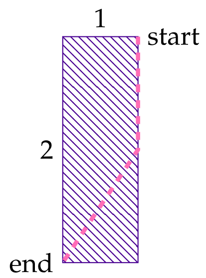
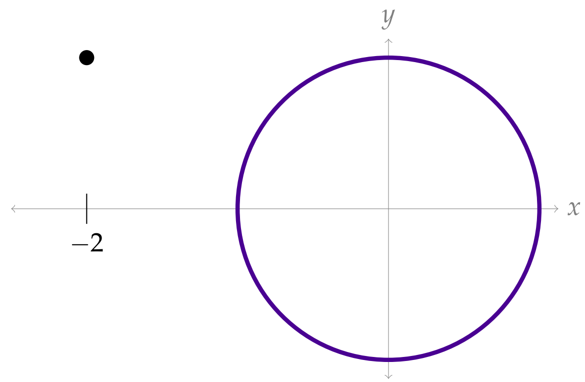


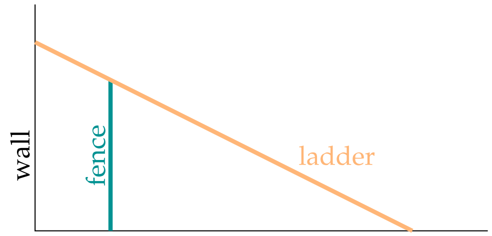
Let C be the circle given by $x^2 + y^2 = 1$. What is the closest point on C to the point $(-2, 1)$?



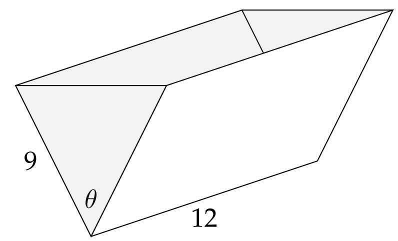
Suppose you want to manufacture a closed cylindrical can on the cheap. If the can should have a volume of one litre (1000 cm^3), what is the smallest surface area it can have?

A cylindrical can is to hold 20π cubic metres. The material for the top and bottom costs \$10 per square metre, and material for the side costs \$8 per square metre. Find the radius r and height h of the most economical can.

Suppose a 2-metre high fence stands 1 metre away from a high wall. What is the shortest ladder that will reach over the fence to the wall?

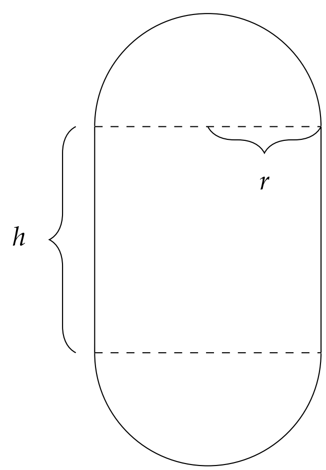


Suppose a file folder is 12 inches long and 9 inches wide. You want to make a box by opening the folder and capping the ends. What angle should you open the folder to, to make the box with the greatest volume?

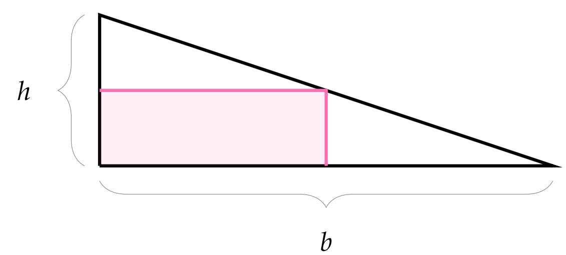


We want to bend a piece of wire into the perimeter of the shape shown below: a rectangle of height h and width $2r$, with a half circle of radius r on the top and bottom.

If you only have 100cm of wire, what values of r and h give the largest enclosed area?

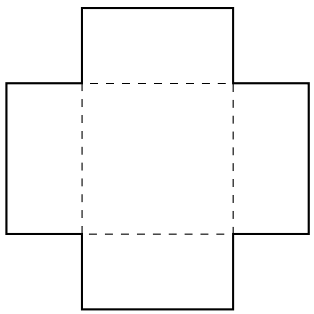


Suppose we take a right triangle, with height h and base b. We inscribe a rectangle in it that shares a right angle, as shown below. What are the dimensions of the rectangle with the biggest area?



ACTIVITY

By cutting out squares from the corners, turn a piece of paper into an open-topped box that holds a lot of beans.



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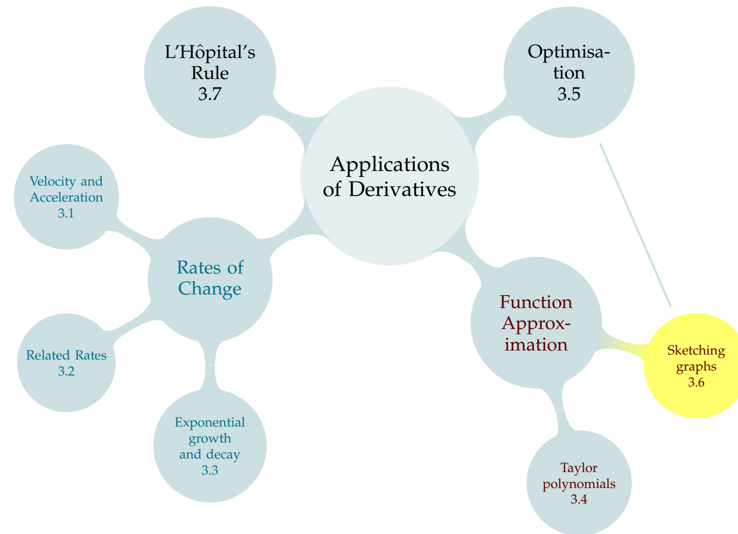


CURVE SKETCHING

Review: find the domain of the following function.

$$f(x) = \frac{\sqrt{3-x^2}}{\log(x+1)}$$

Where might you expect $f(x)$ to have a vertical asymptote? What does the function look like nearby?
(Recall: a vertical asymptote occurs at $x = a$ if the function has an infinite discontinuity at a. That is, $\lim\limits_{x\to a^\pm} f(x) = \pm\infty$.)

Where is $f(x) = 0$?

What happens to $f(x)$ near its other endpoint, $x = -1$?

CURVE SKETCHING

Good things to check:

- Domain
- Vertical asymptotes: $\lim\limits_{x\to a} f(x) = \pm\infty$
- Intercepts: $x = 0, f(x) = 0$
- Horizontal asymptotes and end behavior: $\lim\limits_{x\to\pm\infty} f(x)$

CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{x-2}{(x+3)^2}$$

CURVE SKETCHING

Identify: domain, vertical asymptotes, intercepts, and horizontal asymptotes

$$f(x) = \frac{(x+2)(x-3)^2}{x(x-5)}$$

FIRST DERIVATIVE

Add complexity: Increasing/decreasing, critical and singular points.

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

What does the graph of the following function look like?

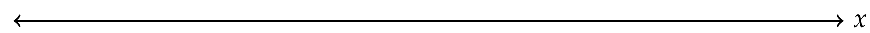
$$f(x) = \frac{1}{3}x^3 + 2x^2 + 4x + 24$$

What does the graph of the following function look like?

$$f(x) = e^{\frac{x+1}{x-1}}$$

Signs of Factored Functions

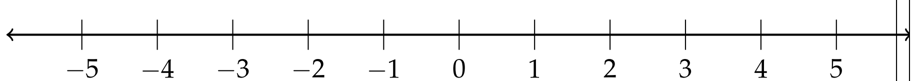
$$f(x) = \quad (x-1) \quad (x-2)^2 \quad (x-3)$$



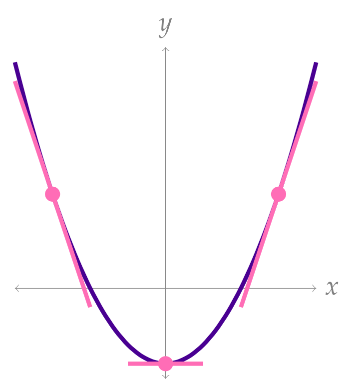
Signs of Factored Functions

$$f(x) = (x-3)(x-1)^2 x(x+2)^3(x+5)^4$$

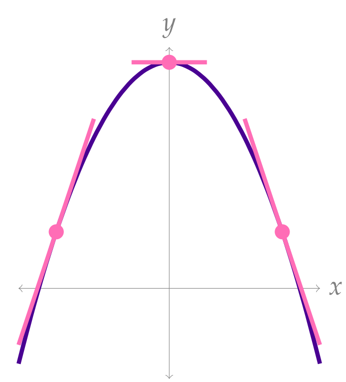
Where is $f(x)$ positive? Where is it negative?



Concavity



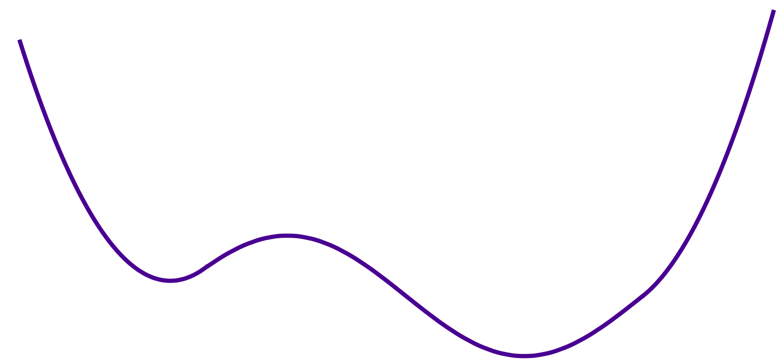
- Slopes are increasing
- $f''(x) > 0$
- "concave up"
- tangent line below curve



- Slopes are decreasing
- $f''(x) < 0$
- "concave down"
- tangent line above curve

Mnemonic

Concavity



Sketch graphs with the following properties, or explain that none exist.

	concave up	concave down
increasing		
decreasing		

Poll Questions

Describe the concavity of the function $f(x) = e^x$.

A. concave up
B. concave down
C. concave up for $x < 0$; concave down for $x > 0$
D. concave down for $x < 0$; concave up for $x > 0$
E. I'm not sure

Is it possible to be concave up and decreasing?

A. Yes B. No C. I'm not sure

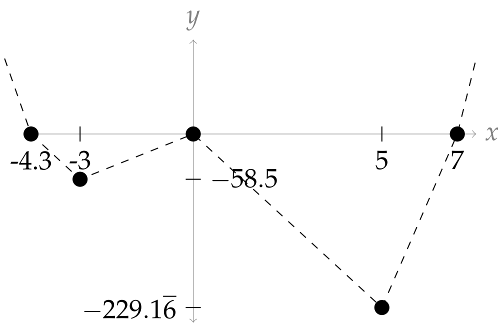
Suppose a function $f(x)$ is defined for all real numbers, and is concave up on the interval $[0, 1]$. Which of the following must be true?

A. $f'(0) < f'(1)$
B. $f'(0) > f'(1)$
C. $f'(0)$ is positive
D. $f'(0)$ is negative
E. I'm not sure

Revisiting a previous example

original example

$$f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - 15x^2$$

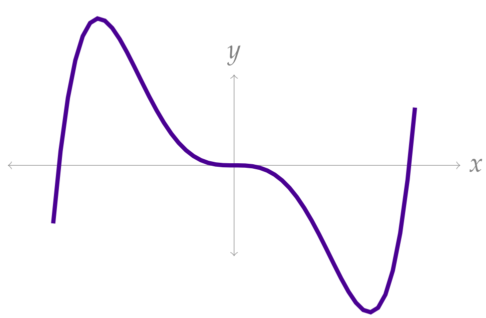


$f''(x) = 6x^2 - 8x - 30 = 2(x-3)(3x+5)$

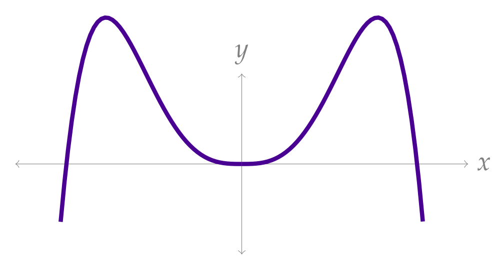
Sketch:

$$f(x) = x^5 - 15x^3$$

Even and Odd Functions



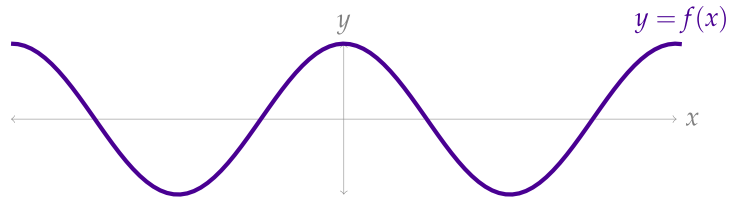
Even and Odd Functions



Even Function – Definition 3.6.6

A function $f(x)$ is even if, for all x in its domain,

$$f(-x) = f(x)$$



Even Functions

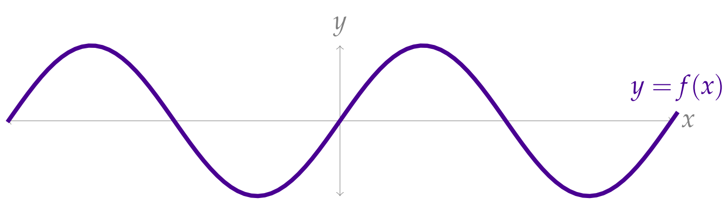
Even Function – Definition 3.6.6

A function $f(x)$ is even if, for all x in its domain,

$$f(-x) = f(x)$$

Examples:
$f(x) = x^2$
$f(x) = x^4$
$f(x) = \cos(x)$
$f(x) = \dfrac{x^4 + \cos(x)}{x^{16} + 7}$

Odd Functions



Suppose $f(1) = 2$. Then $f(-1) =$
Suppose $f(3) = -2$. Then $f(-3) =$

Odd Function – Definition 3.6.7

A function $f(x)$ is odd if, for all x in its domain,

$$f(-x) = -f(x)$$

Odd Functions

Odd Function – Definition 3.6.7

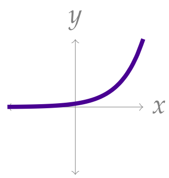
A function $f(x)$ is odd if, for all x in its domain,

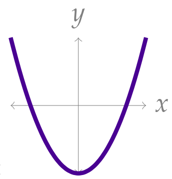
$$f(-x) = -f(x)$$

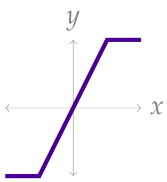
Examples:
$f(x) = x$
$f(x) = x^3$
$f(x) = \sin(x)$
$f(x) = \dfrac{x(1 + x^2)}{x^2 + 5}$

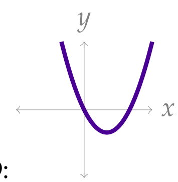
Poll Tiiime

Pick out the odd function.

A: 

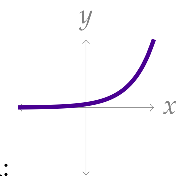
B: 

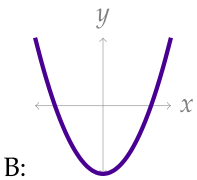
C: 

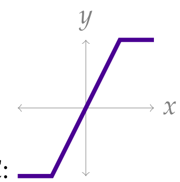
D: 

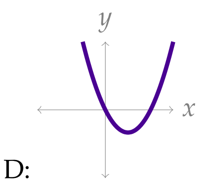
Poll Tiiime

Pick out the even function.

A: 

B: 

C: 

D: 

Even more Poll tiiiiime

Suppose $f(x)$ is an odd function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

A. $f(0) = f(-0)$
B. $f(0) = -f(0)$
C. $f(0) = 0$
D. all of the above are true
E. none of the above are necessarily true

Even more and more Poll tiiiiime

Suppose $f(x)$ is an even function, continuous, defined for all real numbers. What is $f(0)$? Pick the best answer.

A. $f(0) = f(-0)$
B. $f(0) = -f(0)$
C. $f(0) = 0$
D. all of the above are true
E. none of the above are necessarily true

OK OK... LAST ONE

Suppose $f(x)$ is an even function, differentiable for all real numbers. What can we say about $f'(x)$?

A. $f'(x)$ is also even
B. $f'(x)$ is odd
C. $f'(x)$ is constant
D. all of the above are true
E. none of the above are necessarily true

PERIODICITY

Periodic – Definition 3.6.10

A function is periodic with period $P > 0$ if

$$f(x) = f(x + P)$$

whenever x and $x + P$ are in the domain of f, and P is the smallest such (positive) number

Examples: $\sin(x)$, $\cos(x)$ both have period 2π; $\tan(x)$ has period π.

Ignoring concavity, sketch $f(x) = \sin(\sin x)$.

Challenge: ignoring exact locations of extrema, sketch $g(x) = \sin(2\pi \sin x)$.

LET'S GRAPH

$$f(x) = (x^2 - 64)^{1/3}$$

$$f'(x) = \frac{2x}{3(x^2-64)^{2/3}};$$

$$f''(x) = \frac{-2(\frac{1}{3}x^2 + 64)}{3(x^2-64)^{5/3}}$$

Let's Graph

$$f(x) = \frac{x^2+x}{(x+1)(x^2+1)^2}$$

Note: for $x \neq -1, f(x) = \dfrac{x(x+1)}{(x+1)(x^2+1)^2} = \dfrac{x}{(x^2+1)^2}$

$$g(x) := \frac{x}{(x^2+1)^2}$$

$$g'(x) = \frac{1-3x^2}{(x^2+1)^3}$$

$$g''(x) = \frac{12x(x^2-1)}{(x^2+1)^4}$$

Let's Graph

$$f(x) = x(x-1)^{2/3}$$

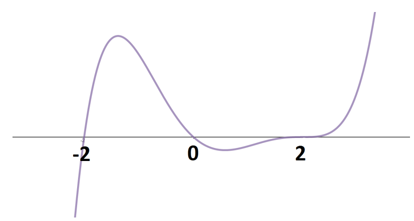
- $f'(x) = \dfrac{5x-3}{3\sqrt[3]{x-1}}$
- $f''(x) = \dfrac{2(5x-6)}{9(\sqrt[3]{x-1})^4}$

- $f(3/5) \approx 0.3$
- $f(6/5) \approx 0.4$

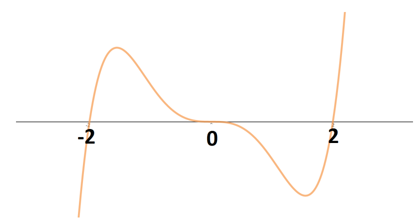
Ch 3.6 Review: matching

Match the Function to its Graph

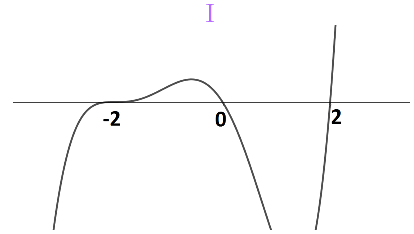
A. $f(x) = x^3(x+2)(x-2) = x^5 - 4x^3$
B. $f(x) = x(x+2)^3(x-2) = x^5 + 4x^4 - 16x^2 - 16x$
C. $f(x) = x(x+2)(x-2)^3 = x^5 - 4x^4 + 16x^2 - 16x$



I



III



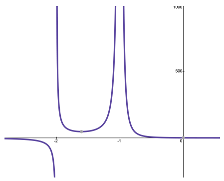
II

A. $f(x) = \dfrac{x-1}{(x+1)(x+2)}$

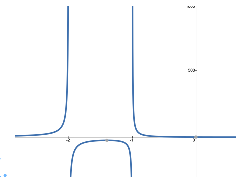
B. $f(x) = \dfrac{(x-1)^2}{(x+1)(x+2)}$

C. $f(x) = \dfrac{x-1}{(x+1)^2(x+2)}$

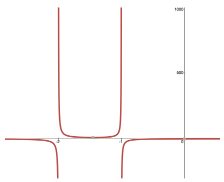
D. $f(x) = \dfrac{(x-1)^2}{(x+1)^2(x+2)}$



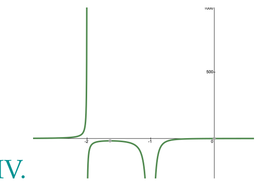
I.



III.



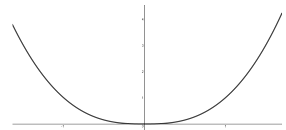
II.



IV.

Match the Function to its Graph

A. $f(x) = |x|^e$ B. $f(x) = e^{|x|}$ C. $f(x) = e^{x^2}$ D. $f(x) = e^{x^4-x}$



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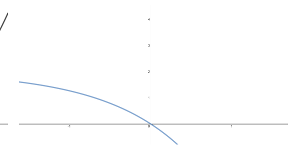
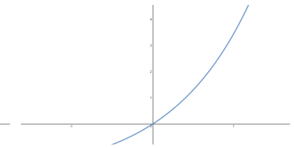


II



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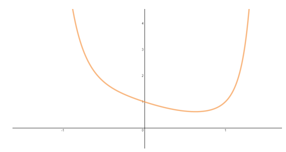

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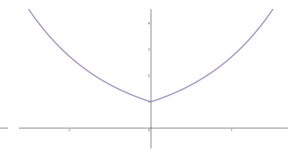
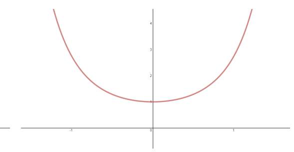
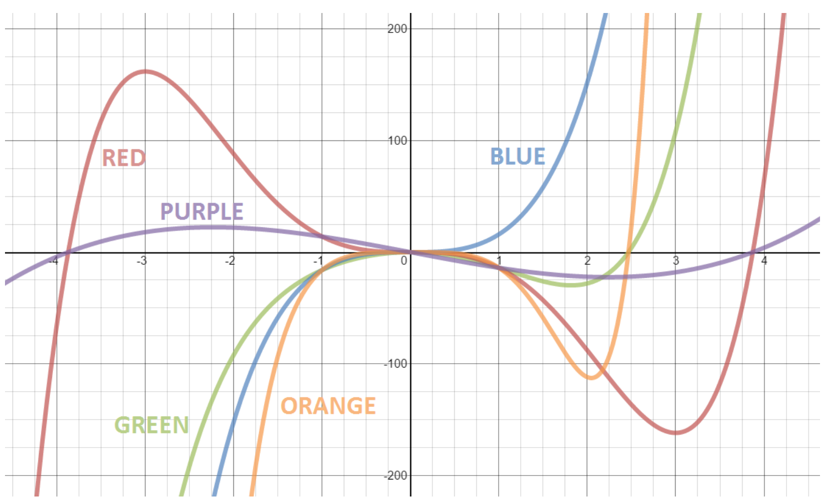


V



VI

A. $f(x) = x^5 + 15x^3$ B. $f(x) = x^5 - 15x^3$ C. $f(x) = x^5 - 15x^2$
D. $f(x) = x^3 - 15x$ E. $f(x) = x^7 - 15x^4$



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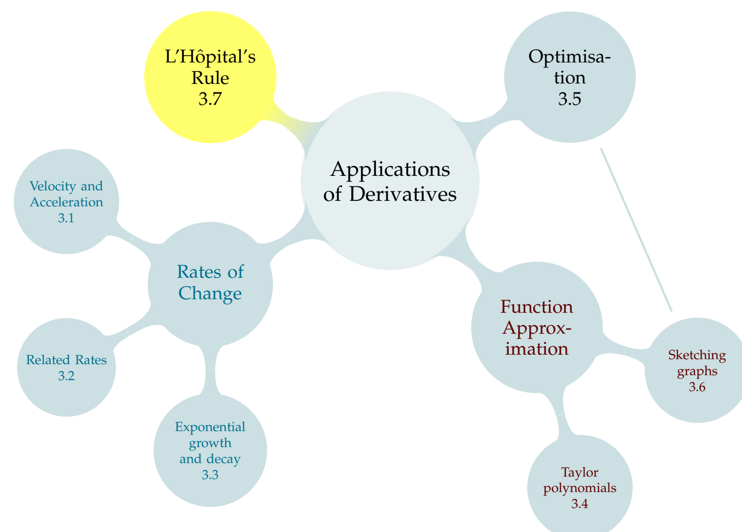

Back to Limits!

$$\lim_{x\to\infty} \frac{x^2}{5} \qquad \lim_{x\to\infty} \frac{5}{x^2} \qquad \lim_{x\to 0} \frac{x^2}{5} \qquad \lim_{x\to 0} \frac{5}{x^2}$$

Indeterminate Forms – Definition 3.7.1

Suppose $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Then the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is an indeterminate form of the type $\frac{0}{0}$.
Suppose $\lim_{x\to a} F(x) = \lim_{x\to a} G(x) = \infty$ (or $-\infty$). Then the limit

$$\lim_{x\to a} \frac{F(x)}{G(x)}$$

is an indeterminate form of the type $\frac{\infty}{\infty}$.

When you see an indeterminate form, you need to do more work.

Indeterminate Forms

$$\lim_{x\to 5} \frac{x^2 - 3x - 10}{x - 5}$$

indeterminate form of the type $\frac{0}{0}$

$$\lim_{x\to\infty} \frac{3x^2 - 4x + 2}{8x^2 - 5}$$

indeterminate form of the type $\frac{\infty}{\infty}$

Indeterminate Forms and the Derivative

$$\lim_{x\to 0} \frac{3\sin x - x^4}{x^2 + \cos x - e^x}$$

indeterminate form of the type $\frac{0}{0}$

L'Hôpital's Rule: First Part – Theorem 3.7.2

Let f and g be functions such that $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$.

If $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$.

If f and g are differentiable on an open interval containing a, and if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

This works even for $a = \pm\infty$.

Extremely Important Note:
L'Hôpital's Rule only works on indeterminate forms.

L'Hôpital's Rule: Second Part – Theorem 3.7.2

Let f and g be functions such that $\lim\limits_{x\to a} f(x) = \infty = \lim\limits_{x\to a} g(x)$.

If $f'(a)$ and $g'(a)$ exist and $g'(a) \neq 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{f'(a)}{g'(a)}$.

If f and g are differentiable on an open interval containing a, and if $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$.

This works even for $a = \pm\infty$.

Extremely Important Note:
L'Hôpital's Rule only works on indeterminate forms.

Evaluate:

$$\lim_{x\to 2} \frac{3x\tan(x-2)}{x-2}$$

LITTLE HARDER

$$\lim_{x\to 0} \frac{x^4}{e^x - \cos x - x}$$

indeterminate form of the type $\frac{0}{0}$

Evaluate:

$$\lim_{x\to\infty} \frac{\log x}{\sqrt{x}}$$

Other Indeterminate Forms

$\lim_{x\to\infty} e^{-x}\log x$ form $0\cdot\infty$

Vote Vote Vote

Which of the following can you immediately apply L'Hôpital's rule to?

A. $\frac{e^x}{2e^x+1}$

B. $\lim_{x\to 0} \frac{e^x}{2e^x+1}$

C. $\lim_{x\to\infty} \frac{e^x}{2e^x+1}$

D. $\lim_{x\to\infty} e^{-x}(2e^x+1)$

E. $\lim_{x\to 0} \frac{e^x}{x^2}$

Votey McVoteface

Suppose you want to use L'Hôpital's rule to evaluate $\lim_{x\to a}\frac{f(x)}{g(x)}$, which has the form $\frac{0}{0}$. How does the quotient rule fit into this problem?

A. You should use the quotient rule because the function you are differentiating is a quotient.

B. You will not use the quotient rule because you differentiate the numerator and the denominator separately

C. You may use the quotient rule because perhaps $f(x)$ or $g(x)$ is itself in the form of a quotient

D. You will not use L'Hôpital's rule because $\frac{0}{0}$ is not an appropriate indeterminate form

E. You will not use L'Hôpital's rule because, since the top has limit zero, the whole function has limit 0

More Questions

Which of the following is NOT an indeterminate form?

A. $\frac{\infty}{\infty}$ for example, $\lim_{x\to\infty}\frac{e^x}{x^2}$

B. $\frac{0}{0}$ for example, $\lim_{x\to 0}\frac{e^x-1}{x}$

C. $\frac{0}{\infty}$ for example, $\lim_{x\to 0^+}\frac{x}{\log x}$

D. $0\cdot\infty$ for example, $\lim_{x\to\infty} x(\arctan(x)-\pi/2)$

E. all of the above are indeterminate forms

I have so many questions

Which of the following is NOT an indeterminate form?

A. 1^∞ for example, $\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^x$

B. 0^∞ for example, $\lim_{x\to\infty}\left(\frac{1}{x}\right)^x$

C. ∞^0 for example, $\lim_{x\to\infty} x^{\frac{1}{x}}$

D. 0^0 for example, $\lim_{x\to 0^+} x^x$

E. all of the above are indeterminate forms

F. none of the above are indeterminate forms

Exponential Indeterminate Forms

$$\lim_{x\to\infty} x^{1/x}$$

Exponential Indeterminate Forms

$$\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{3x}$$

Evaluate:

$$\lim_{x\to\infty}\frac{\log x}{\log\sqrt{x}}$$

$$\lim_{x\to\infty}(\log x)^{\sqrt{x}}$$

$$\lim_{x\to 0}\frac{\arcsin x}{x}$$

More Examples

$$\lim_{x\to\infty} \sqrt{2x^2+1} - \sqrt{x^2+x}$$

$$\lim_{x\to 0} \sqrt[x^2]{\sin^2 x}$$

$$\lim_{x\to 0} \sqrt[x^2]{\cos x}$$

Sketch the graph of $f(x) = x\log x$.
Note: when you want to know $\lim\limits_{x\to 0} f(x)$, you'll need to use L'Hôpital.

Evaluate $\lim\limits_{x\to 0^+} (\csc x)^x$

4.1 Antiderivatives

Basic Question

What function has derivative $f(x)$?

If $F'(x) = f(x)$, we call $F(x)$ an antiderivative of $f(x)$.

Examples

$\frac{\mathrm{d}}{\mathrm{d}x}[x^2] = 2x$, so x^2 is an antiderivative of $2x$.

$\frac{\mathrm{d}}{\mathrm{d}x}[x^2+5] = 2x$, so x^2+5 is (also) an antiderivative of $2x$.

What is the most general antiderivative of $2x$?

ANTIDERIVATIVES

Find the most general antiderivative for the following equations.

$$f(x) = 17$$

$$f(x) = m$$

where m is a constant.

differentiation fact		antidifferentiation fact
$\frac{d}{dx}[x^2] = 2x$	$\Longrightarrow$	antideriv of $2x$:
$\frac{d}{dx}[x^3] = 3x^2$	$\Longrightarrow$	
$\frac{d}{dx}[x^4] = 4x^3$	$\Longrightarrow$	
$\frac{d}{dx}[x^5] = 5x^4$	$\Longrightarrow$	
		antideriv of x^n:

Power Rule for Antidifferentiation

The most general antiderivative of x^n is $\frac{1}{n+1}x^{n+1} + c$ if $n \neq -1$

- $\frac{d}{dx}\Big[\qquad \Big] = x^5$
- $\frac{d}{dx}\Big[\qquad \Big] = x^3$
- $\frac{d}{dx}\Big[\qquad \Big] = \frac{1}{2}x^3$

Power Rule for Antidifferentiation

The most general antiderivative of x^n is $\frac{1}{n+1}x^{n+1} + c$ if $n \neq -1$

- $\frac{d}{dx}\Big[\qquad \Big] = 5x^2 - 15x + 3$
- $\frac{d}{dx}\Big[\qquad \Big] = 13\left(5x^{14} - 3x^{3/7} + 52e^x\right)$

Find the most general antiderivatives.

$$f(x) = \cos x$$

$$f(x) = \sin x$$

$$f(x) = \sec^2 x$$

$$f(x) = \frac{1}{1+x^2}$$

$$f(x) = \frac{1}{1+x^2+2x}$$

Find the most general antiderivatives.

$$f(x) = 17\cos x + x^5$$

$$f(x) = \frac{23}{5+5x^2}$$

$$f(x) = \frac{23}{5+125x^2}$$

Find the most general antiderivatives.

$$f(x) = \frac{1}{x},\ x > 0$$

$$f(x) = 5x^2 - 32x^5 - 17$$

$$f(x) = \csc x \cot x$$

$$f(x) = \frac{5}{\sqrt{1-x^2}} + 17$$

Chose Your Own Adventure

Antiderivative of $\sin x \cos x$:

A. $\cos x \sin x + c$

B. $-\cos x \sin x + c$

C. $\sin^2 x + c$

D. $\frac{1}{2}\sin^2 x + c$

E. $\frac{1}{2}\cos^2 x \sin^2 x + c$

In general, antiderivatives of x^n have the form $\frac{1}{n+1}x^{n+1}$. What is the single exception?

A. $n = -1$

B. $n = 0$

C. $n = 1$

D. $n = e$

E. $n = 1/2$

All the Adventures are Calculus, Though

Suppose the velocity of a particle at time t is given by $v(t) = t^2 + \cos t + 3$. What function gives its position?

A. $s(t) = 2t - \sin t$
B. $s(t) = 2t - \sin t + c$
C. $s(t) = t^3 + \sin t + 3t + c$
D. $s(t) = \frac{1}{3}t^3 + \sin t + 3t + c$
E. $s(t) = \frac{1}{3}t^2 - \sin t + 3t + c$

Suppose the velocity of a particle at time t is given by $v(t) = t^2 + \cos t + 3$, and its position at time 0 is given by $s(0) = 5$. What function gives its position?

A. $s(t) = \frac{1}{3}t^3 + \sin t + 3t$
B. $s(t) = \frac{1}{3}t^3 + \sin t + 3t + 5$
C. $s(t) = \frac{1}{3}t^3 + \sin t + 3t + c$
D. $s(t) = 5t + c$
E. $s(t) = 5t + 5$

Find all functions $f(x)$ with $f(1) = 5$ and $f'(x) = e^{3x+5}$.

Let $Q(t)$ be the amount of a radioactive isotope in a sample. Suppose the sample is losing $50e^{-5t}$ mg per second to decay. If $Q(1) = 10e^{-5}$mg, find the equation for the amount of the isotope at time t.

Suppose $f'(t) = 2t + 7$. What is $f(10) - f(3)$?

This file contains questions spanning CLP-1. It should not be taken as a complete review of the course, but rather as a jumping-off point. If you struggle with one question, go back to review its entire section. Sections are noted at the bottom of each page.

S1

Find all solutions to $x^3 - 3x^2 - x + 3 = 0$

S2

Compute the limit $\displaystyle\lim_{x\to 2} \frac{x-2}{x^2-4}$

S3

Find all values of c such that the following function is continuous:

$$f(x) = \begin{cases} 8 - cx & \text{if} \quad x \le c \\ x^2 & \text{if} \quad x > c \end{cases}$$

Use the definition of continuity to justify your answer.

S4

Compute

$$\lim_{x\to-\infty} \frac{3x+5}{\sqrt{x^2+5}-x}$$

S5

Find the equation of the tangent line to the graph of $y = \cos(x)$ at $x = \frac{\pi}{4}$.

S6

For what values of x does the derivative of $\frac{\sin(x)}{x^2+6x+5}$ exist?

S7

Find $f'(x)$ if $f(x) = (x^2+1)^{\sin(x)}$.

S8

Consider a function of the form $f(x) = Ae^{kx}$ where A and k are constants. If $f(0) = 3$ and $f(2) = 5$, find the constants A and k.

S9

Consider a function $f(x)$ which has $f'''(x) = \dfrac{x^3}{10-x^2}$. Show that when we approximate $f(1)$ using its second Maclaurin polynomial, the absolute error is less than $\frac{1}{50} = 0.02$.

S10

Estimate $\sqrt{35}$ using a linear approximation

S11

Let $f(x) = x^2 - 2\pi x - \sin(x)$. Show that there exists a real number c such that $f'(c) = 0$.

S12

Find the intervals where $f(x) = \frac{\sqrt{x}}{x+6}$ is increasing.

L1

Compute the limit $\lim_{x\to 1} \dfrac{\sqrt{x+2} - \sqrt{4-x}}{x-1}$.

L2

Show that there exists at least one real number c such that $2\tan(c) = c + 1$.

L3

Determine whether the derivative of following function exists at $x = 0$

$$f(x) = \begin{cases} 2x^3 - x^2 & \text{if } x \leq 0 \\ x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x > 0 \end{cases}$$

You must justify your answer using the definition of a derivative.

L4

If $x^2\cos(y) + 2xe^y = 8$, then find y' at the points where $y = 0$. You must justify your answer.

L5

Two particles move in the cartesian plane. Particle A travels on the x-axis starting at $(10, 0)$ and moving towards the origin with a speed of 2 units per second. Particle B travels on the y-axis starting at $(0, 12)$ and moving towards the origin with a speed of 3 units per second. What is the rate of change of the distance between the two particles when particle A reaches the point $(4, 0)$?

L6

Find the global maximum and the global minimum for $f(x) = x^3 - 6x^2 + 2$ on the interval $[3, 5]$.

Included Work

Belk, J. (13 April 2014). *Bad Optimization Problems. I thought that Jack M made an interesting comment about this question*. [Comment on the online forum post *Optimization problems that today's students might actually encounter?*]. Stackexchange. https://matheducators.stackexchange.com/questions/1550/optimization-problems-that-todays-students-might-actually-encounter (accessed October 2019 or earlier), 397

'Water Drop' by hunotika is licensed under CC BY 3.0 (accessed 21 July 2021), 373

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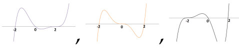
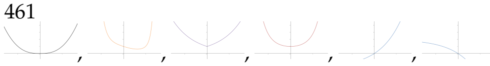
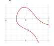
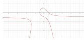
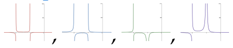
 screenshots of graphs generated using Desmos Graphing Calculator https://www.desmos.com/calculator (accessed 13 November 2015), 461

 screenshots of graphs generated using Desmos Graphing Calculator https://www.desmos.com/calculator, (accessed 13 November 2015), 465

 screenshot of graph using Desmos Graphing Calculator, https://www.desmos.com/calculator (accessed 19 October 2017), 249

 screenshot of graph using Desmos Graphing Calculator, https://www.desmos.com/calculator (accessed 19 October 2017), 249

 screenshots from graphs generated using Desmos Graphing Calculator https://www.desmos.com/calculator (accessed 16 July 2021), 465

 screenshot of graph using Desmos Graphing Calculator, https://www.desmos.com/calculator (accessed 7 July 2021), 317

 screenshot from graphs generated using Desmos Graphing Calculator https://www.desmos.com/calculator, with text added (accessed 13 November 2015), 465

Author unknown. Optimization Problems. (2006). *Drexel University Department of Mathematics, Calculus I Home Page Spring 2006, Calc 1 Spring lecture 6.* https://www.

math.drexel.edu/%7Ejwd25/CALC1_SPRING_06/lectures/lecture9.html (accessed October 2019 or earlier), 397

Vectornaut. (10 May 2015). *When someone swallows a dose of a drug, it doesn't go into their bloodstream all at once.* [Comment on the online forum post *Optimization problems that today's students might actually encounter?*]. Stackexchange. https://matheducators.stackexchange.com/questions/1550/optimization-problems-that-todays-students-might-actually-encounter (accessed October 2019 or earlier), 397

 'Dog' by Vladimir Belochkin is licensed under CC BY 3.0 (accessed 17 June 2021), 25

 'Goose' by Mary B is licensed under CC BY 3.0 (accessed 6 July 2021), 289

 U.S. WHO/NREVSS Collaborating Laboratories and ILNet. 'Stacked Column Chart WHO//NREVSS' Centers for Disease Control and Prevention. No longer available from http://gis.cdc.gov/grasp/fluview/fluportaldashboard.html (accessed 20 October 2015), 329

Alaska Department of Fish and Game, Division of Wildlife Conservation. (April 2007). Wood Bison Restoration in Alaska: A Review of Environmental and Regulatory Issues and Proposed Decisions for Project Implementation, p. 11.

http://www.adfg.alaska.gov/static/species/speciesinfo/woodbison/pdfs/er_no_appendices.pdf (accessed 2015 or 2016), 337

Driver et.al. Stratigraphy, Radiocarbon Dating, and Culture History of Charlie Lake Cave, British Columbia. *ARCTICVOL. 49, no. 3 (September 1996) pp. 265 – 277.* http://pubs.aina.ucalgary.ca/arctic/Arctic49-3-265.pdf (accessed 2015 or 2016), 345

Natasha Deshpande, Anoosha Kumar, Rohini Ramaswami. (2014). The Effect of National Healthcare Expenditure on Life Expectancy, page 12. *College of Liberal Arts - Ivan Allen College (IAC), School of Economics: Econometric Analysis Undergraduate Research Papers.* https://smartech.gatech.edu/handle/1853/51648 (accessed July 2021), 133

Public Domain by Man vyi via https://commons.wikimedia.org/wiki/File:West_Show_Jersey_2010_farrier_f.jpg, accessed October 2015, 337