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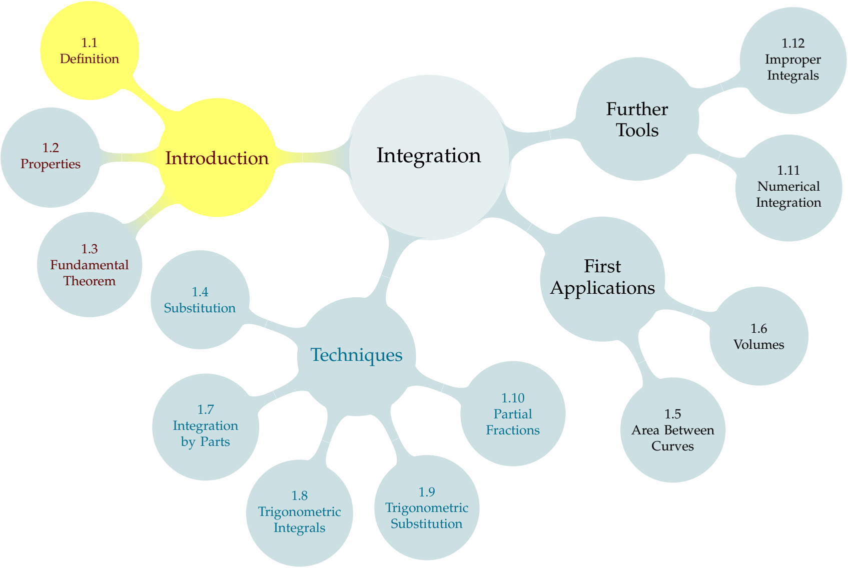


Calculus is build on two operations: differentiation and integration.

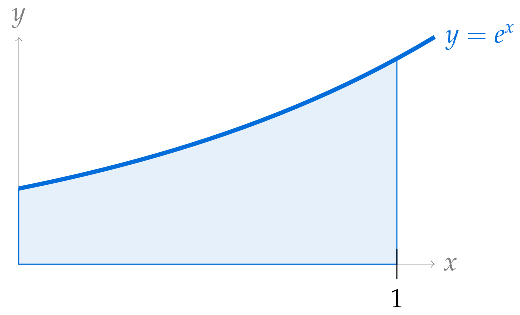
Differentiation

- Slope of a line
- Rate of change
- Optimization
- Numerical Approximations

Integration

- Area under a curve
- “Reverse” of differentiation
- Solving differential equations
- Calculate net change from rate of change
- Volume of solids
- Work (in the physics sense)

Approximate the area of the shaded region using rectangles.



We're going to be doing a lot of adding.

SUMMATION (SIGMA) NOTATION

$$\sum_{i=a}^{b} f(i)$$

- a, b (integers with $a \leq b$) "bounds"
- i "index:" integer which runs from a to b
- $f(i)$ "summands:" compute for every i, add

$$\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + f(a+2) + \cdots + f(b)$$

Sigma Notation

$$\text{Expand } \sum_{i=2}^{4}(2i+5).$$

Sigma Notation

Expand $\displaystyle\sum_{i=1}^{4}(i+(i-1)^2)$.

Q

Write the following expressions in sigma notation:

- $3 + 4 + 5 + 6 + 7$
- $8 + 8 + 8 + 8 + 8$
- $1 + (-2) + 4 + (-8) + 16$

Arithmetic of Summation Notation

Let c be a constant.

- Adding constants: $\sum_{i=1}^{10} c =$
- Factoring constants: $\sum_{i=1}^{10} 5(i^2) =$
- Addition is Commutative: $\sum_{i=1}^{10} (i + i^2) =$

Arithmetic of Summation Notation

Let c be a constant.

- Adding constants: $\sum_{i=1}^{10} c = 10c$
- Factoring constants: $\sum_{i=1}^{10} 5(i^2) = 5\sum_{i=1}^{10}(i^2)$
- Addition is Commutative: $\sum_{i=1}^{10}(i+i^2) = \left(\sum_{i=1}^{10} i\right) + \left(\sum_{i=1}^{10} i^2\right)$

Common Sums

Let $n \geq 1$ be an integer, a be a real number, and $r \neq 1$.

$$\sum_{i=0}^{n} ar^i = a + ar + ar^2 + \cdots + ar^n = a\frac{1-r^{n+1}}{1-r}$$

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Let $n \geq 1$ be an integer, a be a real number, and $r \neq 1$.

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$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Simplify: $\displaystyle\sum_{i=1}^{13}(i^2 + i^3)$

Let $n \geq 1$ be an integer, a be a real number, and $r \neq 1$.

$$\sum_{i=0}^{n} ar^i = a + ar + ar^2 + \cdots + ar^n = a\frac{1-r^{n+1}}{1-r}$$

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Simplify: $\displaystyle\sum_{i=1}^{50}(1 - i^2)$

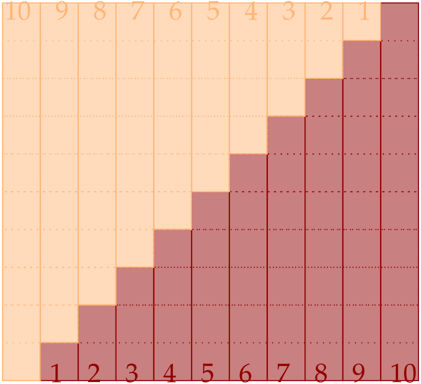
Q

(Optional) Proof of a Common Sum

Here is a derivation of $\displaystyle\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$:

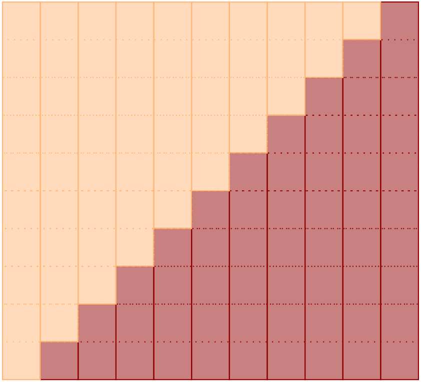
(Optional) Proof of Another Common Sum

$$\sum_{i=1}^{10} i = 1+2+3+4+5+6+7+8+9+10 =$$



(Optional) Proof of a Common Sum

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n =$$



The purpose of these sums is to describe areas.

Notation

The symbol

$$\int_a^b f(x)\,\mathrm{d}x$$

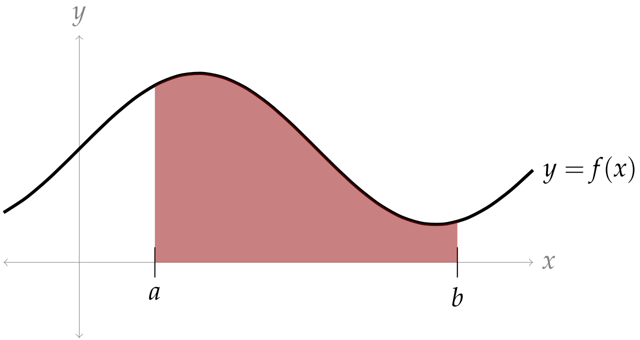
is read "the definite integral of the function $f(x)$ from a to b."

- $f(x)$: integrand
- a and b: limits of integration
- $\mathrm{d}x$: differential

If $f(x) \geq 0$ and $a \leq b$, one interpretation of

$$\int_a^b f(x)\,\mathrm{d}x$$

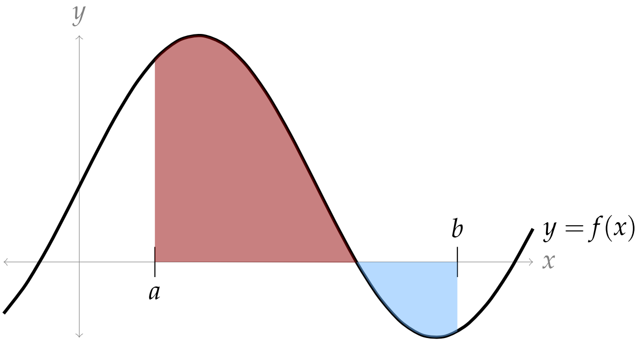
is "the area of the region bounded above by $y = f(x)$, below by $y = 0$, to the left by $x = a$, and to the right by $x = b$."



If ~~$f(x) \geq 0$ and~~ $a \leq b$, one interpretation of

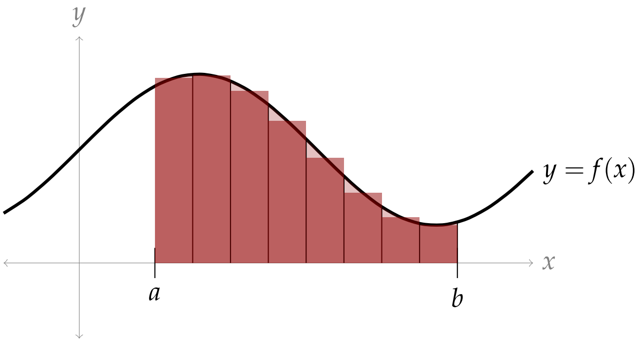
$$\int_a^b f(x)\,\mathrm{d}x$$

is the signed area of the region between $y = f(x)$ and $y = 0$, from $x = a$ to $x = b$. Area above the axis is positive, and area below it is negative.



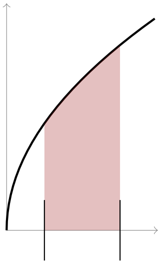
Riemann Sums

A Riemann sum approximates the area under a curve by cutting it into equal-width segments, and approximating each segment as a rectangle.

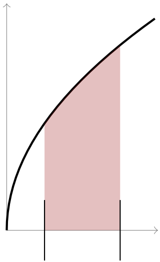


There are different ways to choose the height of each rectangle.

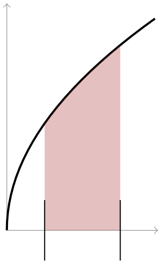
Types of Riemann Sums (RS)



Left RS

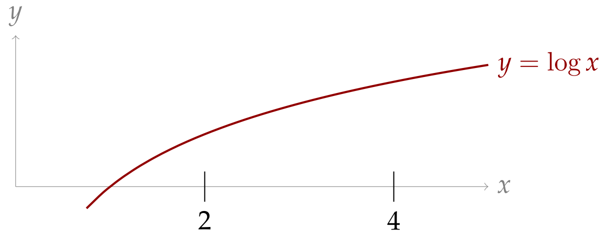


Right RS

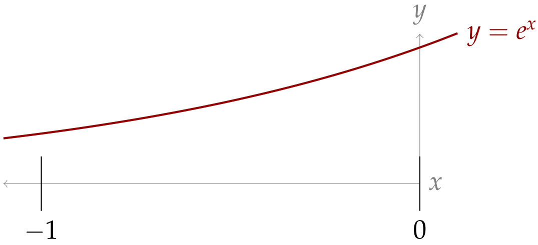


Midpoint RS

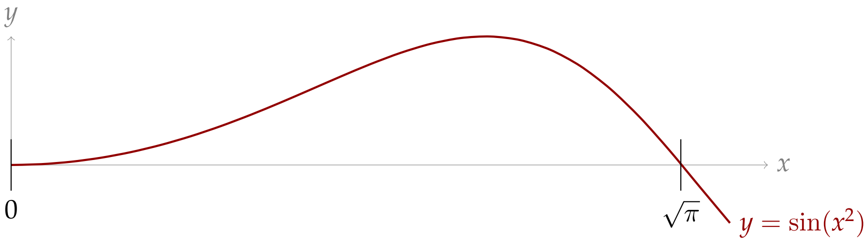
Approximate $\int_2^4 \log(x)\,\mathrm{d}x$ using a right Riemann sum with $n = 4$ rectangles. For now, do not use sigma notation.



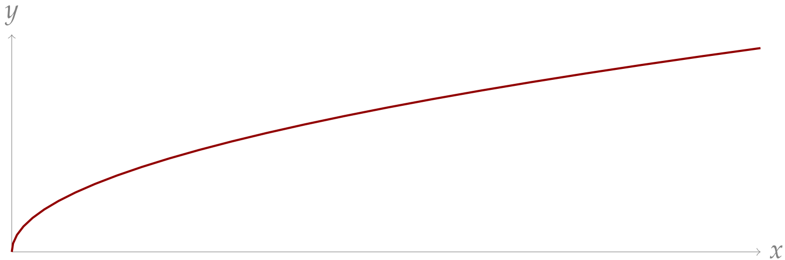
Approximate $\int_{-1}^{0} e^x \, dx$ using a left Riemann sum with $n = 3$ rectangles. For now, do not use sigma notation.



Approximate $\int_0^{\sqrt{\pi}} \sin\left(x^2\right) \mathrm{d}x$ using a midpoint Riemann sum with $n = 5$ rectangles. For now, do not use sigma notation.

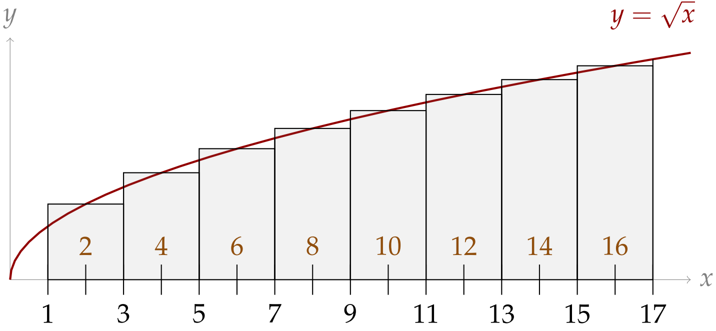


Approximate $\int_1^{17} \sqrt{x}\, \mathrm{d}x$ using a midpoint Riemann sum with 8 rectangles. Write the result in sigma notation.



Q

$$\sum_{i=1}^{8} 2\sqrt{2i} = \underbrace{2\sqrt{2}}_{i=1} + \underbrace{2\sqrt{4}}_{i=2} + \underbrace{2\sqrt{6}}_{i=3} + \underbrace{2\sqrt{8}}_{i=4} + \underbrace{2\sqrt{10}}_{i=5} + \underbrace{2\sqrt{12}}_{i=6} + \underbrace{2\sqrt{14}}_{i=7} + \underbrace{2\sqrt{16}}_{i=8}$$

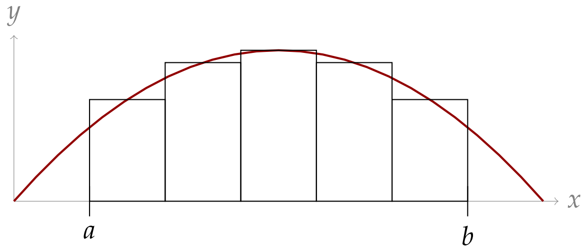


Riemann sum with n rectangles

$$\int_a^b f(x)\,\mathrm{d}x \approx \sum_{i=1}^{n} \Delta x \cdot f(x^*_{i,n})$$

where $\Delta x = \frac{b-a}{n}$ and $x^*_{i,n}$ is an x-value in the ith rectangle.

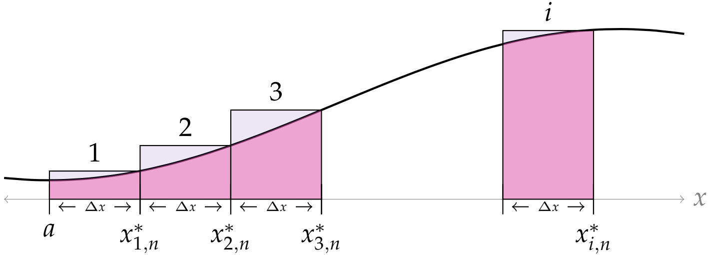
$$\sum_{i=1}^{n} \Delta x \cdot f(x^*_{i,n}) = \Delta x \cdot f\left(x^*_{1,n}\right) + \Delta x \cdot f\left(x^*_{2,n}\right) + \Delta x \cdot f\left(x^*_{3,n}\right) + \cdots + \Delta x \cdot f\left(x^*_{n,n}\right)$$



Right Riemann sum with n rectangles

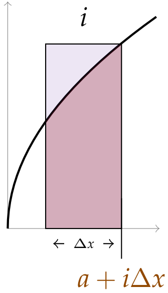
$$\int_a^b f(x)\mathrm{d}x \approx \sum_{i=1}^{n} \Delta x \cdot f\left(\qquad\qquad \right)$$

where $\Delta x = \frac{b-a}{n}$ and $x_{i,n}^* =$

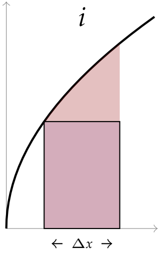


Types of Riemann Sums (RS)

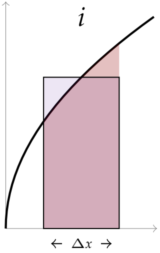
What height would you choose for the ith rectangle?



Right RS



Left RS



Midpoint RS

Riemann sums with n rectangles. Let $\Delta x = \frac{b-a}{n}$

The right Riemann sum approximation of $\int_a^b f(x)\,\mathrm{d}x$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + i\Delta x\right)$$

The left Riemann sum approximation of $\int_a^b f(x)\,\mathrm{d}x$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + (i-1)\Delta x\right)$$

The midpoint Riemann sum approximation of $\int_a^b f(x)\,\mathrm{d}x$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right)$$

Riemann sums with n rectangles: Let $\Delta x = \frac{b-a}{n}$

The right Riemann sum approximation of $\int_a^b f(x)\,\mathrm{d}x$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + i\Delta x\right)$$

Give a right Riemann Sum for the area under the curve $y = x^2 - x$ from $a = 1$ to $b = 6$ using $n = 1000$ intervals.

Riemann sums with n rectangles: Let $\Delta x = \frac{b-a}{n}$

The midpoint Riemann sum approximation of $\int_a^b f(x)\,\mathrm{d}x$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right)$$

Give a midpoint Riemann Sum for the area under the curve $y = 5x - x^2$ from $a = 6$ to $b = 9$ using $n = 1000$ intervals.

Q

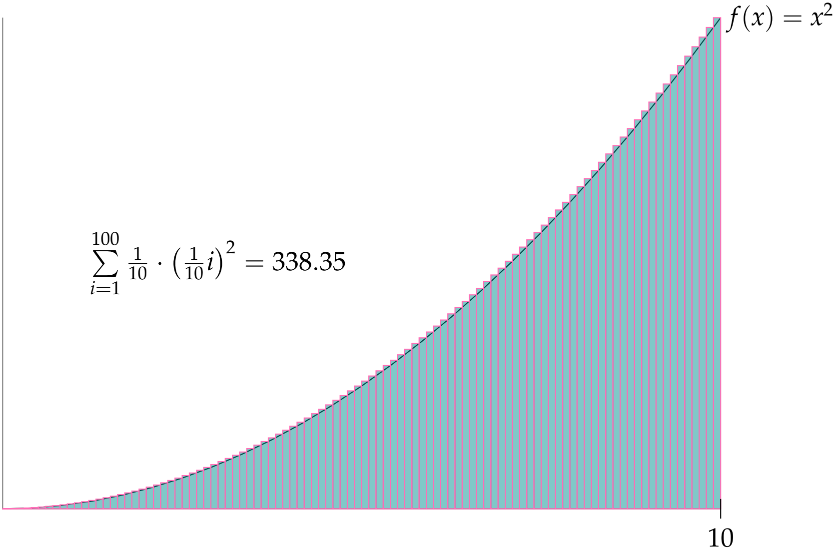
Evaluating Riemann Sums

Skip Riemann evaluations

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Give the right Riemann sum of $f(x) = x^2$ from $a = 0$ to $b = 10$, $n = 100$:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + i\Delta x\right) =$$

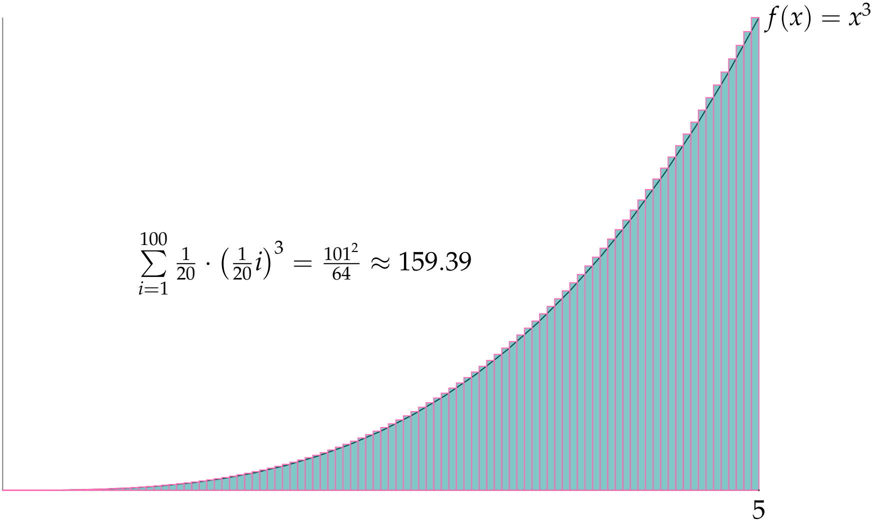


$$\sum_{i=1}^{100} \frac{1}{10} \cdot \left(\frac{1}{10}i\right)^2 = 338.35$$

Evaluating Riemann Sums in Sigma Notation

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Give the right Riemann sum of $f(x) = x^3$ from $a = 0$ to $b = 5$, $n = 100$:



$$\sum_{i=1}^{100} \frac{1}{20} \cdot \left(\frac{1}{20} i\right)^3 = \frac{101^2}{64} \approx 159.39$$

Definition

Let a and b be two real numbers and let $f(x)$ be a function that is defined for all x between a and b. Then we define $\Delta x = \frac{b-a}{N}$ and

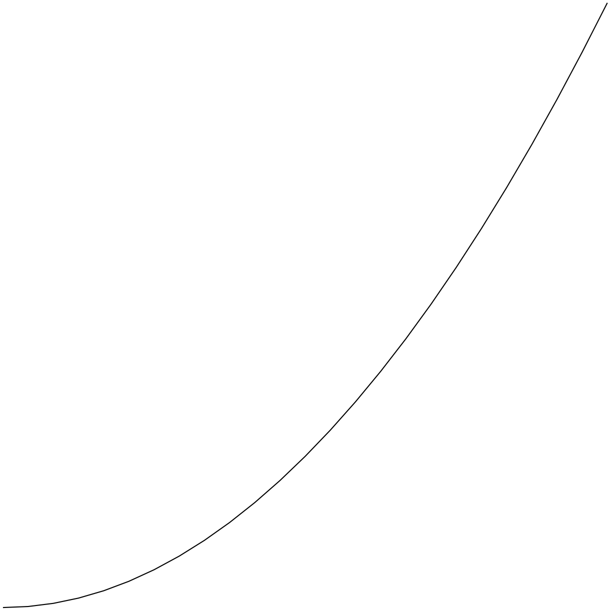
$$\int_a^b f(x)\,\mathrm{d}x = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_{i,N}^*) \cdot \Delta x$$

when the limit exists and when the choice of $x_{i,N}^*$ in the i^{th} interval doesn't matter.

$\sum$, $\int$ both stand for "sum"

Δx, $\mathrm{d}x$ are tiny pieces of the x-axis, the bases of our very skinny rectangles

It's understood we're taking a limit as N goes to infinity, so we don't bother specifying N (or each location where we find our height) in the second notation.



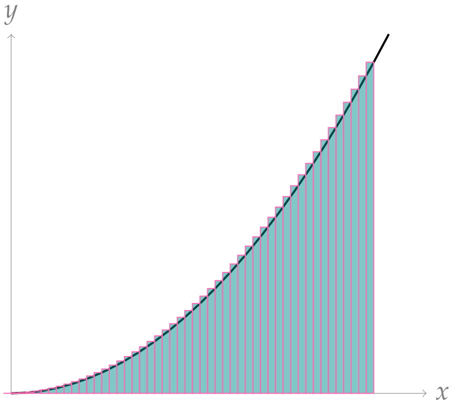
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Give the right Riemann sum of $y = x^2$ from $a = 0$ to $b = 5$ with n slices, and simplify:

We found the right Riemann sum of $y = x^2$ from $a = 0$ to $b = 5$ using n slices:

$$\frac{125}{6} \cdot \frac{2n^2 + 3n + 1}{n^2}$$

Use it to find the exact area under the curve.



REFRESHER: LIMITS OF RATIONAL FUNCTIONS

$$\lim_{n\to\infty} \frac{n^2+2n+15}{3n^2-9n+5} =$$

When the degree of the top and bottom are the same, the limit as n goes to infinity is the ratio of the leading coefficients.

$$\lim_{n\to\infty} \frac{n^2+2n+15}{3n^3-9n+5} =$$

When the degree of the top is smaller than the degree of the bottom, the limit as n goes to infinity is 0.

$$\lim_{n\to\infty} \frac{n^3+2n+15}{3n^2-9n+5} =$$

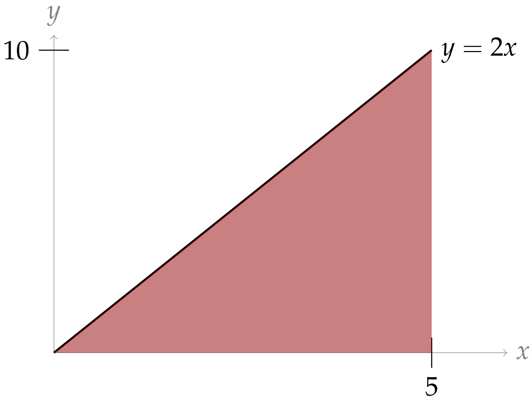
When the degree of the top is larger than the degree of the bottom, the limit as n goes to infinity is positive or negative infinity.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

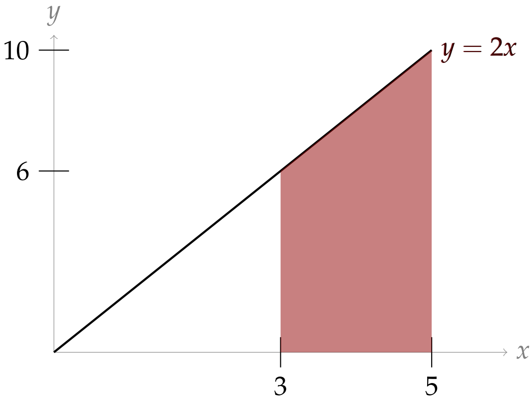
Evaluate $\int_0^1 x^2 \, \mathrm{d}x$ exactly using midpoint Riemann sums.

Let's see some special cases where we can use geometry to evaluate integrals without Riemann sums.

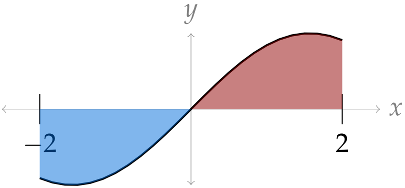
$$\int_0^5 2x\,\mathrm{d}x$$



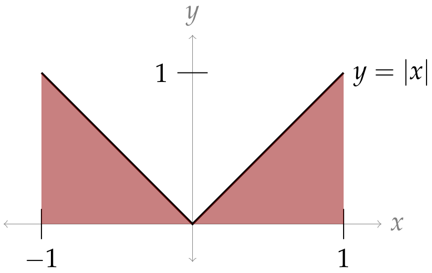
$$\int_3^5 2x\,\mathrm{d}x$$



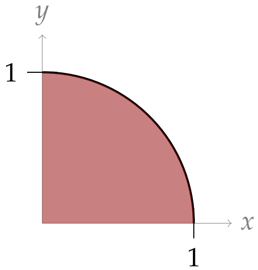
$$\int_{-2}^{2} \sin x \, \mathrm{d}x$$



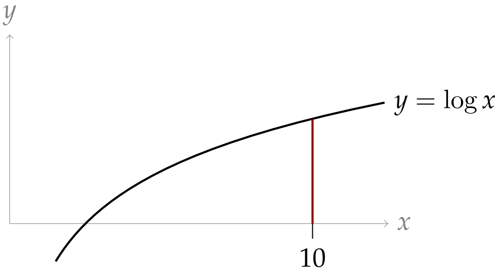
$$\int_{-1}^{1} |x|\, dx$$



$$\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$$



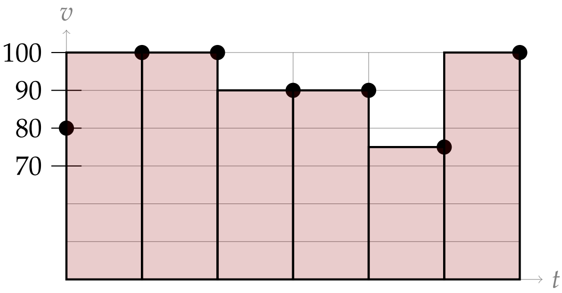
$$\int_{10}^{10} \log x \, dx$$



A car travelling down a straight highway records the following measurements:

Time	12:00	12:10	12:20	12:30	12:40	12:50	1:00
Speed (kph)	80	100	100	90	90	75	100

Approximately how far did the car travel from 12:00 to 1:00?



The computation

$$\text{distance} = \text{rate} \times \text{time}$$

looks a lot like the computation

$$\text{area} = \text{base} \times \text{height}$$

for a rectangle. This gives us another interpretation for an integral.

Another interpretation of the integral

Let $x(t)$ be the position of an object moving along the x-axis at time t, and let $v(t) = x'(t)$ be its velocity. Then for all $b > a$,

$$x(b) - x(a) = \int_a^b v(t)\,\mathrm{d}t$$

That is, $\int_a^b v(t)\,\mathrm{d}t$ gives the *net distance* moved by the object from time a to time b.