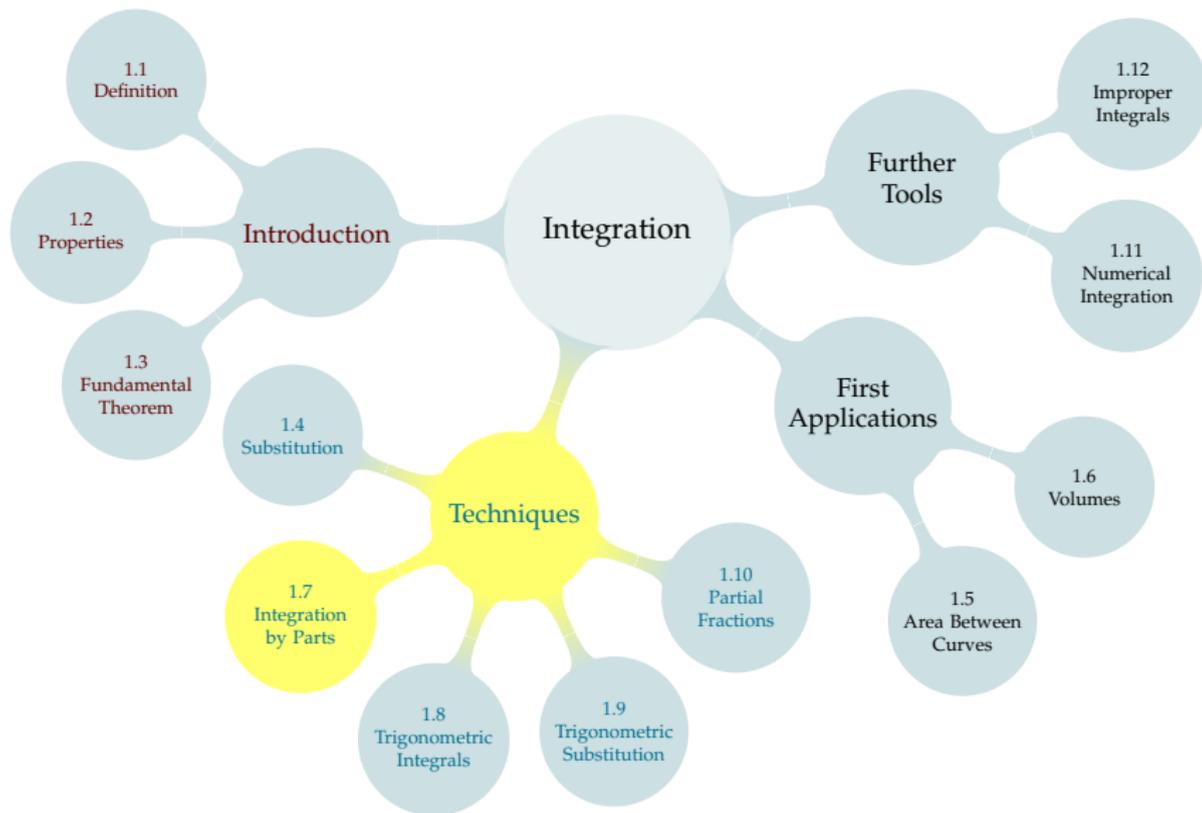


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REVERSE THE PRODUCT RULE

Product Rule:

$$\frac{d}{dx} \{u(x) \cdot v(x)\} = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

Related fact:

$$\int [u'(x) \cdot v(x) + u(x) \cdot v'(x)] dx = u(x) \cdot v(x) + C$$

Rearrange:

$$\implies \int [u'(x)v(x)] dx + \int [u(x)v'(x)] dx = u(x)v(x) + C$$

$$\implies \int [u(x)v'(x)] dx = u(x)v(x) - \int [v(x)u'(x)] dx + C$$

INTEGRATION BY PARTS

$$\int [u(x)v'(x)] dx = u(x)v(x) - \int [v(x)u'(x)] dx$$

Example: $\int xe^x dx$

CHECK OUR WORK

In the previous slide, we evaluated

$$\int xe^x dx = xe^x - e^x + C$$

for some constant C . We can check that this is correct by differentiating.

INTEGRATION BY PARTS (IBP): A CLOSER LOOK

$$\int [u(x)v'(x)] dx = u(x)v(x) - \int [v(x)u'(x)] dx + C$$
$$\underbrace{\int xe^x dx}_{\text{How to integrate??}} = x(e^x) - \underbrace{1 \int e^x dx}_{\text{Easy to integrate!}} + C$$

We start and end with an integral. IBP is only useful if the new integral is somehow an improvement.

We **differentiate** the function we choose as $u(x)$, and **antidifferentiate** the function we choose as $v'(x)$

CHOOSING $u(x)$ AND $v(x)$

$$\int [u(x)v'(x)] dx = u(x)v(x) - \int [v(x)u'(x)] dx + C$$
$$\int [x \sin x] dx =$$

Option A:

$$\begin{array}{l} u(x) = x \\ v'(x) = \sin x \end{array} \quad |$$

Option B:

$$\begin{array}{l} u(x) = \sin x \\ v'(x) = x \end{array} \quad |$$

CHECK OUR WORK

To check our work, we can calculate $\frac{d}{dx} \left\{ -x \cos x + \sin x + C \right\}$. It should work out to be $x \sin x$.

CHOOSING $u(x)$ AND $v(x)$

$$\int [u(x)v'(x)] dx = u(x)v(x) - \int [v(x)u'(x)] dx + C$$
$$\int [x^2 \log x] dx =$$

Option A:

$$\begin{array}{l} u(x) = x^2 \\ v'(x) = \log x \end{array} \quad |$$

Option B:

$$\begin{array}{l} u(x) = \log x \\ v'(x) = x^2 \end{array} \quad |$$



CHECK OUR WORK

To check our work, we can calculate $\frac{d}{dx} \left\{ \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C \right\}$. It should work out to be $x^2 \log x$.

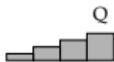
$$\int [u(x)v'(x)] dx = u(x)v(x) - \int [v(x)u'(x)] dx + C$$
$$\int \left[\frac{1}{2}xe^{6x} \right] dx =$$

Option A:

$$\begin{array}{l} u(x) = \frac{1}{2}x \\ v'(x) = e^{6x} \end{array} \quad \left| \right.$$

Option B:

$$\begin{array}{l} u(x) = e^{6x} \\ v'(x) = \frac{1}{2}x \end{array} \quad \left| \right.$$



CHECK OUR WORK

We check that $\int \left[\frac{1}{2} x e^{6x} \right] dx = \frac{1}{12} x e^{6x} - \frac{1}{72} e^{6x} + C$ by differentiating.

MNEMONIC

$$\int [u(x)v'(x)] dx = u(x)v(x) - \int [v(x)u'(x)] dx + C$$
$$\int u dv = uv - \int v du + C$$

We abbreviate:

- ▶ $u(x) \rightarrow u$
- ▶ $u'(x) dx \rightarrow du$
- ▶ $v(x) \rightarrow v$
- ▶ $v'(x) dx \rightarrow dv$

CHOOSING u , dv IN YOUR HEAD

Choose u and dv to evaluate the integral below:

$$\int (3t + 5) \cos(t/4) dt$$

Thoughts: $\int u dv = uv - \int v du$

u gets differentiated, and dv gets antidifferentiated.

Evaluate, using IBP or Substitution

$$\int u dv = uv - \int v du + C$$

▶ $\int xe^{x^2} dx$

▶ $\int x^2 e^x dx$

▶ $\int e^{x+e^x} dx$



DEFINITE INTEGRATION BY PARTS

Method 1: Antidifferentiate first, then plug in limits of integration.

Method 2: Plug as you go.

Evaluate $\int_1^e \log^2 x \, dx$

SPECIAL TECHNIQUE: $v'(x) = 1$

$$\int u \, dv = uv - \int v \, du + C$$

Evaluate $\int \log x \, dx$ using integration by parts.

CHECK OUR WORK

Let's check that $\int \log x \, dx = x \log x - x + C$.

$$\int u \, dv = uv - \int v \, du + C$$

Evaluate $\int \arctan x \, dx$ using integration by parts.

CHECK OUR WORK

Let's check that $\int \arctan x \, dx = x \arctan x - \frac{1}{2} \log |1 + x^2| + C$.

Setting $dv = 1 dx$ is a very specific technique. You'll probably only see it in situations integrating logarithms and inverse trigonometric functions.

$$\int \log x dx, \quad \int \arcsin x dx, \quad \int \arccos x dx, \quad \int \arctan x dx, \quad \text{etc.}$$

Evaluate $\int e^x \cos x \, dx$ using integration by parts.

Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$ and $v = \sin x$:

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Let $u = e^x$ and $dv = \sin x \, dx$. Then $du = e^x \, dx$ and $v = -\cos x$:

$$= e^x \sin x - \left[-e^x \cos x - \int -e^x \cos x \, dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x \, dx = \frac{1}{2} (e^x \sin x + e^x \cos x) + C$$

INTEGRATING AROUND IN A CIRCLE

We can use this technique to antidifferentiate products of two functions that almost, but don't quite, stay the same under (anti)differentiation.

Use integration by parts a number of times, ending up with an expression involving (a scalar multiple of) the original integral.

To do this, **be consistent** with your choice of u and dv .

Evaluate $\int \cos(\log x) dx$.



CHECK OUR WORK

We check that $\int \cos(\log x) dx = \frac{x}{2} [\cos(\log x) + \sin(\log x)] + C$ by differentiating.

Evaluate $\int e^{2x} \sin x \, dx$ using integration by parts.



CHECK OUR WORK

We can check our work by differentiating $\frac{1}{5}e^{2x}[2 \sin x - \cos x] + C$.
We should end up with $e^{2x} \sin x$.