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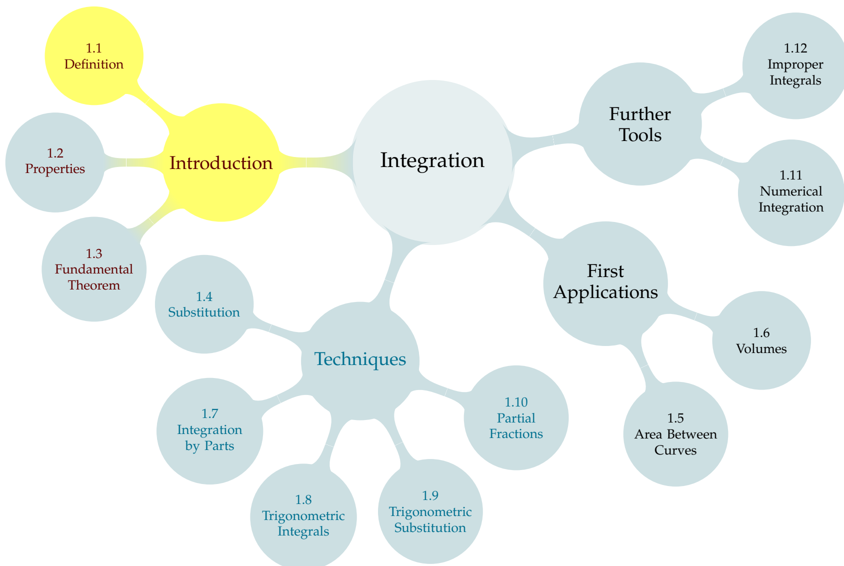


Calculus is build on two operations: differentiation and integration.

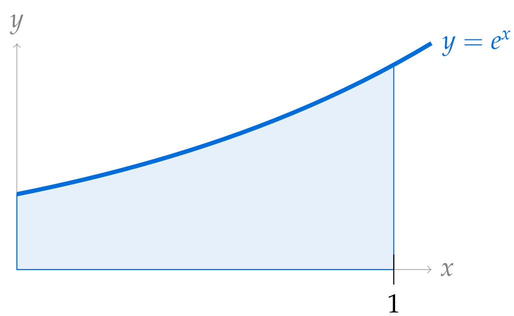
Differentiation

- Slope of a line
- Rate of change
- Optimization
- Numerical Approximations

Integration

- Area under a curve
- "Reverse" of differentiation
- Solving differential equations
- Calculate net change from rate of change
- Volume of solids
- Work (in the physics sense)

Approximate the area of the shaded region using rectangles.



We're going to be doing a lot of adding.

Summation (Sigma) Notation

$$\sum_{i=a}^{b} f(i)$$

- a, b (integers with $a \leq b$) "bounds"
- i "index:" integer which runs from a to b
- $f(i)$ "summands:" compute for every i, add

$$\sum_{i=a}^{b} f(i) = f(a) + f(a+1) + f(a+2) + \cdots + f(b)$$

Sigma Notation

Expand $\sum_{i=2}^{4}(2i+5)$.

$$\sum_{i=2}^{4}(2i+5) = \underbrace{(2\cdot 2+5)}_{i=2} + \underbrace{(2\cdot 3+5)}_{i=3} + \underbrace{(2\cdot 4+5)}_{i=4}$$
$$= 9 + 11 + 13 = 33$$

Sigma Notation

Expand $\sum_{i=1}^{4}(i+(i-1)^2)$.

$$= \underbrace{(1+0^2)}_{i=1} + \underbrace{(2+1^2)}_{i=2} + \underbrace{(3+2^2)}_{i=3} + \underbrace{(4+3^2)}_{i=4}$$
$$= 1 + 3 + 7 + 13 = 24$$

Write the following expressions in sigma notation:

- $3+4+5+6+7$
  $\sum_{i=3}^{7} i$ and $\sum_{i=1}^{5}(i+2)$ are two options (others are possible)
- $8+8+8+8+8$
  $\sum_{i=1}^{5} 8$ is one way (others are possible)
- $1+(-2)+4+(-8)+16$
  $\sum_{i=0}^{4}(-2)^i$ is one way (others are possible)

Arithmetic of Summation Notation

Let c be a constant.

- Adding constants: $\sum_{i=1}^{10} c =$
- Factoring constants: $\sum_{i=1}^{10} 5(i^2) =$
- Addition is Commutative: $\sum_{i=1}^{10}(i+i^2) =$

Arithmetic of Summation Notation

Let c be a constant.

- Adding constants: $\sum_{i=1}^{10} c = 10c$
- Factoring constants: $\sum_{i=1}^{10} 5(i^2) = 5\sum_{i=1}^{10}(i^2)$
- Addition is Commutative: $\sum_{i=1}^{10}(i+i^2) = \left(\sum_{i=1}^{10} i\right) + \left(\sum_{i=1}^{10} i^2\right)$

Common Sums

Let $n \geq 1$ be an integer, a be a real number, and $r \neq 1$.

$$\sum_{i=0}^{n} ar^i = a + ar + ar^2 + \cdots + ar^n = a\frac{1-r^{n+1}}{1-r}$$

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Let $n \geq 1$ be an integer, a be a real number, and $r \neq 1$.

$$\sum_{i=0}^{n} ar^i = a + ar + ar^2 + \cdots + ar^n = a\frac{1-r^{n+1}}{1-r}$$

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Simplify: $\sum_{i=1}^{13}(i^2+i^3) = \sum_{i=1}^{13} i^2 + \sum_{i=1}^{13} i^3 = \frac{13(14)(27)}{6} + \frac{13^2(14^2)}{4}$

Let $n \geq 1$ be an integer, a be a real number, and $r \neq 1$.

$$\sum_{i=0}^{n} ar^i = a + ar + ar^2 + \cdots + ar^n = a\frac{1-r^{n+1}}{1-r}$$

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Simplify: $\sum_{i=1}^{50}(1-i^2) = \sum_{i=1}^{50} 1 - \sum_{i=1}^{50} i^2 = 50 - \frac{50(51)(101)}{6}$

(Optional) Proof of a Common Sum

Here is a derivation of $\displaystyle\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}$:

$$A = 1 + \cancel{r} + \cancel{r^2} + \cdots + \cancel{r^{n-1}} + \cancel{r^n}$$

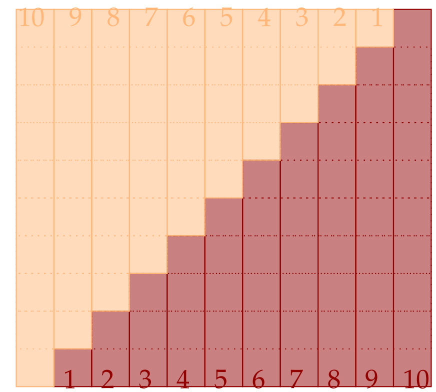
$$rA = \cancel{r} + \cancel{r^2} + \cdots + \cancel{r^{n-1}} + \cancel{r^n} + r^{n+1}$$

subtract $\quad A - rA = 1 - r^{n+1}$

divide across $\quad A = \dfrac{1-r^{n+1}}{1-r}$

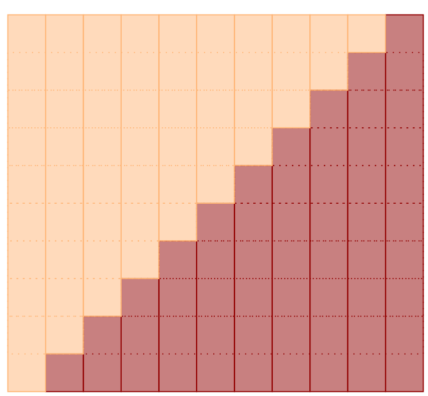
(Optional) Proof of Another Common Sum

$$\sum_{i=1}^{10} i = 1+2+3+4+5+6+7+8+9+10 = \frac{10\cdot 11}{2}$$



(Optional) Proof of a Common Sum

$$\sum_{i=1}^{n} i = 1+2+3+\cdots+n = \frac{n\cdot(n+1)}{2}$$



The purpose of these sums is to describe areas.

Notation

The symbol

$$\int_a^b f(x)\,\mathrm{d}x$$

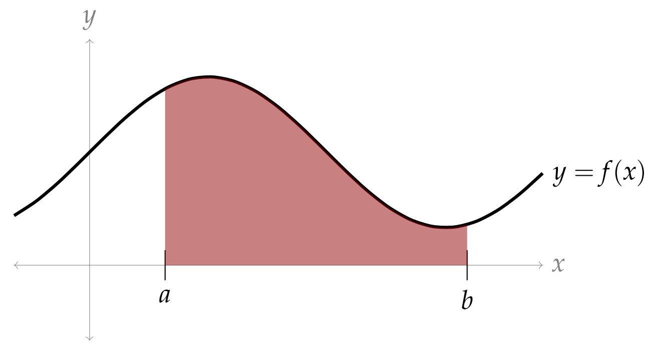
is read "the definite integral of the function $f(x)$ from a to b."

- $f(x)$: integrand
- a and b: limits of integration
- $\mathrm{d}x$: differential

If $f(x) \geq 0$ and $a \leq b$, one interpretation of

$$\int_a^b f(x)\,\mathrm{d}x$$

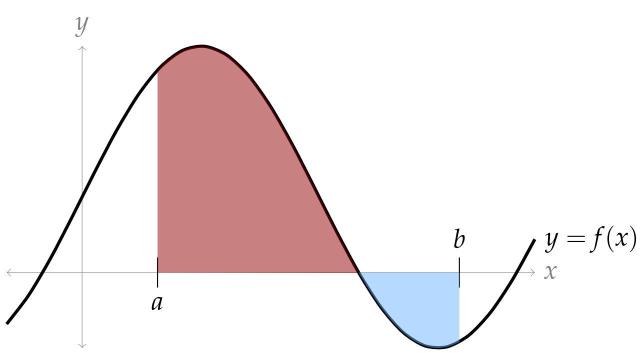
is "the area of the region bounded above by $y = f(x)$, below by $y = 0$, to the left by $x = a$, and to the right by $x = b$."



If ~~$f(x) \geq 0$ and~~ $a \leq b$, one interpretation of

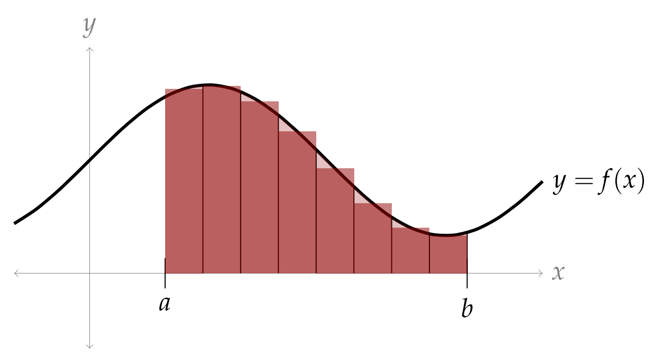
$$\int_a^b f(x)\,\mathrm{d}x$$

is the signed area of the region between $y = f(x)$ and $y = 0$, from $x = a$ to $x = b$. Area above the axis is positive, and area below it is negative.



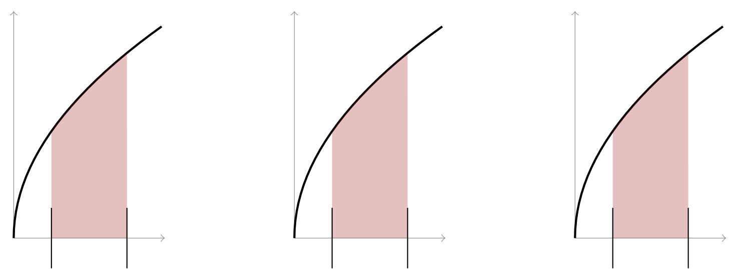
Riemann Sums

A Riemann sum approximates the area under a curve by cutting it into equal-width segments, and approximating each segment as a rectangle.



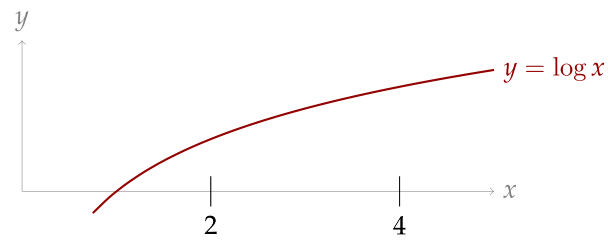
There are different ways to choose the height of each rectangle.

Types of Riemann Sums (RS)

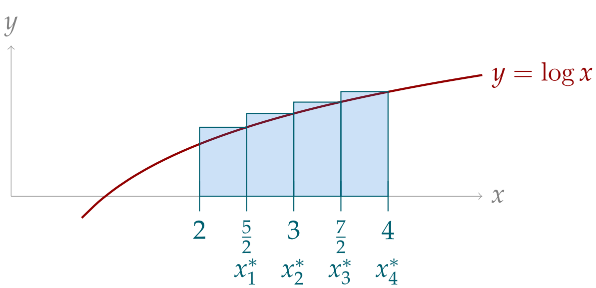


Left RS Right RS Midpoint RS

Approximate $\int_2^4 \log(x)\,dx$ using a right Riemann sum with $n = 4$ rectangles. For now, do not use sigma notation.



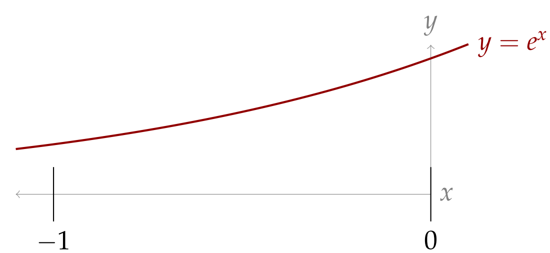
Approximate $\int_2^4 \log(x)\,dx$ using a right Riemann sum with $n = 4$ rectangles. For now, do not use sigma notation.



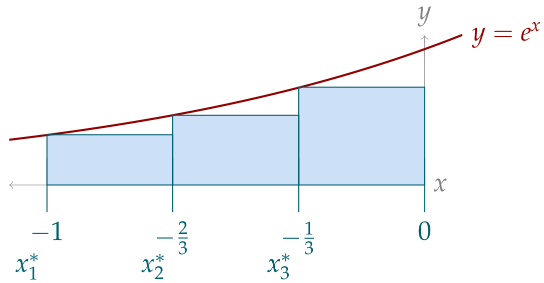
- Width of each rectangle: $\frac{4-2}{4} = \frac{1}{2}$
- Heights taken at right endpoints of rectangles: $x_1^* = \frac{5}{2}$, $x_2^* = 3$, $x_3^* = \frac{7}{2}$, $x_4^* = 4$

$$\int_2^4 \log(x)\,dx \approx \frac{1}{2}\log\left(\frac{5}{2}\right) + \frac{1}{2}\log(3) + \frac{1}{2}\log\left(\frac{7}{2}\right) + \frac{1}{2}\log(4)$$

Approximate $\int_{-1}^0 e^x\,dx$ using a left Riemann sum with $n = 3$ rectangles. For now, do not use sigma notation.



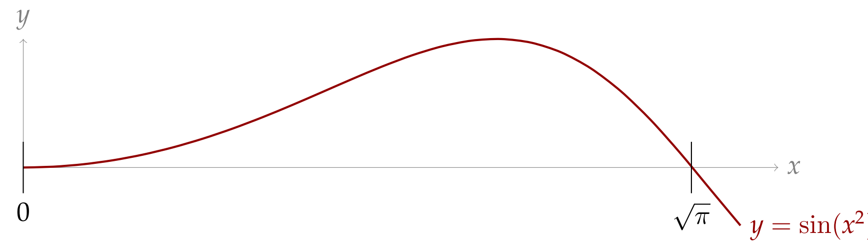
Approximate $\int_{-1}^{0} e^x \, dx$ using a left Riemann sum with $n = 3$ rectangles. For now, do not use sigma notation.



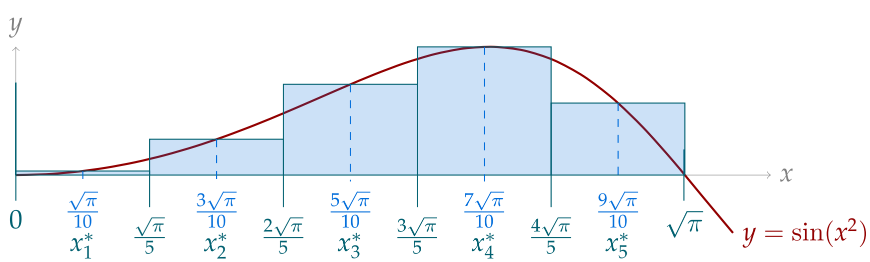
- Width of each rectangle: $\frac{0-(-1)}{3} = \frac{1}{3}$
- Heights taken at left endpoints of rectangles: $x_1^* = -1, x_2^* = -\frac{2}{3}, x_3^* = -\frac{1}{3}$

$$\int_{-1}^{0} e^x \, dx \approx \frac{1}{3}e^{-1} + \frac{1}{3}e^{-2/3} + \frac{1}{3}e^{-1/3}$$

Approximate $\int_0^{\sqrt{\pi}} \sin\left(x^2\right) dx$ using a midpoint Riemann sum with $n = 5$ rectangles. For now, do not use sigma notation.



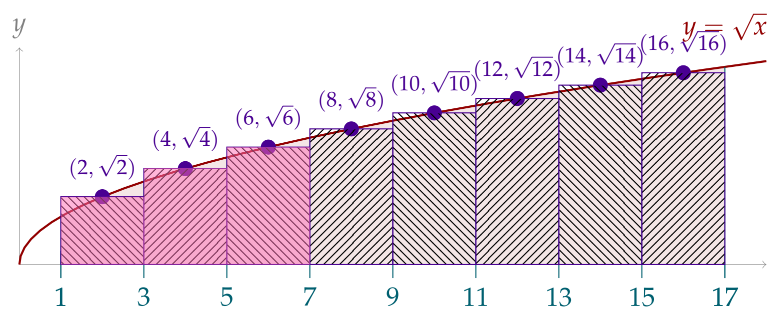
Approximate $\int_0^{\sqrt{\pi}} \sin\left(x^2\right) dx$ using a midpoint Riemann sum with $n = 5$ rectangles. For now, do not use sigma notation.



- Width of each rectangle: $\frac{\sqrt{\pi}-0}{5} = \frac{\sqrt{\pi}}{5}$
- Heights taken at midpoints of rectangles: $x_1^* = \frac{\sqrt{\pi}}{10}, x_2^* = \frac{3\sqrt{\pi}}{10}, x_3^* = \frac{5\sqrt{\pi}}{10}, x_4^* = \frac{7\sqrt{\pi}}{10}, x_5^* = \frac{9\sqrt{\pi}}{10}$

$$\frac{\sqrt{\pi}}{5}\left[\sin\left(\frac{\pi}{100}\right) + \sin\left(\frac{9\pi}{100}\right) + \sin\left(\frac{25\pi}{100}\right) + \sin\left(\frac{49\pi}{100}\right) + \sin\left(\frac{81\pi}{100}\right)\right]$$

Approximate $\int_1^{17} \sqrt{x} \, dx$ using a midpoint Riemann sum with 8 rectangles. Write the result in sigma notation.

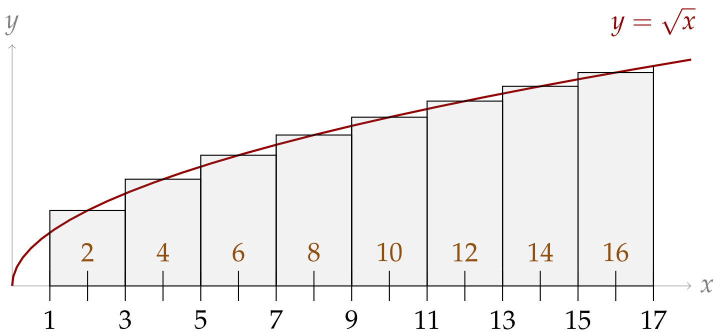


First $i = 1$	Second $i = 2$	Third $i = 3$	$\cdots i$	
Base: 2	Base: 2	Base: 2	$\cdots$ Base: 2	The
Height: $\sqrt{2}$	Height: $\sqrt{4}$	Height: $\sqrt{6}$	$\cdots$ Height: $\sqrt{2i}$	

ith rectangle has base 2 and height $\sqrt{2i}$, so

$$\text{area} \approx \sum_{i=1} 2\sqrt{2i}$$

$$\sum_{i=1}^{8} 2\sqrt{2i} = \underbrace{2\sqrt{2}}_{i=1} + \underbrace{2\sqrt{4}}_{i=2} + \underbrace{2\sqrt{6}}_{i=3} + \underbrace{2\sqrt{8}}_{i=4} + \underbrace{2\sqrt{10}}_{i=5} + \underbrace{2\sqrt{12}}_{i=6} + \underbrace{2\sqrt{14}}_{i=7} + \underbrace{2\sqrt{16}}_{i=8}$$

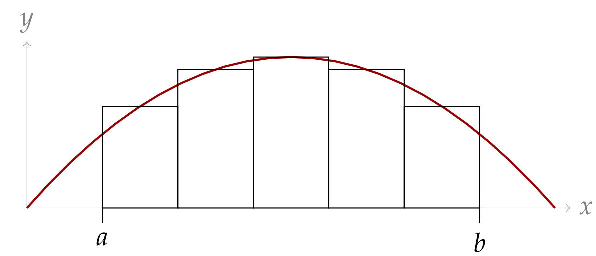


Riemann sum with n rectangles

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} \Delta x \cdot f(x^*_{i,n})$$

where $\Delta x = \frac{b-a}{n}$ and $x^*_{i,n}$ is an x-value in the ith rectangle.

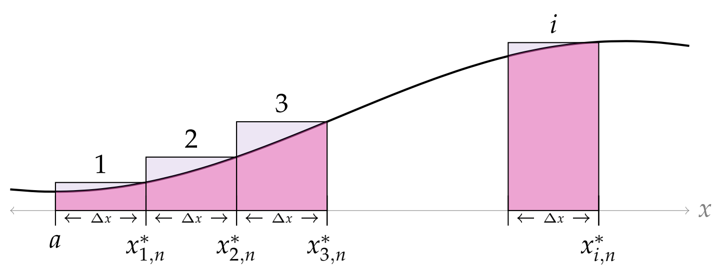
$$\sum_{i=1}^{n} \Delta x{\cdot}f(x^*_{i,n}) = \Delta x{\cdot}f\left(x^*_{1,n}\right) + \Delta x{\cdot}f\left(x^*_{2,n}\right) + \Delta x{\cdot}f\left(x^*_{3,n}\right) + \cdots + \Delta x{\cdot}f\left(x^*_{n,n}\right)$$



Right Riemann sum with n rectangles

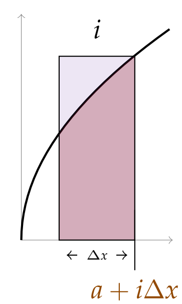
$$\int_a^b f(x)dx \approx \sum_{i=1}^{n} \Delta x \cdot f\left(x^*_{i,n}a + i\Delta x\right)$$

where $\Delta x = \frac{b-a}{n}$ and $x^*_{i,n} =$

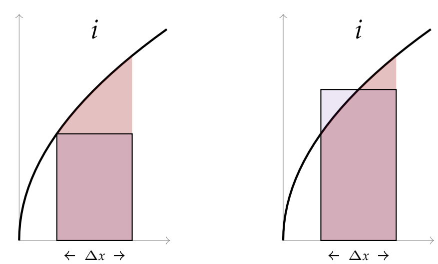


Types of Riemann Sums (RS)

What height would you choose for the ith rectangle?



Right RS



Left RS

Midpoint RS

Riemann sums with n rectangles. Let $\Delta x = \frac{b-a}{n}$

The right Riemann sum approximation of $\int_a^b f(x)\,dx$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f(a + i\Delta x)$$

The left Riemann sum approximation of $\int_a^b f(x)\,dx$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f(a + (i-1)\Delta x)$$

The midpoint Riemann sum approximation of $\int_a^b f(x)\,dx$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right)$$

Riemann sums with n rectangles: Let $\Delta x = \frac{b-a}{n}$

The right Riemann sum approximation of $\int_a^b f(x)\,dx$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f(a + i\Delta x)$$

Give a right Riemann Sum for the area under the curve $y = x^2 - x$ from $a = 1$ to $b = 6$ using $n = 1000$ intervals.

$$\sum_{n=1}^{1000} \frac{5}{1000}\left[\left(1 + \frac{5}{1000}i\right)^2 - \left(1 + \frac{5}{1000}i\right)\right]$$

Riemann sums with n rectangles: Let $\Delta x = \frac{b-a}{n}$

The midpoint Riemann sum approximation of $\int_a^b f(x)\,dx$ is:

$$\sum_{i=1}^{n} \Delta x \cdot f\left(a + \left(i - \frac{1}{2}\right)\Delta x\right)$$

Give a midpoint Riemann Sum for the area under the curve $y = 5x - x^2$ from $a = 6$ to $b = 9$ using $n = 1000$ intervals.

$$\sum_{n=1}^{1000} \frac{3}{1000}\left[5\left(6 + \frac{3}{1000}(i - 1/2)\right) - \left(6 + \frac{3}{1000}(i - 1/2)\right)^2\right]$$

Evaluating Riemann Sums

Skip Riemann evaluations

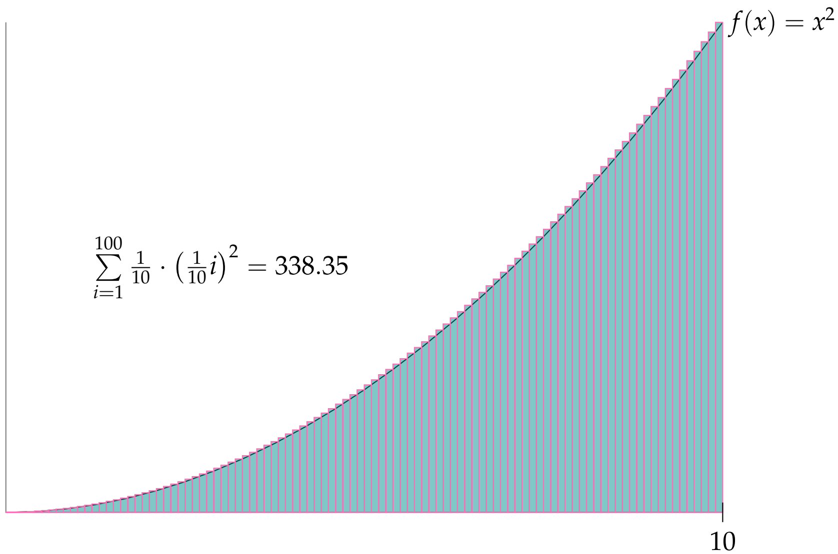
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Give the right Riemann sum of $f(x) = x^2$ from $a = 0$ to $b = 10$, $n = 100$:

$$\sum_{i=1}^{n} \Delta x \cdot f(a + i\Delta x) = \sum_{i=1}^{100} \frac{10}{100} \cdot \left(0 + \frac{10}{100}i\right)^2$$

$$= \sum_{i=1}^{100} \frac{1}{10} \cdot \left(\frac{1}{10}i\right)^2 = \frac{1}{10}\sum_{i=1}^{100} \frac{1}{100}i^2$$

$$= \frac{1}{1000}\sum_{i=1}^{100} i^2 = \frac{1}{1000}\,\frac{100(101)(201)}{6} = \frac{101 \cdot 201}{60}$$



$$\sum_{i=1}^{100} \frac{1}{10} \cdot \left(\frac{1}{10}i\right)^2 = 338.35$$

Evaluating Riemann Sums in Sigma Notation

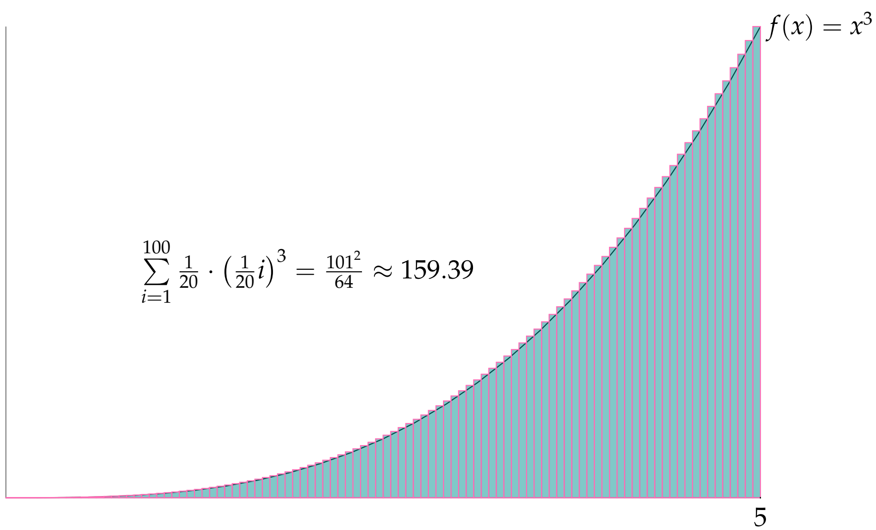
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Give the right Riemann sum of $f(x) = x^3$ from $a = 0$ to $b = 5$, $n = 100$:

$$\sum_{i=1}^{n} \Delta x \cdot f(a + i\Delta x) = \sum_{i=1}^{100} \frac{5}{100} \cdot \left(0 + \frac{5}{100}i\right)^3$$

$$= \sum_{i=1}^{100} \frac{1}{20} \cdot \left(\frac{1}{20}i\right)^3 = \frac{1}{20} \sum_{i=1}^{100} \frac{1}{20^3} i^3$$

$$= \frac{1}{20^4} \sum_{i=1}^{100} i^3 = \frac{1}{20^4} \frac{100^2(101^2)}{4} = \frac{101^2}{64}$$



$$\sum_{i=1}^{100} \frac{1}{20} \cdot \left(\frac{1}{20}i\right)^3 = \frac{101^2}{64} \approx 159.39$$

Definition

Let a and b be two real numbers and let $f(x)$ be a function that is defined for all x between a and b. Then we define $\Delta x = \frac{b-a}{N}$ and

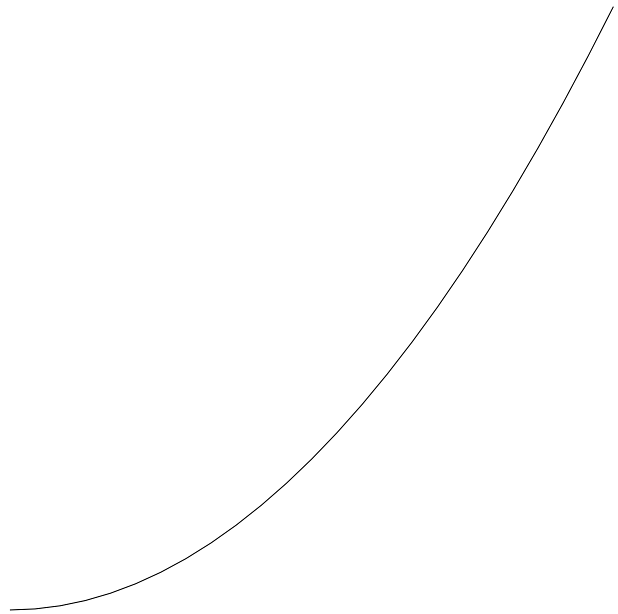
$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x^*_{i,N}) \cdot \Delta x$$

when the limit exists and when the choice of $x^*_{i,N}$ in the i^{th} interval doesn't matter.

$\sum$, $\int$ both stand for "sum"

Δx, dx are tiny pieces of the x-axis, the bases of our very skinny rectangles

It's understood we're taking a limit as N goes to infinity, so we don't bother specifying N (or each location where we find our height) in the second notation.



$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

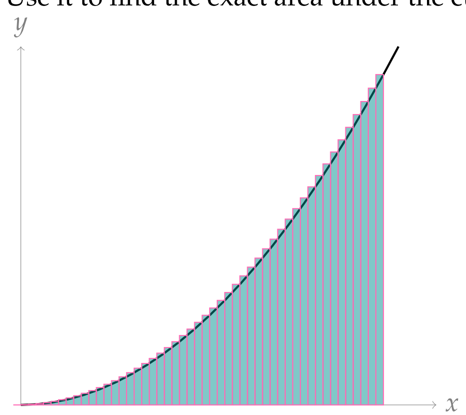
Give the right Riemann sum of $y = x^2$ from $a = 0$ to $b = 5$ with n slices, and simplify:

$$\begin{aligned}\sum_{i=1}^{n} \Delta x \cdot f(a + i\Delta x) &= \sum_{i=1}^{n} \frac{5}{n} \cdot \left(\frac{5}{n} i\right)^2 = \sum_{i=1}^{n} \frac{125}{n^3} i^2 \\ &= \frac{125}{n^3}\left[\sum_{i=1}^{n} i^2\right] = \frac{125}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \\ &= \frac{125}{n^2}\left(\frac{(n+1)(2n+1)}{6}\right) = \frac{125}{6}\left(\frac{2n^2+3n+1}{n^2}\right)\end{aligned}$$

We found the right Riemann sum of $y = x^2$ from $a = 0$ to $b = 5$ using n slices:

$$\frac{125}{6} \cdot \frac{2n^2+3n+1}{n^2}$$

Use it to find the exact area under the curve.



$$\int_0^5 x^2 \, dx = \lim_{n\to\infty}\left[\frac{125}{6} \cdot \frac{2n^2+3n+1}{n^2}\right]$$

125 [3 1]

125 (2) 125

Refresher: Limits of Rational Functions

$$\lim_{n\to\infty} \frac{n^2+2n+15}{3n^2-9n+5} = \lim_{n\to\infty} \frac{1+2/n+15/n^2}{3-9/n+5/n^2} = \frac{1}{3}$$

When the degree of the top and bottom are the same, the limit as n goes to infinity is the ratio of the leading coefficients.

$$\lim_{n\to\infty} \frac{n^2+2n+15}{3n^3-9n+5} = \lim_{n\to\infty} \frac{1/n+2/n^2+15/n^3}{3-9/n^2+5/n^3} = 0$$

When the degree of the top is smaller than the degree of the bottom, the limit as n goes to infinity is 0.

$$\lim_{n\to\infty} \frac{n^3+2n+15}{3n^2-9n+5} = \lim_{n\to\infty} \frac{n+2/n+15/n^2}{3-9/n+5/n^2} = \infty$$

When the degree of the top is larger than the degree of the bottom, the limit as n goes to infinity is positive or negative infinity.

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Evaluate $\int_0^1 x^2 \, dx$ exactly using midpoint Riemann sums.

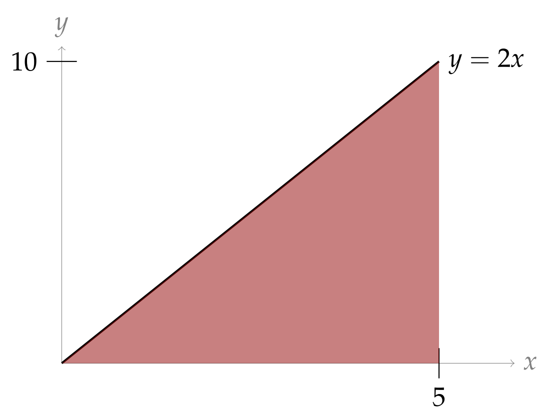
$$\sum_{i=1}^{n} \Delta x \cdot \left(\left(i - \frac{1}{2}\right)\Delta x\right)^2 = \frac{1}{n^3}\sum_{i=1}^{n}\left(i^2 - i + \frac{1}{4}\right) = \frac{1}{n^3}\left[\sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i + \sum_{i=1}^{n}\frac{1}{4}\right]$$
$$= \frac{1}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + \frac{1}{4}n\right]$$
$$= \frac{2n^2+3n+1}{6n^2} - \frac{n+1}{2n^2} + \frac{1}{4n^2}$$

Exact area under the curve:

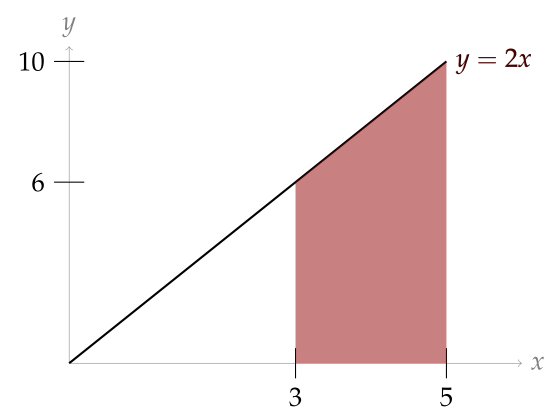
$$\lim_{n\to\infty}\left[\frac{2n^2+3n+1}{6n^2} - \frac{n+1}{2n^2} + \frac{1}{4n^2}\right] = \frac{2}{6} - 0 + 0 = \frac{1}{3}$$

Let's see some special cases where we can use geometry to evaluate integrals without Riemann sums.

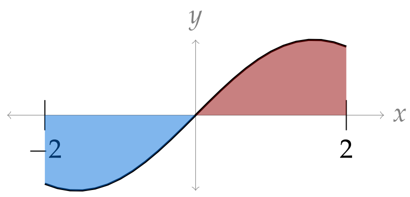
$$\int_0^5 2x \, dx = \frac{1}{2}(5)(10) = 25$$



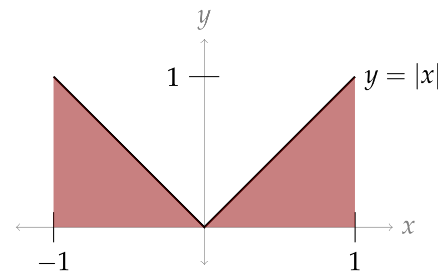
$$\int_3^5 2x \, dx = \frac{1}{2}(5)(10) - \frac{1}{2}(3)(6) = 25 - 9 = 16$$



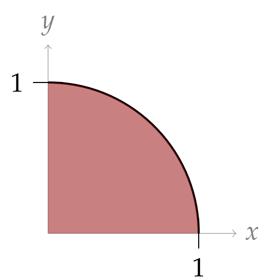
$$\int_{-2}^{2} \sin x \, dx = -A + A = 0$$



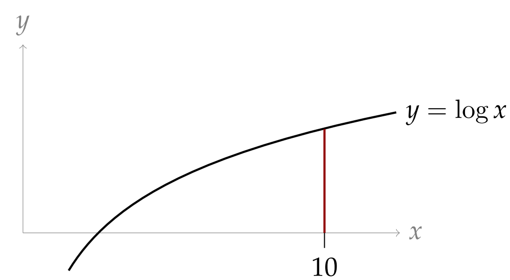
$$\int_{-1}^{1} |x| \, dx = \frac{1}{2}(1)(1) + \frac{1}{2}(1)(1) = 1$$



$$\int_{0}^{1} \sqrt{1 - x^2} \, dx = \frac{1}{4}(\pi \cdot 1^2) = \frac{\pi}{4}$$



$$\int_{10}^{10} \log x \, dx = 0$$



A car travelling down a straight highway records the following measurements:

Time	12:00	12:10	12:20	12:30	12:40	12:50	1:00
Speed (kph)	80	100	100	90	90	75	100

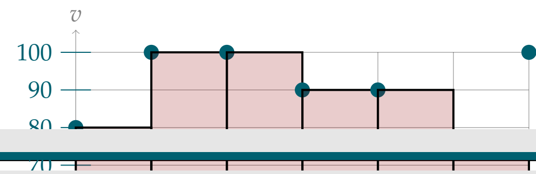
Approximately how far did the car travel from 12:00 to 1:00?

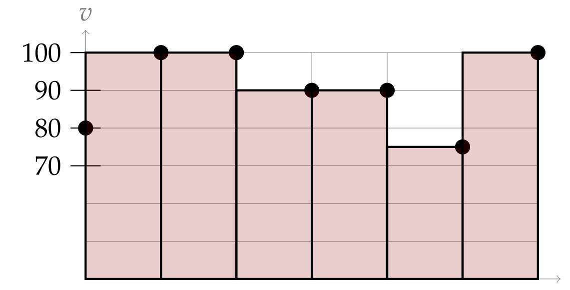
We don't know the speed of the car over the entire hour, so the best we can do is to use the measured speeds as approximations for the speeds the car travelled over 10-minute intervals.

We can use left, right, and midpoint Riemann sums. Note that there are only six 10-minute intervals, but we know seven points. For a midpoint Riemann sum, since we need to know the speed at the midpoint of the interval, we can only use three intervals, not six.

Finally, note that 10 minutes is $\frac{1}{6}$ of an hour, and 20 minutes is $\frac{1}{3}$ of an hour.

Left RS: $\underbrace{80 \cdot \frac{1}{6}}_{12:00-12:10} + \underbrace{100 \cdot \frac{1}{6}}_{12:10-12:20} + \underbrace{100 \cdot \frac{1}{6}}_{12:20-12:30} + \underbrace{90 \cdot \frac{1}{6}}_{12:30-12:40} + \underbrace{90 \cdot \frac{1}{6}}_{12:40-12:50} + \underbrace{75 \cdot \frac{1}{6}}_{12:50-1:00}$





The computation

$$\text{distance} = \text{rate} \times \text{time}$$

looks a lot like the computation

$$\text{area} = \text{base} \times \text{height}$$

for a rectangle. This gives us another interpretation for an integral.

Another interpretation of the integral

Let $x(t)$ be the position of an object moving along the x-axis at time t, and let $v(t) = x'(t)$ be its velocity. Then for all $b > a$,

$$x(b) - x(a) = \int_a^b v(t)\ \mathrm{d}t$$

That is, $\int_a^b v(t)\ \mathrm{d}t$ gives the *net distance* moved by the object from time a to time b.

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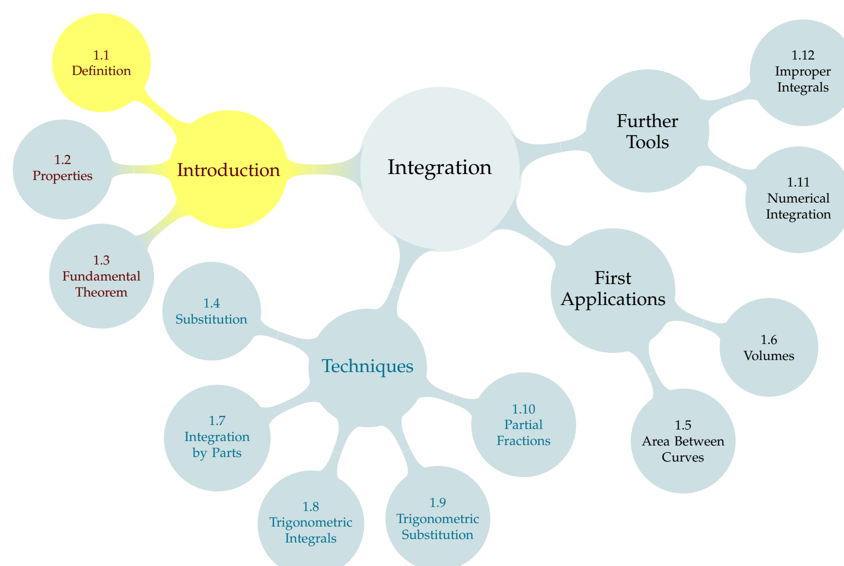


We defined the definite integral as

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{N\to\infty} \sum_{i=1}^{N} \Delta x \cdot f\left(x^*_{i,N}\right)$$

where $\Delta x = \frac{b-a}{N}$ and $x^*_{i,N}$ is a point in the interval $[a+(i-1)\Delta x\,,\ a+i\Delta x]$.

We have seen in previous classes that limits don't always exist. We will verify that this limit does indeed exist, and is equal to the desired area (at least in the most common cases).

We'll start with some general ideas that appear in the proof.

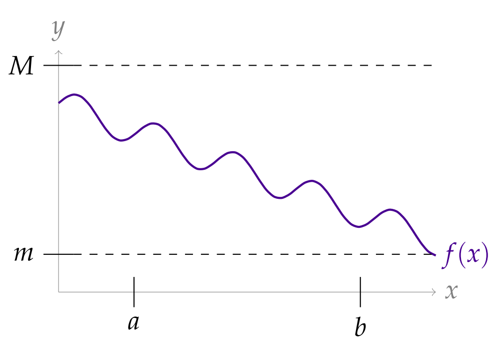
Suppose x and y are both in the interval $[a,b]$. What is the maximum possible value of $|x-y|$?

Proposition 1: distance between two numbers in an interval

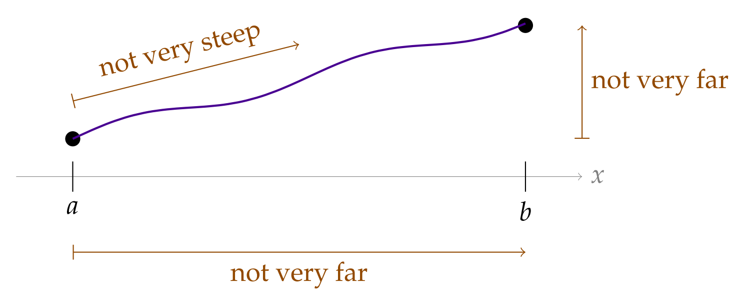
If $a \le x \le b$ and $a \le y \le b$, then $|x-y| \le$

Proposition 2: area inequality

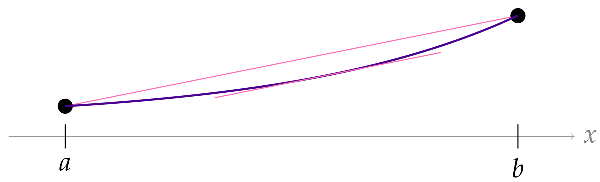
Let $f(x)$ be a function, defined over the interval $[a,b]$. If $m \le f(x) \le M$ over the entire interval $[a,b]$, then the (signed) area between the curve $y=f(x)$ and the x-axis, from a to b, is between $m(b-a)$ and $M(b-a)$.



Intuition: If $f'(x)$ is bounded on (a,b) and $b-a$ is small, then $f(b)-f(a)$ is also small.



The Mean Value Theorem provides a more explicit connection between these quantities.



Mean Value Theorem

Let a and b be real numbers with $a < b$. Let f be a function such that

- $f(x)$ is continuous on the closed interval $a \le x \le b$, and
- $f(x)$ is differentiable on the open interval $a < x < b$.

Then there is a c in (a, b) such that

$$f(b) - f(a) = f'(c)(b-a).$$

Equivalently: $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Triangle Inequality

For any real numbers $x_1,\ x_2,\ \cdots,\ x_n$:

$$\left|\sum_{i=1}^{n} x_i\right| \le \sum_{i=1}^{n} |x_i|$$

Intuition: If some terms are positive and some are negative, they "cancel each other out" and make the overall sum smaller.

$\|1+2\|$	$\|1\|+\|2\|$
$\|1+(-2)\|$	$\|1\|+\|-2\|$
$\|(-1)+(-2)\|$	$\|-1\|+\|-2\|$

Triangle Inequality

For any real numbers $x_1,\ x_2,\ \cdots,\ x_n$:

$$\left|\sum_{i=1}^{n} x_i\right| \le \sum_{i=1}^{n} |x_i|$$

Proof outline:

Let x and y be any real numbers.

- $x \le |x|$ and $y \le |y|$, so $x + y \le |x| + |y|$
- $-x \le |x|$ and $-y \le |y|$, so $-(x+y) = (-x) + (-y) \le |x| + |y|$
- $|x+y| = \begin{cases} x+y & \text{if } x+y \ge 0 \\ -(x+y) & \text{if } x+y < 0 \end{cases} \le |x| + |y|$
- Then $|x+y+z| = |(x+y)+z| \le |x+y| + |z| \le |x| + |y| + |z|$, etc.

Requirements

We will consider

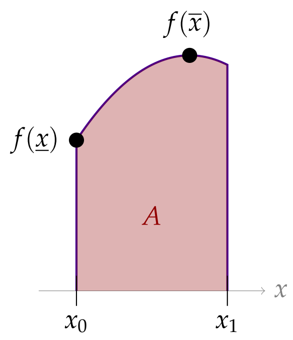
$$\int_a^b f(x)\,\mathrm{d}x$$

where:

- $a < b$
- $f(x)$ is continuous over the interval $[a, b]$
- $f(x)$ is differentiable over the interval (a, b)
- $f'(x)$ is bounded over the interval (a, b). That is, there exists a positive constant number F such that $|f'(x)| \le F$ for all x in the interval (a, b).

Error in a Single Slice

Consider approximating the area of single slice, from x_0 to x_1.

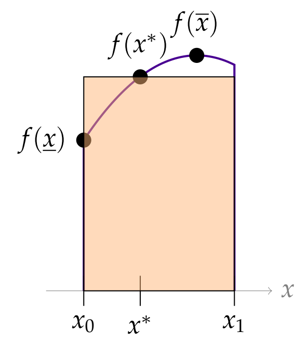


- A is the actual area of the slice
- $f(\overline{x})$ and $f(\underline{x})$ are the largest and smallest function values over the slice
- Our slice has width $x_1 - x_0$

Then we can bound our area:

Error in a Single Slice

Consider approximating the area of single slice, from x_0 to x_1.

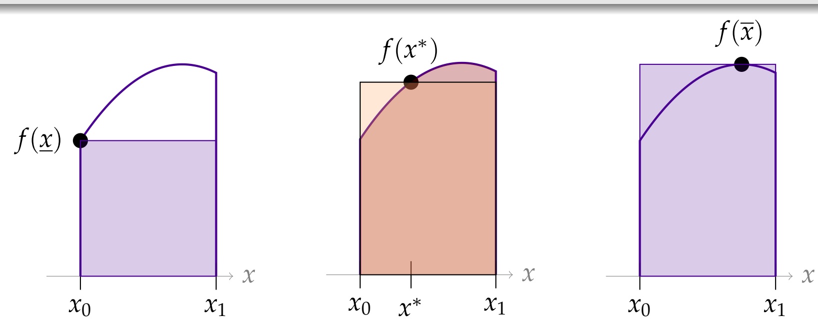


- $f(x^*) \cdot (x_1 - x_0)$ is our approximation of the area of the slice, for some x^* in the interval $[x_0, x_1]$.
- $f(\overline{x})$ and $f(\underline{x})$ are the largest and smallest function values over the slice, so

$$f(\underline{x}) \leq f(x^*) \leq f(\overline{x})$$

Then we can bound our approximation:

Error in a Single Slice



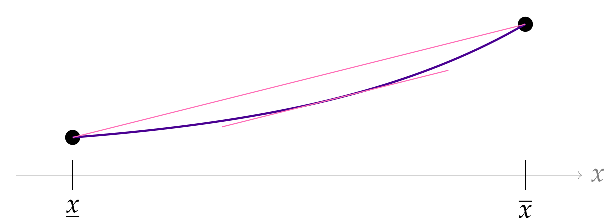
$$f(\underline{x}) \cdot (x_1 - x_0) \leq A \leq f(\overline{x}) \cdot (x_1 - x_0)$$

$$f(\underline{x}) \cdot (x_1 - x_0) \leq f(x^*) \cdot (x_1 - x_0) \leq f(\overline{x}) \cdot (x_1 - x_0)$$

$$\underbrace{|A - f(x^*) \cdot (x_1 - x_0)|}_{\text{error in slice}} \leq$$

Error in a Single Slice

- The error in our single slice is at most $[f(\overline{x}) - f(\underline{x})] \cdot (x_1 - x_0)$
- We want to show that our total error is not too large.
- Intuitively, if $|f'(x)|$ is never very large, and $x_1 - x_0$ is not very large, then $f(\overline{x}) - f(\underline{x})$ is not very large.



Error in a Single Slice

Mean Value Theorem

Let a and b be real numbers with $a < b$. Let f be a function such that

- $f(x)$ is continuous on the closed interval $a \le x \le b$, and
- $f(x)$ is differentiable on the open interval $a < x < b$.

Then there is a c in (a, b) such that

$$f(b) - f(a) = f'(c)(b - a)$$

There exists some c in (x_0, x_1) such that

$$f(\overline{x}) - f(\underline{x}) = f'(c) \cdot (\overline{x} - \underline{x})$$

Since $|f'(x)|$ is never larger than the positive constant F in (a, b),

$$|f(\overline{x}) - f(\underline{x})| \le F \cdot |\overline{x} - \underline{x}| \le F \cdot |x_1 - x_0|$$

Error in a Single Slice

All together,

$$\underbrace{|A - f(x^*) \cdot (x_1 - x_0)|}_{\text{error in slice}} \le [f(\overline{x}) - f(\underline{x})] \cdot (x_1 - x_0)$$

$$\le F \cdot |\overline{x} - \underline{x}| \cdot (x_1 - x_0)$$

$$\le F \cdot (x_1 - x_0) \cdot (x_1 - x_0)$$

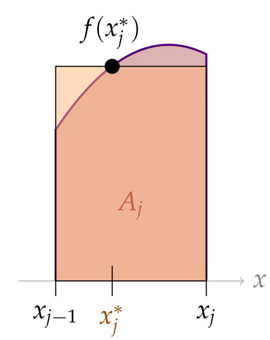
So,

$$\underbrace{|A - f(x^*) \cdot (x_1 - x_0)|}_{\text{error in slice}} \le F \cdot (x_1 - x_0)^2$$

We have shown that the error on a single slice can't be worse than some amount.

Now let's consider adding up slices.

What we did for a single slice, we now do for all slices.
Updated notation for slice j:



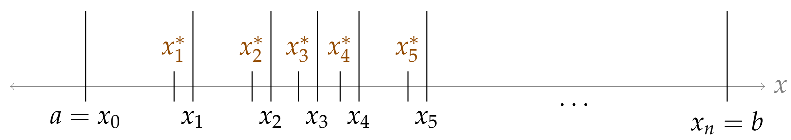
Slice error bound:

$$\left| A_j - f(x_j^*) \cdot (x_j - x_{j-1}) \right| \le F \cdot (x_j - x_{j-1})^2$$

(Possibly Irregular) Partitions

Consider partitioning the interval $[a, b]$ into n subintervals, not necessarily the same size. Let the points at the edges of the slices be $a = x_0, x_1, x_2, \cdots, x_{n-1}, x_n = b$.

In each part, choose a vertex x_i^* to sample the height of the function.



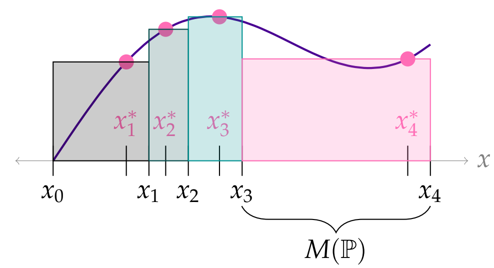
The approximation of $\int_a^b f(x)\,\mathrm{d}x$ depends on how you choose your subintervals, and where you choose your sample points. Let

$$\mathbb{P} = (n, x_1, x_2, \cdots, x_{n-1}, x_1^*, x_2^*, \cdots, x_n^*)$$

denote these choices.

Let $\mathcal{I}(\mathbb{P})$ be the approximation that arises from $\mathbb{P}$:

$$\mathcal{I}(\mathbb{P}) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1})$$



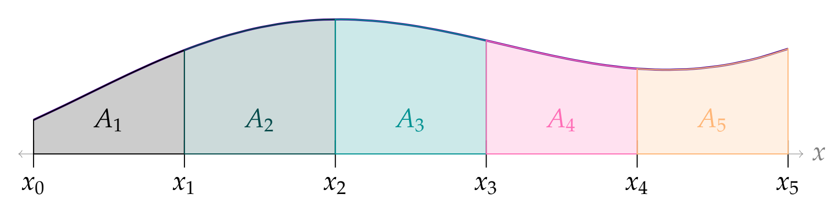
Let $M(\mathbb{P})$ be the maximum width of any subinterval.
If $M(\mathbb{P})$ is small, then *every* subinterval is small (narrow).

Define the integral as the limit

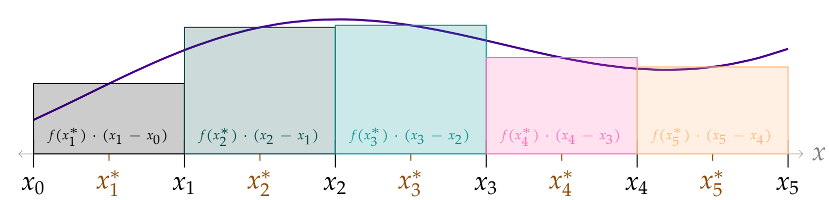
$$\int_a^b f(x)\,\mathrm{d}x = \lim_{M(\mathbb{P})\to 0} \mathcal{I}(\mathbb{P})$$

(Compare to our previous Riemann sum definition.)

We will show that the limit exists and is equal to the signed area under the curve.



Actual area: $\int_a^b f(x)\,\mathrm{d}x = \sum_{i=1}^{n} A_i$



Approximation: $\mathcal{I}(\mathbb{P}) = \sum_{i=1}^{n} f(x_i^*)\cdot(x_i - x_{i-1})$

$$\underbrace{\left|\int_a^b f(x)\,dx - \mathcal{I}(\mathbb{P})\right|}_{\text{overall error}} = \left|\sum_{i=1}^n A_i - \sum_{i=1}^n f(x_i^*)\cdot(x_i - x_{i-1})\right|$$

$$= \left|\sum_{i=1}^n [A_i - f(x_i^*)\cdot(x_i - x_{i-1})]\right|$$

$$\text{(triangle inequality)} \quad \le \sum_{i=1}^n |A_i - f(x_i^*)\cdot(x_i - x_{i-1})|$$

$$\text{(slice error bound)} \quad \le \sum_{i=1}^n F\cdot(x_i - x_{i-1})^2$$

$$= \sum_{i=1}^n F\cdot(x_i - x_{i-1})\cdot(x_i - x_{i-1})$$

$$\le \sum_{i=1}^n F\cdot M(\mathbb{P})\cdot(x_i - x_{i-1})$$

$$= F\cdot M(\mathbb{P})\cdot\sum_{i=1}^n (x_i - x_{i-1})$$

$$= F\cdot M(\mathbb{P})\cdot(b-a)$$

$$0 \le \underbrace{\left|\int_a^b f(x)\,dx - \mathcal{I}(\mathbb{P})\right|}_{\text{overall error}} \le F\cdot M(\mathbb{P})\cdot(b-a)$$

$$\lim_{M(\mathbb{P})\to 0} 0 = 0 \qquad \lim_{M(\mathbb{P})\to 0} [F\cdot M(\mathbb{P})\cdot(b-a)] = 0$$

So, by the squeeze theorem,

$$\lim_{M(\mathbb{P})\to 0} \underbrace{\left|\int_a^b f(x)\,dx - \mathcal{I}(\mathbb{P})\right|}_{\text{overall error}} = 0$$

That is,

$$\lim_{M(\mathbb{P})\to 0} \mathcal{I}(\mathbb{P}) = \int_a^b f(x)\,dx$$

Comparing Definitions

Here, we defined

$$\int_a^b f(x)\,dx = \lim_{M(\mathbb{P})\to 0} \mathcal{I}(\mathbb{P})$$

for "nice" functions $f(x)$.
Originally, we used a slightly different definition:

Definition 1.1.9 (abridged)

For "nice" functions $f(x)$:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_{i,n}^*)\cdot\frac{b-a}{n}$$

when the limit exists and takes the same value for all choices of the $x_{i,n}^*$'s.

Comparing Definitions

We showed that all families of partitions "work," as long as their largest subintervals shrink to length 0.

If all families of partitions "work," then we might as well choose a simple one. The (arguably) simplest choices are regular partitions, cutting the interval $[a, b]$ into n subintervals of length $\frac{b-a}{n}$.

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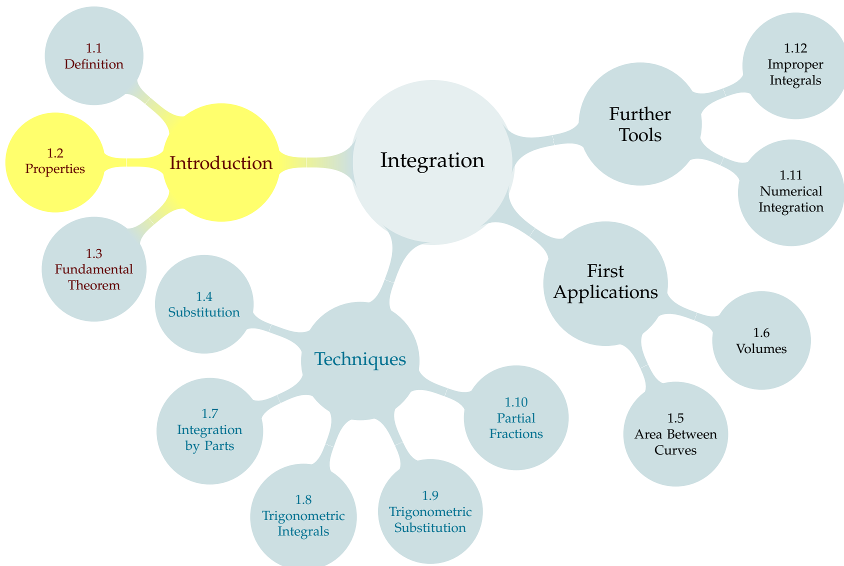


We defined the definite integral using a limit and a sum.

Definition

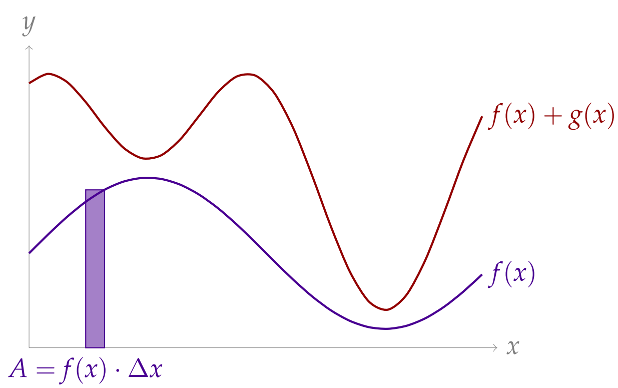
Let a and b be two real numbers and let $f(x)$ be a function that is defined for all x between a and b. Then we define $\Delta x = \frac{b-a}{N}$ and

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{N\to\infty} \sum_{i=1}^{N} f(x^*_{i,N}) \cdot \Delta x$$

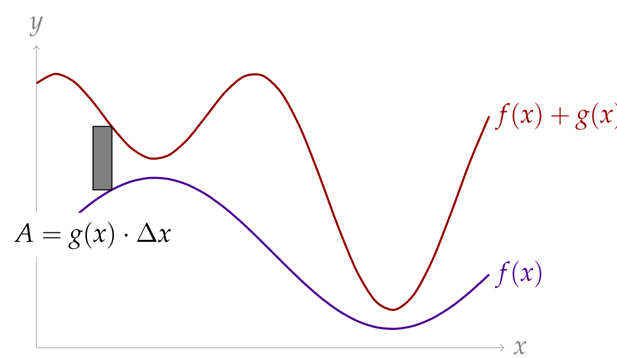
when the limit exists and when the choice of $x^*_{i,N}$ in the i^{th} interval doesn't matter.

Many of the operations that work nicely with sums and limits will also work nicely with integrals.

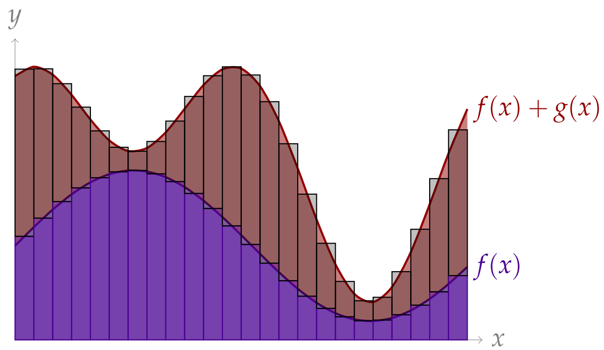
Adding (and Subtracting) Functions



Adding (and Subtracting) Functions

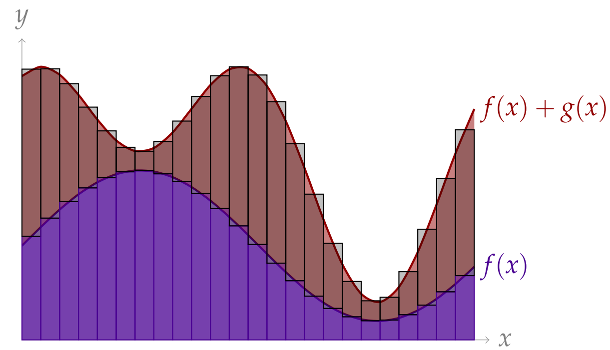


Adding (and Subtracting) Functions



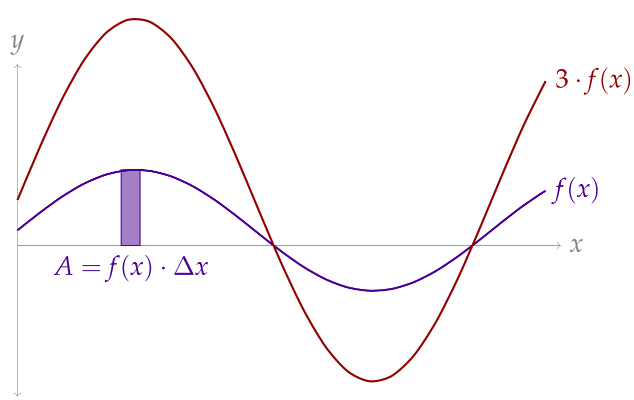
$$\int_a^b \left(f(x)+g(x)\right)\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

Adding (and Subtracting) Functions

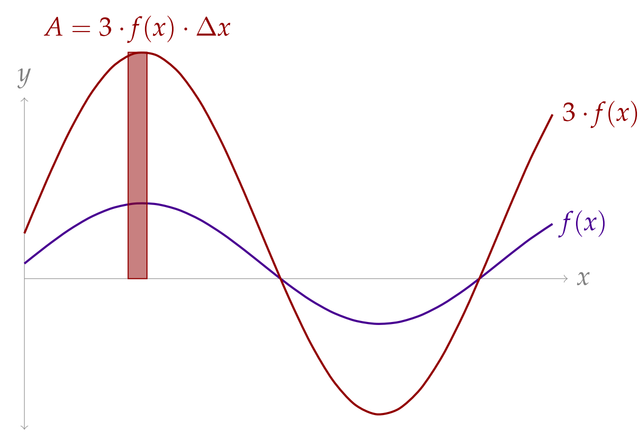


$$\int_a^b \left(f(x)-g(x)\right)\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x - \int_a^b g(x)\,\mathrm{d}x$$

Multiplying a Function by a Constant



Multiplying a Function by a Constant



$$\int_a^b c\cdot f(x)\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x$$

Arithmetic of Integration

When a, b, and c are real numbers, and the functions $f(x)$ and $g(x)$ are integrable on an interval containing a and b:

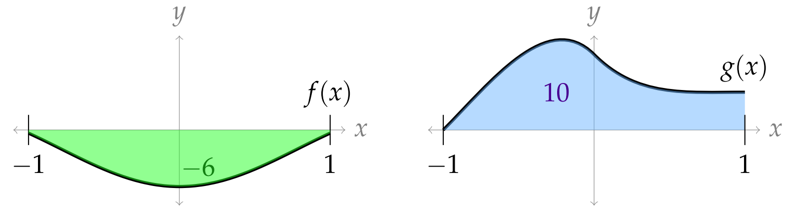
(a) $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

(b) $\displaystyle\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

(c) $\displaystyle\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$ when c is constant

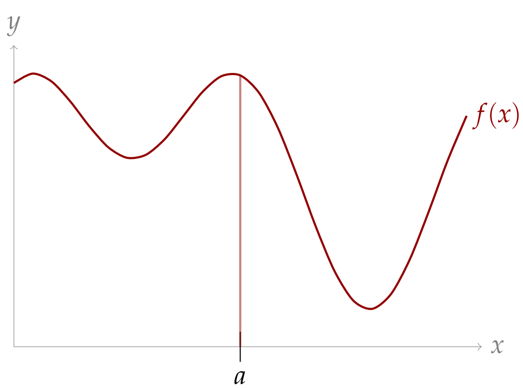
Arithmetic of Integration

Suppose $\displaystyle\int_{-1}^{1} f(x)\,dx = -6$ and $\displaystyle\int_{-1}^{1} g(x)\,dx = 10$.



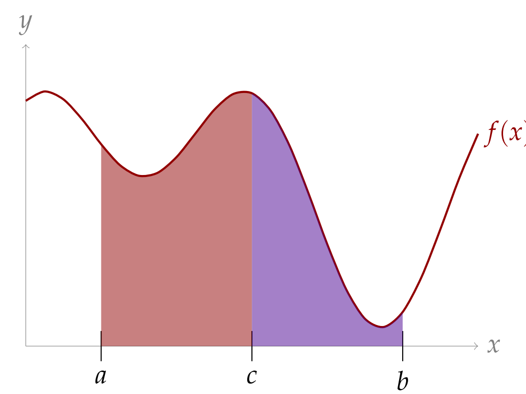
$$\int_{-1}^{1} (2f(x)+g(x))\,dx = 2\int_{-1}^{1} f(x)\,dx + \int_{-1}^{1} g(x)\,dx = 2(-6)+10 = -2$$

Interval of Integration



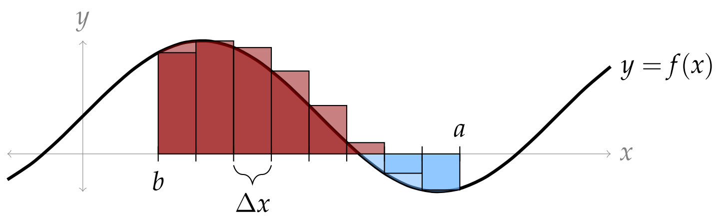
$$\int_a^a f(x)\,dx =$$

Interval of Integration



What rule do you think is being illustrated?

WHAT HAPPENS IN $\int_a^b f(x)\,dx$ WHEN $b < a$?



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i,n}^*) \cdot \frac{b-a}{n}$$

This is the definition of a definite integral *whether or not* $a < b$.

PROPERTY OF DEFINITE INTEGRALS

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

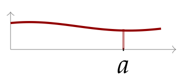
As strictly a measure of area, not usually a super useful fact – but helps later when we do arithmetic with integrals.

It's also useful that the definition works without having to worry about which limit of integration (a or b) is larger.

ARITHMETIC FOR DOMAIN OF INTEGRATION

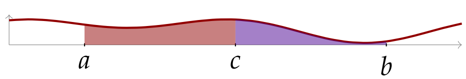
When a, b, and c are constants, and $f(x)$ is integrable over a domain containing all three:

(a) $\displaystyle\int_a^a f(x)\,dx = 0$

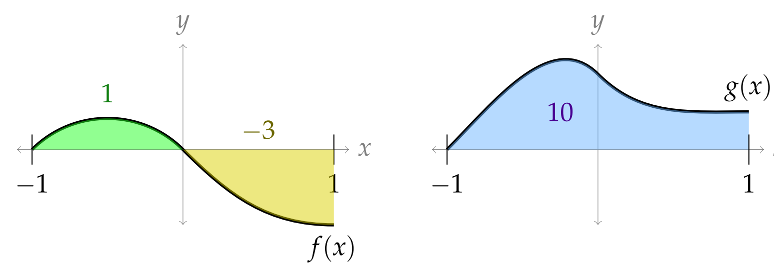


(b) $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ $\qquad \Delta x = \dfrac{b-a}{n} = -\dfrac{a-b}{n}$

(c) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ for constant c

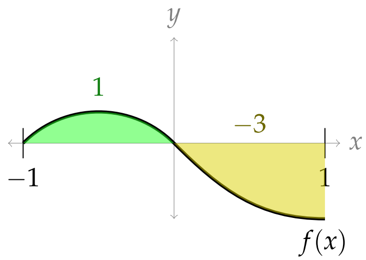


Suppose $\displaystyle\int_{-1}^0 f(x)\,dx = 1$, $\displaystyle\int_0^1 f(x)\,dx = -3$, and $\displaystyle\int_{-1}^1 g(x)\,dx = 10$.



$$\int_{-1}^1 (2f(x) + g(x))\,dx = 2\left[\int_{-1}^0 f(x)\,dx + \int_0^1 f(x)\,dx\right] + \int_{-1}^1 g(x)\,dx$$
$$= 2\,[1-3] + 10 = 6$$

Suppose $\int_{-1}^{0} f(x)\,dx = 1$ and $\int_{0}^{1} f(x)\,dx = -3$.

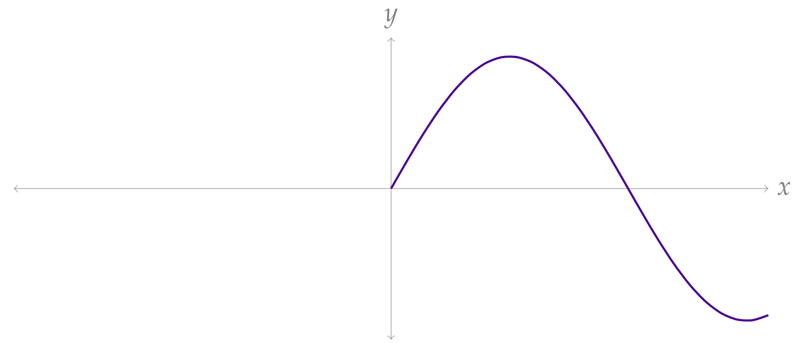


$$\int_{-1}^{3} f(x)\,dx + \int_{3}^{0} f(x)\,dx = \int_{-1}^{0} f(x)\,dx = 1$$

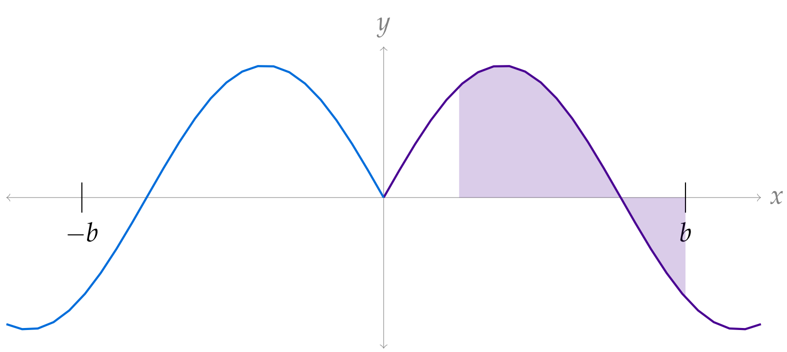
Even and Odd Functions

Let $f(x)$ be a function.

- We say $f(x)$ is even when $f(x) = f(-x)$ for all x, and
- we say $f(x)$ is odd when $f(x) = -f(-x)$ for all x.



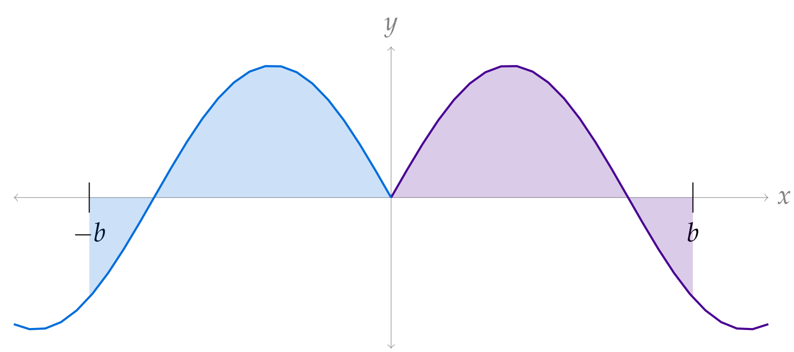
Integrals of Even Functions



Suppose $f(x)$ is even. Then

$$\int_{a}^{b} f(x)\,dx = \int_{-b}^{-a} f(x)\,dx$$

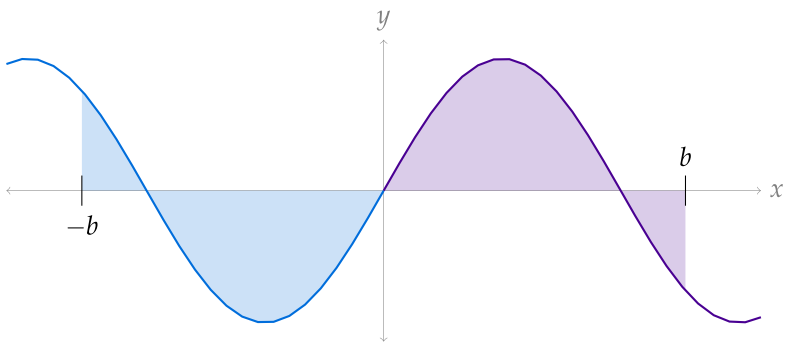
Integrals of Even Functions



Suppose $f(x)$ is even. Then

$$\int_{-b}^{b} f(x)\,dx = 2\int_{0}^{b} f(x)\,dx$$

Integrals of Odd Functions



Suppose $f(x)$ is odd. Then

$$\int_{-b}^{b} f(x)\,\mathrm{d}x = 0$$

Theorem 1.2.12 (Even and Odd)

Let $a > 0$.

(a) If $f(x)$ is an even function, then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_{0}^{a} f(x)\,\mathrm{d}x$$

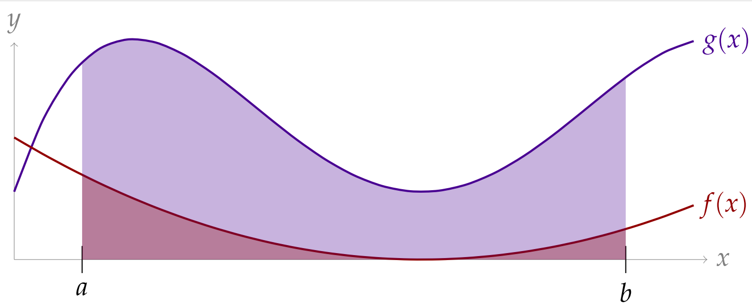
(b) If $f(x)$ is an odd function, then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0$$

Integral Inequality

Let $a \le b$ be real numbers and let the functions $f(x)$ and $g(x)$ be integrable on the interval $a \le x \le b$.
If $f(x) \le g(x)$ for all $a \le x \le b$, then

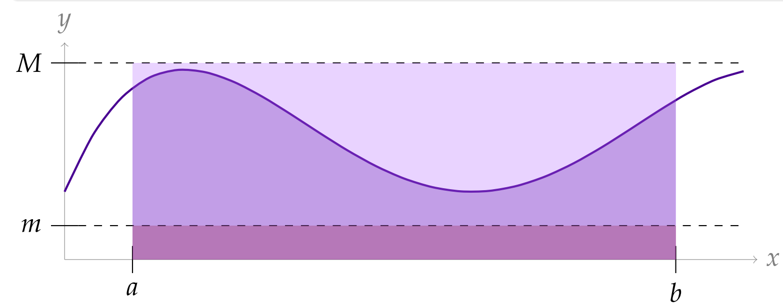
$$\int_{a}^{b} f(x)\,\mathrm{d}x \le \int_{a}^{b} g(x)\,\mathrm{d}x$$



Integral Inequality

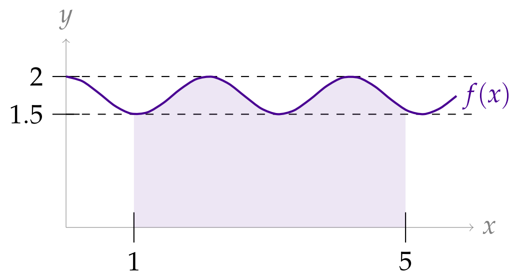
Let $a \le b$ and $m \le M$ be real numbers and let the function $f(x)$ be integrable on the interval $a \le x \le b$.
If $m \le f(x) \le M$ for all $a \le x \le b$, then

$$m(b-a) \le \int_{a}^{b} f(x)\mathrm{d}x \le M(b-a)$$



Find a lower bound c and an upper bound d such that

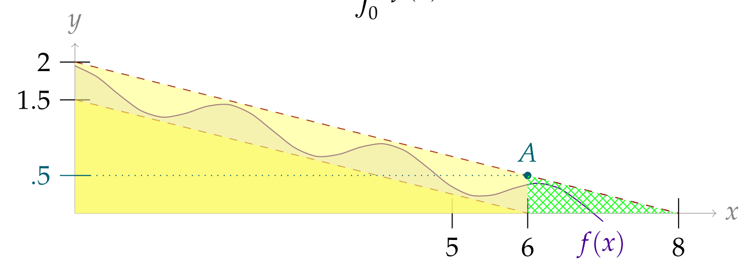
$$c \leq \int_1^5 f(x)\,dx \leq d$$



$$1.5 \leq f(x) \leq 2 \implies \overbrace{1.5(5-1)}^{6} \leq \int_1^5 f(x)\,dx \leq \overbrace{2(5-1)}^{8}$$

Find a lower bound c and an upper bound d such that $d - c \leq 3$ and

$$c \leq \int_0^6 f(x)\,dx \leq d$$



The area under the curve is no smaller than the area of the highlighted triangle.

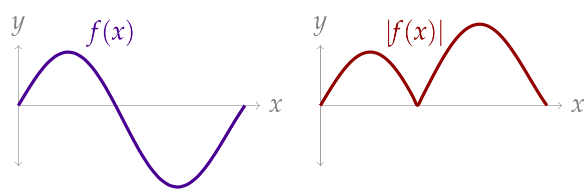
$$\int_0^6 (\text{dashed line})\,dx = \frac{1}{2} \cdot \frac{3}{2} \cdot 6 = \frac{9}{2} \leq \int_0^6 f(x)\,dx$$

The area under the curve is not greater than the area under the solid yellow trapezoid.
Because the dashed line has slope $-\frac{1}{4}$, the y–coordinate of point A is $\frac{1}{2}$.
We can compute the area of the trapezoid as the difference in the area of the triangle under the dotted line, and the green cross-hatched triangle.

$$\int_0^6 f(x)\,dx \leq \int_0^6 (\text{dashed line})\,dx = \frac{1}{2}(8)(2) - \frac{1}{2}(2)\frac{1}{2} = \frac{15}{2}$$

Absolute Values

$f(x) \leq |f(x)|$ for any $f(x)$
$-f(x) \leq |f(x)|$ for any $f(x)$



$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

because $\left|\int_a^b f(x)\,dx\right|$ is either $\int_a^b f(x)\,dx$ or $-\int_a^b f(x)\,dx$.

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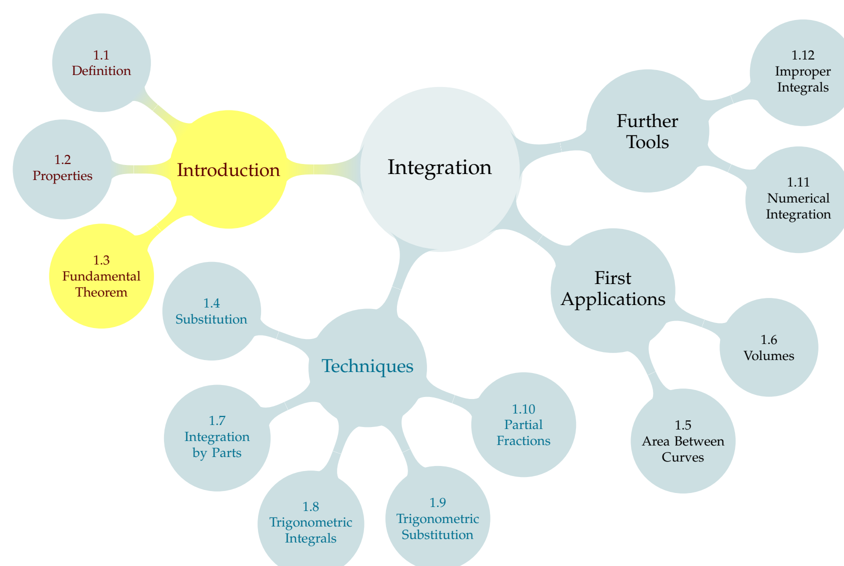
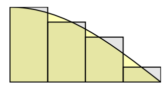


Review: Area under a Curve

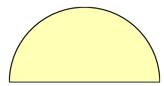
Methods for finding the area under a curve.

- Limit of a Riemann Sum
  - Conceptually easy – cut into rectangles
  - Computationally rough



$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x; \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

- Use Geometry
  - Computationally nice when it's available! (Circles, triangles, symmetry, etc.)
  - Often not available – most functions don't make such nice shapes.



- Up next: Fundamental Theorem of Calculus
  - Conceptually less obvious – we'll spend about a day explaining why it works
  - Computationally generally nicer than Riemann sums
  - Doesn't work for every function

Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

$$A(x) = \int_a^x f(t)\,dt$$

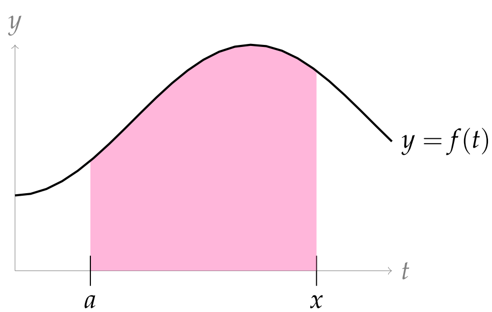
for any x in $[a, b]$. Then the function $A(x)$ is differentiable and

$$A'(x) = f(x)\ .$$

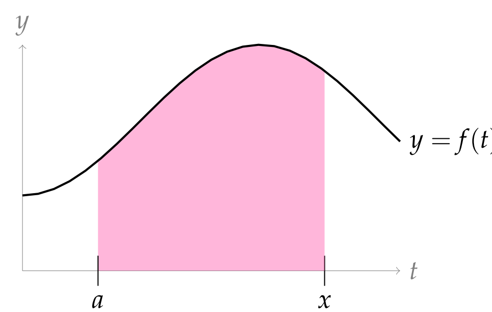
FTC(I) gives us the derivative of a very specific function (subject to some fine print).

It shows a close relationship between integrals and derivatives.

Area Function: $A(x) = \int_a^x f(t)dt$ for $a \le x \le b$



Area Function: $A(x) = \int_a^x f(t)dt$ for $a \le x \le b$



Notation: the function A depends on the variable x.

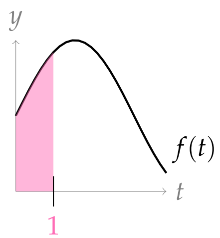
We need to know how the function f behaves on the whole interval $(0, x)$ to find $A(x)$. That's why we use $f(t)$, not $f(x)$.

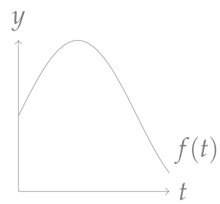
Area function notation

It might look strange at first to see two different variables. Let's consider the alternatives:

$$A(x) = \int_0^x f(t)\,\mathrm{d}t \qquad B(x) = \int_0^x f(x)\,\mathrm{d}t \qquad C(x) = \int_0^x f(x)\,\mathrm{d}x$$

$$A(1) = \int_0^1 f(t)\,\mathrm{d}t \qquad B(1) = \int_0^1 f(1)\,\mathrm{d}t \qquad C(1) = \int_0^1 f(1)\,\underbrace{\mathrm{d}1}_{??}$$





Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

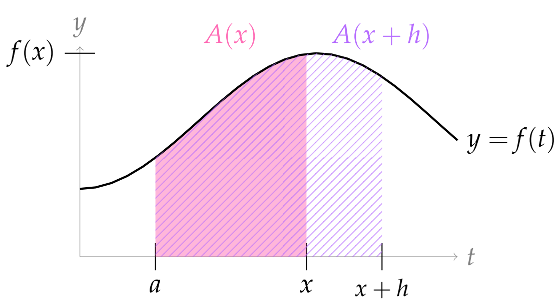
$$A(x) = \int_a^x f(t)\,\mathrm{d}t$$

for any x in $[a, b]$. Then the function $A(x)$ is differentiable and

$$A'(x) = f(x)\ .$$

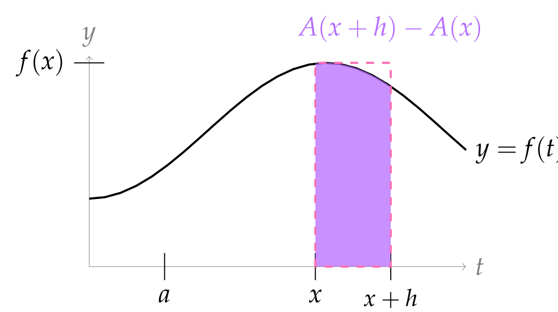
Question: Why is it true?

Derivative of Area Function, $A(x) = \int_a^x f(t)\mathrm{d}t$



$$A'(x) = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \frac{hf(x)}{h} = f(x)$$

Derivative of Area Function, $A(x) = \int_a^x f(t)\mathrm{d}t$



$$A'(x) = \lim_{\Delta x \to 0} \frac{\Delta A}{\Delta x} = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} \frac{hf(x)}{h} = f(x)$$

When h is very small, the purple area looks like a rectangle with base h and height $f(x)$, so $A(x+h) - A(x) \approx hf(x)$ and $\frac{A(x+h)-A(x)}{h} \approx f(x)$. As h tends to zero, the error in this approximation approaches 0.

Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

$$A(x) = \int_a^x f(t)\,\mathrm{d}t$$

for any x in $[a, b]$. Then the function $A(x)$ is differentiable and

$$A'(x) = f(x)\ .$$

Suppose $A(x) = \int_2^x \sin t\,\mathrm{d}t$. What is $A'(x)$?

$A'(x) = \sin x$

Suppose $B(x) = \int_x^2 \sin t\,\mathrm{d}t$. What is $B'(x)$?

$B'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left\{-\int_2^x f(t)\,\mathrm{d}t\right\} = -\sin x$

Fundamental Theorem of Calculus, Part 1

Let $a < b$ and let $f(x)$ be a function which is defined and continuous on $[a, b]$. Let

$$A(x) = \int_a^x f(t)\,\mathrm{d}t$$

for any x in $[a, b]$. Then the function $A(x)$ is differentiable and

$$A'(x) = f(x)\ .$$

Suppose $C(x) = \int_2^{e^x} \sin t\,\mathrm{d}t$. What is $C'(x)$?

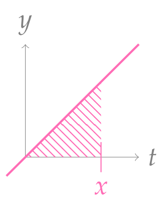
$C'(x) = e^x \sin(e^x)$: if we set $a = 2$, then

$$C(x) = A(e^x)$$

$$\implies C'(x) = A'(e^x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\{e^x\} = \sin(e^x)\cdot e^x$$

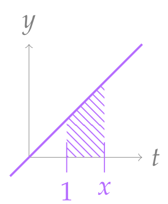
It's possible to have two different functions with the same derivative.

$A(x) = \int_0^x 2t\,\mathrm{d}t = x^2$



$A'(x) = 2x$

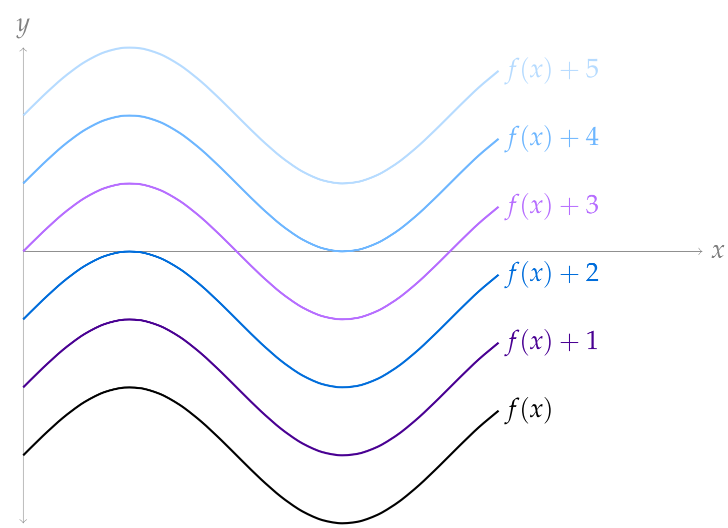
$B(x) = \int_1^x 2t\,\mathrm{d}t = x^2 - 1$



$B'(x) = 2x$

When two functions have the same derivative, they differ only by a constant.

In this example: $B(x) = A(x) - 1$



If two continuous functions have the same derivative, then one is a constant plus the other.

Two clues for finding $A(x) = \int_a^x f(t)\,dt$:

- If $A(x) = \displaystyle\int_a^x f(t)\,dt$, then[1] $A'(x) = f(x)$
- If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C.

$A(x) = \displaystyle\int_a^x e^t\,dt$. What functions could $A(x)$ be?

- $A'(x) = e^x$.
- Guess a function with derivative e^x: $F(x) = e^x$.
- Then $A(x) = e^x + C$ for some constant C.

[1](as long as $f(t)$ is continuous on $[a, x]$)

Two clues for finding $A(x) = \int_a^x f(t)\,dt$:

- If $A(x) = \displaystyle\int_a^x f(t)\,dt$, then[1] $A'(x) = f(x)$
- If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C.

$A(x) = \displaystyle\int_a^x \cos t\,dt$. What functions could $A(x)$ be?

- $A'(x) = \cos x$.
- Guess a function with derivative $\cos x$: $F(x) = \sin x$.
- Then $A(x) = \sin x + C$ for some constant C.

[1](as long as $f(t)$ is continuous on $[a, x]$)

Two clues for finding $A(x) = \int_a^x f(t)\,dt$:

- If $A(x) = \displaystyle\int_a^x f(t)\,dt$, then[1] $A'(x) = f(x)$
- If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C.

$A(x) = \displaystyle\int_{-2}^x 5t^4\,dt$. What functions could $A(x)$ be?

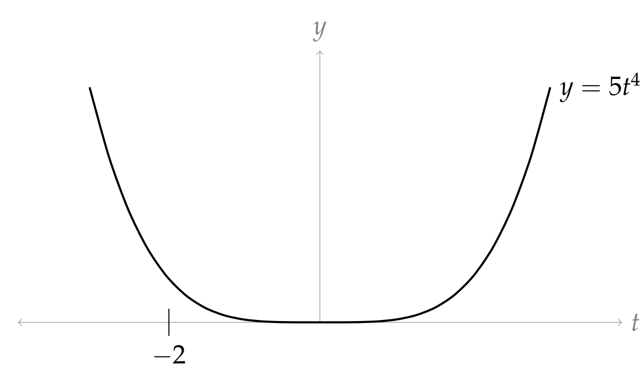
- $A'(x) = 5x^4$.
- Guess a function with derivative $5x^4$: $F(x) = x^5$.
- Then $A(x) = x^5 + C$ for some constant C.
- We ALSO know $A(-2) = \int_{-2}^{-2} 5t^4\,dt = 0$, so we can find C:

$$0 = A(-2) = (-2)^5 + C \implies C = 32$$

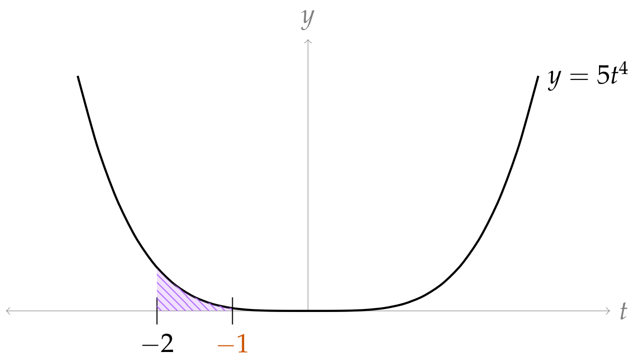
- So, $A(x) = x^5 + 32$

[1](as long as $f(t)$ is continuous on $[a, x]$)

$$A(x) = \int_{-2}^x 5t^4\,dt = x^5 + 32$$

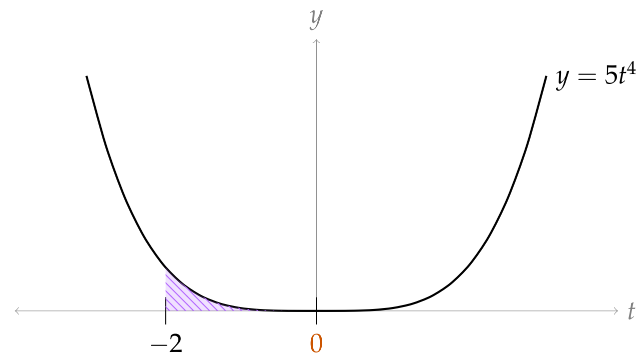


$$A(x) = \int_{-2}^{x} 5t^4 \, \mathrm{d}t = x^5 + 32$$



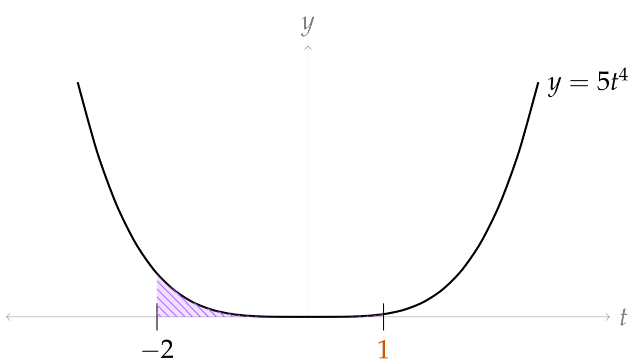
$$A(-1) = \int_{-2}^{-1} 5t^4 \, \mathrm{d}t = (-1)^5 + 32 = 31$$

$$A(x) = \int_{-2}^{x} 5t^4 \, \mathrm{d}t = x^5 + 32$$



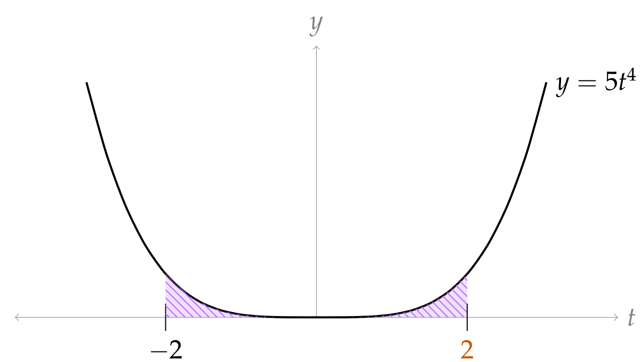
$$A(0) = \int_{-2}^{0} 5t^4 \, \mathrm{d}t = (0)^5 + 32 = 32$$

$$A(x) = \int_{-2}^{x} 5t^4 \, \mathrm{d}t = x^5 + 32$$



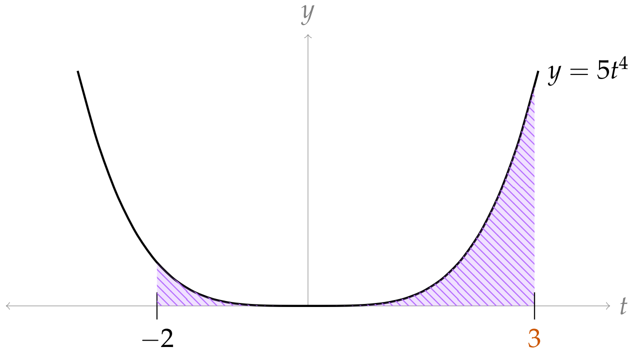
$$A(1) = \int_{-2}^{1} 5t^4 \, \mathrm{d}t = (1)^5 + 32 = 33$$

$$A(x) = \int_{-2}^{x} 5t^4 \, \mathrm{d}t = x^5 + 32$$



$$A(2) = \int_{-2}^{2} 5t^4 \, \mathrm{d}t = (2)^5 + 32 = 64$$

$$A(x) = \int_{-2}^{x} 5t^4 \, dt = x^5 + 32$$



$$A(3) = \int_{-2}^{3} 5t^4 \, dt = (3)^5 + 32 = 275$$

Two clues for finding $A(x) = \int_a^x f(t)\,dt$:

- If $A(x) = \int_a^x f(t)\,dt$, then[1] $A'(x) = f(x)$
- If $F'(x) = A'(x)$, then $A(x) = F(x) + C$ for some constant C.

$A(x) = \int_a^x f(t)\,dt$. What functions could $A(x)$ be?

[1](as long as $f(t)$ is continuous on $[a, x]$)

Two clues for finding $A(x) = \int_a^x f(t)\,dt$:

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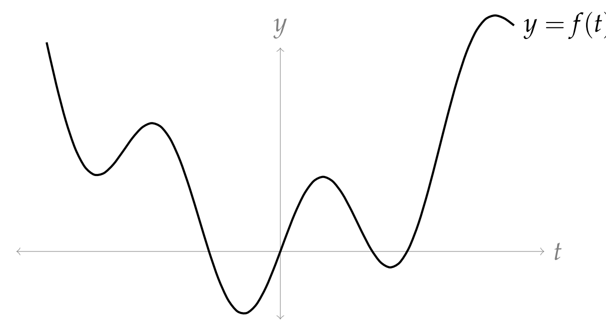
$A(b) = \int_a^b f(t)\,dt$. What functions could $A(b)$ be?

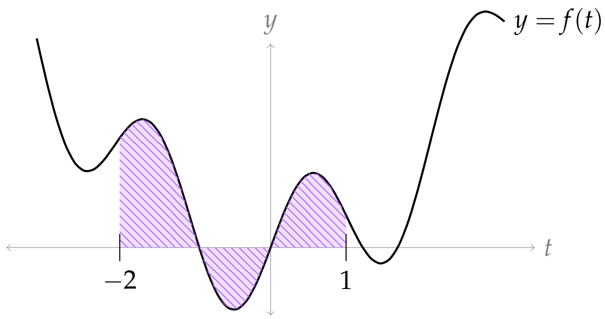
- $A'(x) = f(x)$.
- Guess a function with derivative $f(x)$: $F(x)$.
- Then $A(x) = F(x) + C$ for some constant C.
- Also $A(a) = 0$, so $0 = F(a) + C$, so $C = -F(a)$
- So, $A(b) = F(b) - F(a)$

[1](as long as $f(t)$ is continuous on $[a, x]$)

$$\int_a^b f(t)\,dt = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$

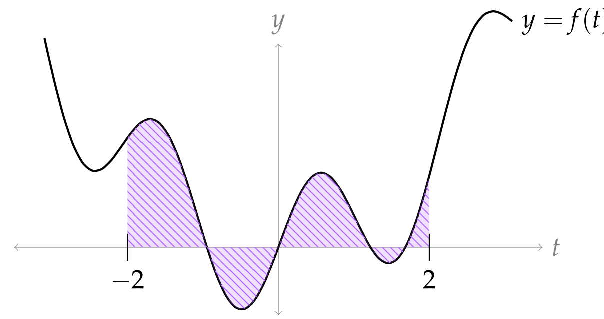


$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$



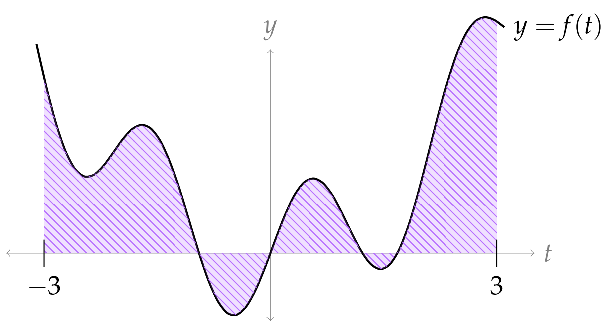
$$\int_{-2}^{1} f(t)\,\mathrm{d}t = F(1) - F(-2)$$

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$



$$\int_{-2}^{2} f(t)\,\mathrm{d}t = F(2) - F(-2)$$

$$\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a) \quad \text{where} \quad F'(x) = f(x)$$



$$\int_{-3}^{3} f(t)\,\mathrm{d}t = F(3) - F(-3)$$

Fundamental Theorem of Calculus, Part 2

Let $F(x)$ be differentiable, defined, and continuous on the interval $[a, b]$ with $F'(x) = f(x)$ for all $a < x < b$. Then

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

Examples:

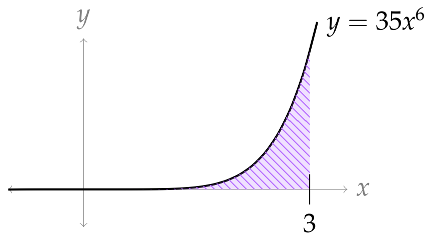
$\frac{\mathrm{d}}{\mathrm{d}x}\left\{5x^7\right\} = 35x^6$, so

$$\int_0^3 35x^6\,\mathrm{d}x = 5x^7\Big|_{x=3} - 5x^7\Big|_{x=0} = 5(3^7) - 5(0^7) = 5 \cdot 3^7$$

$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\tan x\right\} = \sec^2 x$, so

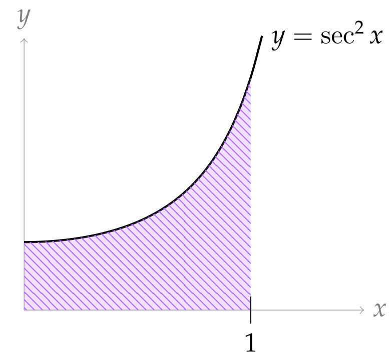
$$\int_0^{\pi/4} \sec^2 x\,\mathrm{d}x = \tan x\Big|_{x=\frac{\pi}{4}} - \tan x\Big|_{x=0} = \tan(\pi/4) - \tan 0 = 1$$

$$\int_0^3 35x^6\,dx = F(b) - F(a) \quad \text{where} \quad F(x) = 5x^7$$



$$\int_0^3 35x^6\,dx = 5(3)^7 - 5(0)^7$$

$$\int_0^{\pi/4} \sec^2 x\,dx = F(b) - F(a) \quad \text{where} \quad F(x) = \tan x$$



$$\int_0^{\pi/4} \sec^2 x\,dx = \tan\left(\frac{\pi}{4}\right) - \tan 0 = 1$$

Relevant Vocabulary

Definition

If $F(x)$ is a function whose derivative is $f(x)$, we call $F(x)$ an **antiderivative** of $f(x)$.

Examples:
The derivative of x^2 is $2x$, so:
x^2 is an **antiderivative** of $2x$.

When $x > 0$, the derivative of $\log x$ is $\frac{1}{x}$, so:
$\frac{1}{x}$ is an **antiderivative** of $\log x$.

For all x, the derivative of $\log|x|$ is $\frac{1}{x}$, so:
$\frac{1}{x}$ is an **antiderivative** of $\log|x|$.

An antiderivative of $\sin x$ is $-\cos x$, because $\frac{d}{dx}\{-\cos x\} = \sin x$.

Convenient Notation

Definition

$$f(x)\Big|_a^b = f(b) - f(a)$$

The function $f(x)$ evaluated from a to b

Examples:

$(5x + x^2)\Big|_1^2 = (10 + 4) - (5 + 1)$

$\frac{x^2}{x+2}\Big|_5^{-1} = \frac{1}{1} - \frac{25}{7}$

FTC Part 2, Abridged

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b$$

where $F(x)$ is an antiderivative of $f(x)$

Definition

The **indefinite integral** of a function $f(x)$:

$$\int f(x)\,\mathrm{d}x$$

means the *most general* antiderivative of $f(x)$.

Examples:

$\int 2x\,\mathrm{d}x = x^2 + C,$ C "arbitrary constant."

$\int \frac{1}{x}\,\mathrm{d}x = \log|x| + C$

Remember: two functions with the same derivative differ by a constant, so we include the "$+C$" for indefinite integrals.

Definite vs Indefinite Integrals

For each pair of properties below, decide which applies to **definite** integrals, and which to **indefinite** integrals.

No limits (or bounds) of integration, $\int f(x)\,\mathrm{d}x$	indefinite
Limits (or bounds) of integration, $\int_a^b f(x)\,\mathrm{d}x$	definite
Area under a curve	definite
Antiderivative	indefinite
Number	definite
Function	indefinite

Antidifferentiation by Inspection

1. $\int e^x\,\mathrm{d}x = e^x + C$
2. $\int \cos x\,\mathrm{d}x = \sin x + C$
3. $\int -\sin x\,\mathrm{d}x = \cos x + C$
4. $\int \frac{1}{x}\,\mathrm{d}x = \log|x| + C$
5. $\int 1\,\mathrm{d}x = x + C$
6. $\int 2x\,\mathrm{d}x = x^2 + C$
7. $\int nx^{n-1}\,\mathrm{d}x = x^n + C$ $(n \neq 0$, constant)
8. $\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + C$ $(n \neq -1$, constant)

Power Rule for Antidifferentiation

$$\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + C$$

if $n \neq -1$ is a constant

Example:

$$\int \left(5x^2 - 15x + 3\right)\,\mathrm{d}x = \frac{5}{3}x^3 - \frac{15}{2}x^2 + 3x + C$$

Antiderivatives to Recognize

- $\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C$
- $\int a \, dx = ax + C$
- $\int e^x \, dx = e^x + C$
- $\int \frac{1}{x} \, dx = \log|x| + C$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \sec x \tan x \, dx = \sec x + C$
- $\int \csc x \cot x \, dx = -\csc x + C$
- $\int \csc^2 x \, dx = -\cot x + C$
- $\int \frac{1}{1+x^2} \, dx = \arctan x + C$
- $\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$

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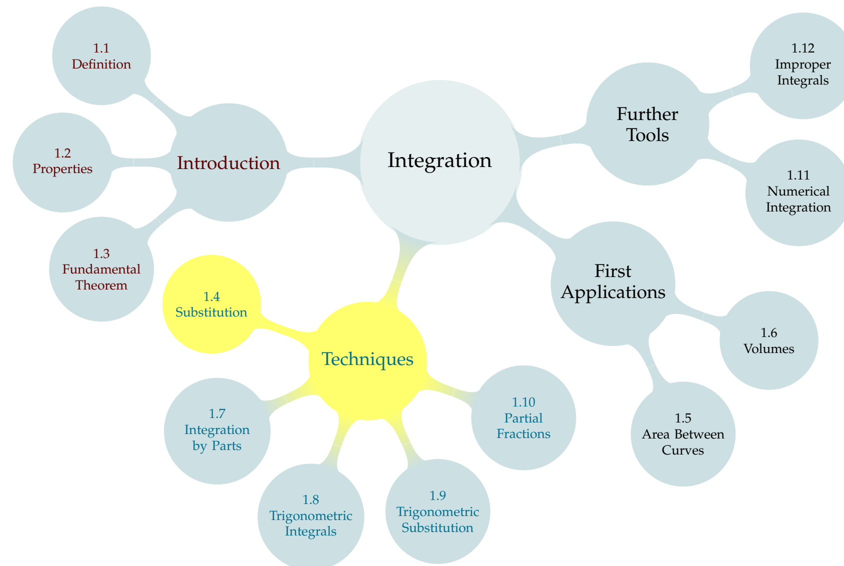

Antiderivatives

Fact:

$$\frac{d}{dx}\left\{\sin\left(x^2+x\right)\right\} =$$

Related Fact:

$$\int (2x+1)\cos(x^2+x)\,dx =$$

Antiderivatives

Chain Rule:

$$\frac{d}{dx}\left\{\sin\left(\underbrace{x^2+x}_{\text{inside function}}\right)\right\} = \left(\underbrace{2x+1}_{\text{derivative of inside function}}\right)\cos\left(\underbrace{x^2+x}_{\text{inside function}}\right)$$

Hallmark of the chain rule: an “inside” function, with that function’s derivative multiplied.

Solve by Inspection

$$\int 2xe^{x^2+1}\,dx = e^{x^2+1} + C$$

$$\int \frac{1}{x}\cos(\log x)\,dx = \sin(\log x) + C$$

$$\int 3(\sin x + 1)^2 \cos x\,dx = (\sin x + 1)^3 + C$$

(Look for an "inside" function, with its derivative multiplied.)

Undoing the Chain Rule

Chain Rule:

$$\frac{d}{dx}\{f(u(x))\} = f'(u(x)) \cdot u'(x)$$

(Here, $u(x)$ is our "inside function")

Antiderivative Fact:

$$\int f'(u(x)) \cdot u'(x)\,dx = f(u(x)) + C$$

Undoing the Chain Rule

Antiderivative Fact:

$$\int f'(u(x)) \cdot u'(x)\,dx = f(u(x)) + C$$

Shorthand: call $u(x)$ simply u;
since $\frac{du}{dx} = u'(x)$, call $u'(x)\,dx$ simply du.

$$\int f'(u(x)) \cdot u'(x)\,dx = \int f'(u)\,du\Big|_{u=u(x)} = f\big(u(x)\big) + C$$

This is the substitution rule.

We saw these integrals before, and solved them by inspection. Now try using the language of substitution.

$$\int 2xe^{x^2+1}\,dx$$

Using u as shorthand for x^2+1, and du as shorthand for $2x\,dx$:
$\int 2xe^{x^2+1}\,dx = \int e^u\,du = e^u + C = e^{x^2+1} + C$

$$\int \frac{1}{x}\cos(\log x)\,dx$$

Using u as shorthand for $\log x$, and du as shorthand for $\frac{1}{x}\,dx$:
$\int \frac{1}{x}\cos(\log x)\,dx = \int \cos(u)\,du = \sin(u) + C = \sin(\log x) + C$

$$\int 3(\sin x + 1)^2 \cos x\,dx$$

Using u as shorthand for $\sin x + 1$, and du as shorthand for $\cos x\,dx$:
$\int 3(\sin x + 1)^2 \cos x\,dx = \int 3u^2\,du = u^3 + C = (\sin x + 1)^3 + C$

$$\int (3x^2)\sin(x^3+1)\,\mathrm{d}x =$$

$$\int (3x^2)\sin(x^3+1)\,\mathrm{d}x = \int \sin(u)\,\mathrm{d}u\Big|_{u=x^3+1}$$

"Inside" function: x^3+1. Its derivative: $3x^2$
Shorthand: $x^3+1 \to u$, $3x^2\,\mathrm{d}x \to \mathrm{d}u$

$$\begin{aligned}\int (3x^2)\sin(x^3+1)\,\mathrm{d}x &= \int \sin(u)\,\mathrm{d}u\Big|_{u=x^3+1}\\ &= -\cos(u)+C\big|_{u=x^3+1}\\ &= \cos(x^3+1)+C\end{aligned}$$

"Inside" function: x^3+1. Its derivative: $3x^2$
Shorthand: $x^3+1 \to u$, $3x^2\,\mathrm{d}x \to \mathrm{d}u$

Warning 1: We don't just change $\mathrm{d}x$ to $\mathrm{d}u$. We need to couple $\mathrm{d}x$ with the derivative of our inside function.
After all, we're undoing the chain rule! We need to have an "inside derivative."

Warning 2: The final answer is a function of x.

We used the substitution rule to conclude

$$\int (3x^2)\sin(x^3+1)\,\mathrm{d}x = -\cos(x^3+1)+C$$

We can check that our antiderivative is correct by differentiating.

We saw:

$$\int 3x^2 \sin(x^3+1)\,dx = -\cos(x^3+1) + C$$

So, we can evaluate:

$$\int_0^1 3x^2 \sin(x^3+1)\,dx = -\cos(x^3+1)\Big|_0^1 = \cos(1) - \cos(2)$$

Alternately, we can put in the limits of integration as we substitute. The bounds are originally given as values of x; we simply change them to values of u.
If $u(x) = x^3+1$, then $u(0) = 1$ and $u(1) = 2$.

$$\underbrace{\int_0^1}_{x\text{-values}} 3x^2 \sin(x^3+1)\,dx = \underbrace{\int_1^2}_{u\text{-values}} \sin(u)\,du = -\cos(2) + \cos(1)$$

Notation: limits of integration

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\,dx$$

Let $u = \sin x$, $du = \cos x\,dx$. Note the limits (or bounds) of integration $\pi/4$ and $\pi/2$ are values of x, not u: they follow the differential, unless otherwise specified.

$$\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\,dx$$

$x = \frac{\pi}{2}$
$x = \frac{\pi}{4}$

True or False?

1. Using $u = x^2$,

$$\int e^{x^2}\,dx = \int e^u\,du$$

   False: missing $u'(x)$.
   $du = (2x\,dx) \neq dx$

2. Using $u = x^2+1$,

$$\int_0^1 x\sin(x^2+1)\,dx = \int_0^1 \frac{1}{2}\sin u\;du$$

   False: limits of integration didn't translate.
   When $x = 0$, $u = 0^2+1 = 1$, and when $x = 1$, $u = 1^2+1 = 2$.

Evaluate $\displaystyle\int_0^1 x^7\left(x^4+1\right)^5\,dx$.

$$u = x^4+1,\ du = 4x^3\,dx$$
$$u(0) = 1,\ u(1) = 2$$
$$x^4 = u-1,\ x^3\,dx = \tfrac{1}{4}\,du$$
$$\int_0^1 x^7\left(x^4+1\right)^5 dx = \int_0^1 (x^4)\cdot(x^4+1)^5\cdot x^3\,dx$$
$$= \int_1^2 (u-1)\cdot u^5\cdot\tfrac{1}{4}\,du$$
$$= \tfrac{1}{4}\int_1^2 (u^6-u^5)\,du$$
$$= \tfrac{1}{4}\left[\tfrac{1}{7}u^7 - \tfrac{1}{6}u^6\right]_1^2$$
$$= \tfrac{1}{4}\left[\tfrac{2^7}{7} - \tfrac{2^6}{6} - \tfrac{1}{7} + \tfrac{1}{6}\right]$$

Time permitting, more examples using the substitution rule

$$\text{Evaluate } \int \sin x \cos x \, dx.$$

Let $u = \sin x$, $du = \cos x \, dx$:

$$\int \sin x \cos x \, dx = \int u \, du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$$

Or, let $u = \cos x$, $du = -\sin x \, dx$:

$$\int \cos x \sin x \, dx = -\int u \, du = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2 x + C$$

Recall $\sin^2 x + \cos^2 x = 1$ for all x, so $\frac{1}{2}\sin^2 x = -\frac{1}{2}\cos^2 x + \frac{1}{2}$. The two answers look different, but they only differ by a constant, which can be absorbed in the arbitrary constant C. If we rename the second C to C' so that the second answer is $-\frac{1}{2}\cos^2 x + C'$, then $C' = C + \frac{1}{2}$.

Check Our Work

We can check that $\int \sin x \cos x \, dx =$ by differentiating.

$$\frac{d}{dx}\left\{\frac{1}{2}\sin^2 x + C\right\} = \frac{2}{2}\sin x \cdot \cos x = \sin x \cos x$$

Our answer works.

We can check that $\int \sin x \cos x \, dx =$ by differentiating.

$$\frac{d}{dx}\left\{-\frac{1}{2}\cos^2 x + C\right\} = -\frac{2}{2}\cos x \cdot (-\sin x) = \sin x \cos x$$

This answer works too.

$$\text{Evaluate } \int \frac{\log x}{3x} \, dx.$$

Let $u = \log x$, $du = \frac{1}{x} \, dx$:

$$\begin{aligned}\int \frac{\log x}{3} \cdot \frac{1}{x} \, dx &= \frac{1}{3}\int u \, du \\ &= \frac{1}{6}u^2 + C \\ &= \frac{1}{6}\log^2 x + C\end{aligned}$$

Check Our Work

We can check that $\int \frac{\log x}{3x}\,dx = \qquad$ by differentiating.

$$\frac{d}{dx}\left\{\frac{1}{6}\log^2 x + C\right\} = \frac{2}{6}\log x \cdot \frac{1}{x} = \frac{\log x}{3x}$$

Our answer works.

Evaluate $\int \frac{e^x}{e^x+15}\,dx$.

Let $u = e^x + 15$, $du = e^x\,dx$

$$\int \frac{e^x}{e^x+15}\,dx = \int \frac{1}{u}\,du = \log|u| + C = \log|e^x+15| + C$$

In this case, since $e^x + 15 > 0$, the absolute values on $|e^x + 15|$ are optional.

Evaluate $\int x^4(x^5+1)^8\,dx$.

Let $u = x^5 + 1$, $du = 5x^4\,dx$. Then, $x^4\,dx = \frac{1}{5}\,du$.

$$\int x^4(x^5+1)^8\,dx = \int \frac{1}{5}(u)^8\,du$$
$$= \frac{1}{5}\cdot\frac{1}{9}u^9 + C = \frac{1}{45}(x^5+1)^9 + C$$

Check Our Work

We can check that $\int \frac{e^x}{e^x+15}\,dx = \qquad$ by differentiating.

$$\frac{d}{dx}\{\log|e^x+15| + C\} = \frac{1}{e^x+15}\cdot e^x = \frac{e^x}{e^x+15}$$

Our answer works.

We can check that $\int x^4(x^5+1)^8\,dx = \qquad$ by differentiating.

$$\frac{d}{dx}\left\{\frac{1}{45}(x^5+1)^9 + C\right\} = \frac{9}{45}(x^5+1)^8\cdot 5x^4 = (x^5+1)^8x^4$$

Our answer works.

Evaluate $\int_4^8 \frac{s}{s-3}\,ds$. Be careful to use correct notation.

Let $u = s - 3$, $du = ds$.
Then $s = u + 3$, $u(4) = 1$ and $u(8) = 5$.

$$\int_4^8 \frac{s}{s-3}\,ds = \int_1^5 \frac{u+3}{u}\,du$$
$$= \int_1^5 \left(1 + \frac{3}{u}\right)du$$
$$= [u + 3\log|u|]_1^5$$
$$= [5 + 3\log 5] - [1 + 3\log 1]$$
$$= 4 + 3\log 5$$

Evaluate $\displaystyle\int x^9(x^5+1)^8\,dx$.

Let $u = x^5+1$, $du = 5x^4\,dx$.
Then $x^4\,dx = \frac{1}{5}\,du$, $x^5 = u-1$.

$$\begin{aligned}\int x^9(x^5+1)^8\,dx &= \int (x^4)\cdot(x^5)\cdot(x^5+1)^8\,dx\\ &= \int\left(\frac{1}{5}\right)\cdot(u-1)\cdot u^8\,du = \frac{1}{5}\int(u^9-u^8)\,du\\ &= \frac{1}{5}\left[\frac{1}{10}u^{10}-\frac{1}{9}u^9\right]+C\\ &= \frac{1}{5}\left[\frac{(x^5+1)^{10}}{10}-\frac{(x^5+1)^9}{9}\right]+C\end{aligned}$$

Q

Check Our Work

We can check that $\displaystyle\int x^9(x^5+1)^8\,dx =$ by differentiating.

$$\begin{aligned}&\frac{d}{dx}\left\{\frac{1}{5}\left[\frac{(x^5+1)^{10}}{10}-\frac{(x^5+1)^9}{9}\right]+C\right\}\\ &= \frac{1}{5}\left[(x^5+1)^9\cdot 5x^4-(x^5+1)^8\cdot 5x^4\right]\\ &= x^4(x^5+1)^9-x^4(x^5+1)^8\\ &= x^4(x^5+1)^8\left[(x^5+1)-1\right]\\ &= x^4(x^5+1)^8[x^5]\\ &= x^9(x^5+1)^8\end{aligned}$$

Our answer works.

Particularly Tricky Substitution

Evaluate $\displaystyle\int \frac{1}{e^x+e^{-x}}\,dx$.

Let $u = e^x$, $du = e^x\,dx$. Then $dx = \frac{du}{e^x} = \frac{du}{u}$.

$$\begin{aligned}\int\frac{1}{e^x+e^{-x}}\,dx &= \int\frac{1}{u+\frac{1}{u}}\,\frac{du}{u}\\ &= \int\frac{1}{u^2+1}\,du\\ &= \arctan(u)+C\\ &= \arctan(e^x)+C\end{aligned}$$

Q

Check Our Work

We can check that $\displaystyle\int\frac{1}{e^x+e^{-x}}\,dx =$ by differentiating.

$$\begin{aligned}\frac{d}{dx}\left\{\arctan(e^x)+C\right\} &= \frac{1}{(e^x)^2+1}\cdot e^x\\ &= \frac{e^x}{(e^x)^2+1}\\ &= \frac{e^x}{(e^x)^2+1}\cdot\frac{e^{-x}}{e^{-x}}\\ &= \frac{1}{e^x+e^{-x}}\end{aligned}$$

Our answer works.

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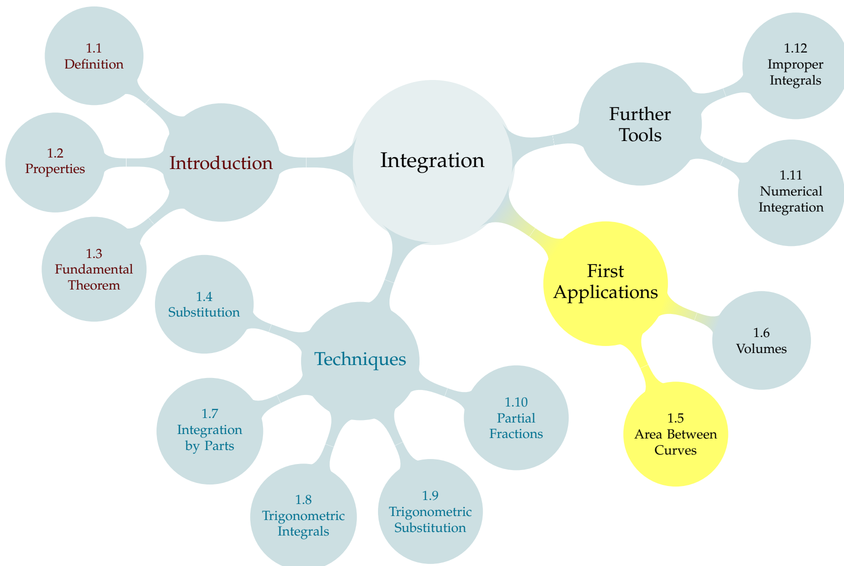
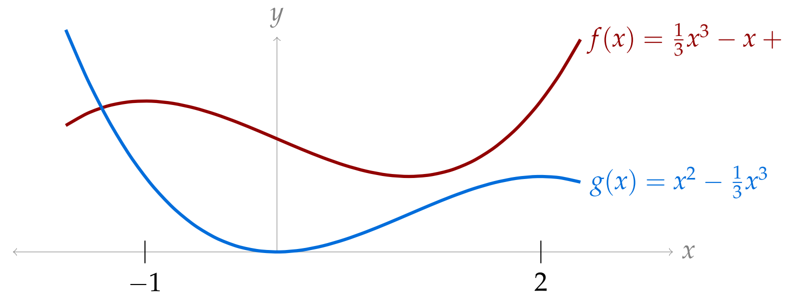


Find the area between $f(x)$ and $g(x)$ from $x = -1$ to $x = 2$.

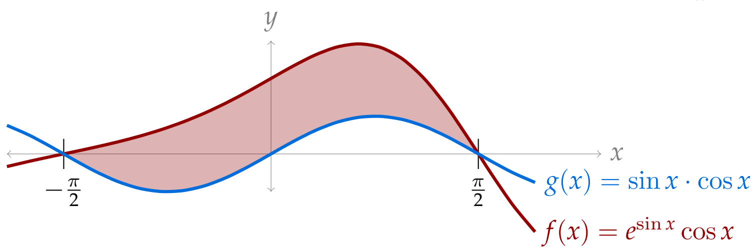


$$= \int_{-1}^{2} \left[\frac{1}{3}x^3 - x + 2 - x^2 + \frac{1}{3}x^3\right] dx$$

$$= \int_{-1}^{2} \left[\frac{2}{3}x^3 - x^2 - x + 2\right] dx$$

$$= \left[\frac{1}{6}x^4 - \frac{1}{3}x^3 - \frac{1}{2}x^2 + 2x\right]_{-1}^{2}$$

Find the (unsigned) area between $f(x)$ and $g(x)$ from $x = -\frac{\pi}{2}$ to $x = \frac{\pi}{2}$.



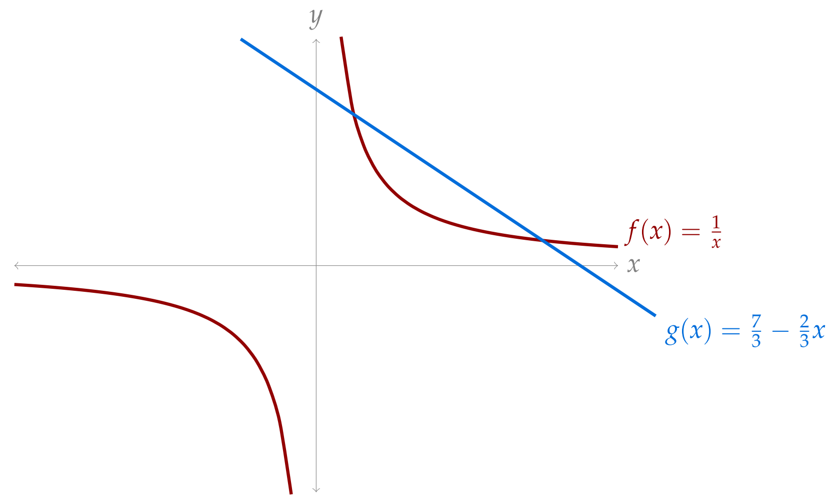
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[f(x) - g(x)\right] dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(e^{\sin x}\cos x - \sin x \cos x\right) dx$$

Let $u = \sin x$.
Then: $du = \cos x \, dx$, $u\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$, $u\left(\frac{-\pi}{2}\right) = \sin\left(\frac{-\pi}{2}\right) = -1$.

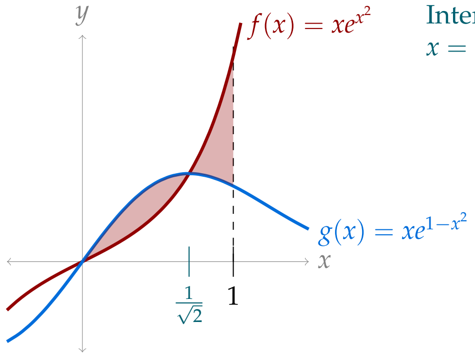
$$= \int_{-1}^{1} \left(e^u - u\right) du$$

$$= \left[e^u - \frac{1}{2}u^2\right]_{-1}^{1}$$

Find the (unsigned) area of the finite region bounded by $f(x)$ and $g(x)$.



Find the (unsigned) area in the figure below between the curves $f(x)$ and $g(x)$ from $x = 0$ to $x = 1$.



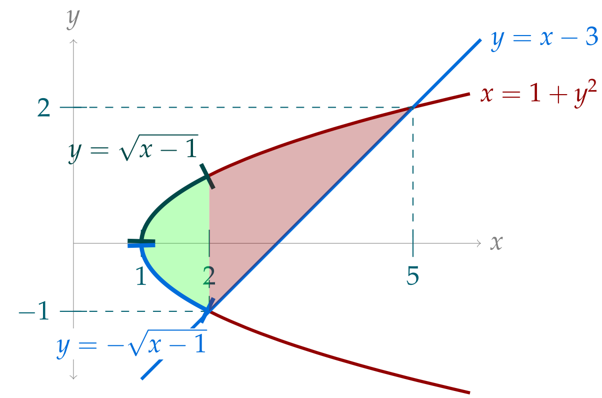
Intersections at $x = 0$ and $x = \pm\frac{1}{\sqrt{2}}$:

$$xe^{x^2} = xe^{1-x^2}$$
$$e^{x^2} = e^{1-x^2} \text{ or } x = 0$$
$$x^2 = 1 - x^2$$
$$x = \pm\frac{1}{\sqrt{2}}$$

$$\text{Area} = \int_0^{\frac{1}{\sqrt{2}}} \left[g(x) - f(x)\right] \mathrm{d}x + \int_{\frac{1}{\sqrt{2}}}^1 \left[f(x) - g(x)\right] \mathrm{d}x$$
$$= \int_0^{\frac{1}{\sqrt{2}}} \left[xe^{1-x^2} - xe^{x^2}\right] \mathrm{d}x + \int_{\frac{1}{\sqrt{2}}}^1 \left[xe^{x^2} - xe^{1-x^2}\right] \mathrm{d}x$$

Q

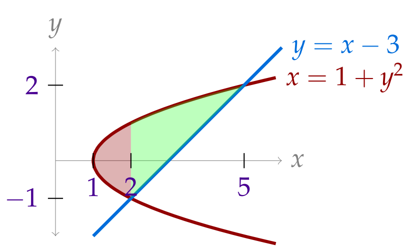
Set up, but do not evaluate, integral(s) to find the (unsigned) area of the finite region bounded by $x = 1 + y^2$ and $y = x - 3$.

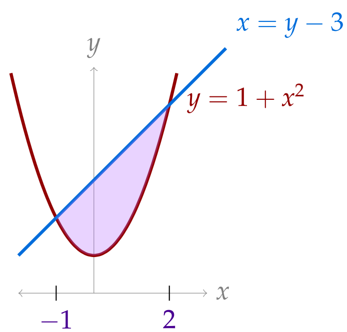


Option 1: $\displaystyle\int_1^2 \left[\sqrt{x-1} - \left(-\sqrt{x-1}\right)\right] \mathrm{d}x + \int_2^5 \left[\sqrt{x-1} - (x-3)\right] \mathrm{d}x$

Q

Set up, but do not evaluate, integral(s) to find the (unsigned) area of the finite region bounded by $x = 1 + y^2$ and $y = x - 3$.





$$\int_{-1}^2 \left[(x+3) - (1+x^2)\right] \mathrm{d}x$$

ans

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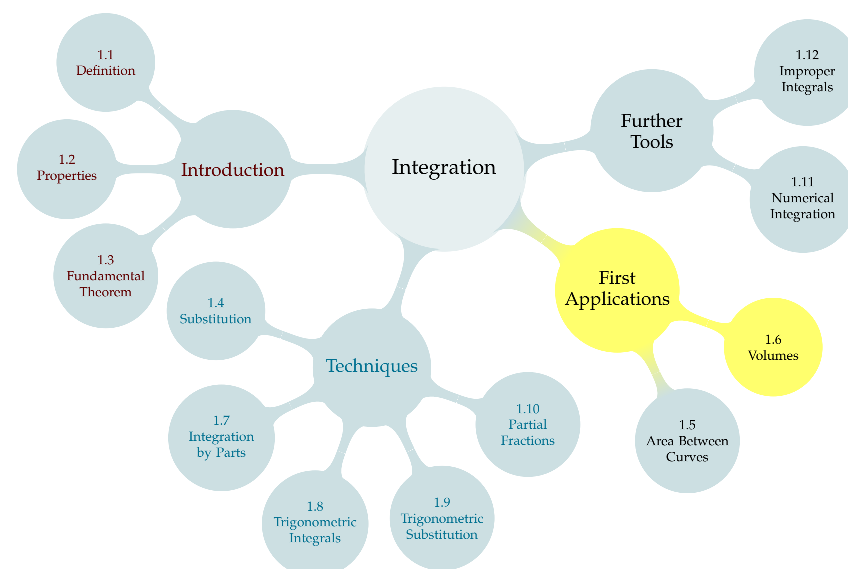
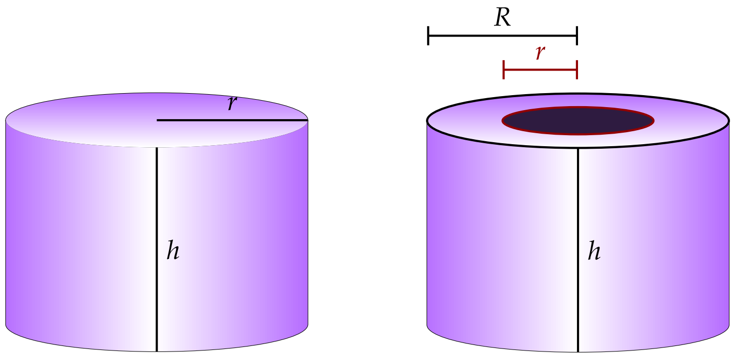




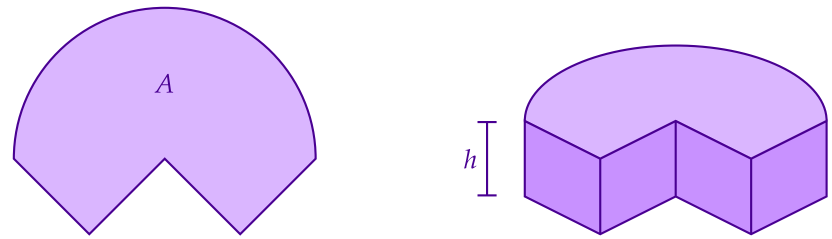
The volume of a cylinder with radius r and height h is:

$\pi r^2 h$

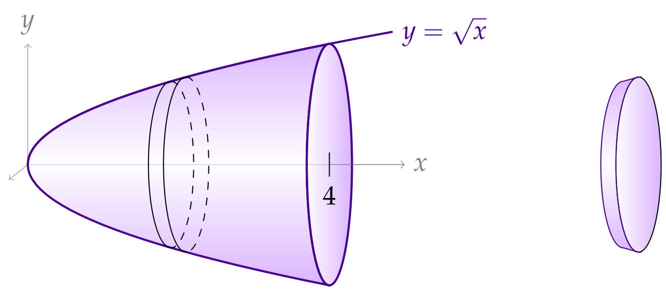
The volume of a washer, with outer radius R, inner radius r, and height h is:

$\left(\pi R^2 h - \pi r^2 h\right) \;=\; \pi h \left(R^2 - r^2\right)$

More generally, if we have a shape of area A, and we extrude it into a solid of height h, the resulting solid has volume: Ah



Consider the volume, V, enclosed by rotating the curve $y = \sqrt{x}$, from $x = 0$ to $x = 4$, around the x-axis.

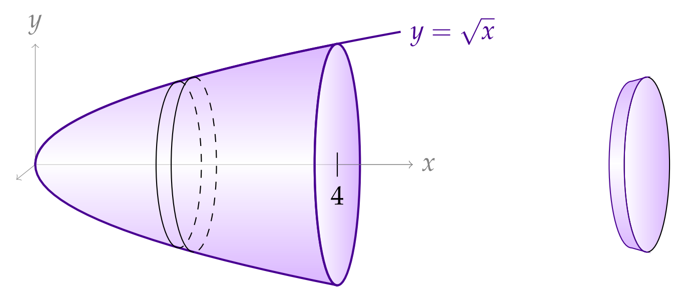


We cut the solid into slices, and approximate the volume of each slice. Each thin slice is *approximately* a cylinder.

If we use n slices, the width of each is: $\frac{4}{n}$.

The radius of the slice at $x = x_i^*$ is: $\sqrt{x_i^*}$.

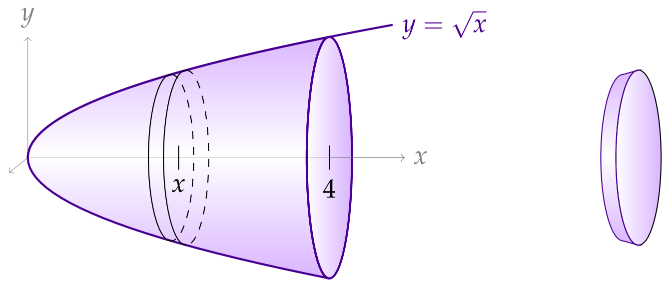
Consider the volume, V, enclosed by rotating the curve $y = \sqrt{x}$, from $x = 0$ to $x = 4$, around the x-axis.



$$V \approx \sum_{i=1}^{n} (\text{volume of each slice}) = \sum_{i=1}^{n} \pi \left(\sqrt{x_i^*}\right)^2 \frac{4}{n} = \sum_{i=1}^{n} \underbrace{\pi x_i^*}_{f(x_i^*)} \underbrace{\frac{4}{n}}_{\Delta x}$$

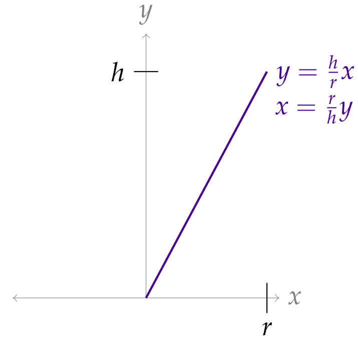
This is a Riemann sum for $\displaystyle\int_0^4 \pi x \, dx$.

Consider the volume, V, enclosed by rotating the curve $y = \sqrt{x}$, from $x = 0$ to $x = 4$, around the x-axis.



Informally, we think of one slice, at position x, as having thickness dx. So, we can write the volume of this slice as:

Summing up the volumes of slices from $x = 0$ to $x = 4$, our total volume is:



Let h and r be positive constants.

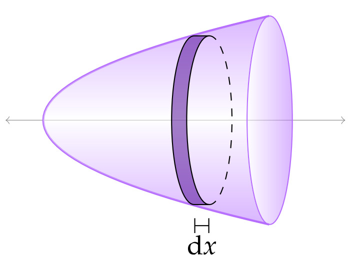
1. What familiar solid results from rotating the line segment from $(0, 0)$ to (r, h) around the y-axis?
2. In the informal manner of the last example, describe the volume of a horizontal slice of the cone taken at height y.
3. What is the volume of the entire cone?

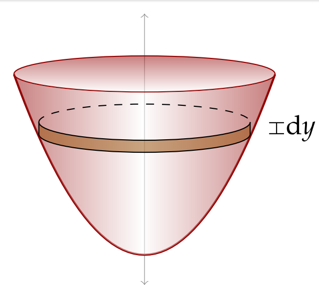
Cone volume: $\int_0^h \pi \left(\frac{r}{h}y\right)^2 dy = \left[\frac{\pi r^2}{3h^2}y^3\right]_{y=0}^{y=h} = \frac{\pi r^2}{3h^2}(h^3 - 0) = \frac{\pi}{3}r^2 h$

Observation

When we rotated around the horizontal axis, the width of our cylindrical slices was dx, and our integrand was written in terms of x.

When we rotated around the vertical axis, the width of our cylindrical slices was dy, and we integrated in terms of y.

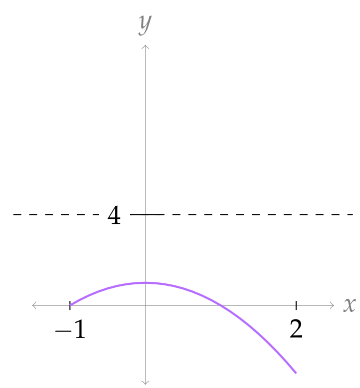




Vertical slices are approximately cylinders

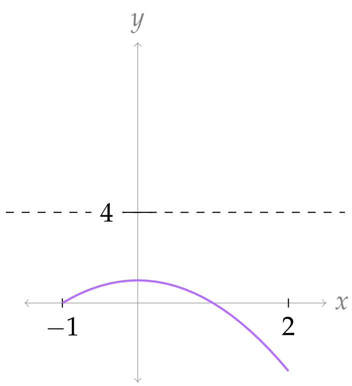
Horizontal slices are approximately cylinders

In this question, we will find the volume enclosed by rotating the curve $y = 1 - x^2$, from $x = -1$ to $x = 2$, about the line $y = 4$.



1. Sketch the surface traced out by the rotating curve.
2. Sketch a cylindrical slice. (Consider: will it be horizontal or vertical?)
3. Give the volume of your slice. Use dx or dy for the width, as appropriate.
4. Integrate (with the appropriate limits of integration) to find the volume of the solid.

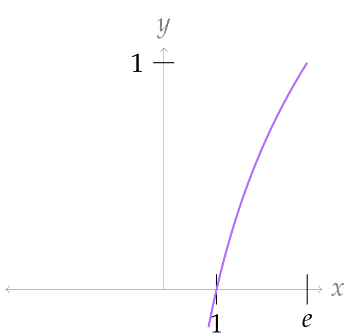
In this question, we will find the volume enclosed by rotating the curve $y = 1 - x^2$, from $x = -1$ to $x = 2$, about the line $y = 4$.



To find the volume of the entire object, we "add up" the slices from $x = -1$ to $x = 2$ by integrating.

$$\begin{aligned}\int_{-1}^{2} \pi(3+x^2)^2 \mathrm{d}x &= \pi \int_{-1}^{2} (9 + 6x^2 + x^4)\, \mathrm{d}x \\ &= \pi \left[9x + 2x^3 + \frac{1}{5}x^5\right]_{-1}^{2} \\ &= \pi \left[\left(18 + 16 + \frac{32}{5}\right) - \left(-9 - 2 - \frac{1}{5}\right)\right] \\ &= \pi \left[\left(40 + \frac{2}{5}\right) + \left(11 + \frac{1}{5}\right)\right] \\ &= 51.6\pi\end{aligned}$$

Let A be the area between the curve $y = \log x$ and the x-axis, from $(1, 0)$ to $(e, 1)$. In this question, we will consider the volume of the solid formed by rotating A about the y-axis.

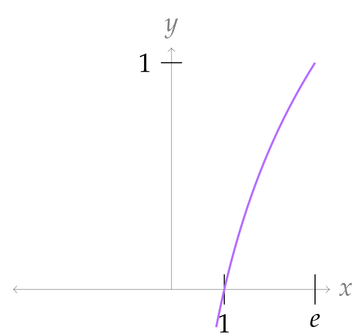


1. Sketch A.
2. Sketch a washer-shaped slice of the solid. (Should it be horizontal or vertical?)
3. Give the volume of your slice. Use $\mathrm{d}x$ or $\mathrm{d}y$ for the width, as appropriate.
4. Integrate to find the volume of the entire solid.

The outer radius is e, while the inner radius at height y is $x = e^y$.

Slice volume at height y: $\pi\left(e^2 - (e^y)^2\right)\mathrm{d}y = \pi\left(e^2 - e^{2y}\right)\mathrm{d}y$

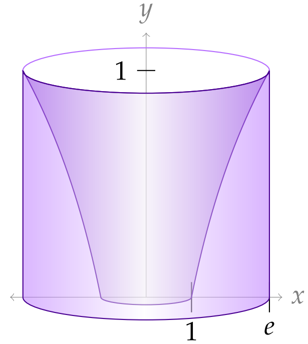
Let A be the area between the curve $y = \log x$ and the x-axis, from $(1, 0)$ to $(e, 1)$. In this question, we will consider the volume of the solid formed by rotating A about the y-axis.



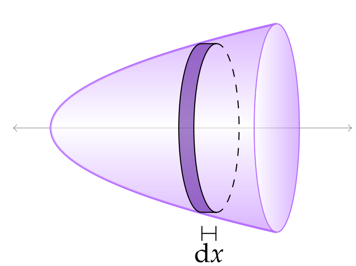
The outer radius is e, while the inner radius at height y is $x = e^y$.

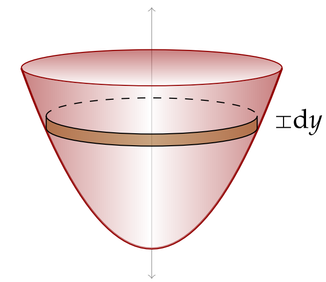
Slice volume at height y: $\pi\left(e^2 - (e^y)^2\right)\mathrm{d}y = \pi\left(e^2 - e^{2y}\right)\mathrm{d}y$

To find the volume of the entire object, we "add up" the slices from $y = 0$ to $y = 1$ by integrating.
Below we use the substitution rule with $u = 2y$ and $du = 2dy$. With practice, you'll probably be able to do this substitution in your head, but we have written it out for clarity



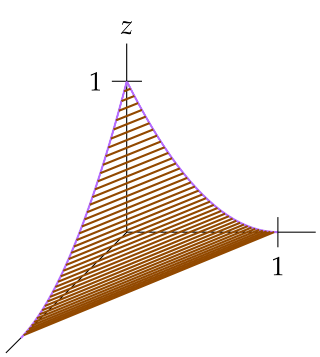
So far, we've found the volume of solids formed by rotating a curve. When a point rotates about a fixed centre, the result is a circle, so we could slice those solids up into pieces that are approximately cylinders.





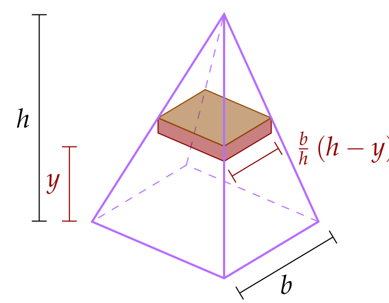
We can find the volumes of other shapes, as long as we can find the areas of their cross-sections.

The corner of a room is sealed off as follows:
On both walls, a parabola of the form $z = (x-1)^2$ is drawn, where z is the vertical axis and x is the horizontal. They start one metre above the corner, and end one metre to the side of the corner.
Taught ropes are strung *horizontally* from one parabola to the other, so the horizontal cross-sections are right triangles. How much volume is enclosed?



At height z, the cross-section is a right triangle. Its side length is the x-value on the parabola.
Solving $z = (x-1)^2$ for x, we find $x = \sqrt{z} + 1$.
So, the area of a cross-section at height z is $\frac{1}{2}\left(\sqrt{z}+1\right)^2$. We call its width dz.
All together, the enclosed volume is $\int_0^1 \frac{1}{2}\left(z + 2\sqrt{z} + 1\right)\,dz = \frac{17}{12}$ cubic metres.

A pyramid with height h metres has a square base with side-length b metres. At an elevation of y metres above the base, $0 \le y \le h$, the cross-section of the pyramid is a square with side-length $\frac{b}{h}(h-y)$.
What is the volume of the pyramid?



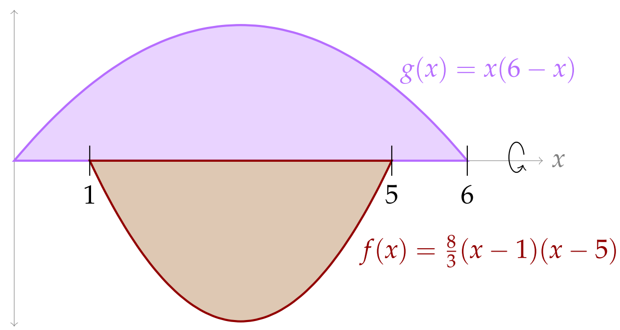
The area of the square cross-section at height y is $\left[\frac{b}{h}(h-y)\right]^2 = \frac{b^2}{h^2}\left(h^2 - 2hy + y^2\right)$.

If we give a horizontal slice width dy, then the slice volume is $\frac{b^2}{h^2}\left(h^2 - 2hy + y^2\right)dy$. So, the total volume of the pyramid is

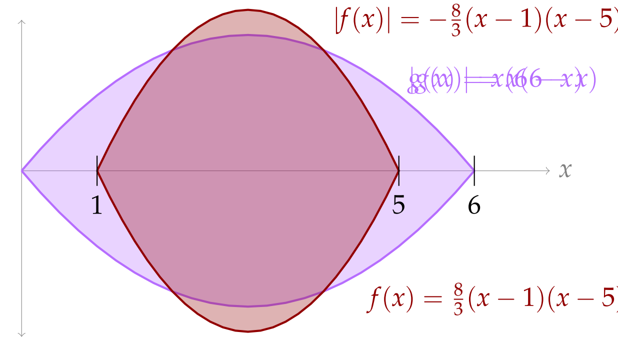
$$\int_0^h \frac{b^2}{h^2}\left(h^2 - 2hy + y^2\right)dy$$
$$= \frac{b^2}{h^2}\left[h^2 y - hy^2 + \frac{1}{3}y^3\right]_{y=0}^{y=h}$$
$$= \frac{b^2}{h^2}\left[h^3 - h^3 + \frac{1}{3}h^3\right] = \frac{1}{3}b^2 h$$

Optional: Challenge Question

A paddle fixed to the x-axis has two flat blades. One blade is in the shape of $f(x) = \frac{8}{3}(x-1)(x-5)$, from $x = 1$ to $x = 5$. The other blade is in the shape of $g(x) = x(6-x)$, $0 \le x \le 6$. The paddle turns through a gelatinous fluid, scraping out a hollow cavity as it turns. What is the volume of this cavity?
You may leave your answer as an integral, or sum of integrals.



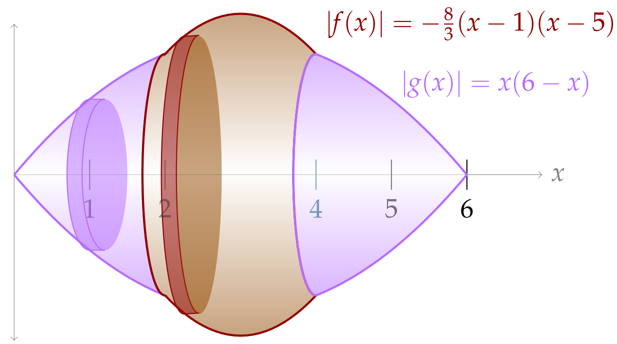
The size of the cavity at a point x along the paddle is determined by the larger of $|f(x)|$ and $|g(x)|$.



The radius of a cylindrical slice is $|g(x)| = x(6-x)$ when $0 < x < 2$ and $4 < x < 6$, and the radius is $|f(x)| = -\frac{8}{3}(x-1)(x-5)$ when $2 < x < 4$.
$|f(x)|^2 = [f(x)]^2$, so we can drop our absolute values in this step.

$$\text{Volume} = \int_0^2 \pi\,(6x-x^2)^2\,dx + \int_2^4 \pi\left(\frac{8}{3}(x^2-6x+5)\right)^2 dx$$
$$+ \int^6 \pi\,(6x-x^2)^2\,dx$$

The size of the cavity at a point x along the paddle is determined by the larger of $|f(x)|$ and $|g(x)|$.



Let's find where $|f(x)| = |g(x)|$:

$$x(6-x) = -\frac{8}{3}(x-1)(x-5)$$
$$6x - x^2 = -\frac{8}{3}(x^2-6x+5) = -\frac{8}{3}x^2 + 16x - \frac{40}{3}$$
$$\frac{5}{3}x^2 - 10x + \frac{40}{3} = 0$$
$$x^2 - 6x + 8 = 0$$
$$(x-2)(x-4) = 0$$
$$x = 2, \quad x = 4$$

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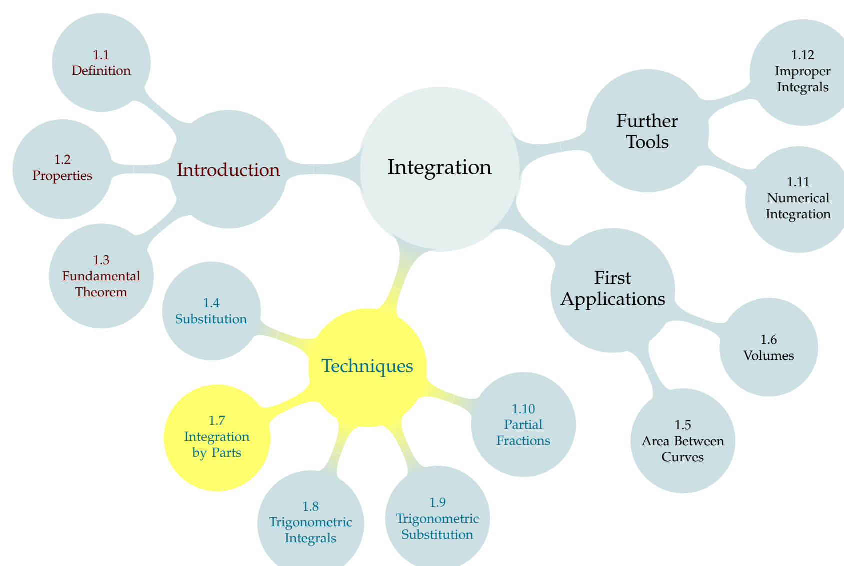

Reverse the Product Rule

Product Rule:

$$\frac{\mathrm{d}}{\mathrm{d}x}\{u(x)\cdot v(x)\} = u'(x)\cdot v(x) + u(x)\cdot v'(x)$$

Related fact:

$$\int \Big[u'(x)\cdot v(x) + u(x)\cdot v'(x)\Big]\mathrm{d}x = u(x)\cdot v(x) + C$$

Rearrange:

$$\implies \int \Big[u'(x)v(x)\Big]\mathrm{d}x + \int \Big[u(x)v'(x)\Big]\mathrm{d}x = u(x)v(x) + C$$

$$\implies \int \Big[u(x)v'(x)\Big]\mathrm{d}x = u(x)v(x) - \int \Big[v(x)u'(x)\Big]\mathrm{d}x + C$$

Integration by Parts

$$\int \Big[u(x)v'(x)\Big]\mathrm{d}x = u(x)v(x) - \int \Big[v(x)u'(x)\Big]\mathrm{d}x$$

Example: $\int xe^x\mathrm{d}x$

Let $u(x) = x$ and $v'(x) = e^x$. (We'll talk **later** about choosing these)
Then $u'(x) = 1$ and $v(x) = e^x$.

$$\int \Big[u(x)v'(x)\Big]\mathrm{d}x = u(x)v(x) - \int \Big[v(x)u'(x)\Big]\mathrm{d}x$$

$$\int \Big[xe^x\Big]\mathrm{d}x = x(e^x) - \int \Big[(e^x)1\Big]\mathrm{d}x + C$$

$$\int xe^x = xe^x - \int (e^x)\mathrm{d}x + C$$

$$= xe^x - e^x + C$$

Check Our Work

In the previous slide, we evaluated

$$\int xe^x\mathrm{d}x = xe^x - e^x + C$$

for some constant C. We can check that this is correct by differentiating.

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big\{xe^x - e^x + C\Big\} = (xe^x + e^x) - e^x = xe^x$$

We used the product rule to differentiate. Remember integration by parts helps us to reverse the product rule.

Integration by Parts (IBP): A Closer Look

$$\int \Big[u(x)v'(x)\Big]\mathrm{d}x = u(x)v(x) - \int \Big[v(x)u'(x)\Big]\mathrm{d}x + C$$

$$\underbrace{\int xe^x\mathrm{d}x}_{\text{How to integrate??}} = x(e^x) - \underbrace{1\int e^x\mathrm{d}x}_{\text{Easy to integrate!}} + C$$

We start and end with an integral. IBP is only useful if the new integral is somehow an improvement.

We differentiate the function we choose as $u(x)$, and antidifferentiate the function we choose as $v'(x)$

Choosing $u(x)$ and $v(x)$

$$\int \left[u(x)v'(x)\right] dx = u(x)v(x) - \int \left[v(x)u'(x)\right] dx + C$$

$$\int \left[x \sin x\right] dx =$$

Option A:

$u(x) = x$ | $u'(x) = 1$

$v'(x) = \sin x$ | $v(x) = -\cos x$

$\rightarrow \int -\cos x \cdot 1 \, dx$

Option B:

$u(x) = \sin x$ | $u'(x) = \cos x$

$v'(x) = x$ | $v(x) = \frac{1}{2}x^2$

$\rightarrow \int \frac{1}{2}x^2 \cdot \cos x \, dx$

Option A:

$$\int x \sin x \, dx = x(-\cos x) - \int -\cos x \, dx = -x\cos x + \sin x + C$$

Fine Print: We can choose any antiderivative of $v'(x)$ to be $v(x)$. So, we omit "+C."

Check Our Work

To check our work, we can calculate $\frac{d}{dx}\left\{-x\cos x + \sin x + C\right\}$. It should work out to be $x \sin x$.

$$\frac{d}{dx}\left\{-x\cos x + \sin x + C\right\} = (-x)(-\sin x) + (\cos x)(-1) + \cos x = x \sin x$$

Our answer works!

Choosing $u(x)$ and $v(x)$

$$\int \left[u(x)v'(x)\right] dx = u(x)v(x) - \int \left[v(x)u'(x)\right] dx + C$$

$$\int \left[x^2 \log x\right] dx =$$

Option A:

$u(x) = x^2$ | $u'(x) = 2x$

$v'(x) = \log x$ | $v(x) = ??$

$\rightarrow \int ?? \cdot 2x \, dx$

Option B:

$u(x) = \log x$ | $u'(x) = \frac{1}{x}$

$v'(x) = x^2$ | $v(x) = \frac{1}{3}x^3$

$\rightarrow \int \frac{1}{3}x^3 \cdot \frac{1}{x} \, dx$

Option B:

$$\int x^2 \log x \, dx = \log x \cdot \frac{1}{3}x^3 - \int \frac{1}{3}x^3 \cdot \frac{1}{x} \, dx$$

$$= \frac{1}{3}x^3 \log x - \frac{1}{3}\int x^2 \, dx = \frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C$$

Check Our Work

To check our work, we can calculate $\frac{d}{dx}\left\{\frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C\right\}$. It should work out to be $x^2 \log x$.

$$\frac{d}{dx}\left\{\frac{1}{3}x^3 \log x - \frac{1}{9}x^3 + C\right\} = x^2 \log x + \frac{1}{3}x^3 \cdot \frac{1}{x} - \frac{3}{9}x^2$$
$$= x^2 \log x$$

Our answer works.

$$\int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x + C$$

$$\int \left[\tfrac{1}{2}xe^{6x}\right]\mathrm{d}x =$$

Option A:

$u(x) = \frac{1}{2}x$ | $u'(x) = \frac{1}{2}$

$v'(x) = e^{6x}$ | $v(x) = \frac{1}{6}e^{6x}$

$\rightarrow \int \frac{1}{6}e^{6x}\cdot\frac{1}{2}\,\mathrm{d}x$

Option B:

$u(x) = e^{6x}$ | $u'(x) = 6e^{6x}$

$v'(x) = \frac{1}{2}x$ | $v(x) = \frac{1}{4}x^2$

$\rightarrow \int \frac{1}{4}x^2\cdot 6e^{6x}\,\mathrm{d}x$

Option A:

$$\int \tfrac{1}{2}x\cdot e^{6x}\,\mathrm{d}x = \frac{1}{2}x\cdot\frac{1}{6}e^{6x} - \int \frac{1}{6}e^{6x}\cdot\frac{1}{2}\,\mathrm{d}x$$

$$= \frac{1}{12}xe^{6x} - \frac{1}{12}\int e^{6x}\mathrm{d}x = \frac{1}{12}xe^{6x} - \frac{1}{72}e^{6x} + C$$

Check Our Work

We check that $\int \left[\frac{1}{2}xe^{6x}\right]\mathrm{d}x = \frac{1}{12}xe^{6x} - \frac{1}{72}e^{6x} + C$ by differentiating.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{12}xe^{6x} - \frac{1}{72}e^{6x} + C\right\} = \frac{1}{12}x\cdot 6e^{6x} + e^{6x}\cdot\frac{1}{12} - \frac{6}{72}e^{6x}$$

$$= \frac{1}{2}xe^{6x} + \frac{1}{12}e^{6x} - \frac{1}{12}e^{6x}$$

$$= \frac{1}{2}xe^{6x}$$

Our answer works.

Mnemonic

$$\int \left[u(x)v'(x)\right]\mathrm{d}x = u(x)v(x) - \int \left[v(x)u'(x)\right]\mathrm{d}x + C$$

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u + C$$

We abbreviate:

- $u(x) \rightarrow u$
- $u'(x)\,\mathrm{d}x \rightarrow \mathrm{d}u$
- $v(x) \rightarrow v$
- $v'(x)\,\mathrm{d}x \rightarrow \mathrm{d}v$

Choosing u, $\mathrm{d}v$ in your head

Choose u and $\mathrm{d}v$ to evaluate the integral below:

$$\int (3t+5)\cos(t/4)\mathrm{d}t$$

Thoughts: $\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$

u gets differentiated, and $\mathrm{d}v$ gets antidifferentiated.

Evaluate, using IBP or Substitution

$$\int u\,dv = uv - \int v\,du + C$$

- $\int xe^{x^2}\,dx$
- $\int x^2e^x\,dx$
- $\int e^{x+e^x}\,dx$

(sub) $\int xe^{x^2}\,dx = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C$

(IBP) $\int \underbrace{x^2}_{u}\underbrace{e^x dx}_{dv} = x^2\cdot e^x - \int e^x \cdot 2x\,dx$

$= x^2e^x - 2\int \underbrace{x}_{u}\underbrace{e^x\,dx}_{dv} = x^2e^x - 2\left[xe^x - \int e^x\,dx\right]$

$= x^2e^x - 2xe^x + 2e^x + C$

(sub) $\int e^{x+e^x}\,dx = \int e^{e^x}\cdot e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$

Definite Integration by Parts

Method 1: Antidifferentiate first, then plug in limits of integration.

Method 2: Plug as you go.

Evaluate $\int_1^e \log^2 x\,dx$

Method 1:

Let $u = \log^2 x$, $dv = 1dx$; $du = 2\log x \cdot \frac{1}{x}dx$, $v = x$

$$\int \log^2 x\,dx = x\log^2 x - \int 2\log x\,dx$$

Now let $u = \log x$, $dv = 2dx$; $du = \frac{1}{x}dx$, $v = 2x$

$$= x\log^2 x - \left[2x\log x - \int 2dx\right] = x\log^2 x - 2x\log x + 2x + C$$

$$\int_1^e \log^2 x\,dx = \left[x\log^2 x - 2x\log x + 2x + C\right]_1^e$$

$$= (e - 2e + 2e + C) - (0 - 0 + 2 + C) = e - 2$$

Method 2:

Let $u = \log^2 x$, $dv = 1dx$; $du = 2\log x \cdot \frac{1}{x}dx$, $v = x$

$$\int_1^e \log^2 x\,dx = \left[x\log^2 x\right]_1^e - \int_1^e 2\log x\,dx = (e-0) - \int_1^e 2\log x\,dx$$

Now let $u = \log x$, $dv = 2\,dx$; $du = \frac{1}{x}dx$, $v = 2x$

$$= e - \left[\left[2x\log x\right]_1^e - \int_1^e 2\,dx\right] = e - (2e - 0) + \left[2x\right]_1^e$$

Special Technique: $v'(x) = 1$

$$\int u\,dv = uv - \int v\,du + C$$

Evaluate $\int \log x\,dx$ using integration by parts.

$$\int \log x\,dx = \int \underbrace{\log x}_{u}\cdot\underbrace{1\,dx}_{dv}$$

$$= \log x \cdot x - \int x \cdot \frac{1}{x}\,dx$$

$$= x\log x - \int 1\,dx = x\log x - x + C$$

Check Our Work

Let's check that $\int \log x \, dx = x \log x - x + C$.

$$\frac{d}{dx}\left\{x \log x - x + C\right\} = x \cdot \frac{1}{x} + \log x - 1 = 1 + \log x - 1 = \log x$$

So we have indeed found the antiderivative of $\log x$.

$$\int u \, dv = uv - \int v \, du + C$$

Evaluate $\int \arctan x \, dx$ using integration by parts.

Hint: $\arctan x = (\arctan x)(1)$, and $\frac{d}{dx}\{\arctan x\} = \dfrac{1}{1+x^2}$

$$\int \underbrace{\arctan x}_{u} \cdot \underbrace{1 \, dx}_{dv} = \arctan x \cdot x - \int x \cdot \frac{1}{1+x^2} \, dx$$

Set $s = 1 + x^2$, $ds = 2x \, dx$.

$$= x \arctan x - \frac{1}{2} \int \frac{1}{s} \, ds$$

$$= x \arctan x - \frac{1}{2} \log |1 + x^2| + C$$

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Check Our Work

Let's check that $\int \arctan x \, dx = x \arctan x - \frac{1}{2} \log |1 + x^2| + C$.

$$\begin{aligned}\frac{d}{dx}\left\{x \arctan x - \frac{1}{2} \log |1+x^2| + C\right\} &= x \cdot \frac{1}{1+x^2} + \arctan x - \frac{1}{2} \cdot \frac{2x}{1+x^2} \\ &= \frac{x}{1+x^2} + \arctan x - \frac{x}{1+x^2} \\ &= \arctan x\end{aligned}$$

So we have indeed found the antiderivative of $\arctan x$.

Setting $dv = 1 \, dx$ is a very specific technique. You'll probably only see it in situations integrating logarithms and inverse trigonometric functions.

$$\int \log x \, dx, \quad \int \arcsin x \, dx, \quad \int \arccos x \, dx, \quad \int \arctan x \, dx, \quad \text{etc.}$$

Evaluate $\int e^x \cos x \, dx$ using integration by parts.

Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$ and $v = \sin x$:

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$$

Let $u = e^x$ and $dv = \sin x \, dx$. Then $du = e^x \, dx$ and $v = -\cos x$:

$$= e^x \sin x - \left[-e^x \cos x - \int -e^x \cos x \, dx \right]$$

$$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$$2 \int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C$$

$$\int e^x \cos x \, dx = \frac{1}{2}\left(e^x \sin x + e^x \cos x\right) + C$$

Integrating around in a circle

We can use this technique to antidifferentiate products of two functions that almost, but don't quite, stay the same under (anti)differentiation.

Use integration by parts a number of times, ending up with an expression involving (a scalar multiple of) the original integral.

To do this, be consistent with your choice of u and dv.

Evaluate $\int \cos(\log x) \, dx$.

Let $u = \cos(\log x)$, $dv = dx$; then $du = -\frac{\sin(\log x)}{x} dx$, $v = x$

$$\int \cos(\log x) dx = x \cos(\log x) - \int \left(-\frac{\sin(\log x)}{x} \right) x \, dx$$

$$= x \cos(\log x) + \int \sin(\log x) \, dx$$

Let $u = \sin(\log x)$, $dv = dx$; then $du = \frac{\cos(\log x)}{x}$, $v = x$

$$= x \cos(\log x) + x \sin(\log x) - \int \cos(\log x) \, dx$$

So, $$2 \int \cos(\log x) \, dx = x \cos(\log x) + x \sin(\log x)$$

$$\int \cos(\log x) \, dx = \frac{x}{2} \left[\cos(\log x) + \sin(\log x)\right] + C$$

Check Our Work

We check that $\int \cos(\log x) dx = \frac{x}{2}\left[\cos(\log x) + \sin(\log x)\right] + C$ by differentiating.

$$\frac{d}{dx}\left\{\frac{x}{2}\left[\cos(\log x) + \sin(\log x)\right] + C\right\}$$

$$= \frac{x}{2}\left[\frac{-\sin(\log x)}{x} + \frac{\cos(\log x)}{x}\right] + \frac{1}{2}\left[\cos(\log x) + \sin(\log x)\right]$$

$$= -\frac{1}{2}\sin(\log x) + \frac{1}{2}\cos(\log x) + \frac{1}{2}\cos(\log x) + \frac{1}{2}\sin(\log x)$$

$$= \cos(\log x)$$

Our answer works.

Evaluate $\int e^{2x}\sin x\,dx$ using integration by parts.

Let $u = e^{2x}$ and $dv = \sin x\,dx$. Then $du = 2e^{2x}\,dx$ and $v = -\cos x$.

$$\int e^{2x}\sin x\,dx = e^{2x}(-\cos x) - \int(-\cos x)2e^{2x}\,dx$$
$$= -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx$$

Let $u = e^{2x}$ and $dv = \cos x\,dx$. Then $du = 2e^{2x}\,dx$ and $v = \sin x$

$$\int e^{2x}\sin x\,dx = -e^{2x}\cos x + 2\left[e^{2x}\sin x - \int 2e^{2x}\sin x\,dx\right]$$
$$= -e^{2x}\cos x + 2e^{2x}\sin x - 4\int e^{2x}\sin x\,dx$$
$$5\int e^{2x}\sin x\,dx = -e^{2x}(\cos x - 2\sin x)$$
$$\int e^{2x}\sin x\,dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$$

Check Our Work

We can check our work by differentiating $\frac{1}{5}e^{2x}[2\sin x - \cos x] + C$. We should end up with $e^{2x}\sin x$.

$$\frac{d}{dx}\left\{\frac{1}{5}e^{2x}(2\sin x - \cos x) + C\right\} = \frac{1}{5}e^{2x}(2\cos x + \sin x) + \frac{2}{5}e^{2x}(2\sin x - \cos x)$$
$$= \frac{2}{5}e^{2x}\cos x + \frac{1}{5}e^{2x}\sin x + \frac{4}{5}e^{2x}\sin x - \frac{2}{5}e^{2x}\cos x$$
$$= e^{2x}\sin x$$

Our answer, strange though it looks, is the correct antiderivative.

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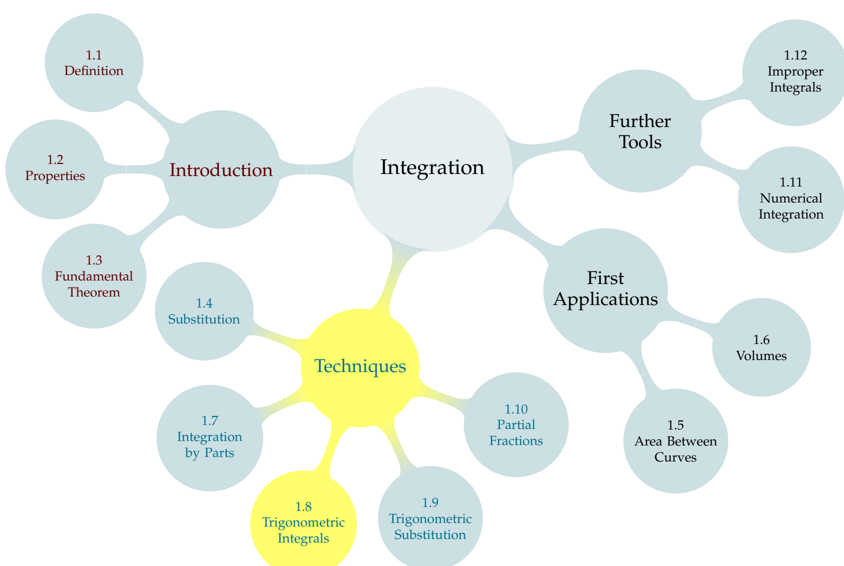

1.8 Trigonometric Integrals

Recall:

- $\sin^2 x + \cos^2 x = 1$
- $\tan^2 x + 1 = \sec^2 x$
- $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$
- $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$
- $\sin(2x) = 2\sin x\cos x$

Integrating Products of Sine and Cosine

Let $u = \sin x$, $du = \cos x\,dx$

$$\int \sin x \cos x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$$

Integrating Products of Sine and Cosine

Let $u = \sin x$, $du = \cos x\,dx$

$$\int \sin x \cos x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$$

Let $u = \sin x$, $du = \cos x\,dx$

$$\int \sin^{10}x \cos x\,dx = \int u^{10}\,du = \frac{1}{11}u^{11} + C = \frac{1}{11}\sin^{11} x + C$$

Check Our Work

If we are correct that $\int \sin x \cos x\,dx =$, then it should be true that $\frac{d}{dx}\left\{ \quad \right\} = \sin x \cos x$.

We differentiate, using the chain rule:

$$\frac{d}{dx}\left\{\frac{\sin^2 x}{2} + C\right\} = \frac{2}{2}\sin x \cos x = \sin x \cos x$$

Our answer works.

Check Our Work

If we are correct that $\int \sin^{10} x \cos x\,dx =$, then it should be true that $\frac{d}{dx}\left\{ \quad \right\} = \sin^{10} x \cos x$.

We differentiate, using the chain rule:

$$\frac{d}{dx}\left\{\frac{\sin^{11} x}{11} + C\right\} = \frac{11}{11}\sin^{10} x \cos x = \sin^{10} x \cos x$$

Our answer works.

Integrating Products of Sine and Cosine

Let $u = \sin x$, $du = \cos x\, dx$

$$\int_0^{\frac{\pi}{2}} \sin^{\pi+1} x \cos x\, dx = \int_{\sin(0)}^{\sin(\pi/2)} u^{\pi+1} du = \frac{1}{\pi+2} u^{\pi+2}\Big|_0^1$$

$$= \frac{1}{\pi+2}$$

Check Our Work

If we are correct that $\int \sin^{\pi+1} x \cos x\, dx = \qquad$, then it should be true that $\frac{d}{dx}\left\{ \qquad \right\} = \sin^{\pi+1} x \cos x$.

We differentiate, using the chain rule:

$$\frac{d}{dx}\left\{\frac{\sin^{\pi+2} x}{\pi+2} + C\right\} = \frac{\pi+2}{\pi+2}\sin^{\pi+1} x \cos x = \sin^{\pi+1} x \cos x$$

Our answer works.

Integrating Products of Sine and Cosine

Let $u = \sin x$, $du = \cos x\, dx$.

$$\int \sin^{10} x \cos^3 x\, dx = \int \sin^{10} x \cos^2 x \cos x\, dx$$

$$= \int \sin^{10} x\,(1 - \sin^2 x) \cos x\, dx$$

$$= \int u^{10}(1-u^2)\, du = \int (u^{10} - u^{12})\, du$$

$$= \frac{1}{11}u^{11} - \frac{1}{13}u^{13} + C = \frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C$$

Check Our Work

If we are correct that $\int \sin^{10} x \cos^3 x\, dx = \qquad$, then it should be true that $\frac{d}{dx}\left\{ \qquad \right\} = \sin^{10} x \cos^3 x$.

We differentiate, using the chain rule:

$$\frac{d}{dx}\left\{\frac{\sin^{11} x}{11} - \frac{\sin^{13} x}{13} + C\right\} = \frac{11}{11}\sin^{10} x \cos x - \frac{13}{13}\sin^{12} x \cos x$$

$$= \sin^{10} x\left(1 - \sin^2 x\right) \cos x = \sin^{10} x \cos^2 x \cos x$$

$$= \sin^{10} x \cos^3 x$$

Our answer works.

Integrating Products of Sine and Cosine

$u = \cos x$, $du = -\sin x\,dx$ $\sin^2 x + \cos^2 x = 1.$

$$\int \sin^5 x \cos^4 x\,dx = \int (\sin^2 x)^2 \cos^4 x \sin x\,dx$$
$$= \int (1-\cos^2 x)^2 \cos^4 x \sin x\,dx$$
$$= -\int (1-u^2)^2 u^4\,du = -\int (1-2u^2+u^4)u^4 du$$
$$= -\int (u^4 - 2u^6 + u^8) du = -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C$$
$$= -\tfrac{1}{5}\cos^5 x + \tfrac{2}{7}\cos^7 x - \tfrac{1}{9}\cos^9 x + C$$

Check Our Work

If we are correct that $\int \sin^5 x \cos^4 x\,dx =$, then it should be true that $\frac{d}{dx}\{$ $\} = \sin^5 x \cos^4 x.$

We differentiate, using the chain rule:

$$\frac{d}{dx}\left\{-\tfrac{1}{5}\cos^5 x + \tfrac{2}{7}\cos^7 x - \tfrac{1}{9}\cos^9 x + C\right\}$$
$$= \frac{5}{5}\cos^4 x \sin x - \frac{2\cdot 7}{7}\cos^6 x \sin x + \frac{9}{9}\cos^8 x \sin x$$
$$= \cos^4 x \sin x \left(1 - 2\cos^2 x + \cos^4 x\right)$$
$$= \cos^4 x \sin x \left(1-\cos^2 x\right)^2 = \cos^4 x \sin x \left(\sin^2 x\right)^2$$
$$= \sin^5 x \cos^4 x$$

Our answer works.

Generalize: $\int \sin^m x \cos^n bx\,dx$

To use the substitution $u = \sin x$, $du = \cos x\,dx$:

- We need to reserve one $\cos x$ for the differential.
- We need to convert the remaining $\cos^{n-1} x$ to $\sin x$ terms.
- We convert using $\cos^2 x = 1 - \sin^2 x$. To avoid square roots, that means $n-1$ should be even when we convert.
- So, we can use this substitution when the original power of cosine, n, is ODD: one cosine goes to the differential, the rest are converted to sines.

Generalize: $\int \sin^m x \cos^n x\,dx$

To use the substitution $u = \cos x$, $du = -\sin x\,dx$:

- We need to reserve one $\sin x$ for the differential.
- We need to convert the remaining $\sin^{m-1} x$ to $\cos x$ terms.
- We convert using $\sin^2 x = 1 - \cos^2 x$. To avoid square roots, that means $m-1$ should be even when we convert.
- So, we can use this substitution when the original power of sine, m, is ODD: one sine goes to the differential, the rest are converted to cosines.

MNEMONIC: "ODD ONE OUT"

Integrating $\int \sin^m x \cos^n x \, dx$

If you want to use $u = \sin x$, there should be an odd power of cosine.

If you want to use $u = \cos x$, there should be an odd power of sine.

Carry out a suitable substitution (but do not evaluate the resulting integral):

- $\int \sin^4 x \cos^7 x \, dx$
- $\int \sin^7 x \cos^4 x \, dx$
- $\int \sin^7 x \cos^7 x \, dx$

$\int \sin^4 x \cos^7 x \, dx$

The power of cosine is odd, so it becomes our differential. That is, we use $u = \sin x$, $du = \cos x \, dx$.

$$\begin{aligned}
&\int \sin^4 x \cos^7 x \, dx \\
&= \int \sin^4 x \, (\cos^2 x)^3 \cos x \, dx \\
&= \int \sin^4 x \, (1 - \sin^2 x)^3 \cos x \, dx \\
&= \int u^4 (1 - u^2)^3 \, du
\end{aligned}$$

$\int \sin^7 x \cos^4 x \, dx$

The power of sine is odd, so it becomes our differential. That is, we

To evaluate $\int \sin^m x \cos^n x \, dx$, we use:

- $u = \sin x$ if n is odd, and/or
- $u = \cos x$ if m is odd

What if n and m are both even?

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2} \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\begin{aligned}
\int \sin^2 x \, dx &= \int \frac{1 - \cos 2x}{2} \, dx \\
&= \frac{1}{2} \int (1 - \cos 2x) \, dx \\
&= \frac{1}{2} \left(x - \frac{1}{2} \sin 2x \right) + C
\end{aligned}$$

$= \int \sin^7 x (1 - \sin^2 x)^3 \cos x \, dx$

$= \int (1 - \cos^2 x)^3 \cos^7 x \sin x \, dx$

Check Our Work

We check that $\int \sin^2 x \, dx =$ by differentiating:

$$\frac{d}{dx}\left\{\frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C\right\} = \frac{1}{2}\left(1 - \frac{1}{2}(\cos 2x)(2)\right)$$
$$= \frac{1-\cos 2x}{2} = \sin^2 x$$

So, our answer works.

$$\cos^2 x = \frac{1+\cos 2x}{2} \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

Evaluate $\int \sin^4 x \, dx$.

$$\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$$
$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) \, dx$$
$$= \frac{1}{4}\int (1 - 2\cos 2x) \, dx + \frac{1}{4}\int \cos^2(2x) \, dx$$
$$= \frac{1}{4}(x - \sin 2x) + \frac{1}{4}\int \left(\frac{1+\cos(4x)}{2}\right) dx$$
$$= \frac{1}{4}(x - \sin 2x) + \frac{1}{8}\left(x + \frac{1}{4}\sin(4x)\right) + C$$
$$= \frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C$$

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Check Our Work

We want to check that $\int \sin^4 x \, dx =$

Note $\sin^2 x = \frac{1-\cos(2x)}{2}$, so $\cos(2x) = 1 - 2\sin^2 x$. Also remember $\frac{1}{2}\sin(2x) = \sin x \cos x$.

$$\frac{d}{dx}\left\{\frac{3}{8}x - \frac{1}{4}\sin(2x) + \frac{1}{32}\sin(4x) + C\right\} = \frac{3}{8} - \frac{2}{4}\cos(2x) + \frac{4}{32}\cos(4x)$$
$$= \frac{3}{8} - \frac{1}{2}\left(1 - 2\sin^2 x\right) + \frac{1}{8}\left(1 - 2\sin^2(2x)\right)$$
$$= \frac{3}{8} - \frac{1}{2} + \sin^2 x + \frac{1}{8} - \frac{1}{4}\sin^2(2x)$$
$$= \sin^2 x - \left(\frac{1}{2}\sin 2x\right)^2 = \sin^2 x - \sin^2 x \cos^2 x$$
$$= \sin^2 x(1 - \cos^2 x) = \sin^2 x(\sin^2 x) = \sin^4 x$$

So, our answer works.

Recall:

- $\frac{d}{dx}\{\tan x\} = \sec^2 x$
- $\frac{d}{dx}\{\sec x\} = \sec x \tan x$
- $\tan^2 x + 1 = \sec^2 x$

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx \qquad u = \cos x \quad du = -\sin x \, dx$$

$$= -\int \frac{1}{u} \, du = -\log|u| + C$$

$$= \log|u^{-1}| + C = \log\left|\frac{1}{\cos x}\right| + C$$

$$= \log|\sec x| + C$$

Check Our Work

Let's check that $\int \tan x dx =$ by differentiating.

$$\frac{d}{dx}\{\log|\sec x| + C\} = \frac{\sec x \tan x}{\sec x} = \tan x$$

So, our answer works.

Optional: A nifty trick – you won't be expected to come up with it. There is some motivation for the trick in Example 1.8.19 in the CLP-2 text.

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$$

$$= \int \left(\frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\right) dx$$

set $u = \sec x + \tan x,\ du = (\sec x \tan x + \sec^2 x)\, dx$

$$= \int \frac{1}{u} \, du = \log|u| + C$$

$$= \log|\sec x + \tan x| + C$$

Useful integrals:

- $\int \tan x \, dx = \log|\sec x| + C$
- $\int \sec x \, dx = \log|\sec x + \tan x| + C$

1. $\int \sec x \tan x \, dx = \sec x + C$

2. $\int \sec^2 x \, dx = \tan x + C$

3. $\int \tan x \, dx = \log|\sec x| + C$

4. $\int \sec x \, dx = \log|\sec x + \tan x| + C$

Evaluate using the substitution rule:

$u = \tan x, \quad du = \sec^2 x \, dx$

$$\int \tan^5 x \sec^2 x \, dx = \int u^5 du = \frac{1}{6}u^6 + C = \frac{1}{6}\tan^6 x + C$$

$u = \sec x, \; du = \sec x \tan x \, dx$

$$\int \sec^4 x \, (\sec x \tan x) \, dx = \int u^4 du = \frac{1}{5}u^5 + C = \frac{1}{5}\sec^5 x + C$$

Check Our Work

Let's check that $\int \tan^5 x \sec^2 x \, dx =$ by differentiating.

$$\frac{d}{dx}\left\{\frac{1}{6}\tan^6 x + C\right\} = \frac{6}{6}\tan^5 x \sec^2 x = \tan^5 x \sec^2 x$$

So, our answer works.

Evaluate using the identity $\sec^2 x = 1 + \tan^2 x$

$$\int \tan^4 x \sec^6 x \, dx =$$

$$\int \tan^3 x \sec^5 x \, dx =$$

Check Our Work

Let's check that $\int \tan^4 x \sec^6 x \, dx =$

$$\frac{d}{dx}\left\{\frac{1}{5}\tan^5 x + \frac{2}{7}\tan^7 x + \frac{1}{9}\tan^9 x + C\right\}$$
$$= \tan^4 x \sec^2 x + 2\tan^6 x \sec^2 x + \tan^8 x \sec^2 x$$
$$= \tan^4 x \sec^2 x(1 + 2\tan^2 x + \tan^4 x) = \tan^4 x \sec^2 x(1+\tan^2 x)^2$$
$$= \tan^4 x \sec^2 x(\sec^2 x)^2 = \tan^4 x \sec^6 x$$

So, our answer works.

Choosing a Substitution: $\int \tan^m x \sec^n x \, dx$

Using $u = \sec x$, $du = \sec x \tan x \, dx$:

- Reserve $\sec x \tan x$ for the differential. (m, n should each be at least 1)
- From the remaining $\tan^{m-1} x \ \sec^{n-1} x$, convert all tangents to secants using $\tan^2 x + 1 = \sec^2 x$. ($m - 1$ should be even, to avoid square roots)

To use the substitution $u = \sec x$, $du = \sec x \tan x \, dx$ to evaluate $\int \tan^m x \sec^n x \, dx$, n should be at least one, and m should be odd.

Choosing a Substitution: $\int \tan^m x \sec^n x \, dx$

Using $u = \tan x$, $du = \sec^2 x \, dx$:

- Reserve $\sec^2 x$ for the differential. ($n \geq 2$)
- From the remaining terms, convert all secants to tangents using $\tan^2 x + 1 = \sec^2 x$. ($n - 2$ should be even, to avoid square roots)

To use the substitution $u = \tan x$, $du = \sec^2 x \, dx$ to evaluate $\int \tan^m x \sec^n x \, dx$, n should be even (and at least 2).

Evaluating $\int \tan^m x \sec^n \, dx$

To evaluate $\int \tan^m x \sec^n \, dx$, we can use:

- $u = \sec x$ if m is odd and $n \geq 1$
- $u = \tan x$ if n is even and $n \geq 2$

Choose a substitution for the integrals below.

- $\int \sec^2 x \tan^3 x \, dx$
- $\int \sec^2 x \tan^2 x \, dx$
- $\int \sec^3 x \tan^3 x \, dx$

$$\int \sec^2 x \tan^2 x \, dx$$

Let $u = \tan x$ and $du = \sec^2 x \, dx$.

$$\int \sec^2 x \tan^2 x \, dx = \int u^2 \, du$$

(the rest you can do)

$$\int \sec^3 x \tan^3 x \, dx$$

Let $u = \sec x$ and $du = \sec x \tan x \, dx$.

$$\begin{aligned}\int \sec^3 x \tan^3 x \, dx &= \int \sec^2 x \tan^2 x \, (\sec x \tan x) \, dx \\ &= \int \sec^2 x(\sec^2 x - 1)(\sec x \tan x) \, dx \\ &= \int u^2(u^2 - 1) \, du\end{aligned}$$

(the rest you can do)

$$\text{Evaluate} \int \tan^3 x \, dx = \int \frac{\sin^3 x}{\cos^3 x} \, dx$$

Let $u = \cos x$, $du = -\sin x \, dx$.

$$\begin{aligned}&= \int \frac{\sin^2 x}{\cos^3 x} \sin x \, dx = \int \frac{1 - \cos^2 x}{\cos^3 x} \sin x \, dx \\ &= -\int \frac{1 - u^2}{u^3} du \\ &= \int \left(\frac{1}{u} - u^{-3}\right) du \\ &= \log|u| + \frac{1}{2}u^{-2} + C \\ &= \log|\cos x| + \frac{1}{2}\sec^2 x + C\end{aligned}$$

Check Our Work

Let's check that $\int \tan^3 x \, dx =$ by differentiating.

$$\begin{aligned}\frac{d}{dx}\left\{\log|\cos x| + \frac{1}{2}\sec^2 x + C\right\} &= \frac{-\sin x}{\cos x} + \frac{1}{2}(2\sec x)\sec x \tan x \\ &= -\tan x + \sec^2 x \tan x \\ &= -\tan x + (\tan^2 x + 1)\tan x \\ &= -\tan x + \tan^3 x + \tan x \\ &= \tan^3 x\end{aligned}$$

So, indeed, $\int \tan^3 x \, dx = \log|\cos x| + \frac{1}{2}\sec^2 x + C$.

Generalizing the last example:

$$\begin{aligned}\int \tan^m x \sec^n x \, \mathrm{d}x &= \int \left(\frac{\sin x}{\cos x}\right)^m \left(\frac{1}{\cos x}\right)^n \mathrm{d}x \\ &= \int \frac{\sin^m x}{\cos^{m+n} x} \mathrm{d}x \\ &= \int \left(\frac{\sin^{m-1} x}{\cos^{m+n} x}\right) \sin x \, \mathrm{d}x\end{aligned}$$

To use $u = \cos x$, $\mathrm{d}u = \sin x \, \mathrm{d}x$: we will convert $\sin^{m-1}(x)$ into cosines, so $m - 1$ must be even, so m must be odd.

Evaluating $\int \tan^m x \sec^n \ \mathrm{d}x$

To evaluate $\int \tan^m x \sec^n \ \mathrm{d}x$, we can use:

- $u = \sec x$ if m is odd and $n \geq 1$
- $u = \tan x$ if n is even and $n \geq 2$
- $u = \cos x$ if m is odd
- $u = \tan x$ if m is even and $n = 0$
  (after using $\tan^2 x = \sec^2 x - 1$, maybe several times)

Evaluate $\displaystyle\int \tan^2 x \, \mathrm{d}x$

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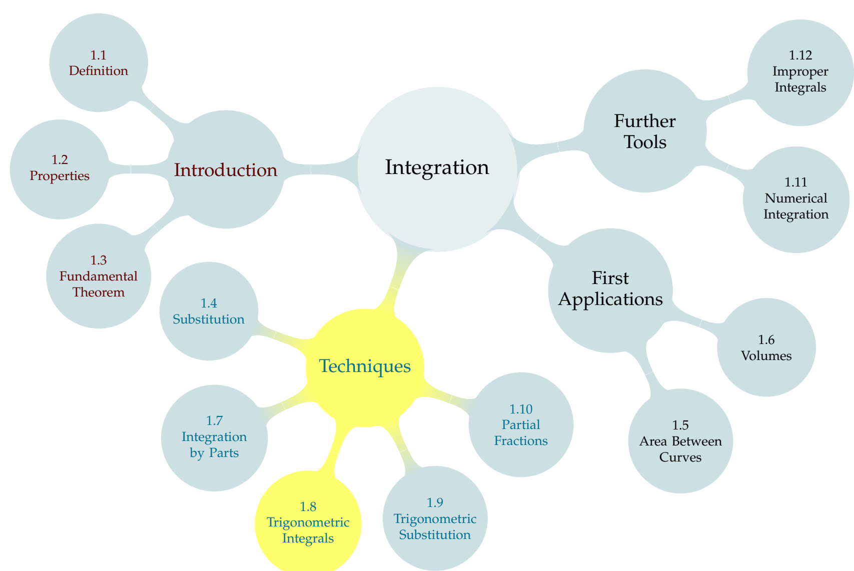

Evaluating $\int \tan^m x \sec^n \ \mathrm{d}x$

To evaluate $\int \tan^m x \sec^n \ \mathrm{d}x$, we can use:

- $u = \sec x$ if m is odd and $n \geq 1$
- $u = \tan x$ if n is even and $n \geq 2$
- $u = \cos x$ if m is odd
- $u = \tan x$ if m is even and $n = 0$
  (after using $\tan^2 x = \sec^2 x - 1$, maybe several times)

Remaining case: n odd and m is even.

The general remaining case is known, but complicated. Instead of treating it exhaustively, we'll show examples of two methods.

$\int \sec x \, dx$

We saw a way of integrating secant with the following trick:

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx$$
$$= \int \frac{1}{u} du \quad \text{with } u = \sec x + \tan x$$

Another trick: this time let $u = \sin x$, $du = \cos x \, dx$:

$$\int \sec x \, dx = \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx$$
$$= \int \frac{1}{1 - \sin^2 x} \cos x \, dx = \int \frac{1}{1 - u^2} du$$

The integrand $\frac{1}{1-u^2}$ is a rational function of u (i.e. a ratio of two polynomials). There is a procedure, called Partial Fractions, that can be used to evaluate all integrals of rational functions. We will learn it in Section 1.10.

$\int \sec^3 x \, dx$

We can integrate around in a circle (with integration by parts) to evaluate $\int \sec^3 x \, dx$. Let $u = \sec x$, $dv = \sec^2 x \, dx$. Then $du = \sec x \tan x \, dx$ and $v = \tan x$.

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$
$$= \sec x \tan x - \int \sec x \left(\sec^2 x - 1 \right) dx$$
$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$
$$= \sec x \tan x - \int \sec^3 x \, dx + \log|\sec x + \tan x| + C'$$
$$2 \int \sec^3 x \, dx = \sec x \tan x + \log|\sec x + \tan x| + C'$$
$$\int \sec^3 x \, dx = \frac{1}{2} \left(\sec x \tan x + \log|\sec x + \tan x| \right) + C$$

with $C = C'/2$.

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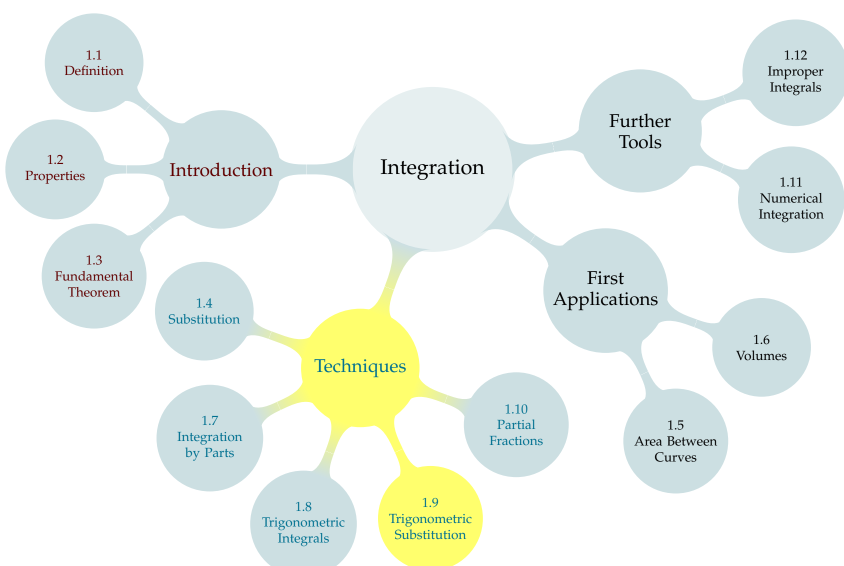


Warmup

Evaluate $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2 + 2x + 1}} dx$.

$$\int_3^7 \frac{1}{\sqrt{x^2 + 2x + 1}} dx = \int_3^7 \frac{1}{\sqrt{(x+1)^2}} dx$$
$$= \int_3^7 \frac{1}{|x+1|} dx$$

When $3 \le x \le 7$, we have $|x+1| = x+1$.

$$= \int_3^7 \frac{1}{x+1} dx$$
$$= \left[\log|x+1| \right]_3^7$$
$$= \log 8 - \log 4 = \log 2$$

Idea: $\sqrt{(\text{something})^2} = |\text{something}|$. We cancelled off the square root.

Evaluate $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$.

We still want to cancel off the square root, but x^2+1 is not obviously of the form $(\text{something})^2$.
Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$.

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sqrt{\tan^2\theta+1}}\sec^2\theta d\theta = \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}d\theta$$
$$= \int \frac{\sec^2\theta}{\sec\theta}d\theta = \int \sec\theta\,d\theta = \log|\sec\theta+\tan\theta| + C$$

We need to get these back in terms of x. From our substitution, we know $\tan\theta = x$. From simplifying our denominator, we also know $\sec\theta = \sqrt{x^2+1}$.

$$= \log\left|\sqrt{x^2+1}+x\right| + C$$

Same idea: $\sqrt{(\text{something})^2} = |\text{something}|$; cancel off the square root.

Check Our Work

Let's verify that $\displaystyle\int \frac{1}{\sqrt{x^2+1}} =$.
Seems improbable, right?

$$\frac{d}{dx}\left[\log\left|\sqrt{x^2+1}+x\right|+C\right] = \frac{1}{\sqrt{x^2+1}+x}\cdot\left(\frac{2x}{2\sqrt{x^2+1}}+1\right)$$
$$= \frac{x+\sqrt{x^2+1}}{(\sqrt{x^2+1}+x)\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}$$

So, our answer works!

Method (one standard case)

- An integrand has the form: $\sqrt{\text{quadratic}}$, and we'd like to cancel off the square root.
- So, we need to write our quadratic expression as a perfect square. Choose a helpful substitution:
  - $x = \sin\theta$, $1-\sin^2\theta = \cos^2\theta$ changes $\sqrt{1-x^2}$ into
  - $x = \tan\theta$, $1+\tan^2\theta = \sec^2\theta$ changes $\sqrt{1+x^2}$ into
  - $x = \sec\theta$, $\sec^2\theta - 1 = \tan^2\theta$ changes $\sqrt{x^2-1}$ into
- After integrating, convert back to the original variable (possibly using a triangle–more details later)

Focus on the Algebra

$1-\sin^2\theta = \cos^2\theta$ $\qquad 1+\tan^2\theta = \sec^2\theta$ $\qquad \sec^2\theta - 1 = \tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2-1}$
  Let $x = \sec\theta$, so $\sqrt{x^2-1}$ becomes $\sqrt{\sec^2\theta-1} = \sqrt{\tan^2\theta} = |\tan\theta|$
- $\sqrt{x^2+1}$
  Let $x = \tan\theta$, so $\sqrt{x^2+1}$ becomes $\sqrt{\tan^2\theta+1} = \sqrt{\sec^2\theta} = |\sec\theta|$
- $\sqrt{1-x^2}$
  Let $x = \sin\theta$ so $\sqrt{1-x^2}$ becomes $\sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = |\cos\theta|$
  (Alternately, $x = \cos\theta$ works as well)

Focus on the Algebra

$1 - \sin^2\theta = \cos^2\theta \qquad 1 + \tan^2\theta = \sec^2\theta \qquad \sec^2\theta - 1 = \tan^2\theta$

Choose a trigonometric substitution that will allow the square root to cancel out of the following expressions:

- $\sqrt{x^2+7}$
  Adjust a given identity by multiplying both sides by 7:
  $7\tan^2\theta + 7 = 7\sec^2\theta$. Now we see we want $x^2 = 7\tan^2\theta$. That is, $x = \sqrt{7}\tan\theta$:
  $\sqrt{x^2+7} = \sqrt{7\tan^2\theta+7} = \sqrt{7(\sec^2\theta)} = \sqrt{7}\,|\sec\theta|$

- $\sqrt{3-2x^2}$
  Adjust a given identity by multiplying both sides by 3:
  $3 - 3\sin^2\theta = 3\cos^2\theta$. Now we see we want $2x^2 = 3\sin^2\theta$, so $x = \sqrt{\frac{3}{2}}\sin\theta$:
  $\sqrt{3-2x^2} = \sqrt{3 - 2\left(\frac{3}{2}\sin^2\theta\right)} = \sqrt{3-3\sin^2\theta} = \sqrt{3\cos^2\theta} = \sqrt{3}\,|\cos\theta|$

Closer look at absolute values

SKIP CLOSER LOOK

Consider the substitution $x = \sin\theta$, $dx = \cos\theta\, d\theta$ for the integral:

$$\int_0^1 \sqrt{1-x^2}\, dx$$

When $x = 0$ (lower limit of integration), what is θ?
When $x = 1$ (upper limit of integration), what is θ?

If $x = 0$, then $\sin\theta = 0$, but there are infinitely many values of θ that could make this true. To use the substitution $x = \sin\theta$, we need the function $x = \sin\theta$ to be invertible. That way, we can unambiguously convert between x and θ. With that in mind, we'll actually set $\theta = \arcsin x$. Now θ is restricted to the interval $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

$$\int_0^1 \sqrt{1-x^2}\, dx = \int_{\arcsin 0}^{\arcsin 1} \sqrt{1-\sin^2\theta}\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta}\cdot\cos\theta\, d\theta$$
$$= \int_0^{\frac{\pi}{2}} |\cos\theta|\cdot\cos\theta\, d\theta$$

For $0 \le \theta \le \frac{\pi}{2}$, we have $\cos\theta \ge 0$, so $|\cos\theta| = \cos\theta$.

Closer look at absolute values

SKIP CLOSER LOOK

More generally, suppose a is a positive constant and we use the substitution $x = a\sin\theta$ for the term $\sqrt{a^2-x^2}$.

Closer look at absolute values

SKIP CLOSER LOOK

Now, consider the substitution $x = a\tan\theta$ for $\sqrt{a^2+x^2}$, where a is a positive constant.

SKIP CLOSER LOOK

Finally, consider the substitution $x = a\sec\theta$ for $\sqrt{x^2 - a^2}$, where a is a positive constant.

From now on, we will assume:

- With the substitution $x = a\sin\theta$ for $\sqrt{a^2 - x^2}$, $|\cos\theta| = \cos\theta$
- With the substitution $x = a\tan\theta$ for $\sqrt{a^2 + x^2}$, $|\sec\theta| = \sec\theta$

Identities

$$1 - \sin^2\theta = \cos^2\theta \qquad \sin(2\theta) = 2\sin\theta\cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sec^2\theta - 1 = \tan^2\theta \qquad \cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate $\displaystyle\int_0^1 (1 + x^2)^{-3/2}\,dx$

Let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$. When $x = 0$, then $\theta = \arctan 0 = 0$; when $x = 1$, then $\theta = \arctan 1 = \frac{\pi}{4}$.

$$\int_0^1 (1 + x^2)^{-3/2}\,dx = \int_{\theta=0}^{\theta=\pi/4} \frac{1}{\sqrt{1 + \tan^2\theta}^3}\sec^2\theta\,d\theta$$

$$= \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{\sec^2\theta}^3}d\theta = \int_0^{\pi/4} \frac{\sec^2\theta}{|\sec\theta|^3}d\theta$$

$$= \int_0^{\pi/4} \frac{1}{|\sec\theta|}\,d\theta = \int_0^{\pi/4} |\cos\theta|\,d\theta$$

Identities

$$1 - \sin^2\theta = \cos^2\theta \qquad \sin(2\theta) = \cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sec^2\theta - 1 = \tan^2\theta \qquad \cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

Evaluate $\displaystyle\int \sqrt{1 - 4x^2}\,dx$

Under the square root, we have "one minus a term with a variable," which matches the identity $1 - \sin^2\theta$. So, we want $4x^2$ to become $\sin^2\theta$. That is, $x = \frac{1}{2}\sin\theta$. Then $dx = \frac{1}{2}\cos\theta\,d\theta$.

$$\int \sqrt{1 - 4x^2}\,dx = \int \sqrt{1 - 4\left(\frac{1}{2}\sin\theta\right)^2} \cdot \frac{1}{2}\cos\theta\,d\theta$$

$$= \frac{1}{2}\int \sqrt{1 - \sin^2\theta} \cdot \cos\theta d\theta = \frac{1}{2}\int \sqrt{\cos^2\theta} \cdot \cos\theta d\theta$$

$$= \frac{1}{2}\int |\cos\theta| \cdot \cos\theta\,d\theta = \frac{1}{2}\int \cos^2\theta\,d\theta$$

$$= \frac{1}{2}\int \left(\frac{\quad}{2}\right) d\theta = \frac{1}{4}\int (1 + \cos(2\theta))d\theta$$

Check Our Work

In the last example, we computed

$$\int \sqrt{1-4x^2}\,dx =$$

To check, we differentiate.

$$\frac{d}{dx}\left\{\frac{1}{4}\left(\arcsin(2x)+2x\sqrt{1-4x^2}\right)+C\right\}$$
$$=\frac{1}{4}\left(\frac{2}{\sqrt{1-(2x)^2}}+2x\frac{-8x}{2\sqrt{1-4x^2}}+2\sqrt{1-4x^2}\right)$$
$$=\frac{1}{4}\left(\frac{2}{\sqrt{1-4x^2}}-\frac{8x^2}{\sqrt{1-4x^2}}+\frac{2(1-4x^2)}{\sqrt{1-4x^2}}\right)$$
$$=\frac{1}{4}\left(\frac{2-8x^2+2-8x^2}{\sqrt{1-4x^2}}\right)=\frac{1-4x^2}{\sqrt{1-4x^2}}=\sqrt{1-4x^2} \qquad \checkmark$$

Identities

$$1-\sin^2\theta=\cos^2\theta \qquad \sin(2\theta)=\cos\theta$$
$$1+\tan^2\theta=\sec^2\theta \qquad \sin^2\theta=\frac{1-\cos(2\theta)}{2}$$
$$\sec^2\theta-1=\tan^2\theta \qquad \cos^2\theta=\frac{1+\cos(2\theta)}{2}$$

Evaluate $\int \frac{1}{\sqrt{x^2-1}}\,dx$

We use the substitution $x=\sec\theta$, $dx=\sec\theta\tan\theta\,d\theta$.

To make the substitution work, we're actually taking $\theta=\arccos\left(\frac{1}{x}\right)$, and so $0\le\theta\le\pi$.

Note that the integrand exists on the intervals $x<-1$ and $x>1$.

- When $x>1$, then $0<\frac{1}{x}<1$, so $0<\arccos\left(\frac{1}{x}\right)<\frac{\pi}{2}$. That is, $0<\theta<\frac{\pi}{2}$, so $|\tan\theta|=\tan\theta$.
- When $x<-1$, then $-1<\frac{1}{x}<0$, so $\frac{\pi}{2}<\arccos\left(\frac{1}{x}\right)<\pi$. That is, $\frac{\pi}{2}<\theta<\pi$, so $|\tan\theta|=-\tan\theta$.

$$\int \frac{1}{\ldots}\,dx=\int\frac{1}{\ldots}\cdot\sec\theta\tan\theta\,d\theta=\int\frac{\sec\theta\tan\theta}{\ldots}\,d\theta$$

Check Our Work

Let's check our result, $\int \frac{1}{\sqrt{x^2-1}}dx =$

$$\frac{d}{dx}\left\{\log\left|x+\sqrt{x^2-1}\right|+C\right\}=\frac{1+\frac{2x}{2\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}=\frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}$$
$$=\frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1}}{\sqrt{x^2-1}}\right)=\frac{\left(\sqrt{x^2-1}+x\right)}{\left(x+\sqrt{x^2-1}\right)\sqrt{x^2-1}}$$
$$=\frac{1}{\sqrt{x^2-1}}$$

So, our answer works.

Completing the Square

Choose a trigonometric substitution to simplify $\sqrt{3-x^2+2x}$.

Identities have two "parts" that turn into one part:

- $1-\sin^2\theta=\cos^2\theta \qquad 4-4\sin^2\theta=4\cos^2\theta$
- $1+\tan^2\theta=\sec^2\theta$
- $\sec^2\theta-1=\tan^2\theta$

But our quadratic expression has *three* parts.

Fact: $3-x^2+2x=4-(x-1)^2$

$$\sqrt{3-x^2+2x}=\sqrt{4-(x-1)^2}$$

We want $(x-1)^2=4\sin^2\theta$, so let $(x-1)=2\sin\theta$

$$=\sqrt{4-4\sin^2\theta}=\sqrt{4\cos^2\theta}=2\cos\theta$$

Completing the Square

$$(x+b)^2 = x^2 + 2bx + b^2$$
$$c - (x+b)^2 = (c - b^2) - x^2 - 2bx$$

Write $3 - x^2 + 2x$ in the form $c - (x+b)^2$ for constants b, c.

1. Find b:
2. Solve for c:
3. All together:

Evaluate $\displaystyle\int \frac{x^2-6x+9}{\sqrt{6x-x^2}}\,dx$.

Identities have two "parts" that turn into one part:

- $1 - \sin^2\theta = \cos^2\theta$
- $1 + \tan^2\theta = \sec^2\theta$
- $\sec^2\theta - 1 = \tan^2\theta$

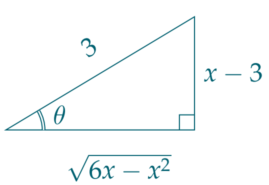
One of those parts is a constant, and one is squared.
Write $6x - x^2$ as $c - (x+b)^2$.

$$c - (x+b)^2 = (c-b^2) - x^2 - 2bx$$
$$6x = -2bx \implies b = -3$$
$$0 = c - b^2 = c - 9 \implies c = 9$$
$$6x - x^2 = 9 - (x-3)^2$$

Evaluate $\displaystyle\int \frac{x^2-6x+9}{\sqrt{6x-x^2}}\,dx = \int \frac{(x-3)^2}{\sqrt{9-(x-3)^2}}dx.$

We use the identity $9 - 9\sin^2\theta = 9\cos^2\theta$.
We want $(x-3)^2 = 9\sin^2\theta$, so take $(x-3) = 3\sin\theta$, $dx = 3\cos\theta\,d\theta$.

$$\int \frac{(x-3)^2}{\sqrt{9-(x-3)^2}}dx = \int \frac{9\sin^2\theta}{\sqrt{9-9\sin^2\theta}}3\cos\theta\,d\theta$$
$$= \int \frac{9\sin^2\theta}{\sqrt{9\cos^2\theta}}3\cos\theta\,d\theta = \int 9\sin^2\theta\,d\theta$$
$$= \frac{9}{2}\int (1-\cos 2\theta)\,d\theta = \frac{9}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$
$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C$$
$$= \frac{9}{2}\left(\arcsin\left(\frac{x-3}{3}\right) - \frac{x-3}{3}\cdot\frac{\sqrt{6x-x^2}}{3}\right) + C$$



Check Our Work

Let's verify that

$$\int \frac{x^2-6x+9}{\sqrt{6x-x^2}} =$$

$$\frac{d}{dx}\left\{\frac{9}{2}\left(\arcsin\left(\frac{x-3}{3}\right) - \frac{x-3}{3}\cdot\frac{\sqrt{6x-x^2}}{3}\right) + C\right\}$$
$$= \frac{9}{2}\left(\frac{1/3}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}} - \frac{x-3}{3}\cdot\frac{3-x}{3\sqrt{6x-x^2}} - \frac{1}{9}\sqrt{6x-x^2}\right)$$
$$= \frac{9}{2}\left(\frac{9}{9\sqrt{6x-x^2}} - \frac{6x-x^2-9}{9\sqrt{6x-x^2}} - \frac{6x-x^2}{9\sqrt{6x-x^2}}\right)$$
$$= \frac{9-6x+x^2}{\sqrt{6x-x^2}}$$

So, our answer works.

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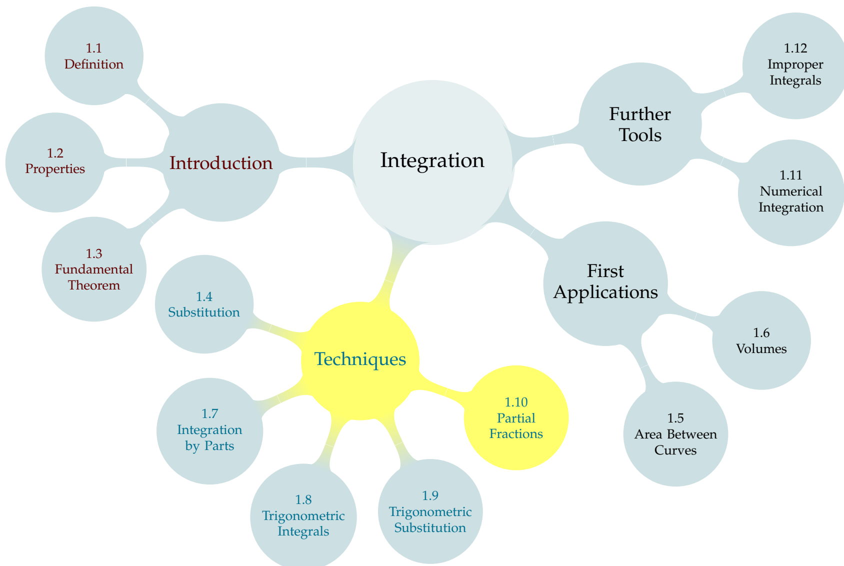


Motivation

How to integrate $\displaystyle\int \frac{x-2}{(x+1)(2x-1)}\,\mathrm{d}x$?

Useful fact: $\displaystyle\frac{x-2}{(x+1)(2x-1)} = \frac{1}{x+1} - \frac{1}{2x-1}$

So:

$$\int \frac{x-2}{(x+1)(2x-1)}\,\mathrm{d}x = \int \frac{1}{x+1}\,\mathrm{d}x - \int \frac{1}{2x-1}\,\mathrm{d}x$$
$$= \log|x+1| - \frac{1}{2}\log|2x-1| + C$$

Method of Partial Fractions: Algebraic method to turn any rational function (i.e. ratio of two polynomials) into the sum of easier-to-integrate rational functions.

Distinct Linear Factors

The rational function

$$\frac{\text{numerator}}{K(x-a_1)(x-a_2)\cdots(x-a_j)}$$

can be written as

$$\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_j}{x-a_j}$$

for some constants $A_1, A_2, \ldots, A_j$, provided

(1) the linear roots $a_1, \cdots a_j$ are distinct, and

(2) the degree of the numerator is strictly less than the degree of the denominator.

Distinct Linear Factors

$$\frac{7x+13}{(2x+5)(x-2)} =$$

To find A and B, simplify the right-hand side by finding a common denominator.

$$\frac{7x+13}{2x^2+x-10} = \frac{A}{2x+5} + \frac{B}{x-2} = \frac{A(x-2)}{(2x+5)(x-2)} + \frac{B(2x+5)}{(2x+5)(x-2)}$$
$$= \frac{A(x-2)+B(2x+5)}{2x^2+x-10}$$

Cancel denominators

$$7x+13 = A(x-2) + B(2x+5)$$

Distinct Linear Factors

We found $7x + 13 = A(x-2) + B(2x+5)$ for some constants A and B. What are A and B?

Method 1: set x to convenient values.

When $x = 2$ (chosen to eliminate A from the right hand side), we have $14 + 13 = B \cdot 9$, so $B = 3$.
If $x = -\frac{5}{2}$ (chosen to eliminate B from the right hand side), then $-\frac{35}{2} + 13 = A\left(-\frac{5}{2} - 2\right)$, so $A = 1$.

Method 2: match coefficients of powers of x.

$7x + 13 = (A+2B)x + (-2A+5B)$, so $7 = A + 2B$ and $13 = -2A + 5B$.
Then $A = 7 - 2B$, so $13 = -2(7-2B) + 5B$.
Then $B = 3$ and $A = 1$.

Distinct Linear Factors

All together:

$$\frac{7x+13}{2x^2+x-10} = \frac{A}{2x+5} + \frac{B}{x-2}$$
$$A = 1, \qquad B = 3$$
$$\frac{7x+13}{2x^2+x-10} = \frac{1}{2x+5} + \frac{3}{x-2}$$
$$\int \frac{7x+13}{2x^2+x-10}\,dx = \int \left(\frac{1}{2x+5} + \frac{3}{x-2}\right) dx$$
$$= \frac{1}{2}\log|2x+5| + 3\log|x-2| + C$$

Check Our Work

We check that $\int \frac{7x+13}{2x^2+x-10} =$ by differentiating.

$$\frac{d}{dx}\left[\frac{1}{2}\log|2x+5| + 3\log|x-2| + C\right] = \frac{1}{2}\cdot\frac{1}{2x+5}\cdot 2 + 3\cdot\frac{1}{x-2}$$
$$= \frac{1}{2x+5}\left(\frac{x-2}{x-2}\right) + \frac{3}{x-2}\left(\frac{2x+5}{2x+5}\right)$$
$$= \frac{(x-2)+(6x+15)}{(x-2)(2x+5)} = \frac{7x+13}{2x^2+x-10}$$

So, our work checks out.

Distinct Linear Factors

$\frac{x^2+5}{2x(3x+1)(x+5)}$ is hard to antidifferentiate, but it can be written as $\frac{A}{2x} + \frac{B}{3x+1} + \frac{C}{x+5}$ for some constants A, B, and C.

Once we find A, B, and C, integration is easy:

$$\int \frac{x^2-24x+5}{2x(3x+1)(x+5)}\,dx$$
$$= \int \left(\frac{A}{2x} + \frac{B}{3x+1} + \frac{C}{x+5}\right) dx$$
$$= \frac{A}{2}\log|x| + \frac{B}{3}\log|3x+1| + C\log|x+5| + D$$

Distinct Linear Factors

$$\frac{x^2+5}{2x(3x+1)(x+5)} = \frac{A}{2x} + \frac{B}{3x+1} + \frac{C}{x+5}$$

Find constants A, B, and C.
Start: make a common denominator

$$= \frac{A(3x+1)(x+5)}{2x(3x+1)(x+5)} + \frac{B(2x)(x+5)}{2x(3x+1)(x+5)} + \frac{C(2x)(3x+1)}{2x(3x+1)(x+5)}$$
$$= \frac{A(3x+1)(x+5) + B(2x)(x+5) + C(2x)(3x+1)}{2x(3x+1)(x+5)}$$

Cancel off denominator

$$x^2+5 = A(3x+1)(x+5) + B(2x)(x+5) + C(2x)(3x+1)$$

Check Our Work

Let's check that

$$\frac{x^2+5}{2x(3x+1)(x+5)} =$$

$$\frac{1}{2x} - \frac{23/14}{3x+1} + \frac{3/14}{x+5}$$
$$= \frac{1(3x+1)(x+5)}{2x(3x+1)(x+5)} - \frac{23/14(2x)(x+5)}{(2x)(3x+1)(x+5)} + \frac{3/14(2x)(3x+1)}{(2x)(3x+1)(x+5)}$$
$$= \frac{(3x^2+16x+5) - (\frac{23}{7}x^2 + \frac{115}{7}x) + (\frac{9}{7}x^2 + \frac{3}{7}x)}{2x(3x+1)(x+3)}$$
$$= \frac{x^2+5}{2x(3x+1)(x+3)}$$

So, our algebra is good.

Distinct Linear Factors

All together:

$$\frac{x^2+5}{2x(3x+1)(x+5)} = \frac{1}{2x} - \frac{23/14}{3x+1} + \frac{3/14}{x+5}$$
$$\int \frac{x^2-24x+5}{2x(3x+1)(x+5)}\,dx = \int \left(\frac{1}{2x} - \frac{23/14}{3x+1} + \frac{3/14}{x+5}\right)\,dx$$
$$= \frac{1}{2}\log|x| - \frac{23}{42}\log|3x+1| + \frac{3}{14}\log|x+5| + C$$

Repeated Linear Factors

A rational function $\frac{P(x)}{(x-1)^4}$, where $P(x)$ is a polynomial of degree strictly less than 4, can be written as

$$\frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x-1)^4}$$

for some constants A, B, C, and D.

$$\frac{5x-11}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

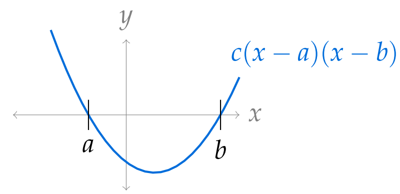
Set up the form of the partial fractions decomposition. (You do not have to solve for the parameters.)

$$\frac{3x+16}{(x+5)^3} = \frac{A}{x+5} + \frac{B}{(x+5)^2} + \frac{C}{(x+5)^3}$$

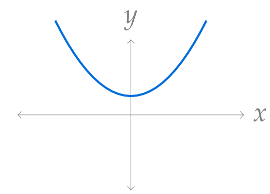
$$\frac{-2x-10}{(x+1)^2(x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1}$$

Irreducible Quadratic Factors

Sometimes it's not possible to factor our denominator into linear factors with real terms.



If a quadratic function has real roots a and b (possibly $a = b$, possibly $a \neq b$), then we can write it as $c(x-a)(x-b)$ for some constant c.



If a quadratic function has no real roots, then it can't be factored into (real) linear factors. It is irreducible.

Irreducible Quadratic Factors

When the denominator has an irreducible quadratic factor x^2+bx+c, we add a term $\dfrac{Ax+B}{x^2+bx+c}$ to our composition. (The degree of the numerator must still be smaller than the degree of the denominator.) Write out the form of the partial fraction decomposition (but do not solve for the parameters):

- $$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$
- $$\frac{3x^2-x+5}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$$

The purpose of the partial fraction decomposition is to end up with functions that we can integrate.

- Recall: $\displaystyle\int \frac{1}{x^2+1}\,dx = \arctan x + C$.
- Evaluate: $\displaystyle\int \frac{1}{(x+1)^2+1}\,dx$

$u = x+1$, $du = dx$:

$$\int \frac{1}{u^2+1}\,du = \arctan u + C = \arctan(x+1) + C$$

Evaluate $\displaystyle\int \frac{4}{(3x+8)^2+9}\,\mathrm{d}x$

$$= \int \frac{4}{9\left(\frac{(3x+8)^2}{9}+1\right)}\mathrm{d}x$$
$$= \frac{4}{9}\int \frac{1}{\left(\frac{3x+8}{3}\right)^2+1}\,\mathrm{d}x$$
$$= \frac{4}{9}\int \frac{1}{\left(x+\frac{8}{3}\right)^2+1}\,\mathrm{d}x$$

$u = x + \frac{8}{3}, \quad \mathrm{d}u = \mathrm{d}x$
$$= \frac{4}{9}\int \frac{1}{u^2+1}\,\mathrm{d}u$$
$$= \frac{4}{9}\arctan u + C$$
$$= \frac{4}{9}\arctan\left(x+\frac{8}{3}\right) + C$$

Check Our Work

We found $\displaystyle\int \frac{4}{(3x+8)^2+9}\mathrm{d}x =$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{4}{9}\arctan\left(x+\frac{8}{3}\right)+C\right\} = \frac{4}{9}\cdot\frac{1}{\left(x+\frac{8}{3}\right)^2+1}$$
$$= \frac{4}{9\left(\left(x+\frac{8}{3}\right)^2+1\right)}$$
$$= \frac{4}{3^2\left(x+\frac{8}{3}\right)^2+9}$$
$$= \frac{4}{(3x+8)^2+9}$$

So, our answer works.

Evaluate $\displaystyle\int \frac{x+1}{x^2+2x+2}\mathrm{d}x$.

(Hint: start by completing the square.)

$$= \int \frac{x+1}{(x+1)^2+1}\mathrm{d}x$$

Let $y = x+1$, $\mathrm{d}y = \mathrm{d}x$:
$$= \int \frac{y}{y^2+1}\mathrm{d}y$$

Let $u = y^2+1$, $\mathrm{d}u = 2y\,\mathrm{d}y$:
$$= \frac{1}{2}\int \frac{1}{u}\,\mathrm{d}u$$
$$= \frac{1}{2}\log|u| + C$$
$$= \frac{1}{2}\log\left|y^2+1\right| + C$$
$$= \frac{1}{2}\log\left|(x+1)^2+1\right| + C$$

Check Our Work

We found $\displaystyle\int \frac{x+1}{x^2+2x+2}\mathrm{d}x =$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{2}\log\left|(x+1)^2+1\right|+C\right\} = \frac{1}{2}\cdot\frac{2(x+1)}{(x+1)^2+1}$$
$$= \frac{x+1}{(x+1)^2+1}$$
$$= \frac{x+1}{x^2+2x+2}$$

So, our answer works.

These rules work only when the degree of the numerator is less than the degree of the denominator.

$$\int \frac{x^3}{(x-2)^2(x-3)(x-4)^2}\,dx \qquad \int \frac{x^5}{(x-2)^2(x-3)(x-4)^2}\,dx$$

If the degree of the numerator is too large, we use polynomial long division.

Evaluate $\displaystyle\int \frac{8x^2+22x+23}{2x+3}\,dx$.

$$\begin{array}{r|l} & \quad 4x \quad +5 \\ \hline 2x+3 & \ 8x^2+22x+23 \\ & -\,8x^2-12x \\ \hline & \qquad\ \ 10x+23 \\ & \qquad -\,10x-15 \\ \hline & \qquad\qquad\quad 8 \end{array}$$

So,

$$\frac{8x^2+22x+23}{2x+3} = 4x+5+\frac{8}{2x+3}$$

$$\int \frac{8x^2+22x+23}{2x+3}\,dx = 2x^2+5x+4\log|2x+3|+C$$

Check Our Work

We computed

$$\int \frac{8x^2+22x+23}{2x+3}\,dx =$$

$$\begin{aligned} &\frac{d}{dx}\left\{2x^2+5x+4\log|2x+3|+C\right\} \\ &= 4x+5+\frac{8}{2x+3} \\ &= \frac{4x(2x+3)+5(2x+3)+8}{2x+3} \\ &= \frac{8x^2+12x+10x+15+8}{2x+3} \\ &= \frac{8x^2+22x+23}{2x+3} \end{aligned}$$

So, our solution works.

Evaluate $\displaystyle\int \frac{3x^3+x+3}{x-2}\,dx$.

$$\begin{array}{r|l} & \qquad 3x^2 \quad +6x+13 \\ \hline x-2 & \ 3x^3 \qquad\qquad +x\ \ +3 \\ & -\,3x^3+6x^2 \\ \hline & \qquad\ 6x^2 \quad +x \\ & \quad -\,6x^2+12x \\ \hline & \qquad\qquad 13x\ \ +3 \\ & \qquad\quad -\,13x+26 \\ \hline & \qquad\qquad\qquad 29 \end{array}$$

So,

$$\int \frac{3x^3+x+3}{x-2}\,dx = \int\left(3x^2+6x+13+\frac{29}{x-2}\right)dx$$
$$= x^3+3x^2+13x+29\log|x-2|+C$$

Check Our Work

We found

$$\int \frac{3x^3 + x + 3}{x-2}\,\mathrm{d}x =$$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^3 + 3x^2 + 13x + 29\log|x-2| + C\right\} \\
&= 3x^2 + 6x + 13 + \frac{29}{x-2} \\
&= \frac{3x^2(x-2) + 6x(x-2) + 13(x-2) + 29}{x-2} \\
&= \frac{3x^3 - 6x^2 + 6x^2 - 12x + 13x - 26 + 29}{x-2} \\
&= \frac{3x^3 + x + 3}{x-2}
\end{aligned}$$

Evaluate $\displaystyle\int \frac{3x^2+1}{x^2+5x}\,\mathrm{d}x$.

$$\begin{array}{r|l}
 & \quad 3 \\ \hline
x^2+5x & 3x^2 \qquad\quad +1 \\
 & -3x^2 - 15x \\ \hline
 & \quad -15x + 1
\end{array}$$

So, $\displaystyle\frac{3x^2+1}{x^2+5x} = 3 + \frac{-15x+1}{x^2+5x}$

Now, we can use partial fraction decomposition.

$$\frac{-15x+1}{x(x+5)} = \frac{A}{x} + \frac{B}{x+5} = \frac{(A+B)x + 5A}{x(x+5)}$$

$$A = \frac{1}{5}, \quad B = -15 - \frac{1}{5} = -\frac{76}{5}$$

$$\begin{aligned}
\int \frac{3x^2+1}{x^2+5x}\,\mathrm{d}x &= \int\left(3 + \frac{1/5}{x} - \frac{76/5}{x+5}\right)\mathrm{d}x \\
&= 3x + \frac{1}{5}\log|x| - \frac{76}{5}\log|x+5| + C
\end{aligned}$$

Check Our Work

We found $\displaystyle\int \frac{3x^2+1}{x^2+5x}\,\mathrm{d}x =$

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left\{3x + \frac{1}{5}\log|x| - \frac{76}{5}\log|x+5| + C\right\} \\
&= 3 + \frac{1}{5x} - \frac{76}{5(x+5)} \\
&= 3\left(\frac{5x(x+5)}{5x(x+5)}\right) + \frac{1}{5x}\left(\frac{x+5}{x+5}\right) - \frac{76}{5(x+5)}\left(\frac{x}{x}\right) \\
&= \frac{(15x^2 + 75x) + (x+5) - (76x)}{5x(x+5)} \\
&= \frac{15x^2+5}{5x(x+5)} = \frac{3x^2+1}{x^2+5x}
\end{aligned}$$

So, our solution works.

Factoring

$$P(x) = x^3 + 2x^2 - 5x - 6$$

- To start, let's guess a root.
  - Since $P(x)$ has integer coefficients, any integer root must divide 6 exactly.
  - So the only possible integer roots are ±1, ±2, ±3, and ±6. We'll try each until one works.
    - $P(1) = -8 \neq 0 \implies$ 1 is not a root
    - $P(-1) = 0 \implies$ -1 is a root. Therefore, $(x+1)$ is a factor.
- Long division gives the rest:

$$\begin{array}{r|l}
 & \qquad\quad x^2 + x - 6 \\ \hline
x+1 & x^3 + 2x^2 - 5x - 6 \\
 & -x^3 - x^2 \\ \hline
 & \qquad\quad x^2 - 5x \\
 & \qquad -x^2 - x \\ \hline
 & \qquad\qquad -6x - 6 \\
 & \qquad\qquad\ \ 6x + 6 \\ \hline
 & \qquad\qquad\qquad 0
\end{array}$$

$P(x) = (x+1)(x^2+x-6) = (x+1)(x-2)(x+3)$

Factoring

$$P(x) = 2x^3 - 3x^2 + 4x - 6$$

Notice that the first two terms and the last two terms have the same ratios: $\frac{2x^3}{-3x^2} = \frac{2x}{-3} = \frac{4x}{-6}$. So, we can factor $2x - 3$ out of both pairs.

$$\begin{aligned} P(x) &= 2x^3 - 3x^2 \qquad +4x - 6 \\ &= (2x-3)(x^2) \qquad +(2x-3)(2) \\ &= (2x-3)(x^2+2) \end{aligned}$$

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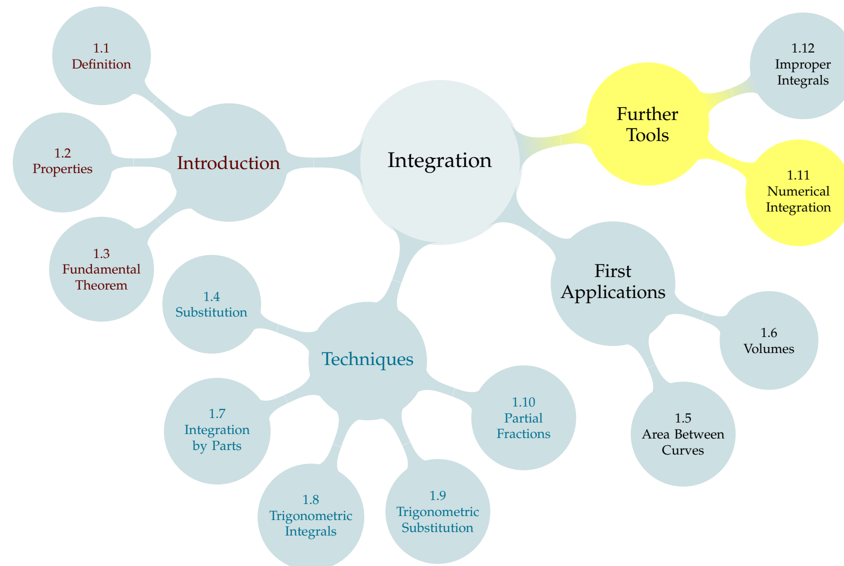


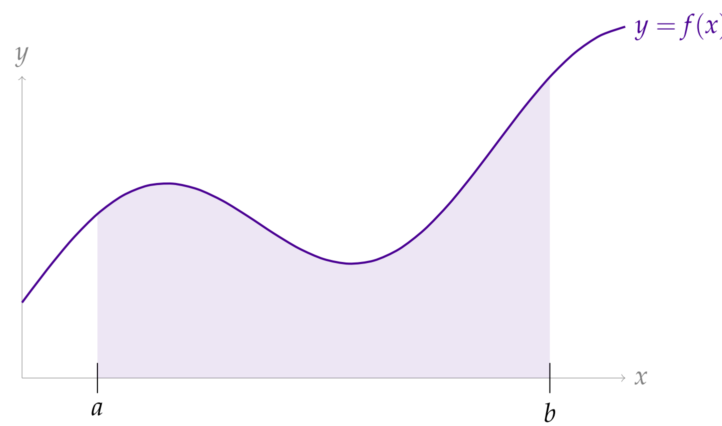
Sometimes, integrals can't be evaluated using the fundamental theorem of calculus:

$$\int_0^1 e^{x^2}\,\mathrm{d}x = ? \qquad \int_0^1 \sin(x^2)\,\mathrm{d}x = ?$$

Sometimes, integrals can be evaluated, but only in terms of complicated constant numbers:

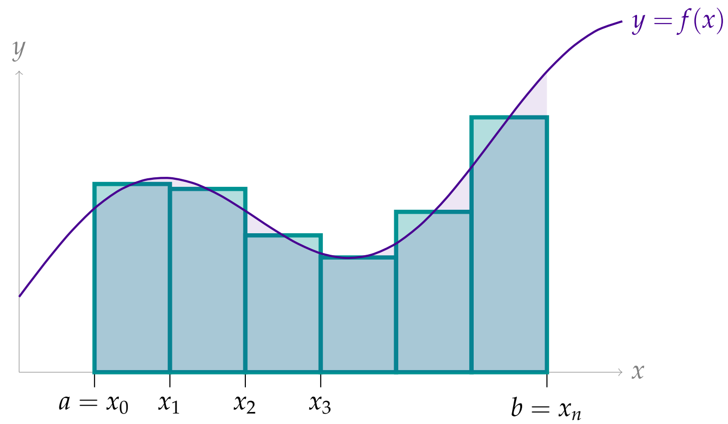
$$\int_0^3 \frac{1}{1+x^2}\,\mathrm{d}x = \arctan(3) = \ldots ?$$

A numerical approximation will give us an approximate number for a definite integral.



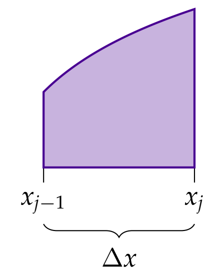
We can approximate the area $\int_a^b f(x)\,\mathrm{d}x$ by cutting it into slices and approximating the area of those slices with a simple geometric figure, such as a rectangle, a trapezoid, or a parabola.

The midpoint rule approximates $\int_a^b f(x)\,\mathrm{d}x$ as its midpoint Riemann sum with n intervals.



Approximate the area under the curve $y = f(x)$ from $x = x_{j-1}$ to $x = x_j$ with a rectangle.

To make our writing cleaner, let $\overline{x_j} = \frac{x_{j-1}+x_j}{2}$



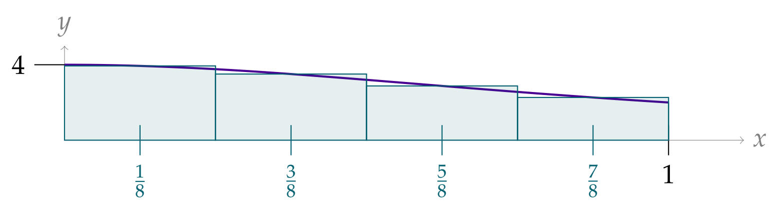
Midpoint Rule

The midpoint rule approximation is

$$\int_a^b f(x)\,\mathrm{d}x \approx \left[f\left(\overline{x_1}\right) + f\left(\overline{x_2}\right) + \cdots + f\left(\overline{x_n}\right)\right]\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_j = a + j\Delta x$

Approximate $\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x$ using the midpoint rule and $n = 4$ slices. Leave your answer in calculator-ready form.



$$\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x \approx \left[\frac{4}{1+\left(\frac{1}{8}\right)^2} + \frac{4}{1+\left(\frac{3}{8}\right)^2} + \frac{4}{1+\left(\frac{5}{8}\right)^2} + \frac{4}{1+\left(\frac{7}{8}\right)^2}\right]\cdot\frac{1}{4}$$

$$\int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x = 4\arctan(1) = 4\cdot\frac{\pi}{4} = \pi$$

Error

$$\pi = \int_0^1 \frac{4}{1+x^2}\,\mathrm{d}x \approx \left[\frac{4}{1+\left(\frac{1}{8}\right)^2} + \frac{4}{1+\left(\frac{3}{8}\right)^2} + \frac{4}{1+\left(\frac{5}{8}\right)^2} + \frac{4}{1+\left(\frac{7}{8}\right)^2}\right]\cdot\frac{1}{4}$$

$$\approx 3.14680$$

Error:

$$|\text{exact} - \text{approximate}|$$

Relative error:

$$\left|\frac{\text{exact}-\text{approximate}}{\text{exact}}\right|$$

Percent error:

$$100\left|\frac{\text{exact}-\text{approximate}}{\text{exact}}\right|$$

Error

A numerical approximation will give us an approximate value for a definite integral.
This is most useful if we know something about its accuracy.

A: approximation E: exact number

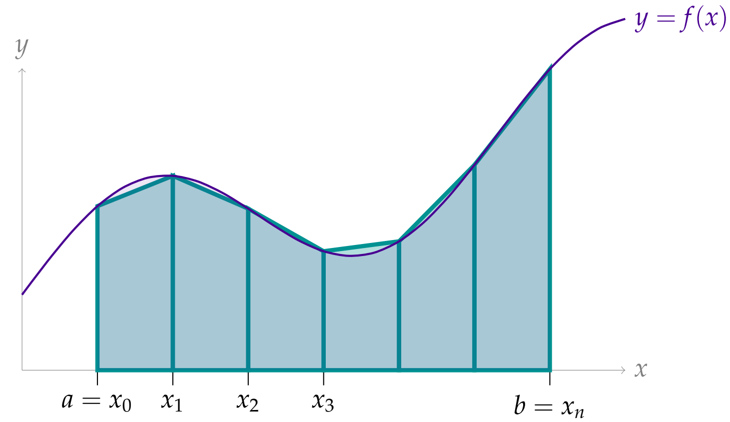
Error: $|A - E|$

Relative Error: $\left|\frac{A - E}{E}\right|$

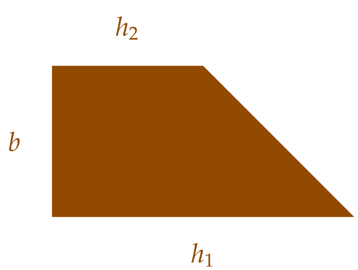
Percent Error: $100\left|\frac{A - E}{E}\right|$

We will discuss error more after we've learned the three approximation rules. For now, we're using error to illustrate that our methods have the potential to produce reasonable approximations without too much work.

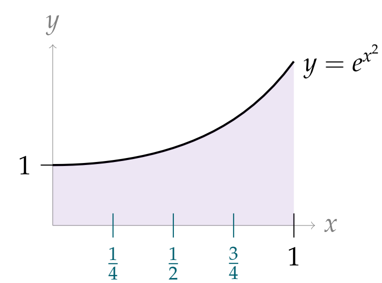
The trapezoidal rule approximates each slice of $\int_a^b f(x)\,dx$ with a trapezoid.



Recall the area of a right trapezoid with base b and heights h_1 and h_2:



Rectangle area: $b(h_1 + h_2)$
Trapezoid area: $\frac{b}{2}(h_1 + h_2)$



Trapezoid area: $\frac{\text{base}}{2}(h_1 + h_2)$

Approximate $\int_0^1 e^{x^2}\,dx$ using $n = 4$ trapezoids.
Leave your answer in calculator-ready form.

$$\int_0^1 e^{x^2}\,dx \approx \frac{1/4}{2}\left(e^0 + e^{\frac{1}{16}} + e^{\frac{1}{16}} + e^{\frac{1}{4}} + e^{\frac{1}{4}} + e^{\frac{9}{16}} + e^{\frac{9}{16}} + e\right)$$
$$= \frac{1/4}{2}\left(e^0 + 2e^{1/16} + 2e^{1/4} + 2e^{9/16} + e\right)$$

Trapezoidal Rule

The trapezoidal rule approximation is

$$\int_a^b f(x)\mathrm{d}x \approx \Delta x \left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]$$

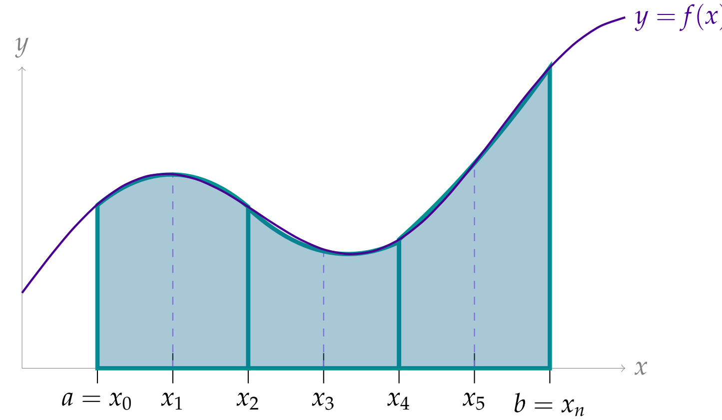
where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$

Using $n = 3$ trapezoids, approximate $\displaystyle\int_1^{10} \frac{1}{x}\,\mathrm{d}x$.

$$\Delta x = \frac{10-1}{3} = 3 \quad x_0 = 1 \quad x_1 = 4 \quad x_2 = 7 \quad x_3 = 10$$

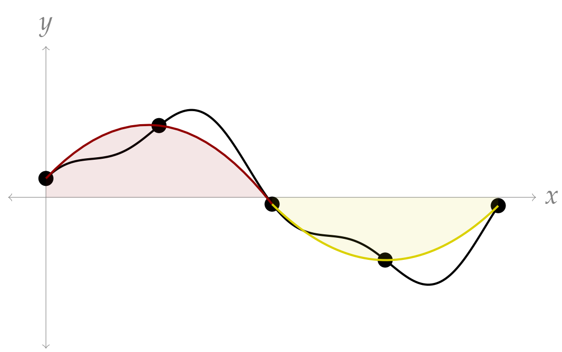
$$\int_1^{10} \frac{1}{x}\,\mathrm{d}x \approx 3\left[\frac{1}{2}(1) + \frac{1}{4} + \frac{1}{7} + \frac{1}{2}\left(\frac{1}{10}\right)\right] = \frac{99}{35}$$

Simpson's rule approximates each *pair of slices* of $\displaystyle\int_a^b f(x)\,\mathrm{d}x$ with a parabola.



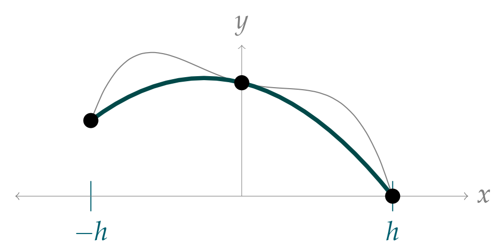
Simpson's Rule

Add up parabolas.



Simpson's Rule Derivation

skip derivation of Simpson's rule



What is the area under the parabola passing through three specified points?

Find

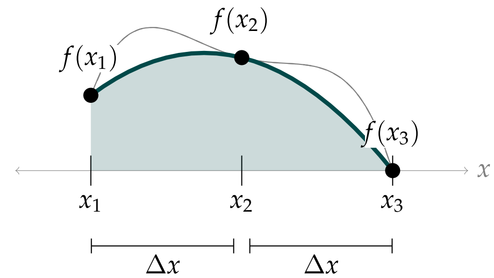
$$\frac{h}{3}\left(2Ah^2 + 6C\right)$$

for A, B, and C such that

$$Ah^2 - Bh + C = f(-h) \quad \text{(E1)}$$
$$C = f(0) \quad \text{(E2)}$$
$$Ah^2 + Bh + C = f(h) \quad \text{(E3)}$$

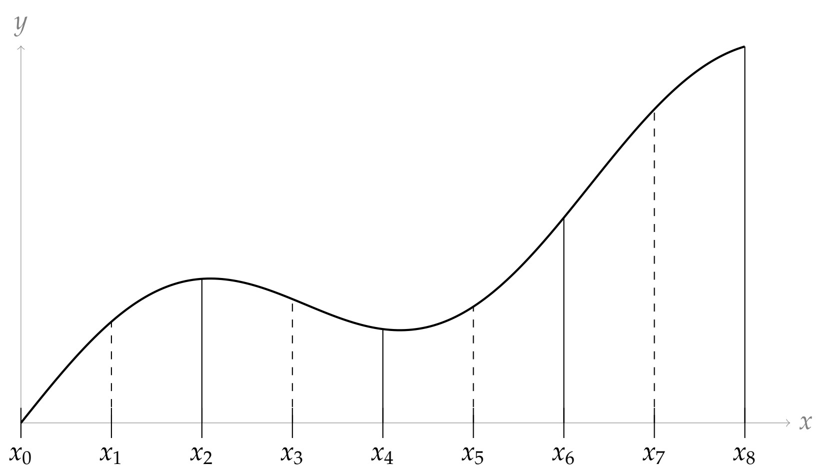
Try $(E1) + 4(E2) + (E3)$:

$$2Ah^2 + 6C = f(-h) + 4f(0) + f(h)$$
$$\text{Area} = \frac{h}{3}\left(2Ah^2 + 6C\right) = \frac{h}{3}\left(f(-h) + 4f(0) + f(h)\right)$$



Area under parabola:

$$\frac{\Delta x}{3}\Big(f(x_1) + 4f(x_2) + f(x_3)\Big)$$



Simpson's Rule

The Simpson's rule approximation is $\displaystyle\int_a^b f(x)\mathrm{d}x \approx$

$$\frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\Big]$$

where n is even, $\Delta x = \frac{b-a}{n}$, and $x_i = a + i\Delta x$

Using Simpson's rule and $n = 8$ (i.e. 4 parabolas), approximate $\displaystyle\int_1^{17} \frac{1}{x}\,\mathrm{d}x$. Leave your answer in calculator-ready form.

$\approx \frac{2}{3}\left[\frac{1}{1} + 4\cdot\frac{1}{3} + 2\cdot\frac{1}{5} + 4\cdot\frac{1}{7} + 2\cdot\frac{1}{9} + 4\cdot\frac{1}{11} + 2\cdot\frac{1}{13} + 4\cdot\frac{1}{15} + \frac{1}{17}\right]$

(We'll call n the number of slices; some people call $n/2$ the number of slices, because that's the number of approximating parabolas.)

The instantaneous electricity use rate (kW/hr) of a factory is measured throughout the day.

time	12:00	1:00	2:00	3:00	4:00	5:00	6:00	7:00	8:00
rate	100	200	150	400	300	300	200	100	150

Use Simpson's Rule to approximate the total amount of electricity you used from noon to 8:00.

We use $n = 8$, with $\Delta x = 1$ hour. Let's re-label the times as $x = 0$ as noon, $x = 1$ as 1 o'clock, etc.

$$\frac{1}{3}\left[f(0) + 4f(1) + 2f(2) + 4f(3) + 2f(4) + 4f(5) + 2f(6) + 4f(7) + f(8)\right]$$
$$= \frac{1}{3}[100 + 800 + 300 + 1600 + 600 + 1200 + 400 + 400 + 150]$$
$$= 1850 \text{ kW}$$

Numerical integration errors

Assume that $|f''(x)| \le M$ for all $a \le x \le b$ and $|f^{(4)}(x)| \le L$ for all $a \le x \le b$. Then

- the total error introduced by the midpoint rule is bounded by $\frac{M}{24}\frac{(b-a)^3}{n^2}$,
- the total error introduced by the trapezoidal rule is bounded by $\frac{M}{12}\frac{(b-a)^3}{n^2}$, and
- the total error introduced by Simpson's rule is bounded by $\frac{L}{180}\frac{(b-a)^5}{n^4}$

when approximating $\int_a^b f(x)\,dx$.

Numerical integration errors

Assume that $|f''(x)| \le M$ for all $a \le x \le b$. Then the total error introduced by the midpoint rule is bounded by $\frac{M}{24}\frac{(b-a)^3}{n^2}$ when approximating $\int_a^b f(x)\,dx$.

Suppose we approximate $\int_0^3 \sin(x)\,dx$ using the midpoint rule and $n = 6$ intervals. Give an upper bound of the resulting error.

If $f(x) = \sin x$, then $f''(x) = -\sin x$. For $0 \le x \le 3$ (indeed, for any x), $|f''(x)| = |-\sin x| \le 1$, so we take $M = 1$.

$$|\text{error}| \le \frac{1}{24}\frac{(3-0)^3}{6^2} = \frac{1}{32}$$

Numerical integration errors

Assume that $|f^{(4)}(x)| \le L$ for all $a \le x \le b$. Then the total error introduced by Simpson's rule is bounded by $\frac{L}{180}\frac{(b-a)^5}{n^4}$ when approximating $\int_a^b f(x)\,dx$.

Suppose we approximate $\int_2^3 \frac{1}{x}dx$ using Simpson's rule with $n = 10$ slices (5 parabolas). Give an upper bound of the resulting error.

If $f(x) = \frac{1}{x}$, then $f^{(4)}(x) = \frac{24}{x^5}$. This is a positive, decreasing function for positive values of x, so its maximum value on the interval $[2, 3]$ is $f^{(4)}(2) = \frac{24}{2^5} = \frac{3}{4}$. So, we take $L = \frac{3}{4}$. Then the error is bounded by

$$\frac{3/4}{180}\frac{1^5}{10^4} = \frac{1}{240 \times 10^4} = \frac{1}{2\,400\,000}$$

Numerical integration errors

Assume that $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then the total error introduced by Simpson's rule is bounded by $\frac{L}{180}\frac{(b-a)^5}{n^4}$ when approximating $\int_a^b f(x)\,dx$.

We will approximate $\int_0^{1/2} e^{x^2}\,dx$ using Simpson's rule, and we need our error to be no more than $\frac{1}{10\,000}$. How many intervals will suffice?

You may use, without proof:

$$\frac{d^4}{dx^4}\left\{e^{x^2}\right\} = 4e^{x^2}\left(4x^4 + 12x^2 + 3\right) \qquad \frac{25\sqrt[4]{e}}{180 \cdot 2^5} < \frac{1}{3^4}$$

Numerical integration errors

Assume that $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then the total error introduced by Simpson's rule is bounded by $\frac{L}{180}\frac{(b-a)^5}{n^4}$ when approximating $\int_a^b f(x)\,dx$.

It can be shown that the fourth derivative of $\frac{1}{x^2+1}$ has absolute value at most 24 for all real numbers x. Using this information, find a rational number approximating $\arctan(2)$ with an error of no more than $\frac{2^6}{3 \cdot 5^5} \approx 0.007$.

First, we'll set up our integral:

$$\int_0^2 \frac{1}{1+x^2}\,dx = \arctan(2) - \arctan(0) = \arctan 2$$

From the given information, we'll use $L = 24$.

$$|\text{error}| \leq \frac{L}{180}\frac{(2-0)^5}{n^4}$$
$$= \frac{24 \cdot 2^5}{180 n^4} = \frac{2^6}{15 n^4}$$
$$\frac{2^6}{15n^4} \leq \frac{2^6}{15 \cdot 5^4}$$
$$\frac{1}{n^4} \leq \frac{1}{5^4}$$
$$n \geq 5$$

Since n must be even, we'll use $n = 6$. Now, we can give the approximation.

$$\arctan(2) = \int_0^2 \frac{1}{1+x^2}dx, \qquad n = 6, \qquad \Delta x = \frac{2-0}{6} = \frac{1}{3}$$

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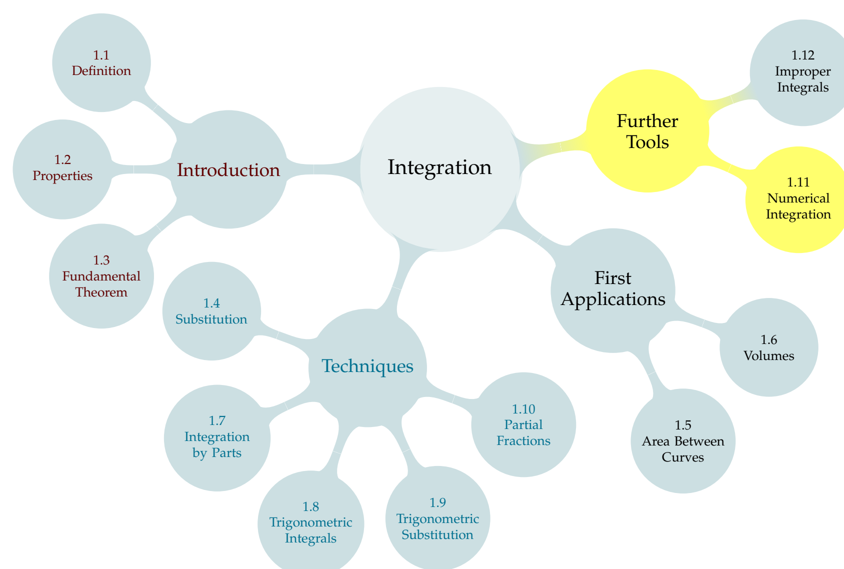

Numerical integration errors

Assume that $|f''(x)| \leq M$ for all $a \leq x \leq b$ and $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then

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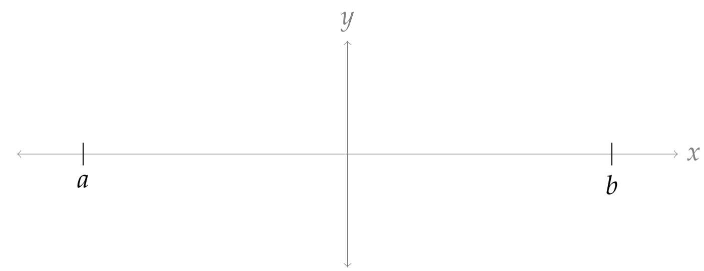
when approximating $\int_a^b f(x)\,\mathrm{d}x$.

WHY THE *second* DERIVATIVE?

The midpoint rule gives the exact area under the curve for

$$f(x) = ax + b$$

when a and b are any constants.



The first derivative can be large without causing a large error.

Numerical integration errors

Assume that $|f''(x)| \leq M$ for all $a \leq x \leq b$ and $|f^{(4)}(x)| \leq L$ for all $a \leq x \leq b$. Then

- the total error introduced by the midpoint rule is bounded by $\frac{M}{24}\frac{(b-a)^3}{n^2}$,
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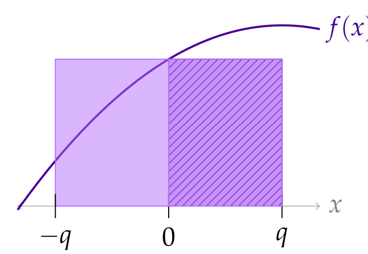
- the total error introduced by Simpson's rule is bounded by $\frac{L}{180}\frac{(b-a)^5}{n^4}$

when approximating $\int_a^b f(x)\,\mathrm{d}x$.

We'll start small: let's consider one-half of a single interval being approximated using the midpoint rule.
To avoid messiness, let's also consider a simplified location:



We want to relate the actual area of this half-slice to its approximate area:

$$\int_0^q f(x)\,\mathrm{d}x \approx q \cdot f(0)$$

$$\int_0^q f(x)\,\mathrm{d}x \approx q \cdot f(0)$$

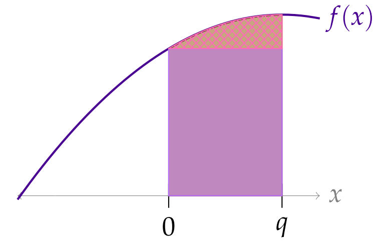
If you squint just right, the right-hand side looks a bit like the "$u \cdot v$" term from integration by parts, where $u = f(x)$ and $\mathrm{d}v = \mathrm{d}x$.

- Set $u = f(x)$ and $\mathrm{d}v = \mathrm{d}x$, so $\mathrm{d}u = f'(x)\,\mathrm{d}x$.
  We choose $v(x) = x - q$, so that $f(v(q)) = f(0)$.

$$\int_0^q f(x)\,\mathrm{d}x = \big[(x-q)f(x)\big]_0^q - \int_0^q (x-q)f'(x)\,\mathrm{d}x$$
$$= q \cdot f(0) - \int_0^q (x-q)f'(x)\,\mathrm{d}x$$

- We know something about the second derivative, not the first, so repeat: set $u = f'(x)$, $\mathrm{d}v = (x-q)\,\mathrm{d}x$; $\mathrm{d}u = f''(x)\,\mathrm{d}x$, $v = \frac{(x-q)^2}{2}$

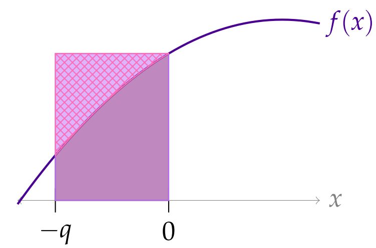
$$\int_0^q f(x)\,\mathrm{d}x = q \cdot f(0) + \frac{q^2}{2} \cdot f'(0) + \int_0^q \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x$$



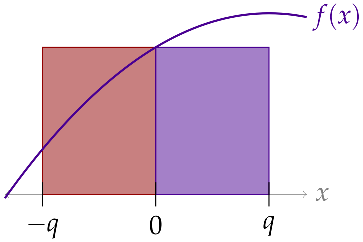
$$\underbrace{\int_0^q f(x)\,\mathrm{d}x}_{\text{exact}} = \underbrace{q \cdot f(0)}_{\text{approximate}} + \underbrace{\frac{q^2}{2} \cdot f'(0) + \int_0^q \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x}_{\pm\ \text{error}}$$

Repeat for the other half of the slice:

$$\int_{-q}^0 \underbrace{f(x)}_{u}\, \underbrace{\mathrm{d}x}_{\mathrm{d}v} = \big[\underbrace{f(x)}_{u} \cdot \underbrace{(x+q)}_{v}\big]_{-q}^0 - \int_{-q}^0 \underbrace{(x+q)}_{v} \cdot \underbrace{f'(x)\,\mathrm{d}x}_{\mathrm{d}u}$$
$$= q \cdot f(0) - \int_{-q}^0 \underbrace{f'(x)}_{\hat{u}} \cdot \underbrace{(x+q)\,\mathrm{d}x}_{\mathrm{d}\hat{v}}$$
$$= q \cdot f(0) - \left[\underbrace{f'(x)}_{\hat{u}} \underbrace{\frac{(x+q)^2}{2}}_{\hat{v}}\right]_{-q}^0 + \int_{-q}^0 \underbrace{\frac{(x+q)^2}{2}}_{\hat{v}} \underbrace{f''(x)\,\mathrm{d}x}_{\mathrm{d}\hat{u}}$$
$$= q \cdot f(0) - \frac{q^2}{2} f'(0) + \int_{-q}^0 \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x$$



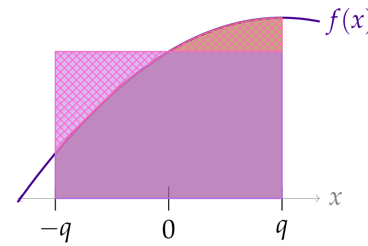
$$\underbrace{\int_{-q}^0 f(x)\,\mathrm{d}x}_{\text{exact}} = \underbrace{q \cdot f(0)}_{\text{approximate}} - \underbrace{\frac{q^2}{2} \cdot f'(0) + \int_{-q}^0 \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x}_{\pm\ \text{error}}$$



$$\int_{-q}^{0} f(x)\,\mathrm{d}x = q \cdot f(0) - \frac{q^2}{2} f'(0) + \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x$$

$$\int_{0}^{q} f(x)\,\mathrm{d}x = q \cdot f(0) + \frac{q^2}{2} \cdot f'(0) + \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x$$

$$\int_{-q}^{q} f(x)\,\mathrm{d}x = 2q \cdot f(0) + \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x + \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x$$



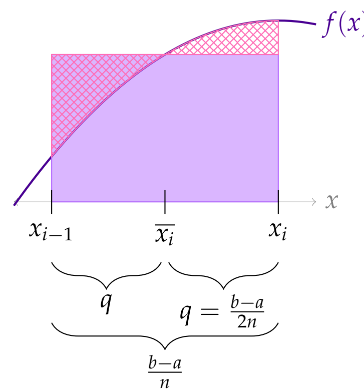
$$\underbrace{\int_{-q}^{q} f(x)\,\mathrm{d}x}_{\text{exact}} = \underbrace{2q \cdot f(0)}_{\text{approximate}} + \underbrace{\int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x + \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x}_{\pm\ \text{error}}$$

We re-arrange to write the error as the difference between the actual area of one slice and its rectangular approximation.

$$\int_{-q}^{q} f(x)\,\mathrm{d}x - 2q \cdot f(0) = \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x + \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x$$

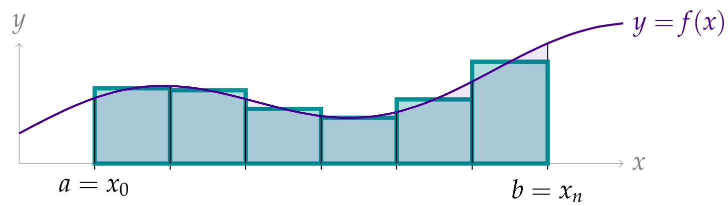
$$\begin{aligned}
\text{error} &= \left| \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x + \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x \right| \\
&\le \left| \int_{-q}^{0} \frac{(x+q)^2}{2} f''(x)\,\mathrm{d}x \right| + \left| \int_{0}^{q} \frac{(x-q)^2}{2} f''(x)\,\mathrm{d}x \right| \\
&\le \int_{-q}^{0} \frac{(x+q)^2}{2} M\,\mathrm{d}x + \int_{0}^{q} \frac{(x-q)^2}{2} M\,\mathrm{d}x \\
&= M \left[\frac{(x+q)^3}{6} \right]_{-q}^{0} + M \left[\frac{(x-q)^3}{6} \right]_{0}^{q} \\
&= \frac{M \cdot q^3}{3}
\end{aligned}$$

Now we can bound the error of a single slice:



$$\left| \int_{-q}^{q} f(x)\,\mathrm{d}x - 2q \cdot f(0) \right| \le \frac{M}{3} \cdot q^3$$

$$\left| \int_{x_{i-1}}^{x_i} f(x)\,\mathrm{d}x - \frac{b-a}{n} \cdot f(\overline{x_i}) \right| \le \frac{M}{3} \left(\frac{b-a}{2n} \right)^3 = \frac{M}{24} \frac{(b-a)^3}{n^3}$$



- The error in each slice is at most $\frac{M}{24}\frac{(b-a)^3}{n^3}$
- There are n slices
- The overall error is at most $n \cdot \frac{M}{24}\frac{(b-a)^3}{n^3} = \frac{M}{24}\frac{(b-a)^3}{n^2}$

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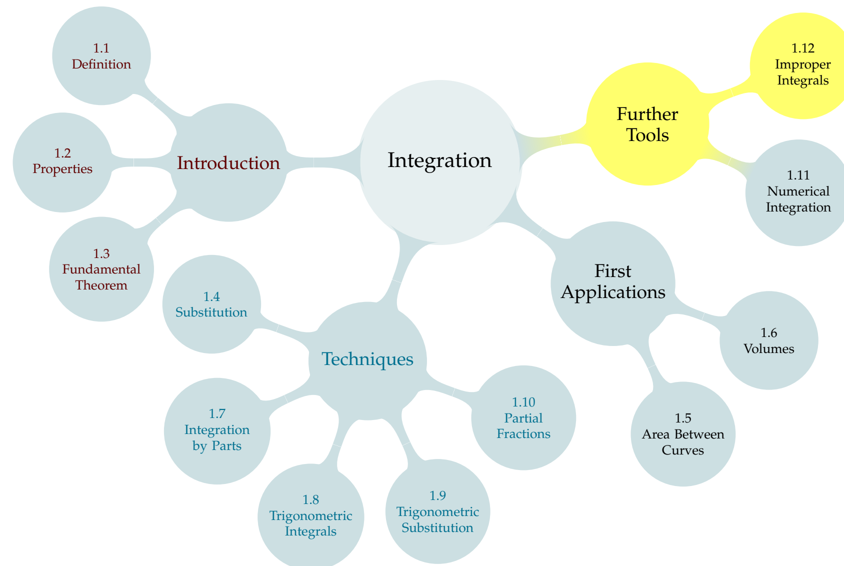
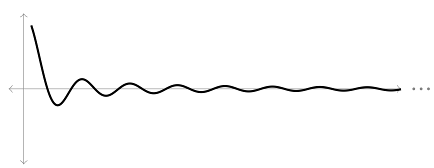


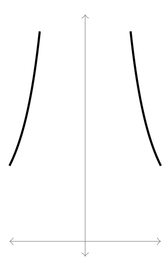
An integral is improper if one or both of the following happen:

- The region of integration is unbounded, e.g. $\int_1^\infty \frac{\sin x}{x}\,dx$



$\Delta x = \frac{b-a}{n} = \frac{\infty}{n}$???

- The integrand is unbounded over the interval, e.g. $\int_{-1}^{1} \frac{1}{x^2}\,dx$



$f(0)\Delta x =$???

Strategy

In both cases, we eliminate the offending parts of the integral using limits.

$$\int_1^\infty \frac{\sin x}{x}\,dx = \lim_{b\to\infty}\left[\int_1^b \frac{\sin x}{x}\,dx\right]$$

$$\int_0^3 \frac{1}{x}\,dx = \lim_{a\to 0^+}\left[\int_a^3 \frac{1}{x}\,dx\right]$$

If the limit doesn't exist, we say the integral diverges. Otherwise it converges.

$$\int_1^\infty \frac{1}{x}\,dx =$$

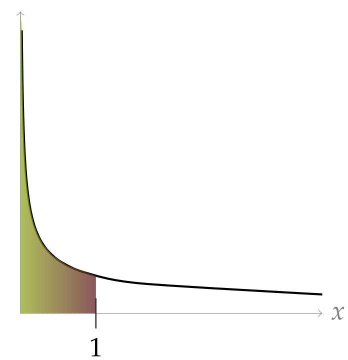
$$\int_1^\infty \frac{1}{x^2}\,dx =$$

Evaluate $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

When an integral has multiple sources of impropriety, we break it up into integrals that have only one source each. If all of them converge, the original integral converges. If any of them diverges, the original integral diverges as well.

$$= \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx$$

Evaluate $\int_0^1 \frac{1}{2\sqrt{x}}\,dx$



Same idea: we solve our problems by ignoring them (temporarily).
Eliminate the problematic part of the integral using a limit.

$$\int_0^1 \frac{1}{2\sqrt{x}}\,dx = \lim_{a\to 0^+}\left[\int_a^1 \frac{1}{2\sqrt{x}}\,dx\right] = \lim_{a\to 0^+}\left[1-\sqrt{a}\right] = 1$$

Evaluate $\displaystyle\int_{-2}^{1} \frac{1}{x^2}\,dx$

$$\int x^{-2}\,dx = -x^{-1} + C = -\frac{1}{x} + C$$

$$\lim_{a\to 0+} \int_a^1 \frac{1}{x^2}\,dx = \lim_{a\to 0+} \left[-\frac{1}{x}\right]_a^1 = \lim_{a\to 0+} \left[-1 + \frac{1}{a}\right] = \infty$$

Once we see that one part of the improper integral diverges, we stop: the entire integral diverges, regardless of what happens to the left of the y-axis.

Evaluate $\displaystyle\int_0^{\infty} \frac{\cos x}{1+\sin^2 x}\,dx$, or show that it diverges.

$$u = \sin x,\ du = \cos x\,dx$$

$$u(0) = 0$$

$$\lim_{b\to\infty}\left[\int_0^b \frac{\cos x}{1+\sin^2 x}\,dx\right] = \lim_{b\to\infty}\left[\int_0^{\sin b} \frac{1}{1+u^2}\,du\right] = \lim_{b\to\infty}\left[\arctan(\sin b) - \arctan(0)\right] = \lim_{b\to\infty}\left[\arctan(\sin b)\right]$$

As b goes to infinity, $\sin b$ oscillates between -1 and 1, so $\arctan(\sin b)$ oscillates between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$. Since its limit does not exist, the integral diverges.

Warning: Sneaky Divergence

If you don't realize that an integral diverges, you can generate answers that look plausible but are secretly nonsense.

For example, attempting to use the Fundamental Theorem of Calculus in the example $\displaystyle\int_{-2}^{1} \frac{1}{x^2}dx$ gives $\displaystyle\left[-\frac{1}{x}\right]_{-2}^{1} = -\frac{3}{2}$: a poor approximation for positive infinity!

Evaluate $\displaystyle\int_0^1 \frac{1}{x^p}dx$ and $\displaystyle\int_1^{\infty} \frac{1}{x^p}dx$ when p is constant.

$$\int \frac{1}{x^p}\,dx = \int x^{-p}\,dx = \begin{cases} \log|x| + C & \text{if } p = 1 \\ \frac{x^{1-p}}{1-p} + C & \text{if } p \neq 1 \end{cases}$$

$$\int_a^b \frac{1}{x^p}\,dx = \begin{cases} \log|b| - \log|a| & \text{if } p = 1 \\ \frac{b^{1-p}-a^{1-p}}{1-p} & \text{if } p \neq 1 \end{cases} \quad \text{if } x = 0 \text{ is not in } [a,b]$$

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \begin{cases} \lim\limits_{b\to\infty} \log|b| & \text{if } p = 1 \\ \lim\limits_{b\to\infty} \left[\frac{b^{1-p}-1}{1-p}\right] & \text{if } p \neq 1 \end{cases} : \begin{cases} \text{divergent} & \text{if } p = 1 \\ \text{divergent} & \text{if } p < 1 \\ \frac{1}{p-1} & \text{if } p > 1 \end{cases}$$

$$\int_0^1 \frac{1}{x^p}\,dx = \begin{cases} \lim\limits_{a\to 0+} -\log|a| & \text{if } p = 1 \\ \lim\limits_{a\to 0+} \left[\frac{1-a^{1-p}}{1-p}\right] & \text{if } p \neq 1 \end{cases} : \begin{cases} \text{divergent} & \text{if } p = 1 \\ \frac{1}{1-p} & \text{if } p < 1 \\ \text{divergent} & \text{if } p > 1 \end{cases}$$

p-test

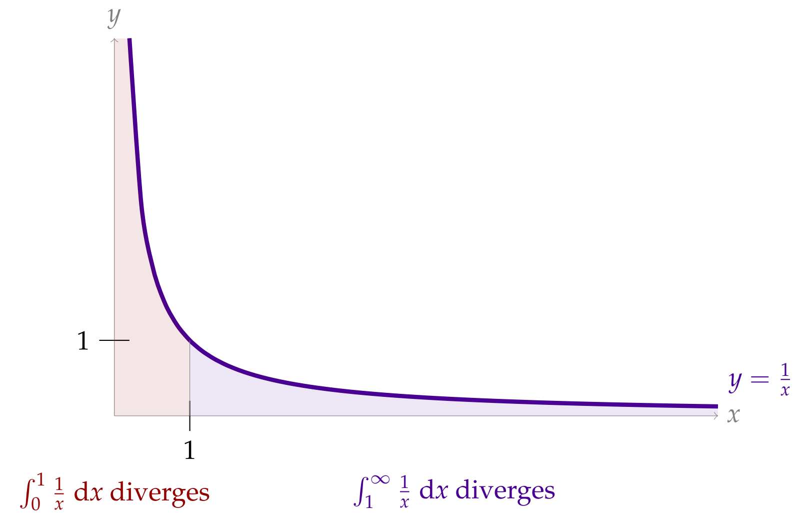
Let p be a constant.

If $p < 1$, then $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges

If $p \geq 1$, then $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ diverges

If $p > 1$, then $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges

If $p \leq 1$, then $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ diverges

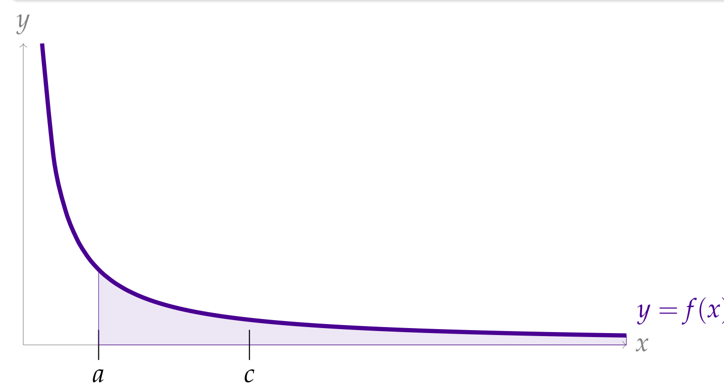


Decide whether each integral converges or diverges.

- $\displaystyle\int_0^1 \frac{1}{x^{1/3}}\,dx$ converges
- $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ converges
- $\displaystyle\int_0^1 \frac{1}{x}\,dx$ diverges
- $\displaystyle\int_0^1 \frac{1}{x^{1.5}}\,dx$ diverges
- $\displaystyle\int_1^\infty \frac{1}{x^{1/3}}\,dx$ diverges
- $\displaystyle\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges
- $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges
- $\displaystyle\int_1^\infty \frac{1}{x^{1.5}}\,dx$ converges

Theorem 1.12.20

Let a and c be real numbers with $a < c$ and let the function $f(x)$ be continuous for all $x \geq a$. Then the improper integral $\int_a^\infty f(x)\,dx$ converges if and only if the improper integral $\int_c^\infty f(x)\,dx$ converges.



Decide whether each integral converges or diverges.

- $\displaystyle\int_0^9 \frac{1}{x^{0.3}}\,\mathrm{d}x$ converges
- $\displaystyle\int_0^{81} \frac{1}{x^{2}}\,\mathrm{d}x$ diverges
- $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{x^{3}}\,\mathrm{d}x$ diverges
- $\displaystyle\int_{15}^{\infty} \frac{1}{x^{0.3}}\,\mathrm{d}x$ diverges
- $\displaystyle\int_{0.4}^{\infty} \frac{1}{x^{2}}\,\mathrm{d}x$ converges
- $\displaystyle\int_{\frac{1}{2}}^{\infty} \frac{1}{x^{3}}\,\mathrm{d}x$ converges

It is very common to encounter integrals that are too complicated to evaluate explicitly. Numerical approximation schemes, evaluated by computer, are often used instead. You want to be sure that at least the integral converges before feeding it into a computer.

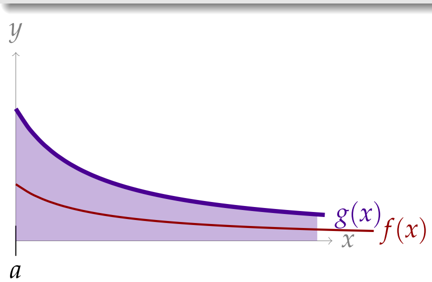
Fortunately it is usually possible to determine whether or not an improper integral converges even when you cannot evaluate it explicitly.

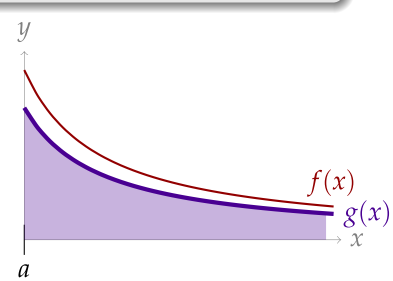
Comparison

Let a be a real number. Let f and g be functions that are defined and continuous for all $x \ge a$ and assume that $g(x) \ge 0$ for all $x \ge a$.

(a) If $|f(x)| \le g(x)$ for all $x \ge a$ and if $\int_a^\infty g(x)\,\mathrm{d}x$ converges, then $\int_a^\infty f(x)\,\mathrm{d}x$ converges.

(b) If $f(x) \ge g(x)$ for all $x \ge a$ and if $\int_a^\infty g(x)\,\mathrm{d}x$ diverges, then $\int_a^\infty f(x)\,\mathrm{d}x$ diverges.





Does the integral $\displaystyle\int_1^\infty e^{-x^2}$ converge or diverge?

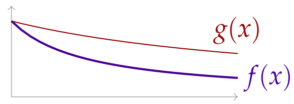
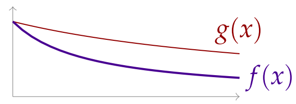
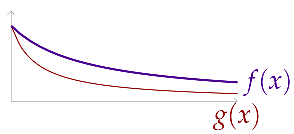
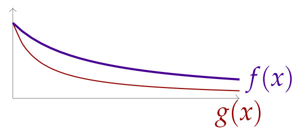
We know from previous examples that we can't evaluate $\int e^{-x^2}\mathrm{d}x$ directly. For $x \ge 1$:

$$x^2 > x \implies -x^2 < -x \implies e^{-x^2} < e^{-x}$$

$$\begin{aligned}\int_1^\infty e^{-x}\,\mathrm{d}x &= \lim_{b\to\infty}\int_1^b e^{-x}\,\mathrm{d}x \\ &= \lim_{b\to\infty}\left[-e^{-x}\right]_1^b \\ &= \lim_{b\to\infty}\left[e^{-b} - e^{-1}\right] \\ &= e^{-1} = \frac{1}{e}\end{aligned}$$

Since $0 \le e^{-x^2} \le e^{-x}$ for $x \ge 1$, and since $\int_1^\infty e^{-x}\,\mathrm{d}x$ converges, by the comparison test we conclude that $\int e^{-x^2}\mathrm{d}x$ converges, as well.

Let functions $f(x)$ and $g(x)$ be positive and continuous for all $x \geq a$.

	$\int_a^\infty g(x)\,dx$ converges	$\int_a^\infty g(x)\,dx$ diverges
$f(x) \leq g(x)$ for all $x \geq a$	 $\int_a^\infty f(x)$ converges	 inconclusive
$f(x) \geq g(x)$ for all $x \geq a$	 inconclusive	 $\int_a^\infty f(x)$ diverges

For each example below, decide whether the statement is a valid use of the comparison theorem.

- $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges and $0 \leq \frac{1}{x^2} \leq \frac{2+\sin x}{x^2}$ for $x \geq 1$. So by the comparison test, $\displaystyle\int_1^\infty \frac{2+\sin x}{x^2}\,dx$ converges as well.
- $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges and $0 \leq \frac{e^{-x}}{x^2} \leq \frac{1}{x^2}$ for $x \geq 1$. So by the comparison test, $\displaystyle\int_1^\infty \frac{e^{-x}}{x^2}\,dx$ converges as well.
- $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges and $-\frac{1}{x} \leq \frac{1}{x^2}$ for $x \geq 1$. So by the comparison test, $\displaystyle\int_1^\infty \frac{-1}{x}\,dx$ converges as well.

Limiting comparison

Let $-\infty < a < \infty$. Let f and g be functions that are defined and continuous for all $x \geq a$ and assume that $g(x) \geq 0$ for all $x \geq a$.
If the limit

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

exists and is nonzero, then either $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ both converge, or they both diverge.

Use limiting comparison to determine whether $\displaystyle\int_1^\infty \frac{1}{x+10}\,dx$ converges or diverges.

An integrand that looks similar and simpler is $\frac{1}{x}$. Since $\frac{1}{x+10} < \frac{1}{x}$ and $\int_1^\infty \frac{1}{x}\,dx$ diverges, we can't directly compare the two series. So, let's use limiting comparison. Set $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x+10}$. Then:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x}{1/(x+10)} = \lim_{x\to\infty} \frac{x+10}{x} = 1$$

Since 1 is nonzero and finite, the integrals either both converge or both diverge. Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, we conclude $\int_1^\infty \frac{1}{x+10}\,dx$ diverges as well.

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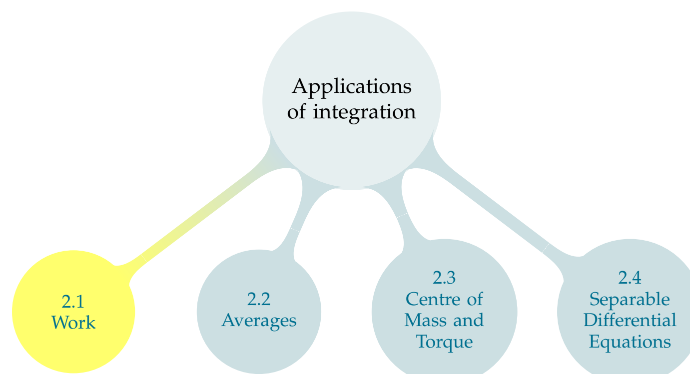


Helpful units

- Force is measured in units of newtons, with $1\text{ N} = 1\,\frac{\text{kg m}}{\text{s}^2}$.
- From its units, we see force looks like (mass)×(acceleration)
- Work is measured in units of joules, with $1\text{ J} = 1\,\frac{\text{kg}\cdot\text{m}^2}{\text{s}^2}$
- From its units, we see work looks like (force)×(distance)

Intuition

Work, in physics, is a way of quantifying the amount of energy that is required to act against a force.

For example:

- An object on the ground is subject to gravity. The force acting on the object is

$$m \cdot g$$

where m is the mass of the object (here, we're using kilograms), and g is the standard acceleration due to gravity (about $9.8\,\frac{\text{kg m}}{\text{s}^2}$ on Earth).
- When you lift an object in the air, you are acting against that force. How much work you have to do depends on how strong the force is (how much mass the object has, and how strong gravity is) and also how far you lift it.

Work

The work done by a force $F(x)$ in moving an object from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx$$

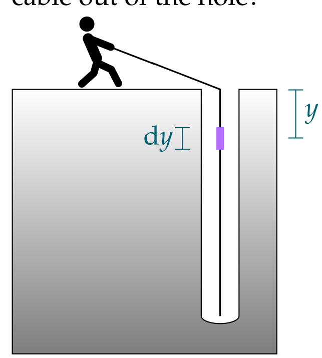
In particular, if the force is a constant F, the work is $F \cdot (b - a)$.

(For motivation of this definition, see Section 2.1 in the CLP–2 text.)

We saw the force of gravity on an object of mass m kg is $m \cdot g$ N. So to lift such an object a distance of y metres requires work of

$$m \cdot g \cdot y \text{ J}$$

A cable dangles in a hole. The cable is 10 metres long, and has a mass of 5 kg. Its density is constant. How much work is done to pull the cable out of the hole?

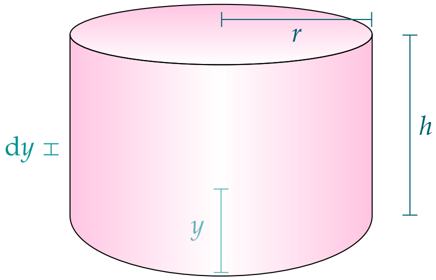


The cable has density $\frac{5\text{ kg}}{10\text{ m}} = \frac{1}{2}\frac{\text{kg}}{\text{m}}$. A slice of length dy has mass $\frac{1}{2}dy$ kg, so it is subject to a downward gravitational force of $\frac{g}{2}dy$ N, where g is the acceleration due to gravity.
A slice y metres below the top of the hole travels y metres to get out of the hole, taking work $\frac{g}{2}y\,dy$. So the work required to life the entire cable out of the hole is:

$$\int_0^{10} \frac{g}{2}y\,dy = \left[\frac{g}{4}y^2\right]_0^{10} = 25g \text{ J}$$

- A piece of the cable near the top of the hole isn't lifted very far.

- Consider a small piece of cable starting y metres from the top.



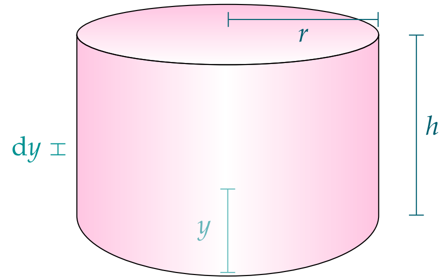
The volume of a cylindrical slice at height y is $\pi r^2\,dy$. If the density of the liquid is ρ, then the mass of liquid in the slice is $\rho \cdot \pi r^2\,dy$. Let g be the acceleration due to gravity. The force of gravity on the slice is $g\rho \cdot \pi r^2\,dy$.

A cylinder is filled with a liquid that we will pump out the top.

- To pump out a molecule from the top of the container, we don't have to work against gravity for very far.
- To pump out a molecule from the bottom of the container, we have to work against gravity for a longer distance.

Liquid in the slice needs to travel to the top of the container, a distance of $h - y$. So the work required to pump out a single slice at height y is $(h-y)g\rho \cdot \pi r^2\,dy$. All together, the work to empty the container is

$$\int^h (h-y)g\rho \cdot \pi r^2\,dy.$$



The volume of a cylindrical slice at height y is $\pi r^2\,dy$. If the density of the liquid is ρ, then the mass of liquid in the slice is $\rho \cdot \pi r^2\,dy$. Let g be the acceleration due to gravity. The force of gravity on the slice is $g\rho \cdot \pi r^2\,dy$.

- Every molecule at the same height has the same distance to travel to reach the top of the container. So, we'll chop up the tank into thin horizontal slices.

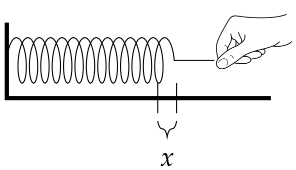
Liquid in the slice needs to travel to the top of the container, a distance of $h - y$. So the work required to pump out a single slice at height y is $(h-y)g\rho \cdot \pi r^2\,dy$. All together, the work to empty the container is

$$\int_0^h (h-y)g\rho \cdot \pi r^2\,dy.$$

$$\int (h-y)g\rho \cdot \pi r^2\,dy = g\rho \cdot \pi r^2 \int (h-y)\,dy$$

Hooke's Law

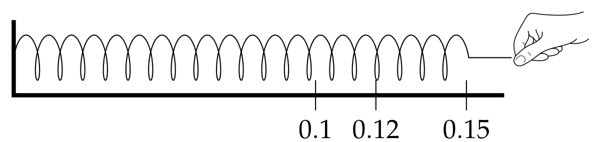
When a (linear) spring is stretched (or compressed) by x units beyond its natural length, it exerts a force of magnitude kx, where the constant k is the spring constant of that spring.



Suppose we want to stretch a string from a units beyond its natural length to b units beyond its natural length. The force of the spring at position x is kx, for some constant k. So, the work required is:

$$\int_a^b kx\,dx = \frac{k}{2}\left(b^2 - a^2\right)$$

A spring has a natural length of 0.1 m. If a 12 N force is needed to keep it stretched to a length of 0.12 m, how much work is required to stretch it from 0.12 m to 0.15 m?



When the spring is stretched to 0.12 m, the force exerted is

$$k(0.12 - 0.1) = 0.02k = 12N$$

So, $k = \frac{12\text{ N}}{0.02\text{ m}} = 600\ \frac{\text{N}}{\text{m}} = 600\ \frac{\text{kg}}{\text{s}^2}$. The spring starts at 0.02 metres beyond its natural length, and ends 0.05 metres beyond its natural length. The work required is:

$$\int_{0.02}^{0.05} kx\,dx = \int_{0.02}^{0.05} 600x\,dx = \left[300x^2\right]_{0.02}^{0.05}$$

$$= 300\left[0.05^2 - 0.02^2\right] = 0.63\ \frac{\text{kg m}^2}{\text{s}^2} = 0.63\text{ J}$$

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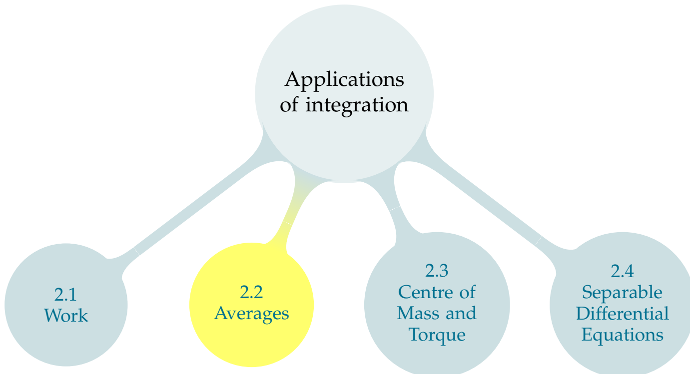
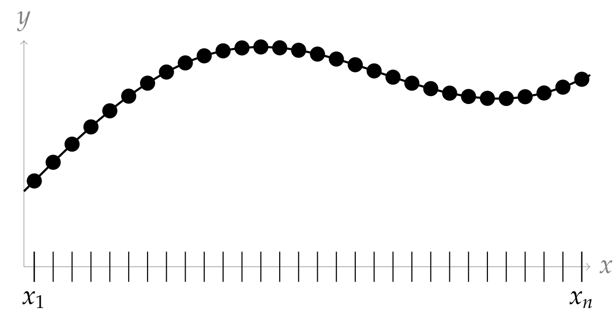




$$\text{Average} \approx \frac{f(x_1)+\cdots+f(x_n)}{n}$$

$$\text{Average} = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i)\right] = \lim_{n\to\infty}\left[\frac{(b-a)}{(b-a)n}\sum_{i=1}^{n} f(x_i)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\Delta x\right] = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$$

Average

Let $f(x)$ be an integrable function defined on the interval $a \le x \le b$. The average value of f on that interval is

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

The temperature in a certain city at time t (measured in hours past midnight) is given by

$$T(t) = t - \frac{t^2}{30}$$

What was the average temperature of one day (from $t = 0$ to $t = 24$)?

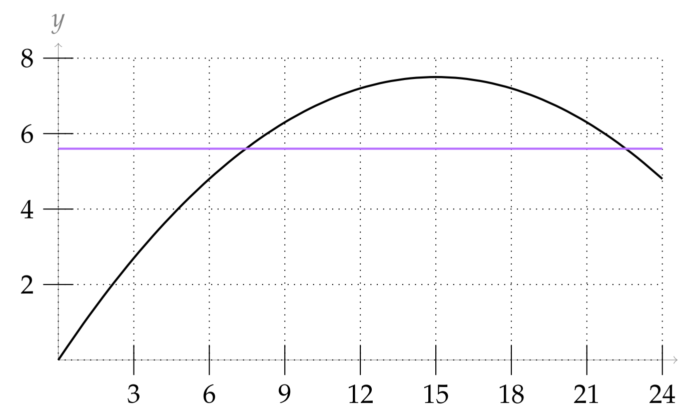
$$\text{Average} = \frac{1}{24}\int_0^{24}\left[t - \frac{t^2}{30}\right]\mathrm{d}t$$

$$= \frac{1}{24}\left[\frac{t^2}{2} - \frac{t^3}{90}\right]_0^{24}$$

$$= \frac{1}{24}\left[\frac{24^2}{2} - \frac{24^3}{90}\right]$$

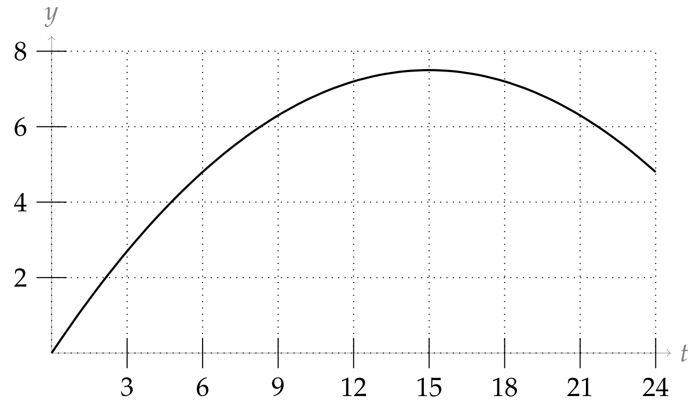
$$= \frac{\quad}{5} = 5.6$$

Let's check that our answer makes some intuitive sense.

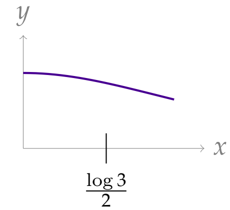


Since the temperature is always between 0 and 8, we expect the average to be between 0 and 8

Let's also recall the motivation for our definition



Find the average value of the function $f(x) = \dfrac{e^x}{e^{2x}+1}$ over the interval $\left[0, \frac{\log 3}{2}\right]$.



Let $u(x) = e^x$. Then $u(0) = 1$ and $u\left(\frac{\log 3}{2}\right) = e^{\frac{\log 3}{2}} = 3^{1/2} = \sqrt{3}$.

$$\frac{1}{\frac{\log 3}{2} - 0} \cdot \int_0^{\frac{\log 3}{2}} \frac{e^x}{e^{2x}+1}\,dx$$
$$= \frac{2}{\log 3}\int_1^{\sqrt{3}} \frac{1}{u^2+1}\,du$$
$$= \frac{2}{\log 3}\left[\arctan(\sqrt{3}) - \arctan(1)\right]$$
$$= \frac{2}{\log 3}\left[\frac{\pi}{3} - \frac{\pi}{4}\right] = \frac{\pi}{6\log 3} \approx 0.477$$

Average Velocity

Let $x(t)$ be the position at time t of a car moving along the x-axis. The velocity of the car at time t is the derivative $v(t) = x'(t)$. The average velocity of the car over the time interval $a \le t \le b$ is:

$$v_{\text{ave}} = \frac{1}{b-a}\int_a^b v(t)\,dt = \frac{1}{b-a}\int_a^b x'(t)\,dt = \frac{x(b)-x(a)}{b-a}$$

That is: $\dfrac{\text{change in distance}}{\text{change in time}}$

Notice that this is exactly the formula we used way back at the start of your differential calculus class to help introduce the idea of the derivative. Of course this is a very circuitous way to get to this formula — but it is reassuring that we get the same answer.

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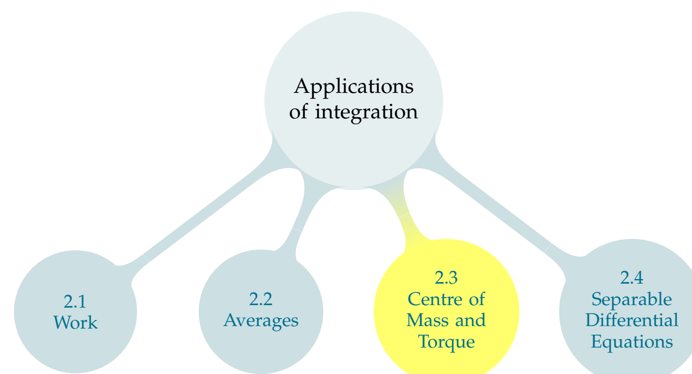
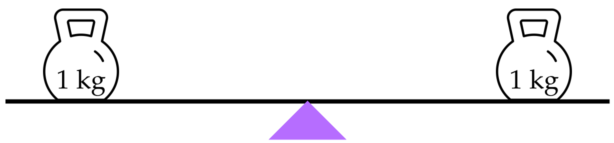
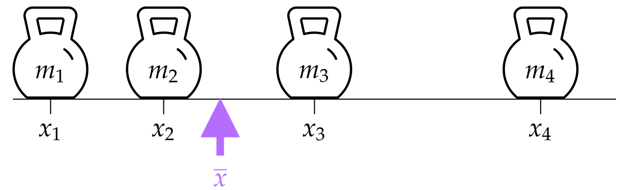


Centre of Mass

If you support a body at its centre of mass (in a uniform gravitational field) it balances perfectly. That's the definition of the centre of mass of the body.



If you support a body at its centre of mass (in a uniform gravitational field) it balances perfectly. That's the definition of the centre of mass of the body.



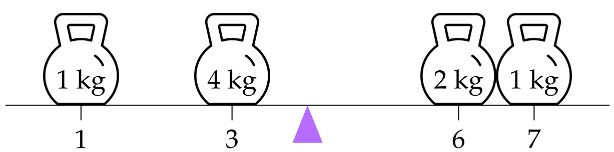
If the body consists of a finite number of masses $m_1, \cdots, m_n$ attached to an infinitely strong, weightless (idealized) rod with mass number i attached at position x_i, then the center of mass is at the (weighted) average value of x:

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

The denominator $m = \sum_{i=1}^{n} m_i$ is the total mass of the body.

An idealized (weightless, unbending) rod has small masses attached to it at the following locations:

- 1 kg at $x = 1$ metre from the left end
- 4 kg at $x = 3$ metres from the left end
- 2 kg at $x = 6$ metres from the left end
- 1 kg at $x = 7$ metres from the left end

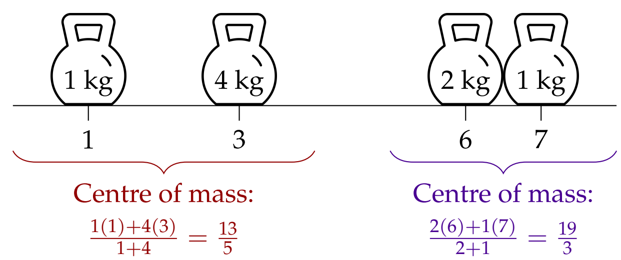


What is the location of its centre of mass?

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{1(1) + 4(3) + 2(6) + 1(7)}{1 + 4 + 2 + 1} = 4$$

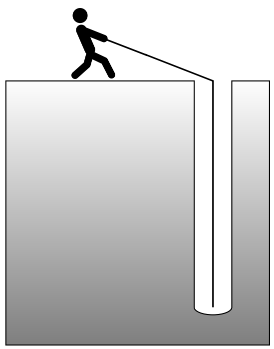
So the centre of mass is 4 metres from the left end of the rod.

We can also group the masses, and treat them as single points of mass at their centres of gravity, without affecting the centre of gravity of the entire object.



Centre of mass of second rod: $\bar{x} = \frac{5\left(\frac{13}{5}\right)+3\left(\frac{19}{3}\right)}{5+3} = 4$

Sometimes we can simplify a physical calculation by treating an object as a point particle located at its centre of mass.
When we were learning about work, we found the following:



A cable dangles in a hole. The cable is 10 metres long, and has a mass of 5 kg. Its density is constant. We found that the work required to pull the cable out of the hole was

$$25g\ J$$

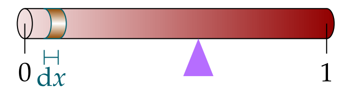
where g is the acceleration due to gravity.

Since the cable has constant density, it should "balance" at its centre (if it were rigid), so its centre of mass starts 5 metres below the ground. It ends up on the ground. If we treat the cable as a point particle of mass 5 kg, moving against gravity for a distance of 5 metres, we find the work done to be

Consider a metre-long rod that is denser on one end than the other, with density

$$\rho(x) = (2x+1)\ \frac{\text{kg}}{\text{m}}$$

at a position x metres from its left end.



What is its centre of mass?

We can use our usual slicing-up procedure. Consider slicing the rod into tiny cross-sections, each with width $\mathrm{d}x$. Then a cross-section at position x is approximately a point mass with position x and mass $\rho(x)\,\mathrm{d}x = (2x+1)\,\mathrm{d}x$. So, using integrals to add up the contributions from the different slices, the centre of mass is:

$$\bar{x} = \frac{\int_0^1 x(2x+1)\,\mathrm{d}x}{\int_0^1 (2x+1)\,\mathrm{d}x} = \frac{\left[\frac{2}{3}x^3 + \frac{1}{2}x^2\right]_0^1}{\left[x^2+x\right]_0^1} = \frac{7/6}{2} = \frac{7}{12}$$

If a body consists of mass distributed continuously along a straight line, say with mass density $\rho(x)$kg/m and with x running from a to b, rather than consisting of a finite number of point masses, the formula for the center of mass becomes

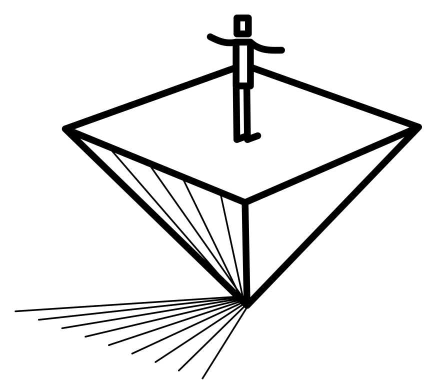
$$\bar{x} = \frac{\int_a^b x\,\rho(x)\,\mathrm{d}x}{\int_a^b \rho(x)\,\mathrm{d}x}$$

Think of $\rho(x)\,\mathrm{d}x$ as the mass of the "almost point particle" between x and $x + \mathrm{d}x$.

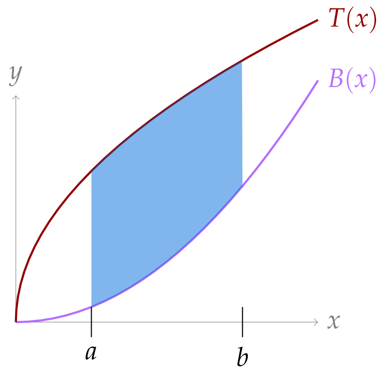
Centre of Mass

If you support a body at its centre of mass (in a uniform gravitational field) it balances perfectly. That's the definition of the center of mass of the body.

Centre of mass isn't just for linear solids: it applies to 2- and 3-dimensional objects as well.

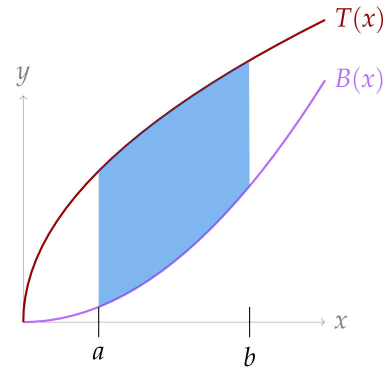


Consider a flat metal plate of uniform density, whose shape is the area below curve $y = T(x)$ and above curve $y = B(x)$, from $x = a$ to $x = b$.



The centre of mass will be a point in the xy-plane, $(\bar{x}, \bar{y})$.
To find $\bar{x}$ and $\bar{y}$, we will treat vertical slices as point particles.

Consider a flat metal plate of uniform density, whose shape is the area below curve $y = T(x)$ and above curve $y = B(x)$, from $x = a$ to $x = b$.

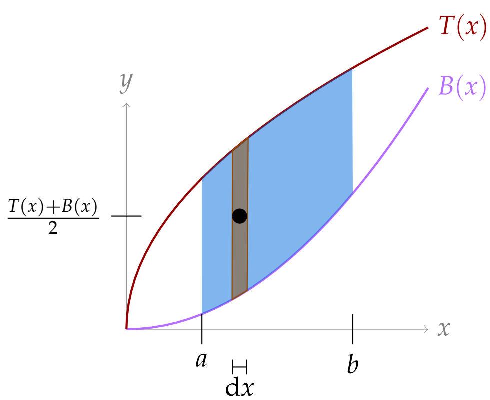


If ρ is the density of the plate, so that a slice of width $\mathrm{d}x$ and height $h = T(x) - B(x)$ has mass $\rho\, h\, \mathrm{d}x = \rho\big(T(x) - B(x)\big)\, \mathrm{d}x$, then:

$$\bar{x} = \frac{\int_a^b \rho(T(x) - B(x)) \cdot x \, \mathrm{d}x}{\int_a^b \rho(T(x) - B(x)) \, \mathrm{d}x} = \frac{\int_a^b (T(x) - B(x)) \cdot x \, \mathrm{d}x}{\int_a^b (T(x) - B(x)) \, \mathrm{d}x}$$

The centre of mass will be a point in the xy-plane, $(\bar{x}, \bar{y})$.
To find $\bar{x}$ and $\bar{y}$, we will treat vertical slices as point particles.

To find $\bar{y}$, note that the y-coordinate of the centre of mass of a slice that is almost a rectangle, and has uniform density, will be halfway up the slice, at $\frac{T(x)+B(x)}{2}$.



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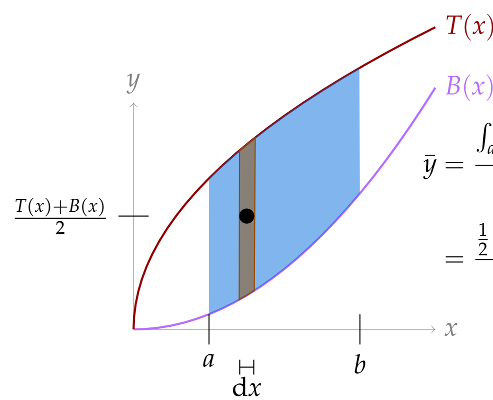
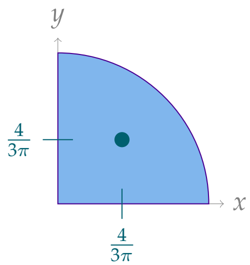


$$\bar{y} = \frac{\int_a^b \left(\frac{T(x)+B(x)}{2}\right) \cdot \rho(T(x) - B(x)) \, \mathrm{d}x}{\int_a^b \rho(T(x) - B(x)) \, \mathrm{d}x} = \frac{\frac{1}{2}\int_a^b (T^2(x) - B^2(x)) \, \mathrm{d}x}{\int_a^b (T(x) - B(x)) \, \mathrm{d}x}$$

Find the centre of mass (centroid) of the quarter circular unit disk $x \geq 0, y \geq 0, x^2 + y^2 \leq 1$.



By symmetry, $\bar{x} = \bar{y}$. Using the equations we developed above with top $y = T(x) = \sqrt{1-x^2}$ and bottom $y = B(x) = 0$:

$$\bar{x} = \frac{\int_0^1 (\sqrt{1-x^2} - 0) \cdot x \, \mathrm{d}x}{\int_0^1 (\sqrt{1-x^2} - 0) \, \mathrm{d}x}$$

For the integral in the numerator, let $u = 1 - x^2$, $\mathrm{d}u = -2x\,\mathrm{d}x$. The denominator is the area of the quarter unit circle.

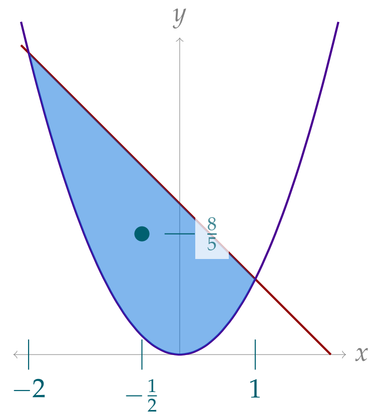
$$= \frac{\int_1^0 -\frac{1}{2} u^{1/2} \, \mathrm{d}u}{\frac{\pi}{4}}$$

$$= \frac{2}{\pi} \int_0^1 u^{1/2} \, \mathrm{d}u$$

$$= \frac{2}{\pi} \left[\frac{2}{3} u^{3/2} \right]_0^1 = \frac{4}{3\pi}$$

$$(\bar{x}, \bar{y}) = \left(\frac{4}{3\pi}, \frac{4}{3\pi} \right)$$

Find the centre of mass (centroid) of a plate of uniform density in the shape of the finite area enclosed by the functions $y = T(x) = 2 - x$ and $y = B(x) = x^2$.



First, we find where the curves intersect.

$$x^2 = 2 - x$$
$$x^2 + x - 2 = 0$$
$$(x-1)(x+2) = 0$$
$$x = -2, \; x = 1$$

The denominator is the same in our $\bar{x}$ and $\bar{y}$ calculations, so let's find that next.

$$\int_{-2}^1 (T(x) - B(x)) \, \mathrm{d}x = \int_{-2}^1 (2 - x - x^2) \, \mathrm{d}x$$

$$= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_{-2}^1$$

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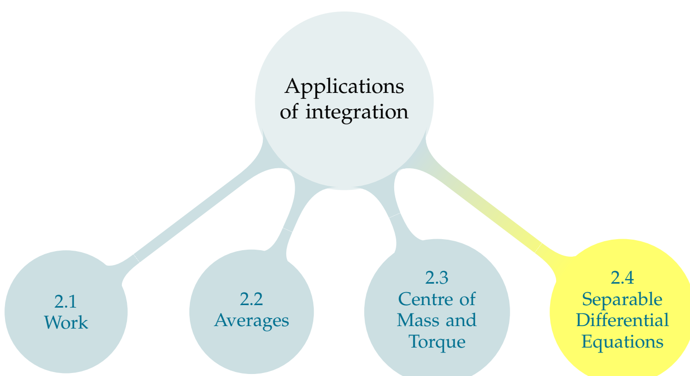

Differential Equation

A differential equation is an equation for an unknown function that involves the derivative of the unknown function.

Differential equations play a central role in modelling a huge number of different phenomena. Here is a table giving a bunch of named differential equations and what they are used for. It is far from complete.

Newton's Law of Motion	describes motion of particles
Maxwell's equations	describes electromagnetic radiation
Navier–Stokes equations	describes fluid motion
Heat equation	describes heat flow
Wave equation	describes wave motion
Schrödinger equation	describes atoms, molecules and crystals
Stress-strain equations	describes elastic materials
Black–Scholes models	used for pricing financial options
Predator–prey equations	describes ecosystem populations
Einstein's equations	connects gravity and geometry
Ludwig–Jones–Holling's equation	models spruce budworm/Balsam fir ecosystem
Zeeman's model	models heart beats and nerve impulses
Sherman–Rinzel–Keizer model	for electrical activity in Pancreatic β–cells
Hodgkin–Huxley equations	models nerve action potentials

Disclaimer:
We are dipping our toes into a vast topic. Most universities offer half a dozen different undergraduate courses on various aspects of differential equations. We will just look at one special, but important, type of equation.

- We will first learn to verify solutions without finding them. (If you learned about differential equations last semester, this will be review.)
- Then, we will learn to solve one particular type of differential equation.

Differential Equations

Definition
A **differential equation** is an equation involving the derivative of an unknown function.

Examples: $\frac{dy}{dx} = 2x$; $\quad x\frac{dy}{dx} = 7xy + y$

Definition
If a function makes a differential equation true, we say it **satisfies** the differential equation, or is a solution to the differential equation.

Example: $y = x^2$ and $y = x^2 + 1$ both satisfy the first differential equation

Verifying Solutions

Consider the equation
$$x + 2 = x^3 - x^2$$
How would you verify whether $x = 1$ satisfies the equation?
How would you verify whether $x = 2$ satisfies the equation?
Plug x into the equation, check whether the left-hand side and the right-hand side are the same **number**.

Verifying Solutions

Consider the differential equation
$$\frac{dy}{dx} = 2y + 4x$$
How would you verify whether $y = e^{2x} - 2x$ satisfies the equation?
How would you verify whether $y = e^{2x} - 2x - 1$ satisfies the equation?
Replace y and $\frac{dy}{dx}$ in the equation, check whether the left-hand side and the right-hand side are the same **function**.

- If $y = e^{2x} - 2x$, then $\frac{dy}{dx} = 2e^{2x} - 2$. Plug these into both sides of the differential equation, replacing anything depending on y:
$$\frac{dy}{dx} = 2y + 4x$$
$$2e^{2x} - 2 \stackrel{?}{=} 2(e^{2x} - 2x) + 4x$$
$$2e^{2x} - 2 \stackrel{?}{=} 2e^{2x}$$
Since the functions on the left and right are not the same function, $y = e^{2x} - 2x$ is not a solution to the differential equation.
- If $y = e^{2x} - 2x - 1$, then $\frac{dy}{dx} = 2e^{2x} - 2$. Plug these into both sides of the differential equation, replacing anything depending on y:
$$\frac{dy}{}$$

Differential equation:

$$x\frac{dy}{dx} = 7xy + y$$

Interpretation:

There is a function $y(x)$ that makes the left-hand side and the right-hand side into the same function.

To check whether a given function satisfies the differential equation, plug it in for everything with a "y": y itself and $\frac{dy}{dx}$.

Is $y = xe^{7x+9}$ a solution to the differential equation?

Which of the following solve the differential equation $\frac{dy}{dx} = \frac{x}{y}$?

A. $y = -x$ B. $y = x + 5$ C. $y = \sqrt{x^2 + 5}$

- If $y = -x$, then $\frac{dy}{dx} = -1$. Plugging into the differential equation yields: $-1 \stackrel{?}{=} \frac{x}{-x}$. Since the left and right are the same function (except for the single point when $x = 0$), we say $y = -x$ solves the differential equation.
- If $y = x + 5$, then $\frac{dy}{dx} = 1$. Plugging into the differential equation yields: $1 \stackrel{?}{=} \frac{x}{x+5}$. Since the left and right are not the same function, $y = x + 5$ does not solve the differential equation.
- If $y = \sqrt{x^2 + 5}$, then $\frac{dy}{dx} = \frac{2x}{2\sqrt{x^2+5}} = \frac{x}{\sqrt{x^2+5}}$. Plugging into the differential equation yields: $\frac{x}{\sqrt{x^2+5}} \stackrel{?}{=} \frac{x}{\sqrt{x^2+5}}$. Since the left and right are the same function, we say $y = \sqrt{x^2 + 5}$ solves the differential equation.

First Example of a Separable DE

Definition

A separable differential equation is an equation for a function y(x) that can be written in the form

$$g(y) \cdot \frac{dy}{dx} = f(x)$$

(It may take some rearranging to get the equation into this form.)

For example:

$$y^2 \cdot \frac{dy}{dx} = 4x$$

$$\int \left(y^2 \cdot \frac{dy}{dx}\right) dx = \int 4x\, dx$$

$$\int y^2\, dy = 2x^2 + C$$

$$\frac{1}{3}y^3 = 2x^2 + C$$

$$y^3 = 6x^2 + 3C$$

$$y(x) = \sqrt[3]{6x^2 + 3C}$$

General Method for Solving Separable DEs

$$g(y) \cdot \frac{dy}{dx} = f(x)$$

$$g(y(x)) \cdot \frac{dy}{dx} = f(x)$$

$$\int \left(g(y(x)) \cdot \frac{dy}{dx}\right) dx = \int f(x)\, dx$$

y-substitution:

$$\int g(y)\, dy = \int f(x)\, dx$$

General Method for Solving Separable DEs

$$g(y)\cdot\frac{dy}{dx}=f(x)$$

$$g(y(x))\cdot\frac{dy}{dx}=f(x)$$

$$\int\left(g(y(x))\cdot\frac{dy}{dx}\right)dx=\int f(x)\,dx$$

y-substitution:

$$\int g(y)\,dy=\int f(x)\,dx$$

Shorthand:

$$g(y)\cdot\frac{dy}{dx}=f(x)$$
$$g(y)\,dy=f(x)\,dx$$
$$\int g(y)\,dy=\int f(x)\,dx$$

$$\frac{dy}{dx}=y^2x$$

1. "Separate" y's from x's.
2. Integrate.
3. Solve explicitly for y.

$$\frac{dy}{dx}=(xy)^4,\qquad y(0)=\frac{1}{2}$$

$$\frac{dy}{dx}=x^4y^4$$
$$y^{-4}\,dy=x^4dx$$
$$\int y^{-4}dy=\int x^4dx$$
$$\frac{1}{-3}y^{-3}=\frac{1}{5}x^5+C$$
$$\frac{1}{y^3}=-3\left(\frac{1}{5}x^5+C\right)$$
$$y=\frac{1}{-\sqrt[3]{3\left(\frac{1}{5}x^5+C\right)}}$$

$$y(0)=-\sqrt[3]{\frac{1}{3(\frac{1}{5}x^5+C)}}\Bigg|_{x=0}$$
$$\frac{1}{2}=-\sqrt[3]{\frac{1}{3C}}$$
$$2=-\sqrt[3]{3C}$$
$$3C=-8$$
$$y(x)=-\sqrt[3]{\frac{1}{\frac{3}{5}x^5-8}}$$
$$=\sqrt[3]{\frac{1}{8-\frac{3}{5}x^5}}$$

$$\frac{dy}{dx}=y(4x^3-1)\qquad y(0)=-2$$

$$\frac{1}{y}\,dy=(4x^3-1)\,dx$$
$$\int\frac{1}{y}\,dy=\int(4x^3-1)\,dx$$
$$\log|y|=x^4-x+C$$
When $x=0$, $\log|-2|=0^4-0+C$
$$C=\log 2$$
$$|y(x)|=e^{x^4-x+\log 2}$$
$$y(x)=e^{x^4-x+\log 2}\quad\text{or}\quad y(x)=-e^{x^4-x+\log 2}$$
$$y(x)=-e^{x^4-x+\log 2}=-2e^{x^4-x}\quad\text{to make } y(0)=-2$$

Let a and b be any two constants. We'll now solve the family of differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(y-b)$$

using our mnemonic device.

$$\frac{\mathrm{d}y}{y-b} = a\,\mathrm{d}x$$
$$\int \frac{\mathrm{d}y}{y-b} = \int a\,\mathrm{d}x$$
$$\log|y-b| = ax + c$$
$$|y-b| = e^{ax+c} = e^c e^{ax}$$
$$y - b = \pm e^c e^{ax} = Ce^{ax}$$

where the constant C can be any real number. (Even $C=0$ works, i.e. $y(x)=b$ solves $\frac{\mathrm{d}y}{\mathrm{d}x} = a(y-b)$.) Note that when $y(x) = Ce^{ax}+b$ we have $y(0) = C+b$. So $C = y(0)-b$ and the general solution is

$$y(x) = \{y(0)-b\}\, e^{ax} + b$$

Linear First-Order Differential Equations

Let a and b be constants. The differentiable function $y(x)$ obeys the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(y-b)$$

if and only if

$$y(x) = \{y(0)-b\}\, e^{ax} + b$$

Find a function $y(x)$ with $y' = 3y+7$ and $y(2)=5$.

To avoid re-inventing the wheel, we'll use our equation. But first, we should re-write our differential equation so the formatting matches.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3\left(y + \frac{7}{3}\right)$$
$$a = 3, \quad b = -\frac{7}{3}$$

Since we aren't given $y(0)$, we can't use the theorem as a shortcut to find C. We'll do it the old-fashioned way.

$$5 = y(2) = Ce^{3(2)} - \frac{7}{3}$$
$$\frac{22}{3} = Ce^6$$
$$C = \frac{22}{\ldots}$$

The rate at which a medicine is metabolized (broken down) in the body depends on how much of it is in the bloodstream. Suppose a certain medicine is metabolized at a rate of $\frac{1}{10}A$ μg/hr, where A is the amount of medicine in the patient. The medicine is being administered to the patient at a constant rate of 2 μg/hr.
If the patient starts with no medicine in their blood at $t=0$, give the formula for the amount of medicine in the patient at time t. What happens to the amount over time?

The rate of change of the amount of medicine in the patient is given by how quickly the medicine is being administered, minus how quickly it is metabolized:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2 - \frac{1}{10}A$$

Linear First-Order Differential Equations

Let a and b be constants. The differentiable function $y(x)$ obeys the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(y-b)$$

if and only if

$$y(x) = \{y(0)-b\}\, e^{ax} + b$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 2 - \frac{1}{10}A = -\frac{1}{10}(A-20) \qquad A(0) = 0$$

$$a = -\frac{1}{10}, \quad b = 20$$
$$A(t) = (A(0)-20)e^{-t/10} + 20$$
$$A(t) = -20e^{-t/10} + 20$$

This is an increasing function, with $\lim\limits_{t\to\infty} A(t) = 20$. So the amount of medicine initially increases, but eventually almost holds steady at 20 μg.

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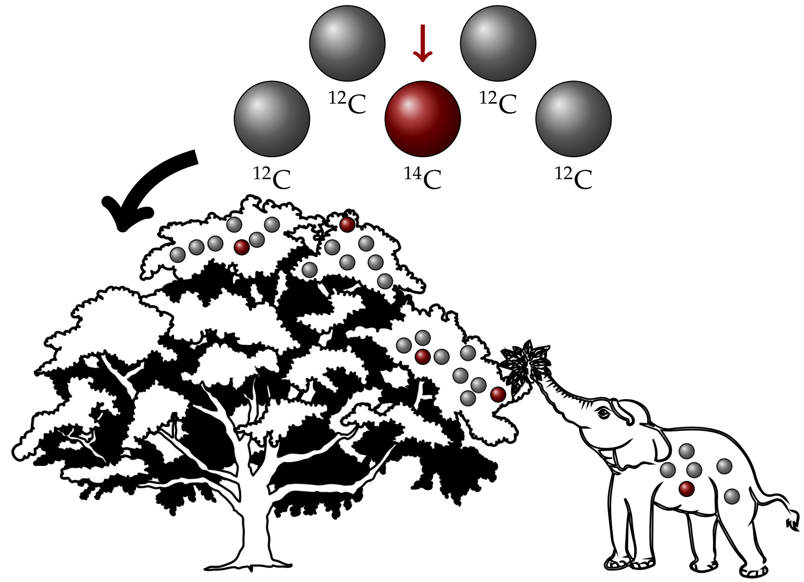
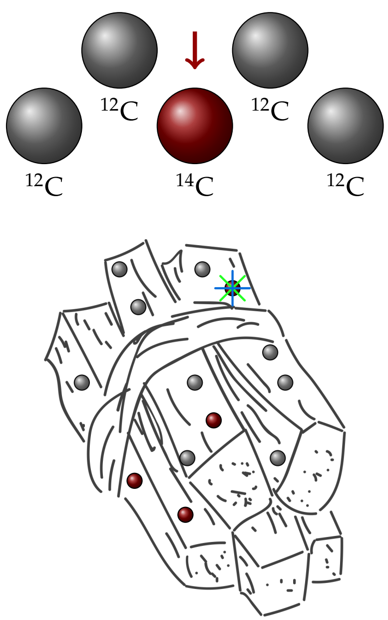




Radioactive Decay

One model for radioactive decay says that the rate at which an isotope decays is proportional to the amount present. So if $Q(t)$ is the amount of a radioactive substance, then

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -kQ(t)$$

for some constant[1] k.

This is a first-order linear differential equation. Its explicit solutions have the form:

$$Q(t) = Ce^{-kt}$$

where $C = Q(0)$.

[1]By including the negative sign, we ensure k will be positive, but of course we could also write "$\frac{\mathrm{d}Q}{\mathrm{d}t} = KQ(t)$ for some [negative] constant K".

Half-life

The half-life of an isotope is the time required for half of that isotope to decay. If we know the half-life of a substance is $t_{1/2}$, and its quantity at time t is given by $Q(0)e^{-kt}$ we can find the constant k:

$$\frac{1}{2}Q(0) = Q(t_{1/2}) = Q(0)e^{-kt_{1/2}}$$
$$\frac{1}{2} = e^{-kt_{1/2}}$$
$$2 = e^{kt_{1/2}}$$
$$\log 2 = kt_{1/2}$$
$$\frac{\log 2}{t_{1/2}} = k$$

Plugging this back in gives us a more intuitive equation for the quantity of a radioactive substance over time:

$$Q(t) = Q(0)e^{-\frac{\log 2}{t_{1/2}}t} = Q(0)\cdot\left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}}$$

So if $t = t_{1/2}$, the initial amount is cut in half; if $t = 2t_{1/2}$, the initial amount is cut in half twice (i.e. quartered), etc.

Radioactive Decay

The function $Q(t)$ satisfies the equation $\frac{dQ}{dt} = -kQ(t)$ if and only if

$$Q(t) = Q(0)\,e^{-kt}$$

The half–life is defined to be the time $t_{1/2}$ which obeys $Q(t_{1/2}) = \frac{1}{2}Q(0)$. The half–life is related to the constant k by $t_{1/2} = \frac{\log 2}{k}$. Then

$$Q(t) = Q(0)\,e^{-\frac{\log 2}{t_{1/2}}t} = Q(0)\cdot\left(\frac{1}{2}\right)^{\frac{t}{t_{1/2}}}$$

If the half-life of ^{14}C is $t_{1/2} = 5730$ years, then the quantity of carbon-14 present in a sample after t years is:

$$Q(t) = Q(0)e^{-\frac{\log 2}{5730}t} = Q(0)\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$

A particular piece of flax parchment contains about 64% as much ^{14}C as flax plants do today. We will estimate the age of the parchment, using 5730 years as the half-life of ^{14}C.

First, a rough estimate: is the parchment older or younger than 5730 years?

Younger: it has *more* that half its ^{14}C left, so it has been decaying for *less* than one half-life.

Let $Q(t)$ denote the amount of ^{14}C in the parchment t years after it was first created.

$$Q(t) = Q(0)\left(\frac{1}{2}\right)^{\frac{t}{5730}}$$
$$0.64 = \left(\frac{1}{2}\right)^{\frac{t}{5730}}$$
$$\log(0.64) = \frac{t}{5730}\log\frac{1}{2} = -\frac{\log 2}{5730}t$$
$$t = -\frac{5730\log(0.64)}{\log 2} \approx 3689$$

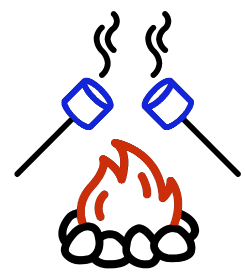
$$Q(t) = Q(0)e^{-\frac{\log 2}{5370}t}$$
$$0.64 = e^{-\frac{\log 2}{5370}t}$$
$$\log(0.64) = -\frac{\log 2}{5730}t$$
$$t = -\frac{5730\log(0.64)}{\log 2} \approx 3689$$

So the parchment was made of plants that died about 3700 years ago.

Newton's law of cooling

The rate of change of temperature of an object is proportional to the difference in temperature between the object and its surroundings.

The temperature of the surroundings is sometimes called the ambient temperature.



$$\frac{dT}{dt} =$$

Linear First-Order Differential Equations

Let a and b be constants. The differentiable function $y(x)$ obeys the differential equation

$$\frac{dy}{dx} = a(y-b)$$

if and only if

$$y(x) = \{y(0)-b\}\, e^{ax} + b$$

Find an explicit formula for functions $T(t)$ solving the differential equation $\frac{dT}{dt} = K(T(t)-A)$ for some constants K and A.

$$T(t) = (T(0)-A)\, e^{Kt} + A$$

The temperature of a glass of iced tea is initially 5°. After 5 minutes, the tea has heated to 10° in a room where the air temperature is 30°. Assume the temperature of the tea as it cools follows Newton's law of cooling,

$$T(t) = (T(0)-A)\, e^{Kt} + A$$

(a) Determine the temperature as a function of time.
(b) When the tea will reach a temperature of 14° ?

The ambient temperature is $A = 30$ and $T(0) = 5$, so we only have to determine K. (Or, more neatly, e^K.)

$$\begin{aligned} T(t) &= (5-30)\, e^{Kt} + 30 \\ &= 30 - 25e^{Kt} \\ 10 = T(5) &= 30 - 25e^{5K} \\ 25e^{5K} &= 20 \end{aligned}$$

$$\begin{aligned} e^{5K} &= \frac{4}{5} \\ e^K &= \left(\frac{4}{5}\right)^{1/5} \\ T(t) &= 30 - 25\left(\frac{4}{5}\right)^{\frac{t}{5}} \end{aligned}$$

A glass of room-temperature water is carried out onto a balcony from an apartment where the temperature is 22°C. After one minute the water has temperature 26°C and after two minutes it has temperature 28°C. Assuming the water warms according to Newton's law of cooling, what is the outdoor temperature?
Assume that the temperature of the water obeys Newton's law of cooling.

$$\begin{aligned} T(t) &= A + (T(0)-A)e^{Kt} \\ &= A + (22-A)e^{Kt} \\ \text{Given:}\quad 26 &= A + (22-A)e^{K} \\ \implies e^K &= \frac{26-A}{22-A} \\ \text{Given:}\quad 28 &= A + (22-A)e^{2K} \\ \implies e^{2K} &= \frac{28-A}{22-A} \end{aligned}$$

$$\begin{aligned} \frac{28-A}{22-A} = \left(e^K\right)^2 &= \left(\frac{26-A}{22-A}\right)^2 \\ (28-A)(22-A) &= (26-A)^2 \\ 28\cdot 22 - 50A + A^2 &= 26^2 - 52A + A^2 \\ 2A &= 26^2 - 28\cdot 22 \\ A &= (26)(13) - (22)(14) \\ &= (26)(13) - (22)(13) - 22 \\ &= 4\cdot 13 - 22 = 30 \end{aligned}$$

Let P the the size of a population, and let K be the carrying capacity of its environment (i.e. the population size that can be sustainably supported).

When P is much less than K, our population has...

A. not enough resources
B. just enough resources
C. extra resources

So when the P is much less than K, we expect the population to...

A. shrink
B. stay the same
C. grow

Malthusian growth

The Malthusian growth model relates population growth to population size:

$$\frac{dP}{dt} = bP(t)$$

where b is a constant representing net birthrate per member of the population.

Let P the the size of a population, and let K be the carrying capacity of its environment (i.e. the population size that can be sustainably supported).

When P is greater than K, our population has...

A. not enough resources

B. just enough resources

C. extra resources

So when the P is greater than K, we expect the population to...

A. shrink

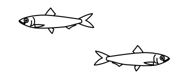
B. stay the same

C. grow

Logistic growth models population growth as:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = b_0\left(1 - \frac{P(t)}{K}\right)P(t)$$

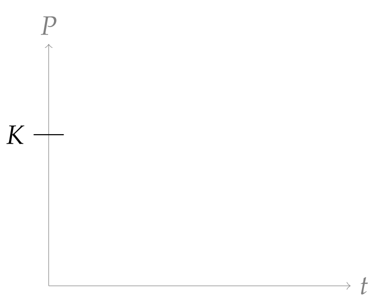
- If $P << K$, then $\frac{\mathrm{d}P}{\mathrm{d}t} \approx$
- If $P \approx K$, then $\frac{\mathrm{d}P}{\mathrm{d}t} \approx$
- If $P > K$, then $\frac{\mathrm{d}P}{\mathrm{d}t}$



Before we solve explicitly, let's sketch some solutions to

$$\frac{\mathrm{d}P}{\mathrm{d}t} = b_0\left(1 - \frac{P(t)}{K}\right)P(t)$$

- If $P(a) = 0$:
- If $0 < P(a) < K$:
- If $P(a) = K$:
- If $K < P(0)$:



Find the explicit solutions to

$$\frac{\mathrm{d}P}{\mathrm{d}t} = b\left(1 - \frac{P(t)}{K}\right)P(t)$$

when b and K are constants.

$$\int b\,\mathrm{d}t = \int \frac{1}{P(1-\frac{1}{K}P)}\,\mathrm{d}P$$

Using partial fractions, we can find $\frac{1}{P(1-\frac{1}{K}P)} = \frac{1}{P} + \frac{1/K}{1-\frac{1}{K}P}$

$$bt + D = \int \left(\frac{1}{P} - \frac{1/K}{1-\frac{1}{K}P}\right)\mathrm{d}P$$

$$= \log|P| - \log\left|1 - \tfrac{1}{K}P\right| = \log\left|\frac{P}{1-\frac{1}{K}P}\right|$$

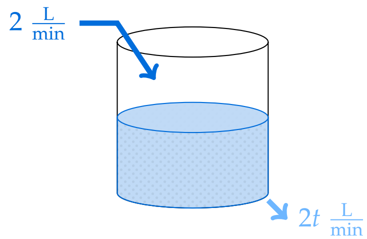
let $C = \pm e^D$:

$$Ce^{bt} = \frac{P}{1-\frac{1}{K}P} \implies P(t) = \frac{Ce^{bt}}{1 + \frac{C}{K}e^{bt}}$$

At time $t = 0$, where t is measured in minutes, a large tank contains 3 litres of water in which 1 kg of salt is dissolved. Fresh water enters the tank at a rate of 2 litres per minute and the fully mixed solution leaks out of the tank at the varying rate of $2t$ litres per minute.

(a) Determine the volume of solution $V(t)$ in the tank at time t.

(b) Determine the amount of salt $Q(t)$ in solution when the amount of water in the tank is at maximum.



We're given information about the rate of change of V: $\frac{dV}{dt} = 2 - 2t$. Then $V(t) = 2t - t^2 + C$. From the initial value $V(0) = 3$, we see

$$V(t) = 2t - t^2 + 3$$

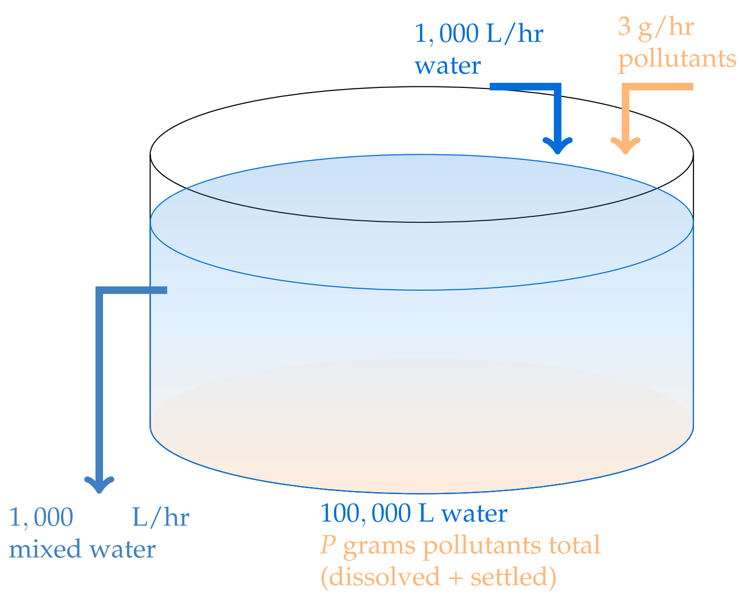
The maximum value of a downwards-facing parabola occurs at its critical point, so the water in the tank is at its highest level when $t = 1$. The amount of salt is decreasing as it leaks out. The

concentration of salt in the tank water at time t is ... If the

A settling tank is filled with 100,000 litres of pure water. Every hour, 1,000 litres of water, containing 3 grams of pollutants, enters the tank.

90% of the pollutants in the settling tank sink to the bottom, with the remaining 10% well-mixed into the water. The tank drains 1,000 litres of this mixed water into the sewer every hour.

In order to drain the water into the local sewer, the concentration of pollutants cannot be more than 1 gram per 1,000 litres. How long can the settling tank take dirty water until the process must be stopped?



Let $P(t)$ be the total amount (in grams) of pollutants in the tank. Pollutants are entering at a rate of 3 grams per hour. How fast are they leaving?

Every hour, the tank drains 1,000 of its 100,000 litres. That is, every hour, it drains $\frac{1}{100}$ of its total volume. So, every hour, it disgorges $\frac{1}{100}$ of its *dissolved* pollutants. The amount of dissolved pollutants in the tank is $\frac{1}{10}P(t)$. So, the rate the tank leaks pollutants is

$$\frac{1}{100} \cdot \frac{1}{10}P = \frac{1}{1,000}P$$

So, the quantity of pollutants in the tank satisfies the differential equation:

$$\frac{dP}{dt} = \text{(rate in)-(rate out)} = 3 - \frac{1}{1000}P$$

$$\implies \log|Q(1)| = \frac{1}{2}\log 2 + \frac{3}{2}\log 2 - \frac{3}{2}\log 3 = 2\log 2 - \frac{3}{2}\log 3$$

You deposit \$$P$ in a bank account at time $t = 0$, and the account pays r% interest per year, compounded n times per year. Your balance at time t is $B(t)$.
If one interest payment comes at time t, then the next interest payment will be made at time $t + \frac{1}{n}$ and will be:

$$\frac{1}{n} \times \frac{r}{100} \times B(t) = \frac{r}{100n} B(t)$$

So, calling $\frac{1}{n} = h$,

$$B(t+h) = B(t) + \frac{r}{100} B(t) h \qquad \text{or} \qquad \frac{B(t+h) - B(t)}{h} = \frac{r}{100} B(t)$$

If the interest is compounded continuously,

$$\frac{\mathrm{d}B}{\mathrm{d}t}(t) = \lim_{h \to 0} \frac{B(t+h) - B(t)}{h} = \frac{r}{100} B(t)$$
$$\implies B(t) = B(0) \cdot e^{rt/100} = P \cdot e^{rt/100}$$

Continuously compounding interest

If an account with balance $B(t)$ pays a continuously compounding rate of r% per year, then:

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{r}{100} B$$
$$B(t) = B(0) \cdot e^{rt/100}$$

You invest \$200 000 into an account with continuously compounded interest of 5% annually. You want to withdraw from the account continuously at a rate of \$$W$ per year, for the next 20 years. How big can W be?

Let $A(t)$ be the balance in the account t years after the initial deposit.

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{5}{100} A - W = \frac{1}{20}(A - 20W)$$
$$A(t) = (200\,000 - 20W)e^{t/20} + 20W$$
$$0 = A(20) = (200\,000 - 20W)e + 20W$$
$$= 200\,000e + 20W(1-e)$$
$$W = \frac{200\,000e}{20(e-1)} = 10\,000 \frac{e}{e-1} \approx 15\,819.77$$

That is, you can withdraw $10\,000\frac{e}{e-1} \approx 15\,819.77$ each year.

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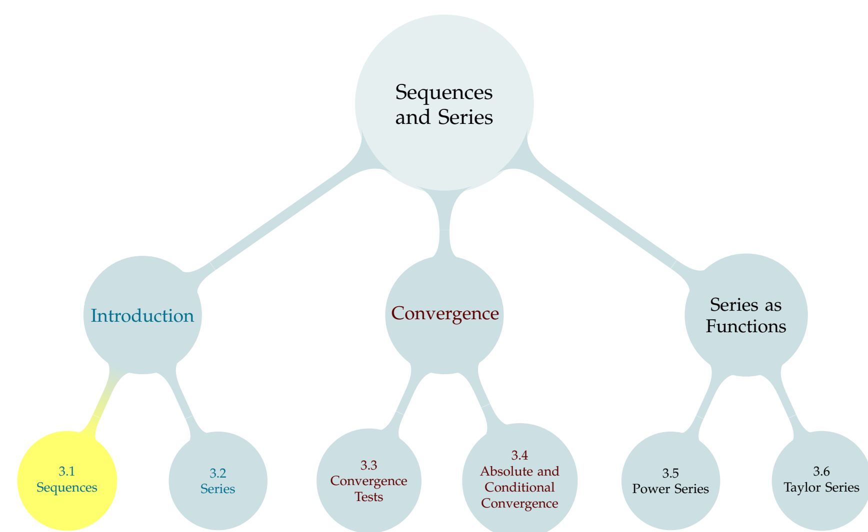

We can imagine the list of numbers below carrying on forever:

$$\begin{aligned} a_1 &= 0.1 \\ a_2 &= 0.01 \\ a_3 &= 0.001 \\ a_4 &= 0.0001 \\ a_5 &= 0.00001 \\ &\vdots \end{aligned}$$

A sequence is a list of infinitely many numbers with a specified order.
It is denoted $\{a_1, \ a_2, \ \cdots, a_n, \ \cdots\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.
Imagine *adding up* this sequence of numbers.
A series is a sum $a_1 + a_2 + \cdots + a_n + \cdots$ of infinitely many terms.

To handle sequences and series, we should define them more carefully. A good definition should allow us to answer some basic questions, such as:

- What does it mean to add up infinitely many things?
- Should infinitely many things add up to an infinitely large number?
- Does the order in which the numbers are added matter?
- Can we add up infinitely many functions, instead of just infinitely many numbers?

Sequence

A sequence is a list of infinitely many numbers with a specified order.

Some examples of sequences:

- $\{1, 2, 3, 4, 5, 6, 7, 8, \cdots\}$ (natural numbers)
- $\{3, 1, 4, 1, 5, 9, 2, 6, \cdots\}$ (digits of π)
- $\{1, -1, 1, -1, 1, \cdots\}$ (powers of $-1 : (-1)^0, (-1)^1, (-1)^2$, etc.)

Sequence

A sequence is a list of infinitely many numbers with a specified order.
It is denoted $\{a_1, \ a_2, \ a_3, \ \cdots, a_n \ \cdots\}$ or $\{a_n\}$ or $\{a_n\}_{n=1}^{\infty}$, etc.

$$\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

- $n = 1$: this is the index of the first term of our sequence. Sometimes it's 0, sometimes something else, for example a year.
- ∞: there is no end to our sequence.
- $\frac{1}{n}$: this tells us the value of a_n.
- Often we omit the limits and even the brackets, writing $a_n = \frac{1}{n}$.

Sequence Notation

For convenience, we write a_1 for the first term of a sequence, a_2 for the second term, etc.

In the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$,
a_3 is another name for $\frac{1}{3}$.

Sometimes we can find a rule for a sequence.
In the above sequence, $a_n = \frac{1}{n}$ (whenever n is a whole number).

We can write $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

Our primary concern with sequences will be the behaviour of a_n as n tends to infinity and, in particular, whether or not a_n "settles down" to some value as n tends to infinity.

Convergence

A sequence $\{a_n\}_{n=1}^{\infty}$ is said to converge to the limit A if a_n approaches A as n tends to infinity. If so, we write

$$\lim_{n\to\infty} a_n = A \qquad \text{or} \qquad a_n \to A \text{ as } n \to \infty$$

A sequence is said to converge if it converges to some limit. Otherwise it is said to diverge.

Convergence

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A sequence is said to converge if it converges to some limit. Otherwise it is said to diverge.

- $\{1, 2, 3, 4, 5, 6, 7, 8, \cdots\}$ (natural numbers)
  This sequence diverges, growing without bound, not approaching a real number.
- $\{3, 1, 4, 1, 5, 9, 2, 6, \cdots\}$ (digits of π)
  This sequence diverges, since it bounces around, not approaching a real number.
- $\{1, -1, 1, -1, 1, \cdots\}$ (powers of $-1 : (-1)^0, (-1)^1, (-1)^2$, etc.)
  This sequence diverges, since it bounces around, not approaching a real number.

Does the sequence $a_n = \dfrac{n}{2n+1}$ converge or diverge?

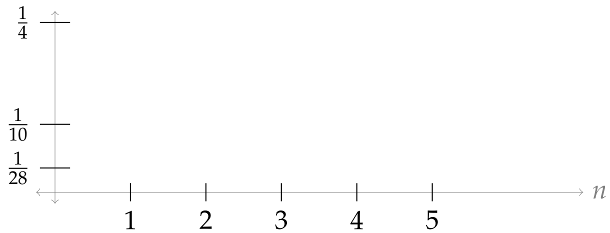
To study the behaviour of $\dfrac{n}{2n+1}$ as $n \to \infty$, it is a good idea to write it as:

$$\frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

As $n \to \infty$, the $\frac{1}{n}$ in the denominator tends to zero, so that the denominator $2 + \frac{1}{n}$ tends to 2 and $\frac{1}{2+\frac{1}{n}}$ tends to $\frac{1}{2}$. So

$$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}} = \frac{1}{2+0} = \frac{1}{2}$$

Consider the sequence $a_n = \dfrac{1}{3^n + 1}$. $\lim\limits_{n\to\infty} a_n = 0$



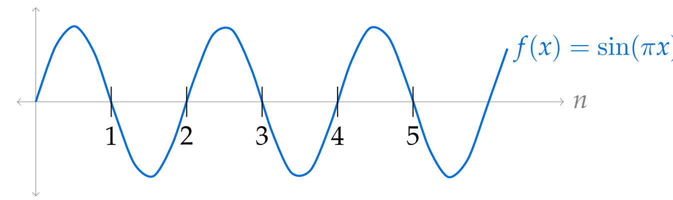
Theorem 3.1.6

If
$$\lim_{x\to\infty} f(x) = L$$
and if $a_n = f(n)$ for all positive integers n, then
$$\lim_{n\to\infty} a_n = L$$

Cautionary Tale

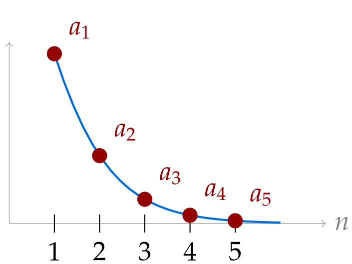
Consider the sequence $b_n = \sin(\pi n) = \{0, 0, 0, 0, 0, \ldots\}$

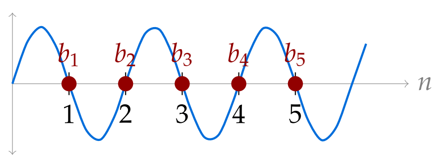
$\lim\limits_{n\to\infty} b_n = 0$ $\lim\limits_{x\to\infty} f(x)$ DNE



Theorem

If $\lim\limits_{x\to\infty} f(x) = L$ and if $a_n = f(n)$ for all natural n, then $\lim\limits_{n\to\infty} a_n = L$.





Arithmetic of Limits

Let A, B and C be real numbers and let the two sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ converge to A and B respectively. That is, assume that
$$\lim_{n\to\infty} a_n = A \qquad \lim_{n\to\infty} b_n = B$$
Then the following limits hold.

(a) $\lim\limits_{n\to\infty} \left[a_n + b_n\right] = A + B$

(b) $\lim\limits_{n\to\infty} \left[a_n - b_n\right] = A - B$

(c) $\lim\limits_{n\to\infty} Ca_n = CA$.

(d) $\lim\limits_{n\to\infty} a_n\, b_n = A\, B$

(e) If $B \neq 0$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$

Evaluate the following limits:

- $\lim_{n\to\infty} e^{-n} = 0$
- $\lim_{n\to\infty} \frac{1+n}{n} = 1$
- $\lim_{n\to\infty} \frac{1}{n^2} = 0$
- $\lim_{n\to\infty} 2n^2 = \infty$
- $\lim_{n\to\infty} \left(\frac{1}{n^2}\right)\left(2n^2\right) = 2$

(As you might guess, the expression "$\lim_{n\to\infty} a_n = \infty$" means that a_n grows without bound as $n \to \infty$.)

Continuous functions of limits

If $\lim_{n\to\infty} a_n = L$ and if the function $g(x)$ is continuous at L, then

$$\lim_{n\to\infty} g(a_n) = g(L)$$

Evaluate $\lim_{n\to\infty} \left[\sin\left(\frac{\pi n}{2n+1}\right)\right]$

$$\lim_{n\to\infty}\left[\frac{\pi n}{2n+1}\right] = \lim_{n\to\infty}\left[\frac{\pi}{2+\frac{1}{n}}\right] = \frac{\pi}{2}$$

$$\lim_{n\to\infty}\left[\sin\left(\frac{\pi n}{2n+1}\right)\right] = \sin\left(\frac{\pi}{2}\right) = 1$$

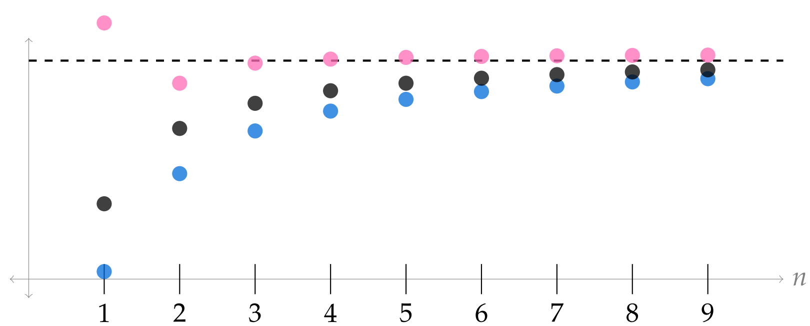
Squeeze Theorem

If $a_n \le c_n \le b_n$ for all sufficiently large natural numbers n, and if

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

then

$$\lim_{n\to\infty} c_n = L$$



Evaluate

$$\lim_{n\to\infty}\left(\frac{2n+\cos n}{n+1}\right)$$

Use squeeze theorem:

$$-1 \le \cos n \le 1$$

$$2n-1 \le 2n+\cos n \le 2n+1$$

$$\frac{2n-1}{n+1} \le \frac{2n+\cos n}{n+1} \le \frac{2n+1}{n+1}$$

$$\lim_{n\to\infty}\frac{2n-1}{n+1} = \lim_{n\to\infty}\frac{2n+1}{n+1} = 2$$

$$2 = \lim_{n\to\infty}\frac{2n+\cos n}{n+1}$$

Let $a_n = (-n)^{-n}$. Evaluate $\lim_{n\to\infty} a_n$.

First, we note $a_n = (-1)^{-n} \cdot (n^{-n}) = \frac{(-1)^n}{n^n}$ because $(-1)^{-n} = \left((-1)^{-1}\right)^n = (-1)^n$.
This sequence alternates between positive and negative terms. We can show that the positive terms tend to zero and the negative terms tend to zero. So, we can apply the squeeze theorem.

$$\text{Set } b_n = \frac{-1}{n^n} \text{ and } c_n = \frac{1}{n^n}$$

$$\text{Then, } b_n < a_n < c_n \text{ for all natural } n$$

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$$

$$\text{So, } \lim_{n\to\infty} a_n = 0$$

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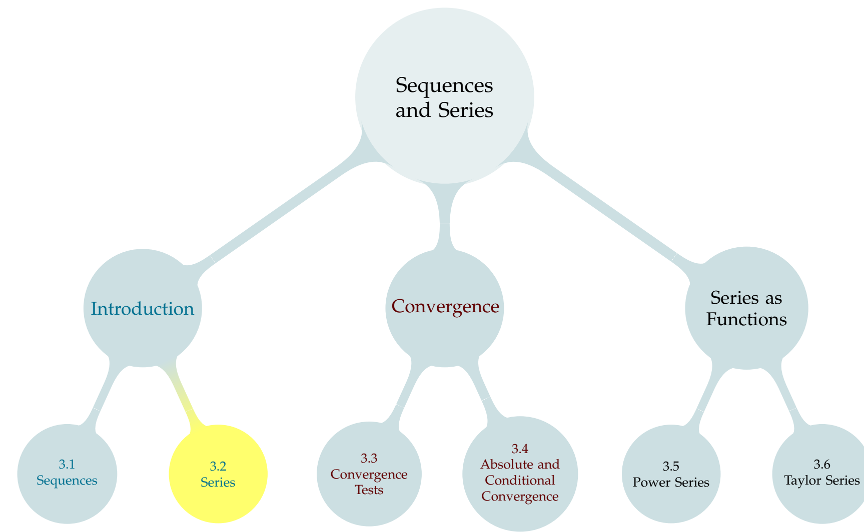


Sequences and Series

A sequence is a list of numbers
A series is the sum of such a list.

Sequences and Series

Sequence

List of numbers,

approaching

Series

Sum of numbers,

approaching

Square of Area 1

Quick Review: Sigma Notation

Recall:

$$\sum_{n=1}^{5} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2}$$

We informally interpret:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \cdots$$

(a more rigorous definition will be discussed soon)

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty} (C \cdot a_n) =$$

A. $\sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$

B. $\sum_{n=1}^{\infty} C + \sum_{n=1}^{\infty} a_n$

C. $C \sum_{n=1}^{\infty} a_n$

D. $a_n \sum_{n=1}^{\infty} C$

E. none of the above

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty} (a_n + b_n) =$$

A. $\sum_{n=1}^{\infty} a_n \cdot \sum_{n=1}^{\infty} b_n$

B. $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

C. $a_n + \sum_{n=1}^{\infty} b_n$

D. $a_n \sum_{n=1}^{\infty} b_n$

E. none of the above

Let a_n and b_n be sequences, and let C be a constant.

$$\sum_{n=1}^{\infty} (a_n)^C =$$

A. $\sum_{n=1}^{\infty} C \cdot \sum_{n=1}^{\infty} a_n$

B. $\left(\sum_{n=1}^{\infty} a_n \right)^C$

C. $C^n \sum_{n=1}^{\infty} a_n$

D. $\sum_{n=1}^{\infty} C(a_n)^{C-1}$

E. none of the above

SERIES PHILOSOPHY

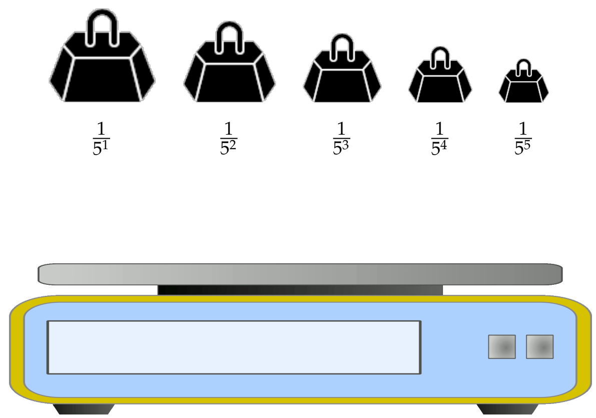
What does it really mean to add up infinitely many things?

$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

We need an unambiguous definition.

HOW CAN WE ADD UP INFINITELY MANY THINGS?

SEQUENCE OF PARTIAL SUMS



$S_1 = 0.2000$

$S_2 = 0.2400$

$S_3 = 0.2480$

$S_4 = 0.2496$

$S_5 = 0.2499$

Partial sums let us think about series (sums) using the tools we've developed for sequences (lists).

$$a_1 = \frac{1}{5} = 0.2 \qquad S_1 = 0.2$$
$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24$$
$$a_3 = \frac{1}{5^3} = 0.008 \qquad S_3 = 0.248$$
$$a_4 = \frac{1}{5^4} = 0.0016 \qquad S_4 = 0.2496$$
$$a_5 = \frac{1}{5^5} = 0.00032 \qquad S_5 = 0.24992$$

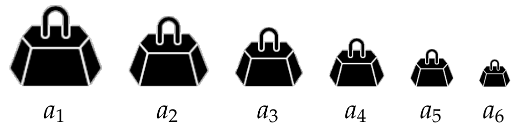
We define $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_N$.

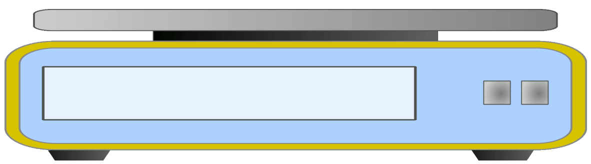
$$a_1 = \frac{1}{5} = 0.2 \qquad S_1 = 0.2 \qquad a_5 = \frac{1}{5^5} = 0.00032 \qquad S_5 = 0.24992$$
$$a_2 = \frac{1}{5^2} = 0.04 \qquad S_2 = 0.24 \qquad a_6 = \frac{1}{5^6} = 0.000064 \qquad S_6 = 0.249984$$
$$a_3 = \frac{1}{5^3} = 0.008 \qquad S_3 = 0.248 \qquad a_7 = \frac{1}{5^7} = 0.0000128 \qquad S_7 = 0.2499968$$
$$a_4 = \frac{1}{5^4} = 0.0016 \qquad S_4 = 0.2496 \qquad a_8 = \frac{1}{5^8} = 0.00000256 \qquad S_8 = 0.24999936$$

From the sequence of partial sums, we guess

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \lim_{N\to\infty} S_N = \frac{1}{4}$$

NOTATION: $S_N = \sum_{n=1}^{N} a_n$





$S_1 = a_1$

$S_2 = a_1 + a_2$

$S_3 = a_1 + a_2 + a_3$

$S_4 = a_1 + \cdots + a_4$

$S_5 = a_1 + \cdots + a_5$

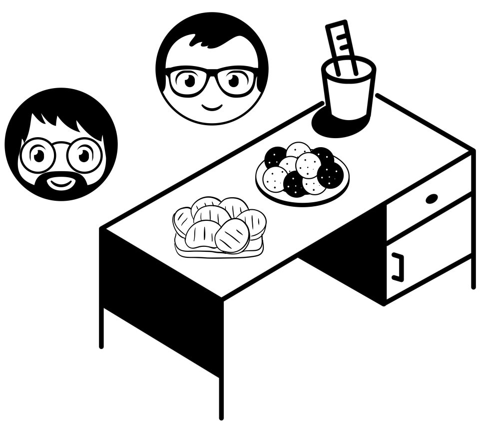
$S_6 = a_1 + \cdots + a_6$

NOTATION PRACTICE

Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^{N} a_n = \frac{N}{N+1}$.

- Evaluate $\sum_{n=1}^{100} a_n$. $\sum_{n=1}^{100} a_n = S_{100} = \frac{100}{101}$
- Evaluate $\sum_{n=1}^{\infty} a_n$. $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{N}{N+1} = 1$

NOTATION PRACTICE



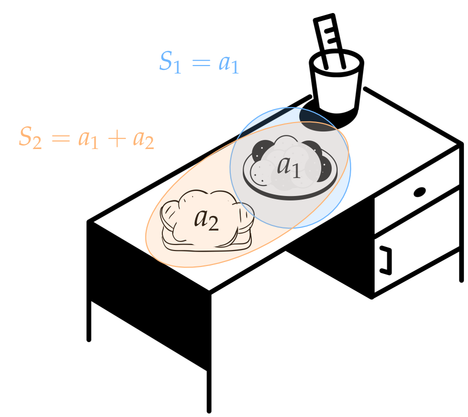
Andrew brings a plate of cookies to the professor's desk. When he puts them down, there are 10 cookies on the desk.

Then, Joel brings a plate of cookies. When he puts them down, there are 19 cookies on the desk.

How many cookies did each person bring?

Andrew brought 10, and Joel brought 19 − 10 = 9.

NOTATION PRACTICE



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Notation Practice

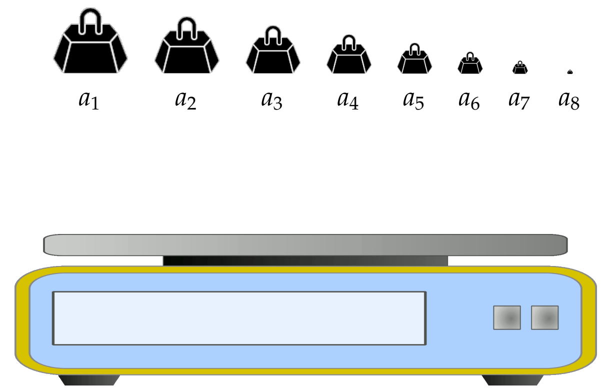
Suppose $\sum_{n=1}^{\infty} a_n$ has partial sums $S_N = \sum_{n=1}^{N} a_n = \dfrac{N}{N+1}$.

- Find a_1. $\quad a_1 = \sum_{n=1}^{1} a_n = S_1 = \dfrac{1}{2}$
- Give an explicit expression for a_n, when $n > 1$.

$$a_n = \left(\sum_{k=1}^{n} a_k\right) - \left(\sum_{k=1}^{n-1} a_k\right) = S_n - S_{n-1}$$

$$= \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)}$$

$S_N = \sum_{n=1}^{N} a_n = \frac{N}{N+1}$



$S_1 = 1/(1{+}1)$

$S_2 = 2/(2{+}1)$

$S_3 = 3/(3{+}1)$

$S_4 = 4/(4{+}1)$

$S_5 = 5/(5{+}1)$

$S_6 = 6/(6{+}1)$

$S_7 = 7/(7{+}1)$

$S_8 = 8/(8{+}1)$

Definition

The N^{th} partial sum of the series $\sum_{n=1}^{\infty} a_n$ is the sum of its first N terms

$$S_N = \sum_{n=1}^{N} a_n.$$

The partial sums form a sequence $\{S_N\}_{N=1}^{\infty}$. If this sequence of partial sums converges $S_N \to S$ as $N \to \infty$ then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S and we write

$$\sum_{n=1}^{\infty} a_n = S$$

If the sequence of partial sums diverges, we say that the series diverges.

Geometric Series

Let a and r be two fixed real numbers with $a \neq 0$. The series

$$a + ar + ar^2 + ar^3 + \cdots$$

is called the **geometric series** with first term a and ratio r.

We call r the *ratio* because it is the quotient of consecutive terms:

$$\frac{ar^{n+1}}{ar^n} = r$$

Another useful way of identifying geometric series is to determine whether all pairs of consecutive terms have the same ratio.

- Geometric: $1 + \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \cdots$
- Geometric: $\sum_{n=0}^{\infty} \frac{1}{2^n}$
- Not geometric: $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots$

Consider the partial sum S_N of a geometric series:

$$\begin{aligned} S_N &= a + ar + ar^2 + ar^3 + \cdots + ar^N \\ rS_N &= \\ rS_N - S_N &= \\ S_N(r-1) &= ar^{N+1} - a \end{aligned}$$

If $r \neq 1$, then

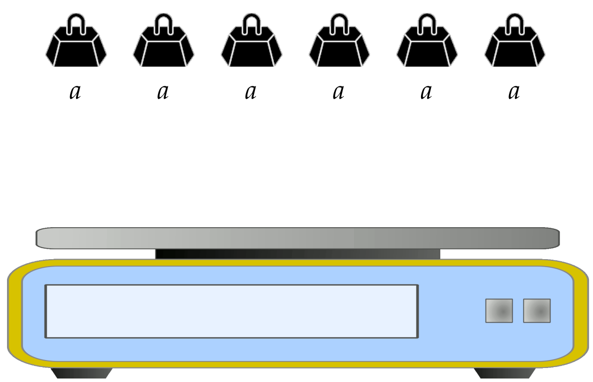
$$S_N = \frac{ar^{N+1} - a}{r-1} = a\frac{r^{N+1}-1}{r-1}$$

Geometric Series and Partial Sums

Let a and r be constants with $a \neq 0$, and let N be a natural number.

- If $r \neq 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N = a\frac{r^{N+1}-1}{r-1}$.
- If $r = 1$, then $a + ar + ar^2 + ar^3 + \cdots + ar^N = (N+1)a$.
- If $|r| < 1$, then $\sum_{n=0}^{\infty} ar^n = \lim_{N\to\infty} a\frac{r^{N+1}-1}{r-1} = a\frac{1}{1-r}$
- If $r = 1$, then $\sum_{n=0}^{\infty} ar^n$ diverges
- If $r = -1$, then $\sum_{n=0}^{\infty} ar^n$ diverges
- If $|r| > 1$, then $\sum_{n=0}^{\infty} ar^n$ diverges

$\sum_{n=0}^{\infty} ar^n$, $r = 1$, $a \neq 0$



$S_0 = a$

$S_1 = 2a$

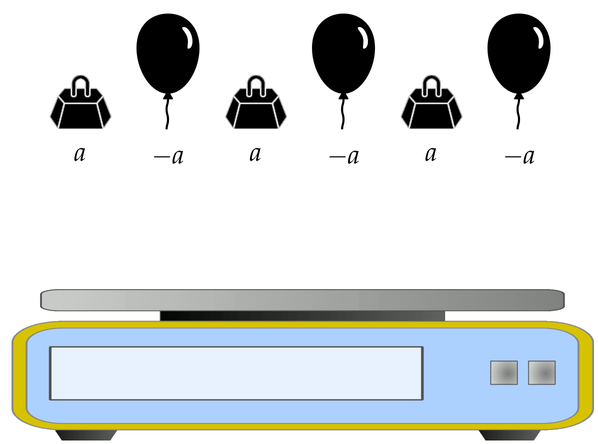
$S_2 = 3a$

$S_3 = 4a$

$S_4 = 5a$

$S_5 = 6a$

$\sum_{n=0}^{\infty} ar^n$, $r = -1$, $a \neq 0$



$S_0 = a$

$S_1 = 0$

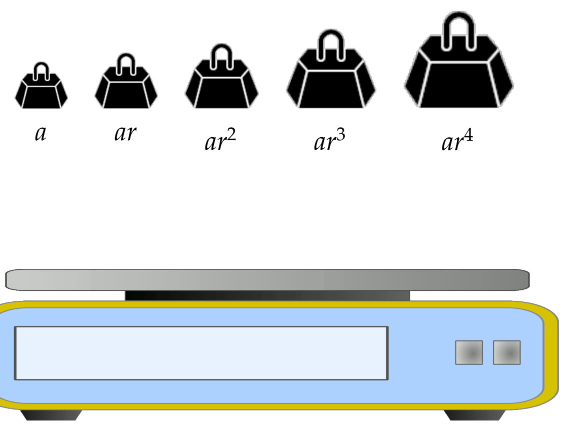
$S_2 = a$

$S_3 = 0$

$S_4 = a$

$S_5 = 0$

$\sum_{n=0}^{\infty} ar^n, r > 1, a \neq 0$



$S_0 = a$

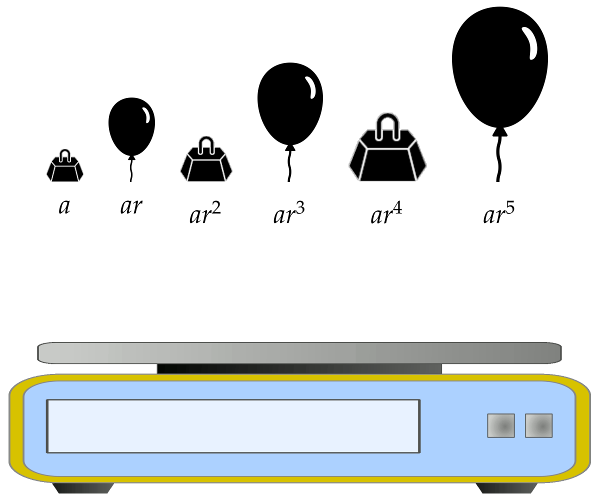
$S_1 = a\frac{r^2-1}{r-1}$

$S_2 = a\frac{r^3-1}{r-1}$

$S_3 = a\frac{r^4-1}{r-1}$

$S_4 = a\frac{r^5-1}{r-1}$

$\sum_{n=0}^{\infty} ar^n, r < -1, a \neq 0$



$S_0 = a$

$S_1 = a\frac{r^2-1}{r-1}$

$S_2 = a\frac{r^3-1}{r-1}$

$S_3 = a\frac{r^4-1}{r-1}$

$S_4 = a\frac{r^5-1}{r-1}$

$S_5 = a\frac{r^6-1}{r-1}$

Geometric Series

New bitcoins are produced when a particular type of computational problem is solved. Every time 210,000 solutions are found, the number of bitcoins each solution can produce is cut in half.

- Each of the first 210,000 solutions can produce 50 bitcoins.
- Each of the next 210,000 solutions can produce $\frac{50}{2}$ bitcoins.
- Each of the next 210,000 solutions can produce $\frac{50}{2^2}$ bitcoins.
- Each of the next 210,000 solutions can produce $\frac{50}{2^3}$ bitcoins.

Assume that this continues forever, and that bitcoins are infinitely divisible.[2] How many bitcoins can possibly be produced?

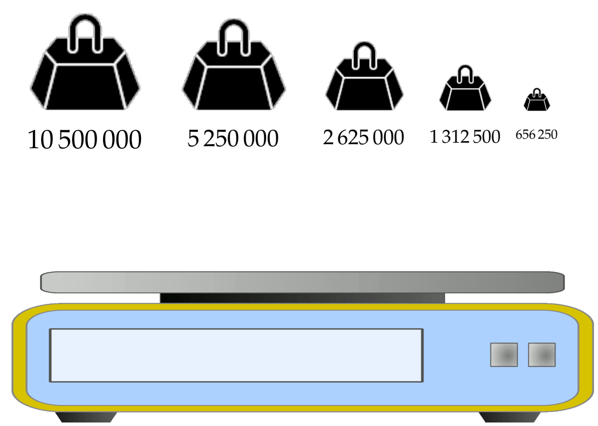
We start by writing the total number of bitcoin produced as a series. Since we want to know an upper bound, we'll assume that infinite solutions can be found and used to make bitcoin.

$$210\,000(50) + 210\,000\left(\frac{50}{2}\right) + 210\,000\left(\frac{50}{2^2}\right) + \cdots = \sum_{n=0}^{\infty}(210\,000)\left(\frac{50}{2^n}\right)$$

$$\sum_{n=0}^{\infty}(210\,000)\left(\frac{50}{2^n}\right) = \sum_{n=0}^{\infty}(210\,000\cdot 50)\left(\frac{1}{2}\right)^n$$

$$= (210\,000\cdot 50)(2)$$

$$\sum_{n=0}^{\infty} 210\,000\left(\frac{50}{2^n}\right) = 21\,000\,000$$



$S_0 = 10\,500\,000$

$S_1 = 15\,750\,000$

$S_2 = 18\,375\,000$

$S_3 = 19\,687\,500$

$S_4 = 20\,343\,750$

Arithmetic of Series

Let S, T, and C be real numbers. Let the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge to S and T respectively. Then

$$\sum_{n=1}^{\infty}[a_n + b_n] = S + T$$

$$\sum_{n=1}^{\infty}[a_n - b_n] = S - T$$

$$\sum_{n=1}^{\infty}[Ca_n] = CS$$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2}{3^n} + \frac{4}{5^n}\right)$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\displaystyle\sum_{n=6}^{\infty}\left(\frac{3^{n-1}}{5^{2n}}\right)$

Geometric Series and Partial Sums

Let a and r be fixed numbers, and let N be a positive integer. Then

$$\sum_{n=0}^{N} ar^n = \begin{cases} a \cdot \frac{1-r^{N+1}}{1-r} & \text{if } r \neq 1 \\ a(N+1) & \text{if } r = 1 \end{cases}$$

so

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ provided } |r| < 1$$

Evaluate $\displaystyle\sum_{n=0}^{\infty}\left(\frac{2^{2n}}{3^n}\right)$

Telescoping Sums

Evaluate $\displaystyle\sum_{n=1}^{800}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

Evaluate $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

$$\begin{aligned}
a_1 &: \tfrac{1}{1} - \tfrac{1}{2}\\
a_2 &: \tfrac{1}{2} - \tfrac{1}{3}\\
a_3 &: \tfrac{1}{3} - \tfrac{1}{4}\\
a_4 &: \tfrac{1}{4} - \tfrac{1}{5}\\
&\vdots\\
a_{N-1} &: \tfrac{1}{N-1} - \tfrac{1}{N}\\
a_N &: \tfrac{1}{N} - \tfrac{1}{N+1}
\end{aligned}$$

$$\begin{aligned}
S_1 &= \tfrac{1}{1} - \tfrac{1}{2}\\
S_2 &= \tfrac{1}{1} - \tfrac{1}{3}\\
S_3 &= \tfrac{1}{1} - \tfrac{1}{4}\\
S_4 &= \tfrac{1}{1} - \tfrac{1}{5}\\
&\vdots\\
S_N &= \tfrac{1}{1} - \tfrac{1}{N+1} = \tfrac{N}{N+1}
\end{aligned}$$

$$\sum_{n=1}^{800}\left(\frac{1}{n}-\frac{1}{n+1}\right) = S_{800} = \frac{800}{801}$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}\right) = \lim_{N\to\infty} S_N = 1$$

Evaluate $\displaystyle\sum_{n=1}^{1000}\log\left(\frac{n+1}{n}\right)$.

Evaluate $\displaystyle\sum_{n=1}^{\infty}\log\left(\frac{n+1}{n}\right)$.

$$\begin{aligned}
a_1 &: \log(2) - \log(1) & S_1 &= \log(2)\\
a_2 &: \log(3) - \log(2) & S_2 &= \log(3)\\
a_3 &: \log(4) - \log(3) & S_3 &= \log(4)\\
&\vdots\\
a_{n-1} &: \log(n) - \log(n-1)\\
a_n &: \log(n+1) - \log(n) & S_n &= \log(n+1)
\end{aligned}$$

So, $\displaystyle\sum_{n=1}^{1000}\log\left(\frac{n+1}{n}\right) = S_{1000} = \log(1001)$ and

$$\sum_{n=1}^{\infty}\log\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}\log(n+1) = \infty$$

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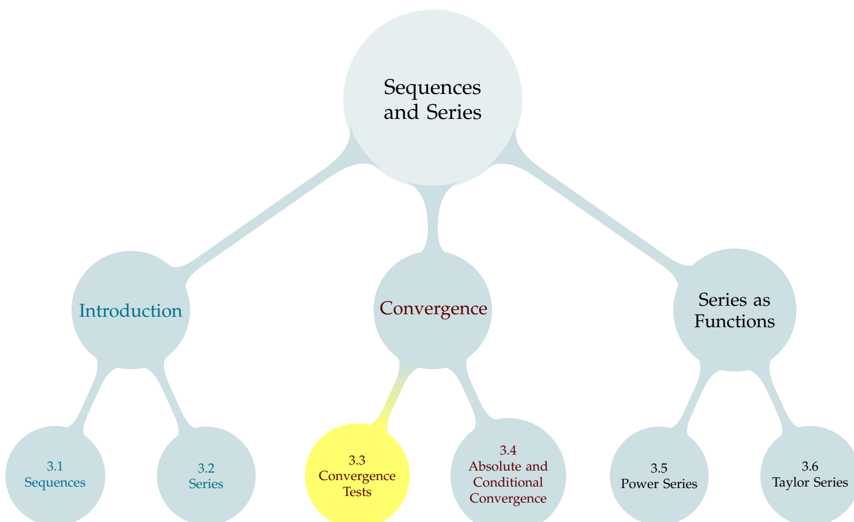
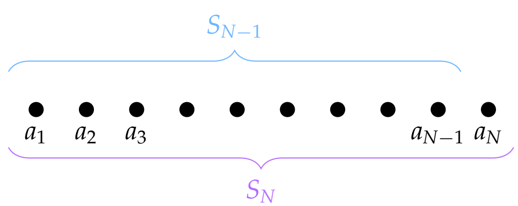


For a convergent geometric or telescoping series, we can easily determine what the series converges *to*.

For other types of series, finding out what the series converges to can be very difficult. It is often necessary to resort to approximating the full sum by, for example, using a computer to find the sum of the first N terms, for some large N. But before we even try to do that, we should at least know *whether or not the series converges*.

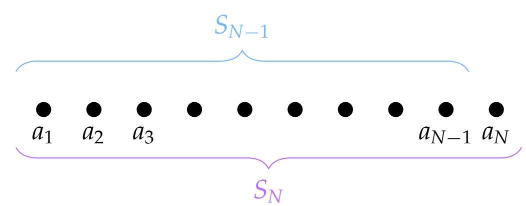
Suppose a series $\sum_{n=1}^{\infty} a_n$ converges to a limit L. Let $S_N = \sum_{n=1}^{N} a_n$.

$$\lim_{N\to\infty} S_N = L$$
$$\lim_{N\to\infty} S_{N-1} = L$$
$$\lim_{N\to\infty} \left[S_N - S_{N-1}\right] = L - L = 0$$
$$\lim_{N\to\infty} a_N = 0$$



Suppose a series $\sum_{n=1}^{\infty} a_n$ converges to a limit L. Let $S_N = \sum_{n=1}^{N} a_n$.

$$\lim_{N\to\infty} S_N = L$$
$$\lim_{N\to\infty} S_{N-1} = L$$
$$\lim_{N\to\infty} \left[S_N - S_{N-1}\right] = L - L = 0$$
$$\lim_{N\to\infty} a_N = 0$$



Every convergent series has its N^{th} term, a_N, tending to 0 as $N \to \infty$.

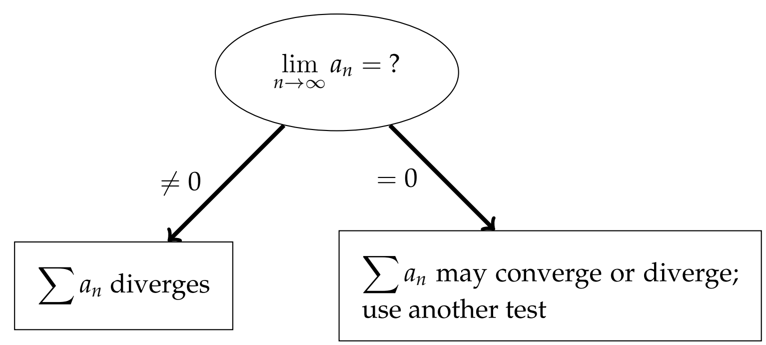
Divergence Test

If the sequence $\{a_n\}_{n=c}^{\infty}$ fails to converge to zero as $n \to \infty$, then the series $\sum_{n=c}^{\infty} a_n$ diverges.

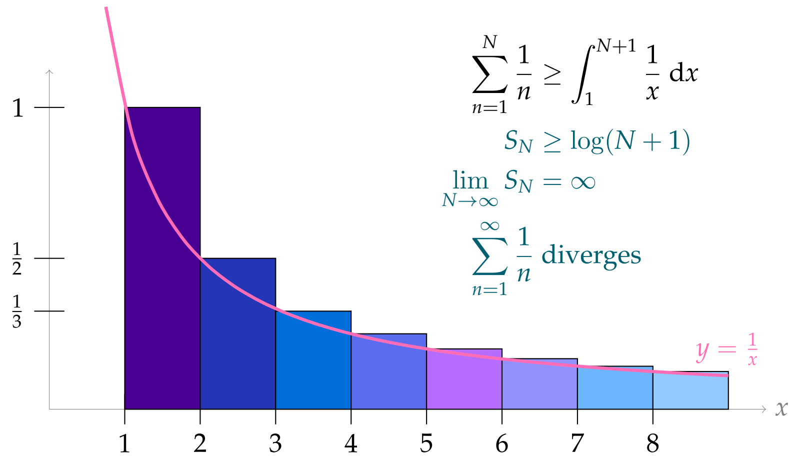
Do the following series diverge?

- $\sum_{n=0}^{\infty} (-1)^n$ — yes, it diverges
- $\sum_{n=10}^{\infty} \left(\frac{1}{10} + \frac{1}{2^n}\right)$ — yes, it diverges
- $\sum_{n=15}^{\infty} \frac{e^n}{2e^n - 1}$ — yes, it diverges
- $\sum_{n=15}^{\infty} \frac{1}{n}$ — at this point, unclear: maybe, maybe not

Using the Divergence Test for $\sum a_n$



Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n}$



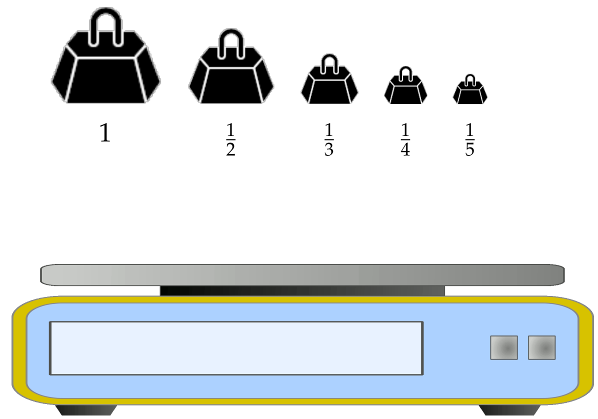
$$\sum_{n=1}^{N} \frac{1}{n} \geq \int_1^{N+1} \frac{1}{x}\,\mathrm{d}x$$

$$S_N \geq \log(N+1)$$

$$\lim_{N\to\infty} S_N = \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES



$S_1 = 1.0000$

$S_2 = 1.5000$

$S_3 = 1.8333$

$S_4 = 2.0833$

$S_5 = 2.2833$

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

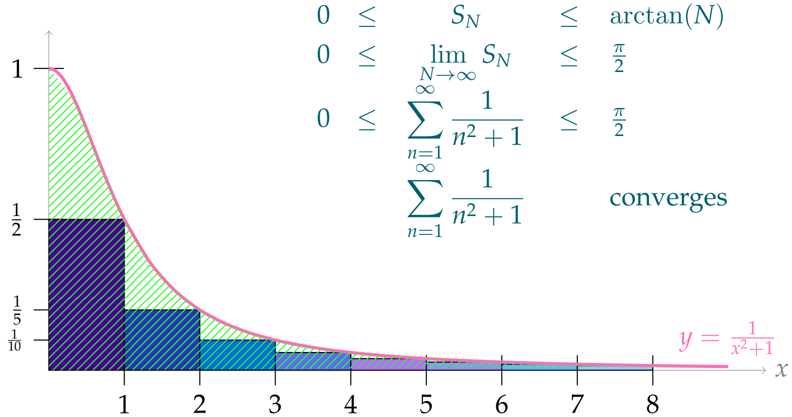
$$0 \leq \sum_{n=1}^{N} \frac{1}{n^2+1} \leq \int_0^N \frac{1}{x^2+1}\,\mathrm{d}x$$

$$0 \leq S_N \leq \arctan(N)$$

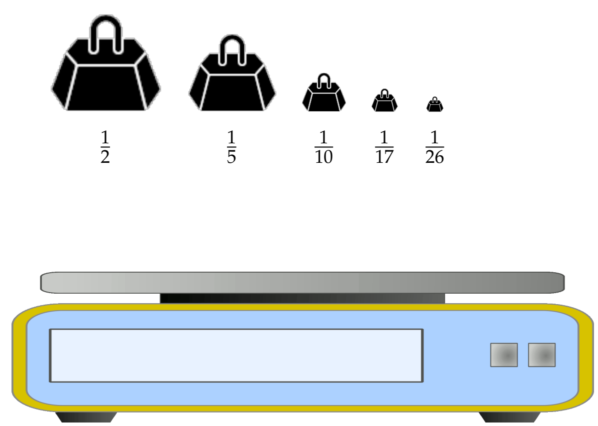
$$0 \leq \lim_{N\to\infty} S_N \leq \frac{\pi}{2}$$

$$0 \leq \sum_{n=1}^{\infty} \frac{1}{n^2+1} \leq \frac{\pi}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} \text{ converges}$$



$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ CONVERGES



$S_1 = 0.5000$

$S_2 = 0.7000$

$S_3 = 0.8000$

$S_4 = 0.8588$

$S_5 = 0.8973$

Integral Test

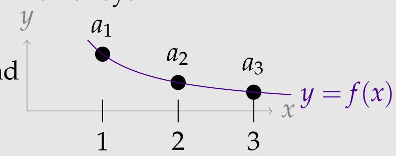
Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \ge N_0$ and which obeys

(i) $f(x) \ge 0$ for all $x \ge N_0$ and

(ii) $f(x)$ decreases as x increases and

(iii) $f(n) = a_n$ for all $n \ge N_0$.

Then



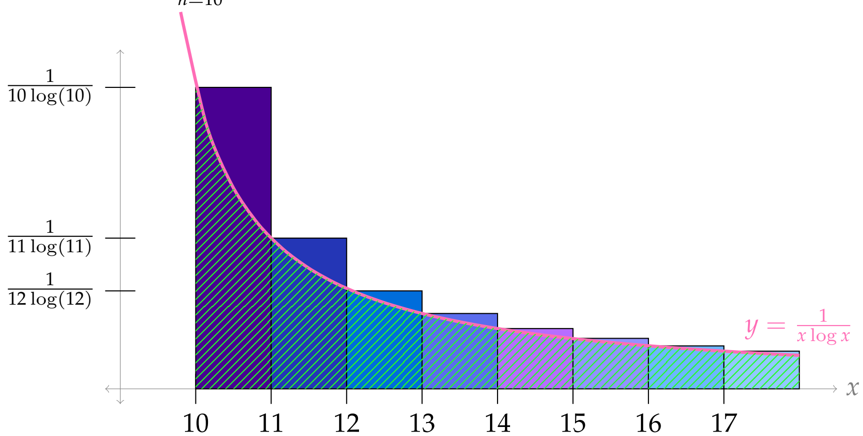
$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x)\, dx \text{ converges}$$

Furthermore, when the series converges, the truncation error satisfies

$$0 \le \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \le \int_N^{\infty} f(x)\, dx \qquad \text{for all } N \ge N_0$$

Does the series $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n \log n}$ converge or diverge?

Does the series $\displaystyle\sum_{n=10}^{\infty} \frac{1}{n \log n}$ converge or diverge?

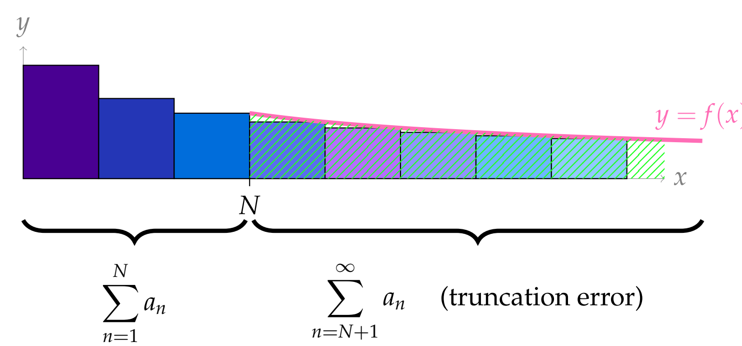


$$\int_{10}^{\infty} \frac{1}{x \log x}\, dx = \infty$$

Integral Test, abridged

... When the series converges, the truncation error satisfies

$$0 \le \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \le \int_N^{\infty} f(x)\, dx$$



Integral Test, abridged

When the series converges, the truncation error satisfies

$$0 \le \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \le \int_N^{\infty} f(x)\,dx$$

We already decided that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges.

Suppose we had a computer add up the terms $n = 1$ through $n = 100$.

Use the integral test to bound the error, $\sum_{n=1}^{\infty} \frac{1}{n^2+1} - \sum_{n=1}^{100} \frac{1}{n^2+1}$.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} - \sum_{n=1}^{100} \frac{1}{n^2+1} \le \int_{100}^{\infty} \frac{1}{x^2+1}\,dx$$

$$= \lim_{b\to\infty} [\arctan(b) - \arctan(100)] = \frac{\pi}{2} - \arctan(100) \approx 0.01$$

By computer, $\sum_{n=1}^{100} \frac{1}{n^2+1} \approx 1.0667$. Using the truncation error of about 0.01, give a (small) range of possible values for $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

$$0 \le \sum_{n=1}^{\infty} \frac{1}{n^2+1} - \sum_{n=1}^{100} \frac{1}{n^2+1} \le \int_{100}^{\infty} \frac{1}{x^2+1}\,dx$$

$$0 \le \sum_{n=1}^{\infty} \frac{1}{n^2+1} - 1.0667 \le 0.01$$

$$1.0667 \le \sum_{n=1}^{\infty} \frac{1}{n^2+1} \le 1.0767$$

p-TEST

Let p be a positive constant. When we talked about improper integrals, we showed:

$$\int_1^{\infty} \frac{1}{x^p}\,dx \quad \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases}$$

Set $f(x) = \frac{1}{x^p}$.

(i) $f(x) \ge 0$ for all $x \ge 1$, and

(ii) $f(x)$ decreases as x increases

$$\sum_{n=1}^{\infty} \frac{1}{n^p}\,dx \quad \begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \le 1 \end{cases}$$

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}.$$

By the p-test, we know this series

How many terms should we add up to approximate the series to within an error of no more than 0.02?

$$\sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{N} \frac{1}{n^3} \le \int_N^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_N^b = \frac{1}{2N^2}$$

$$\frac{1}{2N^2} \le \frac{2}{100} \implies N \ge 5$$

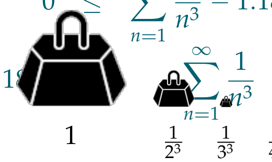
5 terms will suffice.

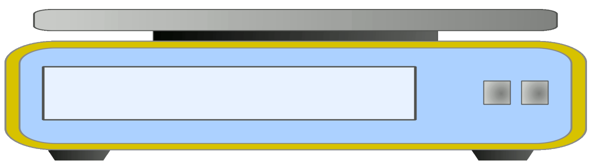
$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges to within 0.02 of $\sum_{n=1}^{5} \frac{1}{n^3}$.

$$0 \leq \sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{5} \frac{1}{n^3} \leq 0.02$$

$$0 \leq \sum_{n=1}^{\infty} \frac{1}{n^3} - 1.1856 \leq 0.02$$

$$1.1\ldots \quad \sum_{n=1}^{\infty} \frac{1}{n^3} \leq 1.2056$$





$S_1 = 1.0000$

$S_2 = 1.1250$

$S_3 = 1.1620$

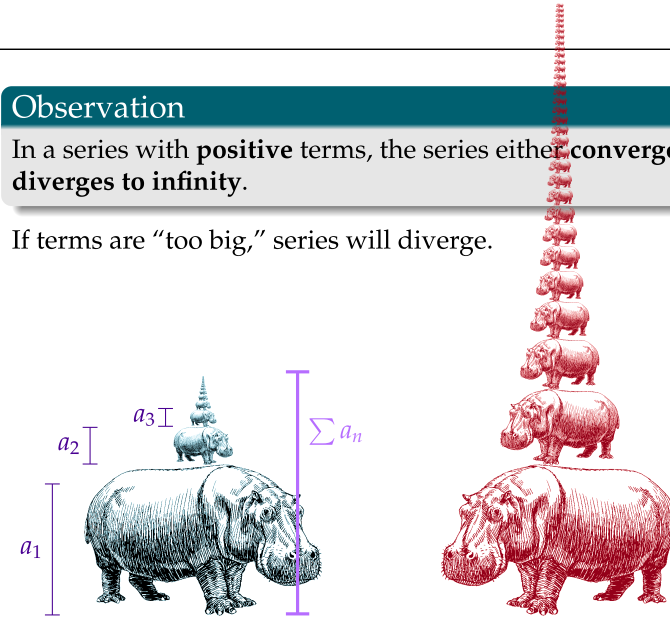
$S_4 = 1.1776$

$S_5 = 1.1856$

Observation

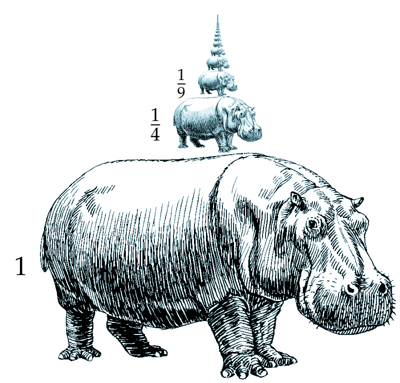
In a series with **positive** terms, the series either **converges**, or **diverges to infinity**.

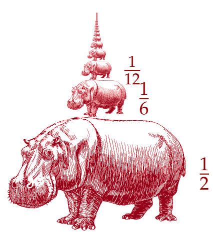
If terms are "too big," series will diverge.



$\sum \frac{1}{n^2}$ converges

$\sum \frac{1}{n^2+n}$ converges, too





Terms are "small enough" for sum to converge

Terms are also "small enough" for sum to converge

The Comparison Test

Let N_0 be a natural number and let $K > 0$.

(a) If $|a_n| \leq Kc_n$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

(b) If $a_n \geq Kd_n \geq 0$ for all $n \geq N_0$ and $\sum_{n=0}^{\infty} d_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

Consider $\sum_{n=1}^{\infty} \frac{1}{n-0.1}$.

- We know $0 < \frac{1}{n} < \frac{1}{n-0.1}$
- We know $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series)
- So, by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{n-0.1}$ diverges as well.

Does the series $\sum_{n=1}^{\infty} \frac{n+\cos n}{n^3-1/3}$ converge or diverge?

Step 1: Intuition.

When n is very large, we expect:

- $n + \cos n \approx n$
- $n^3 + \frac{1}{3} \approx n^3$
- So, we expect $\frac{n+\cos n}{n^3-1/3} \approx \frac{n}{n^3} = \frac{1}{n^2}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ ... converges (by the p-test),

we expect $\sum_{n=1}^{\infty} \frac{n+\cos n}{n^3-1/3}$ to also .... converge.

Does the series $\sum_{n=1}^{\infty} \frac{n+\cos n}{n^3-1/3}$ converge or diverge?

Step 2: Choose comparison series.

The Comparison Test, abridged

Let N_0 be a natural number and let $K > 0$.

If $|a_n| \le K c_n$ for all $n \ge N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

To show that original series converges, we should find a comparison series that also converges and whose terms (times some positive constant) are larger than the original terms. *There are many possibilities.* For $n \ge 1$,

- $n + \cos n < n + n = 2n$
- $n^3 - \frac{1}{3} > n^3 - \frac{n^3}{2} = \frac{1}{2}n^3$
- So $\frac{n+\cos n}{n^3-1/3} < \frac{2n}{\frac{1}{2}n^3} = 4 \cdot \frac{1}{n^2}$

Does the series $\sum_{n=1}^{\infty} \frac{n+\cos n}{n^3-1/3}$ converge or diverge?

Step 3: Verify.

The Comparison Test, abridged

Let N_0 be a natural number and let $K > 0$.

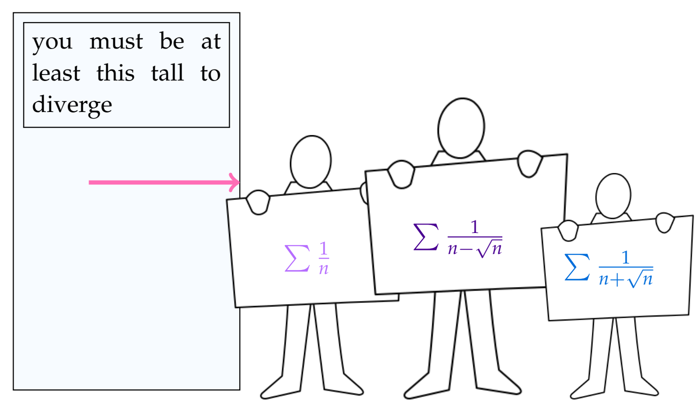
If $|a_n| \le K c_n$ for all $n \ge N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

Set $c_n = \frac{1}{n^2}$ and $K = 4$. Note $\sum_{n=1}^{\infty} c_n$ converges.

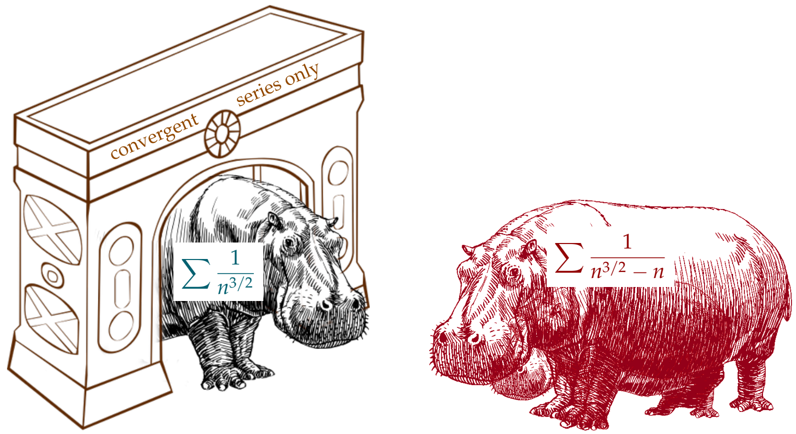
Note also $\left|\frac{n+\cos n}{n^3-1/3}\right| < \frac{n+n}{n^3-\frac{n^3}{2}} = 4 \cdot \frac{1}{n^2}$ for all $n \ge 1$.

By the comparison test, $\sum_{n=1}^{\infty} \frac{n+\cos n}{n^3-1/3}$ converges.

For the comparison test as we have seen it so far, to conclude that a given series diverges, we have to find a divergent comparison series whose terms are smaller than (a positive multiple of) those of our original series .



For the comparison test as we've seen it so far, to conclude that a given series converges, we have to find a convergent comparison series whose terms are larger than (a positive multiple of) those of our original series .



Limit Comparison Theorem

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $b_n > 0$ for all n. Assume that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

exists.

(a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.

(b) If $L \neq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges too.

In particular, if $L \neq 0$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

- For large n, $a_n \approx L \cdot b_n$;
- so we expect $\sum a_n$ to behave roughly like $\sum (L \cdot b_n)$;
- and since $L \neq 0$, we expect $\sum (L \cdot b_n)$ to converge if and only if $\sum b_n$ converges.

By the p-test, $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges.

Can we conclude that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}-n+1}$ also converges?

$$a_n = \frac{1}{n^{3/2}} \qquad b_n = \frac{1}{n^{3/2}-n+1}$$

$$\frac{a_n}{b_n} = \frac{n^{3/2}-n+1}{n^{3/2}} = 1 - \frac{1}{\sqrt{n}} + \frac{1}{n^{3/2}}$$

$$L = \lim_{n\to\infty} \frac{a_n}{b_n} = 1 - 0 + 0 = 1$$

Since L is a nonzero real number, the two series either both converge or both diverge. By the p-test, $\sum \frac{1}{n^{3/2}}$ converges. So, by the limit comparison test, $\sum \frac{1}{n^{3/2}-n+1}$ also converges.

Does the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2-2n+3}$ converge or diverge?

Step 1: Intuition

For large n,

$$\frac{\sqrt{n+1}}{n^2-2n+3} \approx \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}}$$

So, we'll use $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ as our comparison series. Since this converges, we expect our original series to converge as well.

Does the series $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2-2n+3}$ converge or diverge?

Step 2: Verify Intuition

Let $a_n = \frac{\sqrt{n+1}}{n^2-2n+3}$ and $b_n = \frac{1}{n^{3/2}}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{n^2-2n+3}}{\frac{1}{n^{3/2}}} = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{n^2-2n+3}}{\frac{\sqrt{n}}{n^2}}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n+1}\cdot\frac{1}{\sqrt{n}}}{(n^2-2n+3)\cdot\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}}{1-\frac{2}{n}+\frac{3}{n^2}}$$

$$= \frac{\sqrt{1+0}}{1+0+0} = 1$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (by the p-test), the original series converges as well, by the Limit Comparison Theorem.

Comparison Strategies

- Before you can use either comparison test, you need to guess a series to compare.
- The series you guess should be easy to deal with.
  - p-series
  - geometric series
- Common guess (especially if monotone): consider "largest" piece of numerator and denominator
  (constant) $<$ (logarithm) $<$ (polynomial) $<$ (exponential)
- After you guess a comparison series, **show it works** by finding the correct inequality (comparison test), or computing the limit of the ratio (limit comparison test).

Choose a Series to Compare

$\sum_{n=1}^{\infty} \frac{3n}{n^2+1}$ One option: $\sum_{n=1}^{\infty} \frac{3n}{n^2} = \sum_{n=1}^{\infty} \frac{3}{n}$

$\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^5-n}$ One option: $\sum_{n=1}^{\infty} \frac{n^2}{n^5} = \sum_{n=1}^{\infty} \frac{1}{n^3}$

$\sum_{k=1}^{\infty} \frac{k(2+\sin k)}{k^{\sqrt{2}}}$ One option: $\sum_{k=1}^{\infty} \frac{2k}{k^{\sqrt{2}}} = \sum_{k=1}^{\infty} \frac{2}{k^{\sqrt{2}-1}}$

$\sum_{m=1}^{\infty} \frac{3m+\sin\sqrt{m}}{m^2}$ One option: $\sum_{m=1}^{\infty} \frac{3m}{m^2} = \sum_{m=1}^{\infty} \frac{3}{m}$

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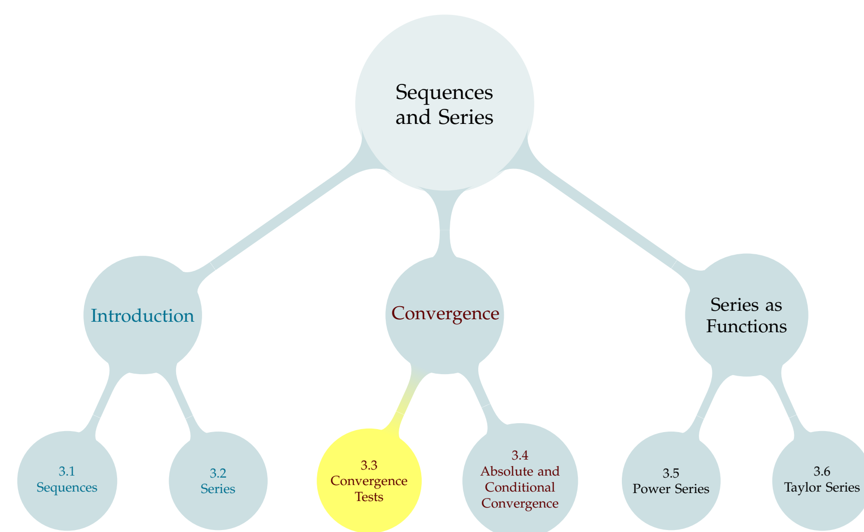


Review

Let $S_N = \sum_{n=1}^{N} a_n$.

Simplify: $S_N - S_{N-1}$. (This will come in handy soon.)

Alternating Series

Alternating Series

The series

$$A_1 - A_2 + A_3 - A_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} A_n$$

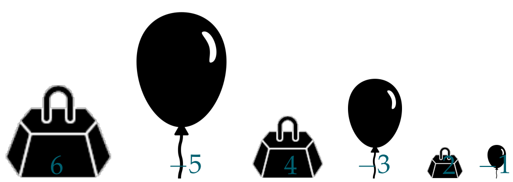
is alternating if every $A_n \geq 0$.

Alternating series:

- $1-2+3-4+5-6+7-8+\cdots$
- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

Not alternating:

- $\cos(1) + \cos(2) + \cos(3) + \cdots$
- $1 - \left(-\frac{1}{2}\right) + \frac{1}{3} - \left(-\frac{1}{4}\right) + \cdots$



6 $\quad -5 \quad 4 \quad -3 \quad 2 \quad -1$

$S_1 = 6.0000$

$S_2 = 1.0000$

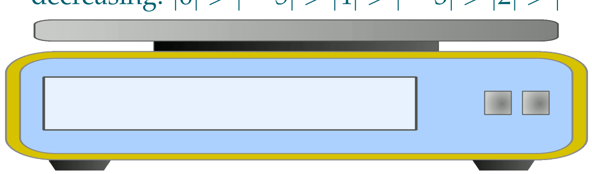
$S_3 = 5.0000$

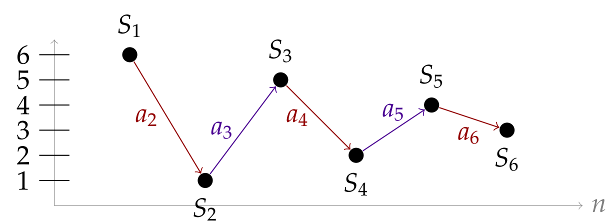
$S_4 = 2.0000$

$S_5 = 4.0000$

$S_6 = 3.0000$

Note: these terms alternate signs, and their magnitudes are decreasing: $|6| > |-5| > |4| > |-3| > |2| > |-1|$

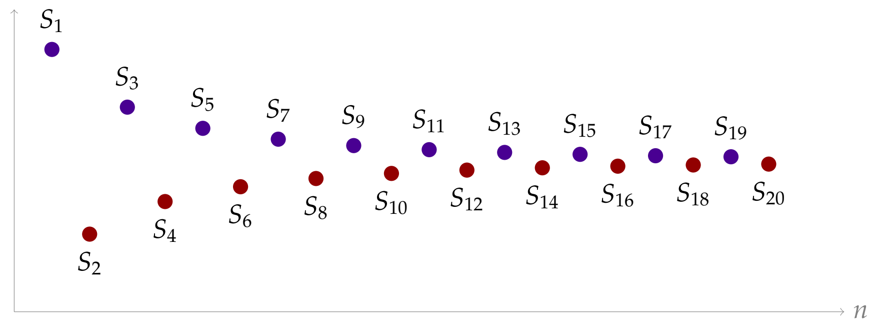




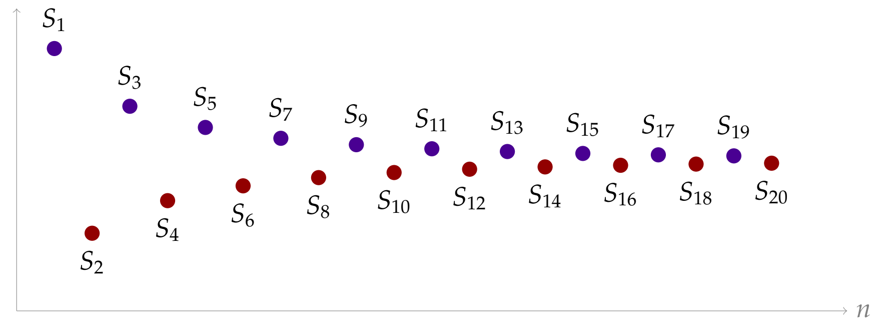
6 $\quad -5 \quad 4 \quad -3 \quad 2 \quad -1$

Note: these terms alternate signs, and their magnitudes are decreasing: $|6| > |-5| > |4| > |-3| > |2| > |-1|$

Consider an alternating series $a_1 - a_2 + a_3 - a_4 + \cdots$, where $\{a_n\}$ is a sequence with positive, decreasing terms and with $\lim_{n\to\infty} a_n = 0$.



Since $a_2 > a_3$, we have $a_1 - (a_2 - a_3) < a_1$, so $S_3 < S_1$.



- For all $n \geq 2$, S_n lies between S_1 and S_2.
- For all $n \geq 3$, S_n lies between S_2 and S_3.
- For all $n \geq 4$, S_n lies between S_3 and S_4.
- For all $n \geq 5$, S_n lies between S_4 and S_5.

The difference between consecutive sums S_n and S_{n-1} is:

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) $a_n \geq 0$ for all $n \geq 1$;

(ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);

(iii) and $\lim_{n\to\infty} a_n = 0$.

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number N, $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the N^{th} partial sum $\sum_{n=1}^{N} (-1)^{n-1} a_n$.

Alternating Series Test (abridged)

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) $a_n \geq 0$ for all $n \geq 1$;

(ii) $a_{n+1} \leq a_n$ for all $n \geq 1$ (i.e. the sequence is monotone decreasing);

(iii) and $\lim_{n\to\infty} a_n = 0$.

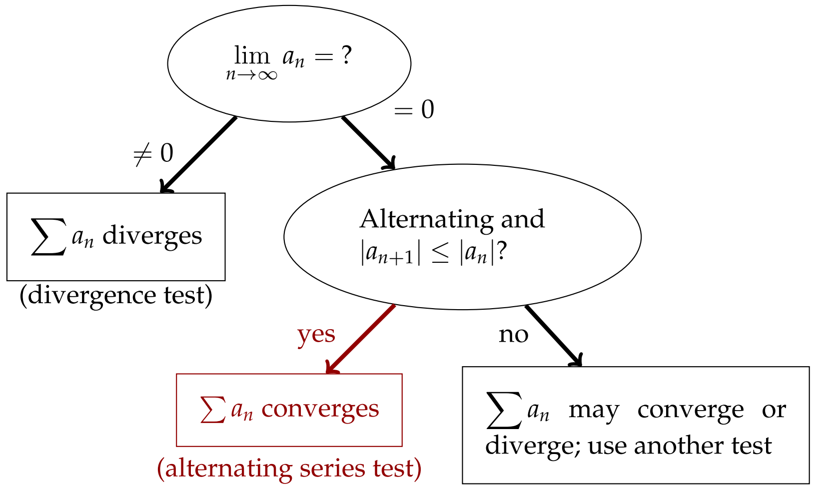
Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges.

- True or false: the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ converges.
- True or false: the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

Divergence Test + Alternating Series Test



Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \ge 0$ for all $n \ge 1$; $a_{n+1} \le a_n$ for all $n \ge 1$; and $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\sum_{n=1}^{99} \frac{(-1)^{n-1}}{n} \approx 0.698$.

How close is that to the value $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$?

$$\frac{-1}{100} = \frac{(-1)^{100-1}}{100} \le \sum_{n=1}^{\infty} \frac{(-1)^n}{n} - \sum_{n=1}^{99} \frac{(-1)^n}{n} \le 0.$$

That is, the actual series has a sum in the interval $[0.688, 0.698]$.

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys $a_n \ge 0$ for all $n \ge 1$; $a_{n+1} \le a_n$ for all $n \ge 1$; and $\lim_{n\to\infty} a_n = 0$. Then $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = S$ converges and $S - S_N$ is between 0 and $(-1)^N a_{N+1}$.

Using a computer, you find $\sum_{n=1}^{19}(-1)^{n-1}\frac{n^2}{n^2+1} \approx 0.6347$.

How close is that to the value $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^2+1}$?

Not close at all: the series is divergent (which we can see by the divergence test).

Recall for a geometric series, the ratios of consecutive terms is constant.

$$\overset{\times\frac{1}{2}}{\frown}\quad \overset{\times\frac{1}{2}}{\frown}\quad \overset{\times\frac{1}{2}}{\frown}\quad \overset{\times\frac{1}{2}}{\frown}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} \cdots$$

$$\frac{1/4}{1/2} = \frac{1/8}{1/4} = \frac{1/16}{1/8} = \frac{1/32}{1/16} = \frac{1}{2}$$

If that ratio has magnitude less then one, then the series converges.
If the ratio has magnitude greater than one, the series diverges.

For series convergence, we are concerned with what happens to terms a_n when n is sufficiently large.
Suppose for a sequence a_n, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ for some constant L.

$$\underbrace{a_n \quad + \quad a_{n+1}}_{\frac{a_{n+1}}{a_n}} \approx \underbrace{a_{n+1} + a_{n+2}}_{\frac{a_{n+2}}{a_{n+1}}} \approx \underbrace{a_{n+2} + a_{n+3}}_{\frac{a_{n+3}}{a_{n+2}}} \approx \underbrace{a_{n+3} + a_{n+4}}_{\frac{a_{n+4}}{a_{n+3}}} \approx \underbrace{a_{n+4} + \cdots}_{\frac{a_{n+5}}{a_{n+4}}} \approx L$$

Like in a geometric series:

If L has magnitude less then one, then the series converges.
If L has magnitude greater than one, the series diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Use the ratio test to determine whether the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

converges or diverges.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \frac{n+1}{n} \cdot \frac{3^n}{3^{n+1}} = \left(1 + \frac{1}{n}\right) \cdot \frac{1}{3}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}$$

Since $\frac{1}{3} < 1$, by the ratio test, $\sum_{n=1}^{\infty} \frac{n}{3^n}$ coverges.

Remark

The series we just considered, $\sum_{n=1}^{\infty} \frac{n}{3^n}$, looks similar to a geometric series, but it is not exactly a geometric series. That's a good indicator that the ratio test will be helpful!

We could have used other tests, but ratio was probably the easiest.

- Integral test: $\int \frac{x}{3^x}\, dx$ can be evaluated using integration by parts.
- Comparison test:
  - $\sum \frac{1}{3^n}$ is not a valid comparison series, nor is $\sum n$.
  - Because $n < 2^n$ for all $n \geq 1$, the series $\sum \left(\frac{2}{3}\right)^n$ will work.
- The divergence test is inconclusive, and the alternating series test does not apply. Our series is not geometric, and not obviously telescoping.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$, or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Let a and x be nonzero constants. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} anx^{n-1}$$

converges or diverges. (This may depend on the values of a and x.)

Let x be a constant. Use the ratio test to determine whether

$$\sum_{n=1}^{\infty} \frac{(-3)^n\sqrt{n+1}}{2n+3}x^n$$

converges or diverges. (This may depend on the value of x.)

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-3)^{n+1}\sqrt{n+2}}{2(n+1)+3}x^{n+1}}{\frac{(-3)^n\sqrt{n+1}}{2n+3}x^n}\right| = \left|\frac{(-3)^{n+1}}{(-3)^n}\cdot\frac{\sqrt{n+2}}{\sqrt{n+1}}\cdot\frac{2n+3}{2n+5}\cdot\frac{x^{n+1}}{x^n}\right|$$

$$= 3\cdot\sqrt{\frac{n+2}{n+1}}\cdot\left(\frac{2n+3}{2n+5}\right)\cdot|x| = 3\sqrt{\frac{1+2/n}{1+1/n}}\cdot\left(\frac{2+3/n}{2+5/n}\right)\cdot|x|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 3\sqrt{\frac{1}{1}}\left(\frac{2}{2}\right)|x| = 3|x|$$

So the series converges when $3|x| < 1$ and diverges when $3|x| > 1$.
So for $|x| < \frac{1}{3}$, the series converges, and for $|x| > \frac{1}{3}$, it diverges.

Fill in in the blanks

Divergence Test

If the sequence $\{a_n\}_{n=c}^{\infty}$ [blank]

then the series $\sum_{n=c}^{\infty} a_n$ diverges.

Ratio Test

Let N be any positive integer and assume that $a_n \neq 0$ for all $n \geq N$.

(a) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ [blank], then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ [blank], or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.

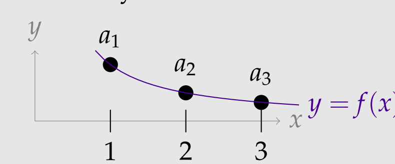
Integral Test

Let N_0 be any natural number. If $f(x)$ is a function which is defined and continuous for all $x \geq N_0$ and which obeys

(i) [blank] and

(ii) [blank] and

(iii) $f(n) = a_n$ for all $n \geq N_0$.



Then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_{N_0}^{\infty} f(x)\,dx \text{ converges}$$

Furthermore, when the series converges, the truncation error satisfies

$$0 \leq \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{N} a_n \leq \int_N^{\infty} f(x)\,dx \qquad \text{for all } N \geq N_0$$

The Comparison Test

Let N_0 be a natural number and let $K > 0$.

(a) If $|a_n|$ ☐ Kc_n for all $n \ge N_0$ and $\sum_{n=0}^{\infty} c_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

(b) If a_n ☐ $Kd_n \ge 0$ for all $n \ge N_0$ and $\sum_{n=0}^{\infty} d_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.

Limit Comparison Theorem

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with $b_n > 0$ for all n. Assume that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

exists.

(a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges too.

(b) If $L \neq 0$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges too.

In particular, if ☐, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.

Alternating Series Test

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers that obeys

(i) ☐

(ii) $a_{n+1} \le a_n$ for all $n \ge 1$ (i.e. the sequence is monotone decreasing);

(iii) and ☐

Then

$$a_1 - a_2 + a_3 - a_4 + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} a_n = S$$

converges and, for each natural number N, $S - S_N$ is between 0 and (the first dropped term) $(-1)^N a_{N+1}$. Here S_N is, as previously, the N^{th} partial sum $\sum_{n=1}^{N} (-1)^{n-1} a_n$.

Divergence Test

When the n^{th} term in the series *fails* to converge to zero as n tends to infinity.
This is a good first thing to check: if it works, it's quick, but it doesn't always work.

Alternating Series Test

- successive terms in the series alternate in sign
- don't forget to check that successive terms decrease in magnitude and tend to zero as n tends to infinity

Integral Test

- works well when, if you substitute x for n in the n^{th} term you get a function, $f(x)$, that you can easily integrate
- don't forget to check that $f(x) \ge 0$ and that $f(x)$ decreases as x increases

List of convergence tests

Ratio Test

- works well when $\frac{a_{n+1}}{a_n}$ simplifies enough that you can easily compute $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$
- this often happens when a_n contains powers, like 7^n, or factorials, like $n!$
- don't forget that $L = 1$ tells you nothing about the convergence/divergence of the series

Comparison Test and Limit Comparison Test

- Comparison test lets you ignore pieces of a function that feel extraneous (like replacing $n^2 + 1$ with n^2) *but* there is a test to make sure the comparison is still valid. Either the limit of a ratio is the right thing, or an inequality goes the right way.
- Limit comparison works well when, for very large n, the n^{th} term a_n is approximately the same as a simpler, nonnegative term b_n

- The integral test gave us the p-test. When you're looking for comparison series, p-series $\sum \frac{1}{n^p}$ are often good choices, because their convergence or divergence is so easy to ascertain.
- Geometric series have the form $\sum a \cdot r^n$ for some nonzero constants a and r. The magnitude of r is all you need to know to deicide whether they converge or diverge, so these are also common comparison series.
- Telescoping series have partial sums that are easy to find because successive terms cancel out. These are less obvious, and are less common choices for comparison series.

Test List

- divergence
- integral
- alternating series
- ratio
- comparison
- limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges or diverges.

The **divergence test** is inconclusive, because $\lim_{n\to\infty} \frac{\cos n}{2^n} = 0$ (which you can show with the squeeze theorem).
The **integral test** doesn't apply, because $f(x) = \frac{\cos x}{2^x}$ is not always positive (and not decreasing).
The **alternating series test** doesn't apply because the signs of the series do not strictly alternate every term.
The **ratio test** does not apply, because $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ does not exist.

Comparison test: Let $a_n = \frac{\cos n}{2^n}$. Note $|a_n| \le \frac{1}{2^n}$, and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (it is a geometric sum with ratio of consecutive terms $\frac{1}{2}$).
So by the comparison test, $\sum_{n=1}^{\infty} \frac{\cos n}{2^n}$ converges.

Limit comparison: Set $a_n = \frac{\cos n}{2^n}$ and $b_n = \left(\frac{2}{3}\right)^n$. Then

$\frac{a_n}{b_n} = \frac{\ \ }{2^n} = \left(\frac{3}{4}\right) \quad \cos n$

Test List

- divergence
- integral
- alternating series
- ratio
- comparison
- limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n \cdot n^2}{(n+5)^5}$ converges or diverges.

The **alternating series test** doesn't apply because the signs of the series do not alternate.
The **integral test** doesn't apply $f(x) = \frac{2^x \cdot x^2}{(x+5)^5}$ is not a decreasing function.
Divergence test: $\lim_{n\to\infty} \frac{2^n \cdot n^2}{(n+5)^5} = \infty$ (which you can see because the numerator is larger than a power function; the denominator is a polynomial; and power functions grow faster than polynomials), so the series diverges by the divergence test.

This is the fastest option, but not the only one.
Ratio test:

$a_n \quad \frac{2^{n+1}\cdot(n+1)^2}{(n+1+5)^5} \quad 2^{n+1} \quad (n+1)^2 \quad (n+5)^5$

Test List

- divergence
- integral
- alternating series
- ratio
- comparison
- limit comparison

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$ converges or diverges.

Hint: If $\theta \geq 0$ then $\sin\theta \leq \theta$.

The **divergence test** is inconclusive because $\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{n} = 0$.

The **alternating series test** does not apply because we are not considering an alternating series.

The **integral test** won't work for us because $\int_1^{\infty} \frac{1}{x}\sin\left(\frac{1}{x}\right)\, dx$ cannot be evaluated with techniques we've learned in class so far.

The **ratio test** is inconclusive because $\lim_{n\to\infty} \frac{\frac{1}{n+1}\sin\left(\frac{1}{n+1}\right)}{\frac{1}{n}\sin\left(\frac{1}{n}\right)} = 1$:

Set $x = \frac{1}{n+1}$. Then $\frac{1}{n} = \frac{x}{1-x}$:

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n+1}\right)}{\frac{1}{\ldots}} = \lim_{x\to 0^+} \frac{\sin x}{\frac{x}{\ldots}} = \lim_{x\to 0^+} (1-x)\frac{\sin x}{x} = 1 \cdot 1 = 1$$

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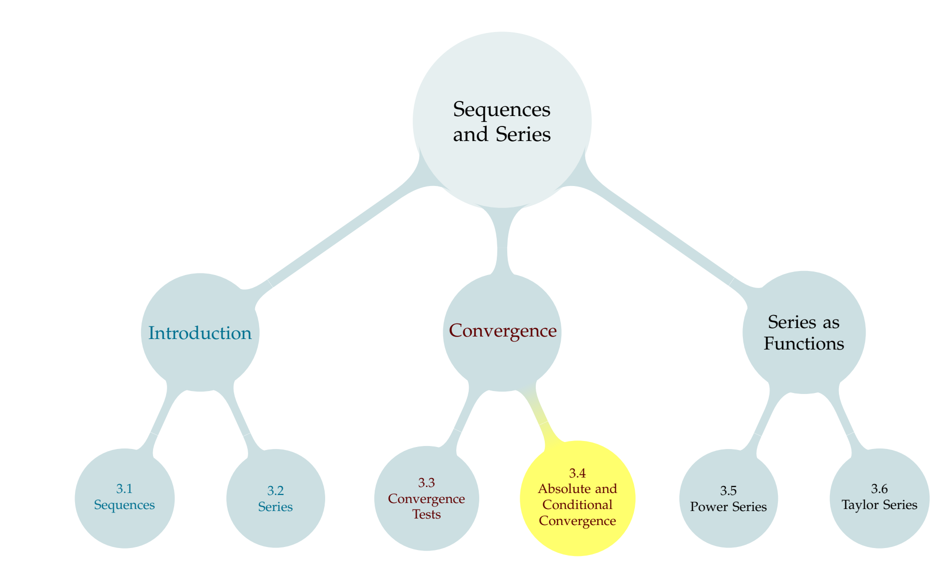

Four Series

Let $a_n = \left(-\frac{2}{3}\right)^n$. Do the following series converge or diverge?

$$\sum_{n=0}^{\infty} a_n \qquad \sum_{n=0}^{\infty} |a_n|$$

converge converge

Let $b_n = \frac{(-1)^n}{n}$. Do the following series converge or diverge?

$$\sum_{n=1}^{\infty} b_n \qquad \sum_{n=1}^{\infty} |b_n|$$

converge diverge

$\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{1}{n}\right)$ converges as well.

The series

$$\sum_{n=0}^{\infty} \left(-\frac{2}{3}\right)^n$$

is called absolutely convergent, because the series converges and if we replace the terms being added by their absolute values, that series *still* converges.

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

is called conditionally convergent, because the series converges, but if we replace the terms being added by their absolute values, that series *diverges*.

Absolute and conditional convergence

(a) A series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely** if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges we say that $\sum_{n=1}^{\infty} a_n$ is **conditionally convergent**.

Theorem

If the series $\sum_{n=1}^{\infty} |a_n|$ converges then the series $\sum_{n=1}^{\infty} a_n$ also converges. That is, absolute convergence implies convergence.

If $\sum a_n$ ...	and $\sum \lvert a_n \rvert$ ...	then we say $\sum a_n$ is ...
converges	converges	
converges	diverges	
diverges	diverges	
diverges	converges	

Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converge or diverge?

Alternating series test:

Let $a_n = \frac{1}{n^2}$. Note a_n has positive, decreasing terms, approaching 0 as n grows. Then $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges by the alternating series test.

Absolute convergence implies convergence:

The series $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^2}\right|$ is the same as the p-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges by the p-test. Then $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ converges absolutely, therefore it converges.

Does the series

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$$

converge or diverge?

The terms of this series are sometimes positive and sometimes negative, but they do not strictly alternate, so the alternating series test does not apply.

Note that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series, and $\frac{|\sin n|}{n^2} \leq \frac{1}{n^2}$ for all n. Then by the comparison test, $\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ converges.

Then $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ converges absolutely, hence it converges.

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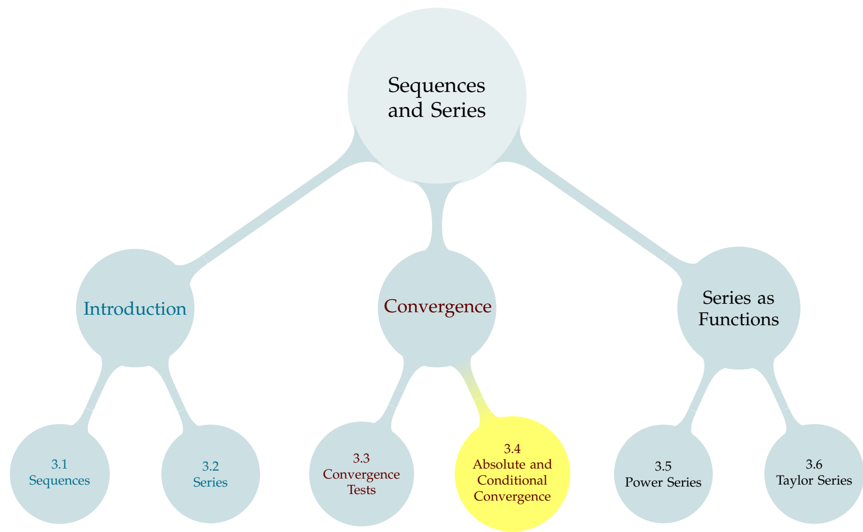


Finite addition is commutative

$$1+2+3+4=4+1+3+2$$

What happens if we re-arrange the terms in a series?

We'll illustrate some possibilities, but first we need to establish some preliminary results.

Preliminary Results

Split up the alternating harmonic series into two series: one with the positive terms, and one with the negative terms.

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}\cdots$$

$$\begin{aligned}&-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}-\cdots\\&=-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)\\&=-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\end{aligned}$$

So, we can make an arbitrarily large negative number by adding up these terms.

$$\begin{aligned}&1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\\&\geq\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots\end{aligned}$$

So, we can make an arbitrarily large positive number by adding up these terms.

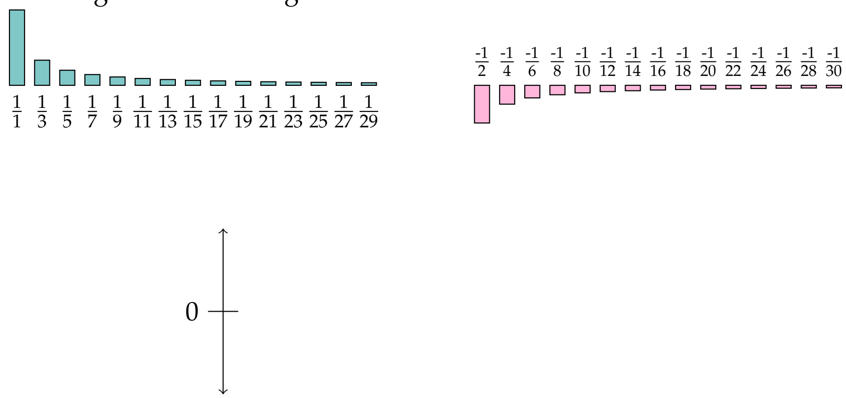
Preliminary Results

We've shown that the alternating harmonic series converges. We don't have the tools to do it just yet, but later we'll be able to compute what it converges to:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}=\log 2$$

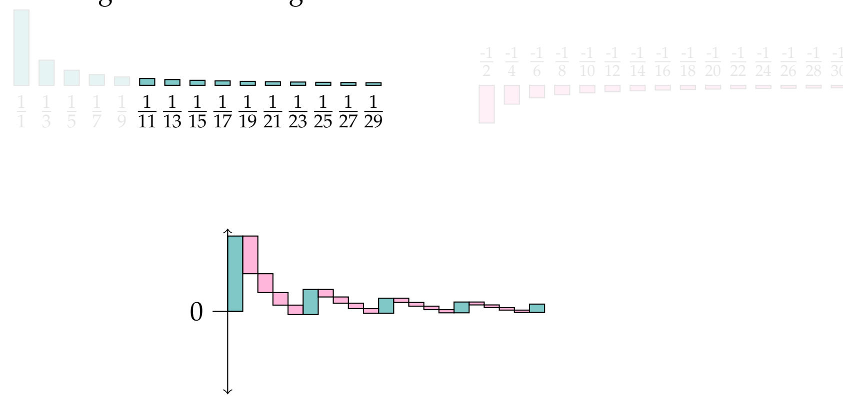
Surprising fact: if we reorder the terms of the series carefully, we can make a new series adding up to any number we want.

Rearrange the alternating harmonic series to sum to 0.



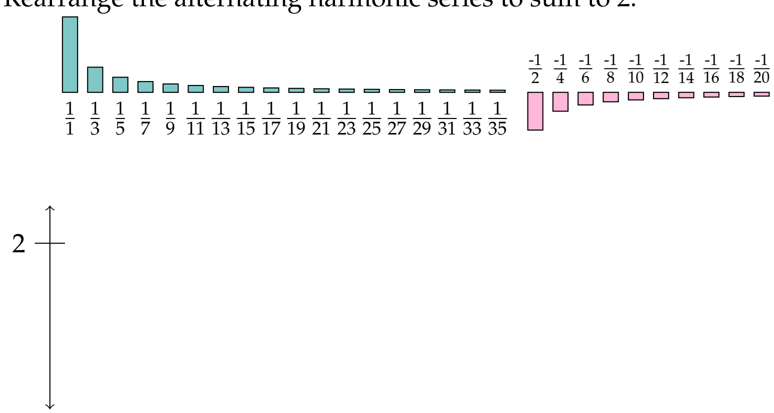
- Add positive terms until the partial sum is greater than 0.
- Add negative terms until the partial sum is less than 0.
- Repeat.

Rearrange the alternating harmonic series to sum to 0.



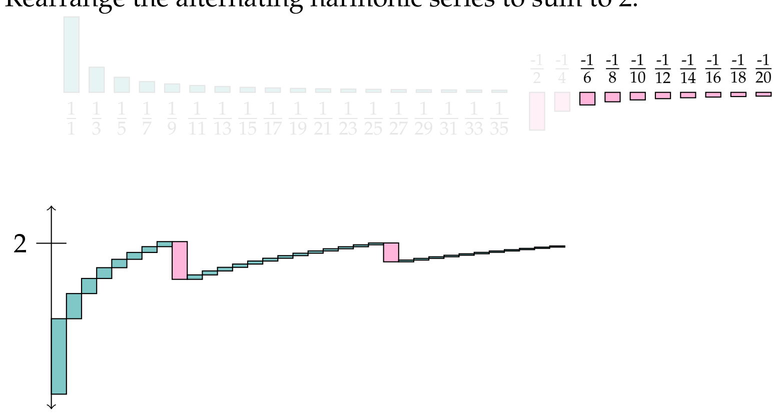
- Add positive terms until the partial sum is greater than 0.
- Add negative terms until the partial sum is less than 0.
- Repeat.

Rearrange the alternating harmonic series to sum to 2.



- Add positive terms until the partial sum is greater than 2.
- Add negative terms until the partial sum is less than 2.
- Repeat.

Rearrange the alternating harmonic series to sum to 2.



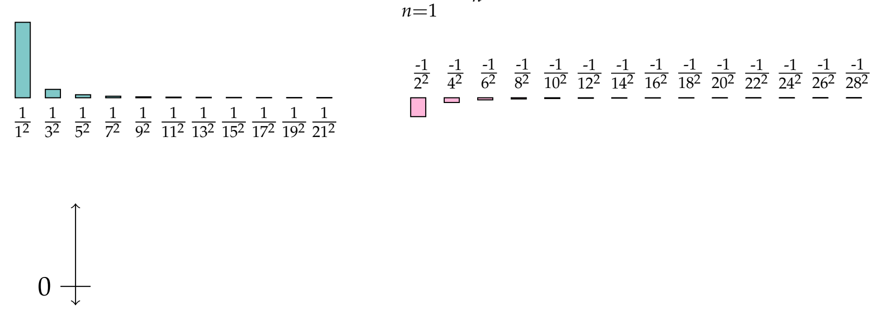
- Add positive terms until the partial sum is greater than 2.
- Add negative terms until the partial sum is less than 2.
- Repeat.

In fact: you can reorder *any* conditionally convergent series to

- add up to *any* number, or
- diverge to infinity, or
- diverge to negative infinity.

This doesn't work with absolutely convergent series.

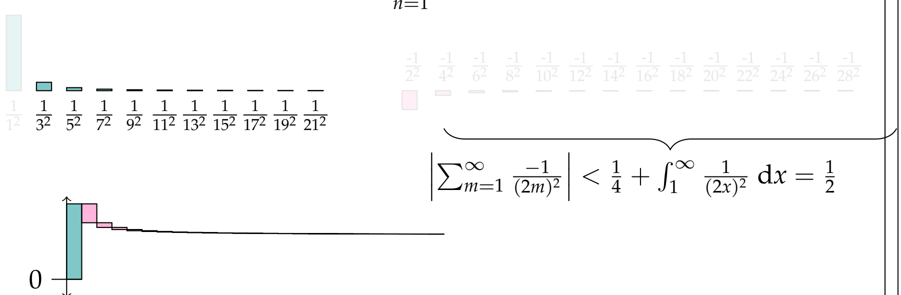
Let's try to rearrange the terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ to add up to 0:



- Add positive terms until the partial sum is greater than 0.
- Add negative terms (those with $n = 2m$, $m = 1, 2, 3, \cdots$) until the partial sum is less than 0.
- Repeat.

This doesn't work with absolutely convergent series.

Let's try to rearrange the terms of $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ to add up to 0:



- Add positive terms until the partial sum is greater than 0.
- Add negative terms (those with $n = 2m$, $m = 1, 2, 3, \cdots$) until the partial sum is less than 0.
- Repeat.

In fact: you can reorder *any* conditionally convergent series to

- add up to *any* number, or
- diverge to infinity, or
- diverge to negative infinity.

Changing the order of terms in an absolutely convergent series does not change its value.

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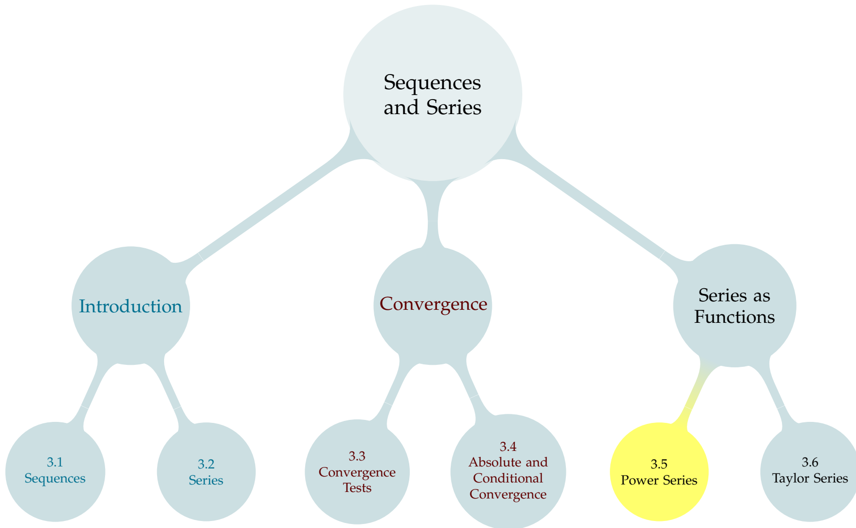


Recall the geometric series: for a constant r, with $|r| < 1$:

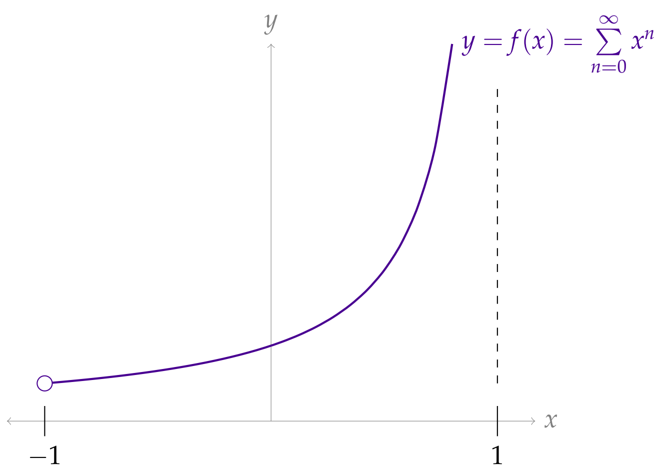
$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

We can think of this as a function. If we set

$$f(x) = \sum_{n=0}^{\infty} x^n$$

and restrict our domain to $-1 < x < 1$, then

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$



Why would we ever prefer to write $\sum_{n=0}^{\infty} x^n$ instead of $\frac{1}{1-x}$?

The function

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots$$

isn't a polynomial, but in certain ways it behaves like one. For $|x| < 1$:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\} = \frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x}\{x^n\}\right) = \sum_{n=0}^{\infty} nx^{n-1}$$

$$\int \frac{1}{1-x}\,\mathrm{d}x = \int \left(\sum_{n=0}^{\infty} x^n\right)\mathrm{d}x = \sum_{n=0}^{\infty}\left(\int x^n\,\mathrm{d}x\right) = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$$

Definition

A series of the form

$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

is called a *power series in* $(x-c)$ or a *power series centered on* c. The numbers A_n are called the coefficients of the power series.

One often considers power series centered on $c = 0$ and then the series reduces to

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots = \sum_{n=0}^{\infty} A_nx^n$$

$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

In a power series, we think of the coefficients A_n as fixed constants, and we think of x as the variable of a function.

Evaluate the power series $\sum_{n=0}^{\infty} A_n(x-c)^n$ when $x = c$:

$$\sum_{n=0}^{\infty} A_n(x-c)^n = A_0 + A_1(x-c) + A_2(x-c)^2 + A_3(x-c)^3 + \cdots$$

$$\sum_{n=0}^{\infty} A_n(c-c)^n = A_0 + A_1\underbrace{(c-c)}_{0} + A_2\underbrace{(c-c)^2}_{0} + A_3\underbrace{(c-c)^3}_{0} + \cdots$$

$$= A_0 \quad \text{(In particular, the series converges when } x = c\text{.)}$$

A fundamental question we want to ask when we see a series is whether it converges or diverges. So, let's find all values of x for which the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges.
This looks somewhat like a geometric series, but not exactly, so the ratio test is a good option.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|\left(\frac{n}{n+1}\right)$$

$$= \lim_{n\to\infty}|x|\left(\frac{n}{n+1}\right) = |x|$$

So the series converges when $|x| < 1$ and diverges when $|x| > 1$. When $x = 1$, we have the harmonic series, which diverges. When $x = -1$, we have the alternating harmonic series, which converges.

So, all together, the series converges when $-1 \le x < 1$, and diverges everywhere else.

-1 0 1

A fundamental question we want to ask when we see a series is whether it converges or diverges. So, let's find all values of x for which the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$$

converges.
This looks somewhat like a geometric series, but not exactly, so the ratio test is a good option.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|\left(\frac{n}{n+1}\right)$$

$$= \lim_{n\to\infty}|x|\left(\frac{n}{n+1}\right) = |x|$$

So the series converges when $|x| < 1$ and diverges when $|x| > 1$. When $x = 1$, we have the harmonic series, which diverges. When $x = -1$, we have the alternating harmonic series, which converges.

So, all together, the series converges when $-1 \le x < 1$, and diverges everywhere else.

-1 0 1

Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty} 2^n (x-1)^n = 1 + 2(x-1) + 2^2(x-1)^2 + 2^3(x-1)^3 + \cdots .$$

This still looks somewhat like a geometric series, so the ratio test is a still good option to start.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-1)^{n+1}}{2^n(x-1)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+1}}{(x-1)^n}\right|\left(\frac{2^{n+1}}{2^n}\right)$$
$$= 2\,|x-1|$$

So we see that the series converges when $|x-1| < \frac{1}{2}$ and diverges when $|x-1| > \frac{1}{2}$.
When $x - 1 = -\frac{1}{2}$, i.e. $x = \frac{1}{2}$, our series is

$$\sum_{n=0}^{\infty} 2^n \left(\frac{1}{2} - 1\right)^n = \sum_{n=0}^{\infty} 2^n \left(-\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} (-1)^n$$

When $x - 1 = \frac{1}{2}$, i.e. $x = \frac{3}{2}$, our series is

$$\sum_{n=0}^{\infty} 2^n \left(\frac{3}{2} - 1\right)^n = \sum_{n=0}^{\infty} 2^n \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} 1$$

In both cases, the series diverge by the divergence test. All together, the

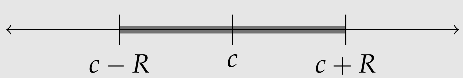
What happens if we apply the ratio test to a generic power series, $\sum_{n=0}^{\infty} A_n (x-c)^n$?

$$\lim_{n\to\infty}\left|\frac{A_{n+1}(x-c)^{n+1}}{A_n(x-c)^n}\right| = \lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}(x-c)\right| = |x-c|\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right|$$

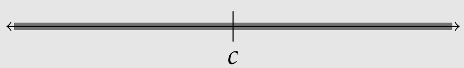
- If $\left|\frac{A_{n+1}}{A_n}\right|$ does not approach a limit as $n \to \infty$, the ratio test tells us nothing. (We should try other tests.)
- If $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = 0$, then
- If $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \infty$, then
- If $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = A$ for some real number A, then

Definition: Radius of Convergence

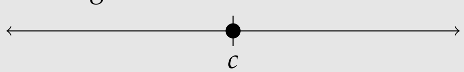
(a) Let $0 < R < \infty$. If $\sum_{n=0}^{\infty} A_n (x-c)^n$ converges for $|x-c| < R$, and diverges for $|x-c| > R$, then we say that the series has radius of convergence R.



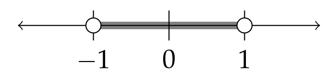
(b) If $\sum_{n=0}^{\infty} A_n (x-c)^n$ converges for every number x, we say that the series has an infinite radius of convergence.



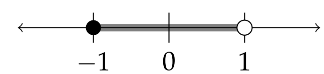
(c) If $\sum_{n=0}^{\infty} A_n (x-c)^n$ diverges for every $x \neq c$, we say that the series has radius of convergence zero.



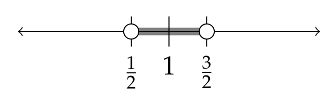
- We saw that $\sum_{n=0}^{\infty} x^n$ converges when $|x| < 1$ and diverges when $|x| > 1$, so this series has radius of convergence $R =$



- We saw that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges when $|x| < 1$ and diverges when $|x| > 1$, so this series also has radius of convergence $R =$



- We saw that $\sum_{n=1}^{\infty} 2^n (x-1)^n$ converges when $|x-1| < \frac{1}{2}$ and diverges when $|x-1| > \frac{1}{2}$, so this series has radius of convergence $R =$



What is the radius of convergence for the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$?

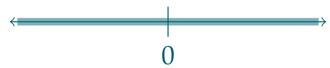
Recall: $n! = (n)(n-1)(n-2)\cdots(2)(1)$.

$$\lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right|\frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty}|x|\frac{(n)(n-1)(n-2)\cdots(2)(1)}{(n+1)(n)(n-1)(n-2)\cdots(2)(1)}$$

$$= \lim_{n\to\infty}\frac{|x|}{n+1} = 0$$

For every real x, the limit is less than one, so the series converges. That is, its radius of convergence is ∞.



What is the radius of convergence for the series $\sum_{n=0}^{\infty} n!\cdot(x-3)^n$?

$$\lim_{n\to\infty}\left|\frac{(n+1)!(x-3)^{n+1}}{(n!)(x-3)^n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!}\left|\frac{(x-3)^{n+1}}{(x-3)^n}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)(n)(n-1)(n-2)\cdots(2)(1)}{(n)(n-1)(n-2)\cdots(2)(1)}|x-3|$$

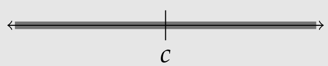
$$= \lim_{n\to\infty}(n+1)|x-3|$$

For every real x except $x = 3$, the limit is greater than one, so the series diverges. The series only converges at $x = 3$. That is, its radius of convergence is 0.

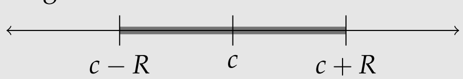
Theorem

Given a power series (say with centre c), one of the following holds.

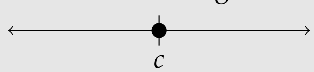
(a) The power series converges for every number x. In this case we say that the radius of convergence is ∞.



(b) There is a number $0 < R < \infty$ such that the series converges for $|x-c| < R$ and diverges for $|x-c| > R$. Then R is called the radius of convergence.



(c) The series converges for $x = c$ and diverges for all $x \neq c$. In this case, we say that the radius of convergence is 0.



We are told that a certain power series with centre $c = 3$ converges at $x = 4$ and diverges at $x = 1$. What else can we say about the convergence or divergence of the series for other values of x?

Operations on Power Series

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n$$

for all x obeying $|x-c| < R$. Let K be a constant. Then:

$$f(x) + g(x) = \sum_{n=0}^{\infty} [A_n + B_n] \, (x-c)^n$$

$$Kf(x) = \sum_{n=0}^{\infty} K \, A_n \, (x-c)^n$$

for all x obeying $|x-c| < R$.

Operations on Power Series

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n$$

for all x obeying $|x-c| < R$. Let K be a constant. Then:

$$\begin{aligned} (x-c)^N f(x) &= \sum_{n=0}^{\infty} A_n \, (x-c)^{n+N} \quad \text{for any integer } N \geq 1 \\ &= \sum_{k=N}^{\infty} A_{k-N} \, (x-c)^k \quad \text{where } k = n+N \end{aligned}$$

for all x obeying $|x-c| < R$.

Operations on Power Series

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n$$

for all x obeying $|x-c| < R$. Let K be a constant. Then:

$$f'(x) = \sum_{n=0}^{\infty} A_n \, n \, (x-c)^{n-1} = \sum_{n=1}^{\infty} A_n \, n \, (x-c)^{n-1}$$

$$\int_c^x f(t) \, \mathrm{d}t = \sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1}$$

$$\int f(x) \, \mathrm{d}x = \left[\sum_{n=0}^{\infty} A_n \frac{(x-c)^{n+1}}{n+1} \right] + C \quad \text{with } C \text{ an arbitrary constant}$$

for all x obeying $|x-c| < R$.

Operations on Power Series

Assume that the functions $f(x)$ and $g(x)$ are given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n (x-c)^n \qquad g(x) = \sum_{n=0}^{\infty} B_n (x-c)^n$$

for all x obeying $|x-c| < R$. Let K be a constant. Then:

for all x obeying $|x-c| < R$.
Differentiating, antidifferentiating, multiplying by a nonzero constant, and multiplying by a positive power of $(x-c)$ do not change the radius of convergence of $f(x)$ (although they may change the interval of convergence).

Given that $\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\} = \frac{1}{(1-x)^2}$, find a power series representation for $\frac{1}{(1-x)^2}$ when $|x|<1$. For $|x|<1$:

$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{1}{1-x}\right\}\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left\{\sum_{n=0}^{\infty} x^n\right\}\\ &= \sum_{n=0}^{\infty}\left(\frac{\mathrm{d}}{\mathrm{d}x}\{x^n\}\right)\\ &= \sum_{n=0}^{\infty} nx^{n-1}\\ &= \sum_{n=1}^{\infty} nx^{n-1}\end{aligned}$$

Find a power series representation for $\log(1+x)$ when $|x|<1$.
First, note $\frac{\mathrm{d}}{\mathrm{d}x}\{\log(1+x)\} = \frac{1}{1+x}$. Our plan is to antidifferentiate a power series representation of $\frac{1}{1+x}$. For $|x|<1$:

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\begin{aligned}\int \frac{1}{1+x}\,\mathrm{d}x &= \int\left(\sum_{n=0}^{\infty}(-1)^n x^n\right)\mathrm{d}x\\ &= \sum_{n=0}^{\infty}\left(\int (-1)^n x^n\,\mathrm{d}x\right)\end{aligned}$$

So, for some constant C,

$$\log(1+x) = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} = C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

To find C, let's plug in a value for x where both sides of the equation are easy to evaluate: $x=0$.

$$\log(1+0) = C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{0^n}{n}$$

Find a power series representation for $\arctan(x)$ when $|x|<1$.
First, note $\frac{\mathrm{d}}{\mathrm{d}x}\{\arctan x\} = \frac{1}{1+x^2}$. To obtain a power series representation of $\frac{1}{1+x^2}$, we'll substitute into the geometric series. Let $y=-x^2$ with $|y|<1$. Then:

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$$

$$\implies \frac{1}{1+x^2} = \sum_{n=0}^{\infty}\left(-x^2\right)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\implies \int\frac{1}{1+x^2}\,\mathrm{d}x = \int\left(\sum_{n=0}^{\infty}(-1)^n x^{2n}\right)\mathrm{d}x = \sum_{n=0}^{\infty}\left(\int(-1)^n x^{2n}\,\mathrm{d}x\right)$$

$$\implies \arctan x = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

for some constant C. To find C, we'll plug in $x=0$, which makes both sides of the last equation easy to evaluate.

$$\arctan 0 = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$

So, $\arctan x = \sum^{\infty}(-1)^n\, x^{2n+1}$

Substituting in a Power Series

Assume that the function $f(x)$ is given by the power series

$$f(x) = \sum_{n=0}^{\infty} A_n x^n$$

for all x in the interval I. Also let K and k be real constants. Then

$$f\left(Kx^k\right) = \sum_{n=0}^{\infty} A_n K^n\, x^{kn}$$

whenever Kx^k is in I. In particular, if $\sum_{n=0}^{\infty} A_n x^n$ has radius of convergence R, K is nonzero and k is a natural number, then $\sum_{n=0}^{\infty} A_n K^n\, x^{kn}$ has radius of convergence $\sqrt[k]{R/|K|}$.

Find a power series representation for $\dfrac{1}{5-x}$ with centre 3.

We know that $\frac{1}{1-(x-3)} = \sum_{n=0}^{\infty}(x-3)^n$ when $|x-3| < 1$. To take advantage of our ability to substitute into power functions, we'd like to write $\frac{1}{5-x}$ in the form $\frac{1}{1-K(x-3)^k}$ for some constant K and some whole number k.

$$\frac{1}{5-x} = \frac{1}{2-(x-3)} = \frac{1}{2}\cdot\frac{1}{1-\left(\frac{x-3}{2}\right)}$$

Set $y = \frac{x-3}{2}$. When $|y| < 1$:

$$\frac{1}{2}\cdot\frac{1}{1-y} = \frac{1}{2}\sum_{n=0}^{\infty} y^n$$

$$\implies \frac{1}{2}\cdot\frac{1}{1-\left(\frac{x-3}{2}\right)} = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x-3}{2}\right)^n$$

$$\implies \frac{1}{5-x} = \sum_{n=0}^{\infty}\frac{(x-3)^n}{2^{n+1}}.$$

The series converges when:

$$|y| < 1$$

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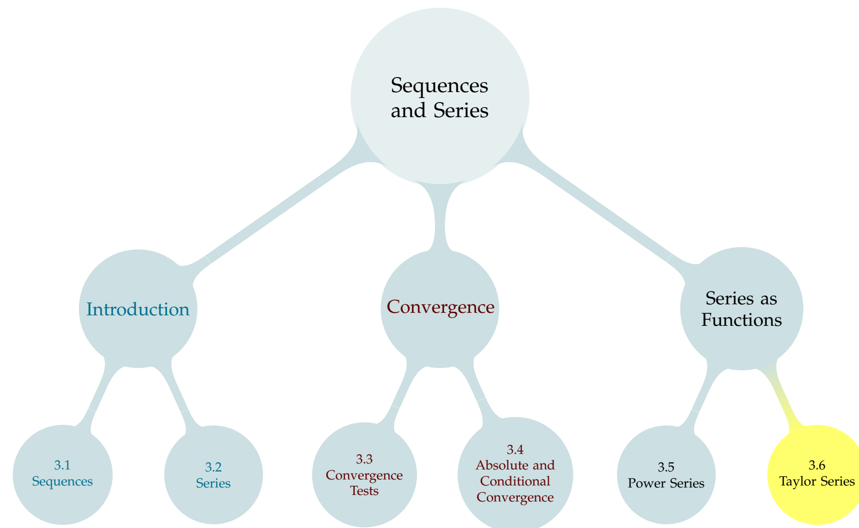


Taylor polynomial

Let a be a constant and let n be a non-negative integer. The n^{th} order Taylor polynomial for $f(x)$ about $x = a$ is

$$T_n(x) = \sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)\cdot(x-a)^k.$$

Taylor series

The Taylor series for the function $f(x)$ expanded around a is the power series

$$\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}(a)\,(x-a)^n.$$

When $a = 0$ it is also called the Maclaurin series of $f(x)$.

Let's compute some Taylor series, using the definition.

The method is nearly identical to finding Taylor *polynomials*, which is covered in CLP–1.

Find the Maclaurin series for $f(x) = \sin x$.

Taylor series

The Taylor series for the function $f(x)$ expanded around a is the power series

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)\,(x-a)^n.$$

When $a = 0$ it is also called the Maclaurin series of $f(x)$.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \end{aligned}$$

The derivatives then repeat. Notice we only have non-zero derivatives for odd orders, and these alternate in sign.
We can write the Maclaurin series as follows:

$$\sin x \approx \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Find the Maclaurin series for $f(x) = \cos x$.

$$\begin{aligned} f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \end{aligned}$$

The derivatives then repeat. Notice we only have non-zero derivatives for even orders, and these alternate in sign.
We can write the Maclaurin series as follows:

$$\begin{aligned} \cos x &\approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \end{aligned}$$

The Maclaurin series for $f(x) = e^x$ is: $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Every derivative of e^x is e^x, so all coefficients $f^{(n)}(0)$ are e^0, i.e. 1.

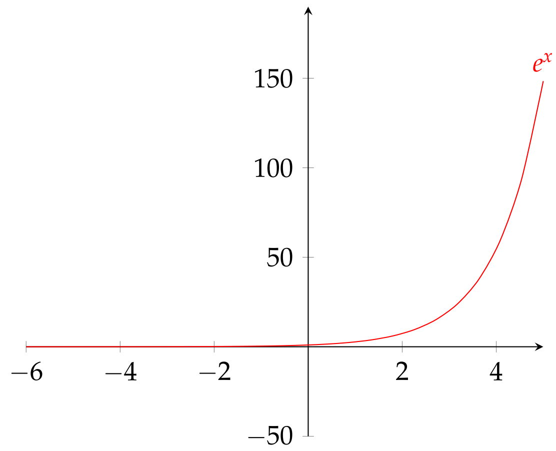
$$\begin{aligned} e^x &\approx 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

Let $T_n(x)$ be the n-th order Taylor polynomial of the function $f(x)$, centred at a.

When we introduced Taylor polynomials in CLP–1, we framed $T_n(x)$ as an approximation of $f(x)$.

Let's see how those approximations look in two cases:

Taylor Polynomials for e^x



It seems like high-order Taylor polynomials do a pretty good job of approximating the function e^x, at least when x is near enough to 0.

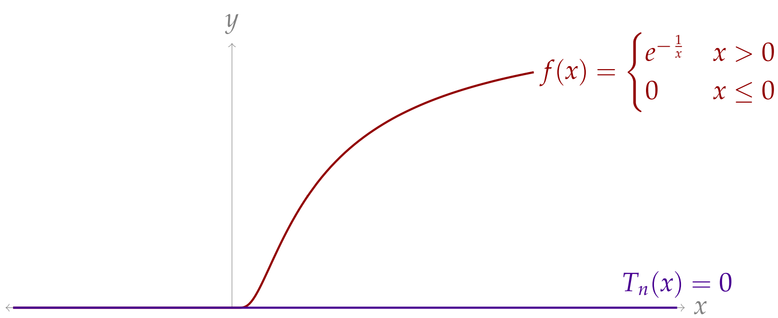
Taylor Polynomials for a different function

But that is not the case for all functions. Define

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \le 0 \end{cases}$$

Using the definition of the derivative and l'Hôpital's rule, one can show that $f^{(n)}(0) = 0$ for all natural numbers n.

Taylor Polynomials for a different function



Taylor polynomial approximations don't always get better as their orders increase – it depends on the function being approximated.

Investigation

- We found the Maclaurin series for $f(x) = e^x$ is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
- But, it's not immediately clear whether $e^x \stackrel{?}{=} \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$.
- We're going to demonstrate that e^x is in fact equal to $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$. The proof involves a particular limit: $\displaystyle\lim_{n\to\infty} \frac{|x|^n}{n!}$. We'll talk about that limit first, so that it doesn't distract us later.

Intermediate result: $\lim_{n\to\infty} \frac{|x|^n}{n!}$, when x is some fixed number.

For large n, we can think of $\frac{|x|^n}{n!}$ as a long multiplication, with decreasing terms. At some point, those terms are all decreasing *and less than 1*.

$$\frac{|x|^n}{n!} = \frac{|x|\cdot|x|\cdot|x|\cdot|x|\cdot|x|\cdot|x|\cdot\dots\cdot|x|}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot\dots\cdot n}$$
$$=$$

 Convenient notation: $\lceil x\rceil$ is the number you get when you round x up to the nearest whole number.

Intermediate result: $\lim_{n\to\infty} \frac{|x|^n}{n!}$, when x is some fixed number.

We're multiplying terms that are closer and closer to 0, so it seems quite reasonable that this sequence should converge to 0.

For a more formal proof, we can use the squeeze theorem to compare this sequence to a geometric sequence.

 Convenient notation: $\lceil x\rceil$ is the number you get when you round x up to the nearest whole number.

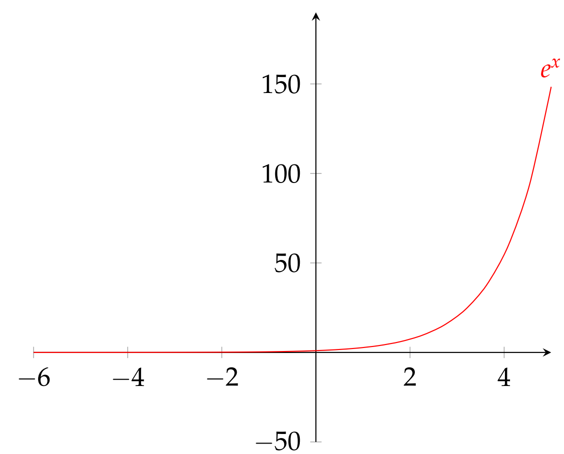
Investigation

- We found the Maclaurin series for $f(x) = e^x$ is $\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$.
- But, it's not immediately clear whether $e^x \stackrel{?}{=} \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!}$.
  How could we determine this?
-

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$
$$\iff 0 = e^x - \sum_{n=0}^{\infty}\frac{x^n}{n} = e^x - \lim_{n\to\infty}\underbrace{\sum_{k=0}^{n}\frac{x^k}{k!}}_{T_n(x)} = \lim_{n\to\infty}\underbrace{\left[e^x - T_n(x)\right]}_{E_n(x)}$$
$$\iff 0 = \lim_{n\to\infty} E_n(x) \quad \text{(for all } x\text{)}$$

Taylor Polynomial Error for $f(x) = e^x$

If $\lim_{n\to\infty} E_n(x) = 0$ for all x, then $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x.
It *looks* plausible, especially when x is close to 0. Let's try to prove it.



Equation 3.6.1-b

Let $T_n(x)$ be the n-th order Taylor approximation of a function $f(x)$, centred at a. Then $E_n(x) = f(x) - T_n(x)$ is the error in the n-th order Taylor approximation.
For some c strictly between x and a,

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) \cdot (x-a)^{n+1}$$

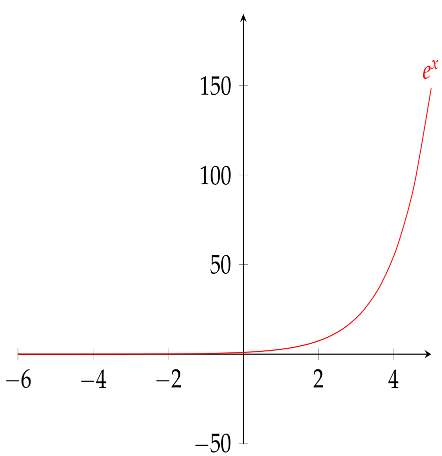
When $f(x) = e^x$,

$$E_n(x) = e^c \frac{x^{n+1}}{(n+1)!}$$

for some c between 0 and x.

$$\begin{aligned} E_n(x) &= e^x - T_n(x) \\ &= e^c \frac{x^{n+1}}{(n+1)!} && \text{for some } c \text{ between 0 and } x \\ 0 \le |E_n(x)| &< \left| e^c \frac{x^{n+1}}{(n+1)!} \right| \\ &\le e^{|x|} \frac{|x|^{n+1}}{(n+1)!} \\ 0 &= \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} && \text{by our previous result} \\ \implies 0 &= \lim_{n\to\infty} |E_n(x)| && \text{by the squeeze theorem} \end{aligned}$$

We found $0 \le |E_n(x)| < e^{|x|}\frac{|x|^{n+1}}{(n+1)!}$ for large n, hence $\lim\limits_{n\to\infty} |E_n(x)| = 0$.



For a particular value of x:

$$\begin{aligned} \text{We saw} \quad 0 &= \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} \\ \text{so} \quad 0 &= \lim_{n\to\infty} E_n(x) \\ \text{That is,} \quad 0 &= \lim_{n\to\infty} [e^x - T_n(x)] \\ \text{So, } e^x &= \lim_{n\to\infty} T_n(x) \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \end{aligned}$$

Taylor polynomial error for sine and cosine

Equation 3.6.1-b

Let $T_n(x)$ be the n-th order Taylor approximation of a function $f(x)$, centred at a. Then $E_n(x) = f(x) - T_n(x)$ is the error in the n-th order Taylor approximation.
For some c strictly between x and a,

$$E_n(x) = \frac{1}{(n+1)!} f^{(n+1)}(c) \cdot (x-a)^{n+1}$$

Suppose $f(x)$ is either $\sin x$ or $\cos x$. Is $f(x)$ equal to its Maclaurin series? In either case, $|f^{(n+1)}(c)|$ is either $|\sin c|$ or $|\cos c|$, so it's between 0 and 1.

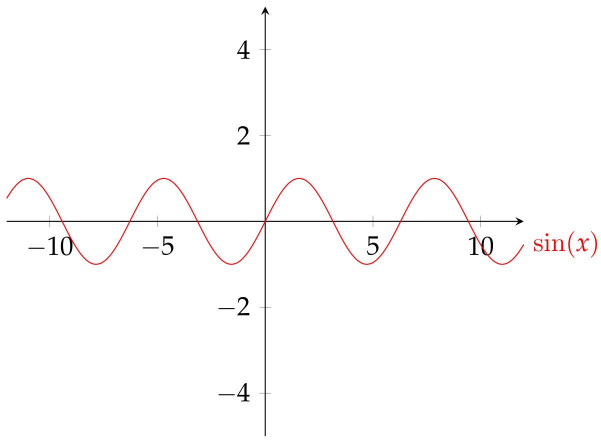
$$|E_n(x)| = \frac{1}{(n+1)!}\left|f^{(n+1)}(c)\right| |x|^{n+1} \le \frac{|x|^{n+1}}{(n+1)!}$$

$$\implies 0 \le |E_n(x)| \le \frac{|x|^{n+1}}{(n+1)!}$$

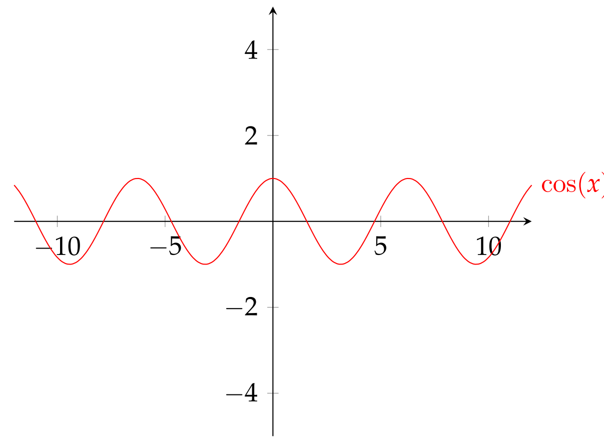
We saw before that $\lim\limits_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$. So, by the squeeze theorem,

$$\lim_{n\to\infty} |E_n(x)| = 0$$

Taylor Polynomials for $\sin(x)$



Taylor Polynomials for $\cos(x)$



Selected Taylor series that equal their functions

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	for all $-\infty < x < \infty$
$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$	for all $-\infty < x < \infty$
$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$	for all $-\infty < x < \infty$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	for all $-1 < x < 1$
$\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$	for all $-1 < x \le 1$
$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	for all $-1 \le x \le 1$

Computing π

Use the fact that $\arctan 1 = \frac{\pi}{4}$ to find a series converging to π whose terms are rational numbers.

For all $-1 \le x \le 1$:

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$4 \arctan x = 4 \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\pi = 4 \arctan 1 = 4 \sum_{n=0}^{\infty} (-1)^n \frac{1^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1}$$

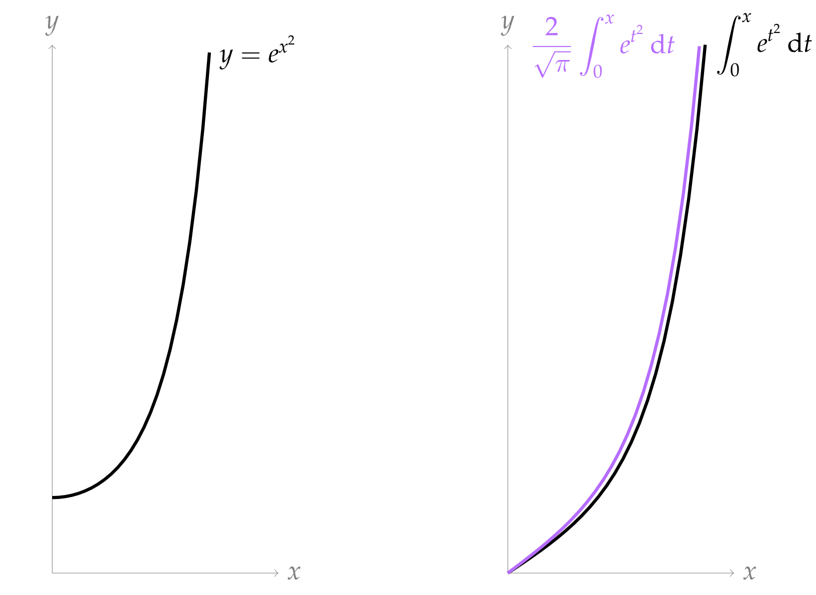
$$= 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$$

Error function

The *error function*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, \mathrm{d}t$$

is used in computing “bell curve” probabilities.



Error function

The *error function*

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, \mathrm{d}t$$

is used in computing “bell curve” probabilities.

The indefinite integral of the integrand e^{-t^2} cannot be expressed in terms of standard functions. But we can still evaluate the integral to within any desired degree of accuracy by using the Taylor expansion of the exponential.

For example, evaluate $\mathrm{erf}\left(\frac{1}{\sqrt{2}}\right)$.

$$\mathrm{erf}\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}}} e^{-t^2}\, \mathrm{d}t = \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}}} \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \Bigg|_{x=-t^2} \mathrm{d}t$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{2}}} \left(\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{n!} \right) \mathrm{d}t = \frac{2}{\sqrt{\pi}} \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)n!} \right]_0^{\frac{1}{\sqrt{2}}}$$

Evaluating a convergent series

Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \cdot 3^n}$

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$	for all $-\infty < x < \infty$
$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$	for all $-\infty < x < \infty$
$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$	for all $-\infty < x < \infty$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$	for all $-1 < x < 1$
$\log(1+x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$	for all $-1 < x \leq 1$
$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$	for all $-1 \leq x \leq 1$

The series most closely resembles the Taylor series $\log(1+x) = \sum_{m=0}^{\infty} (-1)^m \frac{x^{m+1}}{m+1}$. To make that relation clearer, set $m = n - 1$:

Finding a high-order derivative

Let $f(x) = \sin(2x^3)$. Find $f^{(15)}(0)$, the fifteenth derivative of f at $x = 0$.

Differentiating directly gets messy quickly. Instead, let's find the Taylor series. Let $y = 2x^3$:

$$\sin(y) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} y^{2n+1}$$

$$\implies \quad f(x) = \sin(2x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} (2x^3)^{2n+1}$$

$$\implies \quad f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n+1}}{(2n+1)!} x^{6n+3}$$

The coefficients of x^{15} on the left and right series must match for the series to be equal.

When $m = 15$ on the left-hand side, we get the term $\frac{f^{(15)}(0)}{15!} x^{15}$. The right-hand side term corresponding to x^{15} occurs when $6n + 3 = 15$, i.e. when $n = 2$.

Given that $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$, we have a new way of evaluating the familiar limit

$$\lim_{x \to 0} \frac{\sin x}{x}:$$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x}$$

$$= \lim_{x \to 0} \left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots \right]$$

$$= 0$$

This technique is sometimes faster than l'Hôpital's rule.

Evaluate $\lim\limits_{x \to 0} \dfrac{\arctan x - x}{\sin x - x}$.

$$\arctan x - x = \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots \right) - x$$

$$= -\frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

$$\sin x - x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right) - x$$

$$= -\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\lim_{x \to 0} \frac{\arctan x - x}{\sin x - x} = \lim_{x \to 0} \frac{-\frac{x^3}{3} + \frac{x^5}{5} - \cdots}{-\frac{x^3}{3!} + \frac{x^5}{5!} - \cdots} \left(\frac{\frac{1}{x^3}}{\frac{1}{x^3}} \right)$$

$$= \lim_{x \to 0} \frac{-\frac{1}{3} + \frac{x^2}{5} - \cdots}{-\frac{1}{3!} + \frac{x^2}{5!} - \cdots} = \frac{-\frac{1}{3}}{-\frac{1}{6}} = 2$$

Included Work

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