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We defined the definite integral as

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{N\to\infty} \sum_{i=1}^{N} \Delta x \cdot f\left(x_{i,N}^*\right)$$

where $\Delta x = \frac{b-a}{N}$ and $x_{i,N}^*$ is a point in the interval $[a + (i-1)\Delta x\,,\ a + i\Delta x]$.

We have seen in previous classes that limits don't always exist. We will verify that this limit does indeed exist, and is equal to the desired area (at least in the most common cases).

We'll start with some general ideas that appear in the proof.

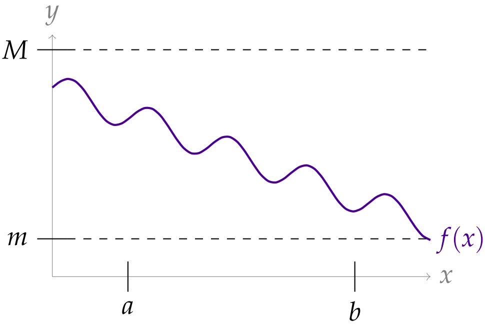
Suppose x and y are both in the interval $[a, b]$. What is the maximum possible value of $|x - y|$?

Proposition 1: distance between two numbers in an interval

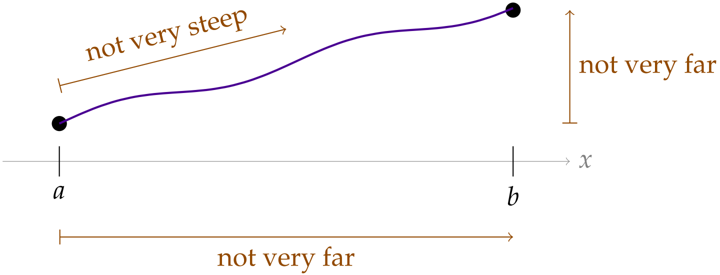
If $a \leq x \leq b$ and $a \leq y \leq b$, then

Proposition 2: area inequality

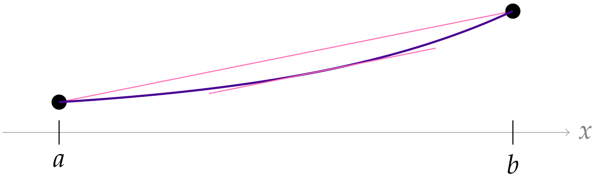
Let $f(x)$ be a function, defined over the interval $[a, b]$. If $m \le f(x) \le M$ over the entire interval $[a, b]$, then the (signed) area between the curve $y = f(x)$ and the x-axis, from a to b, is between $m(b - a)$ and $M(b - a)$.



Intuition: If $f'(x)$ is bounded on (a, b) and $b - a$ is small, then $f(b) - f(a)$ is also small.



The Mean Value Theorem provides a more explicit connection between these quantities.



Mean Value Theorem

Let a and b be real numbers with $a < b$. Let f be a function such that

- $f(x)$ is continuous on the closed interval $a \le x \le b$, and
- $f(x)$ is differentiable on the open interval $a < x < b$.

Then there is a c in (a, b) such that

$$f(b) - f(a) = f'(c)(b - a).$$

Equivalently: $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Triangle Inequality

For any real numbers $x_1,\ x_2,\ \cdots,\ x_n$:

$$\left|\sum_{i=1}^{n} x_i\right| \leq \sum_{i=1}^{n} |x_i|$$

Intuition: If some terms are positive and some are negative, they "cancel each other out" and make the overall sum smaller.

$$|1+2| \qquad |1|+|2|$$

$$|1+(-2)| \qquad |1|+|-2|$$

$$|(-1)+(-2)| \qquad |-1|+|-2|$$

Triangle Inequality

For any real numbers $x_1,\ x_2,\ \cdots,\ x_n$:

$$\left|\sum_{i=1}^{n} x_i\right| \le \sum_{i=1}^{n} |x_i|$$

Proof outline:

REQUIREMENTS

We will consider

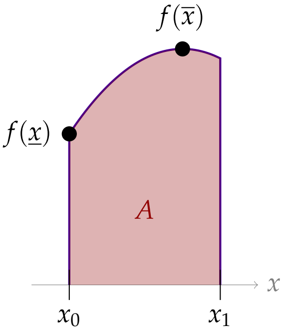
$$\int_a^b f(x)\,\mathrm{d}x$$

where:

- $a < b$
- $f(x)$ is continuous over the interval $[a, b]$
- $f(x)$ is differentiable over the interval (a, b)
- $f'(x)$ is bounded over the interval (a, b). That is, there exists a positive constant number F such that $|f'(x)| \leq F$ for all x in the interval (a, b).

Error in a Single Slice

Consider approximating the area of single slice, from x_0 to x_1.

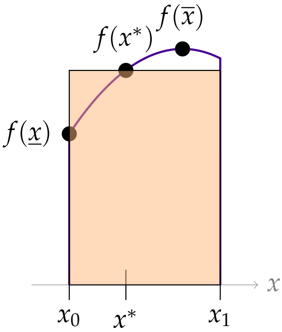


- A is the actual area of the slice
- $f(\overline{x})$ and $f(\underline{x})$ are the largest and smallest function values over the slice
- Our slice has width $x_1 - x_0$

Then we can bound our area:

Error in a Single Slice

Consider approximating the area of single slice, from x_0 to x_1.

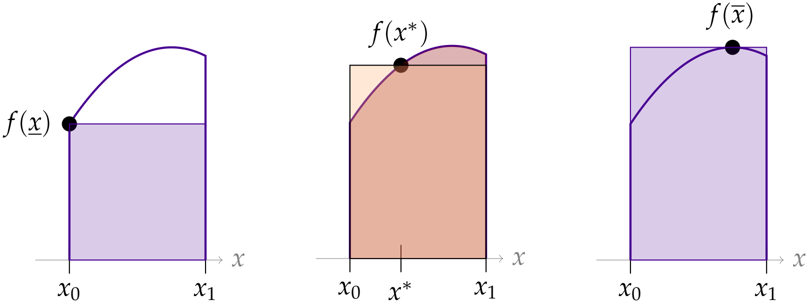


- $f(x^*) \cdot (x_1 - x_0)$ is our approximation of the area of the slice, for some x^* in the interval $[x_0, x_1]$.
- $f(\overline{x})$ and $f(\underline{x})$ are the largest and smallest function values over the slice, so

$$f(\underline{x}) \leq f(x^*) \leq f(\overline{x})$$

Then we can bound our approximation:

Error in a Single Slice



$$f(\underline{x}) \cdot (x_1 - x_0) \quad \leq \quad A \quad \leq \quad f(\overline{x}) \cdot (x_1 - x_0)$$

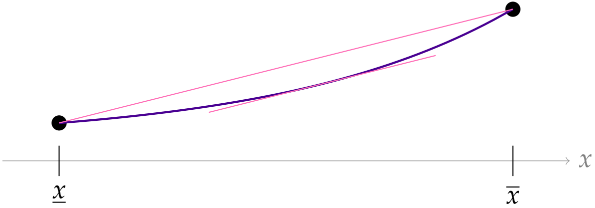
$$f(\underline{x}) \cdot (x_1 - x_0) \quad \leq \quad f(x^*) \cdot (x_1 - x_0) \quad \leq \quad f(\overline{x}) \cdot (x_1 - x_0)$$

$$\underbrace{\left|A - f(x^*) \cdot (x_1 - x_0)\right|}_{\text{error in slice}} \leq$$

using Proposition 1: difference between two numbers in a given interval

Error in a Single Slice

- The error in our single slice is at most $[f(\overline{x}) - f(\underline{x})] \cdot (x_1 - x_0)$
- We want to show that our total error is not too large.
- Intuitively, if $|f'(x)|$ is never very large, and $x_1 - x_0$ is not very large, then $f(\overline{x}) - f(\underline{x})$ is not very large.



Error in a Single Slice

Mean Value Theorem

Let a and b be real numbers with $a < b$. Let f be a function such that

- $f(x)$ is continuous on the closed interval $a \le x \le b$, and
- $f(x)$ is differentiable on the open interval $a < x < b$.

Then there is a c in (a, b) such that

$$f(b) - f(a) = f'(c)(b - a)$$

There exists some c in (x_0, x_1) such that

$$f(\overline{x}) - f(\underline{x}) = f'(c) \cdot (\overline{x} - \underline{x})$$

Since $|f'(x)|$ is never larger than the positive constant F in (a, b),

$$|f(\overline{x}) - f(\underline{x})| \le F \cdot |\overline{x} - \underline{x}| \le F \cdot |x_1 - x_0|$$

Error in a Single Slice

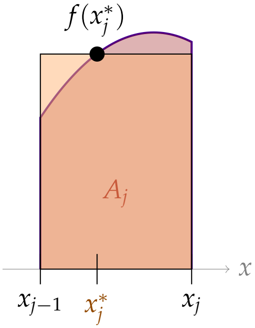
All together,

$$\underbrace{|A - f(x^*) \cdot (x_1 - x_0)|}_{\text{error in slice}} \leq$$

We have shown that the error on a single slice can't be worse than some amount.

Now let's consider adding up slices.

What we did for a single slice, we now do for all slices.
Updated notation for slice j:



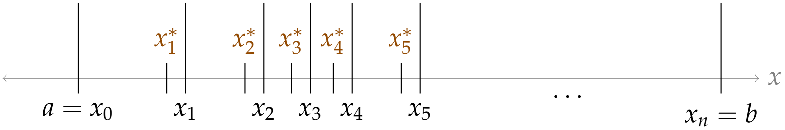
Slice error bound:

$$\left| A_j - f(x_j^*) \cdot (x_j - x_{j-1}) \right| \leq F \cdot (x_j - x_{j-1})^2$$

(Possibly Irregular) Partitions

Consider partitioning the interval $[a, b]$ into n subintervals, not necessarily the same size. Let the points at the edges of the slices be $a = x_0, x_1, x_2, \cdots, x_{n-1}, x_n = b$.

In each part, choose a vertex x_i^* to sample the height of the function.



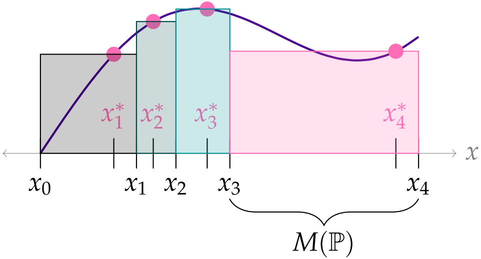
The approximation of $\int_a^b f(x)\,\mathrm{d}x$ depends on how you choose your subintervals, and where you choose your sample points. Let

$$\mathbb{P} = (n, x_1, x_2, \cdots, x_{n-1}, x_1^*, x_2^*, \cdots, x_n^*)$$

denote these choices.

Let $\mathcal{I}(\mathbb{P})$ be the approximation that arises from $\mathbb{P}$:

$$\mathcal{I}(\mathbb{P}) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$



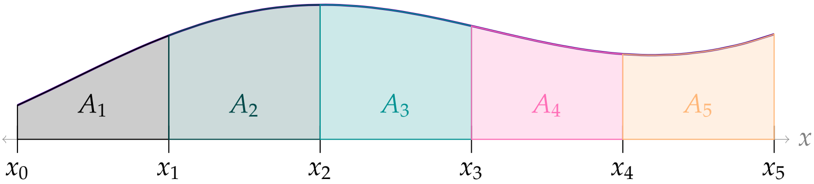
Let $M(\mathbb{P})$ be the maximum width of any subinterval.
If $M(\mathbb{P})$ is small, then *every* subinterval is small (narrow).

Define the integral as the limit

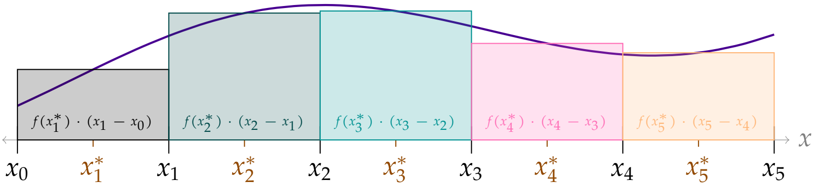
$$\int_a^b f(x)\,\mathrm{d}x = \lim_{M(\mathbb{P})\to 0} \mathcal{I}(\mathbb{P})$$

(Compare to our previous Riemann sum definition.)

We will show that the limit exists and is equal to the signed area under the curve.



Actual area: $\int_a^b f(x)\,\mathrm{d}x = \sum_{i=1}^{n} A_i$



Approximation: $\mathcal{I}(\mathbb{P}) = \sum_{i=1}^{n} f(x_i^*) \cdot (x_i - x_{i-1})$

$$\underbrace{\left|\int_a^b f(x)\,\mathrm{d}x - \mathcal{I}(\mathbb{P})\right|}_{\text{overall error}} = \left|\sum_{i=1}^n A_i \; - \; \sum_{i=1}^n f(x_i^*)\cdot(x_i - x_{i-1})\right|$$

$$= \left|\sum_{i=1}^n \left[A_i - f(x_i^*)\cdot(x_i - x_{i-1})\right]\right|$$

$$0 \quad \leq \quad \underbrace{\left| \int_a^b f(x)\,\mathrm{d}x - \mathcal{I}(\mathbb{P}) \right|}_{\text{overall error}} \quad \leq \quad F \cdot M(\mathbb{P}) \cdot (b-a)$$

$$\lim_{M(\mathbb{P})\to 0} 0 \quad = 0 \qquad\qquad \lim_{M(\mathbb{P})\to 0} \left[F \cdot M(\mathbb{P}) \cdot (b-a)\right] = 0$$

So, by the squeeze theorem,

$$\lim_{M(\mathbb{P})\to 0} \underbrace{\left| \int_a^b f(x)\,\mathrm{d}x - \mathcal{I}(\mathbb{P}) \right|}_{\text{overall error}} = 0$$

That is,

$$\lim_{M(\mathbb{P})\to 0} \mathcal{I}(\mathbb{P}) = \int_a^b f(x)\,\mathrm{d}x$$

Comparing Definitions

Here, we defined

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{M(\mathbb{P})\to 0} \mathcal{I}(\mathbb{P})$$

for "nice" functions $f(x)$.
Originally, we used a slightly different definition:

Definition 1.1.9 (abridged)

For "nice" functions $f(x)$:

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n}^*) \cdot \frac{b-a}{n}$$

when the limit exists and takes the same value for all choices of the $x_{i,n}^*$'s.

Comparing Definitions

We showed that all families of partitions "work," as long as their largest subintervals shrink to length 0.

If all families of partitions "work," then we might as well choose a simple one. The (arguably) simplest choices are regular partitions, cutting the interval $[a, b]$ into n subintervals of length $\frac{b-a}{n}$.