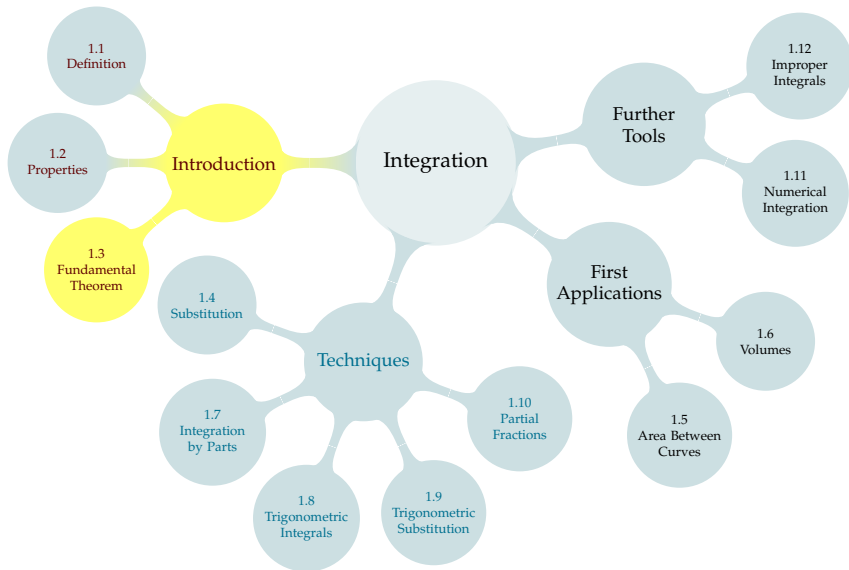


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REVIEW: AREA UNDER A CURVE

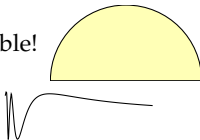
Methods for finding the area under a curve.

- Limit of a Riemann Sum
 - Conceptually easy – cut into rectangles
 - Computationally rough $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)$



$$\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x; \quad \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

- Use Geometry
 - Computationally nice when it's available! (Circles, triangles, symmetry, etc.)
 - Often not available – most functions don't make such nice shapes.



- ▶ Up next: Fundamental Theorem of Calculus
 - ▶ **Conceptually** less obvious – we'll spend about a day explaining why it works
 - ▶ **Computationally** generally nicer than Riemann sums
 - ▶ Doesn't work for every function

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

$$A(x) = \int_a^x f(t) \, dt$$

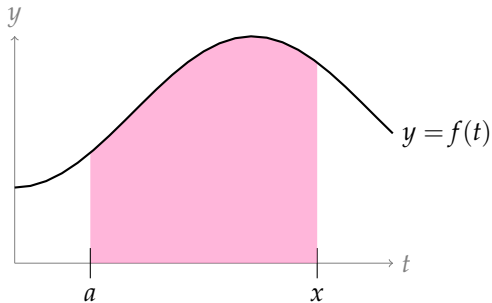
for any x in $[a, b]$. Then the function $A(x)$ is differentiable and

$$A'(x) = f(x) \text{ .}$$

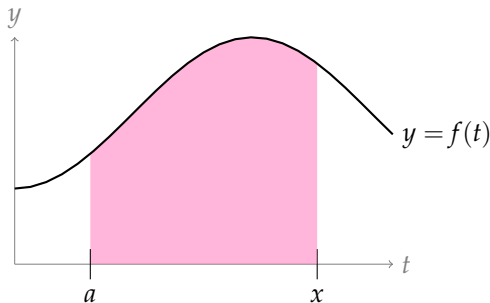
FTC(I) gives us the derivative of a very specific function (subject to some fine print).

It shows a close relationship between integrals and derivatives.

AREA FUNCTION: $A(x) = \int_a^x f(t)dt$ FOR $a \leq x \leq b$



AREA FUNCTION: $A(x) = \int_a^x f(t)dt$ FOR $a \leq x \leq b$



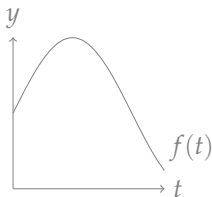
Notation: the function A depends on the variable x .

We need to know how the function f behaves on the whole interval $(0, x)$ to find $A(x)$. That's why we use $f(t)$, not $f(x)$.

It might look strange at first to see two different variables. Let's consider the alternatives:

$$C(x) = \int_0^x f(x) \, dx$$

$$C(\mathbf{1}) = \int_0^{\mathbf{1}} f(\mathbf{1}) \underbrace{d\mathbf{1}}_{??}$$



Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

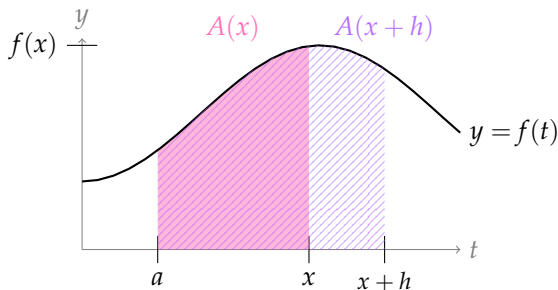
$$A(x) = \int_a^x f(t) \, dt$$

for any x in $[a, b]$. Then the function $A(x)$ is differentiable and

$$A'(x) = f(x) .$$

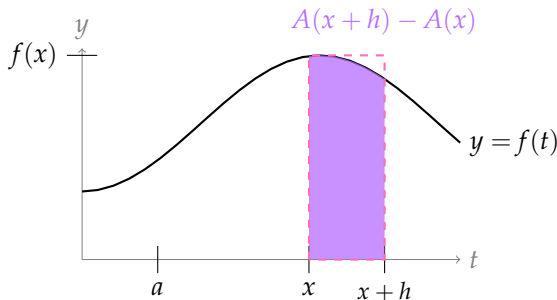
Question: Why is it true?

DERIVATIVE OF AREA FUNCTION, $A(x) = \int_a^x f(t)dt$



$$A'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta A}{\Delta x} = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \rightarrow 0} \frac{hf(x)}{h} = f(x)$$

DERIVATIVE OF AREA FUNCTION, $A(x) = \int_a^x f(t)dt$



$$A'(x) = \lim_{\Delta x \rightarrow 0} \frac{\Delta A}{\Delta x} = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \rightarrow 0} \frac{hf(x)}{h} = f(x)$$

When h is very small, the purple area looks like a rectangle with base h and height $f(x)$, so $A(x+h) - A(x) \approx hf(x)$ and $\frac{A(x+h) - A(x)}{h} \approx f(x)$. As h tends to zero, the error in this approximation approaches 0.

Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

$$A(x) = \int_a^x f(t) \, dt$$

for any x in $[a, b]$. Then the function $A(x)$ is differentiable and

$$A'(x) = f(x) .$$

Suppose $A(x) = \int_2^x \sin t \, dt$. What is $A'(x)$?

Suppose $B(x) = \int_x^2 \sin t \, dt$. What is $B'(x)$?

Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

$$A(x) = \int_a^x f(t) \, dt$$

for any x in $[a, b]$. Then the function $A(x)$ is differentiable and

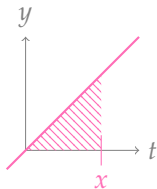
$$A'(x) = f(x) .$$

Suppose $C(x) = \int_2^{e^x} \sin t \, dt$. What is $C'(x)$?



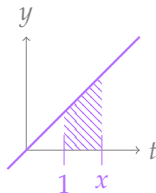
It's possible to have two different functions with the same derivative.

$$A(x) = \int_0^x 2t \, dt = x^2$$



$$A'(x) = 2x$$

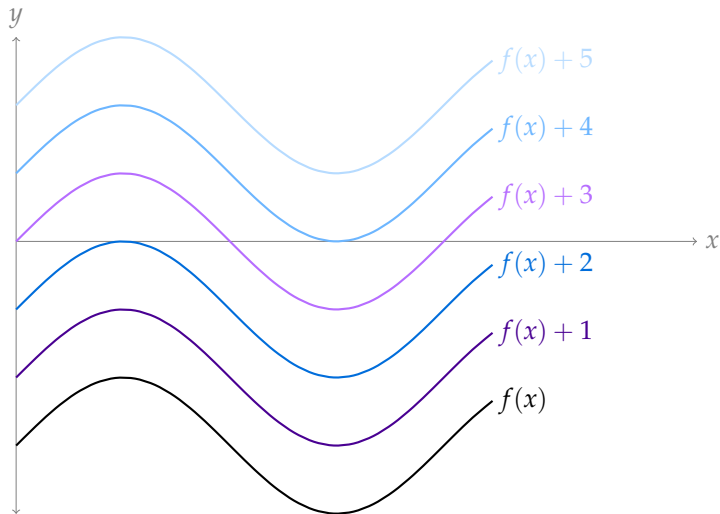
$$B(x) = \int_1^x 2t \, dt = x^2 - 1$$



$$B'(x) = 2x$$

When two functions have the same derivative, **they differ only by a constant.**

In this example: $B(x) = A(x) - 1$



If two continuous functions have the same derivative, then one is a constant plus the other.

Two clues for finding $A(x) = \int_a^x f(t) \, dt$:

- ▶ If $A(x) = \int_a^x f(t) \, dt$, then¹ $A'(x) = f(x)$
- ▶ If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C .

$A(x) = \int_a^x e^t \, dt$. What functions could $A(x)$ be?

¹(as long as $f(t)$ is continuous on $[a, x]$)

Two clues for finding $A(x) = \int_a^x f(t) dt$:

- ▶ If $A(x) = \int_a^x f(t) dt$, then¹ $A'(x) = f(x)$
- ▶ If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C .

$A(x) = \int_a^x \cos t dt$. What functions could $A(x)$ be?

¹(as long as $f(t)$ is continuous on $[a, x]$)

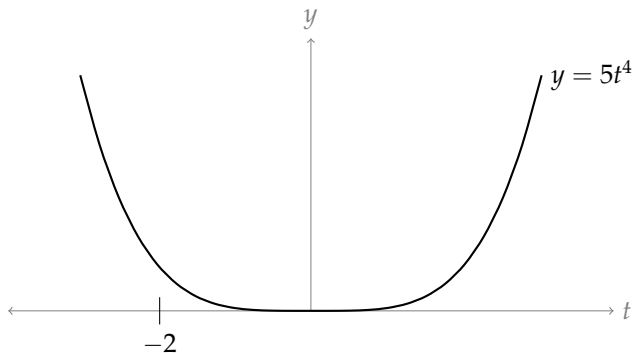
Two clues for finding $A(x) = \int_a^x f(t) \, dt$:

- ▶ If $A(x) = \int_a^x f(t) \, dt$, then¹ $A'(x) = f(x)$
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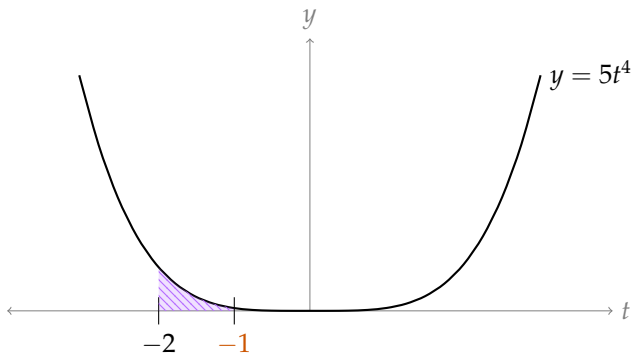
$A(x) = \int_{-2}^x 5t^4 \, dt$. What functions could $A(x)$ be?

¹(as long as $f(t)$ is continuous on $[a, x]$)

$$A(x) = \int_{-2}^x 5t^4 dt = x^5 + 32$$

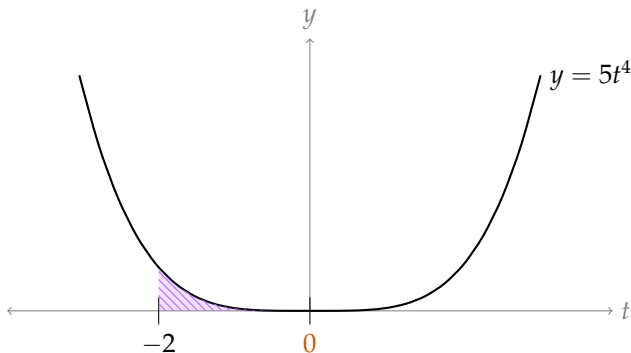


$$A(x) = \int_{-2}^x 5t^4 dt = x^5 + 32$$



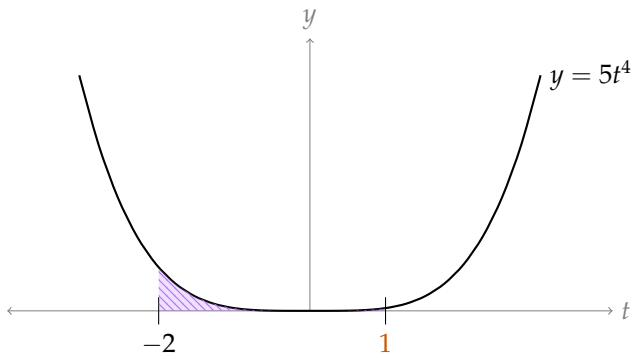
$$A(-1) = \int_{-2}^{-1} 5t^4 dt = (-1)^5 + 32 = 31$$

$$A(x) = \int_{-2}^x 5t^4 \, dt = x^5 + 32$$



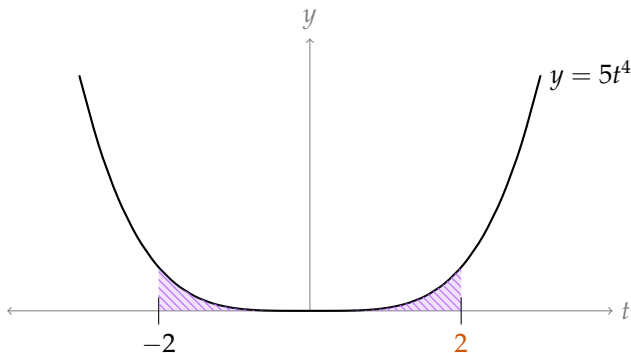
$$A(0) = \int_{-2}^0 5t^4 \, dt = (0)^5 + 32 = 32$$

$$A(x) = \int_{-2}^x 5t^4 dt = x^5 + 32$$



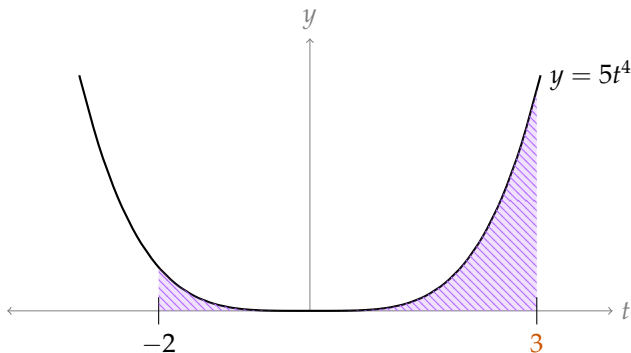
$$A(\textcolor{brown}{1}) = \int_{-2}^{\textcolor{brown}{1}} 5t^4 dt = (\textcolor{brown}{1})^5 + 32 = 33$$

$$A(x) = \int_{-2}^x 5t^4 \, dt = x^5 + 32$$



$$A(2) = \int_{-2}^2 5t^4 \, dt = (2)^5 + 32 = 64$$

$$A(x) = \int_{-2}^x 5t^4 \, dt = x^5 + 32$$



$$A(3) = \int_{-2}^3 5t^4 \, dt = (3)^5 + 32 = 275$$

Two clues for finding $A(x) = \int_a^x f(t) \, dt$:

- ▶ If $A(x) = \int_a^x f(t) \, dt$, then¹ $A'(x) = f(x)$
- ▶ If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C .

$A(x) = \int_a^x f(t) \, dt$. What functions could $A(x)$ be?

¹(as long as $f(t)$ is continuous on $[a, x]$)

Two clues for finding $A(x) = \int_a^x f(t) dt$:

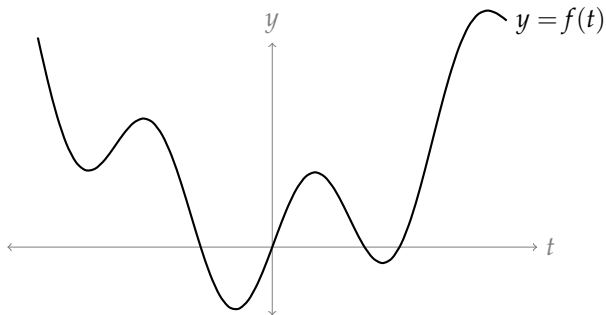
- ▶ If $A(x) = \int_a^x f(t) dt$, then¹ $A'(x) = f(x)$
- ▶ If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C .

$A(b) = \int_a^b f(t) dt$. What functions could $A(b)$ be?

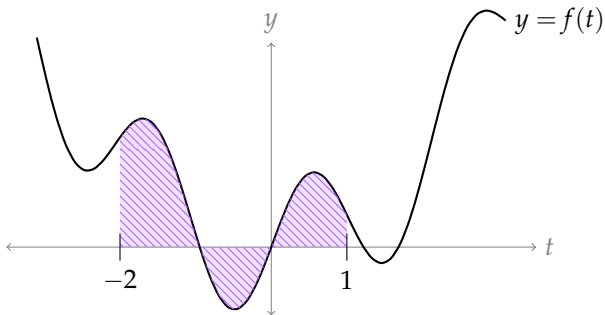
- ▶ $A'(x) = f(x)$.
- ▶ Guess a function with derivative $f(x)$: $F(x)$.
- ▶ Then $A(x) = F(x) + C$ for some constant C .
- ▶ Also $A(a) = 0$, so $0 = F(a) + C$, so $C = -F(a)$
- ▶ So, $A(b) = F(b) - F(a)$

¹(as long as $f(t)$ is continuous on $[a, x]$)

$$\int_a^b f(t) \, dt = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$

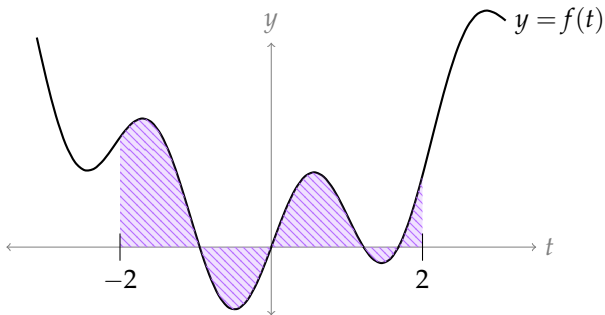


$$\int_a^b f(t) \, dt = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$



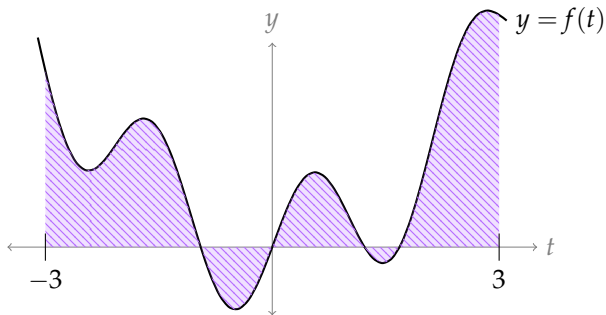
$$\int_{-2}^1 f(t) \, dt = F(1) - F(-2)$$

$$\int_a^b f(t) \, dt = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$



$$\int_{-2}^2 f(t) \, dt = F(2) - F(-2)$$

$$\int_a^b f(t) \, dt = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$



$$\int_{-3}^3 f(t) \, dt = F(3) - F(-3)$$

Fundamental Theorem of Calculus, Part 2

Let $F(x)$ be differentiable, defined, and continuous on the interval $[a, b]$ with $F'(x) = f(x)$ for all $a < x < b$. Then

$$\int_a^b f(x) \, dx = F(b) - F(a)$$

Examples:

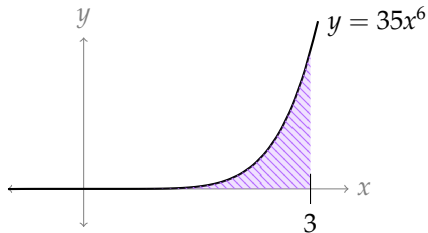
$$\frac{d}{dx} \{5x^7\} = 35x^6, \text{ so}$$

$$\int_0^3 35x^6 \, dx =$$

$$\frac{d}{dx} \{\tan x\} = \sec^2 x, \text{ so}$$

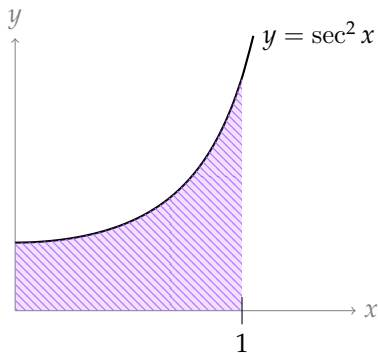
$$\int_0^{\pi/4} \sec^2 x \, dx =$$

$$\int_0^3 35x^6 \, dx = F(b) - F(a) \quad \text{where} \quad F(x) = 5x^7$$



$$\int_0^3 35x^6 \, dx = 5(3)^7 - 5(0)^7$$

$$\int_0^{\pi/4} \sec^2 x \, dx = F(b) - F(a) \quad \text{where} \quad F(x) = \tan x$$



$$\int_0^{\pi/4} \sec^2 x \, dx = \tan\left(\frac{\pi}{4}\right) - \tan 0 = 1$$

RELEVANT VOCABULARY

Definition

If $F(x)$ is a function whose derivative is $f(x)$, we call $F(x)$ an **antiderivative** of $f(x)$.

Examples:

The derivative of x^2 is $2x$, so:

x^2 is an **antiderivative** of $2x$.

When $x > 0$, the derivative of $\log x$ is $\frac{1}{x}$, so:

For all x , the derivative of $\log |x|$ is $\frac{1}{x}$, so:

An antiderivative of $\sin x$ is

CONVENIENT NOTATION

Definition

$$f(x) \Big|_a^b = f(b) - f(a)$$

The function $f(x)$ evaluated from a to b

Examples:

$$(5x + x^2) \Big|_1^2 =$$

$$\frac{x^2}{x+2} \Big|_5^{-1} =$$

FTC Part 2, Abridged

$$\int_a^b f(x) \, dx = F(x) \Big|_a^b$$

where $F(x)$ is an antiderivative of $f(x)$

Definition

The **indefinite integral** of a function $f(x)$:

$$\int f(x) \, dx$$

means the *most general* antiderivative of $f(x)$.

Examples:

$$\int 2x \, dx =$$

$$\int \frac{1}{x} \, dx =$$

Remember: two functions with the same derivative differ by a constant, so we include the “ $+C$ ” for indefinite integrals.

DEFINITE VS INDEFINITE INTEGRALS

For each pair of properties below, decide which applies to **definite** integrals, and which to **indefinite** integrals.

No limits (or bounds) of integration, $\int f(x) \, dx$	
Limits (or bounds) of integration, $\int_a^b f(x) \, dx$	
Area under a curve	
Antiderivative	
Number	
Function	

ANTIDIFFERENTIATION BY INSPECTION

1. $\int e^x \, dx$

2. $\int \cos x \, dx$

3. $\int -\sin x \, dx$

4. $\int \frac{1}{x} \, dx$

5. $\int 1 \, dx$

6. $\int 2x \, dx$

7. $\int nx^{n-1} \, dx \quad (n \neq 0, \text{ constant})$

8. $\int x^n \, dx \quad (n \neq -1, \text{ constant})$



Power Rule for Antidifferentiation

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

if $n \neq -1$ is a constant

Example:

$$\int (5x^2 - 15x + 3) dx =$$



ANTIDERIVATIVES TO RECOGNIZE

- ▶ $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$
- ▶ $\int a dx = ax + C$
- ▶ $\int e^x dx = e^x + C$
- ▶ $\int \frac{1}{x} dx = \log |x| + C$
- ▶ $\int \sin x dx = -\cos x + C$
- ▶ $\int \cos x dx = \sin x + C$
- ▶ $\int \sec^2 x dx = \tan x + C$
- ▶ $\int \sec x \tan x dx = \sec x + C$
- ▶ $\int \csc x \cot x dx = -\csc x + C$
- ▶ $\int \csc^2 x dx = -\cot x + C$
- ▶ $\int \frac{1}{1+x^2} dx = \arctan x + C$
- ▶ $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$