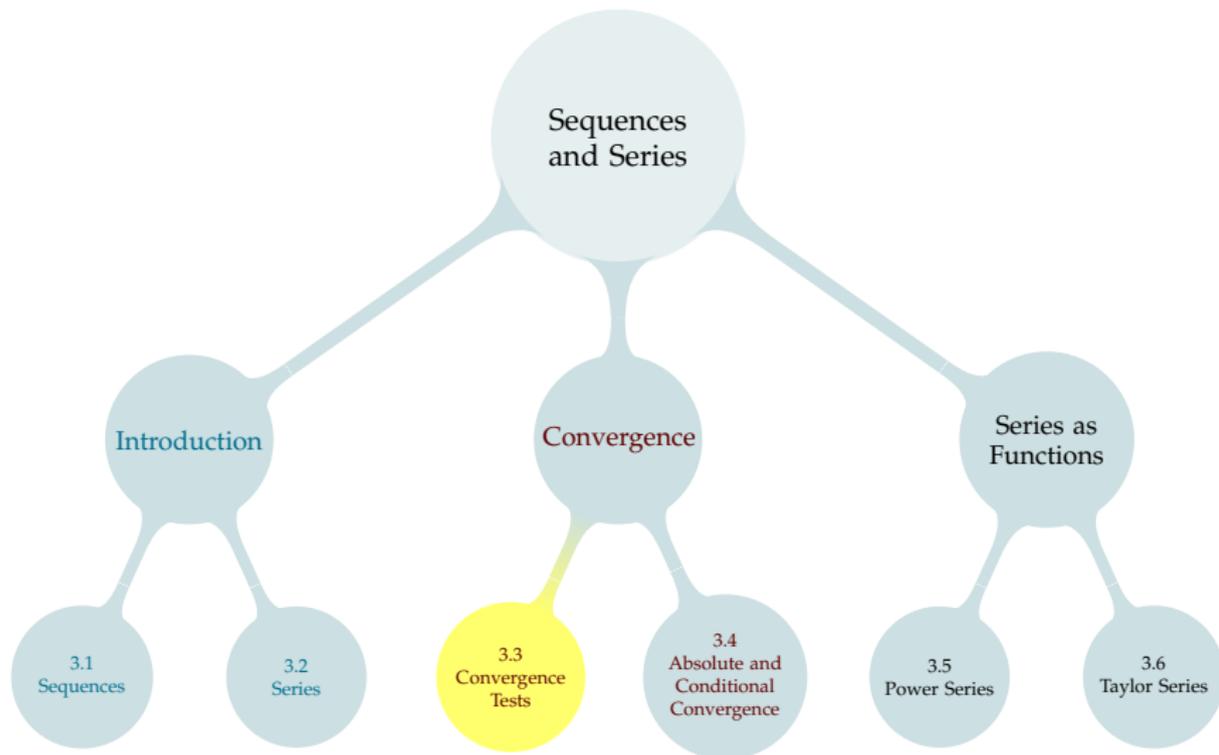


TABLE OF CONTENTS



REVIEW

$$\text{Let } S_N = \sum_{n=1}^N a_n.$$

Simplify: $S_N - S_{N-1}$.

(This will come in handy soon.)

ALTERNATING SERIES

Alternating Series

The series

$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} A_n$$

is alternating if every $A_n \geq 0$.

Alternating series:

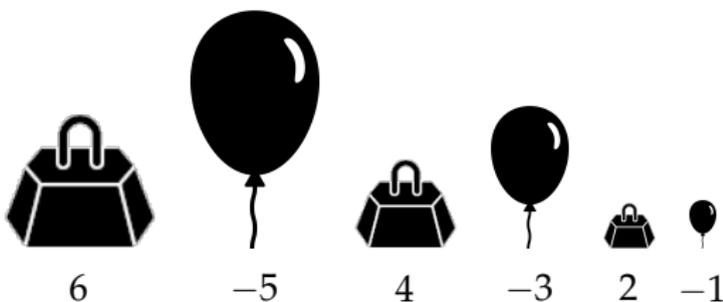
▶ $1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + \cdots$

▶ $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

Not alternating:

▶ $\cos(1) + \cos(2) + \cos(3) + \cdots$

▶ $1 - \left(-\frac{1}{2}\right) + \frac{1}{3} - \left(-\frac{1}{4}\right) + \cdots$



$$S_1 = 6.0000$$

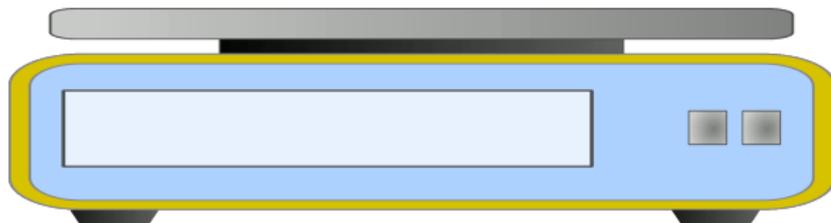
$$S_2 = 1.0000$$

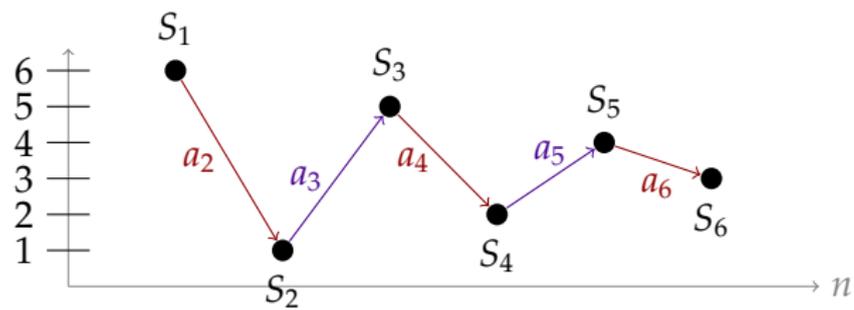
$$S_3 = 5.0000$$

$$S_4 = 2.0000$$

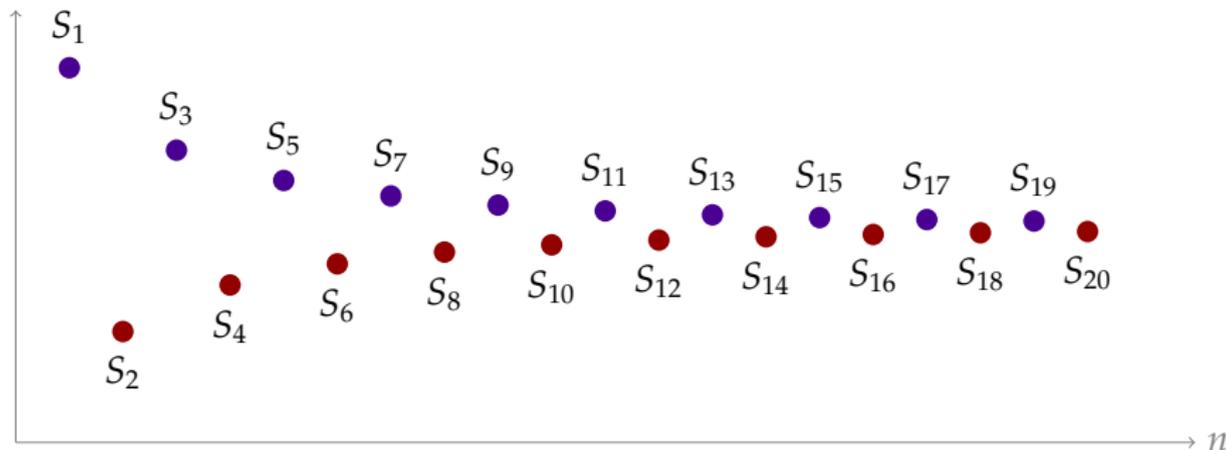
$$S_5 = 4.0000$$

$$S_6 = 3.0000$$

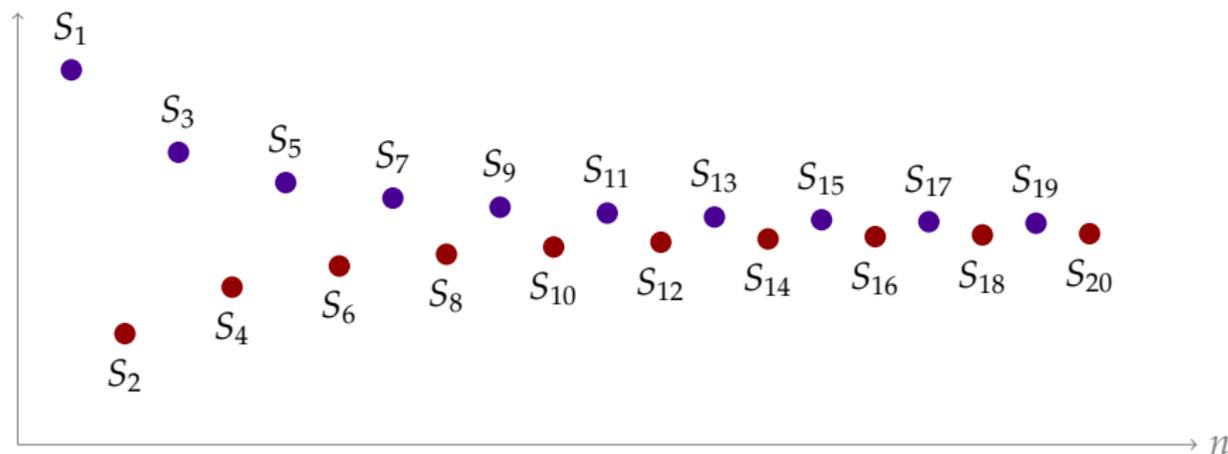




Consider an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $\{a_n\}$ is a sequence with positive, **decreasing** terms and with $\lim_{n \rightarrow \infty} a_n = 0$.



Since $a_2 > a_3$, we have $a_1 - (a_2 - a_3) < a_1$, so $S_3 < S_1$.



- ▶ For all $n \geq 2$, S_n lies between S_1 and S_2 .
- ▶ For all $n \geq 3$, S_n lies between S_2 and S_3 .
- ▶ For all $n \geq 4$, S_n lies between S_3 and S_4 .
- ▶ For all $n \geq 5$, S_n lies between S_4 and S_5 .

The difference between consecutive sums S_n and S_{n-1} is:

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

- (i) $a_n \geq 0$ for all $n \geq 1$;
- (ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);
- (iii) and $\lim_{n \rightarrow \infty} a_n = 0$.

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number N , $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the N^{th} partial sum $\sum_{n=1}^N (-1)^{n-1} a_n$.

Alternating Series Test (abridged)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

- (i) $a_n \geq 0$ for all $n \geq 1$;
- (ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);
- (iii) and $\lim_{n \rightarrow \infty} a_n = 0$.

Then

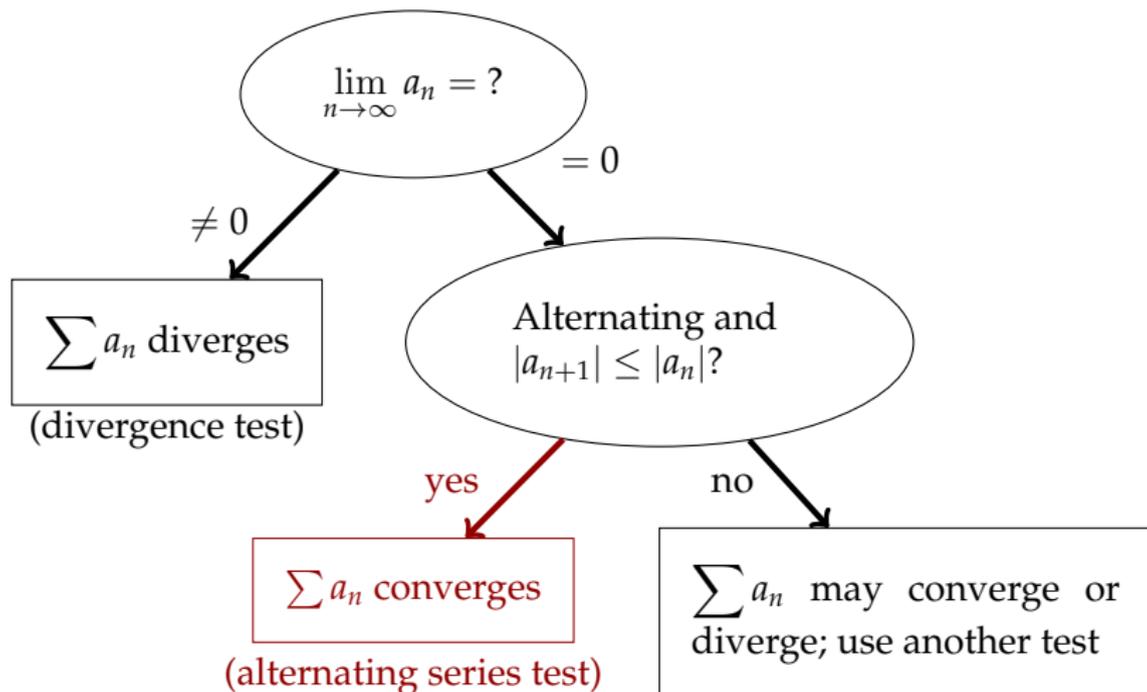
$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

▶ True or false: the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ converges.

▶ True or false: the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

DIVERGENCE TEST + ALTERNATING SERIES TEST



Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \geq 0$ for all $n \geq 1$; $a_{n+1} \leq a_n$ for all $n \geq 1$; and $\lim_{n \rightarrow \infty} a_n = 0$. Then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\sum_{n=1}^{99} \frac{(-1)^{n-1}}{n} \approx 0.698$.

How close is that to the value $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$?

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \geq 0$ for all $n \geq 1$; $a_{n+1} \leq a_n$ for all $n \geq 1$; and $\lim_{n \rightarrow \infty} a_n = 0$. Then $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\sum_{n=1}^{19} (-1)^{n-1} \frac{n^2}{n^2 + 1} \approx 0.6347$.

How close is that to the value $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^2 + 1}$?

Recall for a geometric series, the **ratios of consecutive terms** is constant.

$$\begin{array}{ccccccc}
 & \times \frac{1}{2} & & \times \frac{1}{2} & & \times \frac{1}{2} & & \times \frac{1}{2} \\
 & \underbrace{\hspace{1.5cm}} & \rightarrow & \underbrace{\hspace{1.5cm}} & \rightarrow & \underbrace{\hspace{1.5cm}} & \rightarrow & \underbrace{\hspace{1.5cm}} \\
 \\
 \frac{1}{2} & + & \frac{1}{4} & + & \frac{1}{8} & + & \frac{1}{16} & + & \frac{1}{32} \cdots \\
 \\
 & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & & & & \\
 \frac{1/4}{1/2} & = & \frac{1/8}{1/4} & = & \frac{1/16}{1/8} & = & \frac{1/32}{1/16} & = & \frac{1}{2}
 \end{array}$$

If that ratio has magnitude **less than one**, then the series converges.
 If the ratio has magnitude **greater than one**, the series diverges.

For series convergence, we are concerned with what happens to terms a_n when n is sufficiently large.

Suppose for a sequence a_n , $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L$ for some constant L .

$$\begin{array}{cccccccc}
 a_n & + & a_{n+1} & + & a_{n+2} & + & a_{n+3} & + & a_{n+4} & + & \cdots \\
 \underbrace{\hspace{1.5cm}} & & \\
 \frac{a_{n+1}}{a_n} & \approx & \frac{a_{n+2}}{a_{n+1}} & \approx & \frac{a_{n+3}}{a_{n+2}} & \approx & \frac{a_{n+4}}{a_{n+3}} & \approx & \frac{a_{n+5}}{a_{n+4}} & \approx & L
 \end{array}$$

Like in a geometric series:

If L has magnitude **less than one**, then the series converges.

If L has magnitude **greater than one**, the series diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Use the ratio test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

converges or diverges.

REMARK

The series we just considered, $\sum_{n=1}^{\infty} \frac{n}{3^n}$, looks similar to a geometric series, but it is not exactly a geometric series. That's a good indicator that the ratio test will be helpful!

We could have used other tests, but ratio was probably the easiest.

- ▶ Integral test: $\int \frac{x}{3^x} dx$ can be evaluated using integration by parts.
- ▶ Comparison test:
 - ▶ $\sum \frac{1}{3^n}$ is not a valid comparison series, nor is $\sum n$.
 - ▶ Because $n < 2^n$ for all $n \geq 1$, the series $\sum \left(\frac{2}{3}\right)^n$ will work.
- ▶ The divergence test is inconclusive, and the alternating series test does not apply. Our series is not geometric, and not obviously telescoping.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$, or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Let a and x be nonzero constants. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} anx^{n-1}$$

converges or diverges. (This may depend on the values of a and x .)

Let x be a constant. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} \frac{(-3)^n \sqrt{n+1}}{2n+3} x^n$$

converges or diverges. (This may depend on the value of x .)



FILL IN IN THE BLANKS

Divergence Test

If the sequence $\{a_n\}_{n=c}^{\infty}$ then the series $\sum_{n=c}^{\infty} a_n$ diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$, then $\sum_{n=1}^{\infty} a_n$ converges.

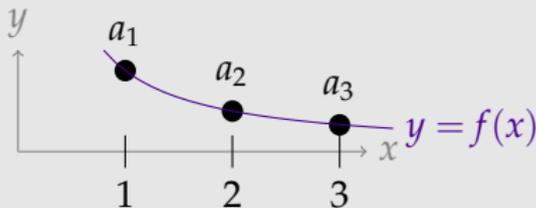
(b) If $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$, or $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Integral Test

Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \geq N_0$ and which obeys

- (i) and
 (ii) and
 (iii) $f(n) = a_n$ for all $n \geq N_0$.

Then



$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x) dx \text{ converges}$$

Furthermore, when the series converges, the truncation error satisfies

$$0 \leq \sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n \leq \int_N^{\infty} f(x) dx \quad \text{for all } N \geq N_0$$

FILL IN IN THE BLANKS

The Comparison Test

Let N_0 be a natural number and let $K > 0$.

(a) If $|a_n| \square Kc_n$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

(b) If $a_n \square Kd_n \geq 0$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} d_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

FILL IN IN THE BLANKS

Limit Comparison Theorem

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $b_n > 0$ for all n . Assume that

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$$

exists.

- (a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.
- (b) If $L \neq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges too.

In particular, if , then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i)

(ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);

(iii) and

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number N , $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the N^{th}

partial sum $\sum_{n=1}^N (-1)^{n-1} a_n$.

LIST OF CONVERGENCE TESTS

Divergence Test

When the n^{th} term in the series *fails* to converge to zero as n tends to infinity.

This is a good first thing to check: if it works, it's quick, but it doesn't always work.

Alternating Series Test

- ▶ successive terms in the series alternate in sign
- ▶ don't forget to check that successive terms decrease in magnitude and tend to zero as n tends to infinity

Integral Test

- ▶ works well when, if you substitute x for n in the n^{th} term you get a function, $f(x)$, that you can easily integrate
- ▶ don't forget to check that $f(x) \geq 0$ and that $f(x)$ decreases as x increases

LIST OF CONVERGENCE TESTS

Ratio Test

- ▶ works well when $\frac{a_{n+1}}{a_n}$ simplifies enough that you can easily compute $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$
- ▶ this often happens when a_n contains powers, like 7^n , or factorials, like $n!$
- ▶ don't forget that $L = 1$ tells you nothing about the convergence/divergence of the series

Comparison Test and Limit Comparison Test

- ▶ Comparison test lets you ignore pieces of a function that feel extraneous (like replacing $n^2 + 1$ with n^2) *but* there is a test to make sure the comparison is still valid. Either the limit of a ratio is the right thing, or an inequality goes the right way.
- ▶ Limit comparison works well when, for very large n , the n^{th} term a_n is approximately the same as a simpler, nonnegative term b_n

- ▶ The integral test gave us the p -test. When you're looking for comparison series, p -series $\sum \frac{1}{n^p}$ are often good choices, because their convergence or divergence is so easy to ascertain.
- ▶ Geometric series have the form $\sum a \cdot r^n$ for some nonzero constants a and r . The magnitude of r is all you need to know to decide whether they converge or diverge, so these are also common comparison series.
- ▶ Telescoping series have partial sums that are easy to find because successive terms cancel out. These are less obvious, and are less common choices for comparison series.

Test List

- ▶ divergence
- ▶ integral
- ▶ alternating series
- ▶ ratio
- ▶ comparison
- ▶ limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges or diverges.

Test List

- ▶ divergence
- ▶ integral
- ▶ alternating series
- ▶ ratio
- ▶ comparison
- ▶ limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

Test List

- ▶ divergence
- ▶ integral
- ▶ alternating series
- ▶ ratio
- ▶ comparison
- ▶ limit comparison

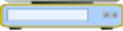
Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin \theta \leq \theta$.



Included Work

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