PLP

An introduction to mathematical proof

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Exercises for PLP

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For our students and all the colleagues and scholars who have freely shared their knowledge with us — SD & AR.

To Elyse for all her love, support, and help with my LaTeX problems — SD.

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The authors would also like to acknowledge that the land UBC Vancouver Campus is on is the traditional, ancestral, and unceded territory of the xʷməθkʷəy̓əm (Musqueam) People. Learn more about UBC's relationship with the Musqueam Nation[3].

[3] aboriginal.ubc.ca/community-youth/musqueam-and-ubc/

Preface

The main idea of this text is to teach you how to write correct and *clear* mathematical proofs. In order to learn to prove things we will study some basic analysis. We will prove many things about the basic properties of numbers sets and functions — like

> There are more real numbers than integers.

In order to make sense of this statement we need to understand how to extend the idea of "more" from the context of finite quantities, where we are used to "more" and "less", to the domain of infinite quantities. We'll have to define ideas about sets and functions, manipulate and combine them with logic. Along the way we will need to think hard about how to communicate the mathematics that we are doing, so that you, and others, can follow what we are doing.

Hence, a critical part of this subject is to learn to communicate mathematics — not just do mathematics. Mathematics is not simply number crunching or using formulas — this is using mathematics. Mathematics is also about understanding and reasoning and most importantly *proving* things. Neither of these aspects is more important than the other. Up until now you have mostly done the former, and the aim of this text is to help you get better at the latter.

It is crucial that we are able to explain to others why what we know is true is actually true — this is what proofs are for. Think of the proof as a dialog between you and the reader — you have to make every (reasonable) effort to be clear, precise and accurate. Always think of the reader when you are writing. It is important to argue and write well — it is a useful skill both at university and beyond in the so-called "real world".

The authors have spend a lot of time reading other people's work (mostly student work, but also articles written by professional mathematicians with years of experience) and puzzling over the lines written on the page — sometimes legible, sometimes not (especially in exams). And finally after sweating for ages you realise what they were trying to say. In some circumstances one can, of course, contact the writer and ask them "What did you mean?" However, this is frequently not the case — all one has is what is written on the page.

> So that's what he meant! Then why didn't he say so?
>
> —Frank Harary

When you do mathematics (and other activities) there is a huge difference between reading and doing. This is especially the case with proofs. So while reading the text is a good way to learn some ideas and get a feeling for some of the stuff, it is really no substitute for *doing* the exercises. That is where you will really learn.

> I write to discover what I know
>
> —Flannery O'Connor

Behind every proof you read (and you write) lies a good bit of work. You cannot generally look at a problem and write out the proof all fine first go. You need to do some rough work to map out the structure of the proof and the details. Then *after* this you write out the proof nicely and neatly. Making sure that you present you work well forces you to think about what you are writing down. The investment in your hard work writing, pays off for the people reading your work.

> Easy reading is damned hard writing.
>
> —Nathaniel Hawthorne

Learning to write proofs takes time and effort. But the rewards are well worth it.

Seçkin Demirbaş and Andrew Rechnitzer

Contents

Chapter 1

Sets

All subjects have to start from somewhere, and we'll start our work at sets. The authors believe that you, the reader, will have all seen some basic bits of set-theory before you got to this text. We hope we can safely assume[4] that you have at least some passing familiarity with sets, intersections, unions, Venn diagrams (those famous overlapping circle pictures), and so forth. Based on this assumption, we will move quite quickly through an introduction to this topic and do our best to get you to new material. We really want to get you proving things as quickly as possible.

Set theory now appears so thoroughly throughout mathematics that it is difficult to imagine how Mathematics could have existed without it. It might be surprising to note that set theory is a much newer part of mathematics than calculus. Set theory (as its own subject) was really only invented in the 19th Century — primarily by Georg Cantor[5] Really mathematicians were using sets well before then, just without defining things quite so formally.

Since it (and logic) will form the underpinning of all the structures we will discuss in this text it is important that we start with some definitions. We should try to make them as firm and formal as we can.

1.1 Not so formal definition

In mathematics and elsewhere[6] we are used to dealing with collections of things. For example

- a family is a collection of relatives.
- hockey team is a collection of hockey players.

[4]Assumptions can be dangerous, and in general we will avoid them, or at least do our best to be honest with the reader that we are making an assumption.

[5]A mathematician we will discuss in much more detail in footnotes much like this much later in the text when we get to the topic of Cardinality in Chapter Chapter 12. We will also try to reduce the overuse of the word "much" as much as possible.

[6]Including the so-called "real life" that non-mathematicians inhabit.

- shopping list is a collection of items we need to buy.

Let us give our first definition for the course. Now this one is not so formal — but it will be enough for our purposes[7].

Definition 1.1.1 (A not so formal definition of sets). A **set** is a collection of objects. The objects are referred to as **elements** or **members** of the set. ♢

One reason to be not-so-formal here, is that while the notion of a set is relatively simple and intuitive, it turns out that making the definition completely rigorous is quite difficult. The interested reader should search-engine their way to discussions of this point.

Now — let's just take a few moments to describe some conventions. There are many of these in mathematics. These are not firm mathematical rules, but rather they are much like traditions. It makes it much easier for people reading your work to understand what you are trying to say.

- Use capital letters to denote sets, A, B, C, X, Y etc.
- Use lower case letters to denote elements of the sets a, b, c, x, y.

So when you are writing a proof or just describing what you are doing then if you stick with these conventions people reading your work (including the person marking your exams) will know — "Oh A is that set they are talking about" and "a is an element of that set.". On the other hand, if you use any old letter or symbol might be correct, but it can be unnecessarily confusing for the reader[8]. Think of it as being a bit like spelling — if you don't spell words correctly people can usually understand what you mean, but it is much easier if you spell words the same way as everyone else[9].

We will encounter more of these conventions as we go — another good one is

- The letters i, j, k, l, m, n usually denote integers.
- The letters x, y, z, w usually denote real numbers.

So — what can we do with a set? There is only thing we can ask of a set:

"Is this object in the set"

and the set will answer

"yes"

[7] Unfortunately the formal theory of sets gets very difficult very quickly and is well beyond the scope of this text. So rather than investing a large amount of time on the precise definition of **set**, we will make do with this one. It is better for us to just get on with learning how to give precise definitions of particular sets and how to work with them.

[8] While obfuscation can be useful in many endeavours, the authors do not know of any good reason to deliberately obfuscate your mathematics.

[9] Okay, maybe Noah Webster had some not completely unreasonable reasons for tweaking English spelling, but this author is not entirely convinced that quite so many z's are needed.

or

"no"

and nothing else. If you want to know more than just "yes" or "no", then you need to use with more complicated mathematical structures (we'll touch on some as we go along).

For example, if A is the set of even numbers we can ask "Is 4 in A" we get back the answer "yes". We write this as

$$4 \in A$$

While if we ask "Is 3 in A?" and we get back the answer "no". Mathematically we would write this as

$$3 \notin A$$

So this symbol "$\in$" is mathematical shorthand for "is an element of", while the same symbol with a stroke through it "$\notin$" is shorthand for "is not an element of". Similar "put a stroke through the symbol to indicate negation"-notation gets used a lot in different contexts and we'll see it throughout this text. While it is arguably not terribly creative, it is effective — perhaps because it isn't too creative.

This is standard notation — it is very important that you learn it and use it. Do not confuse the reader, or the person who marks your tests and exams, by using some variation of this. For instance, some of you may have previously used ε in place of $\in$ — please stop doing so. For most mathematicians, "$4\varepsilon A$" denotes the product of three things, while "$4 \in A$" is a mathematical sentence that tells us that the object "4" is a member of the set A.

1.1.1 Who is this reader you keep on mentioning?

We have referred to a "reader" several times in the text above but not really explained who we mean by "the reader". There are 3 different types of reader that we mean when we say "think about the reader": you, another person, and not-a-real-reader.

- You: Frequently, the only person who will read your mathematics is you. Your lecture notes, your homework drafts, your experimenting, etc — you typically don't show them to other people. For that sort of work ***in isolation*** it doesn't really matter too much if you don't use standard notation, take shortcuts, and a myriad of other things that people typically do to save time. However, if we only think of ourselves when we write then we can form many bad habits that we take with us when we write for other people. These shortcuts can be hard for other people to understand unless we take the time to explain them. Consequently, it is a good idea to avoid these habits even when writing for ourselves; your reader, and even your future self, will thank you.

- Another person: On many occasions another person will read your work — the most obvious being the person who marks your homework, tests and exams. Generally you will not be present while they read (and perhaps grade) your mathematics, so typically they will only be able to mark what you have written on the page; they cannot mark ***what you mean*** by what is on the page. So you need to make sure things are as clear as possible, so that what you have written conveys what you mean. If you are in the habit of using your own shorthand or definitions or notation, then you must make sure these are clearly explained.

- Not-a-real-reader: Finally, we should often think of a reader who isn't really a reader at all, but really just a mechanism we should use to decide if what we are writing is good enough. Our imaginary reader is intelligent, sensible, knows some mathematics (but not everything), and is a bit of an annoying pedant[10]. As we write we should think of this imaginary reader looking over our shoulder asking questions like "Is that the right notation?", "Is that clear enough?", "Does the logic flow in the right direction?" and offering advice like "Add another sentence to the explanation." and "Make sure you define that function."

As we continue along in this text we will keep referring to these readers and reminding you to think of them as you write. Communicating mathematics is a very important part of doing mathematics.

1.2 Describing a set

We really need to be able to describe and define lots of different sets when we are doing mathematics. It must be completely clear from the definition how to answer the question "Is this object in the set or not?"

- "Let A be the set of even integers between 1 and 13." — nice and clear.

- "Let B be the set of tall people in this class room." — not clear.

More generally if there are only a small number of elements in the set we just list them all out

- "Let $C = \{1, 2, 3\}$."

When we write out the list we put the elements inside braces $\{\cdot\}$. Do not use round, square or angle brackets — those things have other mathematical meanings — we must use braces or "curly brackets" if you like. Not that the order we write things in doesn't matter

$$C = \{1, 2, 3\} = \{2, 1, 3\} = \{3, 2, 1\}$$

[10] Is there any other sort of pedant?

because the only thing we can ask is "Is this object an element of C?" We cannot ask more complex questions like "what is the third element of C" — we require more sophisticated mathematical objects to ask such questions and we'll might get around to looking at such things later in the course.

Similarly, it doesn't matter how many times we write the same object in the list

$$C = \{1, 1, 1, 2, 3, 3, 3, 3, 1, 2, 1, 2, 1, 3\} = \{1, 2, 3\}$$

because all we ask is "Is $1 \in C$?". Not "how many times is 1 in C?" (you need a mathematical construction called a multiset to ask and answer this question).

Now — if the set is a bit bigger then we might write do something like this

- $C = \{1, 2, 3, \ldots, 40\}$ the set of all integers between 1 and 40 (inclusive).
- $A = \{1, 4, 9, 16, \ldots\}$ the set of all positive square integers

The "..." (ellipsis) is shorthand for the missing entries and tells us to follow the pattern as long as we can. You must be careful with this as you can easily confuse the reader if the pattern is not clear. That, in turn, that means that your set is not defined sufficiently precisely.

- $B = \{3, 5, 7, \ldots\}$ — is this all odd primes, or all odd numbers bigger than 1 or prime numbers that differ from a power of 2 by exactly 1?

Only use this where it is completely clear by context. A few extra words can save the reader (and yourself) a lot of confusion.

This is perhaps the most important set — many other important objects in mathematics can be built up from this.

Definition 1.2.1 Empty set. The **empty set** (or null set or void set) is the set which contains no elements. It is denoted $\varnothing$. For any object x, we always have $x \notin \varnothing$; hence $\varnothing = \{\}$. ◊

Notice that the empty set is not nothing — you should think of it as an empty bag. Be careful not to confuse it with the empty in the empty bag[11].

Example 1.2.2

- $A = \{1, 2, \varnothing\}$ — this set contains three elements; the numbers one and two and the empty set. A set can contain sets.
- $B = \{\varnothing\}$ — this set is not the empty set — it contains a single element, being the empty set. You can think of this set as being a bag that contains an empty bag.
- $C = \{\varnothing, \{\varnothing\}\}$ — this set contains two elements; the empty set and the set

[11]This potential confusion is akin to that caused by the number zero. How can something, that is "0", denote nothing? We recommend taking a little digression into this topic (with digressions into Parmenides, Leucippus, Democritus, Zeno, horror vacui, and many other topics) with your favourite search engine.

that contains the empty set (our set B above).

□

Now — this is all fine when the set doesn't contain too many elements. But for infinite sets or even just big sets we can't do this and instead we have to give the defining rule. For example the set of all positive even numbers we write as

$$S = \{x \mid x \text{ is even and positive}\} = \{2, 4, 6, 8 \dots\}$$

The second notation is also okay, but you have to be careful to make sure it is completely clear which set you are talking about. The first notation can be read as "S is the set of elements x such that x is even and positive". This is the standard way of writing a set defined by a rule. This sort of notation is sometimes called ***set-builder notation***.

$$\begin{aligned} S &= \{\text{some expression} \mid \text{some rule}\} \qquad \text{or} \\ &= \{\text{a function} \mid \text{a domain}\} \end{aligned}$$

The set of all primes is

$$S = \{p : p \text{ is prime}\}$$

the ":" is read as "such that" or "so that", and you will also often see

$$S = \{p \text{ s.t. } p \text{ is prime}\} = \{p \mid p \text{ is prime}\}$$

This author prefers " | " since it provides a clear (even physical) demarcation. You should recognise all three notations.

While set-builder notation avoids many problems of clarity, it does not avoid all problems. A very famous example is

$$S = \{A \mid A \notin A\}$$

ie. the set of all sets that do not contain themselves. This is a problem, because if $S \in S$, then according to the defining rule, it cannot be. On the other hand, if $S \notin S$ then it must be. Hence the rule is ambiguous. This is Russell's paradox. It is closely related to the sentence

This sentence is false.

One way around these problems is to avoid talking about self-referential objects — but this is way too heavy for the moment, and we should just get back to easier sets[12].

The empty set is one important set, here are a few more. What follows is not really a formal definition of these sets, rather it is here to remind the reader of some sets that they should already know and to highlight some standard notation that people use to refer to them.

[12]The book "Gödel, Escher, Bach: An Eternal Golden Braid" by Douglas Hofstadter is a wonderful exploration of topics related to Russell's paradox and much much more.

Definition 1.2.3 Some other important sets.

- Positive integers $\mathbb{N} = \{1, 2, 3, \dots\}$ — these are usually called the **natural numbers**.
- All **integers** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
- All **rational numbers** (fractions) $\mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{N} \right\}$
- All **real numbers** $\mathbb{R}$
- **Irrational numbers** $\mathbb{I}$ = real numbers that are not rational. Examples of this are $\sqrt{2}, \sqrt{3}, \pi, e$. Hence $\sqrt{2} \in \mathbb{R}$ and $\sqrt{2} \notin \mathbb{Q}$, so $\sqrt{2} \in \mathbb{I}$ (we'll get around to proving this later in the text).

$\Diamond$

Here are some points to note about the above definition.

- Unfortunately there is often confusion as to whether or not zero should be included in the set of "natural numbers". This text will not include zero as is, in the experience of the authors at least, the more common mathematical convention. If you work in formal logic, set theory or computer science, then often zero is included in the set. The number zero has an interesting history in mathematics and the reader should search-engine their way to articles on that history. Often its use in mathematics was complicated by the question "how can nothing be something?"
- We can also define the set of rational numbers as $\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$. The authors prefer the one given in the definition; it feels a little "cleaner" in that we represent a third as $\frac{1}{3}$ rather than $\frac{-1}{-3}$. Of course, we can also represent $\frac{1}{3} = \frac{2}{6}$. We discuss this more in Example 9.2.7.
- We must use "blackboard bold" to denote these sets.

$$\mathbb{N} \neq N, \qquad \mathbb{R} \neq R.$$

  Notice here, these are not written in a normal bold font, but rather they are written in "blackboard bold" so that certain lines, usually the vertical ones, are doubled. This style of writing came from the need to distinguish between regular and bold face letters when writing on blackboards (as mathematicians are want to do). It eventually became standard notation both on boards and in print. Since it is now quite standard notation, please learn it and use it. Do not confuse the reader.
- Also note that $\mathbb{I}$ is not a very standard notation (though we might use it from time to time in this text) — the others are standard. This is in part because the set of irrational numbers has some pretty ugly properties — it is not closed under addition or multiplication; you can take two irrational numbers and add them to get a rational number; you can take two irrational

numbers and multiply them to get a rational number. The other sets are closed under addition and multiplication.

So now using these as standard sets we can start to build more interesting things

- Even integers

$$\begin{aligned} E &= \{n \mid n \text{ is an even integer}\} \\ &= \{n \mid n = 2k \text{ for some } k \in \mathbb{Z}\} \\ &= \{2n \mid n \in \mathbb{Z}\} \end{aligned}$$

- Square integers $S = \{n^2 \mid n \in \mathbb{Z}\}$.

One obvious question that one can ask about a set is "How many elements are there in it?" — there is quite a bit more to this question than you might think.

Definition 1.2.4 For a set S we write $|S|$ to denote the **cardinality** of S or its **cardinal number**. For finite sets, $|S|$ is just the the number of elements in S. We extend this concept to infinite sets in Chapter 12. ◊

Hence $|\varnothing| = 0$, and $|\{1, 2, \{\varnothing\}\}| = 3$. For (small) finite sets we can just list things out, but for larger sets it gets very difficult very quickly, and for infinite sets things become very weird. One thing we will study in this course is the size of sets — in particular we show that

- $|\mathbb{N}| = |\mathbb{Z}|$
- $|E| = |\mathbb{Z}|$
- $|\mathbb{Z}| = |\mathbb{Q}|$
- $|\mathbb{Z}| < |\mathbb{R}|$

These statements are really very strange and we need to build up some mathematical infrastructure to make sense of them. Notice that the first and second statement tell us that there are two infinite sets (positive integers and all integers), where one is a strict subset of the other, but they are actually the same size! The last statement is even stranger — it tells us that there are two infinite sets (integers and reals) that are definitely not the same size. This implies that there is more than one sort of infinity. Before we are done we will actually prove that there are an infinite number of different infinities!

1.3 Onward

Of course there is much more to be done with sets, however we'd really like to get into logic and proving things as quickly as possible. So we'll stop our discussion of sets for now and come back later armed with more logic and some proof ideas.

1.4 Exercises

1. Write the following sets by listing their elements.

(a) $A_1 = \{x \in \mathbb{N} \text{ s.t. } x^2 < 2\}$.

(b) $A_2 = \{x \in \mathbb{Z} \text{ s.t. } x^2 < 2\}$.

(c) $A_3 = \left\{x \in \mathbb{N} \text{ s.t. } x = 3k = \frac{216}{m} \text{ for some } k, m \in \mathbb{N}\right\}$.

(d) $A_4 = \left\{x \in \mathbb{Z} \text{ s.t. } \dfrac{x+2}{5} \in \mathbb{Z}\right\}$.

(e) $A_5 = \{a \in B \text{ s.t. } 6 \leq 4a + 1 < 17\}$, where $B = \{1, 2, 3, 4, 5, 6\}$.

(f) $A_6 = \{x \in B \text{ s.t. } 50 < xd < 100 \text{ for some } d \in D\}$, where $B = \{2, 3, 5, 7, 11, 13, \ldots\}$ is the set of primes and $D = \{5, 10\}$.

(g) $A_7 = \{n \in \mathbb{Z} \text{ s.t. } n^2 - 5n - 16 \leq n\}$.

2. We are going to write the following sets in set builder notation.

(a) $A = \{5, 10, 15, 20, 25, \ldots\}$.

(b) $B = \{10, 11, 12, 13, \ldots, 98, 99, 100\}$.

(c) $C = \{0, 3, 8, 15, 24, 35, ..\}$.

(d) $D = \left\{\ldots, -\frac{3}{10}, -\frac{2}{5}, -\frac{1}{2}, 0, \frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \ldots\right\}$.

(e) $E = \{2, 4, 16, 256, 65536, 4294967296, \ldots\}$.

(f) $F = \{2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 32, 36, \ldots\}$.

3. In each of the following parts, a set is defined in one of three ways: (1) listing elements between braces, (2) using set builder notation, or (3) describing in words. Rewrite each set in the two forms of which it is not already given. For example, since the set in part (a) is given by method (1), write the same set using methods (2) and (3). As another example, since the set in part (c) is given by method (2), write the same set using methods (1) and (3).

(a) $A = \{0, 2, 4, 6, \ldots, 100\}$

(b) $B = \{3, 9, 27, 81, \ldots\}$

(c) $C = \{m \text{ s.t. } m \in \mathbb{Z},\ |m| \leq 3\}$

(d) $D = \{4k + 1 \text{ s.t. } k \in \mathbb{Z}\}$

(e) The set E of all numbers that are the reciprocal of a natural number.

(f) The set F of all integers that are two more than a (possibly negative) multiple of 5.

4. Consider the following ill-defined set: $S = \{2, 4, \dots\}$. Show that the definition of S is ambiguous by providing two different ways that you could interpret its definition.

5. Consider the set

$$\{2n + 1 : n \in \mathbb{N}\}.$$

Explain what is wrong with each of the expressions below and why they should not be used to denote this set.

(a) $A = \{2k + 1\}$

(b) $B = \{2j + 1 : j \in N\}$.

(c) $c = \{2\ell + 1 : \ell \in \mathbb{N}\}$

(d) $D = \{2k + 1 : n \in \mathbb{N}\}$

(e) $E = \{2m + 1 : m\epsilon\mathbb{N}\}$

(f) $F = \{2N + 1 : N \in \mathbb{N}\}$

(g) $G = \{2m + 1 : m\varepsilon\mathbb{N}\}$

(h) H = {2n+1 : n in N}

6. Are the following statements true or false?

(a) $\varnothing = \{0\}$

(b) $\varnothing = \{\varnothing\}$

(c) $|\varnothing| = 0$

(d) $\{\{\varnothing\}\} = \{\varnothing\}$

(e) $\{\varnothing\} = \{\{\}\}$

7. Show that each of the numbers

$$a = 2, \qquad b = 8, \qquad \text{and} \qquad c = -12$$

do not belong to any of the following sets:

$$A = \left\{-\frac{1}{n} \text{ s.t. } n \in \mathbb{N}\right\} \qquad B = \left\{x \in \mathbb{R} \text{ s.t. } x \geq 0,\ x^2 > 100\right\}$$
$$C = \{\{2\}, \{8\}, \{-12\}\} \qquad D = \{4k \text{ s.t. } k \in \mathbb{N},\ k \text{ odd}\}$$

8. Are the following sets equal?

(a) $\mathbb{Z}$ and $\{a : a \in \mathbb{N} \text{ or } -a \in \mathbb{N}\}$

(b) $\{1, 2, 2, 3, 3, 3, 2, 2, 1\}$ and $\{1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3\}$

(c) $\{d : d$ is a day with 40 hours$\}$ and $\{w : w$ is a week with 6 days$\}$

(d) $\{p : p \text{ is prime}, p < 42\}$ and $\{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41\}$

9. Determine which of the following sets are equal to the set $S = \left\{\frac{1}{n} \text{ s.t. } n \in \mathbb{N}\right\}$.

$$A = \left\{\frac{1}{n+1} \text{ s.t. } n \in \mathbb{N}\right\} \qquad B = \left\{\frac{1}{|n|} \text{ s.t. } n \in \mathbb{Z},\ n \neq 0\right\}$$

$$C = \left\{\frac{2}{k} \text{ s.t. } k \in \mathbb{N},\ k \text{ even}\right\} \qquad D = \left\{\frac{a}{b} \text{ s.t. } a, b \in \mathbb{N}\right\}$$

$$E = \left\{\frac{1}{n-1} \text{ s.t. } n \in \mathbb{N},\ n > 1\right\} \qquad F = \left\{\frac{1}{m} \text{ s.t. } m \in \mathbb{Z},\ m > 0\right\}$$

Chapter 2

A little logic

One of the main things we are trying to do in mathematics is prove that a statement is always true. A simple example of this is the sentence

> The square of an even number is even.

More generally we might try to show that

- a particular mathematical object has some interesting properties,
- when an object has property 1 then it always has property 2,
- an object always has property 3 or property 4 but not both at the same time, or
- you cannot find an object that has property 5.

Let's spend a little time on this simple example of squaring even numbers and explore why it is true.

- A number n is even when we can write it as $n = 2k$ where k is an integer.
- This tells us that the square of that number is $n^2 = (2k)^2 = 4k^2$.
- But now we see that n^2 can be written as two times another number: $4k^2 = 2(2k^2)$.
- And since $2k^2$ is an integer, we know that n^2 is also even.

Notice that there is quite a lot going on here — a mixture of definitions, language and logic.

Most obviously (we hope) is that we have to understand what **even** means, so we need to define it. Despite us all being quite familiar with even and odd numbers, we should define it. We should not expect that everyone has exactly the same understanding of **even** since your readers can come from extremely diverse backgrounds. This author has encountered students who were taught at school that the number 0 is neither even nor odd, and others that were taught that only positive numbers can be even or odd. To avoid potential confusion we'll use the following

> A integer is **even** when it is equal to two times another integer, and an integer is odd when it is not even.

We'll come back to this definition in the next chapter after we have done a little more logic. We'll also make it a proper formal definition with bold-text and reference numbers and so on.

Our explanation then consists of sentences asserting bits of mathematics. The sentences are arranged in a particular order and cannot be shuffled around; each one *implies* the next. To be a good explanation we should take care of our reader and use clear language and enough detail so they can follow along. To make the language flow a little more easily we connect the sentences with words and phrases like "hence", "this tells us that", "because of this we can write". These words and phrases are not just there to make the reader feel a little more comfortable, they also help us to emphasise the *logical* connections between the sentences to the reader. At the same time, we can expect our reader to do some work; they should be able to understand standard notation, do basic arithmetic and algebra, etc.

In this chapter we won't do much proving of things, but instead we will focus on basic mathematical sentences and how we combine them together using logic.

2.1 Statements and open sentences

When you pick up a piece of mathematics you find that it is made up of **declarative sentences** — sentences that declare something.

> The number $\sqrt{2}$ is not a rational number.

> The number 17 is even.

The first sentence is **true** (we will prove it later in the text) and the second is **false**. These are sentences that can be assigned a definite **truth value** — they are either true or false. We will usually denote these T and F (and so save ourselves the burden of writing the other 7 characters). A declarative sentence that can be assigned a truth value is called a **statement**.

The reader will have noticed that the definitions in the previous paragraph are not very formal or precise. Since the authors have been emphasising the importance of being careful and precise, this seems a touch hypocritical[13]. However, giving a precise formal definition of mathematical statement turns out to be quite a lot like the problem of giving a precise formal definition of sets — very difficult and lies beyond the scope of this book [14]. So please excuse the (little)

[13] Hypocrisy starts at home.

[14] The interested reader should search-engine their way to articles on the foundations of mathematics, mathematical logic, Richard's paradox, the Berry paradox, and many other interesting topics that, unfortunately, lie outside the scope of this text (since it must have a finite length).

hypocrisy and we'll just stick with this less formal and more intuitive definition of statement.

Sentences like

> I am tall

and

> This sentence is false

are not statements since we cannot decide their truth value — they are neither true or false. In the first case it is because we don't actually know who "I" is. Is it the reader or is it the author? Further, we don't have a very precise definition of "tall" Indeed the notion of what height constitutes "tall" or "short" can vary dramatically between populations [15]. The second sentence is a little more difficult. If it is true, then it tells us it must be false — how can it be both? While if it is false, then that implies it must be true — again, how can it be both? This sort of self-referential sentence is very difficult to work with, so we will avoid them.

Self-referential statements. Self-referential statements are very interesting, but we should really start with simpler and straight-forward statements before rushing into difficult and confusing ones. The interested reader should search-engine their way to the related topic of Russell's paradox. This involves the set

$$R = \{X \mid X \notin X\}.$$

This is the set of all sets that do not contain themselves. The paradox occurs when we ask if R is an element of itself? If it is, then by the definition of R it cannot be. And if it is not, then by definition it must be.

The barber-paradox is similar, but more hirsute.

Here are some simple examples of mathematical statements:

- The 100th decimal digit of π is 7.
- The square of the length of the hypotenuse of a right angle triangle is equal to the sum of the squares of the lengths of the other two sides.
- Every even integer greater than 2 can be written as the sum of two primes.

The first might be true or might be false, but it must either be true or false[16]. The second is perhaps the most famous theorem any of us know. The last statement is **Goldbach's conjecture** and it is not known whether it is true or false. However it is still a statement because it must either be true or it is false.

[15] At the time of typing this author looked up wikipedia to find that the average height of a man in the Netherlands is about 184cm, while in Vietnam it is about 162cm. Quite a sizeable difference. Sorry for the pun.

[16] Actually it is true, though you have to be careful how you count — since the number starts 3.141... we have counted the initial "3" as the first decimal digit. On the other hand, if you start counting from the first "1", then the 100th digit is "9". Definitions matter.

On the other hand, a sentence like

$$x^2 - 5x + 4 = 0$$

is not a statement because its truth value depends on which number x we are discussing. In order to assign a truth value we need to know more about x. Such sentences are called **open sentences**. If we assign a value to x then the open sentence will be either true or false and so become a statement. Usually this variable will come from some predefined set — its "domain". Often this is the integers or the reals, but typically we should make sure it is clear to the reader. This sentence, $x^2 - 5x + 4 = 0$, taken over the domain of the integers is true when $x = 1, 4$ and otherwise false. We'll come back to open sentences in Chapter 6.

We now need to start playing with these statements in a more abstract way. This will allow us to talk more generally about doing operations on statements — either operations that act on a single statement or operations that act on pairs of statements. I won't care too much about the details of the statement ("It is Tuesday" or "I can write with my left-hand"), but rather just its truth value (true and false). So much as we write an integer as n or m, a real number as x or y, I will write a statement as P, Q or R. As for the open sentences, we will use the notation $P(x), Q(x), R(x)$, since their truth values depend[17] on the value of x. This is reminiscent of a function; we put in some value for x and the sentence returns to us a statement. For example, if $P(x) : x^2 - 5x + 4 = 0$, we see that $P(1)$ is true, while $P(2)$ is false.

2.2 Negation

Given a statement, P, we can form a new statement which is the *negation* of the original, which we denote $\sim P$; this little squiggle is called a tilde.

Definition 2.2.1 Let P be a statement. The **negation** of P is denoted $\sim P$. When the original statement P is true, the negation $\sim P$ is false. And when the original statement is false, the negation is true. ◊

You will also see the negation written as $!P$ or $\neg P$. Since all three are quite commonly use, you should recognise all three. To not unduly confuse your reader, you should pick one and stick with it. You should recognise all three notations, as all three are in common use; we'll use the tilde notation in this text[18].

- The negation of "It is Tuesday" is "It is not Tuesday"

[17] When the open sentence depends on more than one variable, say x, k we will write $P(x, k), Q(x, k)$ and so on.

[18] This notation for the negation of a statement goes back at least as far as an Giuseppe Peano (1897) and Bertrand Russell (1908). The use of $\neg$ is due to Arend Heyting (1930) — many thanks to this website. The authors could not track down the earliest use of ! to denote the negation, but we do note that it is very commonly used in programming languages.

- The negation of "I can write with my left hand" is "I cannot write with my left hand". [19]
- The negation of "The integer 4 is even" is "The integer 4 is not even" or better yet "The integer 4 is odd"[20].

For our general statement P we can summarise its truth values and the corresponding truth values of its negation in a table:

P	$\sim P$	$\sim (\sim P)$
T	F	T
F	T	F

This table is called a truth table and we'll use them quite a bit. They can be a bit dull and mechanical to use, but they make the truth values very clear and precise and can help us reduce the problem of understanding the truth value of some complicated combination of statements to a simple procedure of filling in entries of a table.

We have included a column for the double-negation of a statement, $\sim (\sim P)$. Notice that the truth values of the double-negation are the same as those of the original statement. It is related to the law of the excluded middle — a statement is true or its negation is true — there is no third (middle) option. Thus the mean of negations in mathematics is quite different from what can happen in written and spoken English[21]. Also notice that if we only have the negation to play with then we cannot really do very much at all. We need some ways of combining statements. To do this we start with the logical "conjunction" and "disjunction" — "and" and "or".

2.3 Or and And

So the two simplest ways of combining two logical statements are using "or" and "and" . The words "or" and "and" have precise mathematical meanings which sometimes differ from their use in day-to-day language. To avoid conflating these mathematical means with the colloquial meanings we can refer to "or" and "and" by the nicely latin-flavoured words "disjunction" and "conjunction";

[19] The statement "I can write with my right hand" is not the negation of "I can write with my left hand". Just because someone cannot write with the left-hand does not mean that they can write with their right. For most of human history people could not write with either hand.

[20] In this case, because 4 is an integer we know that if it is not even then it must be odd. However, this is not that case for non-integers. For example, the negation of "π is even" is "π is not even" rather than "π is odd". We'll come back to even and odd in Chapter 3.

[21] In written and spoken English a double-negation can sometimes a negation, "We don't need no education."; sometimes it is ambiguous: "I do not disagree."; and sometimes positive: "The time you have is not unlimited.". In many languages a double-negation serves as a means of emphasising the negation. "Yeah, right" is a good example of a double-positive being a (sarcastic) negative.

hopefully we won't need those for very long and we'll get used to being more precise about when we want mathematical "and" and "or" and when we are just being colloquial.

Definition 2.3.1 Let P and Q be statements.

- The **disjunction** of P and Q is the statement "P or Q" and is denoted $P \vee Q$. The disjunction is true if at least one of P and Q are true. The disjunction is only false if both P and Q are false.
- The **conjunction** of P and Q is the statement "P and Q" and is denoted $P \wedge Q$. The conjunction is true when both P and Q are true. It is false if at least one of P and Q are false.

The truth tables of the disjunction and conjunction are

P	Q	$P \vee Q$	$P \wedge Q$
T	T	T	T
T	F	T	F
F	T	T	F
F	F	F	F

$\Diamond$

Be careful to use the correct notation. The symbols $\vee$ and $\wedge$ should not be confused or interchanged with the symbols for unions and intersections, $\cup$ and $\cap$. We'll come back to unions and intersections later in the text.

Notice that this use of "or" defined above is different from how we often use "or" in spoken English. When you are on a flight and the attendant offers you a meal (assuming you are on a long flight that still offers such luxuries) you might be asked

> "Would you like chicken or beef"?

You are not being offered both; you get at most one. This is an example of "exclusive or" — one or the other, but not both. The mathematical "or" we have just described above is "inclusive or" — at least one of the two options. You should assume that when we write "or" in a mathematical context we will mean *inclusive or*. To refer to exclusive or we will typically write "either … or … but not both". If in doubt use more words to clarify things rather then save yourself a few symbols at the expense of your reader's understanding.

The use of "and" in English can also have subtle differences from the mathematical conjunction $\wedge$. For example, "and" can sometimes imply an order: "He lived and he died" is more natural than "He died and he lived". The mathematical and, by contrast, doesn't care about order: $P \wedge Q$ has the same truth table as $Q \wedge P$.

For example, take the statements "7 is prime" and "18 is odd". We can now construct a new statement

7 is prime *and* 18 is odd

Since the first statement is true and the second is false, the conjunction of the two (our new statement) is false. On the other hand

7 is prime and 18 is even

is a true statement. Similarly the statement

7 is prime or 18 is odd

is true.

"Not", "and" and "or" are three logical connectives — or logical operators. They take one or two statements and combine them to make new statements — called "compound statements". Using "not", "and" and "or" you can construct any truth table of two statements you might want (there are $2^4 = 16$ of them). If you have done some computer science you have perhaps heard of NAND (not and), NOR (not or), XOR (exclusive or) and XNOR (exclusive not or). We'll shortly see how to construct such things using the three connectives we have just defined. But first we'll introduce the logical operator that lies at the heart of most of the mathematical proofs that are coming.

2.4 The implication

In mathematics we make many statements of the form

If x is a real number then x^2 is a real number

and a large fraction of the theorems we want to prove are of this form.

Definition 2.4.1 For statements P and Q, the **implications** or **conditional** is the statement

$$\text{if } P \text{ then } Q$$

and is denoted $P \implies Q$. In this context P is called the **hypothesis** and Q is called the **conclusion**. The implication is false when P is true and Q is false; otherwise it is true. The truth table is

P	Q	$P \implies Q$	$(\sim P) \vee Q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

$\Diamond$

Note 2.4.2 It is important to note that $P \implies Q$ has the same truth table as $(\sim P) \vee Q$. Additionally — the truth table is not symmetric in P and Q and

hence the statements $P \implies Q$ and $Q \implies P$ have different truth tables. See the middle two rows of the table below.

P	Q	$P \implies Q$	$Q \implies P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Further note that sometimes the hypothesis is called the **antecedent** while the conclusion is called the **consequent**[22].

Before we get to the peculiarities of the truth table, we should note that the statement "$P \implies Q$" can be read in many different ways. Some of these are (in the opinion of the authors) a little obfuscating, and we recommend to stick with "If P then Q".

- if P then Q
- P implies Q
- whenever P then also Q

The author likes the statements above because the hypothesis comes before the conclusion. One may sometimes also see the implication $P \implies Q$ written as

- P only if Q
- Q if P
- Q whenever P
- Q provided that P

The authors recommend that you avoid these (at least until you have a bit more experience with mathematical proofs) since they write the conclusion before the hypothesis. They make it very easy to confuse the flow of logic. The statement "Q if P" is particularly confusing and, in this author's opinion, should be avoided.

Many mathematicians like to use the terms "necessary" and "sufficient" when writing implications. These terms allow one to put more emphasis on either the hypothesis or conclusion — this can be quite useful depending on the context in which you are writing. Consider again our nonsense example

> If he is Shakespeare then he is dead

It is *sufficient* to check that a given person is Shakespeare to decide that they must be dead. Similarly, someone is *necessarily* dead in order for them to be Shakespeare You may see the implication $P \implies Q$ written in the following ways

[22] In a more linguistic context, the hypothesis is the **protasis** and the conclusion is the **apodosis**.

- P is a sufficient condition for Q,
- P is sufficient for Q,
- Q is a necessary condition for P, or
- Q is necessary for P.

The first two of these emphasise that if you want to know that Q is true, then one need only check (ie it is *sufficient* to check) that P is true. While the last two emphasise that the truth of Q is required (it is*necessary*) for P to be true.

Now back to the truth table. At first glance it can seem a bit strange so to get the feel of it we'll use it on a couple of simple examples and then apply it to something a little larger. Consider the statements $P : 13$ is even and $Q : 7$ is odd. The statement $P \implies Q$

Different conditionals. The interested reader should look up different types of conditionals:

- the material conditional (which is the implication we discuss here)
- the indicative conditional (we really should understand that while sentences like "If 4 is a square then the sky is blue" are true, the truth of the hypothesis has nothing to do with the truth of the conclusion),
- the counterfactual conditional (the dreaded subjunctive mood is lurking here and the author dare not reveal too much of their ignorance).

Of course, one can go a long long way down the rabbit hole when you start looking into this sort of thing (see this[23] and this[24] amongst many other distractions).

> If 13 is even then 7 is odd

is true, while $Q \implies P$

> If 7 is odd then 13 is even

is false.

A more sizeable example which appears in a few textbooks in similar forms. Frequently a student will ask their instructor

> Will I pass this course?

and the author (sometimes) responds

> If you pass the exam, then you will pass the course.

[23] wikipedia.org/wiki/Down_the_Rabbit_Hole
[24] xkcd.com/214/

Under what circumstances is the author lying or telling the truth?

The author has definitely lied if you pass the exam but end up failing to course. The other 3 possible outcomes are consistent with them telling the truth. Let's explore with the caveat that this should not be considered a binding discussion about your actual passing or failing of a given course. Use P to denote "The student passes the exam", and Q to denote "The student passes the course", so we can write my statement as $P \implies Q$:

"If the student passes the exam, then the student passes the course."

(T,T) Say the student passes the exam and passes the course, then clearly $P \implies Q$ is true.

(T,F) Say the student passes the exam, but fails the course. Clearly the statement is wrong and so $P \implies Q$ is false. The author lied!

(F,T) Say they failed the exam, but passed the course. Well this is indeed possible — perhaps the exam was very nasty and the author was very impressed by the only-just-fail (and maybe some good homework) and so gave a passing mark overall. The statement is not false and so must be true. $P \implies Q$ is true.

(F,F) Say the student fails the exam and fails the course. Well the statement is not false and so must be true. Hence $P \implies Q$ is true.

Another good example (which the author has used quite a lot when teaching) is

If he is Shakespeare then he is dead.

—No one ever actually said this (well — except just now)

(T,T) Here is Shakespeare and, sure-enough, he is looking pretty dead [25]. So the implication above is not a lie.

(T,F) I found Shakespeare and he is up and about looking very well! The implication above is wrong and $P \implies Q$ is false. Also we should all learn what his secret is since he is over 450 years old!

(F,T) Here is Chrisopher Marlowe [26] and he is not-alive. This does not actually invalidate the implication — it is still true.

[25] If one is not careful, you can end up on a long digression into dead parrots, pet-shop sketches, and the history of British comedy around here.

(F,F) Consider a very alive modern writer — despite their accomplishments they are not Shakespeare. [27]. Again, this does not invalidate the implcation —so it is still true.

So perhaps the most important thing to re-emphasise at this point is that an implication statement is false only when it fails to deliver it's claim. That is, when we find a situation in which the hypothesis is true but the conclusion is false.

As we noted above, a very large number of mathematical statements we want to prove take the form of an implication. For example:

If n is even then n^2 is even

When we construct the proof of such a statement we need to demonstrate that it is *always true and never false*. To understand how we do so, consider again the four rows of the truth table.

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

Notice that the implication is true in three of those cases and only false in one case — namely when the hypothesis is true and the conclusion is false. So our proof needs to show that this possibility cannot happen.

Hypothesis false We can see from the truth table that the implication is always true. We don't need to know anything about the conclusion since it doesn't matter whether or not the conclusion is true or false. Because of this, a proof doesn't actually have to consider this possibility explicitly. Anyone reading the proof knows[28] the truth-table of the implication and so also knows that the implication is true when the hypothesis is false.

Hypothesis true We will have to work, since the truth-value of the implication will depend on the truth value of the conclusion. Consequently most proofs start with the assumption that the hypothesis is true and then work towards showing that the conclusion must be true.

[26]There is (was?) a theory that Marlowe faked his own death and then started writing under the name "William Shakespeare" — though this is not widely accepted. People who promulgated this theory were called Marlovians.You can search-engine your way to some interesting articles on Marlowe.

[27]I'm sure you can find a few with the help of your favourite search-engine or online purveyor of books. You might even be lucky enough to study on a campus that that has a bookstore that still sells actual books.

[28]Should know.

Note 2.4.3 Note that because we want the implication to be true always, when we have an implication involving open sentences, such as

If n is even then n^2 is even

we want this to be true for every possible choice of n in the domain (in this case, integers). Typically, we do not write

For every possible n, if n is even then n^2 is even

but instead assume that the reader will understand (by context) that we mean for every possible n.

2.5 Modus ponens and chaining implications

2.5.1 Modus ponens

Once we have proved an implication

$$P \implies Q$$

to be always true, then we would like to make use of it. We will (soon) prove that the implication

If n is even then n^2 is even

is always true. Since we know it cannot be false, we'll write down the relevant 3 rows of its truth table and suppress the 1 row corresponding to the implication being false:

P	Q	$P \implies Q$
T	T	T
F	T	T
F	F	T

Notice that if we take a number like $n = 14$, then we know it is even and so the hypothesis is true. By the above truth-table we know that the conclusion must also be true — the number $n^2 = 14^2$ is even. We don't have to do any more work; the truth table, and our proof, ensure that the conclusion is true.

More generally, if we have proved that

$$P \implies Q$$

is always true, then if we know the hypothesis P is true, then the conclusion Q must also be true.

Definition 2.5.1 Modus ponens. The deduction

- P implies Q is true, and
- P is true
- hence Q must be true.

is called **modus ponens**. ◊

This logical deduction was first formalised by Theophrastus [29].

Notice that if we have proved the implication $P \implies Q$ to be true, but the hypothesis P is false, then we ***cannot*** conclude anything about the truth value of the conclusion. When the hypothesis is false, the truth value of the conclusion doesn't matter — the implication is still true. We can verify that by considering the relevant two rows from the truth-table of the implication.

P	Q	$P \implies Q$
F	T	T
F	F	T

Similarly notice that if we have proved the implication $P \implies Q$ to be true, and we have proved the conclusion Q to be true then we cannot conclude anything about the truth value of the hypothesis P. Again, this is easily verified by considering the relevant two rows from the truth-table of the implication.

P	Q	$P \implies Q$
T	T	T
F	T	T

There is, however, one more instance in which we can make a valid conclusion. Consider again the truth-table when the implication $P \implies Q$ is true but the conclusion Q is false:

P	Q	$P \implies Q$
F	F	T

Here the only possibility is that the hypothesis must be false. This allows us to make another valid deduction.

Definition 2.5.2 Modus tollens. The deduction

- P implies Q is true, and
- Q is false
- hence P must be false.

is called **modus tollens**. ◊

So when we know (back to our silly example) that

[29] Theophrastus was a student of Plato, a contemporary of Aristotle, and wrote on everything from botany to logic. Given the fragmented historical record, it is perhaps safer to write that the first record that we have of the formal statement of modus ponens comes from Theophrastus.

> If he is Shakespeare then he is dead.

then we can conclude that any live person is not Shakespeare. We will come back to modus tollens a little later in Section 2.6 when we examine the contrapositive.

2.5.2 Affirming the consequent and denying the antecedent

Misapplication of modus ponens is a frequent source of logical errors. An extremely common one is called "affirming the consequent".

Warning 2.5.3 A common logical error. The ***false*** deduction

- P implies Q is true, and
- Q is true,
- and hence P must be true

is called **affirming the consequent**.

The flow of logic is wrong — check the truth table. Also notice that our arrow notation for the implication, $\implies$, helps to remind us that truth should flow from the hypothesis to the conclusion and not the other way around.

To see just how wrong this can be, consider again the true implication

> If he is Shakespeare then he is dead.

If we were to affirm the consequent, then any dead man must be Shakespeare.

Affirming the consequent does occasionally get used as a rhetorical technique (especially by purveyors of nonsense):

> If they are Galileo then they are supressed. I'm being supressed, so I must be Galileo.

or (when being a bit sorry for oneself)

> If they are a great artist then they are misunderstood. I'm misunderstood so I must be a great artist.

and the social-media comment section fallacy (with thanks to this comic[30]):

> If I tell the truth then I will offend people. I am offending people, so I must be telling the truth.

So be careful of affirming the consequent — it shows up a lot and is always fallacious.

A very similar logical error is called "denying the antecedent"

Warning 2.5.4 Another common logical error. The ***false*** deduction

- P implies Q is true, and

[30] www.smbc-comics.com/comic/the-offensive-truth

- P is false,
- and hence Q must be false

is called **denying the antecedent** and is a misapplication of modus tollens.

Here are some examples:

> If I have been to Toronto then I have visited Canada. I have not been to Toronto. So I have not visited Canada.
>
> If he is Shakespeare then he is dead. Abraham Lincoln is not Shakespeare, so he must be alive.
>
> If tastes bad then it must be healthy. This tastes good, so it must be unhealthy.

2.5.3 Chaining implications together

When we construct a proof that $P \implies Q$ is true, we don't do it in one big leap. Instead we break it down into a sequence of smaller (and easier) implications that we can chain together. To see how this works, consider the following:

Result 2.5.5 *Let P, Q and R be statement. Then the following statement is always true:*

$$\Big((P \implies R) \wedge (R \implies Q)\Big) \implies (P \implies Q).$$

Since we are going to need to refer to this piece of mathematics a few times in this section, we have take the trouble to format it clearly and given it a number.

This result is an example of a tautology, a statement that is always true. We will come back to tautologies later in the text. To show that it is always true we could either build the truth-table, or we can do some reasoning. Both of these methods are *proofs*, but we won't be quite so formal until the next chapter. The truth-table is not hard to construct but a bit tedious; since each of P, Q, R can either be true or false, there are $2^3 = 8$ rows to consider:

P	Q	R	$P \implies R$	$R \implies Q$	$P \implies Q$	The statement Result 2.5.5
T	T	T	T	T	T	T
T	T	F	F	T	T	T
T	F	T	T	F	F	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	T	F	T	T
F	F	F	T	T	T	T

The above is a perfectly reasonable way to show that the statement is always true. However, one can do the same just by a little reasoning; it also has the benefit of improving our understanding of the statement. We'll present the argument in dot-point form:

- The statement is an implication with hypothesis $((P \implies R) \wedge (R \implies Q))$ and conclusion $(P \implies Q)$. An implication is false when the hypothesis is true but the conclusion is false, and otherwise the implication is true.
- Since the conclusion, $(P \implies Q)$, is itself an implication, it can only be false when its hypothesis is true and its conclusion is false. So we must have P is true, but Q is false.
- In order for the hypothesis to be true, both implications, $(P \implies R)$ and $(R \implies Q)$, must be true (since a conjunction of two statements is only true when both statements are true).
  - Since P is true, and we require $(P \implies R)$ to be true, we must have R is true.
  - Since Q is false and we require $(R \implies Q)$ to be true, we must have R is false.

  But since R is a statement it cannot be true and false at the same time.
- So there is no way for us to make the statement false. Since it is never false, it must always be true.

So back to the statement $(P \implies Q)$. We'd like to show it is always true, but we cannot do it in one big leap. Instead, assume that we can make two smaller steps and prove that the two implications $(P \implies R)$ and $(R \implies Q)$ are always true. The conjunction of two implications, $(P \implies R) \wedge (R \implies Q)$, is also true and is exactly the hypothesis in Result Result 2.5.5. Since the implication in Result Result 2.5.5 is always true and its hypothesis is true — modus ponens — its conclusion must be true.

So while we could try to prove $(P \implies Q)$ is true in one big leap, it is sufficient to instead prove it is true in two smaller steps $(P \implies R)$ and $(R \implies Q)$. More generally when we prove $(P \implies Q)$, we will instead prove a sequence of implications:

$$\begin{aligned} P &\implies P_1 && \text{and} \\ P_1 &\implies P_2 && \text{and} \\ P_2 &\implies P_3 && \text{and} \ldots \\ &\vdots \\ P_n &\implies Q \end{aligned}$$

where each of these intermediate implications is easier to prove.

Once we have done that, consider what happens if P is true or P is false:

P is true If P is true, the first implication tells us P_1 is true (modus ponens). Then since P_1 is true, the next implication tells us P_2 is true (again modus ponens). Since P_2 is true, P_3 is true, P_4 is true, and so on until we can conclude Q is true. Since P is true and Q is true, the implication $P \implies Q$ is true.

P is false On the other hand, if P is false then we know, just by looking at the implication truth-table, that the implication $P \implies Q$ is true.

Notice that when P is false, the fact that $(P \implies Q)$ is true is immediate and simply relies on the truth-table of the implication; we don't have to do any work or any reasoning. On the other hand, when P is true, we do need to work to show that $(P \implies Q)$ is true. For this reason almost all of our proofs will start with the *assumption* that P is true. We generally leave the case "P is false" unstated, assuming that our reader knows their truth-tables.

2.6 The converse, contrapositive and biconditional

We really want to get to our first proofs, but we need to do a tiny bit more logic, and define a few terms, before we get there. Consider the following thtree statements derived from implication $P \implies Q$.

Definition 2.6.1 Contrapositive, converse and inverse. Let P and Q be statements, then:

- the statement $(\sim Q) \implies (\sim P)$ is the **contrapositive**,
- the statement $Q \implies P$ is the **converse**, and
- the statement $(\sim P) \implies (\sim Q)$ is the **inverse**

of the implication $P \implies Q$. ◊

The contrapositive and converse appear quite frequently in mathematical writing, but the inverse is rare (in this author's experience at least). The truth-tables of the implication, contrapositive, converse and inverse are:

P	Q	$P \implies Q$	$\sim Q \implies \sim P$	$Q \implies P$	$\sim P \implies \sim Q$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	T	T	F	F
F	F	T	T	T	T

The above tables show that the original implication and the contrapositive have the exactly same truth tables, and that the converse and inverse have the same

tables. However we also see that the original implication does not have the same table as the converse or inverse. The inverse is not very commonly used, however the contrapositive and converse will be very useful for us as we continue.

Remark 2.6.2 Contraposition, conversion and inversion.. Note that the act of forming the contrapositive of $P \implies Q$ is **contraposition**. While forming the converse is (sometimes) called **conversion**, and forming the inverse is called **inversion**. Notice that the inversion is conversion of the contraposition of the implication.

Doing it twice. The inverse is also the contraposition of the conversion of the implication. The contraposition is also the inversion of the conversion. One could make a nice little table of the compositions of contraposition, conversion and inversion. Perhaps that is a good exercise.

While the converse is useful for forming mathematical statements, it can also be the source of bad logic (this is a good moment to go back and look at the warnings Warning 2.5.3 and Warning 2.5.4). The statement

If he is Shakespeare then he is dead.

and its converse

If he is dead then he is Shakespeare.

definitely do not mean the same thing[31]. However, the converse is often a source of interesting mathematics; once we have proved an implication, we should consider whether or not the converse is true. For example, we have already seen that

If n is even then n^2 is even

is true. It's converse is

If n^2 is even then n is even

is also true and will turn out to be quite useful later in the text.

The contrapositive can be extremely helpful — it might be hard to prove the original implication, but much easier to prove the contrapositive. Consider the statement

If n^2 is odd then n is odd

This, it turns out, is awkward to prove as it is stated. Its contrapositive, however, is

If n is not odd then n^2 is not odd.

or equivalently (assuming we are only talking about integers[32])

[31] Not every dead person is Shakespeare — ask any Elvis fan.

[32] Such assumptions happen quite frequently and the reader is often left to infer things from context. Writers do this, not just to be lazy, but so that the text flows and that one is not stating every single assumption in every single statement. That can make reading tedious, toilsome and tiring. Who doesn't like an alliteration.

If n is even then n^2 is even

which, even though we haven't written up the proof formally, we know is true. Since the truth-table of the contrapositive is identical to the original implcation, we now know that

If n^2 is odd then n is odd

must also be true.

Sometimes both an implication and its converse are true. That is

$$(P \implies Q) \wedge (Q \implies P)$$

This really means that whenever P is true, so is Q, and whenever Q is true so is P. It tells us that there is some sort of equivalence between what is expressed by P and Q. We can rewrite the above statement using the symbol $\iff$. It is our last connective and is called the "biconditional".

Definition 2.6.3 The biconditional. Let P and Q be statements. The **biconditional**, $P \iff Q$, read as "P if and only if Q", is true when P and Q have the same truth value and false when P and Q take different truth values. $\Diamond$

The biconditional $P \iff Q$ is also sometimes written as

- P iff Q,
- P is equivalent to Q, or
- P is a necessary and sufficient condition for Q.

Note that "iff" is still read as "if and only if" (and not as "iffffffffff" with a long "f"-noise).

Remark 2.6.4 We noted in the definition above, that $P \iff Q$ is true when P, Q have the same truth-values and false when P, Q have different truth-values. This in turn means that $P \iff Q$ has the same truth table as the statement $(P \implies Q) \wedge (Q \implies P)$.

P	Q	$P \iff Q$	$P \implies Q$	$Q \implies P$	$(P \implies Q) \wedge (Q \implies P)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

One of the first biconditional statements that we'll prove is

n^2 is odd if and only if n is odd.

but we have to walk before we run, so armed with all this logic this lets get on to our first proofs. After that we'll come back to logic.

2.7 Exercises

1. Determine whether or not each of the following is a statement or an open sentence. If it is a statement, determine if it is true or false.

(a) If 13 is prime, then 6 is also prime.

(b) If 6 is prime, then 13 is also prime.

(c) $f(3) = 2$

(d) 13 is prime and 6 is prime.

(e) 13 is prime or 6 is prime.

(f) The circle's radius is equal to 1.

2. Indicate whether the following are true or false.

(a) If today is Saturday, then it is a weekend.

(b) If it is a weekend, then today is Saturday.

(c) If the moon is made of cheese, then every cat in this room is purple.

3. Indicate whether the following are true or false. Explain your answers.

(a) If x is even, then $x \in \{2n : n \in \mathbb{N}\}$.

(b) If x is prime, then $x = 2k + 1$ for some $k \in \mathbb{Z}$.

(c) If $x \in \{3k : k \in \mathbb{Z}\}$ then $x \in \{6k : k \in \mathbb{Z}\}$.

(d) If $x \in \{6k : k \in \mathbb{Z}\}$ then $x \in \{3k : k \in \mathbb{Z}\}$.

4. Indicate whether the following are true or false.

(a) 3 is prime and 3 is even.

(b) 3 is prime or 3 is even.

(c) For $x \in \mathbb{R}$, $x^2 > x$ when $x > 1$, and 18 is composite.

(d) For $x \in \mathbb{R}$, $x^2 > x$ when $x > 1$, or 18 is composite.

5. Write the following sentences in symbolic logic notation. Make sure to note which statements/open sentences are denoted with which letter.

Example: The sentence, "The car is red and blue but not green" can be written as $(P \wedge Q) \wedge (\sim R))$, where P: "The car is red", Q: "The car is blue", and R: "The car is green". Also, the truth value of this sentence depends on the car, so it is an open sentence, not a statement.

(a) 8 is even and 5 is prime.

(b) If n is a multiple of 4 and 6, then it is a multiple of 24.

(c) If n is a not a multiple of 10, then it is a multiple of 2 but is not a multiple of 5.

(d) $3 \leq x \leq 6$.

(e) A real number x is less than -2 or greater than 2 if its square is greater than 4.

(f) If a function f is differentiable everywhere then whenever $x \in \mathbb{R}$ is a local maximum of f we have $f'(x) = 0$.

6. Write the following symbolic statements as English sentences.

(a) $(x \in \mathbb{R}) \implies (x^2 \in \mathbb{R}) \wedge (x^2 \geq 0)$.

(b) $4 \in \{2\ell : \ell \in \mathbb{N}\}$

(c) $(x \in \mathbb{N}) \implies \sim (x^2 = 0)$.

(d) $(x \in \mathbb{Z}) \implies \big(x \in \{2\ell : \ell \in \mathbb{Z}\}\big) \vee \big(x \in \{2k+1 : k \in \mathbb{Z}\}\big)$

7. Let P and Q be statements. Write out the truth tables for

(a) $(\sim P) \implies Q$

(b) $(P \wedge Q) \vee \big((\sim P) \implies Q\big)$

(c) $P \wedge (\sim P)$

(d) $P \vee (\sim P)$

(e) $(P \implies Q) \iff (Q \implies P)$

8. Let P and Q be statements. Show that the truth table for $\sim (P \implies Q)$ is the same as the truth table for $P \wedge \sim Q$.

9. In each of the following situations, determine whether or not it was raining on the given day, or explain why you cannot determine whether or not it was raining. For each situation we give you two pieces of information that are true; one is an implication and one is a statement.

(a) If it rains, then I bring an umbrella to work. I brought an umbrella to work on Monday.

(b) If it rains, then I bring an umbrella to work. I did not bring an umbrella to work on Tuesday.

(c) Whenever I am late for work, it rains. I was late to work on Wednesday.

(d) Whenever I am late for work, it rains. I was not late to work on Thursday.

10. There is an old saying: "Red sky at night, sailor's delight. Red sky at morning, sailors take warning." The phrase tells us that if the sky is red at night, tomorrow's weather will be good for sailing. However, if the sky is red in the morning, there will be a storm that day, and sailors should be prepared.

Assume that the following statement is true:

If the sky is red and it is morning, then sailors should take warning.

Now assume also that ...

(a) the sky is red. What can we conclude?

(b) the sky is red and it is morning. What can we conclude?

(c) sailors should take warning. What can we conclude?

(d) it is not true that (the sky is red and it is morning). That is, the sky is not red or it is not morning. What can we conclude?

(e) sailors should not take warning. What can we conclude?

11. Write the contrapositive of the following statements.

(a) If n is a multiple of 4 and 6, then it is a multiple of 24.

(b) If n is not a multiple of 10, then it is a multiple of 2 but is not a multiple of 5.

(c) A real number x is less than -2 or greater than 2 if its square is greater than 4.

(d) $(x \in \mathbb{R}) \implies (x^2 \in \mathbb{R}) \wedge (x^2 \geq 0)$.

(e) $(x \in \mathbb{N}) \implies \sim (x^2 = 0)$.

(f) $x \in \{3k : k \in \mathbb{Z}\} \implies x \in \{6k : k \in \mathbb{Z}\}$

12. Let $m \in \mathbb{N}$. Then two true statements are:

If m is odd, then m^2 is odd.
If m is even, then m^2 is divisible by 4.

Construct the contrapositive of each implication to give a total of four different implications. Which combinations can you chain together (so that the conclusion of the first is the hypothesis of the second), and what new implications do these combinations form?

13. In Chapter 11, we will prove that the following implication is true for $p = 2$:

If p is prime, then $\sqrt{p}$ is irrational.

In fact, this implication is true for any prime number p.

Write out the contrapositive, converse, and inverse of this implication. Can you determine whether any of these are true or false statements from the fact that the original implication is true?

14. Let P, Q, and R be statements. Suppose that

- "$P \implies (Q \wedge R)$"is false, and
- "$\big((\sim Q) \wedge R\big) \implies (\sim P)$" is true.

Which of P, Q, and R can you determine are true or false?

15. Let P, Q, R, and S be statements. Suppose that

- S is true,
- "$\big(R \vee (\sim P)\big) \implies \big(Q \wedge (\sim S)\big)$" is true, and
- "$P \iff \big(Q \vee (\sim S)\big)$" is true.

Determine the truth values of P, Q, and R.

16. Let P, Q, R, and S be statements. Suppose that

- "$((P \vee Q) \implies R) \iff (Q \wedge S)$" is true,
- "$(P \vee Q) \implies R$" is false, and
- S is true.

Determine the truth values of P, Q, and R.

Chapter 3

Direct proofs

Before we get to actually proving things we should spend a little time looking at how we name and prioritise mathematical statements. Not all things we want to prove are created equal and as a consequence they get different names.

Axioms Axioms are these are statements we accept as true without proof. Clearly they are very important since all our work hangs on them.

Facts We can also state some things as facts — these might be provable from axioms, but for the purposes of our text we don't want to go to the trouble (effort?) of proving them.

In this text we will use the following as an axiom.

Axiom 3.0.1 *Let n and m be integers. Then the following numbers are also integers*

$$-n, \qquad n+m, \qquad n-m \qquad \text{and} \qquad nm.$$

The authors are going to assume that you are familiar with the above properties and we do not need to delve deeper into them. The following is a statement that can be proved from the standard axioms of real numbers — we are not going to prove that, but we will use it. So we'll state it as a fact.

Fact 3.0.2 *Let x be a real number. Then $x^2 \geq 0$.*

Axioms of real numbers. The real numbers have an interesting history and you might be surprised to know that the first rigorous definition was only written down in 1871 by Georg Cantor — about 2 centuries after calculus was discovered by Newton and Leibniz! We'll discuss Cantor quite a bit later in the text. Your favourite search-engine can direct you to the axioms of the real numbers.

Another useful fact is Euclidean Divison, also called the division algorithm by some texts. It will come in very handy when we discuss even and odd numbers (for example).

Fact 3.0.3 Euclidean division. *Given integers a, b with $b > 0$, we can always find unique integers q, r so that*

$$a = bq + r$$

with $0 \leq r < b$.

So axioms and facts are slightly odd in that we don't have to prove them, but lets move onto statements that we do prove to be true.

Theorems A Theorem is a true statement that is important and interesting — Pythagorous' theorem for example. Or Euclid's theorem stating that there are an infinite number of prime numbers. Also, it is sometimes the case that implicit in the use of the word "Theorem" is that this is a result that we will use later to build other interesting results.

Corollary A Corollary is a true statement that is a consequence of a previous theorem. Of course, this makes almost everything a corollary of something else, but we tend to only use the term when the corollary is a useful (and fairly immediate?) consequence of a theorem.

Lemma A Lemma is a true statement that by itself might not be so interesting, but will help us build a more important result (such as a theorem). It is a helping result or a stepping stone to a bigger result[33]. You will occasionally see lemma pluralised as "lemmata".

Result and Proposition Otherwise we might just call a true statement a "Result" (especially if it is just an exercise or an example) or perhaps, if a little more important, a "Proposition".

3.1 Trivialities and vacuousness

As we said previously, most of the statements we want to prove are of the form $P \implies Q$. Before we get into proofs of more substance, we'll look at **trivial proofs** and **vacuous proofs**. These are two special cases that don't show up very often but you should know what they are. Recall that when we wrote out the truth table for $P \implies Q$ there were two observations we made:

- If P is false, then $P \implies Q$ is always true, independent of the truth value of Q.
- If Q is true, then $P \implies Q$ is always true, independent of the truth value of P.

[33]Indeed, the German word for Lemma is "Hilfssatz" — a helping result

The first of these is **vacuously true** — and the second is **trivially true**. They are both direct consequences of the truth table of the implication; no work is required. The results are of little use and so mathematicians use the dismissive terms **trivial** and **vacuous**. Consider:

Result 3.1.1 *Let $x \in \mathbb{R}$. If 8 is prime then $x^3 = 17$.*

So $P(x)$: 8 is prime, and $Q(x) : x^3 = 17$. A quick check shows that the hypothesis is false, so the result is vacuous. Of course we need to explain this to the reader in our proof otherwise its not a proof. It is safe to assume (in the context of writing a proof) that the reader knows their truth-tables. We don't have to explain everything in every proof.

Proof. Since $8 = 2 \times 4$ it is not prime, the hypothesis is false and thus the implications is always true. ■

Now providing the reader knows what a prime number is, and that they recall the truth-table of the implication, then we have clearly demonstrated that the hypothesis is false and so the implication must be true. Thus the reader is now convinced, and all is good.

Prime number? Now the authors are being a little bit sloppy here — we have assumed that the reader knows the definition of **prime**. While this is quite a basic notion of number, this author has been surprised by the very non-standard definitions of **prime number** that some students have been taught at school. Consequently we'll define prime numbers carefully in the next section.

This is an example of a vacuous proof — it is true because the hypothesis is always false. Notice that we cannot use modus ponens with such an implication because the hypothesis will never be true; the implication is true but in a rather useless way.

Despite this being a vacuous proof, we can learn something useful from how it is formatted. It is customary to tell the reader "the proof starts here" and "the proof finishes there", so that they know that all the necessary logic and mathematics is contained within that chunk of text. Typically we'll start a proof by writing "Proof:" (maybe underlined) and then finish it with a little square "□". The little square denotes "End of proof" or "QED" = "quod erat demonstrandum" = "which was to be demonstrated". It is perhaps a little pompous to write "QED" for such a little proof, so it is far more typical to see the little square. Some texts will use a little diamond "♦" or "◊", or a little filled in square "■". Some online-texts will simply enclose the whole proof in a box. In the HTML version of this text we'll enclose the proof in a box and also end it with a little square, while the PDF version of the text will simply have a little square.

Let's look at another example.

Result 3.1.2 *Let $n \in \mathbb{Z}$. If $n^2 < 0$ then $n^3 > 8$.*

Before writing anything down we should really read the hypothesis and conclusion very carefully. Notice that the hypothesis is saying something false. We know that the square of a number cannot be negative (we stated this as Fact 3.0.2), so the hypothesis is false.

Proof. The square of any real number is not negative; since the hypothesis is false, the statement is true. ■

The authors made an assumption about our reader in that proof — we've assumed that the reader knows Fact 3.0.2 well and so doesn't need to be reminded of it in the proof. We could choose to make this more explicit depending on our audience and the context. If you are in doubt as to what your readers know, you should put in more details.

A slightly more explicit proof. By Fact 3.0.2 we know that the square of any real number is not negative. Since the hypothesis is false, the statement is true. ■

There are related (and similarly quite useless) results which come from the conclusion being true independent of the hypothesis. For example:

Result 3.1.3 *Let $x \in \mathbb{R}$. If $x < 3$ then 17 is prime.*

Proof. Since 17 is a prime number the conclusion is always true. Hence the statement is true. ■

Primes and sieves. Here it would be sufficient to show that 17 is not divisible by 2,3 and 5. More generally to show that a number n is prime it suffices to show that it is not divisible by any prime smaller than $\sqrt{n}$ — this is (essentially) the sieve of Eratosthenese. Eratosthenese was also the first person to calculate the circumference of the Earth, invented the leap-day, was the chief librarian at the Library of Alexandria, and invented the study of geography! There are now much more efficient ways for large numbers and a quick bit of search-engineing will direct you to some of them. Anyway, a quick bit of arithmetic gives us $17 = 8 \times 2 + 1 = 5 \times 3 + 2 = 3 \times 5 + 2$, so by Euclidean division (remember Fact 3.0.3), 17 is not divisible by 2, 3 or 5 and hence must be prime.

This is an example of a trivial proof. We could put in more details to prove that 17 really is prime but we are going to assume that our reader knows their times-tables and the first few primes. Notice that since the conclusion is always true, we cannot use modus tollens with this result. Again, the result is true but in a useless way.

Here is another one.

Result 3.1.4 *Let $x \in \mathbb{R}$. If $x < 0$ then $x^2 + 1 > 0$.*

So now this looks a little harder but we can again look at this statement and see what is going on. The square of any number is always bigger or equal to zero (again Fact 3.0.2 is lurking here), so if we add 1 to it then it is definitely bigger than 0. We just need to translate this into mathematical language: Take any real number. Its square is bigger or equal to zero, so when we add 1, it is strictly bigger than 0.

Proof. Let $x \in \mathbb{R}$. Then $x^2 \geq 0$. Hence $x^2 + 1 \geq 1 > 0$. Since the conclusion is always true, the statement is always true. ■

Enough with the vacuous trivialities, it is high time we looked at some real results.

3.2 Direct proofs

A lot of the examples we will see shortly will involve even and odd numbers. Let us define these formally so we have a clear and solid base for our proofs.

Definition 3.2.1 An integer n is **even** if $n = 2k$ for some $k \in \mathbb{Z}$. $\Diamond$

So since $14 = 2 \times 7$ and $7 \in \mathbb{Z}$, we know that 14 is even. Similarly $-22 = 2 \times (-11)$ and $(-11) \in \mathbb{Z}$, so -22 is even.

0 is even. This author has encountered students who were taught that 0 is neither even nor odd — this is false. Since $0 = 2 \times 0$ and $0 \in \mathbb{Z}$, it follows that 0 is most definitely even.

When we first encounter "even" we learn that a number that is not even is "odd". At that time we have not yet encountered fractions or reals, so implicitly we thought all numbers were integers. But we know about rationals and reals (and maybe even complex numbers), so we should define odd numbers a little more carefully.

Definition 3.2.2 An integer n is **odd** if $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$. $\Diamond$

Since $13 = 2 \times 6 + 1$ and $6 \in \mathbb{Z}$ we know that 13 is odd. Similarly, -21 is odd because $-21 = 2 \times (-11) + 1$ and $-11 \in \mathbb{Z}$.

We should make a few observations about definitions before we move on to prove something.

The word "If" Notice that in both definitions we use the word "if" when we really mean "if and only if". If a number n is even then it can be written as $n = 2k$ for some integer k. AND if we can write $n = 2\ell$ for some integer ℓ, then we say that n is even. This becomes quite cumbersome, so it is convention that we write "if" in this context in definitions instead of writing "if and only if". We can safely assume that our reader knows this convention.

Find it easily Also we should make sure our definition is clear on the page. We declare it with "Definition:"! It is poor writing to hide important definitions inside the middle of an otherwise undistinguished paragraph (though this does happen from time to time). Some writers will put the key word being defined in inverted commas, or italics, or bold or underline, to further highlight it. If the definition is of an important object or property, then the reader should be able to find it and read it very easily.

Number it We have numbered[34] the definition so that we can refer to it easily (if necessary).

[34]Assigning numbers to results, lemmas, theorems and so on is standard practice in mathe-

"For some" is coming We have also used the phrase "for some" in both definitions. We haven't yet covered quantifiers in any sort of detail, but we will do so in Chapter 6 after a little more logic and some more proofs.

Definition 3.2.3 We say that two integers have the **same parity** if they are both even or they are both odd. Otherwise we say that the numbers have **opposite parities**. ◊

For a little more practice with definitions, lets extend the idea of "evenness" (being divisible by two):

Definition 3.2.4 Let n and k be integers. We say that k **divides** n if we can find an integer ℓ so that $n = \ell k$. In this case we write $k \mid n$ and say that k is a **divisor** of n and that n is a **multiple** of k. ◊

There is nothing in this definition that you haven't seen before (we hope), but it is worth looking at its structure since it is very typical.

- We start by defining the objects and symbols in our definition (this will necessarily build on previous definitions).
- We then define our main property **divides**.
- We follow up with some additional properties related to the main one.
- We have also used "if" in the definition in the way that we highlighted previously — we really mean "if and only if".

Finally, lets do one more definition related to divisibility — primes.

Definition 3.2.5 Let n be a natural number strictly greater than one. We say that n is **prime** if it cannot be written as the product of two smaller natural numbers. Equivalently n is prime when the only natural numbers that divide it are are 1 and itself.

If a natural number strictly greater than 1 is not prime then we say that it is **composite**. Finally, the number 1 is neither prime nor composite. ◊

Primality of 1. The primality of the number 1 has not always been as clear as it is today. Indeed, many mathematicians in the 19th century considered 1 to be prime and there are lists of prime-numbers published as late as the 1950's that have 1 as the first prime. Today, however, mathematicians treat 1 as a **unit** — a special case that is neither prime nor composite. One very good reason for doing so is that a great many results about prime numbers becomes substantially

matical writing. Unfortunately the topic of numbering equations is much less settled. This author has gotten into some strongly worded "discussions" with their coauthors on exactly this topic. Some authors like to number all equations, some like to only number the equations that get referenced inside the same document, and some like to number only the important equations. See these papers for an good discussion of numbering, Samaritans, Fisher, Occam and Fisher-Occam. It is also a good illustration of how mathematicians like a good argument over tiny details.

simpler and cleaner to state if 1 is not prime.

The interested reader can search engine their way to some interesting articles on this topic including this one[35].

Remark 3.2.6 The importance of being strict or equal. Notice in the above definition the emphasis we have placed on *strictly greater than one*. We really want the reader to realise that we mean "$>$" and not "$\geq$". In general, when writing inequalities in words (rather than symbols) it is a good idea to be very explicit so as to avoid possible confusion:

- $a < b$: "a *strictly* less than b"
- $a \leq b$: "a less than *or equal to* b"

If we were to write "a less than b" we may leave the reader confused as to whether or not a is allowed to be equal to b.

Let us go back to one of our first examples, and now that we have the right definitions and have done the required logic, we can prove it.

Result 3.2.7 *If n is even then n^2 is even.*

Lets think through our truth table again:

- If the hypothesis is false, the implication is true — no work required.
- If the hypothesis is true, then the implication will be true or false depending on the truth value of the conclusion — work required!

Just as we will many many times in the future, *we start by assuming the hypothesis is true.*

> Assume n is an even number.

What are we trying to get to?

> n^2 is an even number.

Know your definitions. It is very important to know definitions precisely. We cannot prove involving an object or property unless we can rigorously and carefully define it. The authors will likely nag you again and again about this.

If you have trouble memorising definitions, then we recommend you search-engine your way to some memorisation tips and tricks. You could also nag the authors back about producing nice auxillary materials that, say, could be easily reviewed on flash-cards or a phone.

At this point we know where we will start and where we need to end up, so a good next step is to flesh out what both the hypothesis and conclusion mean.

- If n is an even number then we can write it as $n = 2k$, where k is an integer.
- If n^2 is an even number then we can write it as $n^2 = 2\ell$, where ℓ is an integer.

[35] cs.uwaterloo.ca/journals/JIS/VOL15/Caldwell2/cald6.html

Remark 3.2.8 Not all even numbers are equal. Notice here that our result involves two even numbers, n and n^2. When I have invoked the definition of even, I have been careful to write $n = 2k$ and $n^2 = 2\ell$ using *different symbols*. This is important because it helps us to avoid making any *additional* assumptions about those numbers. Maybe we'll end up showing they are the same, and maybe we won't.

If we were to *accidentally* use the same symbols (and the authors know you won't do this after this warning), then we would have

- n is even, so $n = 2k$
- n^2 is even, so $n^2 = 2k$
- But then $n^2 = n$ which means $n^2 - n = 0$
- Factoring this gives $n(n-1) = 0$ and so $n = 0, 1$

So by reusing the symbol "k", we have inadvertently assumed that $n = 0, 1$, which is definitely not the result as stated.

Now since we know that $n = 2k$, we also know that $n^2 = (2k)(2k) = 4k^2$. From this we know $n^2 = 4k^2 = 2(2k^2)$. Since k is an integer, k^2 is an integer and $2k^2$ is an intger. So we've shown that n^2 can be written as twice and integer. In other words, we shown that it is even — exactly what we needed to do.

But we're not done. We have (in our work above) worked out ***how*** to prove our result, but we still need to write it up nicely. In this way proving a result really breaks into two parts

Scratch work Scratch work or proof-strategy or exploration or … — this is typically the difficult part, trying to work out what is going on, what does the hypothesis mean, what does the conclusion mean, how do we get from one to the other. What is the idea or path of the proof.

Write-up Once we have all the ideas and parts we still need to write things up nicely. This is typically easier than the scratch work, but it is non-trivial. We'll still need to work to make sure our presentation is clear, precise and easy to follow.

Let's write more nicely (and quite explicitly) with dot-points.

Proof.

- Assume n is even.
- So we can write $n = 2k$, where $k \in \mathbb{Z}$.
- But now, $n^2 = 4k^2$.
- This in turn implies that $n^2 = 2(2k^2)$.
- Since $2k^2$ is an integer, it follows that n^2 is even.

■

Notice that we don't prove that $2k^2 \in \mathbb{Z}$, nor do we have to explain basic facts about multiplication (as stated in Axiom 3.0.1); it is sufficiently obvious[36] that we can assume the reader will follow. Recall that back at the start of this chapter we warned you that we would make such assumptions about our (hypothetical) reader when writing our proofs.

Also notice the structure of the proof.

- P is true — where P is "n is even".
- $P \implies P_1$ is true — where P_1 is "$n = 2k$ for some $k \in \mathbb{Z}$".

  This follows from the definition of even.
- $P_1 \implies P_2$ is true — where P_2 is "$n^2 = 4k^2$".

  This is a basic fact about multiplication. To be more specific, "if $a = b$ then $ac = bc$".
- $P_2 \implies P_3$ is true — where P_3 is "n^2 is twice an integer".

  We are really using the fact that $4 = 2 \times 2$ and that multiplication is associative; we can expect the reader to understand this[37].
- $P_3 \implies Q$ is true.

  This is just the definition of even again.

So we have really shown

$$(P \implies P_1) \wedge (P_1 \implies P_2) \wedge (P_2 \implies P_3) \wedge (P_3 \implies Q)$$

And since we assume P is true, P_1 is true (modus ponens is our friend). Since P_1 is true, P_2 is true. And so forth until we conclude that Q is true. Hence we have shown that when P is true, we must have that Q is true.

This sort of proof in which we start by assuming the hypothesis is true and then work towards the conclusion is called a **direct proof**.

Now of course, when we actually prove something we don't go into this level of lurid detail. But for a first proof its not a bad idea to really see what is going. Let's write the proof more compactly, and a little more naturally:

Proof. Assume n is even. Hence we can write $n = 2k$ where $k \in \mathbb{Z}$. Then $n^2 = 4k^2 = 2(2k^2)$. Since $2k^2 \in \mathbb{Z}$ it follows that n^2 is even. ■

You can see there are some standard phrases that get used again and again

- Hence…
- It follows that…

[36] Well known to those that know it well, as this author's PhD supervisor says.

[37] Or they can quickly recall (or search-engine towards recalling) that the associativity of multiplication is just the fact that $a \times (b \times c) = (a \times b) \times c$.

- So…
- This implies that…
- We can now write…

These serve to make the proof more legible and flow a little more naturally.

Result 3.2.9 *Let n be an integer. If n is odd then $2n + 7$ is also odd.*

We don't leap into the proof; we start with scratch work.

- Assume the hypothesis is true (if it is false, there is nothing to be done).
- The hypothesis means that $n = 2k + 1$ for some integer k.
- The conclusion means that $2n + 7 = 2\ell + 1$ for some $\ell \in \mathbb{Z}$.
- But if $n = 2k + 1$, then $2n + 7 = 2(2k + 1) + 7 = 4k + 9 = 2(2k + 4) + 1$.
- Since $2k + 4 \in \mathbb{Z}$, $2n + 7$ is odd.

So we've got the idea, lets write it up.

Proof. Assume that n is an odd integer and so $n = 2k + 1$ for some $k \in \mathbb{Z}$. So

$$2n + 7 = 2(2k + 1) + 7 = 4k + 9 = 2(2k + 4) + 1.$$

Since k is an integer, $2k + 4$ is also an integer and so $2n + 7$ is odd. ■

Another one — this one for you

Result 3.2.10 *If n is odd then n^2 is odd.*

Do your scratch work before you write up the proof — even if you see the way to prove it. Writing the scratch work really helps to formulate your ideas and makes writing out the proof much easier.

Proof. Assume that n is an odd integer and so $n = 2k + 1$ for some $k \in \mathbb{Z}$. So

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

Since k is an integer, $2k^2 + 2k$ is also an integer and so n^2 is odd. ■

3.3 Proofs of inequalities

Not all mathematics involves integers, nor do all proofs involve equalities. So we should do a few examples of inequalities involving real numbers. This isn't just for variety, but does illustrate an important point about how our scratch work can often be quite different in logical structure from the final proof.

Result 3.3.1 *Let $x, y \in \mathbb{R}$. Then $x^2 + y^2 \geq 2xy$.*

As always we start with some scratch work. We don't know too many facts about inequalities — well, we do, but we haven't stated too many of them as

facts or axioms in this text. We do know that $x^2 \geq 0$ no matter which real number we take for x — this was stated as Fact 3.0.2. So it would suffice to rearrange our inequality to make it look like the square of something.

$$x^2 + y^2 - 2xy \geq 0$$

But this is precisely $(x-y)^2 \geq 0$, which follows from the fact that we are squaring something.

So we see a way to prove things. But we should be very careful of the logical order here — how does the truth flow from one statement to another. Look at the structure of what we have done above.

- We started from the conclusion $x^2 + y^2 \geq 2xy$
- We finished at the fact that the square of any real number is non-negative.

This order is ***not correct***. We know that we must finish at the conclusion, not start at it. But we can reorder our work to give it the correct logical flow:

- Start from the fact that the square of a real number is non-negative.
- $(x - y)$ is a real, so its square is non-negative.
- Expand this expression and rearrange it
- Arrive at the conclusion.

This is quite common when we prove inequalities; the logical flow in scratch work is often the reverse of what is required for the proof. We typically start at the inequality we want to prove and then work our way to something we know — a fact, an axiom, a previous result or theorem. To then present the proof we must start at the axiom, fact or theorem, and then work our way to the result. We can now write things up nicely:

Proof. Let x, y be real numbers. Hence $(x - y)^2 \geq 0$. Expanding this gives $x^2 - 2xy + y^2 \geq 0$. This can then be rewritten as $x^2 + y^2 \geq 2xy$ and so gives the required result. ■

That was very illustrative (which is, of course, why we include this topic). Our scratch work can look very different from the final write up of the proof. We should do a couple more, but first a useful fact about inequalities, multiplication and division.

Fact 3.3.2 *Let $a, b, c \in \mathbb{R}$ with $a \geq b$.*

- *If $c > 0$ then $ac \geq bc$ and $a/c \geq b/c$.*
- *If $c < 0$ then $ac \leq bc$ and $a/c \leq b/c$.*

In proofs we will often need to combine inequalities together to make new inequalities. Very frequently we will make use of the fact that if $a > b$ and $b > c$ then we know that $a > c$. This is the **transitivity** of "$>$" (see Section 9.2). For

example, if we know $a > b > 0$ and $c > d > 0$, then by multiplying the first inequality by c we get $ac > bc$. Similarly, multiplying the second inequality by b we get $bc > bd$. These two inequalities together imply that $ac > bd$.

Result 3.3.3 *Let $x \in \mathbb{R}$. If $x \geq 4$ then $x^2 - 3x + 7 \geq 11$.*

This one isn't too bad and we should break things into pieces.

- We know that $x \geq 4 > 0$, so then multiplying this by x gives us $x^2 \geq 4x$, and similarly, multiplying it by 4 gives us $4x > 16$. Hence we know that $x^2 \geq 16$.
- Similarly, since we know $x \geq 4$ we know that $3x \geq 12$. Ah —now there is a problem — we are about to try to take the difference of inequalities. Bad.
- Instead, go back and write $x^2 - 3x = x(x-3)$. Then since $x \geq 4, (x-3) \geq 1$. Hence $x^2 - 3x = x(x-3) \geq x$. So, because $x \geq 4$ we know that $x^2 - 3x \geq 4$.
- Adding 7 to both sides then gives us $x^2 - 3x + 7 \geq 4 + 7 = 11$.

We should now carefully check the flow of logic. We do indeed start with the hypothesis $x \geq 4$ and arrive at the conclusion. The order is good! Time to write it up.

Proof. Let $x \geq 4$ be a real number. Then we know that $x - 3 \geq 1$, and so $x(x-3) \geq 4$. Thus $x(x-3) + 7 = x^2 - 3x + 7 \geq 11$ as required. ■

In this case the logical flow in our scratch work matched the flow required for the proof. This is different from the previous example. There is not a hard rule that holds for all results. We need to be able to look at our scratch work, see the logical flow and determine how to translate that into a correct proof.

At the end of Chapter 5 we'll prove the triangle inequality. We can't do this just yet as it requires requires the development of a bit little more logical machinery.

3.4 A quick visit to disproofs

Lets look a little way ahead and think about how we might prove an implication to be false. Consider

> If $n \in \mathbb{N}$ then $2^n + 1$ is prime.

How is our theorem true — the conclusion must be true every time the hypothesis is true. Hiding in this is that it must be true every single time the hypothesis is true.

So how could our theorem be false? We need the hypothesis to be true while the conclusion is false. But to be more precise — it only has to fail ***once***. So lets explore a few values of n:

- $n = 1$ then $2^n + 1 = 2 + 1 = 3$ which is prime.

- $n = 2$ then $2^n + 1 = 4 + 1 = 5$ which is prime.
- $n = 3$ then $2^n + 1 = 8 + 1 = 9 = 3 \times 3$ which is not prime.

So since there is a value of n that makes the hypothesis true, but the conclusion false, the implication is false. We are hiding here the idea of **quantifiers**

For all n, $P(n)$

There exists n, $P(n)$

We'll come back to these in Chapter 6, but after we've done a little more logic.

3.5 Exercises

1. If n is even then $n^2 + 3n + 5$ is odd.

2. Prove that the product of two odd numbers is odd.

3. We have already seen a proof that the product of two odd numbers is also odd. We'll now look at the remaining cases for the parity of a product or sum of two integers.

For each of the following cases, determine if the resulting number is even or odd, and prove your statement:

(a) the sum of two odd numbers;

(b) the sum of two even numbers;

(c) the sum of an even and an odd number;

(d) the product of two even numbers;

(e) the product of an even and an odd number.

4. Consider the faulty proof below for the following statement:

Show that if $x + y$ is odd, then either x or y is odd, but not both.

Faulty proof. Assume that either x or y is odd, but not both. Assume that x is odd and y is even (otherwise, switch x and y in the following argument). By the definitions of odd and even numbers, we know that $x = 2n + 1$ and $y = 2m$ for some $n, m \in \mathbb{Z}$. Then

$$x + y = (2n + 1) + (2m) = 2(n + m) + 1.$$

Since $n, m \in \mathbb{Z}$ and the sum of integers is also an integer, we see that $n + m \in \mathbb{Z}$, so that $x + y$ fits the definition of an odd number. ■

Identify any issues with the proof as written above.

5. Consider the faulty proof below for the following statement:

The sum of two odd integers is even.

Faulty proof. Given $a = 2k + 1$ and $b = 2\ell + 1$,

$$a + b = (2k + 1) + (2\ell + 1) = 2(k + \ell + 1).$$

Since $k + \ell + 1 \in \mathbb{Z}$, $a + b$ is even. ■

Identify any issues with the proof as written above, and then give a proper proof of the statement.

6. Let $n, a, b, x, y \in \mathbb{Z}$. If $n \mid a$ and $n \mid b$, then $n \mid (ax + by)$.

7. Let $n, a \in \mathbb{Z}$. Prove that if $n \mid a$ and $n \mid (a + 1)$, then $n = -1$ or $n = 1$.

8. Let $a \in \mathbb{Z}$. If $3 \mid a$ and $2 \mid a$, then $6 \mid a$.

9. Let $n \in \mathbb{Z}$. If $3 \mid (n - 4)$, then $3 \mid (n^2 - 1)$.

10. Consider the faulty proof below for the following statement:

Let a, b, and c be integers. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Faulty proof. Assume a, b, and c are integers such that $a \mid b$ and $b \mid c$. Since a divides b, we have that $b = ka$ for some $k \in \mathbb{Z}$. Moreover, since b divides c, we have that $c = kb$ for some $k \in \mathbb{Z}$. But then

$$c = k(ka) = k^2 a,$$

Since k is an integer, $k^2 \in \mathbb{Z}$, and it follows that a divides c. ■

Identify any issues with the proof as written above, and then give a correct proof of the statement.

11. Consider the faulty proof that $2 = 1$.

Faulty proof. Assume that $x = y$. Then multiplying both sides by x gives

$$\begin{aligned} x^2 &= xy \\ \Rightarrow x^2 - y^2 &= xy - y^2 \\ \Rightarrow (x + y)(x - y) &= y(x - y) \\ \Rightarrow x + y &= y \\ \Rightarrow 2y &= y \end{aligned}$$

Letting $x = y = 1$, we have shown that $2 = 1$. ■

Identify any issues with the proof as written above.

12. The *floor* function, denoted by $\lfloor x \rfloor$, is defined to be the function that takes a real number x and returns the greatest integer less than or equal to x. This is also sometimes called the *greatest integer function*. For example,

$$\lfloor 3.5 \rfloor = 3, \qquad \lfloor -2.5 \rfloor = -3, \qquad \text{and} \qquad \lfloor 7 \rfloor = 7.$$

Using this definition, prove that

$$\lfloor x \rfloor = x \implies x \in \mathbb{Z},$$

and that

$$x \in \mathbb{Z} \implies \lfloor x \rfloor = x.$$

13. *Definition*: We call a number n an *integer root* if $n^k = m$ for some $k \in \mathbb{N}$ and $m \in \mathbb{Z}$.

For example, $\sqrt{7}$ is an integer root because $\left(\sqrt{7}\right)^2 = 7$. However, $\frac{5}{3}$ is not an integer root (but proving that is a little beyond this point in the text).

Use the above definition to show that if a and b are integer roots, then so is ab.

14. Consider the faulty proof below for the following statement:

Let x be a positive real number. If $x < 1$, then $1 < \dfrac{3x+2}{5x}$.

Faulty proof. Let x be positive. Then by multiplying the inequality

$$1 < \frac{3x+2}{5x}$$

by $5x$, which is positive, we obtain

$$5x < 3x + 2.$$

Collecting like terms, we have $2x < 2$, and finally dividing by 2, we have $x < 1$. ■

Identify any issues with the proof as written above, and then give a correct proof of the statement.

15. Consider the faulty proof below for the following statement:

Let x be a negative real number. Show that $-1 < \dfrac{5}{3x-5}$.

Faulty proof. Let x be negative. Then by multiplying the inequality

$$-1 < \frac{5}{3x-5}$$

by $3x - 5$ we obtain

$$-3x + 5 < 5.$$

and therefore $-3x < 0$. Dividing by -3 we end up with $x < 0$, which is true. ■

Identify any issues with the proof as written above, and then give a correct proof of the statement.

16. Let x, y be positive real numbers. Without using Calculus, prove that

$$(x > y) \implies (\sqrt{x} > \sqrt{y})$$

17. Consider the faulty proof below for the following statement:

Let $a, b \in \mathbb{R}$. If $0 < a < b$, then

$$\sqrt{ab} < \frac{a+b}{2}$$

Faulty proof.

$$\begin{aligned}
\sqrt{ab} &< \frac{a+b}{2} \\
ab &< \frac{(a+b)^2}{4} \\
4ab &< a^2 + 2ab + b^2 \\
0 &< a^2 - 2ab + b^2 \\
0 &< (a-b)^2
\end{aligned}$$

■

Identify any issues with the proof as written above and give a correct proof.

18. Let $x, y \in \mathbb{R}$ such that $x, y \geq 0$. Show that $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$.

You may use the following without proof:

$$\text{If } 0 \leq a \leq b, \text{ then } \sqrt{a} \leq \sqrt{b}.$$

Chapter 4

More logic

Before we continue proving things, we need to learn more about how to manipulate logical expressions. We need to be able to rewrite statements as *equivalent* statements — create a new statement with the same truth table as the original. We have already seen an example of this in the contrapositive. In order to do that, we also need to understand how negation interacts with the disjunction, conjunction and implication. Our starting point for all of this is to think about statements which are always true.

4.1 Tautologies and contradictions

If we play around with compound statements and explore what can and cannot happen, we will quickly run into some statements which seem (potentially) rather silly:

$$P \vee (\sim P)$$

This statement is always true — no matter whether P is true or false. Such a statement is called a **tautology**. Why might this be useful? Well — you've seen that when we prove things, we need to use things that are true and the above is always true.

Here is another (more obviously useful) one

$$\sim (P \wedge Q) \iff ((\sim P) \vee (\sim Q))$$

This statement is always true no matter what the truth values of P and Q, so it is a tautology. To see this we could either write up the truth-table, or argue

- The left-hand clause is false only when P and Q are both true. Otherwise it is false.
- The right-hand clause is false only when $\sim P$ and $\sim Q$ are both false. That is, it is only true when both P and Q are false. Otherwise it is true.

- Hence both clauses take the same truth values and so the biconditional is always true.

We'll come back to this expression very shortly.

Just as there are statements that are always true, there are statements such as

$$P \wedge (\sim P)$$

that are always false.This is a **contradiction**. Another example is

$$(P \wedge Q) \wedge ((\sim P) \vee (\sim Q))$$

Lets write these definitions in a proper formal way so that we can refer back to it easily later if we need to do so.

Definition 4.1.1 Tautologies and contradictions. A **tautology** is a statement that is always true, while a **contradiction** is a statement that is always false. ♢

We will use tautologies in the very near future, but contraditions will have to wait until later in the course — there is a proof technique called "proof by contradiction" which relies on us arriving at a contradiction.

4.2 Logical equivalence

Not all tautologies are terribly useful, but we will use one family of tautologies again and again as we write proofs: **logical equivalences**. We have seen that the two statements

$$(P \wedge Q) \qquad \text{and} \qquad (Q \wedge P)$$

have the same truth tables; it only takes a moment to write down the table to convince yourself.

We could write this "have the same truth tables"-fact as follows:

$$(P \wedge Q) \iff (Q \wedge P) \text{ is a tautology}$$

Take a moment to parse this. The biconditional at the heart of the statement must be true, and a quick review of the biconditional tells us that both sides must be true at the same time and false at the same time — exactly what we want to express. This way of writing things is still cumbersome, and mathematicians will always seek out nicer notation if it is available.

Definition 4.2.1 We say that two statements R and S are **logically equivalent** when the statement $R \iff S$ is a tautology. In this case we write $R \equiv S$. ♢

Remark 4.2.2 Equivalent and equal? Note that some texts use "=" to denote logical equivalence, while this author much prefers "≡". One can get into

long debates as to whether or not "=" is equivalent to "≡" despite not being equal. And unfortunately there is no clean and well established convention in the mathematical community. You should, as a reader, recognise both (from context).

Another logical equivalence we've already seen (back in Section 2.4) is

$$(P \implies Q) \equiv ((\sim P) \vee Q)$$

where we have written this down with plenty of brackets to avoid potential ambiguities.

Logical equivalence becomes very useful when we are trying to prove things. If we start with a difficult statement R, and transform it into an easier and logically equivalent statement S, then a proof of S automatically gives us a proof of R.

Here is a list of useful logical equivalences which will be very handy for proving things as we continue in the text. These constitute our first important result and since we will use it frequently we should call it a theorem.

Theorem 4.2.3 Logical equivalences. *Let P, Q and R be statements. Then*

- *Implication:* $(P \implies Q) \equiv ((\sim P) \vee Q)$
- *Contrapositive:* $(P \implies Q) \equiv ((\sim Q) \implies (\sim P))$
- *Biconditional:* $(P \iff Q) \equiv ((P \implies Q) \wedge (Q \implies P))$
- *Double negation:* $\sim (\sim (P)) \equiv P$
- *Commutative laws*
  - $P \vee Q \equiv Q \vee P$
  - $P \wedge Q \equiv Q \wedge P$
- *Associative laws*
  - $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
  - $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$
- *Distributive laws*
  - $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$.
  - $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$.
- *DeMorgan's laws*
  - $\sim (P \vee Q) \equiv (\sim P) \wedge (\sim Q)$
  - $\sim (P \wedge Q) \equiv (\sim P) \vee (\sim Q)$

Proof. These can all be proved in a straightforward, but slightly tedious, manner by computing and comparing truth tables. ■

DeMorgan and camels. De Morgan's laws are named after the 19th century mathematician, Augustus De Morgan, though they were known at least as far back as Aristotle.

This author notes that he has come across many variations of the name, "de Morgan", "De Morgan" and "DeMorgan", but has yet to find anyone writing it with the medial capitalisation so beloved by tech-companies: "deMorgan". Medial capitalisation is very common in computer langauges to make multi-word variable names legible without spaces; in that context it is frequently called camelCase.

By chaining the logical equivalences in Theorem 3 together we can make new ones. For example, we can show the equivalence of the contrapositive as follows:

Example 4.2.4 Show that the contrapositive is logically equivalent to the original implication.

$$\begin{aligned}
(P \implies Q) &\equiv ((\sim P) \vee Q) && \text{implication as or}\\
&\equiv Q \vee (\sim P) && \text{commutation of or}\\
&\equiv \sim (\sim Q) \vee (\sim P) && \text{double negation}\\
&\equiv (\sim Q) \implies (\sim P) && \text{or as implication}
\end{aligned}$$

Arguably this would be easier to do using a truth table, but the above is much more informative. □

Here is a nice, and useful, example.

Example 4.2.5 Prove that $\sim (P \implies Q) \equiv P \wedge (\sim Q)$.

$$\begin{aligned}
\sim (P \implies Q) &\equiv \sim ((\sim P) \vee Q) && \text{rewrite implication as or}\\
&\equiv (\sim (\sim P)) \wedge (\sim Q) && \text{DeMorgan}\\
&\equiv P \wedge (\sim Q) && \text{double negation}
\end{aligned}$$

□

Another nice, and useful example. In fact it is so nice and useful, we probably should have made it an exercise:

Example 4.2.6 Show $\sim (P \iff Q) \equiv (P \wedge \sim Q) \vee (Q \wedge \sim P)$.

$$\begin{aligned}
\sim (P \iff Q) &\equiv \sim ((P \implies Q) \wedge (Q \implies P)) && \text{rewrite biconditional}\\
&\equiv (\sim (P \implies Q) \vee \sim (Q \implies P)) && \text{DeMorgan}\\
&\equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P)) && \text{previous example}
\end{aligned}$$

□

So we now have another useful theorem

Theorem 4.2.7 Negating implications and biconditionals. *For statements P and Q we have*

- $\sim (P \implies Q) \equiv P \wedge (\sim Q)$
- $\sim (P \iff Q) \equiv (P \wedge (\sim Q)) \vee (Q \wedge (\sim P))$

This one will be very useful later on, so we'll call it a lemma. It isn't quite complete, you'll have to finish it off as an exercise later.

Lemma 4.2.8 *Let P, Q and R be statements. Then*

$$((P \vee R) \implies Q) \equiv ((P \implies Q) \wedge (R \implies Q))$$

Proof. We leave the proof as an exercise. ■

Some practice negating things.

Example 4.2.9 What is the negation of

$$(x^2 \geq 4) \wedge (x < 1)$$

Remeber to be careful when we negating inequalities. The negation of $a < b$ is $a \geq b$, and the negation of $a \leq b$ is $a > b$.

Solution.

$$\begin{aligned} \sim ((x^2 \geq 4) \wedge (x < 1)) &\equiv (\sim (x^2 \geq 4) \vee \sim (x < 1)) \\ &\equiv (x^2 < 4) \vee (x \geq 1) \end{aligned}$$

□

Example 4.2.10 Negate the statement

$$(x^2 \geq 1) \implies (x \geq 1).$$

Solution. Remeber to be careful with those inequalities.

$$\begin{aligned} \sim ((x^2 \geq 1) \implies (x \geq 1)) &\equiv (x^2 \geq 1) \wedge \sim (x \geq 1) \\ &\equiv (x^2 \geq 1) \wedge (x < 1) \end{aligned}$$

□

Example 4.2.11 Negate "The integer x is odd if and only if x^2 is odd."

Solution. This is actually a true statement (one we'll prove soon), but we can negate it anyway. We'll make use of the fact that if an integer is not odd, then it must be even (and vice-versa).

$$\begin{aligned} \sim ((x \text{ is odd}) \iff (x^2 \text{ is odd})) &\equiv \Big((x \text{ is odd}) \wedge \sim (x^2 \text{ is odd})\Big) \vee \Big((x^2 \text{ is odd}) \wedge \sim (x \text{ is odd})\Big) \\ &\equiv \Big((x \text{ is odd}) \wedge (x^2 \text{ is not odd})\Big) \vee \Big((x^2 \text{ is odd}) \wedge (x \text{ is not odd})\Big) \\ &\equiv \Big((x \text{ is odd}) \wedge (x^2 \text{ is even})\Big) \vee \Big((x^2 \text{ is odd}) \wedge (x \text{ is even})\Big) \end{aligned}$$

Oof! That is not so pretty. □

We are ready for some more proofs, but first — there are some exercises for you.

4.3 Exercises

1. Use truth tables to determine whether or not the following pairs of statements are logically equivalent.

(a) "$(\sim P) \vee Q$" and "$P \Rightarrow Q$".

(b) "$P \Leftrightarrow Q$" and "$(\sim P) \Leftrightarrow (\sim Q)$".

(c) "$P \Rightarrow (Q \vee R)$" and "$P \Rightarrow ((\sim Q) \Rightarrow R)$".

(d) "$(P \vee Q) \Rightarrow R$" and "$(P \Rightarrow R) \wedge (Q \Rightarrow R)$".

(e) "$P \Rightarrow (Q \vee R)$" and "$(Q \wedge R) \Rightarrow P$".

2. Use the logical equivalences given in Theorem 4.2.3 and Theorem 4.2.7 to negate the following sentences.

(a) 8 is even and 5 is prime.

(b) If n is a multiple of 4 and 6, then it is a multiple of 24.

(c) If n is not a multiple of 10, then it is a multiple of 2 but is not a multiple of 5.

(d) $3 \leq x \leq 6$.

(e) A real number x is less than -2 or greater than 2 if its square is greater than 4.

(f) If a function f is differentiable everywhere then whenever $x \in \mathbb{R}$ is a local maximum of f we have $f'(x) = 0$.

3. Show that the following pairs of statements are logically equivalent using Theorem 4.2.3.

(a) $P \iff Q$ and $(\sim P) \iff (\sim Q)$

(b) $P \implies (Q \vee R)$ and $P \implies ((\sim Q) \implies R)$

(c) $(P \vee Q) \implies R$ and $(P \implies R) \wedge (Q \implies R)$

Chapter 5

More proofs

Now that we've done a little more logic we should get back to proving things. Consider the following examples which (superficially) look quite similar to those we did back in Chapter 3:

Example 5.0.1 Let n be an integer. If $3n + 7$ is even then n is odd.

Scratchwork.

- As always — assume the hypothesis is true.

$$3n + 7 = 2k$$

- Then we try to use this to say something about the conclusion.

$$n = \frac{2k - 7}{3}$$

- And now we are stuck, because it isn't clear that n is an integer and we need to show that

$$\frac{2k - 7}{3} = 2\ell + 1$$

  for some integer ℓ. This doesn't look so obvious. Though it is surprising how many students will try to claim that one can deduce the parity from here.

Urgh. □

Example 5.0.2 If $n \in \mathbb{Z}$ then $n^2 + 5n - 7$ is odd.

Scratchwork. Again, we start as we did previously; assume the hypothesis is true and work towards the conclusion.

- Assume the hypothesis is true — so n is an integer.

- Then $n^2 + 5n - 7 = 2\ell + 1$ means that we need

$$n^2 + 5n - 6 = 2\ell \qquad \text{and so}$$
$$n^2 + 5n = 2(\ell + 3)$$

- So…? Help — I'm stuck.

This would be a lot easier if we knew more about the parity of n. □

In both cases we can make our lives much easier by manipulating the original statement into another form by use of logical equivalences. More precisely

$$(P \implies Q) \equiv (\sim Q \implies \sim P)$$
$$(P \vee R) \implies Q \equiv (P \implies Q) \wedge (R \implies Q)$$

In both cases, proving the statement on the left-hand side of the equivalence is completely logically equivalent to proving the statement on the right-hand side of the equivalence. So if the statement on the right-hand-side is easier, then we should just do that instead. Lets apply these equivalences to the above examples:

Example 5.0.1 starts with the statement

> Let n be an integer. If $3n + 7$ is even then n is odd.

The contrapositive of this is then

> Let n be an integer. If n is not odd then $3n + 7$ is not even.

However since we know that n is an integer, we can clean this up still more

> Let n be an integer. If n is even then $3n + 7$ is odd.

And now we are at a statement that looks exactly like results we proved in Chapter 3. This process of proving the contrapositive of the original statement is called **contrapositive proof** (not such an inventive term, but quite descriptive).

Manipulating Example 5.0.2 requires a little more thought. One of the impediments that we had was that we didn't know about the parity of n. However since n is an integer, we know it must be even or odd. Indeed,

> (n is an integer) $\equiv$ ((n is even) or (n is odd))

Hence we can rewrite Example 5.0.2 as

> If n is even or n is odd, then $n^2 + 3n - 9$ is odd.

We can then use the one of the above logical equivalences to rewrite this as

> (If n is even then $n^2 + 3n - 9$ is odd) and (If n is odd then $n^2 + 3n - 9$ is odd)

Again, we have arrived at statements that look just like those we proved in Chapter 3. This is an example of a **proof by cases**.

5.1 Contrapositive

As we have just seen, when we are presented with an implication to prove we should take a moment to think about the contrapositive of that implication — it might be easier! However, it do that we need to be able to *contrapose* a statement quickly. But that, in turn, requires us to negate statements fluently. Practice is crucial.

Once you have this fluency it only takes a moment to write down the contrapositive when you start your scratch work. Then you can assess what looks easier: the original or the contrapositive. If the contrapositive looks easier to prove, then we should proceed down that path. On the other hand, when it looks harder stick with the original.

It's time to return to Example 5.0.1 and try out a **contrapositive proof**.

Example 5.1.1 Example 5.0.1 redux. Let n be an integer. Prove that if $3n + 7$ is even, then n is odd.

Scratchwork. First up, lets write out some scratch work / explorations.

- We got stuck when we tried a direct proof, so write down the contrapositive:

$$(n \text{ is even}) \implies (3n + 7 \text{ is odd})$$

  This looks easier.

- Assume n is even.

$$n = 2k$$
$$3n + 7 = 6k + 7 = 2(3k + 3) + 1$$

- Since $3k + 3 \in \mathbb{Z}$ we are done.

Now that we've worked out how to make the proof work, we can write it up nicely. Since we are not using a direct proof we should alert the reader that we are going to prove the contrapositive. Otherwise it can look a little strange — I'm not going to assume the hypothesis is true, instead I'm going to assume the conclusion is false. Think about the reader!

Solution.

Proof. We prove the contrapositive. Assume that n is even and hence $n = 2k$ for some integer k. This means that we can write

$$3n + 7 = 6k + 7 = 2(3k + 3) + 1.$$

Since $3k + 3 \in \mathbb{Z}$, it follows that $3n + 7$ is odd as required. ■

□

Remark 5.1.2 Warn your reader. As noted in the example above, when you prove the contrapositive of the result you should warn the reader of what you are going to do. You don't need to write much

- "We prove the contrapositive…",
- "Consider the contrapositive…",
- or even (if, say you are writing a test and running out of time)"Do contrapositive…"

A few words from you can save the reader a lot of confusion "Why are they assuming the conclusion is false?", "What is going on here?", etc.

Of course, there can be more than one way to prove things. Here is another proof of the same result, this one using a direct proof. Is uses a cute little trick that is worth remembering.

Example 5.1.3 Example 5.0.1 redux redux. Let n be an integer. If $3n + 7$ is even, then n is odd.

Solution.

Proof. Assume that $3n + 7$ is even. Hence $3n + 7 = 2\ell$, where ℓ is some integer. Now we can write

$$\begin{aligned} n &= (3n + 7) - (2n + 7) \\ &= 2\ell - 2n - 7 \\ &= 2(\ell - n - 4) + 1 \end{aligned}$$

Since $\ell - n + 4 \in \mathbb{Z}$ it follows that n is odd. ■

□

Another similar example…

Example 5.1.4 Let $n \in \mathbb{Z}$. If $n^2 + 4n + 5$ is odd, then n is even.

Scratchwork. The first thing we should do as part of our scratch work is to write down the contrapositive:

$$(n \text{ is not even}) \implies (n^2 + 4n + 5 \text{ is not odd})$$

We've been a little sloppy here — we didn't write down the "Let $n \in \mathbb{Z}$", but it is scratch work not the actually proof. It is okay to be a *little bit* sloppy. The integerness of n means that we can simplify this further to

$$(n\text{is odd}) \implies (n^2 + 4n + 5 \text{ is even})$$

This is now straight-forward for us.

Solution.

Proof. Let $n \in \mathbb{Z}$ and assume n is odd. Then $n = 2k + 1$ for some integer k. Hence

$$n^2 + 4n + 5 = (2k + 1)^2 + 4(2k + 1) + 5$$

$$= 4k^2 + 4k + 1 + 8k + 4 + 5$$
$$= 4k^2 + 12k + 10 = 2(2k^2 + 6k + 5)$$

Since $2k^2 + 6k + 5$ is an integer, we know that $n^2 + 4n + 5$ is even as required. ■

The above proof can be improved. First up, we forgot to warn the reader "we are going to prove the contrapositive". Second, the reader doesn't need to see those boring algebraic manipulations; we can reasonably assume that the reader can do some basic algebra. So with those things in mind, here is a better proof.

Proof. We will prove the contrapositive, so let $n \in \mathbb{Z}$ and assume n is odd. Then $n = 2k + 1$ for some integer k. Hence

$$n^2 + 4n + 5 = 4k^2 + 12k + 10 = 2(2k^2 + 6k + 5)$$

and since $2k^2 + 6k + 5$ is an integer, we know that $n^2 + 4n + 5$ is even. ■

□

Let us now prove a simple (but useful) biconditional result. We'll call it a result rather than an example, because we'll need to refer back it later. We could even call it a lemma — in fact, let's do that.

Lemma 5.1.5 *Let $n \in \mathbb{Z}$, then n^2 is odd if and only if n is odd.*

It pays to do some exercises. If you haven't done it already, now is a good time to do this exercise 4.3.3.a. It tells us that the above is logically equivalent to "n^2 is even if and only if n is even", which is another useful (equivalent) result.

We've not proved a biconditional before, so where do we start. Our starting point is to rewrite the biconditional as implications because we know what we have to do to prove those. Recall from Theorem 4.2.3 that

$$P \iff Q \equiv (P \implies Q) \wedge (Q \implies P)$$

But how do we prove the conjunction of two implications?

Go back to first principles — we want to show that the statement cannot be false. This means that we must show that *both* implications are true. Hence we have to show that *both*

- If n^2 is odd then n is odd.
- If n is odd then n^2 is odd.

are true and then it follows that the conjunction is true, and so our original biconditional statement is true. Hence our proof consists of two parts (ie sub-proofs).

Of course, we have to tell the reader what we are doing. Statements like "We prove each implication in turn" or "We first prove one direction and then the other" are good ways to warn the reader what structure to expect. Then you can make this structure very easy to read by using dot-points or other formatting tricks.

Proof. We must show both that if n is odd then n^2 is odd, and if n^2 is odd then n is odd.

- Assume n is odd and so $x = 2k+1$ for some $k \in \mathbb{Z}$. Then $x^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$. Since $2k^2+2k$ is an integer, it follows that x^2 is odd.

- To prove that if x^2 is odd then x is odd we will show the contrapositive. Assume x is even and so $x = 2k$. Then $x^2 = 4k^2 = 2(2k^2)$. Since $2k^2$ is an integer, it follows that x^2 is even.

■

So this is pretty much just what we did for the other proofs, we just had to do it twice. First we proved $\implies$ and then we proved $\impliedby$. And we told the reader what we were doing; we didn't leave them to guess.

Parity proofs can get a little dull, so lets prove something a little different. Again, we'll call it a lemma because it might be useful later on.

Lemma 5.1.6 *Let a, b be non-zero integers. If $a \neq \pm b$ then $a \nmid b$ or $b \nmid a$.*

Because this result is about divisibility now is a good time to review Definition 3.2.4. Unfortunately that definition tells us what it means for an integer to *be* divisible by another, but not what is means when one integer is *not* divisible. However, we can turn that around using the contrapositive. The result then becomes

> If $a \mid b$ and $b \mid a$ then $a = \pm b$.

This looks a lot easier. When we assume the hypothesis to be true we can use the definition of divisibility quite directly. This is a good sign that the contrapositive was the right thing to do.

Proof.

- Assume $a \mid b$ and $b \mid a$.

- Hence there are integers k, ℓ so that $b = ka$ and $a = \ell b$.

- Thus we can write $b = ka = k\ell b$.

- Since $b \neq 0$ we can divide both sides by b to get

$$k\ell = 1$$

- Now since $k, \ell \in \mathbb{Z}$ it follows that the only solutions to this are $k, l \in \{1, -1\}$.

- This gives $b = \pm a$ as required.

■

Of course we can also write this out in proper sentences and not point form. Actually, because we wrote things nicely above, we really just need to remove the dots:

Proof. Assume $a \mid b$ and $b \mid a$. Hence there are integers k, ℓ so that $b = ka$ and $a = \ell b$. Thus we can write $b = ka = k\ell b$. Since $b \neq 0$ we can divide both sides by b to get $k\ell = 1$. Now since $k, \ell \in \mathbb{Z}$ it follows that the only solutions to this are $k, l \in \{1, -1\}$. This gives $b = \pm a$ as required. ■

5.2 Proofs with cases

Recall Lemma 4.2.8.

$$((P \vee Q) \implies R) \equiv ((P \implies R) \wedge (Q \implies R))$$

This tells us how to structure a proof when the hypothesis is a disjunction. That is, when the hypothesis can be broken into two (or more) cases.

Take another look at Example 5.0.2. At first glance, the hypothesis is just a single statement "n is an integer", and it is not immediately obvious that it can be broken into separate cases. However there is a clue in the conclusion, "$n^2 + 5n - 7$ is odd"; it tells us to think about parity. Any integer is either even or it is odd, so we can break the hypothesis into two cases

- n is even, or
- n is odd.

Because of this, the original statement can be massaged into the form of Lemma 4.2.8.

$$((n \text{ is even}) \vee (n \text{ is odd})) \implies n^2 + 5n - 7 \text{ is odd}$$

Lemma 4.2.8 then tells us that this is logically equivalent to

$$(n \text{ is even} \implies n^2 + 5n - 7 \text{ is odd}) \wedge (n \text{ is odd} \implies n^2 + 5n - 7 \text{ is odd}).$$

To show that this *conjunction* is true, we just need to show that both parts are true. That is, our proof will split into two **cases**.

1. Prove that (n is even $\implies$ $n^2 + 5n - 7$ is odd).
2. Prove that (n is odd $\implies$ $n^2 + 5n - 7$ is odd).

This is an example of **proof by cases**.

Of course, we don't just leap into things and write "Proof 1" and "Proof 2"; we need to explain to the reader what is happening. We need to explain that the hypothesis breaks into separate cases, and then we should make it clear where each case starts and where it ends. And, it won't hurt to summarise at the end of the proof that since we have proved all the cases, the result is true. Be nice to your reader — an extra sentence or two can make their life much easier.

Self interest? Sometimes the "reader" is a just a device to help us to think about how we are writing, but sometimes the "reader" is the person who marks your homework, tests and exams. Being nice to the reader can be good for the writer too.

Proof. Let $n \in \mathbb{Z}$. Since n can be either even or odd, we consider both cases separately.

- Case 1: Assume n is even. Then $n = 2k$ for some $k \in \mathbb{Z}$ and $n^2 - 3n + 9 = 4k^2 - 6k + 9 = 2(2k^2 - 3k + 4) + 1$. Since $2k^2 - 3k + 4$ is an integer, $n^2 - 3n + 9$ is odd.

- Case 2: Now assume n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$ and $n^2 - 3n + 9 = (4k^2 + 4k + 1) - (6k + 3) + 9 = 4k^2 - 2k + 7 = 2(2k^2 - k + 3) + 1$. Since $2k^2 - k + 3$ is an integer, $n^2 - 3n + 9$ is odd.

Since both cases are true, the result follows. ■

The above shows the very standard structure of **proof by cases** or **proof by case analysis**. Of course, not all such proofs consist of just two cases. More generally the structure will be as follows.

Proof.

- The hypothesis breaks into N cases.
- Here is my proof of case 1.
- Here is my proof of case 2.
- ⋮
- Here is my proof of case N.
- Since I've proved all N cases the proof is done.

■

One of the hardest parts of case-analysis is to be sure that you have found all the cases. And since each case is really its own proof, it is possible that a single case breaks into several sub-cases, each of which requires its own proof. Case analysis is also sometimes called **Proof by exhaustion**! As an extreme example, the original proof of "The four colour theorem" by Appel & Haken in 1976 required the checking of about 1900 different cases. Thankfully that was done by a computer; though that was very controversial at the time.

4 colours by computer. The 4 colour Theorem tells you that any map (eg, a map of countries of the world) can be coloured using no more than 4 colours, so that no two neighbouring regions have the same colour. The interested reader can search-engine their way to many articles on the topic. The original proof by Appel & Haken was one of the first computer-aided proofs. The idea of getting a computer is not without objections — if a proof involves so many logical steps that it cannot be verified by a human, then is it really a proof? Again,

the interested reader should search-engine their way to articles on philosophical questions that emerge from computer-aided proof.

Note that it is significantly easier to prove that you need no more than 5 colours, and we might even do it at the end of this text (assuming the author gets around to it). No computers are required! The proof actually started as a flawed attempt to prove the 4 colour Theorem by Kempe in 1879 which was rescued a decade later by Heawood as the 5 colour Theorem.

Remark 5.2.1 Keep cases just in case "without loss of generality" causes mistakes. You will have noticed that the two cases in the proof above are *very* similar. This is not unusual. It is very often the case that the cases in proof by cases are very similar to each other[38]. Many writers will omit one or more of these similar cases and instead write "The proof of the second case is similar to the proof of the first, so we omit it". You might also see "Without loss of generality (we will only do the first case)", which busy hard-working mathematicians will contract to "WLOG".

"WLOG" is a notoriously dangerous mathematical phrase[39] in mathematics. It sits with phrases such as

- Clearly
- Obviously
- It is easy to show that
- A quick calculation shows

Every mathematician has been caught out by one of these. What we thought would be an easy turned out to be much harder due to that little detail we didn't consider.

> Premature optimisation is the root of all evil

Dictum attribution. This quote is usually attributed to Donald Knuth and is perhaps one of the best rules-of-thumb in programming. Knuth, however referred to it (at least once) as Hoare's dictum, after Tony Hoare. Hoare, however, attributed it to Edsger Dijkstra. Attribution quandaries aside, this makes an important point — make it right, not fast.

Of course, once it is right, then making it a bit faster is a good idea, but not at the expense of introducing errors. The interested reader should search-engine their way to the full quote and discussions thereof.

One should be very careful using WLOG (and its siblings). We should be very sure that the cases really are very similar. Indeed, it is much safer (as a general rule for the inexperienced prover) to actually do all those cases in your scratch work. *Then* determine whether or not they really are similar enough that skipping them is not going to cause any problems. And on then skip them when writting up the proof.

Here are another couple of results to play with

Lemma 5.2.2 *If two integers a and b have opposite parities then their sum is odd.*

Proof. Let a and b be integers and assume they have opposite parities. Now either a is even or a odd; we prove each case in turn.

[38] Enough cases for a luggage related pun if only we could pack one in here. Sorry.

[39] We keep a list. Well — we should keep a list.

- Assume a is even. Since a and b have opposite parities, we know that b is odd. Hence we can write $a = 2k, b = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$. This means that $a + b = 2k + 2\ell + 1 = 2(k + \ell) + 1$. Since $k + \ell \in \mathbb{Z}$ we know that $a + b$ is odd.

- Now assume a is odd. Since a and b have opposite parities, we know that b is even. Hence we can write $a = 2k + 1, b = 2\ell$ for some $k, \ell \in \mathbb{Z}$. This means that $a + b = 2k + 2\ell + 1 = 2(k + \ell) + 1$, and since $k + \ell \in \mathbb{Z}$ we know that $a + b$ is odd.

In both cases, $a + b$ is odd as required. ■

and

Lemma 5.2.3 *Let $a, b \in \mathbb{Z}$. The number ab is even if and only if a is even or b is even.*

Proof. Let a, b be integers. We prove each implication in turn.

- To prove the forward implication we prove the contrapositive. Hence assume that both a, b are odd. So we can write $a = 2k + 1$ and $b = 2\ell + 1$ where $k, \ell \in \mathbb{Z}$. Now $ab = (2k + 1)(2\ell + 1) = 4k\ell + 2k + 2\ell + 1 = 2(2k\ell + k + \ell)$. Since $2k\ell + k + \ell \in \mathbb{Z}$ we know that ab is odd as required.

- The reverse implication breaks into two cases.

  - Assume a is even. Then we can write $a = 2k$ where $k \in \mathbb{Z}$. So $ab = 2kb = 2(kb)$. Since $kb \in \mathbb{Z}$ it follows that ab is even.
  - Now assume that b is even. Then we can write $b = 2\ell$ where $\ell \in \mathbb{Z}$. So $ab = 2a\ell = 2(a\ell)$. Since $a\ell \in \mathbb{Z}$ it follows that ab is even.

  In either case ab is even as required.

■

Remark 5.2.4 Symmetry and WLOG. These last two results display a great deal of symmetry. That is, one can swap a and b without changing the result. That symmetry indicates that it may be possible to shorten the proof, by appealing to that symmetry to justify why a case can be skipped. However, correctly identifying such symmetries and their consequences takes practice. Consequently we still recommend that you avoid WLOG-ing when working through this book, and only WLOG once you have spent many more hours proving things.

Result 5.2.5 *Let $n \in \mathbb{Z}$, then $3 \mid n$ if and only if $3 \mid n^2$.*

In order to prove this we'll make use of **Euclidean division**. We stated this at the beginning of Chapter 3 as Fact 3.0.3. Please revise it before continuing.

Fact 3.0.3 tells us that every integer n can (by dividing by two) be written *uniquely* as either

$$n = 2k \qquad \text{or} \qquad n = 2k + 1$$

for some integer k. That is, every integer is either even or odd. The same result

tells that every integer n can (by dividing by three) be written *uniquely* as

$$n = 3k \qquad \text{or} \qquad n = 3k+1 \qquad \text{or} \qquad n = 3k+2$$

for some integer k. It is this consequence of Euclidean division that will help us prove our result. Time for some scratch work.

It is a biconditional statement so we need to prove both the forward implication and the reverse implication.

- ($\implies$): Assume $3 \mid n$, so $n = 3k$ where k is some integer. Hence $n^2 = 9k^2$ which is a multiple of 3. Not so bad.
- ($\impliedby$): If we try assuming that $3 \mid n^2$ we aren't going to get very far, so instead we look at the contrapositive.

  $$3 \nmid n \implies 3 \nmid n^2$$

  Here is where we can use Euclidean division.

  Any integer n can be written uniquely as one of $n = 3k, n = 3k+1$ or $n = 3k+2$ where $k \in \mathbb{Z}$. Now, we assume that $3 \nmid n$, so we cannot have $n = 3k$. Hence we have 2 cases to explore, namely $n = 3k+1, n = 3k+2$.

  - If $n = 3k+1$ then $n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2+2k) + 1$, and so is not divisible by 3
  - If $n = 3k+2$ then $n^2 = (3k+2)^2 = 9k^2+12k+4 = 3(3k^2+4k+1)+1$, and so is not divisible by 3

  Since both cases work out, we are ready to write things up nicely.

Proof of Result 5.2.5. We prove each implication in turn.

- We start with the forward implication. Assume that $3 \mid n$, so $m = 3k$ where $k \in \mathbb{Z}$. Hence $n^2 = 3(3k^2)$, and since $3k^2 \in \mathbb{Z}$ we know that $3 \mid n^2$.
- To prove the reverse implication we prove the contrapositive. Assume that n is an integer, and that $3 \nmid n$. Hence (by Euclidean division), we know that either $n = 3k+1$ or $n = 3k+2$ where k is some integer.

  - Assume that $n = 3k+1$, then $n^2 = 3(3k^2+2k)+1$, and so is not divisible by 3.
  - Similarly, if we assume that $n = 3k+2$, then $n^2 = 3(3k^2+4k+1)+1$, and so is not divisible by 3.

  In either case we conclude that $3 \nmid n^2$ as required.

■

Remark 5.2.6 The importance of uniqueness. Notice that in the above scratch work and proof we have used the *uniqueness* of Euclidean divison to show

that n^2 is not divisible by 3. Once we have shown (in a case) that $n^2 = 3\ell + 1$ for some integer ℓ, Fact 3.0.3 tells us that there is no other way to write $n^2 = 3q$ with $q \in \mathbb{Z}$. Similarly, in the case where we show that $n^2 = 3\ell + 2$, the uniqueness of Euclidean division means that there is no way for us to write n^2 as 3 times an integer.

We have now had some practice with direct and contrapositive proofs, as well as proof by cases. It will soon be time to go back and do some more logic; we really need to look at quantifiers. But before that, we should do one two more examples of proof by cases — congruence modulo n, and the triangle inequality.

5.3 Congruence modulo n

Many results about divisibility of integers involve a fair bit of tedious work involving proof by cases; Result 5.2.5 is a good example of this. In that case we are interested in divisibility by 3 and so we use Euclidean division to separate the integers into 3 cases depending on their remainder. Much of that work can be simplified by introducing congruence.

Definition 5.3.1 Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$. We say that a is **congruent to** b **modulo** n when $n \mid (a - b)$. The "n" is refered to as the **modulus** and we write the congruence as $a \equiv b \pmod{n}$.

When $n \nmid (a - b)$ we say that a is not congruent to b modulo n, and write $a \not\equiv b \pmod{n}$. ◊

Some simple examples:

- 19 is congruent to 5 modulo 7, since $19 - 5 = 14 = 2(7)$,
- 11 is congruent to 27 modulo 4, since $11 - 27 = -16 = 4(-4)$, and
- 13 is not congruent to 7 modulo 5 since $13 - 7 = 6$ and $5 \nmid 6$.

Also notice that congruence nicely extends parity; we can re-express parity as just congruence modulo 2.

Result 5.3.2 *Let a, b be integers. Then $a \equiv b \pmod{2}$ if and only if a and b have the same parity.*

Proof. We prove both implications in turn. So start by assuming that $a \equiv b \pmod{2}$. Then we know that $2 \mid (a - b)$ and thus $a - b = 2k$ for some $k \in \mathbb{Z}$. Now either a is even or odd.

- When a is even, we know that $a = 2\ell$ for some integer ℓ, and thus $b = 2k + 2\ell$ and so is also even.
- Similarly when a is odd, we know that $a = 2m + 1$ for some integer m, and thus $b = 2k + 2m + 1$ and so is also odd.

Thus being congruent modulo 2 implies that they have the same parity.

Now assume that a, b have the same parity. Then either they are both even or they are both odd.

- When a, b are both even, we can write $a = 2k, b = 2\ell$ and so $a-b = 2(k-\ell)$.
- When a, b are both odd, we can write $a = 2k+1, b = 2\ell+1$ and so $a - b = 2(k - \ell)$.

In both cases the difference $a - b$ is divisible by 2 and so $a \equiv b \pmod 2$ as required. ■

Perhaps the main reason that congruence modulo m is so important is that congruence interacts very nicely with basic arithmetic operations. This gives rise to what is known as **modular arithmetic**.

Theorem 5.3.3 Modular arithmetic. *Let $n \in \mathbb{N}$, and let $a, b, c, d \in \mathbb{Z}$ so that*

$$a \equiv c \pmod n \qquad \textit{and} \qquad b \equiv d \pmod n$$

Then

$$a + b \equiv c + d \pmod n, \qquad a - b \equiv c - d \pmod n \qquad \textit{and}$$
$$ab \equiv cd \pmod n.$$

Proof. Let $n \in \mathbb{N}$, and let $a, b, c, d \in \mathbb{Z}$ so that $a \equiv c \pmod n$ and $b \equiv d \pmod n$. Hence $a = c + nk, b = d + n\ell$ where k, ℓ are integers.

- We can write $(a + b) = (c + d) + n(k + \ell)$, so $(a + b) - (c + d) = n(k + \ell)$. Hence $(a + b) \equiv (c + d) \pmod n$.
- Similarly, we can write $(a-b) = (c-d)+n(k-\ell)$, so $(a-b)-(c-d) = n(k-\ell)$. Hence $(a - b) \equiv (c - d) \pmod n$.
- Now $ab = (c + nk)(d + n\ell) = cd + n(dk + c\ell) + n^2kl$. So $ab - cd = n(dk + c\ell + nkl)$, and hence $ab \equiv cd \pmod n$.

■

We can now use this result to simplify some of our proof-by-cases proofs. Let's start by reproving Result 5.2.5. And we start that by restating the result in terms of congruences:

Another proof of Result 5.2.5. We start by restating the result in terms of congruences:

$$n \equiv 0 \pmod 3 \iff n^2 \equiv 0 \pmod 3,$$

and now we prove each implication in turn

- Assume that $n \equiv 0 \pmod 3$, then by Theorem 5.3.3 we know that $n^2 \equiv 0 \pmod 3$ as required.
- We prove the contrapositive of the reverse implication, so assume that $n \not\equiv 0 \pmod 3$. Thus either $n \equiv 1 \pmod 3$ or $n \equiv 2 \pmod 3$.
  - When $n \equiv 1 \pmod 3$, Theorem 5.3.3 tells us that $n^2 \equiv 1 \pmod 3$.
  - Similarly, when $n \equiv 2 \pmod 3$, we know that $n^2 \equiv 4 \pmod 3$. And since $4 \equiv 1 \pmod 3$, this means that $n^2 \equiv 1 \pmod 3$.

So in both cases, $n^2 \not\equiv 0 \pmod 3$ as required.

■

Notice we are doing something a little sneaky in the proof. We deduced that since $n^2 \equiv 4 \pmod 3$ and $4 \equiv 1 \pmod 3$, we know that $n^2 \equiv 1 \pmod 3$. This is because congruence is **transitive** — a notion we will return to in Chapter 9. But since it is quite useful, let's prove it now.

Result 5.3.4 *Let $n \in \mathbb{N}$, and let $a, b, c \in \mathbb{Z}$ so that $a \equiv b \pmod n$ and $b \equiv c \pmod n$. Then $a \equiv c \pmod n$.*
Proof. Let a, b, c and n be as stated. Then we know that for some $k, \ell \in \mathbb{Z}$

$$a - b = nk \qquad \text{and} \qquad b - c = n\ell.$$

Then $(a - b) + (b - c) = a - c = n(k + \ell)$ and so $a \equiv c \pmod n$. ■

Here is another example; congruence makes this easier to prove.

Result 5.3.5 *Let $a, b \in \mathbb{Z}$. If $3 \nmid a$ and $3 \nmid b$ then $3 \mid (a^2 - b^2)$.*
Proof. Let $3 \nmid a$ and $3 \nmid b$. Then, by Euclidean division, we know that $a \equiv 1 \pmod 3$ or $a \equiv 2 \pmod 3$.

- When $a \equiv 1 \pmod 3$, $a^2 \equiv 1 \pmod 3$.
- And, when $a \equiv 2 \pmod 3$, $a^2 \equiv 4 \pmod 3$. Since $4 \equiv 1 \pmod 3$, we also have that $a^2 \equiv 1 \pmod 3$.

So in either case $a^2 \equiv 1 \pmod 3$. This also implies that $b^2 \equiv 1 \pmod 3$. Thus $a^2 - b^2 \equiv 0 \pmod 3$. So finally we can conclude that $3 \mid (a^2 - b^2)$ as required. ■

We will return to congruences and modular arithmetic in Section 9.4, but now we turn to a very famous inequality.

5.4 Absolute values and the triangle inequality

The triangle inequality is a very simple inequality that turns out to be extremely useful. It relates the absolute value of the sum of numbers to the absolute values of those numbers. So before we state it, we should formalise the absolute value function.

Definition 5.4.1 Let $x \in \mathbb{R}$, then the absolute value of x is denoted $|x|$ and is given by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

◊

How we compute[40] the absolute value of a number depends on whether that

[40]When we first encounter the absolute value, it is often tempting to think of it a function that "removes the minus sign". Such thinking leads to errors. In particular, if x is a negative

number is negative or not — it is an example of a piecewise function.

$$|8| = 8$$
$$|-5| = -(-5) = 5$$

Since the absolute value is defined in two branches like this, it naturally leads to proofs that require cases. The proof of the triangle inequality is a good example of this. Before we state (and prove) the triangle inequality, let's prove a few useful lemmas that describe some useful properties of the absolute value.

Lemma 5.4.2 *Let $x \in \mathbb{R}$, then $|x| \geq 0$.*

We will split the proof into two cases. The first dealing with $x \geq 0$ and the second with $x < 0$. Doing this allows us to rewrite the absolute value $|x|$ as either x or $-x$, and so simplify our analysis.

Proof. Let $x \in \mathbb{R}$ so that either $x \geq 0$ or $x < 0$.

- When $x \geq 0$, we know that $|x| = x$, and so $|x| \geq 0$.
- Now assume that $x < 0$. Then $|x| = -x$. Now since x is negative, it follows that $-x$ is positive, and so $|x| = -x > 0$.

In both cases we have shown that $|x| \geq 0$. ■

To prove the next result it is actually convenient to use a more symmetric definition of the absolute value function that splits into three cases:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases}$$

We do this to take advantage of the fact[41] that $-0 = 0$.

Lemma 5.4.3 *Let $x \in \mathbb{R}$ then $|x| = |-x|$.*

Proof. Let $x \in \mathbb{R}$, then either $x = 0, x > 0, x < 0$.

- When $x = 0$, then $-x = x = 0$ and $|x| = |-x| = 0$.
- Now, let $x > 0$. This means that $|x| = x$. Further $-x < 0$ and so $|-x| = -(-x) = x = |x|$.
- Finally, let $x < 0$, so that $|x| = -x$. Additionally, $-x > 0$ and so $|-x| = (-x) = |x|$.

In all three cases the result holds. ■

We can reuse the basic ideas of this proof to obtain the following

number then $|x| = -x$, and the presence of that minus sign can be very confusing.

[41] In computing and in some applications it can be quite useful to have a "signed zero". The interested reader should search-engine their way to more information on this topic and the related topic of the extended real number system. You have actually already seen some of the ideas when you studied limits to zero and to $\pm\infty$ in Calculus 1.

Lemma 5.4.4 *Let $x \in \mathbb{R}$, then $-|x| \leq x \leq |x|$.*
Proof. Let $x \in \mathbb{R}$ so that either $x \geq 0$ or $x < 0$.

- When $x \geq 0$, we know that $|x| = x$. Now since $x \geq 0$, it follows that $-|x| \leq 0 \leq x = |x|$.
- Now assume that $x < 0$. Then $|x| = -x$, so that $-|x| = x$. Now since $x < 0$ it follows that $-|x| = x < 0 \leq |x|$.

In both cases we have shown that $-|x| \leq x \leq |x|$ as required. ■

We can extend the above lemma a little to get a very useful result. It tells us how to transform bounds on quantity into bounds on its absolute value and vice-versa.

Lemma 5.4.5 *Let $x, y \in \mathbb{R}$, then*

$$|x| \leq y \iff -y \leq x \leq y$$

Proof. We prove each implication in turn.

- Assume that $|x| \leq y$. Now either $x \geq 0$ or $x < 0$.
  - When $x \geq 0$, we know that $|x| = x$. Hence $y \geq |x| = x \geq 0 \geq -y$.
  - On the other hand if $x < 0$, we know that $y \geq |x| = -x > 0$. So, multiplying through by -1 we have $-y \leq x < 0 \leq y$.

  In both cases we have shown that $-y \leq x \leq y$ as required.
- Now assume that $-y \leq x \leq y$. Again, either $x \geq 0$ or $x < 0$.
  - When $x \geq 0$, we know that $x = |x|$. So, by assumption $y \geq x = |x|$.
  - Now, when $x \leq 0$, $|x| = -x$. Transform our assumption $y \geq x \geq -y$, by multiplying everything by -1, giving $-y \leq -x = |x| \leq y$.

  In both cases we have shown that $|x| \leq y$ as required.

■

Okay, now we can prove our main result; it tells us that the absolute value of the sum of two numbers is smaller[42] than the sum of the absolute values. It is a very simple but turns out to be extremely useful[43].

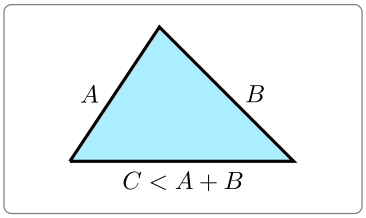
[42] This gives a nice counter example to the misquotation of Aristotle about sums and parts that has been worn out from (over/mis)use.

[43] Mathematicians like results like this one. When we define a new mathematical operation or function, we want to see how it interacts with the ones we already know. When you did Calculus you saw results like this — we have just defined limits, how do limits interact with addition, multiplication and division? There are many other examples including the product rule and the quotient rule and integration by parts. We will see more examples when we get to functions in Chapter 10.

Theorem 5.4.6 The triangle inequality. *Let $x, y \in \mathbb{R}$, then*

$$|x + y| \leq |x| + |y|.$$

The inequality gets its name from a more geometric interpretation[44]. It tells us that the length of the third side of the triangle, C, is bounded by the sum of the lengths of the other two sides, A, B.



Now, rather than leap into a neat proof of this, we should take some time to explore the problem and the various cases that might arise. Since the values of $|x|, |y|$ and $|x + y|$ all depend on whether each quantity is positive or negative[45].

- Notice that if both $x, y \geq 0$, then it is easy to prove since

$$|x| + |y| = x + y$$

- Similarly, if both $x, y < 0$, then we know that $x + y < 0$ and so we have

$$|x| + |y| = (-x) + (-y) = -(x + y) = |x + y|$$

- A little more care is required when (say) $x \geq 0$ but $y < 0$. In this case we have

$$|x| + |y| = x + (-y) = x - y$$

  and it is not immediately obvious how to relate this to $(x + y)$. One way to proceed is to break this case into subcases depending on whether $(x+y) \geq 0$ or $(x + y) < 0$. Urgh.

Let us instead take a step back and start again, but this time from assumptions about $(a + b)$. Either $a + b \geq 0$ or $a + b < 0$.

- In the first case

$$|a + b| = a + b$$

  and we know, from our lemma above, that $a \leq |a|$ and $b \leq |b|$, so

$$|a + b| = a + b \leq |a| + |b|.$$

[44]Indeed it was likely first proved by the ancient Greeks, and appears in Book 1 of Euclid's Elements as Proposition 20. The interested reader should search-engine their way to a discussion of Euclid's elements — likely one of the most influential books ever written.

[45]The pedantic reader will notice that we could have said "whether each quantity is non-negative or negative", since we have defined the absolute value function to lump $x = 0$ together with $x > 0$. We could have split absolute value function in 3 pieces $x > 0, x = 0, x < 0$ as we did in one of the proofs earlier in the section. While this is nicely symmetric, it does make our proofs longer, since we have to consider 3 cases for each quantity.

- Now, the second case gives

$$|a+b| = -(a+b) = -a-b$$

Our lemma above tells us that $-a \leq |a|$ (and similarly that $-b \leq |b|$), so

$$|a+b| = (-a)+(-b) \leq |a|+|b|$$

Oof! Now we can write it up.
Proof. Let $a, b \in \mathbb{R}$. Either $(a+b) \geq 0$ or $(a+b) < 0$.

- When $(a+b) \geq 0$, we know that $|a+b| = a+b$. By Lemma 5.4.4, we know that $a \leq |a|$ and $b \leq |b|$. Thus $|a+b| = a+b \leq |a|+|b|$.
- When $(a+b) < 0$, we have $|a+b| = -a-b = (-a)+(-b)$. Again, by Lemma 5.4.4 $-a \leq |-a| = |a|$ and $-b \leq |-b| \leq |b|$, and so $|a+b| = -(a+b) = (-a)+(-b) \leq |a|+|b|$.

In both cases we have that $|a+b| \leq |a|+|b|$ as required. ■

This is not the only way to prove this result. We can be a little more sneaky via the inequality in Lemma 5.4.5.
Another proof. Let $x, y \in \mathbb{R}$. Then from Lemma 5.4.4we know that $x \leq |x|$, and that $-x \leq |-x| = |x|$, and hence $x \geq -|x|$. Putting those together we have

$$-|x| \leq x \leq |x|$$

Similarly, we know that $-|y| \leq y \leq |y|$. Adding these inequalities together gives

$$-|x|-|y| \leq (x+y) \leq |x|+|y|$$

and so, by Lemma 5.4.5 above

$$|x+y| \leq |x|+|y|$$

as required. ■

A useful corollary of the triangle inequality is a bound on the absolute value of the difference of two numbers. This is often called the reverse triangle inequality.

Corollary 5.4.7 Reverse triangle inequality. *Let $x, y \in \mathbb{R}$ then*

$$|x-y| \geq \Big||x|-|y|\Big|$$

Proof. From the triangle inequality $|x|+|y| \geq |x+y|$ we can arrive at the following two inequalities, by setting $x = a, y = b-a$ and $x = b, y = a-b$

$$|a|+|b-a| \geq |b|$$
$$|b|+|a-b| \geq |a|$$

Rearranging these gives

$$|b-a| \geq |b|-|a|$$

and $|b| - |a| \geq -|a - b|$, which can be rewritten as

$$-(|b| - |a|) \leq |b - a|$$

Putting these together gives

$$-(|b| - |a|) \leq |b - a| \leq |b| - |a|$$

from which the result follows. ■

5.5 Exercises

1. Let $n \in \mathbb{Z}$. Prove that if $n^2 + 4n + 5$ is odd, then n is even.

2. Let $n \in \mathbb{Z}$. Show that if $5 \nmid n^2$, then $5 \nmid n$.

3. Let $n \in \mathbb{Z}$. Prove that if $5 \nmid n$ or $2 \nmid n$, then $10 \nmid n$.

4. Let $n, m \in \mathbb{N}$. Prove that if $n \neq 1$ and $n \neq 2$, then $n \nmid m$ or $n \nmid (m + 2)$.

5. Let $n, m \in \mathbb{Z}$. Prove that if $n^2 + m^2$ is even, then n, m have the same parity.

6. Let $x \in \mathbb{R}$. Show that if $x^3 + 5x \geq x^2 + 1$, then $x > 0$.

7. We say that the pair of numbers a, b are consecutive in the set S when $a < b$ and there is no number $c \in S$ so that $a < c < b$. That is, the number b is the next number in the set after a. For example:

- 5 and 6 are consecutive integers.
- 10 and 12 are consecutive even numbers.
- 25 and 30 are consecutive multiples of 5.

Prove the following statement:

Let $a, b \in \mathbb{Z}$. If $a + b$ is not odd, then a and b are not consecutive.

8. Prove that if n is an even integer then $n = 4k$ or $n = 4k + 2$ for some integer k.

9. Let $n \in \mathbb{Z}$. Show that $2 \mid (n^4 - 7)$ if and only if $4 \mid (n^2 + 3)$.

10. Let $a \in \mathbb{Z}$. Prove that $3 \mid 5a$ if and only if $3 \mid a$.

11. Let $n \in \mathbb{Z}$. Show that $(n^2 - 1)(n^2 + 2n)$ is divisible by 4.

12. Prove the following statement:

If $x + y$ is odd, then either x or y is odd, but not both.

13. Let $n \in \mathbb{Z}$. Prove that if $3 \mid (n^2 + 4n + 1)$, then $n \equiv 1 \pmod 3$.

14. Let $m \in \mathbb{Z}$. Prove that if $5 \nmid m$, then $m^2 \equiv 1 \pmod 5$ or $m^2 \equiv -1 \pmod 5$.

15. Let $q \in \mathbb{Z}$. If $3 \nmid q$, then $q^2 \equiv 1 \pmod 3$.

16. Prove that if $n \in \mathbb{Z}$, then the sum $n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9.

17. Prove that $\forall a \in \mathbb{Z}$, $a^5 \equiv a \pmod 5$.

18. Without using the triangle inequality, prove that if $x \in \mathbb{R}$, then $|x+4| + |x-3| \geq 7$.

19. Let $x \in \mathbb{R}$. Show that if $|x-1| < 1$, then $|x^2 - 1| < 3$. You may use the following result without proof:

$$|ab| = |a| \cdot |b| \text{ for any } a, b \in \mathbb{R}.$$

20. Let $x \in \mathbb{R}$. Show that if $|x-2| < 1$, then $|2x^2 - 3x - 2| < 7$. You may use the following result without proof:

$$|ab| = |a| \cdot |b| \text{ for any } a, b \in \mathbb{R}.$$

21. Prove the reverse triangle inequality. That is, given $x, y \in \mathbb{R}$, prove

$$|x - y| \geq \big||x| - |y|\big|.$$

22. We say that a function f is *decreasing* on its domain D if for all $x, y \in D$, whenever $x \leq y$, we have $f(x) \geq f(y)$. Explain why the following statement is false:

Let $f : \mathbb{R} - \{0\} \to \mathbb{R}$ be defined by $f(x) = 1/x$. Then f is decreasing.

Rewrite the statement to make it true by changing the domain of the function f. Then prove your statement.

Chapter 6

Quantifiers

The authors of this text have aimed to get you started proving things as quickly as possible. This meant that we had to skip over several important topics and return to them later. That is why the text has bounced between logic and proof and logic and proof, and now we return one last time to logic.

The first result we really proved in this text (way back at Result 3.2.7) was

$$(n \text{ is even}) \implies (n^2 \text{ is even}).$$

We approached the proof by thinking about how the implication could possibly be false. That, in turn, led us to assume the hypothesis to be true, and to show that the conclusion could not possibly be false. In so doing, we have hidden something from you, the reader. Sorry, but the authors felt this was a necessary but well-intentioned untruth[46] to achieve their aim of getting you to start proving things as quickly as possible.

Consider the truth-values of hypothesis and conclusion of the above implication carefully. "n is even" and "n^2 is even" are *not* a statement, they are both **open sentences**[47] whose truth values depend on the variable n. We have hidden from you, our reader, is the implicit scope on the variable n. We implied that we want this result to be true *every possible integer* n. To make this implicit explicit:

$$\text{For every integer } n, (n \text{ is even}) \implies (n^2 \text{ is even}).$$

The effect of that extra bit of text is to provide *scope* to the variable n, and so turns the open sentence into a statement with a well-defined truth value. And once we have a statement we can try to prove it.

6.1 Quantified statements

Let us now go back to sentences like

$$x^2 - 5x + 4 = 0.$$

[46] A teenie-tiny one.

[47] The reader who has momentarily forgotten the difference between **statement** and **open sentence** should quickly jump back to Chapter 2

We were unable to assign a truth value to this statement because we had no information about x. It is clear that this sentence is true for some values of x and false for others:

- If $x = 0$ the sentence makes sense but is false.
- If $x = 1$ the sentence makes sense and is true.
- If x is the colour blue, then it doesn't even make sense.

We can write the open sentence above as

$$P(x) : x^2 - 5x + 4 = 0$$

Now we can express things in a more compact (and a little more abstract) way: $P(0)$ is false, while $P(4)$ is true.

Just as Result 3.2.7 was a statement about integers, we might choose to study the open sentence over, say, the set $S = \{0, 1, 2, 3, 4\}$. In this case we would analyse the truth-values of $P(x)$ over the **domain**[48] S. Checking carefully we find that

$P(1), P(4)$ are true and $P(0), P(2), P(3)$ are false.

Such lists are going to become very cumbersome over big domains, let alone infinite[49] domains. However, we could summarise that list by saying that the statement is true sometimes, but not true always. To be a little more precise:

- $P(x)$ is true for some $x \in S$, and
- $P(x)$ is not true for all $x \in S$.

Notice that the extra bits of text "for some $x \in S$" and "for all $x \in S$" place restrictions which values of x we take, and so turn the open sentences into statements. With that extra text we can now assign truth values.

To be even more careful, we can write the above as

- There exists $x \in S$ so that $x^2 - 5x + 4 = 0$, and
- For all $x \in S$, $x^2 + 5x - 4 = 0$

where the first is now a true statement, and the second is a false statement. The extra bits "For all" and "There exists" are called **quantifiers**.

Definition 6.1.1 We typically work with two quantifiers in mathematics — the **universal quantifier** and the **existential quantifier**.

- The **universal quantifier** is denoted $\forall$ and is read as "for all" or "for

[48] Again, this terminology is reminiscent of functions.

[49] Actually we cannot even construct such lists over some types of infinite domains — see Chapter 12.

every". The statement

$$\forall x \in A, P(x)$$

is true provided $P(x)$ is true for every single value of $x \in A$ and otherwise the statement is false.

- The **existential quantifier** is denoted $\exists$ and is read as "there exists". The statement

$$\exists x \in A \text{ so that } P(x)$$

is true provided there is at least one value of $x \in A$ so that $P(x)$ is true, and otherwise the statement is false.

$\Diamond$

Other quantifiers? Sometimes in mathematics we also use the **unique existential quantifier** to indicate that there exists one and only one object of interest. It is sometimes denoted $\exists!$. For example, "the equation $n^3 = -1$ has exactly one solution over the integers". We won't use this particular type of quantifier very often in this course.

Note that one can express the unique existential quantifier in terms of the usual existential quantifier. The interested reader should play around to work out how to do this, or just search-engine their way to it.

The unique existential quantifier is not alone; one can construct an infinite family of quantifiers of the form, "there are exactly 2…", "there are exactly 3…", etc. Further one can also consider quantifiers such as "for all but one", or "for all but a finite number"

Back to our two statements above. We can now write them as

- "$\exists x \in S$ such that $x^2 - 5x + 4 = 0$" or "$\exists x \in S$ s.t. $x^2 - 5x + 4 = 0$".
- "For every $x \in S, x^2 - 5x + 4 = 0$" or "$\forall x \in S, x^2 - 5x + 4 = 0$"

Remark 6.1.2 Punctuation please. Be careful to punctuate these statements nicely — make sure that it is clear to the reader where the quantifier stops and the open sentence begins. In the case of for-all statements we usually just place a comma:

$$\underbrace{\forall x \in S}_{\text{quantifier}} \quad \underbrace{,}_{\text{punctuation}} \quad \underbrace{P(x)}_{\text{open sentence}}$$

For there-exists statements we write in "so that" or "such that", since that is how the statements are typically read. Your busy hard-working mathematician will contract the "so that" to "s.t.":

$$\underbrace{\exists x \in S}_{\text{quantifier}} \quad \underbrace{\text{s.t.}}_{\text{punctuation}} \quad \underbrace{P(x)}_{\text{open sentence}}$$

It is also generally considered bad style to use $\exists$ and $\forall$ in sentences in place of "there exists" and "for all". Mind you, that doesn't stop people doing it, but in general, it is okay to do in a mathematical statement or equation, but you should avoid writing them in the middle of paragraphs (except in scratch work).

Quantifiers are often a point of confusion for students. This can be exacerbated by the number of different ways they can be expressed in written or spoken languauge. For example, the statement "$\exists x \in A$ s.t. $P(x)$" can be read as

- There exists x in A so that $P(x)$ is true.
- There is x in A so that $P(x)$ is true.
- There is at least one x in A so that $P(x)$ is true.
- $P(x)$ is true for at least one value of x from A
- We can find an x in A so that $P(x)$ is true.
- We can always find an x in A that makes $P(x)$ is true.
- At least one x in A exists so that $P(x)$ is true.
- …

The above is just what the author thought of in a couple of minutes.

Similarly, the statement "$\forall x \in A, P(x)$" can be read in many ways:

- For all x in A, $P(x)$ is true.
- For every x in A, $P(x)$ is true.
- No matter which x we choose from A, $P(x)$ is true.
- Every single x in A makes $P(x)$ true.
- $P(x)$ is true for every x in A.
- Every choice of x from A makes $P(x)$ true.
- All the x in A makes $P(x)$ true.
- …

Oof!

We need to add one more to the list of ways to read $\forall x \in A, P(x)$:

- If x is in A then $P(x)$ is true.

This is critically important, because it shows us a link between the universal quantifier and the implication. It shows us that:

$$(\forall x \in A, P(x)) \equiv (x \in A \implies P(x))$$

Thankfully it is not too hard to see why — think about their truth values.

- $\forall x \in A, P(x)$ is true provided $P(x)$ is true for every single x from A. It is false if we can find at least one value of x from A so that $P(x)$ is false.

- On the other hand, the implication $x \in A \implies P(x)$, is false when the hypothesis is true, but the conclusion is false. That is, we can find a value of $x \in A$ so that $P(x)$ is false. Otherwise the implication is true.

More generally when we have a statement like

> If n is even then n^2 is true.

there is an implicit assumption that we actually mean

> For all n, if n is even then n^2 is even.

So it is typically understood that when we write

$$P(x) \implies Q(x)$$

the reader should really read this as

$$\forall x, P(x) \implies Q(x)$$

Of course, we don't really mean for every single possible value of the variable x taken from the set of all possible things in this and every other universe. We actually mean

$$\forall x \in A, P(x) \implies Q(x)$$

where the set A is often inferred by context. So when we are talking about even and odd numbers (as above), we really mean

> For all integers n, if n is even then n^2 is true.

Typically the context is clear, and so it is is just cumbersome[50] to write "$\forall x \in A, \ldots$" before our statements. When it is *us* doing the writing, *we* can look after our reader and try to make sure the context is clear. To this end, a good general rule is:

> If you are worried that the reader might not understand the context or that your statements might be open to misinterpretation, then put in more words and more details.

or, to be a little more to the point:

> If in doubt, put in more details.

Time for a simple example (we'll do more complicated ones in the next section).

Example 6.1.3 Let $P(n)$ be the open sentence "$(7n-6)/3$ is an integer." over

[50] And tedious for the hard-working time-pressed mathematician.

the domain $\mathbb{Z}$. Explain whether the following statements are true

$$\exists n \in \mathbb{Z} \text{ s.t. } P(n)$$
$$\forall n \in \mathbb{Z}, P(n)$$

Solution. Let us think about each in turn.

- In order for this to be true we need to find at least one integer n that makes $P(n)$ to be true. For example, setting $n = 0$ gives

  $$P(0) : -2 \text{ is an integer}$$

  which is true.

  Since we have found at least one value of n to make the open sentence true, the statement is true.

- In order for the second statement to be true, no matter which integer n we choose, the statement $P(n)$ is true. However, if we pick $n = 1$ then

  $$P(1) : \frac{1}{3} \text{ is an integer}$$

  which is clearly false.

  Since we cannot pick whatever integer n we want, and still have $P(n)$ true, it follows that the statement is false. To be more precise, it is false because there is some n so that $P(n)$ is false. In symbols this is:

  $$\exists n \in \mathbb{Z} \text{ s.t. } \sim P(n).$$

□

Notice that in the case of the second statement in the above exercise, we have shown the statement to be false, by demonstrating that its **negation** is true. This brings us to negating quantifiers.

6.2 Negation of quantifiers

This last example brings us to the negation of quantifiers. This isn't difficult, but we should still be careful. Consider the statement

$$\forall n \in \mathbb{N}, n^2 + 1 \text{ is prime.}$$

In order for this to be true, we require that no matter which natural number n, the number $n^2 + 1$ is prime.

$n = 1$ $\quad 1^2 + 1 = 2$ which is prime.

$n = 2$ $\quad 2^2 + 1 = 5$ which is prime.

$$n = 3 \qquad 3^2 + 1 = 10 = 2 \times 5 \text{ is not prime.}$$

Since it fails when $n = 3$, the statement is false.

Think carefully about what we have actually done here. We showed that this statement is false, by demonstrating that we could find $n \in \mathbb{N}$ so that $n^2 + 1$ is not prime. That is, we proved that the statement

$$\exists n \in \mathbb{N} \text{ s.t. } n^2 + 1 \text{ is not prime}$$

is true. What we are really doing here is proving that our original statement is false, by demonstrating that the negation of that statement is true.

Now consider the statement

$$\exists n \in \mathbb{N} \text{ s.t. } n^2 < n$$

You can convince yourself that this is false just by plugging in a few numbers or by drawing some graphs. However *convincing* is not the same as *proving*. In order for this to be false, we need to show that no matter which $n \in \mathbb{N}$ we choose, $n^2 \geq n$. That is, we have to show that

$$\forall n \in \mathbb{N}, n^2 \geq n.$$

We'll do that shortly. But again, we are showing that the original is false by proving the negation to be true.

Notice that our first statement was

$$\forall n \in \mathbb{N}, P(n)$$

and we showed that it was false by proving that

$$\exists n \in \mathbb{N} \text{ s.t. } \sim P(n)$$

is true. And similarly, our second statement was

$$\exists n \in \mathbb{N} \text{ s.t. } Q(n)$$

and to prove it false, we have to show that

$$\forall n \in \mathbb{N}, \sim Q(n)$$

is true.

More generally, when we negate a "for all" we get "there exists" and when we negate "there exists" we have "for all". This is a very important result and we'll summarise it by a theorem.

Theorem 6.2.1 *Let $P(x)$ be an open sentence over the domain A, then*

$$\begin{aligned} \sim (\forall x \in A, P(x)) &\equiv \exists x \in A \text{ s.t. } \sim (P(x)) \\ \sim (\exists x \in A \text{ s.t. } P(x)) &\equiv \forall x \in A, \sim (P(x)) \end{aligned}$$

*Note that the negated statement still has the same **domain**.*

Warning 6.2.2 The domain remains. An extremely common error to make is to assert that

$$\begin{aligned} \sim (\forall x \in A, P(x)) &\equiv \underbrace{\forall x \notin A}_{\text{bad!}} \text{ s.t. } P(x) && \text{do not do this} \\ &\equiv \underbrace{\exists x \notin A}_{\text{also bad!}} \text{ s.t. } P(x) && \text{nor this} \end{aligned}$$

These statements are not equivalent. Nor are

$$\sim (\exists x \in A, P(x)) \equiv \underbrace{\exists x \notin A}_{\text{bad!}} \text{ s.t. } P(x) \qquad \text{don't do this either}$$

The domain of the quantifier remains unchanged when the statement is negated.

For example, consider the statement

$$\forall n \in \mathbb{N} \text{ s.t. } n \geq 0.$$

This is definitely true; every natural number is non-negative. Because of this, the negation of the above must be false. However, if we were to *incorrectly* compute the negation as

$$\exists n \notin \mathbb{N}, n < 0$$

then we have a problem. Since the number $n = -1$ is not a natural number, and is less than 0, the above is also true. So remember

> The domain of a quantified statement does not change when it is negated.

Let us do a couple of simple examples and we'll get to more difficult ones in the next section.

Example 6.2.3 Determine whether or not the following statements are true or false and then prove your answer.

a $\exists n \in \mathbb{Z}$ s.t. $\frac{n+7}{3} \in \mathbb{Z}$.

b $\exists n \in \mathbb{Z}$ s.t. $\frac{n^2+1}{4} \in \mathbb{Z}$.

Solution. We tackle each in turn.

a This is true. It suffices to find one example of an integer n so that $\frac{n+7}{3}$ is an integer. Our proof just has do that — notice that it does not have to say how we found the example, just demonstrate that it works.

Proof. The statement is true. Let $n = 2$. Then $\frac{n+7}{3} = \frac{9}{3} = 3 \in \mathbb{Z}$ as required. ■

b This is a little harder. Notice that the statement is really saying that n^2+1 is divisible by 4. Try a few values of n:

$$n=1: \frac{1^2+1}{4} = \frac{2}{4} = \frac{1}{2} \qquad n=2: \frac{2^2+1}{4} = \frac{5}{4}$$
$$n=3: \frac{3^2+1}{4} = \frac{10}{4} = \frac{5}{2} \qquad n=4: \frac{4^2+1}{4} = \frac{17}{4}$$

Not looking promising. Notice it fails both when n is even and n is odd. We can use that to form our proof. Now, we don't directly prove the statement is false, instead we prove the negation is true. So we can start our proof by saying just that.

Proof. The statement is false; we demonstrate this by proving the negation to be true. The negation is

$$\forall n \in \mathbb{Z}, \frac{n^2+1}{4} \notin \mathbb{Z}$$

Assume $n \in \mathbb{Z}$, then n is either even or odd.

- When n is even, we can write $n = 2k$ for some integer k. Then $n^2+1 = 4k^2+1 = 4(k^2)+1$. Hence n^2+1 is not divisible by 4.
- On the other hand, when n is odd, we can write $n = 2k+1$ for some integer k. Then $n^2+1 = 4k^2+4k+2 = 4(k^2+k)+2$. Hence n^2+1 is not divisible by 4.

In either case $\frac{n^2+1}{4}$ is not an integer. Since the negation is true, the original statement is false. ■

□

Example 6.2.4 Determine whether or not the following statements are true or false and then prove your answer.

a $\forall n \in \mathbb{Z}, \frac{n^2+n}{2} \in \mathbb{Z}$.

b $\forall n \in \mathbb{Z}, n^2 - 8n + 1 < 0$.

Solution. We tackle each in turn.

a This one is quite similar to the second statement in the previous example. It is actually true, and we can leverage the parity of n to get the result we need.

Proof. The statement is true. Let $n \in \mathbb{N}$, then n is even or odd.

- When n is even, $n = 2k$ for some $k \in \mathbb{Z}$. Hence $n^2+n = 4k^2+2k = 2(2k^2+k)$ and so is even.

- When n is odd, $n = 2k + 1$ for some $k \in \mathbb{Z}$. Hence $n^2 + n = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$ and so is even.

Since in both cases $n^2 + n$ is even, it is divisible by 2. The result follows. ■

Alternatively, we might try to prove the result by noticing that $n^2 + n = n(n + 1)$ is the product of successive integers. Hence one if $n, n + 1$ must be even, and the product of any integer and an even number is even, so the result must be even. That would work, excepting that we have not actually proved that "product of any integer and an even number is even". Its not a hard result, but we should prove it before using it.

b If we just try plugging in a few small integers, the result holds. However, we know that n^2 gets bigger much faster than n does, so the result should be false when n is big. Consequently we can show it is false by just plugging in a sufficiently large value of n.

Proof. The result is false. We prove that the negation is true. The negation is

$$\exists n \in \mathbb{Z}, n^2 - 8n + 1 \geq 0.$$

Let $n = 10$ then $n^2 - 8n + 1 = 100 - 80 + 1 = 21 \geq 0$ as required. Since the negation is true, the original statement is false. ■

□

6.3 Nested quantifiers

Example 6.3.1 Rewrite the following using quantifiers, and then write out their negations.

- There is a real number y such that $1/y = y + 1$.
- For every integer z there is a natural number w such that $z^2 < w$.

Solution. Translating the statements into symbols gives

- $\exists y \in \mathbb{R}$ s.t. $1/y = y + 1$
- $\forall z \in \mathbb{Z}, \exists w \in \mathbb{N}$ s.t. $z^2 < w$

The negations are then:

$$\begin{aligned} \sim (\exists y \in \mathbb{R} \text{ s.t. } 1/y = y + 1) &\equiv \forall y \in \mathbb{R} \text{ s.t. } \sim (1/y = y + 1) \\ &\equiv \forall y \in \mathbb{R} \text{ s.t. } 1/y \neq y + 1. \end{aligned}$$

and

$$\sim (\forall z \in \mathbb{Z}, \exists w \in \mathbb{N} \text{ s.t. } z^2 < w) \equiv \exists z \in \mathbb{Z} \text{ s.t. } \sim (\exists w \in \mathbb{N} \text{ s.t. } z^2 < w)$$

$$\equiv \exists z \in \mathbb{Z} \text{ s.t. } \forall w \in \mathbb{N}, \sim (z^2 < w)$$
$$\equiv \exists z \in \mathbb{Z} \text{ s.t. } \forall w \in \mathbb{N}, z^2 \geq w.$$

Notice how we can slide the negation from left to right, and along the way $\forall$ becomes $\exists$ and $\exists$ becomes $\forall$. And, of course, the domains are unchanged by negation. □

The first statement is, in plain(er) english

> You can pick some real number y to make $1/y = y + 1$.

In writing it in this way, the authors have tried to take a mathematical statement into an exercise for the reader. "You — dear reader — can find (go on!) some real number y that has the property that $1/y$ is equal to $1 + y$". Similarly, we can write the statement $\forall z \in \mathbb{R}, z^2 \geq 0$ as

> No matter which real number z you pick, z^2 is non-negative.

Again, we have made this into an exercise for the reader. "You — dear reader — can pick any value of z you want, it will always turn out that $z^2 \geq 0$." So far, nothing too controversial, but lets move on to the second example above since it contains nested quantifiers. But a warning first.

Warning 6.3.2 Quantifiers do not commute. Nested quantifiers do not commute. The statement

$$\forall x, \exists y \text{ s.t. } P(x, y)$$

is not logically equivalent to

$$\exists y \text{ s.t. } \forall x, P(x, y).$$

The second example above contains nested quantifiers. Let us work through it slowly and carefully by writing it as a task for the reader.

> No matter which $z \in \mathbb{Z}$ you pick, there is a choice of $w \in \mathbb{N}$ so that $z^2 < w$.

This statement is true. To see why, we should think of it as a 2-player game. Player 1 picks any z they want, and Player 2 has to choose a value of w so that $z^2 < w$. Player 1 goes first and Player 2 goes second.

- Player 1 picks some integer z.
- Player 2 knows the value of z and can make their choice accordingly. With a little thought, Player 2 realises that choosing $w = z^2 + 1$ will work nicely (though they should be careful to make sure their choice is from the correct domain).

This is then the basis for proving the statement is true.

Proof. Let $z \in \mathbb{Z}$. Then choose $w = z^2 + 1$. Since $z \in \mathbb{Z}$ we know that w is a positive integer and so $w \in \mathbb{N}$. Then $z^2 < z^2 + 1 = w$. Hence the statement is

true. ■

What if we reverse the order of the quantifiers?

$$\exists w \in \mathbb{N} \text{ s.t. } \forall z \in \mathbb{Z}, z^2 < w.$$

Again, we translate this into an exercise for the reader

> You can choose some $w \in \mathbb{N}$, so that no matter what integer z is then chosen, $z^2 < w$.

This is false. Player 1 now has to pick w first, and Player 2 picks z second. Player 1 knows nothing about what Player 2 is going to do. So Player 1 might think "I'll pick a really big number, say $w = 100$", but then Player 2 knows this value of w and can just pick a large value of z.

The best way (arguably) to prove the statement false, is to show that its negation is true. So a **disproof** of the statement is just a proof of the negation of the statement. So write down the negation:

$$\forall w \in \mathbb{N}, \exists z \in \mathbb{Z} \text{ s.t. } z^2 \geq w.$$

and again write it as a task for the reader:

> No matter which $w \in \mathbb{N}$ you pick, there will always be some choice of $z \in \mathbb{Z}$ so that $z^2 \geq w$.

Again, Player 1 goes first and Player 2 goes second. Player 1 knows nothing about what Player 2 will do, but Player 2 knows what Player 1 did.

- Player 1 picks some $w \in \mathbb{N}$.
- Player 2 knows the value of w, and then (after some thought) picks $z = w+1$

It is now sufficient to check that Player 2 has chosen z from the correct domain, and that $z^2 = (w^2 + 2w + 1) \geq w$ (which it is since $w \geq 1$).

Proof. The original statement is false, so we prove the negation to be true. The negation is

$$\forall w \in \mathbb{N}, \exists z \in \mathbb{Z} \text{ s.t. } z^2 \geq w.$$

Let $w \in \mathbb{N}$ and then choose $z = w+1$. Since $w \in \mathbb{N}$, we know that $z \in \mathbb{Z}$. Further since $w \geq 1$ we know that

$$z^2 = (w+1)^2 = w^2 + 2w + 1 = (w^2 + w + 1) + w \geq w$$

as required. Since the negation is true, the original is false. ■

Remark 6.3.3 Context and dropping domain. Mathematicians sometimes are a little sloppy with the way they write quantifiers, especially when the domain of those quantifiers is known by context. For example, if we are doing calculus, then most of our variables will be real numbers, while if we are working on a number theory problem, then our variables are likely to be integers. If the context

is clear the domains will often be dropped.

So one might see the statement

$$\begin{aligned} \forall x \in \mathbb{R}, \text{ if } x < -2 \text{ then } x^2 > 4 \qquad & \text{written as} \\ \forall x, \text{ if } x < -2 \text{ then } x^2 > 4 \qquad & \text{or even as} \\ \text{if } x < -2 \text{ then } x^2 > 4 \qquad & \end{aligned}$$

In this last statement we have dropped the quantifier completely. The universal quantifier is implicit — the statement has to be true for all x in the domain. So, when you see an implication

$$P(x) \implies Q(x)$$

it is really

$$\forall x,\, P(x) \implies Q(x).$$

Your hard-working time-pressed mathematician has just dropped a couple of symbols to save time. This is a little sloppy, but really quite standard.

When students encounter nested quantifiers for the first time they typically find them quite difficult. They are quite difficult but they do become easier with practice. Accordingly, the following example is one of this author's favourites. It, and its kin, have appeared on nearly every exam the author has given for this topic. The author's students are likely to have chosen a different type of question as their favourite. We will provide lots of similar exercises for you — dear reader — to practice, so that any dislike of nested quantifiers can be dispelled.

Example 6.3.4 Consider the following four statements:

(a) $\exists x \in \mathbb{R}$ s.t. $\exists y \in \mathbb{R}$ s.t. $xy = x + y$

(b) $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, xy = x + y$

(c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $xy = x + y$

(d) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, xy = x + y$.

Determine the truth value of the following four statements and prove your answers.

We will work through the statements in order, thinking about them as two-player games as we did above. We will also drop "$\in \mathbb{R}$" from the quantifiers; the reader should understand, by context, that all our variables are selected from $\mathbb{R}$.

Finally — remember to be careful of the order of the quantifiers.

Scratchwork.

(a) This is true.

- Player 1 only has to choose one number. They choose $x = 0$.
- Player 2 only has to choose one number and they know Player 1's choice. So they also choose $y = 0$

Then $xy = 0 = 0 + 0 = x + y$.

(b) Again think about the 2 players:

- Player 1 gets to choose only one number x
- Player 2 then has to be able to choose any real number y so that the equation holds.

This feels unlikely to work because we can "solve" the equation for y to get $y = \frac{x}{x-1}$. That is, for any given x-value there is going be exactly one y-value. So it feels like it will likely be false.

To see that it *is* false, write down the negation:

$$\forall x, \exists y \text{ s.t. } xy \neq x + y.$$

- Player 1 can choose any real number x they want.
- Player 2 then has to be able to choose a real number y so that $xy \neq x + y$. A good choice is $y = 1$ (though there are infinitely many other good choices)

Now no matter what x-value, $xy = x$ and $x + y = x + 1$, and $x \neq x + 1$. We just have to write it up.

(c) This one is a bit tricky. Again, think about our two players.

- Player 1 chooses whatever x-value they feel like.
- Player 2 knows that value of x and based on that has to make a very careful choice of y. But that just requires Player 2 to solve $x + y = xy$. Easy!

$$\begin{aligned} x + y &= xy \\ y - xy &= -x \\ y &= \frac{-x}{1-x} = \frac{x}{x-1} \end{aligned}$$

All good? Not quite — things go wrong when $x = 1$.

When $x = 1$, our equation is $y + 1 = y$. There is no real number y that makes that true.

So everything seemed fine until we considered $x = 1$. So perhaps it is false?

You know the drill now; write down the negation and think about our two players:

$$\exists x \text{ s.t. } \forall y, xy \neq x + y.$$

- Player 1 makes a single careful choice of $x = 1$
- Then no matter what y-value Player 2 chooses, $xy = y$ and $x+y = y+1$, so $xy \neq x + y$.

So it is false. We just need to write out the proof.

(d) This one is also false, and it is easy to see why from the negation:

$$\exists x \text{ s.t. } \exists y \text{ s.t. } xy \neq x + y.$$

It suffices for Player 1 to pick $x = 1$, and Player 2 to then pick $y = 1$. Then $xy = 1$ and $x + y = 2$.

Solution.

Proof. The statement is true. Pick $x = y = 0$ then $xy = 0$ and $x + y = 0$. ■

We give two proofs of (b), depending on how we choose y.

Proof. We prove this to be false by showing the negation is true. The negation is

$$\forall x, \exists y \text{ s.t. } xy \neq x + y.$$

Pick any $x \in \mathbb{R}$, and then set $y = 1$. Since $xy = x$ and $x + y = x + 1$, we have that $xy \neq x + y$ as required. Since the negation is true, the original statement is false. ■

Proof. The statement is false. Let x be any real number. Then either $x = 0$ or $x \neq 0$.

- When $x = 0$ set $y = 1$. Then $xy = 0$ and $x + y = 1$.
- When $x \neq 0$ set $y = 0$. Then $xy = 0$ and $x + y = x$.

In both cases $xy \neq x + y$ as required. ■

Proof. The statement is false. We prove the negation

$$\exists x \text{ s.t. } \forall y, xy \neq x + y.$$

is true. Pick $x = 1$, then no matter what y is chosen, $xy = y$ and $x + y = y + 1$, and Consequently $y \neq y + 1$. Since the negation is true, the original statement is false. ■

Proof. The statement is false. We prove the negation:

$$\exists x \text{ s.t. } \exists y \text{ s.t. } xy \neq x + y.$$

Pick $x = 1$ and $y = 1$, then $xy = 1$ and $x + y = 2$. Since the negation is true, the original is false. ■

□

I hope the reader can see why the above example makes for a good exam question. It is a good mathematical and logical workout. With that in mind, we (and the author really means you) should do another one. But before we leap into another fun example, a quick warning about negating implications:

Warning 6.3.5 Common implication errors. Remember that

$$\sim (P \implies Q) \quad \equiv \quad (P \wedge \sim Q).$$

The negation of $P \implies Q$ is ***definitely not***

$$\underbrace{(P \implies \sim Q)}_{\text{bad!}} \qquad \text{don't do this!}$$

This is a surprisingly common error. Please memorise Theorem 4.2.7.

It's also important to remember that:

- if the hypothesis of an implication is false, the implication is true, and
- if the conclusion of an implication is true, the implication is true.

On with the fun!

Example 6.3.6 Determine the truth value of the following four statements and prove your answers.

- $\exists x \in \mathbb{R}$ s.t. $\exists y \in \mathbb{R}$ s.t. $(y \neq 0) \implies xy = 1$
- $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, (y \neq 0) \implies xy = 1$
- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $(y \neq 0) \implies xy = 1$
- $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (y \neq 0) \implies xy = 1$

Remember that we must pick the value of x before we pick the value of y in every single case. You might find it helpful to write out the negations before deciding how to approach these.

Scratchwork.

(a) As suggested, we'll write out the negation:

$$\forall x, \forall y, (y \neq 0) \wedge (xy \neq 1)$$

where we have dropped the "$\in \mathbb{R}$", assuming the reader will understand by context.

Now, in order to make an implication true, we just have to make the hypothesis false and that is quite easy in this case. Pick any x you want and then choose $y = 0$. That's enough to make a proof (it also gives us a hint for one of the other statements).

(b) Consider the original statement and what our two players have to do.

- Player 1 has to choose some x
- Now, no matter what Player 2 picks, the implication must be true.

Let us assume Player 1 has chosen some x we don't know what it is yet, but let us assume they have chosen it. What can player 2 do to make the implication true. There are three ways to make it true: (hypothesis, conclusion) = (T,T), (F,T), (F,F). When Player 2 picks $y = 0$, the hypothesis is false making the implication true. However, when they pick something else, the conclusion is only true when $y = 1/x$. But this means for every other choice of y, the conclusion will be false. It really looks like this statement is false.

Time to think about the negation:

$$\forall x, \exists y \text{ s.t. } (y \neq 0) \wedge (xy \neq 1)$$

What do our players do?

- Player 1 can pick whatever x they like, so they do.
- Player 2 knows the value of x and so can choose y to make the conjunction true. In particular, if Player 1 picked $x = 0$, then Player 2 can choose $y = 1$. On the other hand, if Player 1 picked any other value of x, then Player 2 can choose $y = -x$.

In both cases, $y \neq 0$ and $xy = 0$ or $xy = -x^2 \neq 1$. Now just write it up!

(c) Think about our two players.

- Player 1 can pick any x they want.
- Player 2 just needs to make the implication true by careful choice of y. By choosing $y = 0$ they can make the hypothesis false.

Since Player 2 can always make the hypothesis false, the implication is true.

(d) Last one. Oof.

Again, let us think about our two players.

- Player 1 picks whatever x they want
- Player 2 picks whatever y they want.

So, no matter what x was chosen, when Player 2 chooses $y = 0$, the implication is true. But what about all the other choices? This feels unlikely.

Look at the negation:

$$\exists x \text{ s.t. }, \exists y \text{ s.t. } (y \neq 0) \wedge xy \neq 1$$

Ah - this is much easier. Player 1 can choose $x = 0$ and Player 2 can choose $y = 1$. Then $y \neq 0$ and $xy = 0 \neq 1$. There are, many other choices that would also work.

Solution.

Proof. The statement is true. Take $x = 0, y = 0$. Since the hypothesis is false, the implication is true. ■

Proof. The statement is true. Take $x = y = 2$, then the hypothesis and conclusion are both true, so the implication is true. ■

Proof. The statement is false, so we prove the negation:

$$\forall x, \exists y \text{ s.t. } (y \neq 0) \wedge (xy \neq 1)$$

Let x be any real number. Then either $x = 0$ or $x \neq 0$.

- If $x = 0$ then pick $y = 1$. Then $xy = 0$
- On the other hand, if $x \neq 0$ then pick $y = -x$, so that $xy = -x^2$

In both cases, $y \neq 0$ and $xy \neq 1$. Since the negation is true, the original statement is false. ■

Proof. We prove the negation. Let x be any real number. Either $x = 1$ or $x \neq 1$. If $x = 1$ then pick $y = 2$, so that $xy = 2$. On the other hand, if $x \neq 1$, then pick $y = 1$ so that $xy = x \neq 1$. In both cases we have that $y \neq 0$ and $xy \neq 1$ as required. Since the negation is true, the original is false. ■

Proof. The statement is true. Let x be any real number and then choose $y = 0$. Since the hypothesis is false, the implication is true. ■

Proof. Let x be any real number. Either $x = 0$ or $x \neq 0$. If $x = 0$ then pick $y = 0$ making the hypothesis false. On the other hand, if $x \neq 0$ then set $y = \frac{1}{x} \neq 0$. In that case both the hypothesis and conclusion are true. In either case the implication is true. ■

Proof. We prove the negation:

$$\exists x \text{ s.t. }, \exists y \text{ s.t. } (y \neq 0) \wedge xy \neq 1.$$

Pick $x = 0, y = 1$. Then $xy = 0 \neq 1$ and $y \neq 0$. Hence the negation is true and the original statement is false. ■

□

6.4 Quantifiers and rigorous limits

Quantifiers appear all over mathematics, and it is essential that you become comfortable reading, understanding and applying them. One particularly important application of quantifiers is to make the notions of limits rigorous. You have[51]

[51] At least we think you should have. The authors are making an assumption about your mathematical education here, and we appologise if you have, in fact, not encountered limits before now.

encountered the idea of a **limit** when you studied differential calculus; the idea that the value of a function gets "closer and closer" to a particular value as we take the argument of that function "closer and closer" to (say) zero. Quantifiers allow us to make "closer and closer" rigorous[52]. We will start with the idea of the limit of a sequence, and then move on to limits of functions. So we begin with the definition of a sequence.

6.4.1 Convergence of sequences

Definition 6.4.1 A **sequence** is an ordered list of real numbers. It is typically denoted

$$(x_n)_{n\in\mathbb{N}} = (x_1, x_2, x_3, \dots)$$

The numbers $x_1, x_2, \cdots$ are the **terms** of the sequence. You will also sometimes see alternate notation such as

$$(x_n)_n, \qquad (x_n)_{n\geq 1} \qquad \text{or} \qquad (x_n)$$

In some texts you will also see a sequence denoted with braces, $\{x_n\}$; we will not use that notation to avoid confusion with set notation. ♢

An alternate way to think of a sequence is as a function that takes natural numbers and maps them to real numbers. So, for example, the sequence $\left(\frac{1}{n}\right)_n$ is just the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = 1/n$. We will come back to functions in Chapter 10. We also note that one can generalise this definition to sequences of other types of numbers or objects, but we will focus on sequences of real numbers.

We will typically define a particular sequence either by giving the first few terms,

$$(x_n) = (2, 3, 5, 7, 11, \dots)$$

or by giving a formula for the n^{th} term in the sequence:

$$(x_n) = \left(\frac{1}{n}\right)$$

ie, the sequence $(1, 1/2, 1/3, 1/4, \dots)$. Just as was the case with defining sets, we must make sure that we give the reader enough information to understand our definition. In this way, giving a formula for the n^{th} term is typically preferred[53].

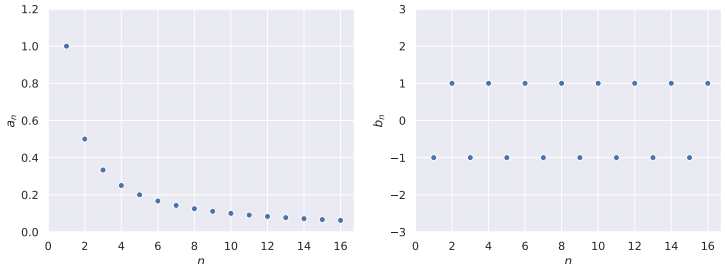
[52] We recommend looking up the history of **infinitesimals** and **fluxions** which predate the rigorous definition of limits we give here. The calculus of Newton and Leibniz used infinitesimals to understand limits and derivatives. These ideas were attacked by Berkeley in his 1734 book "The Analyst" in which he refers to infinitesimals as "the ghosts of departed quantities". The rigorous definition of limits, and so a rigorous foundation for calculus, was given almost 150 years later by Cauchy, Bolzano and Weierstrass. There is much of interest here for a motivated reader with a good search engine. We also recommend a little digression into **surreal numbers**, **hyperreal numbers** and **nonstandard analysis** — things can get pretty weird.

[53] The reader should assume that the sequence $(2, 3, 5, 7, 11, \dots)$ is just the prime numbers, but

Typically one can compute the first few terms of a sequence by hand, and then the next many terms by computer. However, we are very often interested in the behaviour of the terms of the sequence as n becomes very large. So, for example, consider the sequences

$$(a_n) = \left(\frac{1}{n}\right)_n = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$$
$$(b_n) = \left((-1)^n\right)_n = (1, -1, 1, -1, \dots)$$

Here are plots of the first few terms to see how they behave.



Notice that in the first sequence, as n gets larger and larger, the term a_n gets closer and closer to 0. In this case we will say that "a_n **converges** to 0". In the second case, the sequence simply bounces back and forth between $+1$ and -1 and does not appear to "settle" to any particular value. In this case we will say that "b_n diverges". Let's see how we can turn this intuitive (but imprecise) understanding of convergence and turn it into a rigorous, precise definition.

To start towards that definition, let us rephrase convergence as a sort of two player game as we did for some of the nested quantifier examples earlier in this chapter, and focus on the example of $(a_n) = \left(\frac{1}{n}\right)$. In this game, Player 1 has to choose a small positive number, and then Player 2 has to work out how big does n have to be so that we can guarantee that the distance between a_n and 0 is smaller than Player 1's number. So, for example,

- Player 1 says "Make the distance between a_n and 0 smaller than 0.01."
- Player 2 does some thinking and replies "Choose any $n > 100$ and then it will work."

We can verify this by computing the distance[54] between two numbers using the

a little digging in the Online Encyclopedia of Integer Sequences shows a few other possibilities (some reasonable and some weird) including "partitions" (the number of ways of writing a given integer as a sum of smaller positive integers), "additive primes" (the sum of the digits is also prime), "absolute primes" (every permutation of the digits is also a prime) and lengths of "Farey sequences" (the reader should search-engine this one). Of course, if we are doing a good job as an author then we will have provided the reader with enough context to determine the sequence correctly.

[54]The absolute value is an example of a **metric** or **distance function**. You already know other examples of metrics — for example, the distance between points (x, y) and (z, w) on the Cartesian plane is given by $d = \sqrt{(x-z)^2 + (y-w)^2}$; this is called the Euclidean metric. Another way to compute distances on the Cartesian plane is the taxicab metric $d = |x - z| + |y - w|$. A search-engine will guide you to more on this topic.

absolute value, and then noting that

$$n > 100 \qquad \Longrightarrow \qquad \frac{1}{n} < \frac{1}{100} = 0.01 \qquad \Longrightarrow \qquad \left|\frac{1}{n} - 0\right| < 0.01.$$

This is just one instance, and Player 1 could have chosen 0.01 or 2^{-30}. Indeed, Player 1 can choose ***any arbitrarily small*** positive number ε, and then Player 2 can always work out how big to make n so that the distance between a_n and 0 is guaranteed to be smaller than ε.

- Player 1 says "I have chosen $\varepsilon > 0$, please make the distance between a_n and 0 smaller than ε."
- Player 2 does some thinking and replies "Choose any $n > \frac{1}{\varepsilon}$ and then it will work."

Again, we can verify this:

$$n > \frac{1}{\varepsilon} \qquad \Longrightarrow \qquad \frac{1}{n} < \varepsilon \qquad \Longrightarrow \qquad \left|\frac{1}{n} - 0\right| < \varepsilon$$

That is, Player 2 can set some point in the sequence $N = \frac{1}{\varepsilon}$, so that for every value of $n > N$, we can guarantee that $|a_n - 0| < \varepsilon$. We can rephrase this outcome as

> No matter which $\varepsilon > 0$ that Player 1 chooses, Player 2 can always find some N so that if $n > N$ then $|a_n - 0| < \varepsilon$.

If we now try to play the same game with the second sequence $(b_n) = ((-1)^n)$, then we will quickly see that it does not work. Let us attempt to show that the sequence converges to 1 using the same game

- Player 1 says "Make the distance between b_n and 1 smaller than 0.01."
- Player 2 does some thinking and replies "Well, if n is even then this will be true, but it will fail for every single odd value of n. I cannot give you a guarantee."

Player 2 cannot win. When n is even, $b_n = 1$ and so $|b_n - 1| = 0 < 0.01$, but when n is odd, $b_n = -1$ and so $|b_n - 1| = 2 > 0.01$. So there is no way to make n sufficiently big, ie all n bigger than some N, so that $|b_n - 1| < 0.01$. We can rephrase this outcome as

> There is a choice of ε so that no matter which N Player 2 chooses, there will always be some $n > N$ so that $|b_n - 1| > \varepsilon$.

This example will help us understand how to turn the intuitive idea of convergence into the rigorous definition.

6.4.1.1 Quantifying towards a definition

To start towards our definition, we need to first describe all the objects involved, and then we explain that convergence means that the objects satisfy certain conditions. In this case, we will start with something like

> Let $(x_n)_n$ be a sequence of real numbers, that is, $x_n \in \mathbb{R}$, for all $n \in \mathbb{N}$. Then we say that (x_n) converges to a real number L when …

and now we need to explain those conditions. Let us start with our intuitive idea

> Let $(x_n)_n$ be a sequence of real numbers, that is, $x_n \in \mathbb{R}$, for all $n \in \mathbb{N}$. Then we say that (x_n) converges to a real number L when x_n gets ***closer and closer*** to L as n becomes ***larger and larger***.

While this is quite descriptive, it is not actually all that precise. Let us rewrite it as follows:

> Let $(x_n)_n$ be a sequence of real numbers, that is, $x_n \in \mathbb{R}$, for all $n \in \mathbb{N}$. Then we say that (x_n) converges to a real number L when x_n moves ***arbitrarily close*** to L as n becomes ***larger and larger***.

Now this means that in order for x_n to converge to L, it must be possible to bring the number x_n as close to L as we want by making n really big. This is exactly like the two player game we described above: Player 1 can make any choice of ε and then Player 2 has to work out how large to make n. Let us write our definition again:

> Let $(x_n)_n$ be a sequence of real numbers, that is, $x_n \in \mathbb{R}$, for all $n \in \mathbb{N}$. Then we say that (x_n) converges to a real number L when ***no matter what positive*** ε ***we choose***, then ***for all*** n ***sufficiently large***, we have $|x_n - L| < \varepsilon$.

Now, in our game, Player 2 established just how big n needs so that we can guarantee the sequence terms are close enough to L. We denoted that threshold by N, and it will almost always depend on ε. Of course, since Player 1 chooses ε first, Player 2 can pick N with full knowledge of what Player 1 did. Time for another attempt at the definition:

> Let $(x_n)_n$ be a sequence of real numbers, that is, $x_n \in \mathbb{R}$, for all $n \in \mathbb{N}$. Then we say that (x_n) converges to a real number L when ***for all*** $\varepsilon > 0$, ***there is*** $N \in \mathbb{N}$, so that ***for all natural numbers*** $n > N$ we have $|x_n - L| < \varepsilon$.

This is now pretty good. All that remains is to tweak the phrasing a little, clean it up and make it into a formal definition[55].

[55]This is, more or less, the definition given by Bolzano in 1816.

Definition 6.4.2 Let (x_n) be a sequence of real numbers. We say that (x_n) has a **limit** $L \in \mathbb{R}$ when

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, (n > N) \implies (|x_n - L| < \varepsilon).$$

In this case we say that the sequence **converges** to L and write

$$x_n \to L \qquad \text{or} \qquad \lim_{n\to\infty} x_n = L.$$

If the sequence doesn't converge to any number L, we say that the sequence **diverges**. ♢

6.4.1.2 Some examples

Time to put our newly rigorised[56] understanding of limits to work.

[56]The authors were quite sure that "rigorised" is not a real word and were surprised to find it in the dictionary.

Example 6.4.3 Let $c \in \mathbb{R}$. Show that the constant sequence $(x_n)_{n\in\mathbb{N}} = (c)_{n\in\mathbb{N}}$ converges to c.

Scratchwork. We know, at least intuitively, that the constant sequence must converge to that constant value its elements take. It makes sense. But, how can we prove it rigorously using the definition of sequence convergence.

Again, think of the two player game[57]. We need to show that no matter which $\varepsilon > 0$ that Player 1 chooses, Player 2 can always find a threshold $N \in \mathbb{N}$ (which, in general will depend on ε), so that whenever $n > N$ we have $|x_n - c| < \varepsilon$.

Now assume that Player 1 picked an $\varepsilon > 0$ and try to understand what Player 2 needs to show. They want to show that with the right choice of $N \in \mathbb{N}$, we can keep $|x_n - c| < \varepsilon$. But since x_n constant $x_n = c$, we can simplify the inequality that needs to be satisfied:

$$|x_n - c| < \varepsilon \qquad \text{becomes} \qquad |c - c| = 0 < \varepsilon.$$

But since we know $\varepsilon > 0$, this will be true independent of n. This, in turn, means that Player 2 is free to choose any $N \in \mathbb{N}$. Now Player 2 could be mysterious and pick any random value (eg $N = 98127$) but let's force Player 2 to be a bit nicer to the audience (ie to the reader) and get them to pick a sensible value like $N = 1$. Then for all $n > N = 1$, we have $|x_n - c| = |c - c| = 0 < \varepsilon$ as required.

All that remains is to write this nice and tidy in a proof.

Solution.

Proof. Let $\varepsilon > 0$ be given and pick $N = 1$. Then, for all $n > N$, we have

$$|x_n - c| = |c - c| = 0 < \varepsilon.$$

Therefore we conclude that (x_n) converges to c as required. ■

□

Notice that the proof in our example does not have to explain ***how*** we came up with the choice of N, it merely has to show that it works. Our definition of convergence requires that "there exists N", not that "there exists N with a nice explanation of how it was found". This can sometimes make a nicely written proof seem much more clever than it really is. The reader can be left thinking "How on earth did they know how to choose that?". But you should remember that the author of the proof probably did a lot of careful scratch work to work out how to make the proof nice and neat; that scratch work is not required for the proof to be valid. Of course, if the author knows their audience well, and knows they might need some help, then the author will hopefully leave some explanation in the text nearby.

Example 6.4.4 Show that the sequence $(a_n)_{n\in\mathbb{N}} = (\frac{1}{n})_{n\in\mathbb{N}}$ converges to 0.

Scratchwork. This is precisely the example we did above to introduce the idea of convergence. We have even discussed how to think of proving the convergence

[57] We'll dispense with this analogy soon.

as a two player game. So, let us dispense with the game analogy.

We are going to start by choosing some arbitrary $\varepsilon > 0$, and then we need to find a threshold N so that for every $n > N$, we know that $|a_n - 0| < \varepsilon$. Now, our argument from the previous example won't work here, since $|a_n - 0| = \frac{1}{n}$ is no longer constant and varies with n. Let us try rewriting things a little to help us think.

We've chosen some arbitrary $\varepsilon > 0$, and we need to show that $|a_n - 0| = \frac{1}{n} < \varepsilon$ when n is big enough, ie for $n > N$. But the inequality

$$\frac{1}{n} < \varepsilon \qquad \Longleftrightarrow \qquad n > \frac{1}{\varepsilon}$$

since both n, ε are positive. So this means that given our ε, we need to work out how to make sure that $n > \frac{1}{\varepsilon}$, because then we know that $\frac{1}{n} < \varepsilon$.

Well, if we set $N = \frac{1}{\varepsilon}$, then when we require $n > N$, then we have that $n > N > \varepsilon$ and so $\frac{1}{n} < \frac{1}{N} < \frac{1}{\varepsilon}$ implying that $\frac{1}{n} < \frac{1}{\varepsilon}$. The only hitch is that our definition requires that N be a natural number. But that is easy to fix, instead of setting $N = \frac{1}{\varepsilon}$, we can just take N to be the next integer bigger than $\frac{1}{\varepsilon}$. That is, we set N to be the **ceiling** of $\frac{1}{\varepsilon}$ which we write as $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$.

Here, we should mention that even though any $N > \frac{1}{\varepsilon}$ would work for our arguments above and that we have infinitely many possible choices for N it is important to either picking a specific N as we did in this scratch work, or prove the existence of such an N without giving it explicitly — using results like the Archimedean property[58] or similar. Simply saying that such number *should* exist is not an adequate justification. With that caveat out of the way, let's tidy this up and write the proof.

Solution.

Proof. Let $\varepsilon > 0$. Then pick $N = \left\lceil \frac{1}{\varepsilon} \right\rceil$, so that $N \geq \frac{1}{\varepsilon}$. Then for all $n > N$, we have that

$$|a_n - 0| = \frac{1}{n} < \frac{1}{N} = \varepsilon$$

Therefore we see that (a_n) converges to 0. ■

□

Example 6.4.5 Show that the sequence $(x_n) = \left(\frac{2n+4}{n+1}\right)$ converges to 2.

Scratchwork. Again, let's start with the scratch work.

Our zeroth[59] step is to pick some arbitrary $\varepsilon > 0$. Now, as we saw in the previous example, we need to understand how to choose n so that we can guarantee

$$|x_n - L| = \left|\frac{2n+4}{n+1} - 2\right| < \varepsilon$$

[58]This says, roughly, that given any two positive real numbers, x, y, we can always find an integer n, so that $nx > y$. This appears in a more geometric guise in Book V of Euclid's Elements, and Archimedes attributes it to Eudoxus.

To help us understand this, we should clean up the inequality and simplify the expression inside the absolute value. Simplifying shows us that we actually want

$$\left|\frac{2n+4}{n+1}-2\right| = \left|\frac{2n+4-(2n+2)}{n+1}\right| = \left|\frac{2}{n+1}\right| < \varepsilon.$$

At this point we should reiterate — we have not proved this yet, this is just what we *want* to be true. We still need to work out how we choose n to make sure this is true.

We can even go a little further. Since we know n is a natural number, we know that $\frac{2}{n+1} > 0$ and so we can write

$$\left|\frac{2}{n+1}\right| = \frac{2}{n+1} < \varepsilon.$$

This is a little easier to manipulate, and we can quickly isolate n:

$$\frac{2}{n+1} < \varepsilon \qquad \iff \qquad n+1 > \frac{2}{\varepsilon} \qquad \iff \qquad n > \frac{2}{\varepsilon} - 1$$

So, this means that if we have $n > \frac{2}{\varepsilon} - 1$, then we know (moving back along our chain of reasoning) that $|x_n - 2| < \varepsilon$. So it makes sense for us to choose $N = \left\lceil \frac{2}{\varepsilon} \right\rceil - 1$. Again, we make use of the ceiling function to ensure that N is an integer.

Oops! But be careful — what happens if $\varepsilon = 1000$? Then we choose $N = 0$ which is not a natural number. Thankfully this is easily fixed, let us just take N to be a little bit larger. Indeed, we can set $N = \left\lceil \frac{2}{\varepsilon} \right\rceil$ and everything works out nicely. More generally, in these types of proofs, once you have worked out an N, one is free to make it *larger* (you should ask yourself why). One could also argue that the choice $N = \left\lceil \frac{2}{\varepsilon} \right\rceil$ is a little neater[60] and does not change the proof very much.

Solution.

Proof. Let $\varepsilon > 0$, set $N = \lceil (2/\varepsilon) \rceil$ and let $n > N$. By our choice of n, we know that

$$n > N > \frac{2}{\varepsilon}$$

From this we know that $n + 1 > n > \frac{2}{\varepsilon}$ and so

$$\frac{2}{n+1} < \varepsilon$$

Hence

$$|x_n - 2| = \left|\frac{2n+4}{n+1} - 2\right| = \left|\frac{2}{n+1}\right| = \frac{2}{n+1} < \varepsilon,$$

as required. ■

□

[59]This is barely a step at all really. But we do need ε.

Of course we also know that not every sequence converges — we have already seen an example of this above. Let us redo that example using our rigorous definition of convergence.

Example 6.4.6 Show that the sequence $(b_n) = ((-1)^n)$ does not converge to 1.

Scratchwork. Even though we wish to show that the sequence (b_n) does not converge to 1, our starting point will still be the definition of convergence. Recall that the sequence (b_n) converges to 1 when

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}(n > N) \implies (|b_n - 1| < \varepsilon).$$

We want to show that this is false, and we do so by showing that the negation is true. The negation is

$$\exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N}, \exists n \in \mathbb{N} \text{ s.t. } (n > N) \wedge (|(-1)^n - 1| \geq \varepsilon).$$

So, if we can show that this is true, then the original statement is false, and so the sequence (b_n) does not converge to 1.

Now, let's try to understand this statement. It says that we need to show that there is an $\varepsilon > 0$ such that no matter what $N \in \mathbb{N}$ we choose, there is always at least one $n > N$ such that $|(-1)^n - 1|$ is greater than ε. Now, notice that we don't have to show this for all ε (indeed we can't), we just need to find one ε that makes things work.

As we saw above, sequence (b_n) alternates between -1 and 1.

- For n even, $b_n = (-1)^n = 1$ and so $|b_n - 1| = 0 < \varepsilon$. So this is true for all $\varepsilon > 0$ and all even n.
- On the other hand, for n odd, we have $b_n = (-1)^n = -1$, and so $|b_n - 1| = |-1-1| = 2$. So, as long as we choose $0 < \varepsilon < 2$, this case will fail for all odd $n \in \mathbb{N}$.

So to make the proof work, we can choose, say, $\varepsilon = 1$ and then show that things go wrong for odd n. Time for the proof.

Solution.

Proof. We show that the sequence $(b_n) = ((-1)^n)$ doesn't converge to 1. To this we show that

$$\exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N}, \exists n \in \mathbb{N} \text{ s.t. } (n > N) \wedge (|(-1)^n - 1| \geq \varepsilon).$$

Let $\varepsilon = 1$ and let N be any natural number. Now set $n = 2N + 1$, so that $b_n = (-1)^n = -1$. Then

$$|b_n - 1| = |-1-1| = 2 > \varepsilon.$$

Therefore we conclude that the sequence does not converge to 1. ■

[60] Mathematicians generally find "neatness" to be desirable in a proof. Of course, one should not make things so neat that the logic is obscured.

Notice that we set $n = 2N + 1$ in our proof and everything worked out nicely. We can actually choose any odd number larger than N. In fact, we could change the wording of the proof to say "Let n be any odd number larger than N" and it would be correct. But, since we can can make a simple explicit choice, we should do so. □

In general, when we talk about the divergence of a sequence, we don't say that the sequence does not converge to a given specific number L. Rather, we typically want to prove[61] that it does not converge to ***any*** number L. This is a much stronger statement. Written as a quantified statement, it is

$$\forall L \in \mathbb{R}, \exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N}, \exists n \in \mathbb{N} \text{ s.t. } (n > N) \wedge (|x_n - L| \geq \varepsilon),$$

and says that for any number L, the sequence doesn't converge to that number.

Let's do an example where we show that a sequence actually diverges. That is, it does not converge to any number $L \in \mathbb{R}$.

Example 6.4.7 Show that the sequence $(x_n) = (n)$ diverges.

Scratchwork. First, let's write down what we want to show:

$$\forall L \in \mathbb{R}, \exists \varepsilon > 0 \text{ s.t. } \forall N \in \mathbb{N}, \exists n \in \mathbb{N} \text{ s.t. } (n > N) \wedge (|n - L| \geq \varepsilon).$$

Since we need to make this work for every possible L, we let L be an arbitrary real number. Now, what we want is to satisfy $|n - L| \geq \varepsilon$ for some ε and some n. It can be a little intimidating to try to this for an arbitrary L, so perhaps it is better to think about a few different L-values.

- When $L = 0$, the we can simplify $|n - L| = |n| = n$. Now since, $n \in \mathbb{N}$, we know that $n \geq 1$. So if we pick $\varepsilon = 1$, then we will have $|n - L| \geq \varepsilon$ for all $n \in \mathbb{N}$. Now, it is easy to also enforce the requirement that $n > N$, no matter what N is chosen, just pick $n = N + 1$.

- Similarly, if we set $L = -1$, then we have $|n - L| = |n - (-1)| = n + 1$. So again, we can pick $\varepsilon = 1$ and then $|n - L| \geq \varepsilon$ for all $n \in \mathbb{N}$. This reasoning will actually work for any $L \leq 0$. Again, to also enforce the requirement that $n > N$, we can just pick $n = N + 1$.

- What about when, say, $L = 17$, we will have $|n - L| = |n - 17|$. Now, provided $n > 17$, this will be bigger than zero. In particular, if we set $n \geq 18$, then we will have $|n - 17| > 1$. ***However***, this is not quite right. We not only need that $|n - 17| \geq \varepsilon$ but we also need that $n > N$, no matter what choice of $N \in \mathbb{N}$. Thankfully this is easily fixed, just choose $n = \max\{17, N\} + 1$. Alternatively, since we know $N \geq 1$, we can choose $n = N + 17$.

- More generally, if $L > 0$, then we can choose $n \geq \lceil L \rceil + 1$, and then $|n - L| \geq 1$. Just as in the previous point, we need to satisfy $n > N$, so

[61] There are exceptions to this. For example, one easy way to show that a *series* diverges is to show that summands do not converge to zero.

pick $n = \max\{\lceil L \rceil, N\} + 1$. Notice that a similar choice works when $L \leq 0$, just take $n = \max\{\lceil |L| \rceil, N\} + 1$.

We are now ready to write the proof.

Solution.

Proof. Let L be an arbitrary real number and set $\varepsilon = 1$. Then for any $N \in \mathbb{N}$, set $n = \max\{N, \lceil |L| \rceil\} + 1$. Then this gives

$$n > N \qquad \text{and} \qquad |n - L| > 1 = \varepsilon$$

Therefore the sequence $(x_n) = (n)$ diverges. ■

Notice how short the proof is compared to the scratch work. This is not unusual. A nice neat proof can hide a lot of work. □

6.4.2 The limit of a function

Note that in this section of the text we restrict ourselves to real-valued functions. That is, functions that take a real number as input and return a real number as output, just like those you worked with in Calculus courses. We do look at more general functions in Chapter 10, but not their limits.

We define the limits of functions in very much the same way as the limits of sequences. The definition is more general, as now we can talk of the limit of a function as its argument approaches more general points, while for a sequence, we only talked of its behaviour as $n \to \infty$. Let's give the definition and then we'll explain it.

Definition 6.4.8 Let $a, L \in \mathbb{R}$ and let f be a real-valued function. We say that the **limit** of f as x approaches a is L when

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } (0 < |x - a| < \delta) \implies (|f(x) - L| < \varepsilon).$$

In this case we write

$$\lim_{x \to a} f(x) = L \qquad \text{or sometimes} \qquad f(x) \xrightarrow[x \to a]{} L$$

and say that f **converges** to L as x approaches a. We also sometimes say the limit of f as x goes to a is L, which we denote by

$$f(x) \to L \text{ as } x \to a.$$

If f does not converge to any finite limit L as x approaches a, then we say that f **diverges** as x approaches a. ◊

This definition may look[62] more complicated than the equivalent definition for the convergence of a sequence, Definition 6.4.2. But if we do some reverse

[62] In fairness, it is a little more complicated. But it is not *that* much more complicated.

engineering (much as we did for sequence convergence above, but in reverse) then we can read the definition as

- **For all** positive ε, **there is some** positive δ so that **if** the distance between x and a is less than delta (but not zero), **then** the distance between $f(x)$ and L is less than ε.

We can rephrase these quantifiers as little:

- **No matter which** positive ε, **we can always find some** positive δ so that **if** the distance between x and a is less than delta (but not zero), **then** the distance between $f(x)$ and L is less than ε.

This is telling us that if we need to make the distance between the function and its limit very small, we can always find some δ so that we just need to make the distance $|x - a| < \delta$. That is

- We can make the distance between $f(x)$ and L **as small as we want**, by making the distance between x and a **arbitrarily small** (but not zero).

So we reach

- We can make $f(x)$ **closer and closer** to L by taking x **closer and closer** to a (but not actually equal).

This is probably, more or less, the working definition of a limit of a function you used in your first Calculus course. This gives reasonable intuition, but the power of quantifiers is to make everything precise and eliminate misunderstandings.

Remark 6.4.9 Why exclude $x = a$. Notice that the definition of convergence of a function says that given any ε we can find δ so that when

$$0 < |x - a| < \delta$$

we know that the distance between the function and its limit is smaller than ε. This hypothesis tells us that the distance between x and a has to be small, but not zero — that is we do not require the function be close to its limit exactly at $x = a$. This is because the definition of **limit** has been crafted to tell us how the function behaves as it *approaches* $x = a$. We do not care about what happens exactly at $x = a$, and indeed we do not even require that the function be defined there. In fact, many important applications of limits — such as derivatives — would not work if we extended this hypothesis to include $x = a$.

Notice also, that our definition of limits of sequences had a similar quirk. We defined the limit in terms of the behaviour of the sequence terms as n became very very large. We did not care about the "infiniteth" term in the sequence — if such a thing were defined.

Let's put this definiton to work by considering the limit of a simple function.

Example 6.4.10 Show that for any $a \in \mathbb{R}, \lim_{x \to a} x = a$.

Scratchwork. In this example we want to show that the function $f(x) = x$ converges to the limit a as x goes to a. Even though this feels more like a tautology than an example, it is a good exercise in applying the limit definition.

To prove that $\lim_{x \to a} x = a$ the definition tells us that we need to show that

$$\forall \varepsilon > 0, \exists \delta \text{ s.t. } 0 \text{ s.t. } (0 < |x - a| < \delta) \implies (|f(x) - L| < \varepsilon).$$

Now, this simplifies immediately since we have $f(x) = x$ and $L = a$:

$$\forall \varepsilon > 0, \exists \delta \text{ s.t. } 0 \text{ s.t. } (0 < |x - a| < \delta) \implies (|x - a| < \varepsilon).$$

So, given an arbitrary $\varepsilon > 0$, we need a $\delta > 0$ such that

$$0 < |x - a| < \delta \implies |x - a| < \varepsilon.$$

That is, whatever positive δ we pick, whenever $0 < |x - a| < \delta$, it implies that $|x - a| < \varepsilon$. A good[63] choice for δ is simply $\delta = \varepsilon$.

Now, let's write this in a proof.

Solution.

Proof. Suppose ε is any positive real number. Then pick $\delta = \varepsilon$. Then whenever $|x - a| < \delta$, then we know that $|f(x) - a| = |x - a| < \delta = \varepsilon$ as required. ■

□

That one is arguably a little too simple. Here is a slightly more complicated one.

Example 6.4.11 Let $a \in \mathbb{R}$. Prove that

$$\lim_{x \to a} 3x + 5 = 3a + 5.$$

Scratchwork. Again, our starting point is to look at the definition. We need to prove that

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } (0 < |x - a| < \delta) \implies (|(3x + 5) - (3a + 5)| < \varepsilon).$$

Before we go much further, we should clean this up a little. We can simplify that last inequality. That is $|(3x + 5) - (3a + 5)| = |3x - 3a| = 3|x - a|$, so

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } (0 < |x - a| < \delta) \implies (3|x - a| < \varepsilon).$$

Now this is looking pretty similar to the previous example. Given any ε, we need to pick δ, so that when $0 < |x - a| < \delta$, we guarantee that $3|x - a| < \varepsilon$. If we rearrange this last inequality, we want

$$|x - a| < \frac{\varepsilon}{3}.$$

[63] There are an infinite number of possible correct choices of δ. Indeed, $\delta = \varepsilon/n$ for any $n \in \mathbb{N}$ works. But the choice of $\delta = \varepsilon$ is good because it works, while being neat and simple.

And thus we pick $\delta = \frac{\varepsilon}{3}$.

Alternatively, if we assume that we have $0 < |x - a| < \delta$, then multiplying everything by 3 gives:

$$0 < 3|x - a| < 3\delta$$

and thus we need $3\delta \leq \varepsilon$. So again, we reach the neat choice of $\delta = \frac{\varepsilon}{3}$. Time for the proof.

Solution.

Proof. Let ε be any positive real number. Then pick $\delta = \frac{\varepsilon}{3}$. Then as long as $|x - a| < \delta$, we have that

$$|(3x + 5) - (3a + 5)| = |3x - 3a| = 3|x - a| < 3\delta = \varepsilon.$$

And thus $(3x + 5)$ converges to $3a + 5$ as x approaches a. ■

As you can see, the proof of the statement is very short, clean, and doesn't omit any necessary information. And, as was the case with our proofs above, we don't explain to the reader how we come up with the choice of $\delta = \varepsilon/3$, we just have to prove that it works. □

Let's ratchet up the difficultly a little.

Example 6.4.12 Show that $\lim_{x \to 2} \left(\frac{1}{x} \right) = \frac{1}{2}$.

Scratchwork. Just like in our previous example(s) we start with the definition of convergence and adapt it to the problem at hand. We need to show that

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } (0 < |x - 2| < \delta) \implies \left(\left| \frac{1}{x} - \frac{1}{2} \right| < \varepsilon \right).$$

Again, we can clean up and simplify the final inequality, since

$$\left| \frac{1}{x} - \frac{1}{2} \right| = \left| \frac{2 - x}{2x} \right| = \frac{|x - 2|}{2|x|}.$$

Thus we need to show that

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } (0 < |x - 2| < \delta) \implies \left(\frac{|2 - x|}{2|x|} < \varepsilon \right).$$

Now that we know what we need to show, let $\varepsilon > 0$ be arbitrary. Then, we want a $\delta > 0$ such that if we assume $0 < |x - 2| < \delta$, we can guarantee that $\frac{|x-2|}{2|x|} < \varepsilon$. Well, if we know that $|x - 2| < \delta$, then we can write

$$\frac{|x - 2|}{2|x|} < \frac{\delta}{2|x|} < \varepsilon$$

and so we need

$$\delta < 2|x|\varepsilon.$$

This is not quite enough — our choice of δ should not depend on x. We need some bound on x.

Recall our intuitive idea of the limit, as x gets very close to 2, the function $\frac{1}{x}$ gets very close to $\frac{1}{2}$. We don't really care what happens when x is a long way from 2, and only on what happens when x is very close to 2. Thus we should be able to focus on the region around $x = 2$, say, $1 < x < 3$, or equivalently, $|x - 2| < 1$.

How[64] does this help us? Well, if we know that $1 < x < 3$, then we know[65] that $|x| > 1$, and so

$$\frac{|x-2|}{2|x|} < \frac{\delta}{2|x|} < \frac{\delta}{2} \leq \varepsilon$$

And thus we need to ensure that

$$\delta \leq 2\varepsilon.$$

At this point it seems that we can choose any $\delta \leq 2\varepsilon$, but this is not quite right. Say, we chose a large value of ε, like $\varepsilon = 3$, and so we could pick $\delta = 2\varepsilon = 6$. With that choice of ε and δ, the implication at the heart of the definition of convergence becomes

$$(|x-2| < 6) \implies \left(\left|\frac{1}{x} - \frac{1}{2}\right| < 3\right).$$

Unfortunately this is false. We could take, say $x = \frac{1}{10} = 0.1$, and then the hypothesis is true, since $|x - 2| = 1.9 < 6$, but the conclusion is false since $\left|\frac{1}{x} - \frac{1}{2}\right| = |10 - 0.5| = 9.5 > 3$.

What went wrong? Remember to make our bound on $|x|$ we required that $|x - 2| < 1$. This is the same as requiring that $\delta \leq 1$. So we have actually imposed two requirements on δ. We need both $\delta \leq 1$ *and* $\delta \leq 2\varepsilon$. To enforce both of these we can pick

$$\delta = \min\{1, 2\varepsilon\}.$$

Now we can finally write up the proof.

Solution.

Proof. Let $\varepsilon > 0$ and set $\delta = \min\{1, 2\varepsilon\}$. Now assume that $0 < |x - 2| < \delta$. Since $\delta \leq 1$, we know that $|x - 2| < 1$ and so $1 < |x| < 3$ and thus $2|x| > 2$.

Then

$$\left|\frac{1}{x} - \frac{1}{2}\right| = \frac{|x-2|}{2|x|} < \frac{|x-2|}{2} < \frac{\delta}{2}.$$

Since $\delta < 2\varepsilon$, we know that

$$\left|\frac{1}{x} - \frac{1}{2}\right| < \varepsilon$$

and so $\frac{1}{x} \to \frac{1}{2}$ as $x \to 2$. ■

□

[64] Should one ask rhetorical questions in a textbook?

6.5 (Optional) Properties of limits

6.5.1 (Optional) Some properties of limits of sequences

When we work with sequences, it is not convenient to prove sequence convergence for each and every sequence individually. We can make use of some more general properties of limits of sequences to simplify our work. You will have already seen some "limit laws" when you studied calculus. We will prove some similar results in this section.

Theorem 6.5.1 Basic properties of limits of sequences. *Let (x_n) and (y_n) be sequences so that*

$$\lim_{n\to\infty} x_n = a \qquad \text{and} \qquad \lim_{n\to\infty} y_n = b$$

Additionally let $c, d \in \mathbb{R}$. Then

(a) The limit of a sequence is unique

(b) Linearity of limits: $\lim_{n\to\infty} (c \cdot x_n + d \cdot y_n) = c \cdot a + d \cdot b.$

(c) Product of limits: $\lim_{n\to\infty} (x_n \cdot y_n) = a \cdot b.$

(d) Reciprocal of limit: $\lim_{n\to\infty} \frac{1}{y_n} = \frac{1}{b}$ *as long as* $b \neq 0$

(e) Ratio of limits: $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{a}{b}$ *as long as* $b \neq 0$

Notice that for the sequences $(1/b_n)$ and (a_n/b_n) to be defined for all n we need $b_n \neq 0$, but we have not stated that in the theorem. This is because the condition that the limit $b_n \to b \neq 0$ implies that when n is *large enough*[66] we know that $b_n \neq 0$ — this is a consequence of Lemma 6.5.5 below. This is enough to tell us that when n is large everything is defined, and, typically, we don't worry about what happens when n is small.

6.5.1.1 Uniqueness of limits

To prove the first property — uniqueness of limits — we need to do some scratch work to build up our intuition. A very standard approach to proving uniqueness is to assume that we have two objects satisfying the property and then show that those two things must actually be the same.

So, let (x_n) be a convergent sequence and that

$$\lim_{n\to\infty} x_n = K \qquad \text{and also} \qquad \lim_{n\to\infty} x_n = L$$

[65] Be careful with the inequalities here.

[66] To be more precise, we can find some N_0 so that when $n > N_0$ we know that $|b_n| > 0$.

ie, there are two limits. Of course, we don't want these limits to *actually* be different, even though we've labelled them by different variables. We want to show that they are the same, that is $K = L$. In other words,

$$(\text{the limit is unique}) \equiv \left((\lim_{n\to\infty} x_n = K) \wedge (\lim_{n\to\infty} x_n = L) \implies (K = L) \right).$$

Assume the hypothesis is true. So

$$\lim_{n\to\infty} x_n = K \qquad \text{and also} \qquad \lim_{n\to\infty} x_n = L$$

and then we try to show that $K = L$. Intuitively this makes sense. Since $\lim_{n\to\infty} x_n = K$, we know that we can make x_n arbitrarily close to K by making n large enough. Similarly, we can make x_n arbitrarily close to L. The only way this can happen is if K and L are also arbitrarily close to each other. And the only way that can happen is if they are actually the same.

This is an important point that we will have to prove. Namely, we are claiming that if two numbers are arbitrarily close to each other, then they must be equal. Rewriting this with quantifiers gives

$$(\forall \varepsilon > 0, |K - L| < \varepsilon) \implies (K = L).$$

At first glance this might look a little hard to prove, but think about its contrapositive:

$$(K \neq L) \implies (\exists \varepsilon > 0 \text{ s.t. } |K - L| \geq \varepsilon).$$

So if two numbers are different, then we can find some positive number ε so that the distance between those two numbers is bigger. That doesn't sound so bad. It is a useful result, so we'll make it into a lemma.

Lemma 6.5.2 *Let $K, L \in \mathbb{R}$. If for every $\varepsilon > 0$ we have that $|K - L| < \varepsilon$, then we must have that $K = L$.*

Proof. We prove the contrapositive. Let $K, L \in \mathbb{R}$ so that $K \neq L$. Then set $\varepsilon = \frac{|K-L|}{2}$. Since $K \neq L$ we know that $\varepsilon > 0$. Then we have that $|K-L| = 2\varepsilon > \varepsilon$ and so the result holds. ■

Okay, to recap, we have assumed that $x_n \to K$ and $x_n \to L$. This means that

- for all $\varepsilon_K > 0$, there is some $N_K \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ if $n > N_K$ then $|x_n - K| < \varepsilon_K$, and
- for all $\varepsilon_L > 0$, there is some $N_L \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ if $n > N_L$ then $|x_n - L| < \varepsilon_L$.

Notice that we have carefully used different symbols for the ε and N to describe the convergence of $x_n \to K$ and $x_n \to L$. We do this so that we are not accidentally assuming anything extra[67] about how x_n converges to K or L. Now, this tells us that when n is big enough — ie $n > \max\{N_K, N_L\}$, that

$$|x_n - K| < \varepsilon_K \qquad \text{and} \qquad |x_n - L| < \varepsilon_L.$$

[67] We saw something like this back in Remark 3.2.8 — we recommend that the reader quickly review that remark.

But how do we use this, and the lemma above, to tell us about the size of $|K-L|$?

There is a really nice trick using the Theorem 5.4.6 and a little algebra. First, we add zero in a sneaky way that allows us to rewrite $K-L$ in terms of $(K-x_n)$ and $(L-x_n)$:

$$|K-L| = |K-L+0| = |K-L+\underbrace{(x_n-x_n)}_{=0}| = |(K-x_n)+(x_n-L)|$$

Now apply the triangle inequality:

$$|K-L| = |(K-x_n)+(x_n-L)| \leq |K-x_n|+|x_n-L| = |x_n-K|+|x_n-L|$$

This gives us a way to bound the distance between K and L in terms of the distances between x_n and K and between x_n and L. But our assumption about the convergence of x_n gives us exactly that information. That is

$$|K-L| \leq |x_n-K|+|x_n-L| < \varepsilon_K+\varepsilon_L$$

Now, given any ε, we can[68] choose $\varepsilon_K = \varepsilon_L = \frac{\varepsilon}{2}$. Then

- since $x_n \to K$, we know that there is some N_K so that when $n > N_K$, we have that $|x_n-K| < \frac{\varepsilon}{2}$.
- Similarly, since $x_n \to L$, we know that there is some N_L so that when $n > N_L$, we have that $|x_n-L| < \frac{\varepsilon}{2}$.

Then our reasoning above tells us that $|K-L| < \varepsilon$ providing $n > \max\{N_K, N_L\}$. And finally we can use Lemma 6.5.2 to complete the result.

Oof!

Proof of uniqueness of limits. Let (x_n) be a convergent sequence. We will prove that its limit is unique. To do so we prove that if $x_n \to K$ and $x_n \to L$ then we must have that $K = L$.

So assume that $x_n \to K$ and $x_n \to L$, and let $\varepsilon > 0$.

- Since $x_n \to K$, there is some $N_K \in \mathbb{N}$ so that for all $n > N_K$ we have that $|x_n-K| < \frac{\varepsilon}{2}$.
- And, since $x_n \to L$, there is some $N_L \in \mathbb{N}$ so that for all $n > N_L$ we have that $|x_n-L| < \frac{\varepsilon}{2}$.

So if we pick $N = \max\{N_K, N_L\}$ then for all $n > N$ the triangle inequality implies that

$$|K-L| = |(K-x_n)+(x_n-L)| \leq |x_n-K|+|x_n-L| \leq \varepsilon$$

Note that K, L are constants so the inequality $|K-L| < \varepsilon$ must hold independently of the value of n. And since it holds for any $\varepsilon > 0$ Lemma 6.5.2 implies that $K = L$ as required. ■

[68] There are lots of potential choices here. For example, we can also pick $\varepsilon_K = \alpha\varepsilon, \varepsilon_L = \beta\varepsilon$ with $\alpha, \beta > 0$ and $\alpha+\beta \leq 1$, so that $\varepsilon_K+\varepsilon_L \leq \varepsilon$.

6.5.1.2 Linearity of limits

No time to rest! Let's get working on the linearity of limits. We prove this by breaking the result down into two simpler lemmas.

Lemma 6.5.3 *Let $a, c \in \mathbb{R}$ and let (x_n) be a sequence that converges to a. The sequence $(c \cdot x_n)$ converges to $c \cdot a$.*

Lemma 6.5.4 *Let $a, b \in \mathbb{R}$ and let (x_n) and (y_n) be sequences so that $x_n \to a$ and $y_n \to b$. The sequence $(z_n) = (x_n + y_n)$ converges to $a + b$.*

Once we prove both of these, the linearity of limits follows quite directly:

$$\begin{aligned}\lim_{n\to\infty}(c \cdot x_n + d \cdot y_n) &= \lim_{n\to\infty}(c \cdot x_n) + \lim_{n\to\infty}(d \cdot y_n) \\ &= c \cdot \lim_{n\to\infty}(x_n) + d \cdot \lim_{n\to\infty}(y_n).\end{aligned}$$

The first of these lemmas is a little easier than the second, so we'll start there. And, as usual, we start with scratch work. Notice that when $c = 0$ the result simplifies down to the statement that the constant sequence $x_n = 0$ converges to 0. This is just Example 6.4.3 and we can recycle that proof. So since we know how to prove the case $c = 0$, we can now work on $c \neq 0$.

Notice that the statement is really a conditional. If $x_n \to a$ then $c \cdot x_n \to c \cdot a$. We'll assume that $x_n \to a$ and then work towards showing that $c \cdot x_n \to c \cdot a$. To do this we have to prove that for all $\varepsilon > 0$, there is some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ if $n > N$ then $|cx_n - ca| < \varepsilon$. Let's manipulate this inequality a little:

$$|cx_n - ca| = |c||x_n - a|$$

and so it suffices for us to show that $|x_n - a| < \frac{\varepsilon}{|c|}$.

Well, now we can put our assumption that $x_n \to a$ to use. That assumption tells us that for *any* $\varepsilon_x > 0$, there is $N_x \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ when $n > N_x$ then $|x_n - a| < \varepsilon_x$. We are being careful to label those constants with the subscript x to help remind us that those constants describe the convergence of $x_n \to a$.

Since this works for *any* ε_x, we are free to set $\varepsilon_x = \frac{\varepsilon}{|c|}$. Then we know there is N_x so that if $n > N_x$ then $|x_n - a| < \frac{\varepsilon}{|c|}$ and thus $|c||x_n - a| < \varepsilon$, just as we need. All that remains is to write it up as a neat proof.

Proof of Lemma 6.5.3. Let $\varepsilon > 0$ and assume that $x_n \to a$. We split the proof into two cases, $c = 0$ and $c \neq 0$.

When $c = 0$, then we have that $c \cdot x_n = 0$, and hence we trivially have

$$|c \cdot x_n - c \cdot a| = |0 - 0| < \varepsilon$$

Thus $0x_n \to 0$.

So now assume that $c \neq 0$. Since $x_n \to a$, we know that there exists $N_x \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ when $n > N_x$, we have $|x_n - a| < \frac{\varepsilon}{|c|}$. Let $N = N_x$ and then provided $n > N$,

$$|c \cdot x_n - c \cdot a| = |c||x_n - a| < \varepsilon.$$

And thus $c \cdot x_n \to c \cdot a$ as required. ■

We can actually clean this proof up and write it as a single case. We had to separate our the case $c = 0$ so that we did not divide ε by 0. However, we should remember that we do have some flexibility. Here is an alternate, slightly cleaner proof.

Second proof of Lemma 6.5.3. Let $\varepsilon > 0$ and assume that $x_n \to a$. We know that there exists $N_x \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ when $n > N_x$, we have $|x_n - a| < \frac{\varepsilon}{|c|+1}$. Let $N = N_x$ and then provided $n > N$,

$$|c \cdot x_n - c \cdot a| = |c||x_n - a| < \frac{|c|\varepsilon}{|c| + 1} < \varepsilon.$$

And thus $c \cdot x_n \to c \cdot a$ as required. ■

Let us now turn to Lemma 6.5.4. Notice that, again, it is really a conditional: "if those sequences converge to a and b, then their sum converges to $a + b$." So our proof will start by assuming the hypothesis is true and then working our way to the conclusion. We start by assuming that $x_n \to a$ and $y_n \to b$ and, as is always the case, it is a good idea to write down the meaning of the things that we have assumed and also to write down the meaning of what we want to show.

Our assumptions that $x_n \to a$ and $y_n \to b$ mean[69][70]:

- for any $\varepsilon_x > 0$ there is some $N_x \in \mathbb{N}$ so that for all $n \in \mathbb{N}$, if $n > N$ then $|x_n - a| < \varepsilon_x$, and
- for any $\varepsilon_y > 0$ there is some $N_y \in \mathbb{N}$ so that for all $n \in \mathbb{N}$, if $n > N$ then $|y_n - b| < \varepsilon_y$.

And we wish to show that

- for any $\varepsilon > 0$ there is some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ if $n > N$ then

$$|(x_n + y_n) - (a + b)| < \varepsilon.$$

The triangle inequality, Theorem 5.4.6, helps us here. It tells us how to bound the quantity $|(x_n + y_n) - (a + b)|$ by $|x_n - a|$ and $|y_n - b|$:

$$|(x_n + y_n) - (a + b)| = |(x_n - a) + (y_n - b)| \leq |x_n - a| + |y_n - b|$$

And then since we have assumed that $(x_n), (y_n)$ converge to a, b, we know that by making n very big, we can make both $|x_n - a|$ and $|y_n - b|$ very small. This then implies that we can make $|(x_n + y_n) - (a + b)|$ very small. In particular, if we can make both $|x_n - a|$ and $|y_n - b|$ smaller than $\frac{\varepsilon}{2}$, then the triangle inequality tells us that $|(x_n + y_n) - (a + b)|$ is smaller than ε. This is precisely what we need to prove the result.

[69] Know your definitons!

[70] Again, we are careful to use different ε and different N for each convergence statement in order to avoid making accidental extra assumptions.

Time to use our assumptions $x_n \to a$ and $y_n \to b$. Since[71] the definition of convergence works for *any* choice of ε, we can pick $\varepsilon_x = \varepsilon_y = \frac{\varepsilon}{2}$. Then

- there is N_x so that when $n > N_x$, $|x_n - a| < \varepsilon_x = \frac{\varepsilon}{2}$, and
- there is N_y so that when $n > N_y$, $|x_n - b| < \varepsilon_y = \frac{\varepsilon}{2}$.

This means that for any $n > \max\{N_x, N_y\}$ we have $|x_n - a| + |x_n - b| < \varepsilon$, which, in turn, guarantees that $|(x_n + y_n) - (a + b)| < \varepsilon$. Now we just have to tidy it up and write it in a nice proof.

Proof of Lemma 6.5.4. Assume that $x_n \to a$ and $y_n \to b$. We will show that $z_n = x_n + y_n \to a + b$.

Let $\varepsilon > 0$. Then since $x_n \to a$, we know that there exists $N_x \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ if $n > N_x$ then $|x_n - a| < \frac{\varepsilon}{2}$. Similarly, since $y_n \to b$, we know that there exists $N_y \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ if $n > N_y$ then $|y_n - b| < \frac{\varepsilon}{2}$.

Now pick $N = \max\{N_x, N_y\}$. Then for all $n \in \mathbb{N}$ with $n > N$, we have

$$\begin{aligned} |z_n - (a + b)| &= |x_n - a + y_n - b| \\ &\leq |x_n - a| + |y_n - b| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

and thus $z_n \to (a + b)$ as required. ■

Now that we have proved these two lemmas we can complete our proof of the linearity of limits:

Proof of the linearity of limits. Let $a, b, c, d \in \mathbb{R}$ and let (x_n) and (y_n) be sequences so that

$$\lim_{n\to\infty} x_n = a \qquad \text{and} \qquad \lim_{n\to\infty} y_n = b.$$

Then by Lemma 6.5.3:

$$\lim_{n\to\infty} c \cdot x_n = c \cdot a \qquad \text{and} \qquad \lim_{n\to\infty} d \cdot y_n = d \cdot b.$$

and then by Lemma 6.5.4:

$$\lim_{n\to\infty} (c \cdot x_n + d \cdot b_n) = \lim_{n\to\infty} c \cdot x_n + \lim_{n\to\infty} d \cdot b_n = c \cdot a + c \cdot b$$

as required.

Notice that by working in this order, we have been careful to first establish the convergence of the sequences (cx_n) and (dy_n), via Lemma 6.5.3, before establishing the convergence of their sum. This is necessary because Lemma 6.5.4 only works for the sum of convergent sequences. ■

[71] We used a very similar idea in our proof of uniqueness of limits.

6.5.1.3 Product of limits

Again, the statement is really an impliction: "if $x_n \to a$ and $y_n \to b$ then $x_n \cdot y_n \to a \cdot b$". So we assume that $x_n \to a$ and $y_n \to b$. This means, roughly speaking, that when n is really big, we know that $|x_n - a|$ and $|y_n - b|$ are small. And from that we need to show that $|x_n \cdot y_n - a \cdot b|$ is also small.

So we have to somehow express

$$|x_n \cdot y_n - a \cdot b| \qquad \text{in terms of} \qquad |x_n - a| \text{ and } |y_n - b|$$

and we can do it by carefully adding and subtracting terms.

$$\begin{aligned}(x_n \cdot y_n - a \cdot b) &= (x_n \cdot y_n - a \cdot b) + \underbrace{(x_n \cdot b - x_n \cdot b)}_{=0} \\ &= x_n(y_n - b) + b(x_n - a)\end{aligned}$$

So then, a little application of the triangle inequality gives

$$\begin{aligned}|x_n \cdot y_n - a \cdot b| &= |x_n(y_n - b) + b(x_n - a)| \\ &\leq |x_n(y_n - b)| + |b(x_n - a)| \\ &= |x_n| \cdot |y_n - b| + |b| \cdot |x_n - a|\end{aligned}$$

Similar to the argument we used to prove Lemma 6.5.4, we see that if we can keep $|x_n| \cdot |y_n - b| < \varepsilon/2$ and $|b| \cdot |x_n - a| < \varepsilon/2$, then we are done. But, how can we do that? Well, we can recycle the ideas from the proof of Lemma 6.5.3 to keep $|b| \cdot |x_n - a| < \varepsilon/2$, i.e. $|x_n - a| < \frac{\varepsilon}{2|b|+1}$, since $|b|$ is a constant. But that argument doesn't work for the other term, $|x_n| \cdot |y_n - b|$ since x_n need not be a constant.

However, we do know when n is very large that x_n must be close a, its limit. So we should be able to bound $\frac{|a|}{2} \leq |x_n| \leq \frac{3|a|}{2}$ for some sufficiently large n. This, in turn, would allow us to bound

$$\frac{|a|}{2}|y_n - b| \leq |x_n| \cdot |y_n - b| \leq \frac{3|a|}{2}|y_n - b|$$

And now, we use our control[72] over $|y_n - b|$, to make sure that $\frac{3|a|}{2}|y_n - b| < \varepsilon/2$.

Let us make this intermediate result bounding $|x_n|$, into a lemma. It takes a little careful juggling of inequalities and the reverse triangle inequality, Corollary 5.4.7, helps us. Then we can use the lemma to finish our proof.

Lemma 6.5.5 *Let $a \in \mathbb{R}$ and let (x_n) be a sequence that converges to a. Then there is some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$, when $n > N$, we have*

$$\frac{|a|}{2} \leq |x_n| \leq \frac{3|a|}{2}.$$

[72] That is, since $y_n \to b$ we know that we can make $|y_n - b|$ as small as we need, just by making n sufficiently large. In this way, our knowledge that y_n converges, gives us some control over the size of that term, $|y_n - b|$.

Proof. Let a and (x_n) be as given, and let $\varepsilon = \frac{|a|}{2}$. Then since $x_n \to a$, we know that there is $N \in \mathbb{N}$ so that for all integer $n > N$,

$$|x_n - a| < \frac{|a|}{2}.$$

The reverse triangle inequality, Corollary 5.4.7 then tells us that

$$|x_n - a| \geq ||x_n| - |a||$$

and hence we know that

$$||x_n| - |a|| < \frac{|a|}{2}$$

This is equivalent (see Lemma 5.4.5) to the statement

$$-\frac{|a|}{2} < |x_n| - |a| < \frac{|a|}{2}$$

from which the result quickly follows by adding $|a|$ to both sides. ■

Proof of the product of limits. Let (x_n) and (y_n) be sequences so that $x_n \to a$ and $y_n \to b$. Let $\varepsilon > 0$. Then, since those sequences converge we know that

- there is some N_x so that for all $n > N_x$ we have $|x_n - a| < \frac{\varepsilon}{2|b|+1}$, and
- there is some N_y so that for all $n > N_y$ we have $|y_n - b| < \frac{\varepsilon}{3|a|+1}$.

Notice that we have chosen denominators $2|b| + 1$ and $3|a| + 1$ to avoid the possibility of dividing by zero when a or b is zero. We also know

- by Lemma 6.5.5 there is some N_a so that for all $n > N_a$, we have $|x_n| < \frac{3|a|}{2}$.

Now assume that $n > \max\{N_x, N_y, N_a\}$, then

$$\begin{aligned}
|x_n y_n - ab| &= |x_n(y_n - b) + b(x_n - a)| \\
&\leq |x_n||y_n - b| + |b||x_n - a| \\
&\leq \frac{3|a|}{2}|y_n - b| + |b||x_n - a| && \text{by bound on } |x_n| \\
&< \frac{3|a|}{2} \cdot \frac{\varepsilon}{3|a| + 1} + |b| \cdot \frac{\varepsilon}{2|b| + 1} && \text{convergence of } x_n, y_n \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

and thus $x_n \cdot y_n \to a \cdot b$ as required. ■

6.5.1.4 Ratio of limits

We are on the home stretch now. We can prove the last property — the ratio of limits e — by combining the third and forth properties — the product of limits c

and the reciprocal of a limit d. So we now just[73] need to prove the reciprocal of limits.

Again, Item d is a conditional statement, so to prove it, we assume the hypothesis, $(y_n) \to b$ and $b \neq 0$, and then show that $\left(\frac{1}{y_n}\right) \to \frac{1}{b}$.

- So the assumption tells us that $b \neq 0$ and for all $\varepsilon_y > 0$, there is some $N_y \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ when $n > N_y$ then $|y_n - b| < \varepsilon_y$.
- While to prove the conclusion we need to show that for all $\varepsilon > 0$ there is some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ when $n > N$ then $\left|\frac{1}{y_n} - \frac{1}{b}\right| < \varepsilon$.

Obviously[74] we need to somehow relate this final inequality, $\left|\frac{1}{y_n} - \frac{1}{b}\right| < \varepsilon$, to the inequality we get from the convergence of $y_n \to b$, namely $|y_n - b| < \varepsilon_y$. So, time to do some rewriting:

$$\begin{aligned}\left|\frac{1}{y_n} - \frac{1}{b}\right| &= \left|\frac{(b - y_n)}{b \cdot y_n}\right| \\ &= \frac{|b - y_n|}{|b \cdot y_n|} = \frac{1}{|b|} \cdot \frac{1}{|y_n|} \cdot |y_n - b|\end{aligned}$$

And hence we need to choose N so that we can guarantee that

$$\left|\frac{1}{y_n} - \frac{1}{b}\right| = \frac{1}{|b|} \cdot \frac{1}{|y_n|} \cdot |y_n - b| < \varepsilon,$$

or equivalently:

$$|y_n - b| < |b| \cdot |y_n| \cdot \varepsilon.$$

Now since we have control[75] over the size of $|y_n - b|$, we can make it really small. But, just as was the case when we proved the product of limits c, we first have to bound $|y_n|$. Thankfully we did all that hard work already when we proved Lemma 6.5.5. That lemma tells us that there is some N_b so that when $n > N_b$ we know that

$$\frac{|b|}{2} < |y_n| < \frac{3|b|}{2}$$

Thus when $n > N_b$, we know that $\frac{|b|^2}{2} \cdot \varepsilon < |b| \cdot |y_n| \cdot \varepsilon$ So, if we can guarantee that

$$|y_n - b| < \frac{|b|^2}{2} \cdot \varepsilon$$

[73] When someone says that you "just" need to do something, you are right to be skeptical. "Just" can be a very dangerous word.

[74] Another dangerous word. Sorry. Better to say something like "Similarly to our earlier proofs in this section". The point of this footnote is to draw the reader's attention to the fact that words like "obviously", "clearly", or "just", are very subjective and should generally be avoided. But, we hope that it is clear to the reader that instructions like this are obviously to be disregarded from time to time. All things in moderation.

[75] That is, since $y_n \to b$, we know that we can make $|y_n - b|$ as small as we want by making n sufficiently large.

then we have

$$|y_n - b| < \frac{|b|^2}{2} \cdot \varepsilon < |b| \cdot |y_n| \cdot \varepsilon$$

and so $\left|\frac{1}{y_n} - \frac{1}{b}\right| < \varepsilon$ as required. Therefore we set $\varepsilon_y = \frac{|b|^2}{2} \cdot \varepsilon$.

The proof is ready to go. We just have to tidy things up and be careful of our various N's and ε's.

Proof of the reciprocal of a limit. Let $b \in \mathbb{R}$ with $b \neq 0$ and let (y_n) be a sequence that converges to b.

Now let $\varepsilon > 0$. Since $y_n \to b$, Lemma 6.5.5 implies that there is $N_b \in \mathbb{N}$ so that for all integer $n > N_b$

$$\frac{|b|}{2} < |y_n|.$$

Additionally, since $y_n \to b$, we can find $N_y \in \mathbb{N}$ so that for all integer $n > N_y$,

$$|y_n - b| < \frac{|b|^2}{2} \cdot \varepsilon.$$

Thus, if we pick $N = \max\{N_b, N_y\}$, then for all integer $n > N$, we have

$$\begin{aligned}\left|\frac{1}{y_n} - \frac{1}{b}\right| &= \frac{|y_n - b|}{|b| \cdot |y_n|} \\ &< |y_n - b| \cdot \frac{2}{|b|^2} \\ &< \frac{|b|^2}{2} \cdot \varepsilon \cdot \frac{2}{|b|^2} = \varepsilon\end{aligned}$$

And therefore the result follows. ■

Now that we have proved both the product of limits property and the reciprocal of limits property, we get the ratio of limits property quite directly.

Proof of the ratio of limits. Let x_n and y_n be sequences so that $x_n \to a$ and $y_n \to b \neq 0$. Then by the reciprocal of limits d we know that $\frac{1}{y_n} \to \frac{1}{b}$. And so, by the product of limits c, we know that

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} x_n \cdot \frac{1}{y_n} = \frac{a}{b}$$

as required. ■

6.5.2 (Optional) Some properties of limits of functions

The basic properties of limits of functions are very similar to those satisfied by the limits of sequences and should be familiar to the reader who has taken a Calculus course.

Theorem 6.5.6 Basic properties of limits of functions. *Let* $a, K, L \in \mathbb{R}$

and let f and g be real valued functions so that

$$\lim_{x\to a} f(x) = K \qquad \text{and} \qquad \lim_{x\to a} g(x) = L.$$

Additionally let $c, d \in \mathbb{R}$. Then

(a) The limit of a function at a given point is unique.

(b) Linearity of limits: $\lim_{x\to a}(c \cdot f(x) + d \cdot g(x)) = cK + dL.$

(c) Product of limits: $\lim_{x\to a} f(x) \cdot g(x) = K \cdot L.$

(d) Reciprocal of limit: $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{L}$ *as long as* $L \neq 0.$

(e) Ratio of limits: $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{K}{L}$ *as long as* $L \neq 0.$

Notice that the properties of limits of sequences are very similar to the properties of limits of sequences. The proofs are actually very similar as well. The main difference is that instead of picking some threshold $N \in \mathbb{N}$ we need to pick δ. Further, where we picked N to be at least as large as the other N's used to ensure that all inequalities are satisfied (eg the proof of Item c in Theorem 6.5.1), we will need to pick δ to be smaller than all the other δ's used. Because of these similarities we are going to give the proofs without scratch-work; we recommend the reader refer back to Subsection 6.5.1 for the ideas underlying behind the proofs.

Also notice that for the reciprocal $1/g(x)$ and ratio $f(x)/g(x)$ to be defined we require that $g(x) \neq 0$ but we have not stated this in the theorem. This is very similar to the situation for Theorem 6.5.1 above. The condition that $L \neq 0$ tells that when x is *close enough*[76] to a that $g(x) \neq 0$ — this is a consequence of Lemma 6.5.9 below. Since we are typically only interested in what happens when x is close to a, the condition that $L \neq 0$ ensures that $1/g(x)$ and ratio $f(x)/g(x)$ are defined.

6.5.2.1 Uniqueness of limits

Proof of the uniqueness of limits. To show that the limit of a function is unique, we prove that if

$$\lim_{x\to a} f(x) = K \qquad \text{and also} \qquad \lim_{x\to a} f(x) = L$$

then $K = L$.

So now assume that $\lim_{x\to a} f(x) = K$ and $\lim_{x\to a} f(x) = L$, and moreover let $\varepsilon > 0$.

- Since $\lim_{x\to a} f(x) = K$, we see that $\exists \delta_K > 0$ so that when $0 < |x - a| < \delta_K$

[76] That is, we can find some $c > 0$ so that when $|x - a| < c$, we know $|g(x)| > 0$.

we have $|f(x) - K| < \frac{\varepsilon}{2}$.

- Similarly, since $\lim_{x\to a} f(x) = L$, we see that $\exists \delta_L > 0$ so that when $0 < |x - a| < \delta_L$ we have $|f(x) - K| < \frac{\varepsilon}{2}$.

Thus, if we pick $\delta = \min\{\delta_K, \delta_L\}$ then when $0 < |x - a| < \delta$ we know that

$$|K - L| = |(K - f(x)) + (f(x) - L)| \leq |f(x) - K| + |f(x) - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This, by Lemma 6.5.2, implies that $K = L$, and therefore the limit of a function at a point is unique. ■

6.5.2.2 Linearity of limits

We prove the linearity of limits via two simpler lemmas.

Lemma 6.5.7 *Let $a, K, c \in \mathbb{R}$ and let f be a real value function so that*

$$\lim_{x\to a} f(x) = K.$$

Then

$$\lim_{x\to a} c \cdot f(x) = c \cdot K.$$

Proof. Let a, c, K, f be as in the statement of the lemma. Now let $\varepsilon > 0$, so that by the convergence of f we know that there is some δ_K so that when $0 < |x - a| < \delta_K$ we have that

$$|f(x) - K| < \frac{\varepsilon}{|c| + 1}.$$

Notice that this choice avoids any problems that might arise in the case that $c = 0$.

Now let $\delta = \delta_K$, so that when $0 < |x - a| < \delta = \delta_K$ we know that

$$|c \cdot f(x) - c \cdot K| < |c| \cdot |f(x) - K| < \frac{|c|}{|c| + 1}\varepsilon < \varepsilon$$

and thus $cf(x) \to cK$ as $x \to a$ as required. ■

Lemma 6.5.8 *Let $a, K, L \in \mathbb{R}$ and let f and g be real valued functions so that*

$$\lim_{x\to a} f(x) = K \qquad \text{and} \qquad \lim_{x\to a} g(x) = L.$$

Then

$$\lim_{x\to a} f(x) + g(x) = K + L.$$

Proof. Let a, K, L, f, g be as in the statement of the lemma, and let $\varepsilon > 0$. Then

- since $f(x) \to K$ we know that there is some δ_K so that when $0 < |x-a| < \delta_K$ we have that $|f(x) - K| < \frac{\varepsilon}{2}$,

- and similarly, since $g(x) \to L$ we know that there is some δ_L so that when $0 < |x - a| < \delta_L$ we have that $|g(x) - L| < \frac{\varepsilon}{2}$.

Pick $\delta = \min\{\delta_K, \delta_L\}$, so that for all x with $0 < |x - a| < \delta$ we know that

$$\begin{aligned} |(f(x) + g(x)) - (K + L)| &= |(f(x) - K) + (g(x) - L)| \\ &\leq |f(x) - K| + |g(x) - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus $(f(x) + g(x)) \to (K + L)$ as $x \to a$ as required. ■

Equipped with these two lemmas, the proof of the linearity of limits of functions is quite straightforward.

Proof of the linearity of limits. Let f and g be functions so that

$$\lim_{x \to a} f(x) = K \qquad \text{and} \qquad \lim_{x \to a} g(x) = L.$$

Moreover, let $c, d \in \mathbb{R}$. Then using Lemma 6.5.7 we know that

$$\lim_{x \to a} c \cdot f(x) = c \cdot K \qquad \text{and} \qquad \lim_{x \to a} d \cdot g(x) = d \cdot L.$$

And then Lemma 6.5.8 we get

$$\lim_{x \to a} (c \cdot f(x) + d \cdot g(x)) = c \cdot K + d \cdot L$$

as desired. ■

6.5.2.3 Product of limits

As was the case for sequences, our proof of the product of limits (and also the reciprocal of limits) relies the "trick" of rewriting

$$\begin{aligned} |f(x) \cdot g(x) - K \cdot L| &= |f(x) \cdot g(x) - f(x) \cdot L + f(x) \cdot L - K \cdot L| \\ &= |f(x)(g(x) - L) + L(f(x) - K)| \\ &\leq |f(x)| \cdot |g(x) - L| + |L| \cdot |f(x) - K|. \end{aligned}$$

So we again require some control over the size of the function close to $x = a$. Consequently we need lemma analogous to Lemma 6.5.5 that gives us a rigorous bound on $f(x)$ when x is close to a.

Lemma 6.5.9 *Let $a, K \in \mathbb{R}$ and let $f(x)$ be a function that converges to K as x approaches a. Then, there is some $\delta > 0$ so that when $0 < |x - a| < \delta$, we have*

$$\frac{|K|}{2} \leq |f(x)| \leq \frac{3|K|}{2}.$$

Now that we have this lemma we can proceed with the proof.

Proof of the product of limits. Let a, K, L, f, g be as in the statement of the lemma, and let $\varepsilon > 0$. Then we assemble the following three facts:

- Since $f(x) \to K$ as $x \to a$, there is some $\delta_K > 0$ so that when $0 < |x-a| < \delta_K$ we know that $|f(x) - K| < \frac{\varepsilon}{2|L|+1}$.
- Similarly, since $g(x) \to L$ as $x \to a$, there is some $\delta_L > 0$ so that when $0 < |x-a| < \delta_L$ we know that $|g(x) - L| < \frac{\varepsilon}{3|K|+1}$.
- Finally, since $f(x) \to L$ as $x \to a$, Lemma 6.5.9 tells us that there is some δ_f so that when $0 < |x-a| < \delta_f$ we know that $|f(x)| < \frac{3|K|}{2}$.

Notice that we have chosen denominators of $2|L|+1$ and $3|K|+1$ to avoid any problems that could arise if we had $L = 0$ or $K = 0$.

Now let $\delta = \min\{\delta_K, \delta_L, \delta_f\}$. Then when $0 < |x-a| < \delta$ we know that

$$|f(x)-K| < \frac{\varepsilon}{2|L|+1} \qquad \text{and} \qquad |g(x)-L| < \frac{\varepsilon}{3|K|+1} \qquad \text{and} \qquad |f(x)| < \frac{3|K|}{2}.$$

Then:

$$\begin{aligned} |f(x)\cdot g(x) - K\cdot L| &= |f(x)(g(x)-L) + L(f(x)-K)| \\ &\leq |f(x)|\cdot|g(x)-L| + |L|\cdot|f(x)-K| \\ &\leq \frac{3|K|}{2}\cdot\frac{\varepsilon}{3|K|+1} + |L|\cdot\frac{\varepsilon}{2|L|+1} \\ &\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence the result follows. ■

6.5.2.4 Ratio of limits

As was the case for limits of sequences, we prove the limit of ratios of functions by first proving the limit of the reciprocal of a function and then using the above result on the limit of products to complete the result.

Proof of the reciprocal of limits. Let a, L and g be as stated, and let $\varepsilon > 0$ be arbitrary. Then

- since $\lim_{x\to a} g(x) = L$, there is some δ_L so that when $0 < |x-a| < \delta_L$, we know that
$$|g(x) - L| < \varepsilon\frac{|L|^2}{2},$$
- and similarly, since $\lim_{x\to a} g(x) = L$, Lemma 6.5.9 implies that there is some δ_g so that when $0 < |x-a| < \delta_g$, we know that $\frac{|L|}{2} < |g(x)|$.

Now pick $\delta = \min\{\delta_L, \delta_g\}$. Then whenever $0 < |x-a| < \delta$ we get

$$\left|\frac{1}{g(x)} - \frac{1}{L}\right| = \left|\frac{(L-g(x))}{L\cdot g(x)}\right|$$

$$= \frac{1}{|L|} \cdot \frac{1}{|g|} \cdot |g(x) - L|$$
$$< \frac{2}{|L|^2} \cdot \varepsilon \cdot \frac{|L|^2}{2} = \varepsilon.$$

Therefore the result follows. ■

Putting this result together with the result for the product of limits gives us the ratio of limits.

Proof of the ratio of limits. Let f and g be functions so that $\lim_{x\to a} f(x) = K$ and $\lim_{x\to a} g(x) = L \neq 0$. Then from Item d we see that

$$\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{L}$$

and then by Item c we get

$$\lim_{x\to a} \left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} \left(\frac{1}{g(x)}\right)$$
$$= K \cdot \frac{1}{L} = \frac{K}{L}$$

Therefore the result follows. ■

6.6 Exercises

1. Prove that for all $n \in \mathbb{Z}$, $3 \mid (n^3 - n)$.

2. Prove that for all $n, k \in \mathbb{Z}$, if $k \mid (2n+1)$ and $k \mid (4n^2+1)$, then $k \in \{-1, 1\}$.

3. For all $n \in \mathbb{Z}$, prove that if there exists $a, b \in \mathbb{Z}$ such that $a^2 + b^2 = n$, then $n \not\equiv 3 \pmod 4$.

4. Prove or disprove the following statement:

$$\forall a, b \in \mathbb{Z}, \text{ if } 3 \mid (a^2 + b^2), \text{ then } 3 \mid a \text{ and } 3 \mid b.$$

5. Let $n, a, b \in \mathbb{Z}$. We know that if $n \mid a$ or $n \mid b$, then $n \mid ab$. Is the converse true as well?

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. We say that f is

- even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$;
- odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. If $f + g$ is odd, are f and g also odd?

7. Determine whether the following four statements are true or false. Explain your answers.

(a) $\exists x \in \mathbb{Z}$ s.t. $\exists y \in \mathbb{Z}$ s.t. $x + y = 3$.

(b) $\exists x \in \mathbb{Z}$ s.t. $\forall y \in \mathbb{Z}, x + y = 3$.

(c) $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ s.t. $x + y = 3$.

(d) $\forall x \in \mathbb{Z}, \forall y \in \mathbb{Z}, x + y = 3$.

8. Determine whether the following four statements are true or false. Explain your answers.

(a) $\exists x \in \mathbb{R}$ s.t. $\exists y \in \mathbb{R}$ s.t. $x^2 < y$.

(b) $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, x^2 < y$.

(c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $x^2 < y$.

(d) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 < y$.

9. Let $P \subset \mathbb{N}$ be the set of prime numbers $P = \{2, 3, 5, 7, 11, \ldots\}$. Determine whether the following statements are true or false. Prove your answers ("true" or "false" is not sufficient).

(a) $\forall x \in P$, $\forall y \in P$, $x + y \in P$.

(b) $\forall x \in P$, $\exists y \in P$ such that $x + y \in P$.

(c) $\exists x \in P$ such that $\forall y \in P$, $x + y \in P$.

(d) $\exists x \in P$ such that $\exists y \in P$, $x + y \in P$.

10. Given the sets,

$$A = \{n \in \mathbb{Z} \text{ s.t. } 3 \mid n\}, B = \{n \in \mathbb{Z} \text{ s.t. } 4 \nmid n\}, \text{ and } C = \{n \in \mathbb{Z} \text{ s.t. } 6 \nmid n\}$$

determine whether the following statements are true or false, and justify your answer.

(a) $\forall a \in A$, $\exists b \in B$ such that $a + b \in C$.

(b) $\exists a \in A$ such that $\forall b \in B$, $a + b \in C$.

(c) $\forall a \in A$, $\forall b \in B$, $a + b \in C$.

(d) $\exists a \in A$ and $\exists b \in B$ such that $a + b \in C$.

11. Determine whether the following statements are true or false. Explain your answers.

(a) $\exists x \in \mathbb{R}$ s.t. $\exists y \in \mathbb{R}$ s.t. $(xy > 0) \implies (x + y > 0)$.

(b) $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, (xy > 0) \implies (x + y > 0)$.

(c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $(xy > 0) \implies (x + y > 0)$.

(d) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (xy > 0) \implies (x + y > 0)$.

(e) $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, (xy \geq 0) \implies (x + y \geq 0)$.

(f) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $(xy \geq 0) \implies (x + y \geq 0)$.

12. Determine whether the following four statements are true or false. Explain your answers.

(a) $\forall y \in \mathbb{R}, \forall z \in \mathbb{R}, \exists x \in \mathbb{R}$ s.t. $x + y = z$.

(b) $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, \forall z \in \mathbb{R}, x + y = z$.

(c) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $\exists z \in \mathbb{R}$ s.t. if $z > y$ then $z > x + y$.

(d) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $\forall z \in \mathbb{R}$, if $z > y$ then $z > x + y$.

13. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions. For just this question,

- we call f *type A*, if $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ such that $y \geq x$ and $|f(y)| \geq 1$, and
- we say that g is *type B* if $\exists x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}$, if $y \geq x$, then $|g(y)| \geq 1$.

Prove or find a counterexample for the following statements.

(a) If a function is type A, then it is type B.

(b) If a function is type B, then it is type A.

14. Write down the negation of each of the following statements. Then determine whether each statement is true or false. Justify your answers.

(a) $\exists x \in \mathbb{Z}$ such that $(x > 84)$ and $x \equiv 75 \pmod{84}$.

(b) $\exists x, y \in \mathbb{Z}$ such that if $1 \geq x^2 \geq y^2$, then $x \geq y$.

(c) $\forall z \in \mathbb{N}$, $\exists x, y \in \mathbb{Z}$ such that $z = x^2 + y^2$.

(d) $\exists a \in \mathbb{R}$ such that $a > 0$ and $\forall x \in \mathbb{R}$, if $x \geq a$, then $2^{-x} < \dfrac{1}{100}$.

(e) $\forall n \in \mathbb{R}$, n is even if and only if n^2 is even.

15. Prove or disprove the following statements.

(a) $\forall a \in \mathbb{N}$, $\forall b \in \mathbb{N}$ if $b < a$, then $b - b^2 < a$.

(b) $\forall p \in \mathbb{N}$, $\forall q \in \mathbb{N}$, if $\sqrt{\dfrac{p}{q}} \in \mathbb{N}$, then $\sqrt{p} \in \mathbb{N}$ and $\sqrt{q} \in \mathbb{N}$.

(c) $\forall a, b \in \mathbb{R}$, $\exists c, d \in \mathbb{R}$, such that if $ab \geq cd$, then $a \geq c$ and $b \geq d$.

(d) $\forall a, b \in \mathbb{N}$, if $\exists x, y \in \mathbb{Z}$ and $\exists k \in \mathbb{N}$ such that $ax + by = k$, then $(k \mid a)$ and $(k \mid b)$.

16. Show that $\lim\limits_{x \to 4} (-3x + 5) = -7$.

17. Show that $\lim_{x\to 1} x^2 = 1$.

18. Show that $\lim_{n\to\infty} 1/n^2 = 0$.

19. Let $f : \mathbb{R} - \{0\} \to \mathbb{R}$ be defined by

$$f(x) = 6x \sin\left(\frac{1}{x}\right).$$

Show that $\lim_{x\to 0} f(x) = 0$.

20. Prove that the sequence $(x_n)_{n\in\mathbb{N}} = \left((-1)^n + \frac{1}{n}\right)_{n\in\mathbb{N}}$ does not converge to 0.

21. Prove that

$$\lim_{n\to\infty}\left(1 - \frac{2}{n^2} - \frac{3}{n^3}\right) = 1.$$

You may use the following fact: Let x, y be positive real numbers. Then

$$\sqrt{x} < \sqrt{y} \iff x < y.$$

22. Let $(s_n)_{n\in\mathbb{N}}$ be a sequence. We say that $\lim_{n\to\infty} s_n = +\infty$ if the following holds:

$$\forall M > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, n \geq N \implies s_n \geq M.$$

(a) In words, explain what the definition above means.

(b) Negate the statement above in order to describe what $\lim_{n\to\infty} s_n \neq +\infty$ means.

(c) Show that $\lim_{n\to\infty} \sqrt{n} = +\infty$

(d) Show that $\lim_{n\to\infty} (-1)^n\sqrt{n} \neq +\infty$

(e) Show that $\lim_{n\to\infty} (n^2 - 100n) = +\infty$

23. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence. We say that a sequence $\{a_n\}_{n\in\mathbb{N}}$ is *bounded* if there is some $M \in \mathbb{R}$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

(a) Show that if $\lim_{n\to\infty} a_n = L$ for some $L \in \mathbb{R}$, then $\{a_n\}_{n\in\mathbb{N}}$ is bounded.

(b) Show that the converse of the statement in (a) is false.

24. Before completing this question, you should look over Exercise 6.6.23.

A sequence $(a_n)_{n\in\mathbb{N}}$ is *bounded* if there is some $M \geq 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. If no such M exists, then we say $(a_n)_{n\in\mathbb{N}}$ is *unbounded*.

We say that $(a_n)_{n\in\mathbb{N}}$ is *increasing* if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$.

(a) Show that there is an unbounded sequence $(a_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} a_n \neq +\infty$.

(b) Show that there is an increasing sequence $(a_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} a_n \neq +\infty$.

(c) Show that if $(a_n)_{n\in\mathbb{N}}$ is unbounded and increasing, then $\lim_{n\to\infty} a_n = +\infty$.

Parts (a) and (b) tell us that when proving the statement in part (c), we need to use both conditions given.

25. Typically, we define the distance between two real numbers via the absolute value function: $d(x, y) = |x - y|$. This means that we can see the distance as a function on real numbers that takes two numbers and gives us a non-negative number as the distance. This helps us generalize the definition of a "distance". However a function needs to satisfy more than being non-negative to be called a "distance".

Indeed, we define a *distance* (or *metric*) on $\mathbb{R}$, as a function $d : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, that satisfies the properties,

- $\forall x, y \in \mathbb{R}$, $d(x, y) = 0$ if and only if $x = y$.
- $\forall x, y \in \mathbb{R}$, $d(x, y) = d(y, x)$.
- $\forall x, y, z \in \mathbb{R}$, $d(x, y) \leq d(x, z) + d(z, y)$ (this property is also known as, as you guessed it, the "triangle inequality").

Now, we can define the function $D : \mathbb{R} \times \mathbb{R} \to [0, \infty)$, as,

$$D(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

(a) We see that D satisfies the first two properties. Show that D satisfies the triangle inequality to conclude that D is a distance.

(b) Given a distance function, d, we can define sequence convergence as follows,

A sequence $(x_n)_{n\in\mathbb{N}}$ converges to a number L if and only if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n > N$, $d(x_n, L) < \varepsilon$.

Using this definition and the distance function, D, as above, show that a sequence $(x_n)_{n\in\mathbb{N}}$ converges to L implies that the set $\{n \in \mathbb{N} : x_n \neq L\}$ is finite.

Chapter 7

Induction

In this text we started by doing a little bit of sets and a little logic. The aim of these brief introductions was to get you — our reader — started proving things as quickly as possible. We continued with a some basic logical equivalence to show us how to prove cases, biconditionals and contrapositives, and how to work with quantified statements. We have put these methods to work on some basic results on parity and divisibility; while those result were not terribly deep, their simplicity made them good places to learn our proof basics. We will soon apply all[77] of proof skills to more general mathematical problems, but we will first introduce a more specialised, but still extremely useful proof method. **Mathematical induction** is a method for proving statements of the form

$$\forall n \in \mathbb{N}, P(n).$$

The archetypal induction-proof example is the result

$$\text{For all } n \in \mathbb{N}, 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

or, to write it in a slightly more compact way

$$\forall n \in \mathbb{N}, \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Gauss learns to sum. Carl Friedrich Gauss (1777 – 1855) had a huge impact on many areas of mathematics; the interested reader should search-engine their way to more information. When he was in primary school his teacher (no doubt wanting to occupy their students for a while) set the problem of adding up $1 + 2 + \cdots + 99 + 100$. Gauss worked this out extremely quickly by realising that you can add numbers in pairs

$$1 + 100 = 101$$

[77] We still need to learn about **proof by contradiction**, but this author likes to leave that topic until the reader has had a chance to practice other proof methods.

$$2 + 99 = 101$$
$$3 + 98 = 101$$
$$\vdots$$

There are $n/2$ pairs, each adding to $n + 1$. This trick gives the above result and can be generalised, but best we still learn induction — it has uses way beyond sums.

Warning 7.0.1 The law of the instrument. Like any tool, induction is not guaranteed to work everywhere. However, like any tool, it is useful to have and part of learning to use the tool is to understand when it might help and when it might not. There is a tendency when one learns a nice new mathematical method that one tries to apply it to every problem that one comes across. This is often summarised as the **law of the instrument** — being the tendency to always use the same tool, even when it isn't the best tool for the job [78]. Anyway, with that caveat, the authors should get on with it and start talking about induction.

7.1 Induction

Induction, once you get past some of the technicalities, is a very intuitive idea. This author likes to think of induction like climbing an infinite ladder and an induction argument is really just a proof that we can climb that ladder. The proof is in 3 parts:

- If you can step onto the ladder, and
- from the current step you can reach the next step, then
- you can climb the ladder as high as you want.

As more concrete mathematical example, say we wish to prove

$$\text{For all } n \in \mathbb{N}, n^2 + 5n - 7 \text{ is odd.}$$

The dedicated reader may recall a very similar example in Chapter 5 and realise that we can do this using proof by cases, however, it is a good example for proof by "ladder idea".

- Step onto the ladder. When $n = 1$ the polynomial is $1 + 5 - 7 = -1$ which is odd. So the assertion is true for the very smallest value of n.
- Climb from one rung to the next. We wish to show that if the assertion is true for the current value of $n = k$, then it is also true for $n = k + 1$. That

[78] There are also variations of this rule that make reference to hammers — when you have a fancy new hammer, everything is a nail.

is, if we are on the current step, we can move up to the next step. So, to be more precise, we want to show that

$$(k^2 + 5k - 7 \text{ is odd}) \implies ((k+1)^2 + 5(k+1) - 7 \text{ is odd})$$

So this is just a little direct proof hiding inside our argument.

Proof: Assume $k^2 + 5k - 7 = 2\ell + 1$. Then

$$\begin{aligned}(k+1)^2 + 5(k+1) - 7 &= k^2 + 2k + 1 + 5k + 5 - 7 \\ &= \underbrace{k^2 + 5k - 7}_{=2\ell+1} + (2k + 6) \\ &= 2(\ell + k + 3) + 1.\end{aligned}$$

Hence $(k+1)^2 + 5(k+1) - 7$ is odd as required.

- Conclusion. So how do we put these facts together? The first fact guarantees that our result is true for $n = 1$. Then the second fact tells us that since it is true for $n = 1$, it is also true for $n = 2$. But then since it is true for $n = 2$, it true for $n = 3$ (by the same fact). Applying that fact again and again gives us $n = 4, n = 5, n = 6, \cdots$.

Strictly speaking we have to be a bit more careful with that final "…", but we'll see how to do that below.

This proof method, **mathematical induction**, is formalised in the next theorem.

Theorem 7.1.1 The principle of mathematical induction. *For every $n \in \mathbb{N}$ let $P(n)$ be a statement. If*

- *$P(1)$ is a true statement, and*
- *the implication $P(k) \implies P(k+1)$ is true for all $k \in \mathbb{N}$*

then $P(n)$ is true for all $n \in \mathbb{N}$.

We can't quite give a complete proof of this theorem, though we can give almost everything that is needed. Call this a "proof sketch" — the core ideas are there, but we'll skip over a technicality and also write it a little informally.

Proof-sketch. Start by assuming the hypotheses of the theorem are true. That is, we assume that

- $P(1)$ is true, and
- the implication $P(k) \implies P(k+1)$ is true for all natural numbers k.

Then the core idea of the proof is construct two sets

- The "good" set:
$$G = \{n \in \mathbb{N} \mid P(n) \text{ is true}\}$$
This is the set of all n-values for which our statement is true. By assumption, we know that $1 \in G$, and $G \neq \varnothing$.

- The "bad" set:
$$B = \{n \in \mathbb{N} \mid P(n) \text{ is false}\}.$$
This is the set of all n-values for which our statement is false.

As is often the case, the "good" isn't as interesting as the "bad", and we'll focus on the set B. It is either empty or non-empty.

If B is empty, then we are done, because then $P(n)$ must be true for every single $n \in \mathbb{N}$ as required.

- As B is not empty, we can look for the smallest element in B. And since B is a non-empty set of natural numbers we can always find its smallest smallest element. However we do note that this is not possible for all sets — we'll come back to this point shortly.
- Let ℓ be this smallest number in B.
- From the hypothesis in the theorem, we know that $\ell > 1$ (since $P(1)$ is true).
- By the way we constructed B and ℓ, we know that $P(n)$ is true for all $1 \leq n < \ell$.
- Now we have a problem:
  - we know that $P(\ell - 1)$ is true, but
  - we also know that the implication $P(\ell - 1) \implies P(\ell)$ is true (since that is one of the hypotheses).
  - But if $P(\ell - 1)$ is true, and $P(\ell)$ is false, then that implication is false.
- This is a **contradiction** — our assumption is contradicted by $P(\ell)$ being false.
- The only way to resolve this situation is to conclude that no such ℓ exists — that is, B is empty.

Hence we must have $P(n)$ is true for all natural numbers n ■

There are two sneaky things going on in this proof. Firstly, we are doing a **proof by contradiction** by stealth; don't worry too much about this yet as we'll return to those proofs later in the text. Second, and more important in the short term, is that the proof relies on a delicate point

> since B is just a non-empty set of natural numbers, we can always find the smallest number in that set.

We are using here something called the **well ordering principle**. This guarantees us that we can always find the smallest element in a set of natural numbers. This is very definitely false for other sets of numbers — if we build a set of integers, or rational numbers, or real numbers, then it might not have a minimum.

Example 7.1.2 Sets without minima. Consider the following sets

$$\{2k \mid k \in \mathbb{Z}\} \qquad \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\} \qquad (0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}.$$

None of these sets have minimum elements; you can always find another element that is smaller.

Solution. Note that this argument requires proof by contradiction and if the reader is confused by the details of this argument, we suggest that they wait until we have covered that topic and then return to this discussion. In fact, making this an example in that section is a good idea and the authors might do exactly that.

Consider the set $\{2k \mid k \in \mathbb{Z}\}$.

- Either this set has a minimum or it does not.
- If it does have a minimum, call that minimum ℓ.
- However, the number $\ell - 2$ is also in the set, and $\ell - 2 < \ell$.
- So ℓ cannot be the minimum.

So the set cannot have a minimum.

Similarly, if $(0,1)$ has a minimum then call it q. But then since $0 < q < 1$, we know that $0 < q/2 < q$. So $q/2$ is in the set and smaller than q, so q cannot be a minimum. Hence no such minimum exists. The same argument works for the set $\{1/n \mid n \in \mathbb{N}\}$. □

Now that we are armed with a formal statement of **mathematical induction**, we should make use of it. The typical first example is the simple summation we saw in the introduction to this chapter. However (arguably) this can give the wrong idea of what induction is. To (hopefully) avoid this potential trap, let us do an example about divisibility.

Result 7.1.3 *For every natural number n, $3|(4^n - 1)$.*

So our statement is $P(n)$ is "$3|(4^n - 1)$". To prove this true for all naturals n using induction, we need to show

- The statement $P(1)$ is true, and
- The implications $P(k) \implies P(k+1)$ is true for all $k \in \mathbb{N}$.

In most cases (but definitely not all), showing that $P(1)$ is true is easy, and proving the inductive step requires work.

- Setting $n = 1$ in our statement gives

  $$3|(4^1 - 1)$$

  which is true (since $4 - 1 = 3$).

- Now the harder part — the inductive step: We have to prove that

$$(3|(4^k - 1)) \implies (3|(4^{k+1} - 1))$$

for all $k \in \mathbb{N}$. That is, the inductive step requires us to prove an implication. As we have done many times before (and will do many times in the future), we assume the hypothesis is true and work our way to the conclusion. So we start by assuming that $3|(4^k - 1)$, which means that

$$4^k - 1 = 3\ell \qquad \text{for some } \ell \in \mathbb{Z}$$

and we want to arrive at

$$4^{k+1} - 1 = 3m \qquad \text{so that} \qquad 3|(4^{k+1} - 1)$$

We need to think about how we can construct $4^{k+1} - 1$ from $4^k - 1$. Multiplying by 4 is a good way to go:

$$\begin{aligned} 4^k - 1 &= 3\ell \\ 4^k &= 3\ell + 1 \\ 4^{k+1} &= 12\ell + 4 \\ 4^{k+1} - 1 &= 12\ell + 3 = 3(4\ell + 1) \end{aligned}$$

And since $4\ell + 1 \in \mathbb{Z}$ we have reached the conclusion we want.

Now that we know how to prove the base case (that was easy) and we know how to prove the inductive step (not too bad), we can put things together into a proof. Remember to tell the reader what we are doing.

Proof. We proceed by induction.

- Base case — When $n = 1$ the result is true since $3|(4 - 1)$.
- Inductive hypothesis — Assume $P(k)$ is true. Then $4^k - 1 = 3\ell$ for some $\ell \in \mathbb{Z}$. Then

$$\begin{aligned} 4^{k+1} - 1 &= 4(4^k) - 1 \\ &= 4(4^k - 1) + 4 - 1 \\ &= 4 \cdot 3\ell + 3 = 3(4\ell + 1) \end{aligned}$$

Thus if $P(k)$ then $P(k + 1)$.

By the principle of mathematical induction the statement is true for all naturals n. ■

Remark 7.1.4 Make structure obvious to the reader. Notice that we are making the structure of the proof very clear. We start by telling reader that we are using induction. We have then labelled the two conditions "base case" and "inductive hypothesis" or "inductive step", so that it is very clear how the

proof is being constructed. We then have a little summarising sentence at the end saying "by induction it is true!" It means that a reader who is familiar with induction can readily check off each part of the induction proof against what they know about induction as they read the proof. We make it easy for them to read and verify the proof. (This is an especially good idea when someone has to mark your work, not just read it!)

We can rewrite our above simple parity example with this same structure. It is good practice.

Example 7.1.5 For all $n \in \mathbb{N}, n^2 + 5n - 7$ is odd.

Solution.

Proof. We use induction to prove the result.

- Base case. When $n = 1$, the polynomial gives $1 + 5 - 7 = -1$. Hence the base case is true.
- Inductive step. Assume that $k^2 + 5k - 7$ is odd, and so $k^2 + 5k - 7 = 2\ell + 1$. Then

$$(k+1)^2 + 5(k+1) - 7 = 2k + 6 + (k^2 + 5k - 7) = 2(k+3) + 2\ell + 1$$

  and hence is also odd.

By mathematical induction the result holds for all $n \in \mathbb{N}$. ■

□

Extend to all $n \in \mathbb{Z}$. If we wish to prove this same result by induction but for all integers n, we could split the problem into 3. The first we just did, we could then prove it holds for $n = 0$ —easy! And then for negative integers we could rewrite the result as

$$\forall n \in \mathbb{N}, (-n)^2 + 5(-n) - 7 \text{ is odd.}$$

To avoid doing the very standard summation examples for a touch longer, we'll do an inequality. This particular example is another classic induction problem[79].

Result 7.1.6 *Let $n \in \mathbb{N}$ and let $x \in \mathbb{R}$ with $x > -1$. Then*

$$(1 + x)^n \geq 1 + nx.$$

If we choose to use induction (and we will), then it is a good idea to clarify what the actual statement is. Rewriting things carefully, we can write

$$P(n) : \text{if } (x > -1) \text{ then } (1 + x)^n \geq (1 + nx).$$

- First up, we verify the base case; setting $n = 1$ gives

$$(x > -1) \implies (1 + x)^1 \geq 1 + x$$

[79] Like many of the authors "jokes" it is an antique and should be handled with care.

Since $(1+x)^1 = 1+x$, the conclusion is always true, so the implication is true, and the base case is true.

- Now the induction step. We assume that $P(k)$ is true. That is

$$(x > -1) \implies (1+x)^k \geq 1+kx$$

is a true statement, and we then need to prove that

$$(x > -1) \implies (1+x)^{k+1} \geq 1+(k+1)x.$$

To prove this second implication, we do as we have done so many times, and will do many times in the future, we assume the hypothesis is true. So we assume $(x > -1)$. From the first implication (and our assumption that $x > -1$) we know that $(1+x)^k \geq 1+kx$. We now need to prove that $(1+x)^{k+1}$ is bigger than $1+(k+1)x$.

A good place to start is to take the inequality we know and try to make it look like the inequality we want.

$$\begin{aligned} (1+x)^k &\geq 1+kx && \text{multiply by } (1+x) \\ (1+x)^{k+1} &\geq (1+kx)(1+x) \\ &= 1+(k+1)x+kx^2 \end{aligned}$$

Nearly there, we just need to get rid of this kx^2 term. We know $x^2 \geq 0$, so $kx^2 \geq 0$ and hence

$$(1+x)^{k+1} \geq 1+(k+1)x+kx^2 \geq 1+(k+1)x$$

as required.

But there is a problem: where have we used the assumption $x > -1$?

When you are doing mathematics and answering problems, you should keep your eyes open for this sort of thing. If you haven't used one of the hypotheses in your proof then there is very likely to be an error somewhere!

Now — our work above is not actually wrong, but that is more by luck than by design[80]. Where we multiplied by $1+x$ we didn't examine whether quantity is positive or negative. When we multiply an inequality by something positive (say $+2$), then inequality keeps the same sign:

$$2 \leq 3 \qquad \implies 4 \leq 6.$$

but when we multiply by something negative (say -2), the sign flips:

$$2 \leq 3 \qquad \implies (-4) \geq (-6)$$

[80] Of course the author actually designed the example to illustrate this luck. So perhaps it is designed luck?

Thankfully since $x > -1$, we know that $1 + x > 0$ and so we are not multiplying by a negative number. We don't have to go back and fix our scratch work, but we should make sure that we highlight this point when we write up the proof. We help our reader understand important subtle details by pointing them out.

Proof. We use induction.

- Based case: when $n = 1$, the statement is true since $(1 + x)^1 = 1 + 1 \cdot x$.
- Induction step. We assume that $x > -1$ and that $(1 + x)^k \geq 1 + kx$. Since $x > -1$, we know that $1 + x > 0$ and so multiplying through by $1 + x$ doesn't change the sign of the inequality:

$$\begin{aligned} (1+x)^k(1+x) &\geq (1+kx)(1+x) && \text{factor left and expand right} \\ (1+x)^{k+1} &\geq 1 + (k+1)x + kx^2 && \text{since } x^2 \geq 0 \\ &\geq 1 + (k+1)x \end{aligned}$$

  as required.

So by induction the inequality is true for all $n \in \mathbb{N}$. ■

Now we are ready to do the very standard summation example. But first a warning[81]:

Warning 7.1.7 Induction is not summation. Induction requires us to prove both that

- the base case, $P(1)$, is true, and
- the inductive hypothesis, $P(k) \implies P(k+1)$, is true for all $k \in \mathbb{N}$.

Mathematical induction is not simply adding the next term to both sides of the equation.

With that out of the way:

Result 7.1.8 *For every natural number n, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$*

- The base case is very easy (as it often is). When $n = 1$ we have:

$$\begin{aligned} \sum_{k=1}^{n} k &= \frac{1 \cdot 2}{2} && \text{which is} \\ 1 &= 1\checkmark. \end{aligned}$$

- As is typical, the inductive step needs a bit more work. We need to prove that

$$\left(1 + 2 + \cdots + k = \frac{k(k+1)}{2}\right) \implies \left(1 + 2 + \cdots + k + k + 1 = \frac{(k+1)(k+2)}{2}\right)$$

[81] A nag?

Like most proofs of implications, we start by assuming the hypothesis is true and try to work our way to the conclusion. In this case, it is pretty clear that we can get from the first statement to the second just by adding $(k+1)$ to both sides. Note that "adding $k+1$ to both sides" is ***not*** the induction step. The induction step requires us to prove that $P(k) \implies P(k+1)$, and we prove that ***in this particular example*** by adding $k+1$ to both sides. This is a very common error (and more than a little frustrating to mark[82]).

We are now ready to write things up nicely.

Proof. We proceed by induction.

- Base case — When $n = 1$ the statement is true since $1 = \frac{1 \cdot 2}{2}$.
- Inductive hypothesis — Assume the statement is true for $n = k$, that is

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}$$

Adding $k+1$ to both sides gives

$$\begin{aligned} 1 + 2 + \cdots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{1}{2}\left(k^2 + k + 2k + 2\right) \\ &= \frac{1}{2}(k+1)(k+2). \end{aligned}$$

Hence if the statement is true for $n = k$ then it must be true for $n = k+1$.

By the principle of mathematical induction it is true for all natural numbers n. ■

It is a good idea to see a few false induction proofs. Each of these examples contains a flaw.

Example 7.1.9 Polynomial primes. For all $n \in \mathbb{N}, n^2 - n + 41$ is prime.

We call the following argument a "misproof" since it looks like a proof, but is incorrect. In the solution below we explain why it is wrong.

[82] Be nice to your marker?

Misproof. We check the first values of n:

- $n = 1$: $1 - 1 + 41 = 41$ which is prime.
- $n = 2$: $4 - 2 + 41 = 43$ which is prime.
- $n = 3$: $9 - 3 + 41 = 47$ which is prime.
- $n = 4$: $16 - 4 + 41 = 53$ which is prime.
- $n = 5$: $25 - 5 + 41 = 61$ which is prime.

Since it is true for the first few values of n, it is true for all n. ■

Solution. This is a very wrong proof. ***Examples are not proof.***

Surprisingly this polynomial does give you prime numbers for $n = 1, \cdots, 40$. This was first noticed by Euler[83]. However when $n = 41$ it is clearly not prime, since it gives $41^2 - 41 + 41 = 41^2$. □

Example 7.1.10 Everything you know about 2 is wrong? For all $n \in \mathbb{N}$, $n + 1 < n$.

We give the flawed proof and then explain what is wrong with it in the solutions below.

Misproof. We prove this by induction. So assume that the statement is true for $n = k$. That is

$$k + 1 < k.$$

Then we can write, by adding 1 to both sides of the above inequality

$$(k + 2) < (k + 1)$$

And thus the statement is true for $n = k + 1$. By mathematical induction, the statement is true for all $n \in \mathbb{N}$. ■

Solution. This proof of the *inductive step* is perfectly correct. We have correctly shown that

$$(k + 1 < k) \implies (k + 2 < k + 1)$$

However this is a vacuous statement since the hypothesis is never true. This is the flaw in our proof — we did not check the base-case. When $n = 1$ the statement is false.

Since the base case fails, we cannot use mathematical induction. Both the base case and the inductive step must be true for an induction proof to work. □

Example 7.1.11 Pólya's colourless horses. This is George Pólya's proof that "All horses have the same colour".

Again, we'll first give the flawed proof, and then show you where it goes wrong.

[83]The interested reader should search-engine their way to "Euler's lucky numbers". There is actually some quite deep number theory lurking in side that polynomial.

Misproof. We proceed by induction.

- Base case: When there is only 1 horse, it has the same colour as itself. So the statement is true when there is exactly 1 horse.
- Inductive step: We assume the statement is true when there are $n = k$ horses. Now consider a set of $n = k + 1$ horses. Exclude one of those horses from the set to obtain a set of k horses. By assumption all those horses have the same colour. Put that excluded horse back into the set and remove another. Now the set again has k horses and so they all have the same colour. This means that all the $k + 1$ horses (including the two we temporarily removed) must have the same colour.

By induction the statement is true for all $n \in \mathbb{N}$. ■

Solution. So what is wrong here? There is a subtle error in the inductive step. Remember that the inductive step has to be true for all $k \in \mathbb{N}$, however it contains the tacit assumption that k is not too small. To see why, step through the argument carefully when $k = 1$.

Assume that the statement is true when $n = k = 1$. Then consider a set of $n = k + 1 = 2$ horses: call them Alice and Bob[84]. Now we remove one horse (Alice), and then every horse remaining in the set (which is just Bob) has the same colour. Now we put Alice back and take Bob out. So Alice has the same colour as every horse in the set — which is just Alice. At no point do we ever compare the colour of Alice or Bob with a third horse[85]. Consequently we cannot infer any relationship between the colours of equine Alice or Bob. □

7.2 More general inductions

7.2.1 A little more general

There was nothing special about starting at $n = 1$ in our proof-sketch of the principle of mathematical induction. All that was required was that we could find the smallest integer in a set of natural numbers. We can quite readily generalise the proof to other sets of integers — as long as we can find the smallest integer in that set. So for example, instead of considering $P(n)$ for all natural numbers n, we could consider $P(n)$ for all integer $n \geq -3$, or all integers $n \geq 7$ and so forth.

This idea leads to a slightly more general form of mathematical induction. The only differences are that the base case need not be at $n = 1$, and that the inductive step must now be true for all integer k starting from the value in the base case. This is sometimes called **extended mathematical induction**.

Theorem 7.2.1 Principle of mathematical induction. *For a fixed integer ℓ, let $S = \{n \in \mathbb{Z} \mid n \geq \ell\}$. For each integer $n \in S$, let $P(n)$ be a statement. If*

[84] Very traditional names for horses with side-interests in cryptography.
[85] A stalking horse? A dark horse?

- *$P(\ell)$ is true, and*
- *if the implication $P(k) \implies P(k+1)$ is true for all $k \in S$*

then $P(n)$ is true for all $n \in S$.

We should put this theorem to work:

Result 7.2.2 *For every integer $n \geq 5$, $2^n \geq n^2$.*

The proof of this result is nearly the same as our previous induction proofs; it consists of a basis case and an inductive step. The difference is that the basis step starts at $n = 5$.

- When $n = 5$ the inequality is $2^5 = 32 \geq 5^2 = 25$, which is true.
- For the inductive step we need to prove that

$$\left(2^k > k^2\right) \implies \left(2^{k+1} > (k+1)^2\right)$$

So we assume the first inequality, $2^k > k^2$, and have to work our way to the second, $2^{k+1} > (k+1)^2$. Multiplying the original inequality by 2 might be a good place to start.

$$2 \cdot 2^k = 2^{k+1} > 2k^2$$

So we need to show that $2k^2 \geq (k+1)^2$ or $k^2 \geq 2k+1$. Well since $k \geq 5$ we have $k^2 \geq 5k = 2k + 3k$ and since $k > 1$, $3k > 1$ and we are done.

Now we can write it up nicely.
Proof. We proceed by induction.

- Basis step — when $n = 5$ we have $2^5 = 32 > 5^2 = 25$. Hence the statement is true for $n = 5$.
- Inductive hypothesis — assume $2^k > k^2$. Then $2^{k+1} = 2 \cdot 2^k > 2k^2$. Since $k \geq 5$ we have $k^2 \geq 5k$, and hence

$$2^{k+1} > 2k^2 = k^2 + k^2 = k^2 + 5k = k^2 + 2k + 3k.$$

Now since $k \geq 5$, we know that $3k > 1$, and so

$$2^{k+1} > k^2 + 2k + 1 = (k+1)^2$$

Thus if $P(k)$ is true then $P(k+1)$ is true.

By the principle of mathematical induction the statement is true for all $n \geq 5$. ■

It's not a discussion of induction without some summation example.

Result 7.2.3 *Let $a \in \mathbb{Z}$, then*

$$a + (a+1) + \cdots + n = \sum_{j=a}^{n} j = \frac{(n+a)(n+1-a)}{2}$$

Proof. We prove this by induction. When $n = a$ we have $\frac{(a+a)(a+1-a)}{2} = a$ as required, so the base case holds.

Now assume that the result holds for $n = k$. Hence

$$\begin{aligned} a + (a+1) + \cdots + k &= \frac{(k+a)(k+1-a)}{2} \qquad \text{and so} \\ a + (a+1) + \cdots + k + (k+1) &= \frac{(k+a)(k+1-a)}{2} + (k+1) \\ &= \frac{1}{2}\left(k^2 + 3k + 2 + a - a^2\right) \\ &= \frac{1}{2}(k+1+a)(k+2-a)) \end{aligned}$$

as required. Since both the base case and induction step hold, the result follows by induction. ■

Here is another inequality result

Result 7.2.4 *For any integer $n \geq 4$, $n! > 2^n$.*
Proof. We prove the result by induction. When $n = 4$ we have

$$4! = 24 > 16 = 2^4$$

so the base case is true.

Let us now assume that $k! > 2^k$. Then

$$\begin{aligned} k! &> 2^k \\ (k+1)k! &> (k+1)2^k \\ (k+1)! &> (2 + (k-1))2^k \\ &= 2^{k+1} + (k-1)2^k > 2^{k+1} \end{aligned}$$

where we have used the fact that $(k+1) > 0$ and $(k-1) \geq 0$.

Since both the base case and induction hypothesis are true, the result follows by the principle of mathematical induction. ■

These inequality induction problems tend to be the most tricky of these 3 standard induction questions. Induction has its uses way beyond summation, divisibility and inequalities. Here are some more diverse examples.

Result 7.2.5 *For any integer $n \geq 3$, one can always find n distinct natural numbers $\{a_1, a_2, \cdots, a_n\}$ so that*

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

As an example:

$$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

Proof. We prove this result by induction. When $n = 3$, the result holds (as was stated above) with the set $\{2, 3, 6\}$.

Now assume the result holds for $n = k$. Hence there is a set $\{a_1, a_2, \cdots, a_k\}$ so that

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_k}$$

Notice two things. First, none of the a_i can be one, because otherwise the sum would be too large. Second, by dividing by two we get

$$\frac{1}{2} = \frac{1}{2a_1} + \frac{1}{2a_2} + \cdots + \frac{1}{2a_k}$$

So we can make a new set of $k+1$ numbers $\{2, 2a_1, 2a_2, \ldots, 2a_k\}$ so that the sum of their reciprocals is

$$\frac{1}{2} + \underbrace{\frac{1}{2a_1} + \frac{1}{2a_2} + \cdots + \frac{1}{2a_k}}_{=\frac{1}{2}} = 1$$

as required.

Since the base case is true, and the induction hypothesis is true, the result follows by mathematical induction.

Note that instead of dividing everything by two, one could also "split" the largest number in the set via

$$\frac{1}{x} = \frac{1}{1+x} + \frac{1}{x(x+1)},$$

since

$$\frac{1}{1+x} + \frac{1}{x(x+1)} = \frac{x}{x(1+x)} + \frac{1}{x(x+1)} = \frac{x+1}{x(x+1)} = \frac{1}{x}.$$

■

Egyptian fractions. The result above is an example of writing a rational number as an **Egyptian fraction**. More generally, an Egyptian fraction is a sum of reciprocals of integers. For example

$$\frac{17}{12} = \frac{1}{1} + \frac{1}{3} + \frac{1}{12}.$$

The interested reader should work out how to write any given rational number in this way.

The next two examples are calculus-flavoured and so we assume some understanding[86] of results on derivatives and integrals. Those ideas are not covered in

[86]That is, some standard results from typical Calculus-1 and -2 courses on differential and

this text (except as a way of constructing some nice examples and exercises).

Result 7.2.6 *Let $n \in \mathbb{N}, f(x) = x\log(x)$ and use $f^{(n)}(x)$ to denote the n^{th} derivative of $f(x)$. Then for any $n \geq 2$*

$$f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}}$$

Note that we are using $\log x$ to denote the ***natural logarithm****.*

Proof. We prove the result by induction. Let f be as defined in the statement of the result.

First note that

$$\begin{aligned} f(x) &= x\log x \\ f'(x) &= 1 + \log x \\ f''(x) &= \frac{1}{x} \end{aligned}$$

and hence the result holds for $n = 2$.

Assume the result holds for $n = k$, then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x} f^{(k)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(-1)^k \frac{(k-2)!}{x^{k-1}} \\ &= (-1)^k (k-2)! \cdot (-1)(k-1)x^{-k} \\ &= (-1)^{k+1} \frac{(k-1)!}{x^k} \end{aligned}$$

as required. Since the base case and inductive step hold, the result follows by induction. ■

Result 7.2.7 *For every integer $n \geq 0$,*

$$n! = \int_0^\infty x^n e^{-x} \mathrm{d}x$$

To prove this result we'll make use of the following fact (which you probably covered in a calculus course).

Fact 7.2.8 *For any $n \in \mathbb{Z}$,*

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

Proof of Result 7.2.7. We prove this by induction.

- When $n = 0$ we have

$$\begin{aligned} \int_0^\infty e^{-x} \mathrm{d}x &= \left[-e^{-x}\right]_0^\infty \\ &= 1 - \lim_{b\to\infty} e^{-b} \end{aligned}$$

integral calculus.

$$= 1 - 0 = 1 = 0!$$

where we have used Fact 7.2.8 to evaluate the limit. Hence the base case is true.

- Assume that the result holds for $n = k$. Then we evaluate the integral for $n = k + 1$ using integration by parts.

$$\begin{aligned}\int_0^\infty x^{k+1}e^{-x}\mathrm{d}x &= \left[-e^{-x}x^{k+1}\right]_0^\infty + \int_0^\infty (k+1)x^k e^{-x}\mathrm{d}x \\ &= (k+1)\int_0^\infty x^k e^{-x}\mathrm{d}x + 0 - \lim_{b\to 0} e^{-x}x^{k+1} \\ &= (k+1)\int_0^\infty x^k e^{-x}\mathrm{d}x \\ &= (k+1)k! = (k+1)!\end{aligned}$$

as required. Again we have used Fact 7.2.8 to evaluate the limits.

By the principle of mathematical induction, the result holds for all $n \geq 0$. ■

7.2.2 More general and yet equivalent

We can generalise induction yet further with a *stronger* hypothesis in the induction step. In particular, we can set up induction so that we use all of the earlier statements $P(\ell), \cdots, P(k)$ to prove the next $P(k+1)$. This stronger hypothesis gives the result its name, rather than the actual result being stronger. We have called this a corollary because one can prove it from regular induction (and we will). In fact, it is equivalent to regular induction, which we can show by proving regular induction from strong induction.

Corollary 7.2.9 Strong induction. *For a fixed integer ℓ, let $S = \{n \in \mathbb{Z} \mid n \geq \ell\}$. For each integer $n \in S$, let $P(n)$ be a statement. If*

- *$P(\ell)$ is true, and*
- *if the implication $P(\ell) \wedge P(\ell+1) \wedge \cdots \wedge P(k) \implies P(k+1)$ is true for all $k \in S$*

then $P(n)$ is true for all $n \in S$.

Proof. As always, we start by assuming the hypotheses are true and work our way to the conclusion. So assume that $P(\ell)$ is true, and that $P(\ell) \wedge P(\ell+1) \wedge \cdots \wedge P(k) \implies P(k+1)$ is true for all $k \in S$.

To use regular induction we use a little trick. Let $Q(n)$ be the statement

$$P(\ell), P(\ell+1), \cdots, P(n) \text{ are all true}$$

Then since $P(1)$ is true, we know that $Q(1)$ is true. Further, since we know that

$$P(\ell) \wedge P(\ell+1) \wedge \cdots \wedge P(k) \implies P(k+1)$$

then if $Q(k)$ is true, then $Q(k+1)$ is true. Hence $Q(k) \implies Q(k+1)$ is true. By (regular) mathematical induction, we know that $Q(n)$ is true for all $n \in S$, and so $P(n)$ is true for all $n \in S$ as required. ■

We can actually go further and prove that strong induction implies regular induction. This proves that the two types of induction are actually equivalent to each other. Anything you can prove with one, you can prove with the other; though that does not say the proofs would be equally appealing.

Corollary 7.2.10 Strong implies weak. *Strong induction is equivalent to regular induction.*

Proof. We are actually proving here that Strong Induction (Corollary 7.2.9) if and only if Mathematical Induction (Theorem 7.2.1). We have already proved that regular induction implies strong induction in the proof of Corollary 7.2.9, so it is sufficient for us to prove that strong induction implies regular induction.

Assume the hypothesis of regular induction. That is $P(\ell)$ is true and $P(k) \implies P(k+1)$ for any $k \in S$. Again a small trick can be used:

$$P(\ell) \wedge P(\ell+1) \wedge \cdots \wedge P(k) \implies P(k)$$

The only way the above can be false is if the hypothesis is true but the conclusion is false. That cannot happen because $P(k)$ appears in the conjunction of the hypothesis (and so must be true) and as the conclusion. Put this above together with the assumption $P(k) \implies P(k+1)$:

$$P(\ell) \wedge P(\ell+1) \wedge \cdots \wedge P(k) \implies P(k) \implies P(k+1)$$

and we have satisfied the hypotheses of strong induction. Hence $P(n)$ is true for all $n \in S$. ■

Result 7.2.11 *Any integer $n \geq 12$ can be written as a sum of 3's and 7's. That is we can find non-negative integer a, b so that $n = 3a + 7b$.*

Proof. We prove this by strong induction.

- When $n = 12$, we can set $a = 4, b = 0$. Then since $12 = 3 \times 4$ the result holds.
- The inductive step is simpler if we also show that the result holds for $n = 13, 14$. In particular since $13 = 2 \times 3 + 7$ and $14 = 2 \times 7$, the result is true for $n = 12, 13, 14$.
- Now assume that the result holds for all $n = 12, 13, 14 \ldots, k$, with $k \geq 14$, and so we can write $k-2 = 3a+7b$. Then it follows that $k+1 = 3+(k-2) = 3(a+1)+7b$ and so has the required form. That is, since the result holds for all $n = 12, \ldots, k$, we know that it holds for $n = k+1$.

So by strong induction, the result holds for all integer $n \geq 12$. ■

Result 7.2.12 *Let $z \in \mathbb{R}$ be any real number so that $z + z^{-1} \in \mathbb{Z}$. Then for any integer n, $z^n + z^{-n} \in \mathbb{Z}$*

Proof. To simplify the discussion, let q_n denote $z^n + z^{-n}$. By assumption we have that $q_1 \in \mathbb{Z}$. Further note that $q_{-n} = q_n$ so it is sufficient to prove that $q_n \in \mathbb{Z}$ for all integer $n \geq 0$; we do so by induction.

When $n = 0$, we have

$$z^0 + z^{-0} = q_0 = 2 \in \mathbb{Z}$$

so the result holds.

Now assume that the result holds for $0 \leq n \leq k$. So we know that q_1, q_{k-1}, q_k are integers. Hence

$$\begin{aligned} q_k q_1 &= (z^k + z^{-k})(z + z^{-1}) \\ &= z^{k+1} + z^{k-1} + z^{1-k} + z^{-k-1} \\ &= \underbrace{z^{k-1} + z^{1-k}}_{=q_{k-1}} + \underbrace{z^{k+1} + z^{-1-k}}_{=q_{k+1}} \end{aligned}$$

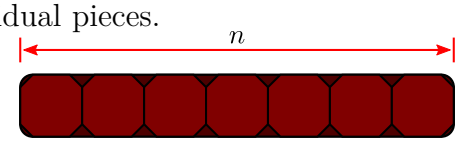
So we have

$$q_{k+1} = q_k q_1 - q_{k-1}$$

Since $q_1, q_{k-1}, q_k \in \mathbb{Z}$ we know that $q_{k+1} \in \mathbb{Z}$ as required. The result then follows by mathematical induction. ■

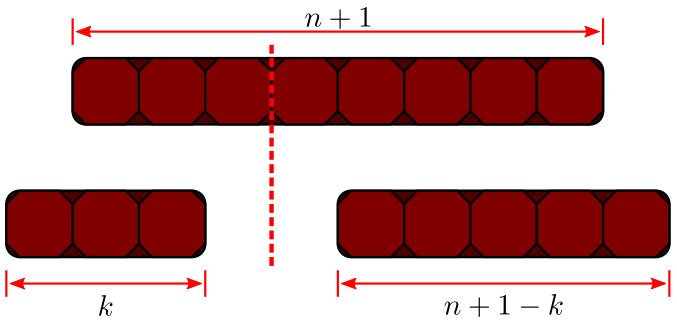
A chocolate bar example.

Example 7.2.13 Breaking a bar of chocolate. Say we have a bar of chocolate as shown below — a line of n blocks. Prove that it takes $n - 1$ breaks to split it into n individual pieces.



We prove this by strong induction.

- First, when $n = 1$, there is nothing to do; it takes 0 breaks to split it into 1 block. The result holds.
- Now assume that the result holds for bars of $1, 2, \ldots, n$ blocks, and then consider a bar of $n + 1$ blocks. Say we split it into two pieces, one of size k, and one of size $n + 1 - k$ (with $k = 1, 2, \ldots, n$). This takes 1 break, and we are left with the problem of splitting up the bar of k blocks and the bar of $n + 1 - k$ blocks (as shown below).



However, by assumption, we know that splitting a bar of k blocks takes $k-1$ breaks, and splitting a bar of $n+1-k$ blocks takes $n-k$ breaks. So, in total we'll need $1+k-1+n-k=n$ breaks. Notice that this is independent of our choice k. Hence the result holds for a bar of $n+1$ blocks.

By strong induction the result holds for all n.

A slightly higher dimensional analogue of this problem involves cutting a pizza using n straight slices. The interested reader should search-engine their way to the "lazy caterer sequence", or (equivalently) the central polygonal numbers. Going up one more dimension one arrives at the "cake numbers" which define the number of chunks of cake you can make when slicing a cube with n-planes. □

Here is a more gamey example.

Example 7.2.14 A game of taking away. Let n be a natural number. Two players place n balls in a box. The players take turns removing either one, two or three balls from the box. The player that removes the last ball loses.

So for example, if there are 13 balls in the box, the game could play out as

- Player 1 removes 3 balls, leaving 10
- Player 2 then removes 2 balls, leaving 8
- Player 1 removes another 3 balls, leaving 5
- Player 2 removes 2 balls, leaving 3
- Player 1 removes 2 balls leaving 1
- Player 2 now must remove the last ball and so loses the game.

We can prove that if $n \equiv 1 \pmod 4$, then Player 2 can always win (if they are careful). We prove this by strong induction.

When the game starts with $n = 1 = 4 \times 0 + 1$ balls, Player 1 must remove the only ball from box and so loses.

Now assume that Player 2 can always win when $n = 4\ell+1$ for $n = 1, 5, \ldots, 4\ell+1$. Now consider what happens when the box starts with $n = 4\ell + 5$ balls. Since Player 2 knows how to win if they leave Player 1 with $4\ell + 1$ balls, they will respond to Player 1's move by trying to get to $4\ell + 1$ balls.

- If Player 1 removes 1 ball, then this gives $4\ell + 4$, and so Player 2 takes 3 balls
- If Player 1 removes 2 balls, then this gives $4\ell + 3$, and so Player 2 takes 2 balls
- If Player 1 removes 3 balls, then this gives $4\ell + 2$, and so Player 2 takes 1 ball

So, no matter what Player 1 does, Player 2 can choose their move so as to leave Player 1 with $4\ell + 1$ balls. From that point, by assumption, Player 2 has a winning strategy.

So by strong induction, Player 2 has a winning strategy whenever $n = 4\ell + 1$. □

Here are a couple of examples involving the Fibonacci numbers. Even those these are well known we'll take a moment to define them formally.

Definition 7.2.15 The **Fibonacci numbers** are defined recursively[87] by

$$F_0 = 0 \qquad F_1 = 1 \qquad F_{n+1} = F_n + F_{n-1}.$$

The first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, \ldots$ ◊

The Fibonacci numbers have many interesting properties and patterns. They also show up all over mathematics and one can find (via the reader's favourite search engine) many examples of Fibonacci numbers in the "real world". We'll start off with an easy result that doesn't require strong induction and then work up towards some harder ones.

Result 7.2.16 *The Fibonacci numbers satisfy*

$$\sum_{\ell=1}^{n} F_\ell = F_{n+2} - 1.$$

Proof. We prove this by induction. When $n = 1$ the statement becomes

$$F_1 = F_3 - 1$$

and since $F_1 = 1, F_3 = 2$ it is true.

Now assume that the result holds for $n = k$. That is

$$F_1 + F_2 + \cdots + F_k = F_{k+2} - 1$$

Then we have

$$\begin{aligned} F_1 + F_2 + \cdots + F_k + F_{k+1} &= F_{k+1} + F_{k+2} - 1 \\ &= F_{k+3} - 1 \end{aligned}$$

as required. Since both the base case and inductive step are true, the result follows by induction. ■

[87] i.e. we give the first few Fibonacci numbers, and then we define later Fibonacci numbers in terms of earlier Fibonacci numbers.

Result 7.2.17 *The Fibonacci numbers satisfy*

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

This is known as Cassini's identity.
Proof. We prove this by induction. When $n = 1$:

$$F_2F_0 - F_1^2 = 0 - 1 = -1$$

as required.

Now assume that it holds for $n = k$, so that

$$F_{k+1}F_{k-1} - F_k^2 = (-1)^k$$

Now consider

$$\begin{aligned} F_{k+2}F_k - F_{k+1}^2 &= (F_{k+1} + F_k)F_k - (F_k + F_{k-1})F_{k+1} \\ &= \underbrace{F_{k+1}F_k} + F_k^2 - \underbrace{F_kF_{k+1}} - F_{k-1}F_{k+1} \\ &= F_k^2 - F_{k-1}F_{k+1} \\ &= -(F_{k+1}F_{k-1} - F_k^2) \\ &= (-1)^{k+1} \end{aligned}$$

as required. Since both the base case and inductive step hold, the result follows by induction. ■

Result 7.2.18 *Let $q \in \mathbb{N}$. Then $5 \mid F_{5n}$ for all $n \in \mathbb{N}$.*
Proof. Fix $q \in \mathbb{N}$. We prove the result by induction. When $n = 1$ the result holds since $F_5 = 5$.

Now assume that $5 \mid F_{5k}$. Now

$$\begin{aligned} F_{5k+5} &= F_{5k+4} + F_{5k+3} \\ &= (F_{5k+3} + F_{5k+2}) + F_{5k+3} = 2F_{5k+3} + F_{5k+2} \\ &= 2(F_{5k+2} + 2F_{5k+1}) + F_{k+2} = 3F_{5k+2} + 2F_{5k+1} \\ &= 3(F_{5k+1} + F_{5k}) + 2F_{5k+1} = 5F_{5k+1} + 3F_{5k} \end{aligned}$$

Now since $5 \mid F_{5k}$ it follows that $5 \mid F_{5k+5}$. The result follows by induction. ■

Result 7.2.19 *The Fibonacci numbers satisfy*

$$\begin{aligned} F_{2n-1} &= F_n^2 + F_{n-1}^2 && \text{and} \\ F_{2n} &= F_{n+1}^2 - F_{n-1}^2. \end{aligned}$$

Proof. Induction is not the easiest way to prove these, but we will press on. The results hold when $n = 1$ since

$$\begin{aligned} F_1 = 1 = 1^2 + 0^2 = F_1^2 + F_0^2 \\ F_2 = 1 = 1^2 - 0^2 = F_2^2 - F_0^2 \end{aligned}$$

Now assume the result holds for all $k \leq n$. Now from the recurrence that defines the Fibonacci numbers:

$$\begin{aligned} F_{2k+1} &= F_{2k} + F_{2k-1} \\ &= (F_{k+1}^2 - F_{k-1}^2) + (F_k^2 + F_{k-1}^2) \qquad \text{by assumption} \\ &= F_{k+1}^2 + F_k^2 \end{aligned}$$

as required. We do similarly for the other equation (though more gymnastics are required).

$$\begin{aligned} F_{2k+2} &= F_{2k+1} + F_{2k} \\ &= (F_{k+1}^2 + F_k^2) + (F_{k+1}^2 - F_{k-1}^2) \end{aligned}$$

Try to make the first bracket look like $F_{k+2}^2 = (F_{k+1} + F_k)^2$

$$= (F_{k+1}^2 \underbrace{+2F_{k+1}F_k}_{+} + F_k^2) + (F_{k+1}^2 - F_{k-1}^2) \underbrace{-2F_kF_{k+1}}_{-}$$

Now manipulate the second bracket using difference of squares and then $F_{k+1} - F_{k-1} = F_k$

$$\begin{aligned} &= (F_{k+1} + F_k)^2 + (F_{k+1} - F_{k-1})(F_{k+1} + F_{k-1}) - 2F_kF_{k+1} \\ &= F_{k+2}^2 + F_k(F_{k+1} + F_{k-1} - 2F_{k+1}) \\ &= F_{k+2}^2 - F_k(F_{k+1} - F_{k-1}) \\ &= F_{k+2}^2 - F_k^2 \end{aligned}$$

as required.

Since the base case is true and the inductive step is true, the result follows by induction. ■

Next is an important example that proves every integer bigger than 2 can be written as a product of prime numbers. This statement is part of the Fundamental Theorem of Arithmetic, which also says that the product is *unique*. Proving uniqueness takes more work — see Section 9.6. This result focuses on proving the *existence* of at least one factorisation into primes.

This is also a really good example of a result that we all know, but we haven't proved. When we do sit down to try to prove such results it can be easy to get confused or disoriented because the result seems so obvious; we've known and worked with the result for years! Some such results may indeed be trivial while some need more work. It is always a good idea to go back to the basic definitions and methods to help work your way to a proof.

Example 7.2.20 There exists a prime factorisation. Let $n \in \mathbb{N}$, $n \geq 2$. Then n is a product of prime numbers.

Scratchwork. Notice that the statement includes the case where n is prime, whence n is the product of a single prime — itself! Then, any integer greater than

2 is prime or not prime. If it is not prime, what can we say about its divisors?

Let's try to prove the statement using induction on n. The base case should be $n = 2$. In this case, the statement is true, since 2 is prime, and therefore a product of prime numbers (in this case, the product of a single prime number, 2). Now suppose the result holds for $n = k$, so that $n = k$ is a product of prime numbers. We need to show that $k + 1$ is a product of prime numbers. We can try to prove this by considering two cases: $k + 1$ is prime, or $k + 1$ is not prime, one of which must be true.

- $k + 1$ is prime: then the result holds for $n = k + 1$, so we're done.
- $k + 1$ is not prime: here we can't conclude anything immediately. Instead we'll try to use the inductive hypothesis to show that the result holds. By definition, $k + 1$ not being prime means that there are $a, b \in \mathbb{N}$, $1 < a, b < k + 1$, such that $k + 1 = ab$.

  Now, if we knew a and b could be written as a product of prime numbers, then we'd also have that $k+1$ is a product of prime numbers. But we don't know this, since for the inductive hypothesis, we only assumed that $n = k$ could be written as a product of prime numbers. Indeed, it is not clear how we can get from $n = k$ to $n = k + 1$ in our inductive step since k and $k + 1$ don't share common factors other than ± 1 (see Exercise 3.5.7). This suggests that we use strong induction instead of regular induction.

Let's try again using strong induction. The base case is the same as before. This time, for the inductive hypothesis, instead of just assuming that $n = k$ is a product of prime numbers, we assume that for any $2 \leq n \leq k$, n is a product of prime numbers. We need to show that $k + 1$ is a product of prime numbers. Again we can split this into two cases.

- $k + 1$ is prime: then the result holds for $n = k + 1$, so we're done.
- $k + 1$ is not prime: By definition, $k + 1$ not being prime means that there are $a, b \in \mathbb{N}$, $1 < a, b < k + 1$, such that $k + 1 = ab$. By the inductive hypothesis, both a and b can be written as a product of prime numbers; that is $a = p_1 \cdots p_r$ and $b = q_1 \cdots q_s$ for prime numbers $p_1, \ldots, p_r, q_1, \ldots, q_s$. (Note that these prime numbers are not necessarily distinct, and if a or b were itself prime, then $r = 1$ or $s = 1$, respectively.) Then

  $$k + 1 = ab = p_1 \cdots p_r q_1 \cdots q_s$$

  and so $k + 1$ may be written as a product of prime numbers.

In either case, we've concluded that $k + 1$ is a product of prime numbers, and so the inductive step is complete. Let's write up our argument.

Solution.

Proof. Let $n \in \mathbb{N}$, $n \geq 2$. We proceed by strong induction on n. For the base case, suppose $n = 2$. Then n is prime, and the result holds.

Now suppose that the result holds for all $2 \leq n \leq k$; that is, if $n \leq k$, then n is a product of prime numbers. Either $k+1$ is prime, or not prime. If $k+1$ is prime, then the result holds. So suppose that $k+1$ is not prime. Then, by definition, $k+1 = ab$ for some $a, b \in \mathbb{N}$, $1 < a, b < k+1$. By the inductive hypothesis, both a and b can be written as a product of prime numbers; that is $a = p_1 \cdots p_r$ and $b = q_1 \cdots q_s$ for prime numbers $p_1, \ldots, p_r, q_1, \ldots, q_s$. Then

$$k+1 = ab = p_1 \cdots p_r q_1 \cdots q_s$$

and so $k+1$ may be written as a product of prime numbers. Thus the result holds for $k+1$. Thus by strong induction, any $n \in \mathbb{N}$, $n \geq 2$ is a product of prime numbers. ■

□

Finally, a nice trigonometric example[88]. We'll assume that you remember (or can quickly revise) the formulas

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

Result 7.2.21 *Let $\theta \in \mathbb{R}$ be fixed. Let $p_0 = 1$ and $p_1 = \cos\theta$ and define $p_n = 2p_1 p_{n-1} - p_{n-2}$. Prove that $p_n = \cos(n\theta)$ for all integer $n \geq 0$.*
Proof. We prove the result by induction. When $n = 0$ we have $p_0 = \cos 0 = 1$, so the base case is true. We now turn to the inductive step.

Assume that the result hold for $n = 0, 1, \cdots, k$. In particular,

$$p_1 = \cos\theta \qquad\qquad p_k = \cos k\theta$$

Now it suffices to show that $p_{k+1} = 2p_1 p_k - p_{k-1} = \cos((k+1)\theta)$. Substitute in the values of p_1, p_k, p_{k-1}:

$$p_{k+1} = 2\cos\theta\cos(k\theta) - \cos((k-1)\theta)$$

Recall (from the formulas above) that

$$\cos((k+1)\theta) = \cos\theta\cos k\theta - \sin\theta\sin k\theta \qquad \text{and}$$
$$\cos((k-1)\theta) = \cos\theta\cos k\theta + \sin\theta\sin k\theta$$

so that

$$\cos((k+1)\theta) + \cos((k-1)\theta) = 2\cos\theta\cos k\theta$$

Substituting this into the expression for p_{k+1} above then gives

$$p_{k+1} = \cos((k+1)\theta)$$

as required. ■

[88]We hope the reader didn't think they could avoid trigonometry just because this isn't a calculus text. Take this as a sine of how important the topic is (and its related puns).

7.3 Exercises

1. Prove, using induction, that $\forall n \in \mathbb{N}$, $3 \mid (n^3 - n)$.

2. Let $n \in \mathbb{N}$ so that $n \geq 2$. Use mathematical induction to prove that

$$n! \leq n^n$$

Recall that $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.

3. Let $n \in \mathbb{N}$. Prove that $\forall n \geq 7$, $n! > 3^n$.

4. Let $n \in \mathbb{N}$. Prove by induction on n, that $\exists x, y, z \in \mathbb{Z}$ such that $x \geq 2$, $y \geq 2$, and $z \geq 2$ and satisfy $x^2 + y^2 = z^{2n+1}$.

5. Let $m \in \mathbb{N}$, with m even. Show that 8 divides $3^m - 1$.

6. The *distributive law* says that for any real numbers a, b, c, we have

$$a \cdot (b + c) = a \cdot b + a \cdot c.$$

Use induction to show that for any $n \geq 2$ and any real numbers $a, b_1, b_2 \ldots, b_n$

$$a \cdot (b_1 + b_2 + \cdots + b_n) = a \cdot b_1 + a \cdot b_2 + \cdots + a \cdot b_n.$$

7. Let $n \in \mathbb{N}$. Using induction, prove that $2^n \geq 2n$.

8. Let $n \in \mathbb{N} \cup \{0\}$. Show that 5 divides $2^{2n+1} + 3^{2n+1}$.

9. Let $n \in \mathbb{N}$, $n \geq 2$. Suppose that $x_1, x_2, \ldots, x_n \in \mathbb{Q}$. Show by induction that $x_1 + x_2 + \cdots + x_n \in \mathbb{Q}$.

10. Prove that, $\forall n \in \mathbb{N}$, $\displaystyle\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$.

11. Prove that $\displaystyle\sum_{j=1}^{n} j^3 > \frac{1}{4}n^4$ for all $n \in \mathbb{N}$.

12. Let r be a real number so that $r \neq 1$. Use induction to show that

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$$

for $n \in \{0, 1, 2, \ldots\}$.

Note: you may have seen a proof of this that does not use induction. Make sure your proof here uses induction.

13. Let $k \in \mathbb{N}$. Compute the k^{th} derivative of the following functions. Use induction to prove that your answer is correct.

(a) $f(x) = x^n$ for $n \in \mathbb{N}$ with $0 \leq k \leq n$.

(b) $g(x) = x^{-n}$ for $n \in \mathbb{N}$.

(c) $h(x) = \frac{1}{\sqrt{9-2x}}$.

You may use the chain and power rules in your solutions.

14. Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

15. Let $n \in \mathbb{N}$. Show that

$$\sum_{k=1}^{n} \frac{1}{2^k} < 1$$

16. Determine why the proposed proof of the following statement is incorrect:

Every non-empty subset A of the natural numbers has a maximum element. That is, there is some $a \in A$ such that $b \leq a$ for all $b \in A$.

Faulty proof. Let $A \subseteq \mathbb{N}$ be non-empty. We proceed with induction on $|A|$.

If $|A| = 1$, then the single element of A is its maximum.

Now suppose that $|A| = n + 1$ for some $n \geq 1$, and that any subset of $\mathbb{N}$ of size n has a maximum element. Choose some $a \in A$, and let $B = A \setminus \{a\}$. Then $B \subseteq \mathbb{N}$, and $|B| = n$. By the inductive hypothesis, B has a maximum element, say b. Therefore $c \leq b$ for all $c \in A \setminus \{a\}$. If $a \leq b$ as well, then b is the maximum element of A. If $b \leq a$, then $c \leq b \leq a$ for all $c \in A \setminus \{a\}$. Therefore a is the maximum element of A.

Taking $n \to \infty$, we see that any non-empty $A \subset \mathbb{N}$ has a maximum. ■

17. Let $n \in \mathbb{Z}$, $n \geq 0$.

(a) Use induction and l'Hôpital's rule to show that

$$\lim_{x \to \infty} x^n e^{-x} = 0.$$

(b) Show that

$$n! = \int_0^\infty x^n e^{-x} \mathrm{d}x.$$

For this question

- recall that $0! = 1$, and
- you may use basic facts about limits and integration from your Calculus-1 course.

18. Let $a_1, a_2, \ldots$ be real numbers. Prove, using induction, that for all $n \in \mathbb{N}$,

$$\left| \sum_{k=1}^{n} a_k \right| \leq \sum_{k=1}^{n} |a_k|.$$

You may assume the triangle inequality, $|x + y| \leq |x| + |y|$ for all real numbers x and y.

19. All numbers of the form $1007, 10017, 100117, 1001117, 10011117, \ldots$ are divisible by 53.

20. Let F_k be the k^{th} Fibonacci number, and let $q \in \mathbb{N}$. Show that $F_q \mid F_{qn}$ for all $n \in \mathbb{N}$. This generalises Result 7.2.18.

21. Let $n, r \in \mathbb{Z}$ so that $0 \leq r \leq n$. We define the binomial coefficient

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

(a) Prove that the binomial coefficients satisfy Pascal's identity:

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1} \qquad \text{for } 0 < r \leq n$$

(b) and so prove, using induction, the Binomal Theorem:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

for any $a, b \in \mathbb{R}$ and any $n \in \mathbb{N}$.

22. Let $s \in \mathbb{R}$, $s > 0$. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that

(a) for any $n \in \mathbb{N}$, the n^{th} derivative $f^{(n)}$ exists and is continuous,

(b) the limit $\lim_{x\to\infty} e^{-sx} f(x) = 0$, and

(c) for any $n \in \mathbb{N}$, $\lim_{x\to\infty} e^{-sx} f^{(n)}(x) = 0$

From f we can define a new *transformed* function

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(x) e^{-sx} \, \mathrm{d}x.$$

Prove that for any $k \in \mathbb{N}$, that

$$\mathcal{L}\left\{f^{(k)}\right\}(s) = s^k \mathcal{L}\{f\}(x) - \sum_{i=0}^{k-1} s^{k-1-i} f^{(i)}(0)$$

This result tells us that the transform of any derivative is simply related to the transform of the original function. That is, the differential equation in f turns into an algebraic equation in $\mathcal{L}(f)$ This sort of result can come in very handy when studying differential equations.

23. Let $\alpha \in \mathbb{R}$ such that $\alpha + \frac{1}{\alpha} \in \mathbb{Z}$. Prove that $\alpha^n + \frac{1}{\alpha^n} \in \mathbb{Z}$ for any $n \in \mathbb{N} \cup \{0\}$.

24. Show that every natural number, n, can be written as

$$n = 3^m a,$$

where $m \in \mathbb{Z}$ such that $m \geq 0$, $a \in \mathbb{N}$ and $3 \nmid a$.

25. You go on vacation to a foreign country. The local currency only has 3 and 5 dollar bills, and locals will only give items a price $p \in \mathbb{N}$ such that $p \geq 8$. Assume that you have access to an unlimited supply of 3 and 5 dollar bills. Can you buy any souvenir you want? Give a proof or a counterexample.

26. Let $a_0, a_1, a_2, \ldots$ be a sequence recursively defined as $a_0 = 2$, $a_1 = 1$, and

$$a_n = a_{n-1} + 6a_{n-2} \quad \text{for } n \geq 2.$$

Prove by induction that

$$a_n = (-2)^n + 3^n \quad \text{for all } n \geq 0.$$

27. Show that every $n \in \mathbb{N}$ can be written in binary. That is, for all $n \in \mathbb{N}$, there exists an $m \in \mathbb{Z}$ with $m \geq 0$ and constants $c_0, c_1, c_2, \ldots, c_m \in \{0, 1\}$ such that

$$a = c_m \cdot 2^m + c_{m-1} \cdot 2^{m-1} + \cdots + c_1 \cdot 2 + c_0.$$

For example,

$$537 = 512 + 16 + 8 + 1 = 2^9 + 2^4 + 2^3 + 2^0,$$

so $m = 9$ and $(c_0, c_1, c_2, \ldots, c_9) = (1, 0, 0, 1, 1, 0, 0, 0, 0, 1)$.

28. For $n \in \mathbb{N}$, consider the recurrence

$$T(1) = 1 \qquad \text{and for } n \geq 2 \qquad T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + n$$

The first few values of $T(n)$ are

$$1, 4, 5, 12, 13, 16, 17, 32, 33, \ldots$$

Use strong induction to prove that for $n \geq 1$, $T(n)$ satisfies the bound

$$T(n) \leq n \log_2 n + n.$$

Note that the logarithm has base 2, and that the floor function, $\lfloor x \rfloor$, gives the largest integer smaller or equal to x.

Recurrences such as this one appear very frequently in the analysis of "divide and conquer" algorithms. That class of algorithms (roughly speaking) work by repeatedly splitting a larger problem into smaller pieces until they can be solved trivially. The interested reader should search-engine their way to more information.

29. Use strong induction to prove the following:

Suppose you begin with a pile of n stones ($n \geq 2$) and split this pile into n separate piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have p and q stones in them, respectively, you compute pq. Show that no matter how you split the piles (eventually into n piles of one stone each), the sum of the products computed at each step equals $\dfrac{n(n-1)}{2}$.

For example — say with start with 5 stones and split them as follows:

$$(5) \to \underbrace{(3)(2)}_{=6} \to \underbrace{(2)(1)}_{=2}\underbrace{(1)(1)}_{=1} \to \underbrace{(1)(1)}_{=1}(1)(1)(1).$$

Then, we get, $6+2+1+1 = 10 = \frac{5\times 4}{2}$ ✓.

Chapter 8

Return to sets

As we mentioned early, we have changed the order of topics a little in this text so that we can get to proving things as quickly as possible and get you more comfortable with basic proof methods. This meant that we skipped several basic aspects of sets that we should now cover. Thankfully we can now do a bit more with these basic bits of set theory — indeed we can prove some things!

8.1 Subsets

When we defined sets way back at the beginning of the course, we saw that the only thing we can ask a set is

> Is this object in the set?

and the set can only respond to us with either

> Yes.

or

> No.

We can make this simple structure much more interesting by enriching it using some of the logic we have learned. Consider the sets

$$A = \{1, 2, 3\} \qquad\qquad B = \{0, 1, 2, 3, 4\}\,.$$

We see that every single element of A is contained in the set B. So rather than asking one-by-one whether or not individual elements of A are contained in B, we can instead ask if all the elements of A are contained in B. Equivalently, we can ask "Is A contained in B". This is the idea of A being a subset of B.

Definition 8.1.1 Let A and B be sets.

- We say that A is a **subset** of B if every element of A is also an element of B. We denote this $A \subseteq B$.

- If A is not a subset of B, then we denote this as $A \not\subseteq B$.
- Further, if A is a subset of B but there is at least one element of B that is not in A then we say that A is a **proper subset** of B. We denote this by $A \subset B$.
- If $A \subseteq B$ then B is a **superset** of A which we can write as $B \supseteq A$. Similarly, if $A \subset B$ then B is a **proper superset** of A which write as $B \supset A$

Finally,

- Two sets A and B are equal if A is a subset of B and B is a subset of A. That is

$$(A = B) \equiv ((A \subseteq B) \wedge (B \subseteq A))$$

$\Diamond$

Some things to note

- The empty set is a subset of every set. That is, for any set A, we always have that $\varnothing \subseteq A$.
- That $A \subseteq B$ is equivalent to saying that if we take any element of A then it is also an element of B. That is

$$(A \subseteq B) \equiv (\forall a \in A, a \in B) \equiv (a \in A \implies a \in B).$$

- If $A \not\subseteq B$, then there is at least one element of A that is not in B. That is

$$(A \not\subseteq B) \equiv (\exists a \in A \text{ s.t. } a \notin B)$$

- Since two sets are equal when they are subsets of each other, we prove set equality using a two part proof. The first part proving that $A \subseteq B$ and then the second part showing that $B \subseteq A$.

Example 8.1.2 Let $S = \{1, 2\}$. What are all the subsets of S?

- The subsets are $\varnothing, \{1\}, \{2\}, \{1, 2\}$.

$\square$

Result 8.1.3 *Let* $A = \{n \in \mathbb{Z} \mid 6|n\}$ *and* $B = \{n \in \mathbb{Z} \mid 2|n\}$. *Then* $A \subseteq B$.
First lets write a few elements of these sets

$$A = \{\ldots, -18, -12, -6, 0, 6, 12, 18, \ldots\} \quad B = \{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$$

The result we are trying to prove is that $A \subseteq B$. This is equivalent to showing that if $a \in A$ then $a \in B$ (which is essentially saying that if an integer is divisible by 6 then it is divisible by 2). Hence to prove the result we are going to assume that $a \in A$ and then work our way to proving that $a \in B$.

- First up we make sure that the reader knows we are going to take A, B to be the sets in the statement of the result.
- Assume $a \in A$.
- Then this means that a is an integer that is divisible by 6. That is $a = 6\ell$ for some $\ell \in \mathbb{Z}$.
- We need to show that $a \in B$. This is equivalent to showing that we can write $a = 2k$ where k is some integer.
- But we know that $a = 6\ell = 2(3\ell)$ and since $3\ell \in \mathbb{Z}$ we are done.

Of course, we aren't really done until we write thing up nicely.

Proof. Let A, B be the sets defined in the result. Assume that $a \in A$. Hence we can write $a = 6\ell$ where $\ell \in \mathbb{Z}$. But now since $6\ell = 2(3\ell)$ and 3ℓ is an integer, we know that $a \in B$. Thus $A \subseteq B$. ■

The following result expresses that being a subset is a transitive relation. We'll see much more about transitivity in the near future.

Result 8.1.4 *Let A, B, C be sets. Then if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.*

We should start by breaking things down carefully because there are a few nested implications hidden inside this result.

- The hypothesis is $A \subseteq B$ and $B \subseteq C$. We can, in turn, write this as
$$(a \in A \implies a \in B) \wedge (b \in B \implies b \in C)$$
- The conclusion says $A \subseteq C$. This can similarly be written as
$$a \in A \implies a \in C.$$
- Since we are trying to prove an implication, we start by assuming the hypothesis is true. So we assume that
$$(a \in A \implies a \in B) \wedge (b \in B \implies b \in C)$$
Notice this we do *not* assume that $a \in A$ and $b \in B$, rather we are assuming that the above implications are both true[89].
- Now we want to prove that the conclusion is true, namely that
$$a \in A \implies a \in C.$$
This we can do by assuming that the hypothesis is true and showing the conclusion.[90] That is, assume that $a \in A$ and then work towards showing that $a \in C$.

[89] And, as you have now memorised, an implication is only false when its hypothesis is true and its conclusion false.

[90] Equivalently, we know that either $a \in A$ or $a \notin A$. When $a \in A$ we have precisely this case. On the other hand, if $a \notin A$ then the implication $(a \in A) \implies (a \in C)$ is true since the hypothesis is false. So it is only when $a \in A$ that there is work to do.

- Since now we have assumed that $a \in A$ and $a \in A \implies a \in B$, we know that $a \in B$. And then, in turn, our assumption that $a \in B \implies a \in C$, we know that $a \in C$. Precisely what we need!

Oof! The above analysis becomes much easier with practice.
Proof. Assume that $A \subseteq B$ and $B \subseteq C$. We wish to show that now $A \subseteq C$, so let $a \in A$. Since we know that $A \subseteq B$, we know that $a \in B$. And since we know that $B \subseteq C$, we know that $a \in C$. Hence we have shown that $A \subseteq C$. ■

So even though our scratch work took some twists and turns, our proof is quite succinct.

Once you start thinking about subsets of a set A, it is quite natural to ask[91] "What are all the subsets of a given set?" The resulting *set of sets* is called the **power set**. This set will be very important when we look at the cardinality of sets, particularly the cardinality of infinite sets. But first, we should really give it a precise definition.

Definition 8.1.5 Let A be a set. The **power set** of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A. ◊

Note that the elements of $\mathcal{P}(A)$ are themselves sets, and

$$X \in \mathcal{P}(A) \iff X \subseteq A.$$

When we get to the chapter on cardinality we'll prove that if A is a finite set with $|A| = n$ then $|\mathcal{P}(A)| = 2^n$. We will also prove an important result about the power set of an infinite set.

Example 8.1.6 What are $\mathcal{P}(\{1,2,3\}), \mathcal{P}(\varnothing)$ and $\mathcal{P}(\mathcal{P}(\varnothing))$?

$$\begin{aligned} \mathcal{P}(\{1,2,3\}) &= \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\} \\ \mathcal{P}(\varnothing) &= \{\varnothing\} \\ \mathcal{P}(\mathcal{P}(\varnothing)) &= \mathcal{P}(\{\varnothing\}) = \{\varnothing, \{\varnothing\}\} \end{aligned}$$

□

Result 8.1.7 *Let A, B be sets. Then $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.*

Since this is a biconditional we need to prove both directions. Lets start with the forward implication and then turn to the converse.

- We want to show that $A \subseteq B \implies \mathcal{P}(A) \subseteq \mathcal{P}(B)$.
- So we assume that $A \subseteq B$ and we want to prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
- This is equivalent to showing that $X \in \mathcal{P}(A) \implies X \in \mathcal{P}(B)$ — note that X is now itself a set (which is why[92] we've used a capital X rather than a lower case x).

[91] Well, the author thinks it is a natural question and hopes you think it is one too.
[92] The power of good notational conventions!

- So just as was the case in the previous proof, we assume that the hypothesis, $X \in \mathcal{P}(A)$ is true. This means that $X \subseteq A$.
- But our previous[93] result — Result 8.1.4 — tells us that if $X \subseteq A$ and $A \subseteq B$ then $X \subseteq B$.
- This, in turn, implies that $X \in \mathcal{P}(B)$, and so $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Okay, now lets look at the converse.

- We want to show that $\mathcal{P}(A) \subseteq \mathcal{P}(B) \implies A \subseteq B$.
- So we assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
- But now $A \in \mathcal{P}(A)$, and so $A \in \mathcal{P}(B)$.
- Hence A must be a subset of B — by definition of the power set.

As always, once the scratch work is done, its time to write up.
Proof. We prove each implication in turn.

- Assume that $A \subseteq B$, and let $X \in \mathcal{P}(A)$. Hence $X \subseteq A$. By our result above, this, in turn, implies that $X \subseteq B$. By the definition of the power set, this tells us that $X \in \mathcal{P}(B)$. Thus we have shown that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.
- Now turn to the converse and assume that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \in \mathcal{P}(A)$, this implies that $A \in \mathcal{P}(B)$. But, by the definition of the power set, this tells us that $A \subseteq B$, and we are done.

Since we have proved both implications we have proved the result. ■

We can think of forming the power set as an operation on a set. Since it is the only operation we have discussed so far, we could keep applying it to see what sorts of things we get. That is, what is the power set of a power set of a power set of a… This is not an entirely silly idea, and we will look at this when we get to the chapter on cardinality, but exploring other set operations is a better use of our time at present.

8.2 Set operations

The first two set operations we'll discuss are union and intersection. Rather than having more discursive blurb, the authors should get on with it and just define them.

Definition 8.2.1 Let A and B be sets. The **union** of A and B, denoted $A \cup B$,

[93] We could refer to this by number as well — that is probably a good idea if the result isn't so close by within the document. However, it is just up there so referring to it as "previous result" is fine.

is the set of all elements in A or B.

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

$\Diamond$

A couple of things to note here. First — the symbol "$\cup$" is not the letter "u". Second — we are using the word "or" in the definition in its inclusive mathematical sense. Indeed we could rewrite the above definition as

$$(x \in A \cup B) \iff (x \in A) \vee (x \in B)$$

We'll draw some pictures to help illustrate things shortly. But first, another definition.

Definition 8.2.2 Let A and B be sets. The **intersection** of A and B, denoted $A \cap B$, to be the set of elements that belong to both A and B.

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

If the intersection $A \cap B = \varnothing$, then we say that A and B are disjoint. $\Diamond$

Again, the symbol "$\cap$" is not an upside-down letter "u", and we are using the word "and" in this definition in its precise mathematical meaning. Just as we rewrote the definition of union, we can rewrite the definition of intersection as

$$(x \in A \cap B) \iff (x \in A) \wedge (x \in B)$$

Indeed there are many parallels between how the operators union and intersection act on sets and how the logical operators "or" and "and" act on mathematical statements. These parallel are reinforced by the similarity in notations.

Warning 8.2.3 The right notation in the right place.. Please be careful to not confuse set and logical operations. When A and B are sets, we cannot take their conjunction or disjunction: "$A \vee B$" "$A \wedge B$" do not make sense. Similarly, if P and Q are statements we cannot take their union or intersection: "$P \cup Q$" and "$P \cap Q$" do not make sense.

Example 8.2.4 Let $A = \{1, 2, 3, 4\}$, $B = \{p : p \text{ is prime}\}$, $C = \{5, 7, 9\}$ and $D = \{\text{even positive integers}\}$. Then

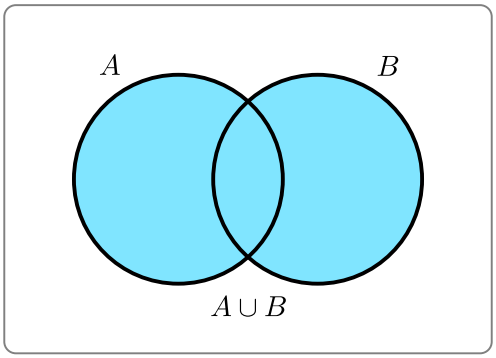
$$\begin{aligned} A \cap B &= \{2, 3\} \\ B \cap D &= \{2\} \\ A \cup C &= \{1, 2, 3, 4, 5, 7, 9\} \\ A \cap C &= \varnothing \end{aligned}$$

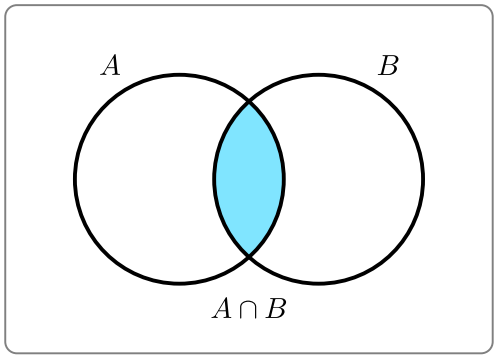
$\square$

We can visualise the operations of union and intersection using a Venn diagram[94]. These diagrams might seem like an obvious and simple idea, however

[94]John Venn was a 19th century English logician and mathematician. He introduced "Eulerian

it is much more recent than the cartesian plane. Venn diagrams are 19th century mathematics, while the cartesian plane is 17th century mathematics[95].





Remark 8.2.5 Long equations, sides, and short-hands. Some of the results we want to prove (like the one coming shortly) involve showing that a long expression on one side of an equality is actually the same as the long expression on the other side. To ease both the writer and the reader of the resulting calculations, we refer to the expressions on either side[96] as "the left-hand side" and "the right-hand side". In typical mathematician style, even this is too long and they are almost always contracted further to "LHS" and "RHS". This shorthand allows us to avoid writing (and reading) chunks of text (with the associated potential errors), but, perhaps more importantly, allows us to indicate that one side of an equation remains fixed unchanged while we work on the other.

Here is a good example adapted from one in "The book of proof" by Richard Hammack[97].

Example 8.2.6 Let

$$A = \{n \in \mathbb{Z} \ : \ 15|n\} \qquad B = \{n \in \mathbb{Z} \ : \ 3|n\} \qquad C = \{n \in \mathbb{Z} \ : \ 5|n\}$$

Prove that $A = B \cap C$.

Scratchwork. So — scratch work first. To prove the equality we need to show that the LHS is a subset of the RHS and vice versa. So we start by proving that $A \subseteq B \cap C$.

- Let $a \in A$, so a is an integer divisible by 15.
- Hence we can write $a = 15k$.
- We now need to show that $a \in B$ and $a \in C$, so that $a \in B \cap C$.

circles" (which later became to be known as Venn diagrams) in a paper "On the Diagrammatic and Mechanical Representation of Propositions and Reasoning". He also built a machine to bowl cricket balls.

[95] Or, if you look into the history a little more, 14th century.

[96] This is, because mathematics is most commonly written sinistrodextrally (left-to-right). Though of course, we could use similar shorthand if we wrote dextrosinistrally or even boustrophedonically! It seems unlikely that anyone would write mathematics vertically. And yes, this footnote is here mostly just to use the word "boustrophedonically".

[97] `www.people.vcu.edu/~rhammack/BookOfProof/`

- Well, since $a = 15k$ we know that $a = 3(5k)$, and because $5k \in \mathbb{Z}$, a is divisible by 3. Hence $a \in B$.
- Similarly, since $a = 15k$ we know that $a = 5(3k)$, and because $3k \in \mathbb{Z}$, a is divisible by 5. Hence $a \in C$.
- Since $a \in B$ and $a \in C$, we know that $a \in B \cap C$.

Thus $A \subseteq B \cap C$.

Now we must argue that $B \cap C \subseteq A$.

- Let $x \in B \cap C$. Notice that we're calling it x rather than b or c since that is a more neutral letter, while b or c suggest that the element might belong to B or C but not both.
- Since $x \in B \cap C$ we know that $x \in B$ and that $x \in C$.
- Because $x \in B$, $x = 3k$ for some integer k.
- Similarly, because $x \in C$, we know that $x = 5\ell$ for some integer ℓ. Notice that we did not write $x = 5k$ since we have already used k to express that x is divisible by 3.
- This implies that $x = 3k = 5\ell$, and we are close to being done.
- Since $5\ell = 3k$ we know that 5ℓ is divisible by 3. We'd like to show that ℓ is a multiple by 3 (which would make sense because 3 and 5 are prime numbers[98]). Perhaps the easiest way to do this is to investigate what happens if ℓ is a multiple of 3 or not. There are 3 possibilities, $\ell = 3j, 3j+1, 3j+2$, for some integer j (we are using Euclidean division here).
  - If $\ell = 3j$ then $5\ell = 15j$ which is precisely what we want.
  - If $\ell = 3j + 1$ then $5\ell = 15j + 5 = 3(5j + 1) + 2$, and so 5ℓ is not divisible by 3. But this contradicts the fact that $x \in B$.
  - Similarly $\ell = 3j + 2$ then $5\ell = 15j + 10 = 3(5j + 3) + 1$, and so 5ℓ is not divisible by 3. Again this contradicts the fact that $x \in B$.

  Hence the only possibility is that ℓ is a multiple of 3. Thus we can write $x = 5\ell = 5 \cdot 3 \cdot j$ for some integer j.
- Hence x is divisible by 15 and $x \in A$.

Thus $B \cap C \subseteq A$.

Oof! Everything is there, we just have to write it up nicely.

Solution.

Proof. Let A, B, C be defined as above. We prove the equality by first proving that the LHS is a subset of the RHS and then vice-versa.

- Let $a \in A$. Then $a = 15m$ for some $m \in \mathbb{Z}$. This means that $a = 3(5m)$ and $a = 5(3m)$. Since both $3m, 5m \in \mathbb{Z}$, it follows that $a \in B$ and $a \in C$. Thus $a \in B \cap C$ as required.

- Now let $x \in B \cap C$. Then $x = 3k$ and $x = 5\ell$ for some integers k, ℓ, so we must have that $3k = 5\ell$. Since $\ell \in \mathbb{Z}$ we know it can be written as $\ell = 3j, 3j+1$ or $3j+2$ (by Euclidean division).

  - If $\ell = 3j$ for some integer j, then $x = 5\ell = 15j$ and so is a multiple of 15.
  - If $\ell = 3j+1$ then $x = 15j+5 = 3(5j+1)+2$, while if $\ell = 3j+2$ then $x = 15j+10 = 3(5j+3)+1$. In either of these cases, we contradict our assumption that x is a multiple of 3.

  Hence we must have that $\ell = 3j$. Hence $x = 5\ell = 5 \cdot 3j = 15(j)$ for some integer j, and so x is divisible by 15. Hence $x \in A$ as required.

So we have shown that $A = B \cap C$. ■

Here is an alternative way to show that $B \cap C \subseteq A$. It relies on the fact that $5 = 6 - 1$.

Proof. Assume that $x \in B \cap C$, and so $x \in B$ and $x \in C$. Then we know that there are $k, \ell \in \mathbb{Z}$ so $x = 3k$ and $x = 5\ell$. Thus we know that $3k = 5\ell$. From this

$$\begin{aligned} 3k &= 6\ell - \ell && \text{and thus} \\ \ell &= 6\ell - 3k = 3(2\ell - k) \end{aligned}$$

That is, ℓ must be divisible by 3. But now $x = 5\ell = 15(2\ell - k)$ and so is divisible by 15. Thus $x \in A$ as required. ■

□

Given two sets A, B we sometimes might need to construct a new set which contains all the elements of A that are not in B. This is the set-difference.

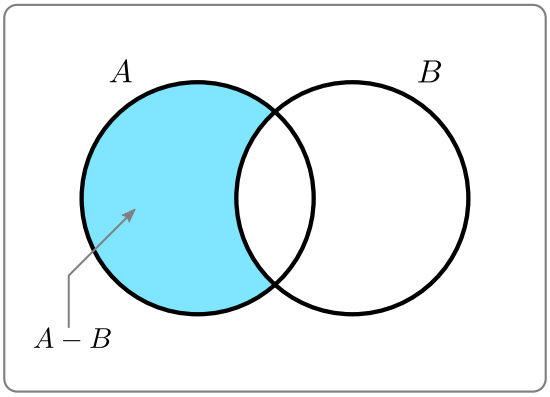
Definition 8.2.7 Let A and B be sets. Then the **difference**, $A - B$ (which is also written $A \setminus B$) is the elements in A that are not in B

$$A - B = \{x \in A \mid x \notin B\}$$

This new set is sometimes called the "**relative complement** of B in A". ◊

Another picture:

[98] This proof can be made a bit more direct if we make use of the fact that every integer has a unique prime factorisation, but we haven't proved that yet.



Example 8.2.8 If we reuse the sets we just defined in the example Example 8.2.4 then,

$$A - D = \{1, 3\}$$
$$C - A = \{5, 7, 9\}$$

□

To define our next set operation we first need the **univeral set** U. This is the set from which we are taking objects to make our sets — it (essentially) tells us what domain we are working in. For many of our results U will be the set of natural numbers, for some other results U might be the set of reals. It depends on context. But given the universal set we can define our last operation.

Definition 8.2.9 Given a **universal set** U and a set $A \subset U$, we define the **complement** of A, denoted $\bar{A}$ to be the set of elements not in A.

$$\bar{A} = \{x \in U | x \notin A\}$$

or equivalently:

$$x \in \bar{A} \iff x \notin A$$

◊

Note

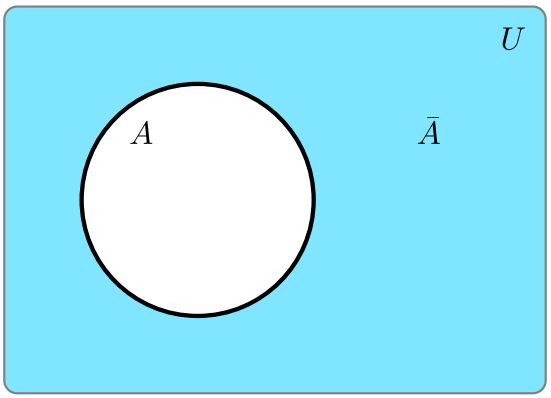
- While it is a nice thing to say, "compl-I-ment" is not the same as "compl-E-ment".
- The complement acts on sets in much the same way that negation acts on logical statements. The complement of the complement is the original set:

$$\bar{\bar{A}} = A.$$

- We can express $A - B$ using the complement as

$$A - B = A \cap \bar{B}$$

Another descriptive picture:



Example 8.2.10 Let $U = \{1, 2, \ldots, 20\}$, $A = \{2, 3, 5, 7, 11, 13, 17, 19\}$ and $B = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$. Determine the following sets

$$\bar{B} \qquad A - B \qquad A \cap \bar{B} \qquad \bar{\bar{B}}$$

Solution. We can just chug through the definitions carefully to get

$$\begin{aligned} \bar{B} &= \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\} \\ A - B &= \{3, 5, 7, 11, 13, 17, 19\} \\ A \cap \bar{B} &= \{3, 5, 7, 11, 13, 17, 19\} = A - B \\ \bar{\bar{B}} &= B \end{aligned}$$

□

8.3 Cartesian products of sets

Now when we were defining sets we used lists $A = \{1, 2, 3\}$ and the order in which we put things in the list didn't matter. On some occasions we really need to put things into some order; we need a way to write an ordered pair of elements in which one of the pair is the *first* and the other is the *second*. Coordinates of points, (x, y), in the plane are a very good example of this: the first number is the x-coordinate (horizontal position) and the second is the y-coordinate (vertical position), and we should not mix them up[99]. The point $(1, 13)$ on the plane is not the same as the point $(13, 1)$. We are used to this notation, "(x, y)", but we should define it before we go any further.

[99]Mind you, people rarely call the parts of an x, y-coordinate by their correct names. The x (ie the first of the pair) is called the **abscissa** and its use goes back at least as far as Fibonacci. The y (the second of the pair) is the **ordinate**. These terms are not so common in modern English and people typically just call them x-coordinate and y-coordinate (which is a little jarring to the ear of the pedant).

Definition 8.3.1 An **ordered pair** of elements is an ordered list of two elements. We write this as (a, b) with round brackets rather than braces. Ordered pairs have the properties that

- $(a, b) \neq (b, a)$ unless $a = b$, and
- $(a, b) = (c, d)$ only when $(a = c)$ and $(b = d)$.

$\Diamond$

Given two sets A, B, the set of all possible ordered pairs is the **Cartesian product** of those sets. To be more precise:

Definition 8.3.2 Cartesian product. Let A, B be sets. The **Cartesian product**,or just **product**, of A and B is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$. We write this as $A \times B$.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

Note that $A \times B \neq B \times A$, unless $A = B$. $\Diamond$

Example 8.3.3 If we set $A = \{a, b, c\}$ and $B = \{1, 2\}$ then

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$$

$\square$

So we are used to playing with cartesian products in the context of functions — $\mathbb{R} \times \mathbb{R}$ is the whole **cartesian plane**[100] and functions we are used to are just subsets of this. For example, the parabola $y = x^2$ is the set

$$\left\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2\right\}$$

which is a subset of $\mathbb{R} \times \mathbb{R}$.

8.4 Some set-flavoured results

There are quite a few standard results that describe how the set operations defined above interact with each other. We'll state some of the more important ones as a theorem (or two (or three)). But first, let us summarise what we have learned above.

Remark 8.4.1 A set of results about sets seen earlier. Let A, B be sets. Then

- $(A \subseteq B) \equiv (\forall x \in A, x \in B) \equiv (x \in A \implies x \in B)$.

[100]While this is named for the French mathematician and philosopher Rene Descartes (1596 – 1650), it was also invented by Pierre de Fermat (1601 – 1665), and even earlier by Nicole Oresme (1325 – 1382). Fermat is famous for writing his "Last Theorem" in the margin of a book, and Oresme was the first to prove that the infinite series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ does not converge.

- $(A = B) \equiv ((A \subseteq B) \wedge (B \subseteq A)) \equiv ((x \in A) \iff (x \in B))$
- $(x \in A \cap B) \equiv ((x \in A) \wedge (x \in B))$
- $(x \in A \cup B) \equiv ((x \in A) \vee (x \in B))$
- $(x \in \bar{A}) \equiv (x \notin A) \equiv \sim (x \in A)$
- $(x \in A - B) \equiv ((x \in A) \wedge (x \notin B)) \equiv ((x \in A) \wedge \sim (x \in B))$

Using these we can prove the following theorems

Theorem 8.4.2 DeMorgan's laws. *Let A, B be sets contained in a universal set U. Then*

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$
$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

Theorem 8.4.3 Distributive laws. *Let A, B, C be sets then*

$$A \cup (B \cap C) = (A \cup C) \cap (A \cup B)$$
$$A \cap (B \cup C) = (A \cap C) \cup (A \cap B)$$

and

$$A \times (B \cap C) = (A \times C) \cap (A \times B)$$
$$A \times (B \cup C) = (A \times C) \cup (A \times B)$$

The above are not so hard to understand. You can see that the distributive laws for sets look very much like the distributive laws for additional and multiplication. Laws for set differences are less obvious and we certainly wouldn't expect you to remember them. However they do make very good exercises! And, as you'll see in the examples below, a quick sketch of Venn diagrams can help a lot. ***But*** a Venn diagram is not a proof [101].

Theorem 8.4.4 Set differences. *Let A, B, C be sets contained in a universal set U. Then*

$$A - B = A \cap \bar{B}$$
$$A - (B \cap C) = (A - B) \cup (A - C)$$
$$A - (B \cup C) = (A - B) \cap (A - C)$$
$$A - (B - C) = (A \cap C) \cup (A - B)$$

When we prove the above theorems we'll make use of some simple little results that follow quite directly from the definitions of our set operations. Rather than explaining those results again and again in each proof (where the reader has to

[101] The Venn diagram is good scratch work for a proof, but really what one is doing is showing that for a particular choice of sets (ie the ones drawn) everything works out. To be a proof it has to work for every choice.

check the reasoning each time), we'll put them in a separate little result of their own. Since it is a helpful result we'll call it a lemma.

Lemma 8.4.5 *Let A, B be sets.*

- *If $x \in A$ then $x \in A \cup B$.*
- *If $x \in A \cap B$ then $x \in A$.*
- *If $x \notin A \cup B$ then $x \notin A$ and $x \notin B$.*
- *If $x \notin A \cap B$ then $x \notin A$ or $x \notin B$.*

The contrapositives of these statements also turn out to be useful in certain circumstances, so we state them explicitly.

- *If $x \notin A \cup B$ then $x \notin A$*
- *If $x \notin A$ then $x \notin A \cap B$*
- *If $x \in A$ or $x \in B$ then $x \in A \cup B$.*
- *If $x \in A$ and $x \in B$ then $x \in A \cap B$.*

These are straight forward to prove using the logic we know and the definitions of union and intersection.

Proof. We prove each in order.

- Assume $x \in A$. Then $x \in A$ or $x \in B$. Hence $x \in A \cup B$.
- Assume $x \in A \cap B$. Hence $x \in A$ and $x \in B$. Thus we have that $x \in A$ as required.
- The contrapositive of the statement is the definition of intersection, so the statement is true.
- The contrapositive of the statement is the definition of union, so the statement is true.

■

Remark 8.4.6 Careful choice of B. Take a careful look at the first of the statements in the lemma above. Observe that the hypothesis does not mention the set B that is in the conclusion. This is a useful observation. It means that we can apply the result by choosing B to be some set we are interested in. That is:

> If $x \in A$ then x is in the union of A and *any set we choose*.

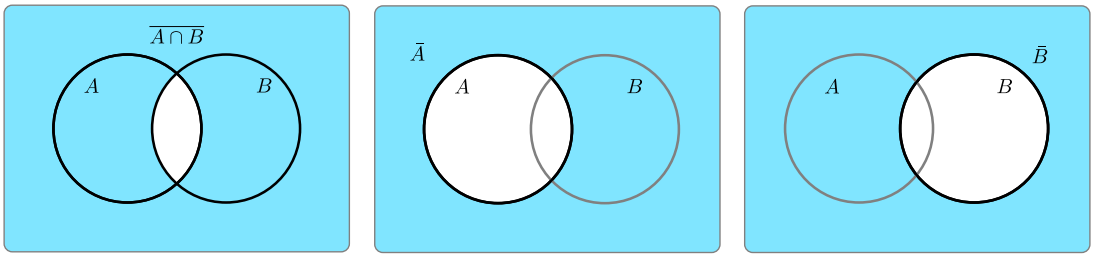
and similarly the second (its contrapositive) says

> If $x \notin A$ then x is not in the intersection of A and *any set we choose*.

Okay, primed with these results let's prove one of DeMorgan's laws

Example 8.4.7 Let A, B be sets included in a universal set U, then $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Scratchwork. First we should draw a Venn diagram describing the sets $\overline{A \cap B}, \bar{A}$ and $\bar{B}$:



Again, note that though Venn diagrams are useful tools for to understand and explore problems like these, they are not proofs. You should think of them as tools to help your scratch work.

Its an equality so we have to prove that the LHS is a subset of the RHS and vice-versa. Let's do those in order.

- Assume $x \in \overline{A \cap B}$.
- By definition of set complement this means that $x \notin (A \cap B)$.
- Now our helpful lemma comes to our aid. It tells that this implies that $x \notin A$ or $x \notin B$.
    - If $x \notin A$ then $x \in \bar{A}$. As per our helpful lemma, if x is an element of a particular set, then it is an element of the union of that set and any other set we choose. Hence, if $x \in \bar{A}$ then $x \in \bar{A} \cup \bar{B}$.
    - Similarly, if $x \notin B$ then $x \in \bar{B}$, and so $x \in \bar{B} \cup \bar{A}$.
- In either case $x \in \bar{A} \cup \bar{B}$ as required.

And the other inclusion:

- Assume $x \in \bar{A} \cup \bar{B}$. Hence $x \in \bar{A}$ or $x \in \bar{B}$.
    - If $x \in \bar{A}$ then $x \notin A$. As above, our helpful lemma comes to our aid. If we know that x is not an element of a particular set, then it is not in the intersection of that set and any other set we choose. Hence, if $x \notin A$ then $x \notin A \cap B$.
    - If $x \in \bar{B}$ then $x \notin B$, and by a similar argument, $x \notin A \cap B$.
- In either case $x \notin A \cap B$, so $x \in \overline{A \cap B}$ as required.

Now we can write this as a proper proof.

Solution.

Proof. We first prove that $\overline{A \cap B} \subset \bar{A} \cup \bar{B}$ and then the reverse inclusion.

- Let $x \in \overline{A \cap B}$, and hence $x \notin (A \cap B)$. This implies that $x \notin A$ or $x \notin B$. If $x \notin A$ then $x \in \bar{A}$, and so $x \in \bar{A} \cup \bar{B}$. Similarly, if $x \notin B$ then $x \in \bar{B}$, and so $x \in \bar{B} \cup \bar{A}$. In either case, $x \in \bar{A} \cup \bar{B}$ as required.

- Now assume that $x \in \bar{A} \cup \bar{B}$, so $x \in \bar{A}$ or $x \in \bar{B}$. If $x \in \bar{A}$ then $x \notin A$, and so $x \notin A \cap B$. Similarly if $x \in \bar{B}$ then $x \notin B$, and so $x \notin B \cap A$. In either case $x \notin A \cap B$, and so $x \in \overline{A \cap B}$.

Since we have shown both inclusions, $\overline{A \cap B} = \bar{A} \cup \bar{B}$ ■

□

What about a distributive law or two?

Example 8.4.8 Let A, B, C be non-empty sets, then $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Notice that this result still holds when the sets are allowed to be empty, but the proof of the result is a little messier and involves more cases.

Scratchwork. Unfortunately it isn't so easy to draw a useful picture here[102], but thankfully this isn't too hard to prove. It is a set-equality, so we need to prove the LHS is a subset of the RHS and vice-versa.

- Let $p \in A \times (B \cup C)$. Notice that since A, B, C are non-empty, we know that p exists. Since the LHS is the cartesian product of two sets, this element p is really an ordered pair of elements. It is probably a bit easier to follow what is going to happen if we make this clear from the start. With this in mind, let us restart things.

- Let $(x, y) \in A \times (B \cup C)$.

- This means that $x \in A$ and $y \in B \cup C$.

- Hence $y \in B$ or $y \in C$.

  - Let $y \in B$, then since $x \in A$, we know that $(x, y) \in A \times B$. Thus $(x, y) \in (A \times B) \cup$(any set we choose), and so $(x, y) \in (A \times B) \cup (A \times C)$.
  - Similarly, let $y \in C$. Since $y \in C$ and $x \in A$, we know that $(x, y) \in A \times C$, and so $(x, y) \in (A \times B) \cup (A \times C)$.

- So $(x, y) \in RHS$.

And the other way around…

- Let $(x, y) \in (A \times B) \cup (A \times C)$, so $(x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$. Again, since A, B, C are non-empty we know that such an (x, y) exists.

  - If $(x, y) \in (A \times B)$ then $x \in A$ and $y \in B$. Hence $y \in B \cup$ (any set we choose), and so $y \in B \cup C$.
  - Similarly, if $(x, y) \in A \times C$ then $x \in A$ and $y \in C$. Hence $y \in C \cup B$.

- In either case $x \in A$ and $y \in B \cup C$, so $(x, y) \in LHS$.

Now we clean it up and make it a proof.

Solution.

Proof. We prove each inclusion in turn. Assume that $(x, y) \in A \times (B \cup C)$, so that $x \in A$ and $y \in B \cup C$. Since the sets are non-empty, such a (x, y) exists. Since $y \in B \cup C$, we know that $y \in B$ or $y \in C$.

- If $y \in B$, then since $x \in A$, we know that $(x, y) \in A \times B$.
- Similarly, if $y \in C$ we know that $(x, y) \in A \times C$.

In both cases we have that $(x, y) \in (A \times B) \cup (A \times C)$, and so $LHS \subseteq RHS$.

Now assume that $(x, y) \in RHS$, so $(x, y) \in A \times B$ or $(x, y) \in A \times C$. Again, since the sets are non-empty, such an (x, y) exists.

- If $(x, y) \in A \times B$, then $x \in A$ and $y \in B$. Hence $y \in B \cup C$.
- Similarly, if $(x, y) \in A \times C$, then $x \in A$ and $y \in C$. Thus $y \in C \cup B$.

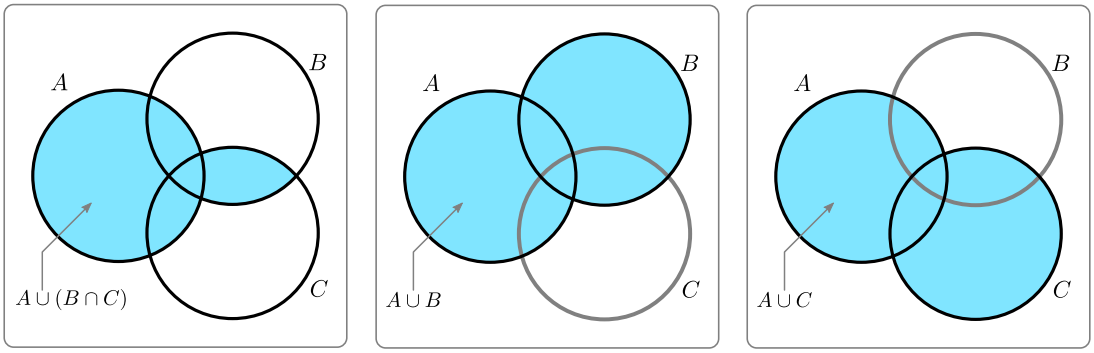
In either case we know that $x \in A$ and $y \in B \cup C$, so $(x, y) \in A \times (B \cup C)$ as required. ■

□

A more challenging distributive law.

Example 8.4.9 Let A, B, C be sets, then $A \cup (B \cap C) = (A \cup C) \cap (A \cup B)$.

Scratchwork. Again, a picture helps.



Lets prove the inclusions one at a time. First LHS is a subset of the RHS.

- Let $x \in A \cup (B \cap C)$, so that $x \in A$ or $x \in (B \cap C)$.
  - Assume $x \in A$.
  - Hence $x \in A \cup B$ and $x \in A \cup C$ (again because we know that if $x \in A$ then it is in the union of A and any set we choose).

[102] Well … I guess we could draw A, B, C as one-dimensional sets lying on different axes, and then the product would be some rectangular region in the plane? Something like that might work. A good exercise for the eager reader.

- Thus $x \in (A \cup B) \cap (A \cup C)$.

- On the other hand, assume that $x \in B \cap C$
  - Hence $x \in B$ and $x \in C$.
  - Since $x \in B$, we know that $x \in B \cup A$.
  - Similarly, since $x \in C$, we know that $x \in C \cup A$
  - Because both $x \in A \cup B$ and $x \in A \cup C$, we have that $x \in (A \cup B) \cap (A \cup C)$.

- In both cases, $x \in (A \cup B) \cap (A \cup C)$.

Now we prove that the RHS is a subset of the LHS.

- Let $x \in (A \cup B) \cap (A \cup C)$, so that $x \in A \cup B$ and $x \in A \cup C$.

- At this point it is a good idea to explore the 4 possibilities that this suggests.
  - $(x \in A) \wedge (x \in A)$ — this is easy because we have that $x \in A \cup$ (any set we choose)
  - $(x \in A) \wedge (x \in C)$ — this is also easy because we have that $x \in A \cup$ (any set we choose)
  - $(x \in B) \wedge (x \in A)$ — similarly we have that $x \in A \cup$ (any set we choose)
  - $(x \in B) \wedge (x \in C)$ — not too hard because $x \in B \cap C$

  So these four possibilities really break down into two cases. Either $x \in A$ or $x \notin A$.
  - If $x \in A$, then we know that $x \in A \cup$ anything and so $x \in A \cup (B \cap C)$.
  - Otherwise, if $x \notin A$ then we must have that $x \in B$ and $x \in C$, so $x \in B \cap C$. Then since $x \in B \cap C$, we also have that $x \in (B \cap C) \cup A$.

- In either case we have that $x \in A \cup (B \cap C)$.

Solution.

Proof. We prove each inclusion in turn. Start by assuming that $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. We consider each case in turn.

- Assume $x \in A$. Then $x \in A \cup B$ and $x \in A \cup C$ and so $x \in RHS$.

- Assume now that $x \in B \cap C$. Then $x \in B$ and $x \in C$, so $x \in B \cup A$ and $x \in C \cup A$. Thus $x \in RHS$.

In either case we have $x \in RHS$.

Now assume that $x \in (A \cup B) \cap (A \cup C)$, and so $x \in A \cup B$ and $x \in A \cup C$.

- If $x \in A$ then $x \in A \cup (B \cap C)$.

- If $x \notin A$ then since $x \in A \cup B$, we must have $x \in B$. Similarly since $x \in A \cup C$ we have $x \in C$. Since $x \in B$ and $x \in C$, $x \in B \cap C$. Thus $x \in A \cup (B \cap C)$.

Thus if $x \in LHS$ then $x \in RHS$. And so $LHS \subseteq RHS$. ■

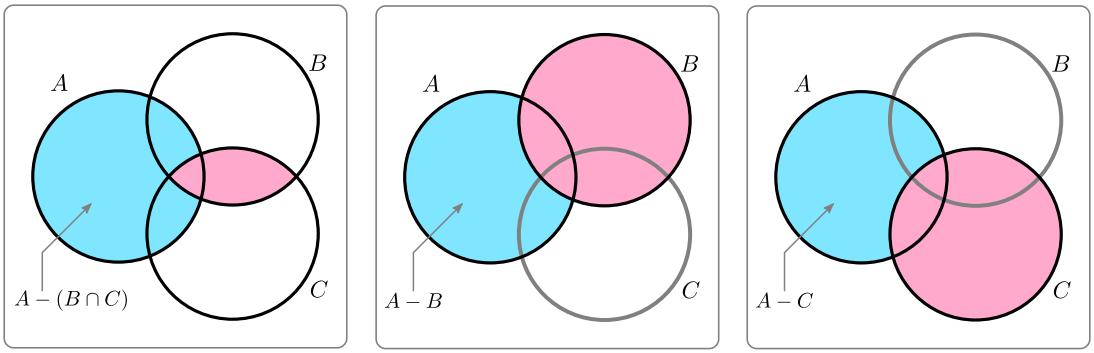
□

We'll prove that $A - (B \cap C) = (A - B) \cup (A - C)$ using the distributive and DeMorgans laws.

Example 8.4.10 Let A, B, C be sets included in a universal set U. Show that

$$A - (B \cap C) = (A - B) \cup (A - C).$$

Scratchwork. A picture can help us understand this



Let us start with the LHS and we'll use DeMorgan's laws and distributive laws to arrive at the RHS.

$$\begin{aligned} A - (B \cap C) &= A \cap \overline{(B \cap C)} && \text{set-difference as intersection} \\ &= A \cap (\bar{B} \cup \bar{C}) && \text{DeMorgan} \\ &= (A \cap \bar{B}) \cup (A \cap \bar{C}) && \text{distributive} \\ &= (A - B) \cup (A - C) && \text{intersection as set-difference} \end{aligned}$$

That is much much easier than proving it from first principles — that is proving it as we proved the couple of examples above. Of course, this argument things requires that we have already done the hard work of proving those other results.

Now the above calculation is all but a proof, we just need a few words. Also, we probably don't need to annotate our calculation as we have done above — though it wouldn't hurt. If we don't then we should, at a minimum, tell the reader what set laws we have used.

Solution.

Proof. Let A, B, C be sets, then

$$\begin{aligned} A - (B \cap C) &= A \cap \overline{(B \cap C)} \\ &= A \cap (\bar{B} \cup \bar{C}) \\ &= (A \cap \bar{B}) \cup (A \cap \bar{C}) \end{aligned}$$

$$= (A - B) \cup (A - C)$$

where we have used DeMorgans' laws, the distributive law and the fact that $A - B = A \cap \bar{B}$. ■

□

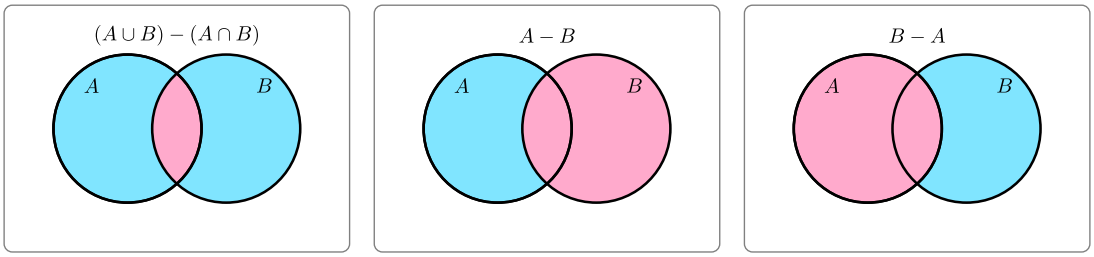
A couple more examples.

Example 8.4.11 Let A, B be sets included in a universal set U. Show that

$$(A \cup B) - (A \cap B) = (A - B) \cup (B - A).$$

This quantity is called the symmetric difference of A and B and is sometimes denoted $A \Delta B$.

Scratchwork. So we can start by trying to rewrite set differences as intersections and see what happens. Actually, we should start by drawing a picture of what is happening.



Now lets try

$$\begin{aligned}
(A \cup B) - (A \cap B) &= (A \cup B) \cap \overline{(A \cap B)} && \text{DeMorgan it} \\
&= (A \cup B) \cap (\bar{A} \cup \bar{B}) && \text{Distribute} \\
&= \big(A \cap (\bar{A} \cup \bar{B})\big) \cup \big(B \cap (\bar{A} \cup \bar{B})\big) && \text{expand more} \\
&= (A \cap \bar{A}) \cup (A \cap \bar{B}) \cup (B \cap \bar{A}) \cup (B \cap \bar{B}) && \text{now clean up} \\
&= \varnothing \cup (A \cap \bar{B}) \cup (B \cap \bar{A}) \cup \varnothing && \text{rewrite} \\
&= (A - B) \cup (B - A)
\end{aligned}$$

Oof! Here we have used that $C \cap \bar{C} = \varnothing$ and $C \cup \varnothing = C$ for any set C.

What if we try starting with the right and work to the left?

$$\begin{aligned}
(A - B) \cup (B - A) &= (A \cap \bar{B}) \cup (B \cap \bar{A}) && \text{expand} \\
&= \big((A \cap \bar{B}) \cup B\big) \cap \big((A \cap \bar{B}) \cup \bar{A}\big) && \text{expand more} \\
&= \big((A \cup B) \cap (B \cap \bar{B})\big) \cap \big((A \cup \bar{A}) \cap (\bar{B} \cup \bar{A})\big) && \text{clean up} \\
&= \big((A \cup B) \cap U\big) \cap \big(U \cap (\bar{B} \cup \bar{A})\big) && \text{clean more} \\
&= (A \cup B) \cap (\bar{B} \cup \bar{A}) && \text{deMorgan it} \\
&= (A \cup B) \cap \overline{(A \cap B)} \\
&= (A \cup B) - (A \cap B)
\end{aligned}$$

Not much difference really. Here we have used that $C \cup \bar{C} = U$ and $C \cap U = C$.

Solution.

Proof. Let A, B be sets. Then

$$\begin{aligned}
(A \cup B) - (A \cap B) &= (A \cup B) \cap \overline{(A \cap B)} && \text{DeMorgans law} \\
&= (A \cup B) \cap (\bar{A} \cup \bar{B}) && \text{distributive law} \\
&= \big(A \cap (\bar{A} \cup \bar{B})\big) \cup \big(B \cap (\bar{A} \cup \bar{B})\big) && \text{distributive law again} \\
&= (A \cap \bar{A}) \cup (A \cap \bar{B}) \cup (B \cap \bar{A}) \cup (B \cap \bar{B}) \\
&= \varnothing \cup (A \cap \bar{B}) \cup (B \cap \bar{A}) \cup \varnothing \\
&= (A - B) \cup (B - A) && \text{as required.}
\end{aligned}$$

Notice that in the last couple of steps we have used the following two facts

$$C \cap \bar{C} = \varnothing \qquad \text{and} \qquad C \cup \varnothing = C$$

for any set C. ■

We can also prove this result from first principles — by showing each side is a subset of the other. We don't present any scratch work here, rather we just leap into the proof. You can see that it is not really any cleaner than the proof we have presented above, but it is always worth seeing things from a different perspective.

Proof. We start by proving the left-hand side is a subset of the right-hand side and then the reverse inclusion. We also make use of the following two facts

$$(x \notin A \cap B) \implies (x \notin A) \vee (x \notin B) \quad \text{and} \quad (x \notin A) \implies (x \notin A \cap B)$$

which are simply the contrapositives of the following statements

$$(x \in A) \wedge (x \in B) \implies (x \in A \cap B) \quad \text{and} \quad (x \in A \cap B) \implies (x \in A)$$

- Assume $x \in LHS$, so $x \in (A \cup B)$ and $x \notin (A \cap B)$. This second fact implies that $x \notin A$ or $x \notin B$.
  - If $x \notin A$ then since $x \in A \cup B$, it follows that we must have $x \in B$. Since $x \in B$ and $x \notin A$, we have that $x \in B - A$.
  - Similarly, if $x \notin B$ then since $x \in A \cup B$, we must have $x \in A$. Hence $x \in A - B$.

  In either case $x \in (A - B) \cup (B - A)$, so $LHS \subseteq RHS$.

- Now assume that $x \in RHS$, so $x \in A - B$ or $x \in B - A$.
  - If $x \in A - B$, then $x \in A$ but $x \notin B$. Since $x \in A$, $x \in A \cup B$. And since $x \notin B$, we know that $x \notin B \cap A$. Hence $x \in (A \cup B) - (A \cap B)$.
  - Similarly, if $x \in B - A$, then $x \in B$ but $x \notin A$. Since $x \in B$, $x \in B \cup A$. And since $x \notin A$, we know that $x \notin A \cap B$. Hence $x \in (A \cup B) - (A \cap B)$.

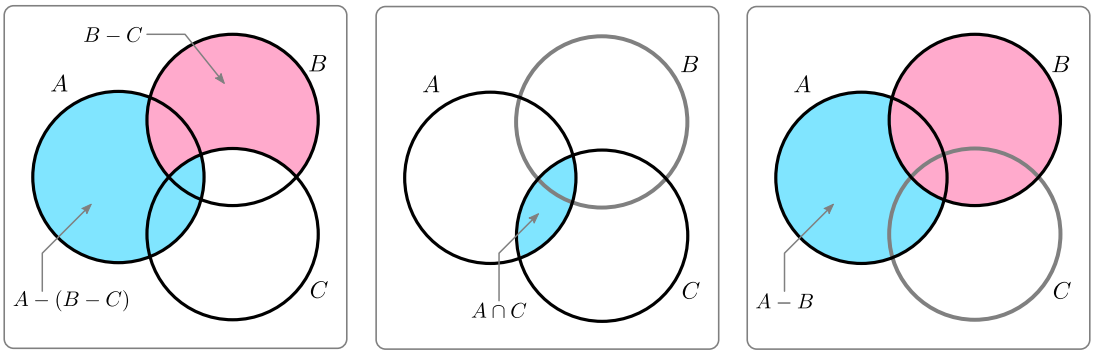
  Thus we know that $RHS \subseteq LHS$. Consequently $LHS = RHS$ as required.

■

□

Example 8.4.12 Let A, B, C be sets included in a universal set U. Prove that $A - (B - C) = (A \cap C) \cup (A - B)$.

Scratchwork. Again, we start with a picture of what is going on:



And now we should play around with some of what we already know

$$\begin{aligned} A - (B - C) &= A \cap \overline{(B - C)} && \text{set difference as intersection} \\ &= A \cap \overline{B \cap \bar{C}} && \text{set difference as intersection} \\ &= A \cap (\bar{B} \cup \bar{\bar{C}}) && \text{DeMorgan} \\ &= A \cap (\bar{B} \cup C) && \text{double complement} \\ &= (A \cap \bar{B}) \cup (A \cap C) && \text{distributive law} \\ &= (A \cap C) \cup (A - B) && \text{intersection as set difference} \end{aligned}$$

oof!

Solution.

Proof. Let A, B, C be sets as given in the statement of the result. Then

$$\begin{aligned} A - (B - C) &= A \cap \overline{(B - C)} \\ &= A \cap \overline{B \cap \bar{C}} \\ &= A \cap (\bar{B} \cup C) \\ &= (A \cap \bar{B}) \cup (A \cap C) \\ &= (A \cap C) \cup (A - B) \end{aligned}$$

where we have used the fact that $A - B = A \cap \bar{B}$, DeMorgans laws and distributive laws. ■

□

8.5 Indexed sets

Our usual set notation works well when we have a small number of sets; if our work involves three sets, we can just denote them A, B and C. However, this quickly becomes inconvenient when we have even a moderate number of sets; what should we use to denote the 17^{th} set? One solution it to use indices to

distinguish between sets:

$$A_1, A_2, A_3, \ldots$$

More generally, our indices do not need to be natural numbers, but instead come from any set. So, for example, we could define a family of sets indexed by elements of another set S

$$\{A_\alpha \text{ s.t. } \alpha \in S\}$$

In this context[103] we call S the **indexing set**, and the sets A_α the **indexed sets**. Typically, the indexing set is a subset of the natural numbers.

Now, recall that we defined the intersection of sets A and B to be the set of all elements that are in both sets. Similarly the union of A and B is the set of all elements that are in at least one of the sets. It is not hard to extend this to the intersection and union of three sets:

$$A \cap B \cap C = \{x \text{ s.t. } x \text{ is every one of } A, B, C\}$$

and

$$A \cup B \cup C = \{x \text{ s.t. } x \text{ is in at least one of } A, B, C\}$$

Writing the intersection and union this way, shows how we can extend them to intersections and unions of large numbers of sets. Indeed we define,

Definition 8.5.1 Let $N \in \mathbb{N}$ and let $A_1, A_2, \ldots, A_N$ be a collection of sets. We define the intersection of $A_1, A_2, \ldots, A_N$ as

$$\bigcap_{k=1}^{N} A_k = \{x \text{ s.t. } x \in A_j \text{ for all } j \in \{1, 2, \ldots, N\}\}.$$

We similarly define their union to be

$$\bigcup_{k=1}^{N} A_k = \{x \text{ s.t. } x \in A_j \text{ for some } j \in \{1, 2, \ldots, N\}\}.$$

$\Diamond$

This notation is very useful when we have to work with lots of sets[104]. We can generalise it further. For example, these unions and intersections need not start at index 1. Nor do they need to be unions and intersections over finite collections of sets.

Definition 8.5.2 Let $m, M \in \mathbb{N}$ with $m \leq M$, and let A_k be a set for all $k = m, m+1, \ldots, M$. Then

$$\bigcap_{k=m}^{M} A_k = \{x \text{ s.t. } \forall j \in \{m, m+1, \ldots, M\}, x \in A_j\}$$

[103] We hope the reader will forgive the authors for not making a formal definition here.

[104] As you can see, this notation is very similar to the Σ, Π notation we use to denote sums and products.

$$\bigcup_{k=m}^{M} A_k = \{x \text{ s.t. } \exists j \in \{m, m+1, \ldots, M\} \text{ s.t. } x \in A_j\}$$

Further, let $B_m, B_{m+1}, B_{m+2}, \ldots$ be sets, then

$$\bigcap_{k=m}^{\infty} B_k = \{x \text{ s.t. } \forall j \in \mathbb{N} \text{ with } j \geq m, x \in B_j\}$$
$$\bigcup_{k=m}^{\infty} B_k = \{x \text{ s.t. } \exists j \in \mathbb{N} \text{ with } j \geq m \text{ s.t. } x \in B_j\}$$

$\Diamond$

These definitions tell us that if $x \in \bigcap_{n=m}^{\infty} A_n$, then $x \in A_k$ for every single $k \geq m$. On the other hand, if we know that $x \notin \bigcap_{n=m}^{\infty} A_n$, then there must be at least one index $k \geq m$ so that $x \notin A_k$. Similarly, if $x \in \bigcup_{n=m}^{\infty} A_n$, then there is at least one index $k \geq m$ so that $x \in A_k$. And if $x \notin \bigcup_{n=m}^{\infty} A_n$, then $x \notin A_k$ for every single $k \geq m$.

Example 8.5.3 Consider the indexed sets, $A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n \in \mathbb{N}$. So that, for example,

$$A_2 = \left(-\frac{1}{2}, \frac{1}{2}\right) \qquad \text{and} \qquad A_{27} = \left(-\frac{1}{27}, \frac{1}{27}\right).$$

Then

$$0 \in \bigcap_{n=1}^{\infty} A_n$$

but

$$\frac{1}{e} \notin \bigcap_{n=1}^{\infty} A_n.$$

The first of these is true since $0 \in A_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$. The second follows[105] since $e < 3$ and so $\frac{1}{e} > \frac{1}{3}$ and so $\frac{1}{e} \notin A_3$; since it is not in one of the sets, it cannot be in the intersection of those sets.

Actually we have something a little stronger here. We have $\frac{1}{e} \notin \left(-\frac{1}{n}, \frac{1}{n}\right)$ for any $n \geq 3$. But since we want to show that $\frac{1}{e}$ is not in the intersection, it is enough to show that it is not in at least one of the intersecting sets. $\square$

In general, we don't have to take our indices from the natural numbers. We can index sets with rational numbers, real number, matrices, colours — just about any set. For example, for any real number $x \in (0, 10)$ we can define

$$A_x = [-x, x^2] = \left\{t \in \mathbb{R} \text{ s.t. } -x \leq t \leq x^2\right\}.$$

[105] We are assuming that we know $e = 2.71828\ldots$. It is not too hard to show that $2 < e < 3$, but it does take some calculus.

So that $A_5 = [5, 25]$ and $A_\pi = [-\pi, \pi^2]$. So the A_x form a family of sets indexed by real numbers. Sometimes we might need to take the union and intersection of all the sets in such a family, and so we generalise the notation even more.

Definition 8.5.4 Let $\mathcal{S}$ be a set, and let A_s be a set for all $s \in \mathcal{S}$. Then we can define the intersection and union over all of these sets:

$$\bigcap_{s\in\mathcal{S}} A_s = \{x \text{ s.t. } \forall s \in \mathcal{S}, x \in A_s\},$$

and

$$\bigcup_{s\in\mathcal{S}} A_s = \{x \text{ s.t. } \exists s \in \mathcal{S} \text{ s.t. } x \in A_s\},$$

$\Diamond$

Remark 8.5.5 When we do have sets indexed by the natural numbers, say $A_m, A_{m+1}, A_{m+2}, \ldots$ etc, it can be convenient to write their intersection and union as

$$A_m \cap A_{m+1} \cap A_{m+2} \cap \cdots \qquad \text{and} \qquad A_m \cup A_{m+1} \cup A_{m+2} \cup \cdots$$

It is arguably better to avoid this notation in favour of the more precise notation above. We should make sure that things are clear for our reader.

A more subtle, but very important point, is that when sets are indexed by real numbers it can, in fact, be impossible to write the union or intersection of those sets in this way. This is because the real numbers are not **countable**. There is simply no way to write out the real numbers in an ordered list in this way. The interested reader should jump ahead to Chapter 12 for more on the uncountability of the reals.

Example 8.5.6 Determine whether or not the following statements are true or false.

(a) $\displaystyle\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \emptyset$

(b) $\displaystyle 3 \in \bigcup_{n=3}^{\infty} \left(2 + \frac{1}{n}, 3 - \frac{1}{n}\right]$

(c) Given the indexed sets $A_x = [-x, x^2]$, for $x \in \mathbb{R}, x > 0$, we have

$$0 \in \bigcap_{x\in(0,1)} A_x.$$

(d) Given the indexed sets $A_y = \left(-e^y, 1 - \frac{1}{y}\right]$, for $y \in \mathbb{R}$, we have

$$1 \in \bigcup_{y\in(1,\infty)} A_y.$$

(e) $[-1,0) \subseteq \bigcup_{n=2}^{\infty} \left[-1, -\frac{1}{n}\right]$

(f) $\bigcap_{n=5}^{\infty} \left[0, 1+\frac{1}{n}\right) \subseteq [0,1)$

Solution.

(a) This statement is false since for all $n \in \mathbb{N}$, we have $0 \in \left(-\frac{1}{n}, \frac{1}{n}\right)$. Therefore $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) \neq \emptyset$.

Notice that we see that our indexed sets are getting smaller and smaller as n grows larger and larger. We also see that

$$\bigcap_{n=1}^{k} \left(-\frac{1}{n}, \frac{1}{n}\right) = \left(-\frac{1}{k}, \frac{1}{k}\right).$$

One might take this to suggest that we can understand $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$, by just looking at what happens to A_k as k goes to infinity. This is a dangerous line of reasoning. Since $-\frac{1}{k}$ and $\frac{1}{k}$ both go to 0 as k goes to infinity, we are tempted to say that $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = (0,0) = \emptyset$. But we have already shown the intersection is not empty since it contains 0. It is important to use the definitions carefully to determine what the intersections or unions contain.

(b) We see that this statement is false since in order for 3 to be in $\bigcup_{n=3}^{\infty} \left(2+\frac{1}{n}, 3-\frac{1}{n}\right]$, it must be in at least one of the sets $\left(2+\frac{1}{n}, 3-\frac{1}{n}\right]$ for $n \in \{3,4,5,\ldots\}$.

But, we see that for all $n \in \{3,4,5,\ldots\}$, $3-\frac{1}{n} < 3$, which implies that $3 \notin \left(2+\frac{1}{n}, 3-\frac{1}{n}\right]$ for any $n \in \{3,4,5,\ldots\}$. Therefore $3 \notin \bigcup_{n=3}^{\infty} \left(2+\frac{1}{n}, 3-\frac{1}{n}\right]$.

(c) We see that for all $x \in (0,1)$, we have $-x < 0$ and $x^2 > 0$. Thus, this statement is true since we have $0 \in [-x, x^2]$, for all $x \in (0,1)$.

(d) This statement is false since for all $y \in (1,\infty)$, we see $1-\frac{1}{n} < 1$, and hence $1 \notin \left(-e^y, 1-\frac{1}{y}\right]$ for any $y \in (1,\infty)$.

(e) This statement is true. To show it let $x \in [-1, 0)$. Then, all we have to do is to show that $x \in \left[-1, -\frac{1}{n}\right]$ for at least one $n \in \{2, 3, 4, \ldots\}$. For that, it is sufficient to show that $x \leq -\frac{1}{n}$ for some $n \in \{2, 3, 4, \ldots\}$ (since we already know that $x \geq -1$).

We can find such n by choosing $N = \left\lceil \frac{1}{|x|} \right\rceil$, where $\lceil t \rceil$ denotes the ceiling function — ie the first integer greater or equal to t. By using the ceiling function we are really just rounding up the value of $\frac{1}{|x|}$. With that choice of N, we know that $x \leq -\frac{1}{N}$ and hence $x \in [-1, \frac{1}{N}]$. Therefore $[-1, 0) \subseteq \bigcup_{n=2}^{\infty} \left[-1, -\frac{1}{n}\right]$.

(f) Since $1 + \frac{1}{n}$ goes to 1 as n goes to infinity, it is tempting to say that the statement is true. Unfortunately[106], this it not correct.

This statement is actually false because, similar to the argument we used in Item a, $1 \in \bigcap_{n=5}^{\infty} \left[0, 1 + \frac{1}{n}\right)$. Thus, $1 \in \bigcap_{n=5}^{\infty} \left[0, 1 + \frac{1}{n}\right)$, but $1 \notin [0, 1)$. Therefore $\bigcap_{n=5}^{\infty} \left[0, 1 + \frac{1}{n}\right) \not\subseteq [0, 1)$.

□

Before we go on to the next example, let's prove a useful lemma.

Lemma 8.5.7 *Let $x \in \mathbb{R}$. If for all $k \in \mathbb{N}$, $x < \frac{1}{k}$ then $x \leq 0$.*

Proof. We prove the contrapositive of this statement. Namely, if $x > 0$ then there is some $k \in \mathbb{N}$ so that $x \geq \frac{1}{k}$. So assume that $x > 0$. Then $0 < \frac{1}{x} \leq \lceil \frac{1}{x} \rceil = \ell$, where $\ell \in \mathbb{N}$. Then $x \geq \frac{1}{\ell}$ as required. ■

Example 8.5.8 For $k \in \mathbb{N}$ let A_k be the intervals $A_k = [\frac{1}{k+1}, 1 + \frac{1}{k+1})$.

(a) Show that $\bigcup_{k=1}^{\infty} A_k = (0, \frac{3}{2})$.

(b) Show that $\bigcap_{k=1}^{\infty} A_k = [\frac{1}{2}, 1]$

Solution.

Proof. Let us show the two inclusions in turn.

- Let $x \in \bigcup_{k=1}^{\infty} A_k$. Then $x \in A_k$ for some $k \in \mathbb{N}$ so that $\frac{1}{k+1} \leq x < 1 + \frac{1}{k+1}$ from which we get that $0 < x < \frac{3}{2}$, so that $x \in (0, \frac{3}{2})$.

[106] Well, actually it is fortunate. The authors have constructed this example with precisely that in mind. The reader should not give in to the temptation of a quick limit to solve all their set problems.

- Now let $x \in (0, \frac{3}{2})$. Then let us consider two cases, $x \geq \frac{1}{2}$ or $x < \frac{1}{2}$.

  1. First case: Suppose that $x \geq \frac{1}{2}$. Then because of the assumption on x we have $\frac{1}{2} \leq x < \frac{3}{2}$, and so $x \in A_1$, so $x \in \bigcup_{k=1}^{\infty} A_k$.
  2. Second case: Suppose that $x < \frac{1}{2}$. Then if we let $n = \left\lceil \frac{1}{x} \right\rceil$ then we have $\frac{1}{x} < n + 1$ and so $x > \frac{1}{n+1}$. This is enough to show that $x \in A_n$ with $n \in \mathbb{N}$ and so $x \in \bigcup_{k=1}^{\infty} A_k$.

■

Proof. Again, we prove the two inclusions in turn.

- Let $x \in \bigcap_{k=1}^{\infty} A_k$. Then for any $k \in \mathbb{N}$ we have $x \in A_k$ so that the inequality $\frac{1}{k+1} \leq x < 1 + \frac{1}{k+1}$ holds for any $k \in \mathbb{N}$. In particular, $x \geq \frac{1}{2}$.

  Now we can use Lemma 8.5.7 to show that $x \leq 1$. Since we know that $x - 1 < \frac{1}{k+1}$, Lemma 8.5.7 implies that $x - 1 < 0$ and hence $x \leq 1$.

  So we have now shown that $x \geq \frac{1}{2}$ and $x \leq 1$ and thus $x \in [\frac{1}{2}, 1]$.

- Now let $x \in [\frac{1}{2}, 1]$. So $\frac{1}{2} \leq x \leq 1$. Let $k \in \mathbb{N}$, we can check that $\frac{1}{k+1} \leq \frac{1}{2}$ and $1 + \frac{1}{k+1} > 1$, hence $\frac{1}{k+1} \leq x < 1 + \frac{1}{k+1}$ so $x \in A_k$. In the end, for any $k \in \mathbb{N}$ we have $x \in A_k$, so that $x \in \bigcap_{k=1}^{\infty} A_k$.

■

□

8.6 Exercises

1. Consider the following sets

- $A_1 = \{x \in \mathbb{Z} \colon x^2 < 2\}$,
- $A_2 = \{x \in \mathbb{N} \colon (3 \mid x) \wedge (x \mid 216)\}$,
- $A_3 = \left\{x \in \mathbb{Z} \colon \dfrac{x+2}{5} \in \mathbb{Z}\right\}$,
- $A_4 = \{a \in B \colon 6 \leq 4a + 1 < 17\}$, where $B = \{1, 2, 3, 4, 5, 6\}$,
- $A_5 = \{x \in C \colon 50 < xd < 100 \text{ for some } d \in D\}$, where $C = \{2, 3, 5, 7, 11, 13, \ldots\}$ and $D = \{5, 10\}$,
- $A_6 = \{5, 10, 15, 20, 25, \ldots\}$,
- $A_7 = \{2, 3, 4, 6, 8, 9, 12, 16, 18, 24 \ldots\}$,

Write down the sets below by listing their elements.

(a) $A_2 - A_3$,

(b) $A_5 \cap A_6$,

(c) $\mathcal{P}(A_1))$,

(d) $\mathcal{P}(\mathcal{P}(A_1 - \{-1\}))$,

(e) $A_3 \cap A_4$,

(f) $A_3 - A_7$,

(g) $A_5 \cup A_2$,

(h) $A_2 \cap A_7$.

(i) $A_5 \cap \overline{A_2}$, given the universal set $U = \mathbb{R}$.

(j) Verify whether $(A_4 \times B) \cap (B \times A_4) = A_4 \times A_4$ and $(A_4 \times B) \cup (B \times A_4) = B \times B$.

2. Write down the set $F \cap G$ where

$$F = \left\{(x, x^2 - 3x + 2) \in \mathbb{R}^2 : x \in \mathbb{R}\right\} \qquad \text{and} \qquad G = \left\{(a, a + 2) \in \mathbb{R}^2 : a \in \mathbb{R}\right\}$$

by listing out all of its elements. Prove your answer.

3. Prove or disprove the following statement:

Suppose A, B and C are sets. If $A = B - C$, then $B = A \cup C$.

4. Suppose $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$ satisfying, $x, y > 0$ and $x^k = y$. Then prove that $\{x^a \text{ s.t. } a \in \mathbb{Q}\} = \{y^a \text{ s.t. } a \in \mathbb{Q}\}$.

5. Prove or disprove the following statement:

If $m, n \in \mathbb{N}$, then $\{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\} \subseteq \{x \in \mathbb{Z} : mn \mid x\}$.

6. Prove or disprove the following statement:

Let $m, n \in \mathbb{Z}$. Then $\{x \in \mathbb{Z} : mn \mid x\} \subseteq \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$.

7. Let A be a set. Prove or disprove the following statements. If the statement is false in general, determine if there are any sets for which the statement is true.

(a) $A \times \emptyset \subseteq A$.

(b) $A \times \emptyset = A$.

8. Suppose that $A, B \neq \emptyset$, and C are sets such that $A \subseteq B$.

(a) Prove that $A \times C \subseteq B \times C$.

(b) Suppose we have a strict containment $A \subset B$ instead. What additional constraints do we need (if any) to show that

$$A \times C \subset B \times C?$$

Prove your claim.

9. Prove or disprove the following statement:

If A and B are sets, then $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

10. Prove or disprove the following statement:

If A and B are sets, then $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$.

11. Let A be a finite set with $|A| = n$. Prove that $|\mathcal{P}(A)| = 2^n$.

12. Let A and B be sets. Prove or disprove the following statements:

- $\mathcal{P}(A - B) \subseteq \mathcal{P}(A) - \mathcal{P}(B)$, and
- $\mathcal{P}(A) - \mathcal{P}(B) \subseteq \mathcal{P}(A - B)$.

13. Prove that

(a) $\displaystyle\bigcup_{n=3}^{\infty} \left(\frac{1}{n}, 1 - \frac{1}{n}\right) = (0, 1)$

(b) $\displaystyle\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0, 1]$

14. Determine what each of the following unions is equal to, and prove your answer.

(a)
$$\bigcup_{n \in \mathbb{N}} [-n, n]$$

(b)
$$\bigcup_{r \in \mathbb{R}, r > 0} B_r$$

where
$$B_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 < r\}.$$

15. Let $S \subset \mathbb{R}$. We say $b \in \mathbb{R}$ is an *upper bound* of S if $s \leq b$ for every $s \in S$. Further, we say $a \in \mathbb{R}$ is the *supremum* (or the *least upper bound*) of S, denoted by $\sup(S)$, if

- a is an upper bound for S, and
- if b is an upper bound for S, then $a \leq b$.

We also call $c \in S$ the *maximum element* of S, denoted by $\max(S)$, if it is the largest element in S. So, $\max(S)$ belongs to S, and is an upper bound of S.

For each of the following sets, determine its maximum and supremum, if they exist. Justify your answers.

(a) $[1, 3] = \{x \in \mathbb{R} \ : \ 1 \leq x \leq 3\}$

(b) $(1,3) = \{x \in \mathbb{R} \ : \ 1 < x < 3\}$

(c) $\{m \in \mathbb{Z} \ : \ |2(m-4)| \leq 15\}$

(d) $\left\{2 - \frac{1}{n} \ : \ n \in \mathbb{N}\right\}$

(e) $\{x \in \mathbb{R} \ : \ \cos(2x) = 1\}$

16. This question involves the supremum, which we first introduced in a previous exercise, Exercise 8.6.15. We recommend that you complete that question before you attempt this one.

Let $S \subset \mathbb{R}$. We say $b \in \mathbb{R}$ is an *upper bound* of S if $s \leq b$ for every $s \in S$. Further, we say $a \in \mathbb{R}$ is the *supremum* (or the *least upper bound*) of S, denoted by $\sup(S)$, if

- a is an upper bound for S, and
- if b is an upper bound for S, then $a \leq b$.

Suppose that S, T are non-empty subsets of $\mathbb{R}$, and $s = \sup(S)$, $t = \sup(T)$, where $s, t \in \mathbb{R}$.

(a) Show that $\sup(S \cup T) = \max\{s, t\}$.

(b) Can you determine $\sup(S \cap T)$?

(c) Define $S + T = \{s + t : s \in S, t \in T\}$. Show that $\sup(S + T) = s + t$.

17. Before completing this question you should look at Exercise 8.6.15 and Exercise 8.6.16. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence such that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, and such that

$$a = \sup\{a_n : n \in \mathbb{N}\}$$

exists as a real number. Show that

$$\lim_{n \to \infty} a_n = a.$$

Chapter 9

Relations

A great number of equations that we see in mathematics tell us about the relationship between two objects. For example

- $a < b$ — the number a is strictly less than the number b.
- $a|b$ — the number a is a divisor of the number b.
- $a = b$ — the quantities a and b are equal.
- $a \in B$ — the object a is a member of the set B.
- $a \equiv b \pmod{n}$ — a and b have the same remainder when divided by n.

We haven't seen this last one yet, but we will soon — it is a way of generalising the notions of odd and even numbers.

These symbols

$$< \qquad | \qquad = \qquad \in \qquad \equiv$$

all encode relationships between objects. Of course, all the relationships we've stated above are quite different however, there is something to be gained from developing a general theory of how **relations** behave and underline what they have in common.

Consider the symbol that we use to denote divisibility. We say that

$$a|b \iff b = ak \text{ where } k \in \mathbb{Z}$$

You can think of "|" as some sort of operator that takes two objects and declares some relationship between them. Here, both of those objects come from the same set — namely the integers.

Similarly, the symbol we use to denote set membership

$$a \in B$$

tells us that the object a lies inside the object B. Again, this symbol "$\in$" can be an operator that takes two objects and declares a relationship between them. In

this case, in contrast with the previous case, the objects need not come from the same set — a could be an integer and B a subset of integers.

In each case, the symbol is declaring a relationship between two objects.

One more. Consider the set $A = \{1, 2, 4, 8\}$, and all the comparisons we can make between those numbers using "is divisible by". Now since there are 4 numbers, we can make 4^2 comparisons:

$1 \mid 1$	$1 \mid 2$	$1 \mid 4$	$1 \mid 8$
$2 \nmid 1$	$2 \mid 2$	$2 \mid 4$	$2 \mid 8$
$4 \nmid 1$	$4 \nmid 2$	$4 \mid 4$	$4 \mid 8$
$8 \nmid 1$	$8 \nmid 2$	$8 \nmid 4$	$8 \mid 8$

Notice a couple of things here:

- The comparisons require an ordered pair of numbers,
- When the relation is true we use "$\mid$", and when it is false, we just draw a line through it and write "$\nmid$".
- The valid comparisons are just a set of ordered pairs — in particular it forms a subset of $A \times A$.

We could form a similar set of comparisons using the "is less than" relation:

$1 \not< 1$	$1 < 2$	$1 < 4$	$1 < 8$
$2 \not< 1$	$2 \not< 2$	$2 < 4$	$2 < 8$
$4 \not< 1$	$4 \not< 2$	$4 \not< 4$	$4 < 8$
$8 \not< 1$	$8 \not< 2$	$8 \not< 4$	$8 \not< 8$

or with the rather silly "starts with an earlier letter when written in German" comparison[107]. We don't have a ready symbol for this comparison, so we'll just use $\mathcal{R}$ and put a slash through it, $\not\mathcal{R}$, to show when the comparison does not hold.

$1 \not\mathcal{R} 1$	$1 \mathcal{R} 2$	$1 \mathcal{R} 4$	$1 \not\mathcal{R} 8$
$2 \not\mathcal{R} 1$	$2 \not\mathcal{R} 2$	$2 \not\mathcal{R} 4$	$2 \not\mathcal{R} 8$
$4 \not\mathcal{R} 1$	$4 \mathcal{R} 2$	$4 \not\mathcal{R} 4$	$4 \not\mathcal{R} 8$
$8 \mathcal{R} 1$	$8 \mathcal{R} 2$	$8 \mathcal{R} 4$	$8 \not\mathcal{R} 8$

In each case the pairs of numbers that satisfy the relation form a subset of $A \times A$.

Since this set is small and finite we could also represent the relation pictorially. Consider the following figure

[107] For the non-germanophone or those without ready access to dictionary or online translation service: 1 = ein, 2=zwei, 4=vier, 8=acht.

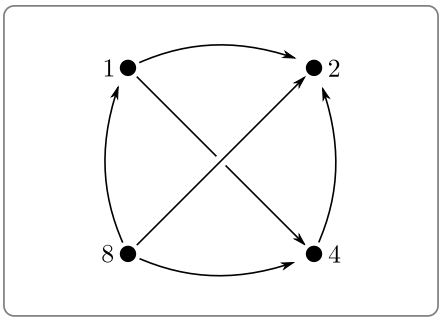


Figure 9.0.1 A pictorial depiction of a silly relation

The dots here denote the elements of the set $A = \{1, 2, 3, 4\}$, and an arrow from x to y denotes $x \mathcal{R} y$. This can be a handy way to visualise things, but really only works when the underlying set is small.

Looking at the above pairs, it should be clear that we can specifiy or *define* the relation using the subset:

$$R = \{(1, 2), (1, 4), (4, 2), (8, 1), (8, 2), (8, 4)\}.$$

More generally we can define *any* relation in this way. This leads us to the formal definition of relation in the next section.

9.1 Relations

Definition 9.1.1 Let A be a set. Then a relation, R, on A is a subset $R \subseteq A \times A$. If the ordered pair $(x, y) \in R$, we denote this as $x \mathcal{R} y$, while if $(x, y) \notin R$ we write $x \not\mathcal{R} y$. ♢

So while we introduced relations in this chapter by starting with relations you already knew — like "is a divisor of" and "is an element of" — the above defines relations as subsets. This allows us much more flexibility and we can use set-builder notation to define specific relations. For example

$$R = \{(x, x) \in \mathbb{R} \times \mathbb{R} \mid x \in \mathbb{R}\}$$

is the relation "=" on the set of real numbers. While

$$S = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x - y \in \mathbb{N}\}$$

is the relation ">" on the set of integers.

The definition of relation allows us to take any subset of $A \times A$ to be a relation. For example,

- $R = \varnothing$ gives the relation in which $x \not\mathcal{R} y$ for all $x, y \in A$.
- $R = A \times A$ gives the relation in which $x \mathcal{R} y$ for all $x, y \in A$.

The first of these is sometimes called the **empty relation**, while the latter is called the **universal relation**. Unsurprisingly, the vast majority of other subsets $R \subset A \times A$, won't be particularly interesting or useful or have especially natural or nice definitions. We will, shortly, start to impose additional requirements or properties onto the relations in order to make them more useful and interesting.

For a great many purposes, we define relationships between objects coming from the same set. However, there are situations in which we want to describe relationships between objects from different sets. For example the relation "is an element of", one object will be an element, while the other will be a set. Or, as another example, we could have a set of children and a set of parents, with the relation "is a child of". Consequently, the above definition is often generalised as follows:

Definition 9.1.2 Definition Definition 9.1.1 generalised. Let A, B be sets. Then a relation, R, between A and B is a subset $R \subseteq A \times B$. If the ordered pair $(x, y) \in R$, we denote this as $x \mathcal{R} y$, while if $(x, y) \notin R$ we write $x \not\mathcal{R} y$. ◊

Though the above generalisation will be important when we start to consider **functions**, most of what we will do in this chapter concerns relations that compare elements from a single set, rather than between two sets.

9.2 Properties of relations

Since a relation on A is just a subset of $A \times A$, there are a huge number of possible relations on any given set. As noted above, the vast majority of these will unstructured and be neither useful nor interesting. Typically we require relations to have some additional structure.

Consider the relation "is divisible by" on the set of integers. This has some very useful properties:

- For every $n \in \mathbb{Z}$, it is always true that $n \mid n$,
- If $a \mid b$ and $b \mid c$ then we must have $a \mid c$.

The relation "is less than" on the set of reals, on the other hand, satisfies the second of these, but not the first. These properties (and others) are useful and interesting so let's formalise them.

Definition 9.2.1 Let R be a relation on a set A.

- We say that the relation R is **reflexive** when $a \mathcal{R} a$ for every $a \in A$.
- The relation R is **symmetric** when for any $a, b \in A$, $a \mathcal{R} b$ implies $b \mathcal{R} a$.
- The relation R is **transitive** when for any $a, b, c \in A$, $a \mathcal{R} b$ and $b \mathcal{R} c$ implies $a \mathcal{R} c$.

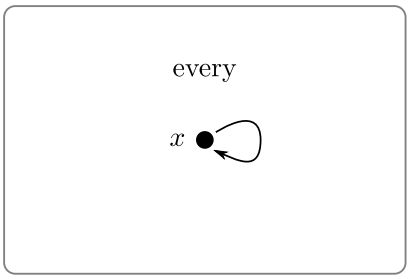
Notice that in the definition of transitive we do not require that a, b, c are different. ◊

Notice that we can write these using quantifiers quite nicely:

reflexive:	$\forall a \in A, (a \mathcal{R} a)$
symmetric:	$\forall a, b \in A, (a \mathcal{R} b) \implies (b \mathcal{R} a)$
transitive:	$\forall a, b, c \in A, (a \mathcal{R} b) \wedge (b \mathcal{R} c) \implies (a \mathcal{R} c)$

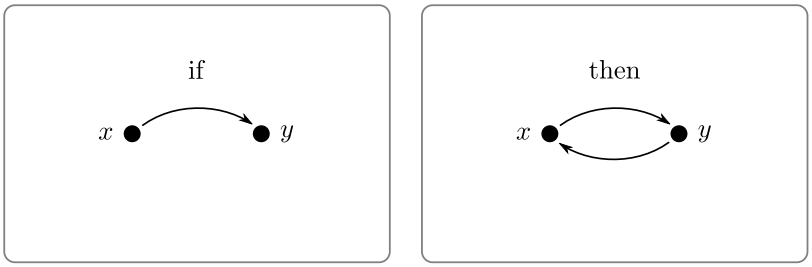
We can visualise these using dot-arrow diagrams (as in Figure 9.0.1). Note that these diagrams are not supposed to represent the entire relation, but rather just enough to illustrate the definitions.

- Reflexive:



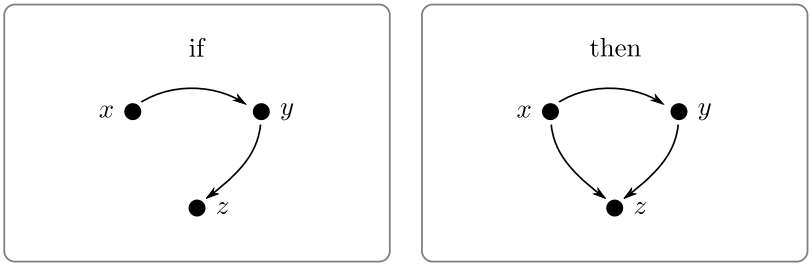
  Every element x must have a loop from x back to itself.

- Symmetric:



  If there is an arc from x to y then there must also be an arc from y to x.

- Transitive:



  If there is an arc from x to y and another from y to z then there must also be an arc from x to z.

With a little work (and some proofs!) we can show that the following table holds for the relations $<, \leq, =, |$ on the set of integers.

R	$<$	$\leq$	$=$	$\mid$
Reflexive	false	true	true	true
Symmetric	false	false	true	false
Transitive	true	true	true	true

Example 9.2.2 Let R be the relation "has the same parity as" on the set of integers. This relation is defined by

$$\begin{aligned} R &= \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid a, b \text{ both even or } a, b \text{ both odd}\} \\ &= \{(a,b) \in \mathbb{Z} \times \mathbb{Z} \mid 2 \mid (a-b)\} \end{aligned}$$

Notice that the second definition is equivalent to the first since

$$(a, b \text{ have same parity}) \iff (2 \mid (a-b)).$$

Prove that this relation is reflexive, symmetric and transitive.

Scratchwork. We explore each in turn:

- Reflexive. We need to show that for every single $a \in \mathbb{Z}$ that $a \mathcal{R} a$. Hence we need to show that for every integer a, that a has the same parity as a. This is pretty obvious, but to write it a little more mathematically — since $a - a = 0$ and $2|0$, we know that $a \mathcal{R} a$.

- Symmetric. We need to show that if $a \mathcal{R} b$ then $b \mathcal{R} a$. So we start by assuming that $a \mathcal{R} b$, so $2 \mid (a-b)$. We need to show that $2 \mid (b-a)$. This, again, is quite obvious, but we can make it more mathematical by saying something like — since $2 \mid (a-b)$ we know that $a - b = 2k$, so $b - a = 2(-k)$, and thus $2 \mid (b-a)$

- Transitive. We need to show that if $a \mathcal{R} b$ and $b \mathcal{R} c$ then $a \mathcal{R} c$. So we assume that $a \mathcal{R} b$ and $b \mathcal{R} a$. This means that $2 \mid (a-b)$ and $2 \mid (b-c)$. So there are integers k, ℓ so that

  $$a - b = 2k \qquad\qquad b - c = 2\ell.$$

  Now we need to show something about $a - c$. It is easy to isolate $a = 2k + b$ and $c = 2\ell + b$, so $a - c = 2k - 2\ell$. Hence $2 \mid (a-c)$ as we need.

We are not done until we write it up nicely. So we do that now.

Solution.

Proof. Let the relation be as defined in the statement. We prove each property in turn.

- Reflexive — Let $a \in \mathbb{Z}$. Since $a - a = 0$ and $2 \mid 0$, we know that $a \mathcal{R} a$.

- Symmetric — Let $a, b \in \mathbb{Z}$ so that $a \mathcal{R} b$. We know that $2 \mid a - b$ and so we can write $a - b = 2k$ for some $k \in \mathbb{Z}$. But then $b - a = 2(-k)$, and so $2 \mid (b-a)$ and thus $b \mathcal{R} a$.

- Transitive — Let $a, b, c \in \mathbb{Z}$ with $a \mathcal{R} b$ and $b \mathcal{R} c$. Thus we know that $a - b = 2k$ and $b - c = 2\ell$ for some integer k, ℓ. But then $a - c = 2(k + \ell)$ and thus $2 \mid (a - c)$. Hence $a \mathcal{R} c$ as required.

■

□

More examples:

Example 9.2.3 Let A be the set of students at UBC, and consider the relation "attended highschool with".

- Reflexive — It is reflexive since every student went to highschool with themselves.
- Symmetric — It is symmetric, because if student a went to highschool with b, then b went to the same highschool as a.
- Transitive — It is not transitive, just because a went to school with b and b went to school with c, it does not mean that a went to school with c. It is possible that b went to two different highschools, one they attended with a and the second they attended with c.

□

Another student flavoured example.

Example 9.2.4 Let A be the set of students in the student union building at 1pm, and let R be the relation "is within 2 metres of".

- Reflexive — It is reflexive since each student is less than 2m from themselves.
- Symmetric — It is symmetric, because if student a is less than 2m from student b, then b is less than 2m from a.
- Transitive — It is not transitive, just because a is 2m from b and b is 2m from c, it does not mean that a is 2m from c. If the students are arranged in a line with 1.5m between each of them, then a is 3m from c.

□

Example 9.2.5 Let R be the relation "is a subset of" on the set of all subsets of the integers.

- Reflexive — The relation is reflexive since for all sets X, we have $X \subseteq X$.
- Symmetric — The relation is not symmetric. Let $X = \varnothing, Y = \{1\}$, then $X \subseteq Y$ but $Y \not\subseteq X$.
- Transitive — The relation is transitive. Assume $A \subseteq B$ and $B \subseteq C$. Now let $a \in A$. Since $A \subseteq B$ we know $a \in B$. And since $B \subseteq C$, we know $a \in C$. Hence $A \subseteq C$ as required.

□

These aren't the only interesting properties of relations. Here are some others we won't really use, except maybe for some examples and exercises.

- A relation is **total** (also called **connex**) when

$$\forall a, b \in A, (a \mathcal{R} b) \vee (b \mathcal{R} a).$$

  The relation $\leq$ on the set of reals is total.

- A relation is **trichotomous** when

$$\forall a, b \in A, \text{ exactly one of } (a \mathcal{R} b) \text{ or } (b \mathcal{R} a) \text{ or } (a = b).$$

  The relation $<$ is trichotomous.

- A relation is **anti-symmetric** when

$$\forall a, b \in A, (a \mathcal{R} b) \wedge (b \mathcal{R} a) \implies a = b.$$

  The relation $\subseteq$ is anti-symmetric, so is $|$ on natural numbers. However $|$ is not anti-symmetric on the set of all integers. If $b = -a$, then $a \mid b$ and $b \mid a$, but $a \neq b$.

- A relation is **dense** when

$$\forall a, b \in A, \exists c \in A \text{ s.t. } (a \mathcal{R} c) \wedge (c \mathcal{R} b).$$

One very important relation on the integers is congruence - recall Definition 5.3.1. It is not hard to prove that congruence has nice properties.

Theorem 9.2.6 *Let $n \in \mathbb{N}$ then congruence modulo n is reflexive, symmetric and transitive.*

Proof. Let n be a fixed natural number. We prove each of the properties in turn.

- Reflexive — Let $a \in \mathbb{Z}$. Then since $a - a = 0$ and $n \mid 0$, we know that $a \equiv a \pmod{n}$.

- Symmetric — Let $a, b \in \mathbb{Z}$ and assume that $a \equiv b \pmod{n}$. Hence we know that $n \mid (a - b)$, and so we can write $a - b = nk$ for some integer k. Thus $b - a = n(-k)$ and so $n \mid (b - a)$ and $b \equiv a \pmod{n}$.

- Transitive — Let $a, b, c \in \mathbb{Z}$ and assume that $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$. This implies that $n \mid (a - b)$ and $n \mid (b - c)$, and so we can write $(a - b) = nk, (b - c = n\ell)$ for some integers k, l. Consequently $(a - c) = (a - b) + (b - c) = n(k + \ell)$ and $n \mid (a - c)$. Hence $a \equiv c \pmod{n}$.

■

Notice that one immediate consequence of this theorem is that since congruence modulo n is symmetric, we can be a little bit more relaxed when discussing it. In particular, in the examples above we have been very careful to say

- "19 is congruent to 5 modulo 7"
- "11 is congruent to 27 modulo 4"
- "13 is congruent to 7 modulo 5"

where there is a definite *first* number and a definite *second* number in the relation. However since congruence is symmetric, we can instead say

- "19 and 5 *are* congruent modulo 7"
- "11 and 27 *are* congruent modulo 4"
- "13 and 7 *are not* congruent modulo 5"

where the order of the two numbers in the relation no longer matters. Of course we must still be careful with the modulus; we cannot mix that up with the other numbers.

We will return to congruences in more detail shortly.

Another example:

Example 9.2.7 Fractions. Let F be the set of all fractions:

$$F = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}.$$

Consider the following fractions that are *different* elements of F:

$$\frac{2}{4} \qquad \frac{3}{6} \qquad \frac{-7}{-14} \qquad \frac{9}{18}$$

All of these are just different ways of writing "one half" and so correspond to the same single element of $\mathbb{Q}$. We typically avoid this duplication by *representing* each rational number by a single reduced fraction. This is an example of an **equivalence class**, but we will get to those in the next section.

But before that, let us formalise how two fractions are related. We've been doing that since primary school — they are related when they are the same rational number. That is

$$\frac{a}{b} \mathcal{R} \frac{c}{d} \iff \frac{a}{b} = \frac{c}{d}$$

Notice the "=" in the equation on the right *does not* mean that the fractions are *identical* rather it means that they represent the same number. We can make this condition a little more comfortable by writing it as

$$\frac{a}{b} \mathcal{R} \frac{c}{d} \iff ad = bc$$

Let us show that this relation is reflexive, symmetric and transitive.

Proof. We show that the relation on the set of fractions defined above is reflexive, symmetric and transitive.

- Reflexive — Let $\frac{a}{b} \in F$. Then since $ab = ab$, it follows that $\frac{a}{b} \mathcal{R} \frac{a}{b}$. Hence the relation is reflexive.
- Symmetric — Let $\frac{a}{b}, \frac{c}{d} \in F$. Assume that $\frac{a}{b} \mathcal{R} \frac{c}{d}$. Hence $ad = bc$, and so $bc = ad$. Thus $\frac{c}{d} \mathcal{R} \frac{a}{b}$. So the relation is symmetric.
- Transitive — Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$. Assume that $\frac{a}{b} \mathcal{R} \frac{c}{d}$ and $\frac{c}{d} \mathcal{R} \frac{e}{f}$. Hence we know that
$$ad = bc \qquad \text{and} \qquad cf = de.$$
Multiply the first of these equations by f and the second by b. This gives
$$adf = bcf \qquad \text{and} \qquad bcf = deb$$
Using the transitivity of equality we know that
$$adf = deb$$
and since $d \neq 0$ we have
$$af = eb$$
Thus $\frac{a}{b} \mathcal{R} \frac{e}{f}$ and so the relation is transitive.

■
□

9.3 Equivalence relations and equivalence classes

An important class of relations are those that are similar to "=". We know that "is equal to" is reflexive, symmetric and transitive. Any relation that has these properties acts something like equality does — we call these relations equivalence relations.

Definition 9.3.1 Let R be a relation on a set A. If R is reflexive, symmetric and transitive then R is an equivalence relation. ◊

From our work in the previous section we know that the following relations are equivalence relations

- "is equal to"
- "has same parity as"
- "is congruent to"

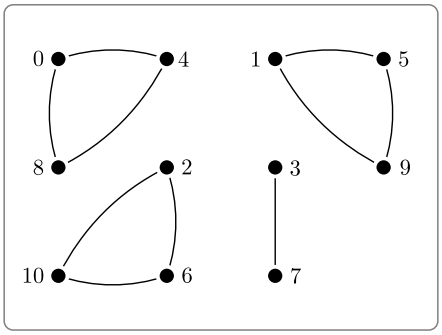
Notice that these other two relations are weaker than equality — the underlying objects do not have to be the same in order to be equivalent.

Example 9.3.2 Let A be the set of students at a particular university. Show that the relation "has the same birthday as" is an equivalence relation.
Proof. We need to prove that the relation is reflexive, symmetric and transitive; we prove each in turn.

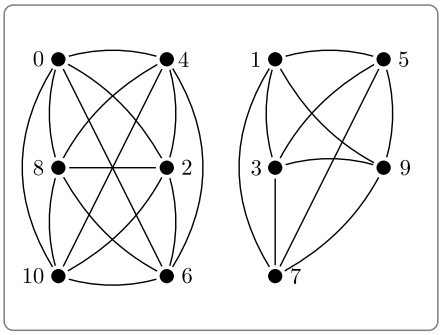
- Reflexive — Since any person has the same birthday as themselves, the relation is reflexive.
- Symmetric — Let a have the same birthday as b. Then b has the same birthday as a. Hence the relation is symmetric.
- Transitive — Let a have the same birthday as b and b have the same birthday as c. Then it follows that a and c must be born on the same day of the year. Hence a has the same birthday as c.

■
□

Consider now the set $A = \{0, 1, 2, 3, \ldots, 10\}$ and consider congruence modulo 4. As we have done a few times, we'll draw a picture of the relation on this set. Not, there could, potentially be a lot of arrows in this figure. To save some space, we can take advantage of the fact that congruence is reflexive, and so we know that if there is an arrow from a to b, there must be one back from b to a. So, instead of drawing two arrows between a and b, we'll just draw a single arc.



We can do exactly the same thing, but now with the relation "has the same parity as"



In each case we should notice that the set of nodes in the pictures fall into a small number of subsets, in which each node is connected to every other node. These connected subsets are examples of equivalence classes.

Definition 9.3.3 Given an equivalence relation R defined on a set A, we define the equivalence class of $x \in A$ (with respect to R) to be the set of elements related to x:

$$[x] = \{y \in A \ : \ y \mathcal{R} x\}$$

This is sometimes also written as "E_x". $\Diamond$

So in our "congruent modulo 4" example above, the equivalence classes are

$$\begin{array}{llll} [0] = \{0,4,8\} & [1] = \{1,5,9\} & [2] = \{2,6,10\} & [3] = \{3,7\} \\ [4] = \{0,4,8\} & [5] = \{1,5,9\} & [6] = \{2,6,10\} & [7] = \{3,7\} \\ [8] = \{0,4,8\} & [9] = \{1,5,9\} & [10] = \{2,6,10\} & \end{array}$$

while in our "has the same parity as" example we get

$$\begin{array}{ll} [0] = \{0,2,4,6,8,10\} & [1] = \{1,3,5,7,9\} \\ [2] = \{0,2,4,6,8,10\} & [3] = \{1,3,5,7,9\} \\ [4] = \{0,2,4,6,8,10\} & [5] = \{1,3,5,7,9\} \\ [6] = \{0,2,4,6,8,10\} & [7] = \{1,3,5,7,9\} \\ [8] = \{0,2,4,6,8,10\} & [9] = \{1,3,5,7,9\} \\ [10] = \{0,2,4,6,8,10\} & \end{array}$$

First notice that there are no empty equivalence classes. This follows from the fact that equivalence relations are reflexive:

Lemma 9.3.4 *Let R be an equivalence relation on a set A. Then for any $x \in A$, $x \in [x]$.*

Proof. Let $x \in A$. We know that R is reflexive, so $x \mathcal{R} x$. Since we define

$$[x] = \{a \in A \ : \ a \mathcal{R} x\}$$

we must have that $x \in [x]$. ■

Also notice that there is a lot of repetition in our lists of equivalence classes. Indeed, we could have been listed them as:

$$\begin{array}{ll} [0] = [4] = [8] = \{0,4,8\} & [1] = [5] = [9] = \{1,5,9\} \\ [2] = [6] = [10] = \{2,6,10\} & [3] = [7] = \{3,7\} \end{array}$$

In fact it looks exactly like

$$[x] = [y] \iff x \mathcal{R} y$$

This is an important (and true!) result, so let's call it a theorem and prove it.

Theorem 9.3.5 *Let R be an equivalence relation on A and let $x, y \in A$. Then*

$$x \mathcal{R} y \iff [x] = [y]$$

Proof. We prove each implication in turn.

- ($\Leftarrow$) Assume $[x] = [y]$. From our lemma above, we know $x \in [x]$. Hence $x \in [y]$. But we define $[y] = \{a \in A \ : \ a \mathcal{R} y\}$, and since x is in this set we know that $x \mathcal{R} y$ as required.

- ($\Rightarrow$) Assume $x \mathcal{R} y$. In order to prove that $[x] = [y]$, we prove that each is a subset of the other.

  - Let $a \in [x]$, and so $a \mathcal{R} x$. Now since $x \mathcal{R} y$ and R is transitive, we know that $a \mathcal{R} y$. Consequently $a \in [y]$, and thus $[x] \subseteq [y]$.
  - Now let $b \in [y]$, so $b \mathcal{R} y$. Since R is symmetric, we know that $y \mathcal{R} x$, and because of transitivity, this implies that $b \mathcal{R} x$. Hence $b \in [x]$ and so $[y] \subseteq [x]$

  Thus $[x] = [y]$.

■

Remark 9.3.6 Representing classes — be kind. This result tells us that given any two related elements $x \mathcal{R} y$, their equivalence classes are the same. This can help us choose how we might represent an equivalence class to our reader; since $[x] = [y]$, we might as well choose to write it using which ever of x and y is simpler for the reader. That is, we'll write the class using its simplest **representative**.

In the above example we saw that $[0] = [4] = [8]$, $[1] = [5] = [9]$ and so forth, so when discussing these classes we should pick to represent them as

$$[0] \qquad [1] \qquad [2] \qquad \text{and} \qquad [3].$$

In general it is a good idea to be kind to your reader (and yourself) by representing your equivalence classes using simple members of those equivalence classes.

We can push the above theorem further to show that any two equivalence classes are disjoint or equal. Equivalence classes cannot overlap "just a little" — it is all or nothing. We'll call this result a corollary.

Corollary 9.3.7 *Let R be an equivalence class on A and let $x, y \in A$. Then either $[x] = [y]$ or $[x] \cap [y] = \varnothing$.*

Proof. Let R be an equivalence relation on A and let $x, y \in A$. Now form the set $B = [x] \cap [y]$. Either this set is empty or not.

- If $B = \varnothing$ then there is nothing left to prove.

- On the other hand, if B is non-empty, then there must be some element $b \in B$. Hence $b \in [x]$, so $b \mathcal{R} x$ and by symmetry of R we know $x \mathcal{R} b$. Now, since $b \in [y]$, we have $b \mathcal{R} y$. By transitivity of R, $x \mathcal{R} y$. The previous theorem then ensures that $[x] = [y]$ as required.

■

Before we go on, let us look at some more examples of equivalence classes.

Example 9.3.8 Let R be congruence modulo 5 on the set of integers. We know from Theorem Theorem 9.2.6 above, that this is an equivalence relation. Now, using Euclidean division Fact 3.0.3, and dividing by 5, we see that any integer n can be written as

$$n = 5q + r \qquad \text{where } r \in \{0, 1, 2, 3, 4\}$$

Hence $n - r = 5q$. So n must be congruent to one of $0, 1, 2, 3, 4$ modulo 5. Thus our equivalence classes are exactly

$$[0] \qquad [1] \qquad [2] \qquad [3] \qquad [4].$$

□

Example 9.3.9 Let S be the set of all students at University of British Columbia. We define a relation on S by

$$a \mathcal{R} b \qquad \Longleftrightarrow \qquad a \text{ has the same age as } b.$$

Now this definition is still a little sloppy around the edges. To make it more precise:

- S is the set of all students enrolled on the 1st of October 2019.
- By "age" we mean the age in years rounded down to the nearest integer.

Show that this defines an equivalence relation and determine the equivalence classes.

Solution. We'll first show that this is reflexive, symmetric and transitive and then look at the equivalence classes.

- Reflexive — Let a be a student. Then since a has the same age as a, it follows that $a \mathcal{R} a$ as required.
- Symmetric — Let a, b be students so that a has the same age as b. This means that b has the same age as a and hence $b \mathcal{R} a$ as required.
- Transitive — Let a, b, c be students so that a has the same age as b and b has the same age as c. Then a has the same age as c and so $a \mathcal{R} c$. Hence the relation is transitive.

The equivalence classes are just sets of students with the same age (in years rounded to nearest integer). So there will be an equivalence class full of 18 year-olds, another of 19 year-olds, and so on. We should be a little careful, there may[108] be a very small number of 15 year-olds, 16 year-olds and on up to 75 year old students. □

Example 9.3.10 Let R be a relation defined on $\mathbb{R}^2$ by

$$(a, b) \mathcal{R} (c, d) \iff a + d = c + b$$

Show that is an equivalence relation and determine its equivalence classes.

Solution. We first show that it is an equivalence relation and then we'll look at its equivalence classes.

- Reflexive — Let $(a, b) \in \mathbb{R}^2$, then since $a + b = a + b$, it follows that $(a, b) \mathcal{R} (a, b)$. Hence the relation is reflexive.
- Symmetric — Let $(a, b), (c, d) \in \mathbb{R}^2$ and assume that $(a, b) \mathcal{R} (c, d)$. This implies that $a + d = c + b$. Since equality is symmetric, we know that $c + b = a + d$ and so $(c, d) \mathcal{R} (a, b)$.

[108] See here.

- Transitive — Let $(a,b),(c,d),(e,f) \in \mathbb{R}^2$ and assume that

$$(a,b)\ \mathcal{R}\ (c,d) \qquad \text{and} \qquad (c,d)\ \mathcal{R}\ (a,b).$$

Hence we know that

$$a+d=c+b \qquad \text{and } c+f=e+d.$$

Rearrange the second of these to give $c=e+d-f$. Substitute this into the first to get

$$\begin{aligned} a+d &= c+b \\ &= \underbrace{e+d-f}_{=c}+b \qquad\qquad \text{and so} \\ a+f &= e+b. \end{aligned}$$

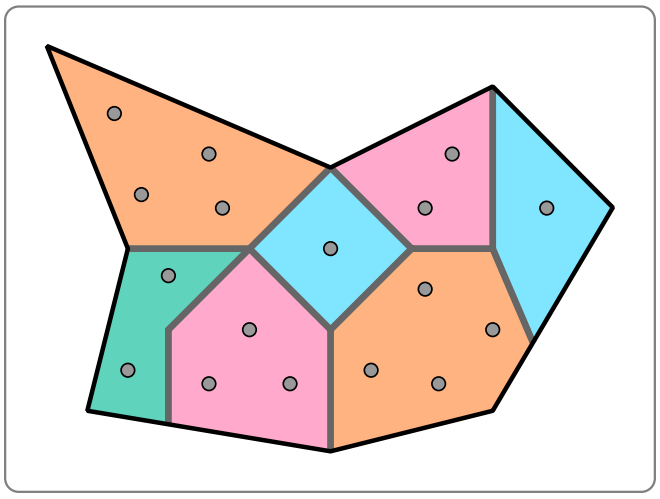
Hence $(a,b)\ \mathcal{R}\ (e,f)$ as required.

So what do the equivalence classes look like? Theorem Theorem 9.3.5 tells us that two elements are in the same equivalence class if and only if they satisfy the relation. So let $(a,b) \in \mathbb{R}^2$ and let us examine its equivalence class. The pair $(x,y) \in [(a,b)]$ if and only if $(a,b)\ \mathcal{R}\ (x,y)$. That is we must have $a+y=x+b$

$$y=x+(b-a).$$

Hence the equivalence class of (a,b) is the set of points lying on the line with gradient 1 that passes through (a,b). □

Our results above tell us that an equivalence relation cuts a set up into non-empty disjoint pieces, such as is depicted below.



Here we have coloured the different equivalence classes and you can think of the dots in the different subsets as being some elements in those equivalence classes. Notice there is no overlap between the subsets, and the entire set is

covered by the subsets — no piece is missing. This separation[109] of a set into disjoint pieces is called a set partition.

Definition 9.3.11 A partition of a set A is a collection $\mathcal{P}$ of non-empty subsets of A, so that

- if $x \in A$ then there exists $X \in \mathcal{P}$ so that $x \in X$, and
- if $X, Y \in \mathcal{P}$, then either $X \cap Y = \varnothing$ or $X = Y$

The elements of $\mathcal{P}$ are then called blocks, parts or pieces of the partition.

An equivalent definition is that a partition of a set A is a collection $\mathcal{P}$ of non-empty subsets of A, so that

- $\bigcup_{X \in \mathcal{P}} X = A$, and
- if $X, Y \in \mathcal{P}$, then either $X \cap Y = \varnothing$ or $X = Y$

Where the union $\bigcup_{X \in \mathcal{P}} X$ is the union of all the sets in the partition $\mathcal{P}$. ◊

Consider our equivalence class examples above, and notice that in each case the equivalence classes form partitions of the underlying set. Rather than just "notice" let us do one (new) example more explicitly.

Given an equivalence relation we can prove that its equivalence classes form a set partition.

Theorem 9.3.12 *Let R be an equivalence relation on A. The set of equivalence classes of R forms a set partition of A.*

Proof. Let $\mathcal{P} = \{[x] \mid x \in A\}$ be the set of equivalence classes.

- Let $x \in A$. Then by Lemma Lemma 9.3.4 above we know that $x \in [x]$, and by definition $[x] \in \mathcal{P}$. Hence for any $x \in A, x \in X$ for some $X \in \mathcal{P}$.
- Let $X, Y \in \mathcal{P}$. Then by Corollary Corollary 9.3.7, we know that either $X = Y$ or $X \cap Y = \varnothing$.

Hence the set of equivalence classes forms a set partition. ■

So an equivalence relation gives equivalence classes that define a set partition. We can also go backwards. A set partition can be used to define equivalence classes that in turn define an equivalence relation. To be more precise, take a set partition $\mathcal{P}$ of a set A. For any two elements $x, y \in A$ we can define

$$x \mathcal{R} y \iff x, y \text{ are elements of the same part of the partition } \mathcal{P}$$

It is not too hard to prove that this relation is an equivalence relation.

[109] Alternatively, we might say that such a *partitioning* of a set is called a set partition. However that sentence is a bit self-referential. At any rate, the term **set partition** is another on point naming by mathematicians.

Theorem 9.3.13 *Let $\mathcal{P}$ be a set partition on the set A, and define a relation R by*

$$x \mathcal{R} y \iff \big(x, y \in X \text{ for some } X \in \mathcal{P}\big)$$

That is $x \mathcal{R} y$ if and only if they belong to the same piece of the partition.

Then the relation R is an equivalence relation.

Proof. We need to show that R is reflexive, symmetric and transitive.

- Reflexive — Let $x \in A$. Since $\mathcal{P}$ is a set partition, we know that x is an element of some piece of the partition. Hence x is in the same piece of the partition as itself, and so $x \mathcal{R} x$ as required.
- Symmetric — Let $x \mathcal{R} y$, so we know that x, y lie in the same piece of the partition. Hence we must also have $y \mathcal{R} x$.
- Transitive — Let $x \mathcal{R} y$ and $y \mathcal{R} z$. Then there must be $X, Y \in \mathcal{P}$ so that $x, y \in X$ and $y, z \in Y$. Hence $X \cap Y \neq \varnothing$ (since it contains y). Since $\mathcal{P}$ is a partition, we know that $X \cap Y = \varnothing$ or $X = Y$. Hence $X = Y$ and so $x, z \in X$. Thus $x \mathcal{R} z$ as required.

■

9.4 Congruence revisited

One very important relation is congruence modulo n. It is not too hard to show that it is an equivalence relation, and its equivalence classes have some very useful properties. The reader should quickly revisit Section 5.3, Definition 5.3.1 and Theorem 5.3.3.

By Theorem Theorem 9.2.6, congruence modulo n is an equivalence relation and so has equivalence classes. For example, if we fix the modulus as 4, then we can write down the four equivalence classes of integers modulo 4:

$$[0] = \{\ldots, -8, -4, 0, 4, 8, \ldots\} \qquad [1] = \{\ldots, -7, -3, 1, 5, 9, \ldots\}$$
$$[2] = \{\ldots, -6, -2, 2, 6, 10, \ldots\} \qquad [3] = \{\ldots, -5, -1, 3, 7, 11, \ldots\}$$

Notice that we are *representing* each equivalence class by the smallest non-negative integer in that class.

Warning 9.4.1 Modulo relation and operator. Many of you who have programmed will have encountered the modulo operator; it is usually denoted by the percentage sign, "%". This operator is more correctly called the "integer remainder operator". In particular, if (via Euclidean division) we know that $a = kn + r$ where $0 \leq r < n$, then

$$a\%n = r$$

When we represent the equivalence classes modulo n we typically[110] represent them by the smallest non-negative member of the equivalence class, which is

the remainder when divided by n. This can lead to a confusion between the equivalence classes and the remainders.

So while equivalence classes modulo n are related to the effect of this integer remainder operator % — they are not the same. You should avoid thinking of "modulo n" as an operation that is done on integers.

One reason that this equivalence relation, congruence modulo m, is so important is that its equivalence classes interact very nicely with arithmetic giving rise to **modular arithmetic**. We saw this idea back in Theorem 5.3.3. Let us rewrite that result in terms of equivalence classes.

Theorem 9.4.2 Modular arithmetic redux. *Let $n \in \mathbb{N}$, and let $a, b, c, d \in \mathbb{Z}$ so that*

$$c \in [a] \qquad \text{and} \qquad d \in [b],$$

where $[x]$ denotes the equivalence class of x under congruence modulo n. Then

$$c + d \in [a + b], \qquad c - d \in [a - b] \qquad \text{and}$$
$$c \cdot d \in [a \cdot b]$$

Proof. The proof of this statement is essentially identical to the proof of Theorem 5.3.3. ■

As an illustration of this result, consider the numbers 5 and 7. When we multiply them together we get 35. But now, think about these numbers modulo 4, and the equivalence classes they lie in:

$$5 \in [1] \qquad 7 \in [3] \qquad 35 \in [3]$$

The above theorem tells us that since $5 \equiv 1 \pmod 4$ and $7 \equiv 3 \pmod 4$, their product

$$\underbrace{35}_{5\times 7} \equiv \underbrace{3}_{3\times 1} \pmod 4$$

Notice that this will hold no matter which elements of [1] and [3] we multiply together, we will always get an element of [3]. This suggests (with a little bending of notation and avoiding any confusion with cartesian products):

$$[1] \cdot [3] = [3]$$

and similarly

$$[1] + [3] = [4] = [0].$$

This can be made to work more generally, but first we should define things carefully and then prove things carefully.

Definition 9.4.3 Let $n \in \mathbb{N}$, and let $a, b \in \mathbb{Z}$. Then we define the following arithmetic operations on the equivalence classes modulo n:

$$[a] + [b] = [a + b]$$

[110] This author has gotten into difficulties in a piece of research when their coauthor (for very valid reasons) instead chose to represent equivalence classes by the integer closest to zero.

$$[a] - [b] = [a - b]$$
$$[a] \cdot [b] = [a \cdot b]$$

where we have used "$\cdot$" to denote multiplication to avoid confusion with the cartesian product of sets. ◊

There is a small problem with this definition — we need to make sure that it makes sense and is **well defined** and that our choice of representative elements a, b does not change the results $[a + b], [a - b], [a \cdot b]$. For example, in our modulo 4 example, we know that

$$[1] = [5] = [-3] \qquad \text{and } [3] = [7] = [-1]$$

so we need to be sure that

$$\underbrace{[1] + [3]}_{[4]} = \underbrace{[5] + [7]}_{[12]} = \underbrace{[-3] + [-1]}_{[-4]} = [0]$$

Thankfully, this is a simple corollary of Theorem 5.3.3 (or Theorem 9.4.2).

Corollary 9.4.4 Modular arithmetic. *Let $n \in \mathbb{N}$, and let $a, b \in \mathbb{Z}$. Then the sets*

$$[a + b] \qquad [a - b] \qquad [a \cdot b]$$

are well defined and do not depend on the choice of representative elements.

Proof. Let a, b, n be as in the statement of the result, and let $c \in [a], d \in [b]$.

By definition $[c] + [d] = [c + d]$. It suffices to show that $[c + d] = [a + b]$. By Theorem Theorem 5.3.3, we know that $c + d \in [a + b]$ and so by Corollary Corollary 9.3.7 $[c + d] = [a + b]$ as required.

The same argument shows that $[c] - [d] = [c - d] = [a - b]$, and that $[c] \cdot [d] = [c \times d] = [a \cdot b]$. ■

This result turns out to be very useful. It tells us that

$$[7] \cdot [9] = [3] \cdot [1] = [3 \cdot 1] = [3].$$

Of course, that one is easy enough to check in our head:

$$7 \times 9 = 63 \qquad \text{and} \qquad 63 \equiv 3 \pmod 4.$$

But (minutely) less easy:

$$[19] \cdot [11] = [3] \cdot [3] = [9] = [1]$$

(which is true since $19 \times 11 = 209 = 208 + 1 = 52 \times 4 + 1$).

Now, just before we switch gears to modulo 10, we should make some notation to emphasise the modulus so that we don't mix equivalence classes from different equivalence relations[111]. We can rewrite the above equivalence classes with the modulus as a subscript:

$$[7]_4 \cdot [9]_4 = [3]_4$$

[111] In some situations it is very helpful to mix results from different moduli, but one does it very deliberately, and not by accident. For a good example of carefully mixing moduli, see Example 9.4.6 below.

This makes it much easier to talk about different modulii without confusing the reader. Of course, if the modulus is clear by context, then we don't need the subscript.

Another, more complicated one, For example, let $a = 17321289, b = 23492871$ and we compute their product as

$$17321289 \times 23492871 = 406926808030718.$$

If you look at this for amoment, you realise that the product is wrong — the last number definitely should not be an "8". We can show this by computing the product modulo 10:

$$[17321289]_{10} \cdot [23492871]_{10} = [9]_{10} \cdot [1]_{10} = [9]_{10}.$$

This is a very simple example of using modular arithmetic as a way of error-checking. Some simple techniques based on modular arithmetic are often used in credit-card numbers and similar.

Example 9.4.5 Checking long numbers — Luhn algorithm. The author is going to assume that everyone has had to either write down a telephone number or a credit card number and messed it up. Very common mistakes are

- Single digit error: $1 \mapsto 2$
- Transposition: $12 \mapsto 21$
- Phonetic errors: $60 \mapsto 16$

Everyone has dialed a wrong number[112], but thankfully, credit card numbers have some built-in checks that prevent simple errors like the above. In particular, a credit card has a "check digit".

Since people rarely mess transcribing up the first or last digit of a long number, one can exploit the last digit to check up on the others. So say you have a long number (like a credit card or phone number), that you want to protect against simple errors, then you should append a single special digit to the end of that number. For example, say we have the first 8 digits of a number "31415926" and we want to make sure that it can be checked when copied down by someone else. A really simple check is to sum up the digits modulo ten:

$$3 + 1 + 4 + 1 + 5 + 9 + 2 + 6 = 31 \equiv 1 \pmod{1}0$$

So we can append "1" to the end of the number to get "31415926***1***".

Now when we read it out to a friend over the phone, they copy down "313159261". They can then compute the check number:

$$3 + 1 + 3 + 1 + 5 + 9 + 2 + 6 = 30 \equiv 0 \pmod{1}0$$

Since $0 \neq 1$ they know there is an error. This will proof against a single-digit substitution error, but it will not be safe against a transposition. If they wrote down "314***51***9261" then the check gives

$$3 + 1 + 4 + 5 + 1 + 9 + 2 + 6 = 31 \equiv 1 \pmod{1}0$$

So the error goes undetected.

However there is a better way. The Luhn algorithm[113] is used in many applications including in credit cards.

- Start with our number $3, 1, 4, 1, 5, 9, 2, 6$.
- Double every second digit (starting from the rightmost): $3, 2, 4, 2, 5, 18, 2, 12$. Notice that our list of digits becomes longer
- Now sum *all*:

$$3 + 2 + 4 + 2 + 5 + (1 + 8) + 2 + (1 + 2) = 30$$

- Multiply the result by 9 and compute the result modulo ten:

$$30 \times 9 = 270 \equiv 0 \pmod{1}0$$

- Append the result as the check digit: "31415926***0***"

Let us try this against the two transcription errors above:

- "31***3***159260" becomes "3,2,3,2,5,18,2,12" which gives

$$3 + 2 + 3 + 3 + 2 + 5 + (1 + 8) + 2 + (1 + 2) = 32.$$

  Multiply by 9 to get $32 \times 9 = 288$ giving check-digit "8". Error detected!

- "314***51***9261" becomes "3,2,4,10,1,18,2,12" which gives

$$3 + 2 + 4 + (1 + 0) + 1 + (1 + 8) + 2 + (1 + 2) = 25.$$

  Multiply by 9 to get 225 giving check-digit "5". Error detected!

This is not too hard to do by hand, but it is very easy for a computer.

This is not perfect, some errors will go undetected. The interested reader should search-engine their way to more information. □

Example 9.4.6 Chinese remainder theorem. You can push this idea further, and instead of doing arithmetic with very large numbers, you can do the same arithmetic modulo several different primes and then reconstruct the big-number result using the results modulo each prime. The piece of mathematics that tells you how to reconstruct the result is called the Chinese Remainder Theorem. It was first stated in the 3rd century AD by the Chinese mathematician Sunzi, but the first real method for this reconstruction is due to Aryabhata (an Indian mathematician of the 6th century AD). It seems to have been rediscovered a few times and seems to have entered European mathematics via Fibonacci in the

[112] I think? Well, the author definitely has

[113] Named for the computer scientist Hans Peter Luhn (1896–1964).

12th century. It's not quite clear when it got its name, but perhaps sometime between 1850 and 1930?

We won't do the Chinese remainder theorem in this course (its a good one to do in a number theory course), but we'll do a few more applications of congruences. □

Example 9.4.7 No integer solutions. Show that the equation

$$x^2 - 4y^2 = 3$$

has no integer solutions.

Scratchwork. A sneaky way to look at this is to realise that if there is an integer solution, then if you look at that solution modulo, say 7, then it will still be a solution. ie

$$(x^2 - 4y^2) \pmod 7 = 3 \pmod 7.$$

Hence if we can prove that the equation has no solution modulo (again say) 7, then it cannot have a solution over the integers (by taking the contrapositive). Notice that the converse is very false — just because we can find a solution modulo 7, does not mean there is an integer solution.

Now, because one of the coefficients in the equation is a 4, it simplifies considerably modulo 4. In particular

$$4y^2 \pmod 4 = 0$$

no matter what y we put in. So modulo 4 the equation simplifies to

$$(x^2 - 4y^2) \pmod 4 = x^2 \pmod 4 = 3 \pmod 4$$

So now we look at squares modulo 4.

$$\begin{array}{lll} 0^2 \pmod 4 = 0 & 1^2 \pmod 4 = 1 & 2^2 \pmod 4 = 0 \\ 3^2 \pmod 4 = 1 & 4^2 \pmod 4 = 0 & 5^2 \pmod 4 = 1 \end{array}$$

So — it looks like squares modulo 4 are either 0 or 1, depending on their parity. If this is true then we are done — there is no solution modulo 4, so there cannot be an integer solution.

Solution.

Proof. The equation either has a solution $x, y \in \mathbb{Z}$ or it does not. If the equation does have a solution, then x is either even or odd.

- If x is even then $x = 2k$ for some $k \in \mathbb{Z}$, and so

  $$x^2 - 4y^2 = 4(k^2 - y^2)$$

  and so $x^2 - 4y^2 \equiv 0 \pmod 4$.

- On the other hand, if x is odd, then $x = 2k + 1$ for some $k \in \mathbb{Z}$ and so

$$x^2 - 4y^2 = 4k^2 + 4k + 1 - 4y^2 = 4(k^2 + 4k - y^2) + 1$$

  and so $x^2 - 4y^2 \equiv 1 \pmod 4$.

In either case $x^2 - 4y^2$ is not congruent to 3 modulo 4. Hence there cannot be a solution. ■

□

Now, noticeably absent from Theorem Theorem 5.3.3 above is any discussion of division. Consider the equation $x = \frac{13}{2}$; here we are really solving the equation $2x = 13$. Consider this equation modulo 4 — find any x so that

$$[2]_4 \cdot [x]_4 = [1]_4$$

(since $[13]_4 = [1]_4$). Similarly modulo 5 we get:

$$[2]_5 \cdot [x]_5 = [1]_5.$$

We can try to solve these equations by brute-force by just writing out the multiplication tables modulo 4 and 5:

·	[0]	[1]	[2]	[3]
[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]
[2]	[0]	[2]	[0]	[2]
[3]	[0]	[3]	[2]	[1]

·	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]
[2]	[0]	[2]	[4]	[1]	[3]
[3]	[0]	[3]	[1]	[4]	[2]
[4]	[0]	[4]	[3]	[2]	[1]

Now - we see that modulo 5 we have a solution since:

$$[2]_5 \cdot [3]_5 = [1]_5$$

but modulo 4 there is no solution since

$$[2]_4 \cdot [x]_4 = [0]_4 \text{ or } [2]_4.$$

The existence of solutions to the equation

$$[a]_n \cdot [x]_n = [1]_n$$

depends on the divisors of a and n.

Lemma 9.4.8 *Let $a \in \mathbb{Z}, n \in \mathbb{N}$. If $d > 1$ divides both a, n then the equation*

$$[a]_n \cdot [x]_n = [1]_n$$

does not have a solution.

Proof. Say that the equation

$$[a]_n \cdot [x]_n = [1]_n$$

has a solution $x = b$. Then for some $k \in \mathbb{Z}$

$$ab + kn = 1.$$

Now if, $d > 1$ divides both a, n the left-hand side is divisible by d. However this does not make sense because the right-hand side is only divisible by ± 1. Hence no such solution can exist. ■

Remark 9.4.9 Bezout and Euclid. The converse of this lemma states that if a, n have no common divisors, then the equation has a solution. To prove this we need to show that if a, n have no common divisors, then we can find b, k so that

$$ab + kn = 1.$$

This is (essentially) Bezout's identity. The values b, k can be computed using the extended Euclidean algorithm. We will discuss all of this in the optional section below.

Example 9.4.10 Pseudo random numbers. Consider the following sequence of numbers

$$1, 10, 7, 8, 4, 9, 0, 3, 2, \ldots$$

The numbers bounce around and look fairly ``random". However, they are not actually random at all; they satisfy the simple relation

$$x_{k+1} \equiv (7x_k + 3) \pmod{11},$$

or, using the % operator (ie integer remainder operator)

$$x_{k+1} = (7x_k + 3)\%11.$$

This is an example of a linear congruent generator; the next term in the sequence is determined by a simple linear equation involving a congruence:

$$x_{k+1} = ax_k + c \pmod{n}$$

If one chooses a and c carefully, then the resulting sequence of numbers will look quite random. However, since the numbers are not actually random, they are usually called **pseudo random** numbers.

Many computer algorithms for generating pseudo random numbers are based on this idea. Of course, one does need to be quite careful and make sure that your pseudo-random numbers are actually fairly random. For example, if we choose $a = 7, c = 5, n = 11$ in the above, then we get the sequence

$$1, 1, 1, 1, \ldots$$

since $7 \cdot 1 + 5 = 12 \equiv 1 \pmod{11}$. Not very random at all.

There is a lot of interesting mathematics to be found in generating pseudo-random numbers and testing their randomness; we recommend that the interested reader take a trip to their favourite search engine. □

9.5 Greatest divisors, Bézout and the Euclidean algorithm

In our (brief) discussion above of division in modular arithmetic we came across the problem of computing **common divisors**. You probably first came across the question of computing common divisors when trying to simplify fractions:

$$\frac{6}{15} = \frac{3 \times 2}{3 \times 5} = \frac{2}{5}$$

We can simplify this fraction because 3 is a divisor of both 6 and 15. In this context we typically want to find the **greatest common divisor** of the numerator and denominator and then factor that out.

Definition 9.5.1 Let $a, b \in \mathbb{Z}$ be non-zero. Then

- The **greatest common divisor** of a, b, denoted $\gcd(a, b)$, is the largest positive integer that divides both a and b.
- If $\gcd(a, b) = 1$ then we say that a and b are **coprime**, or **relatively prime**.
- The **least common multiple** of a, b, denoted $\mathrm{lcm}(a, b)$, is the smallest positive integer that is divisible by both a and b.
- By symmetry, $\gcd(a, b) = \gcd(b, a)$ and $\mathrm{lcm}(a, b) = \mathrm{lcm}(b, a)$.
- Finally, "greatest common divisor" and "least common multiple" are frequently abbreviated to **gcd** and **lcm**.

◊

Remark 9.5.2 What about zero? Notice that in the above defintion if a, b are non-zero then the $\gcd(a, b)$ and $\text{lcm}(a, b)$ are well-defined. We should note that since $0 \times a = 0$, the definition extends to

$$\gcd(a, 0) = a.$$

When both a, b are zero then there is no greatest common divisor since $0 \times n = 0$ for any n. That being said, in some contexts[114] $\gcd(0, 0)$ is *defined* to be zero.

For non-zero a, the $\text{lcm}(a, 0)$ is not well defined since we cannot divide by zero. However, in some contexts $\text{lcm}(a, 0)$ is *defined* to be zero since the only common multiple of a and 0 is zero.

With all of that said, we will only consider $\gcd(a, b)$ when at least one of a, b are non-zero, and only consider $\text{lcm}(a, b)$ when a, b are both non-zero.

Remark 9.5.3 Notice that by definition if c divides both a, b then $c \leq \gcd(a, b)$:

$$(c \mid a) \wedge (c \mid b) \implies c \leq \gcd(a, b).$$

Similarly, if a and b divide m then $\text{lcm}(a, b) \leq m$:

$$(a \mid m) \wedge (b \mid m) \implies m \geq \text{lcm}(a, b).$$

When a, b are small numbers it is not too hard to compute the gcd by hand, typically by thinking about factors. However, there is a much better (and faster) way called Euclid's algorithm. It is based on the following observation.

Lemma 9.5.4 *Let $a, b \in \mathbb{N}$, with $a \geq b$*

$$\gcd(a, 0) = a$$
$$\gcd(a, b) = \gcd(b, a - b).$$

Finally, if $a = qb + r$ for some $q, r \in \mathbb{Z}$, then

$$\gcd(a, b) = gcd(b, r).$$

Remark 9.5.5 Antisymmetry helping equality. To prove the remaining two points we take advantage of the fact that the relation "$\leq$" is antisymmetric. That is

$$(x \leq y) \wedge (y \leq x) \implies x = y$$

This means that we can break down the proof of an equality into proofs of two (potentially easier) inequalities. We have already done this proving set equalities using

$$(A \subseteq B) \wedge (B \subseteq A) \implies A = B.$$

Proof of Lemma 9.5.4. We prove each point in turn.

- Since $a = a \cdot 1$ and $0 = a \cdot 0$, it follows that a is a divisor of both $a, 0$. No number larger than a can divide a, so a must be the greatest common divisor.

[114] Many computer algebra systems define $\gcd(0, 0) = 0$.

As noted above, it is sufficient to show that (say) $\gcd(a,b) \geq \gcd(b, a-b)$ and $\gcd(a,b) \leq \gcd(b, a-b)$ in order to prove that $\gcd(a,b) = \gcd(b, a-b)$.

- We start by showing that $\gcd(a,b) \leq \gcd(b, a-b)$. To do this, we prove that $\gcd(a,b)$ divides both b and $(a-b)$, and hence is a common divisor of b and $(a-b)$. Since it is a common divisor, it cannot be bigger than the largest common divisor (by definition).

  Let $d = \gcd(a,b)$. Since $d \mid a$ and $d \mid b$, we can write $a = kd, b = \ell d$ for some $k, \ell \in \mathbb{Z}$. Thus $b - a = d(k - \ell)$ and so it follows that $d \mid (a-b)$. Hence d is a divisor of b and $a-b$, and so it must be no bigger than the gcd of $b, a-b$.

  $$d \leq \gcd(b, a-b).$$

  Now let us reuse this argument, but now starting from $c = \gcd(b, a-b)$. Since $c \mid b$ and $c \mid (b-a)$, there are $k, \ell \in \mathbb{Z}$ so that $b = ck, (b-a) = c\ell$. It follows that $a = b - (b-a) = c(k-\ell)$ and thus $c \mid a$. And because $c \mid a$ and $c \mid b$ we have

  $$c \leq \gcd(a,b)$$

  But now we have both

  $$\gcd(a,b) \leq \gcd(b, a-b) \qquad \text{and} \qquad \gcd(b, a-b) \leq \gcd(a,b)$$

  and hence $\gcd(a,b) = \gcd(b, a-b)$.

- That $\gcd(a,b) = \gcd(b,r)$ follows by a nearly identical argument. If $d \mid a$ and $d \mid b$, then

  $$d \mid (a - qb) = r$$

  Hence d divides both b and r, and hence

  $$d \leq \gcd(b,r).$$

  And since $c = \gcd(b,r)$ divides both b and r, it must also satisfy

  $$d \mid (qb + r) = a$$

  Thus c divides both b and a, and so

  $$c \leq \gcd(a,b)$$

  Since $\gcd(a,b) \leq \gcd(b,r)$ and $\gcd(b,r) \leq \gcd(a,b)$, we must have that they are equal.

■

Now why is this useful? Well, say $a > b$, then we can compute

$$\gcd(a,b) = \gcd(b, a-b)$$

and now the number $a - b$ is smaller than a. If we keep repeating this, then the numbers in our gcd keep getting smaller and smaller until we have to compute a really simple gcd. This is even faster if we write $a = qb + r$, because then

$$\gcd(a, b) = \gcd(b, r).$$

For example

$$\begin{aligned} \gcd(268, 120) &= \gcd(120, 28) && \text{since } 28 = 268 - 2 \times 120 \\ &= \gcd(28, 8) && \text{since } 8 = 120 - 4 \times 28 \\ &= \gcd(8, 4) && \text{since } 4 = 28 - 3 \times 8 \\ &= \gcd(4, 0) && \text{since } 0 = 8 - 2 \times 4 \\ &= 4 \end{aligned}$$

This method of computing the gcd is known as the Euclidean algorithm.

Algorithm 9.5.6 The Euclidean algorithm. *Let $a, b \in \mathbb{Z}$ with $|a| \geq |b|$ and at least one of a, b non-zero. Then $\gcd(a, b)$ can be computed using the following steps:*

1. *If $b = 0$ then the gcd is a, otherwise go on to (2)*
2. *Compute the remainder of a divided by b, that is $a = qb + r$ with $0 \leq r < |b|$.*
3. *Go back to (1) with (a, b) replaced by (b, r).*

Proof. Assume $a, b \in \mathbb{Z}$ as stated.

- If $b = 0$ then we are done since $\gcd(a, 0) = a$ by Lemma Lemma 9.5.4.
- So now we can assume that $b \neq 0$. By Fact 3.0.3 we know that

  $$a = qb + r \qquad \text{with } 0 \leq r < |b|.$$

  By Lemma Lemma 9.5.4 we know that $\gcd(a, b) = \gcd(a, r)$. Hence computing $\gcd(b, r)$ is the same as computing $\gcd(a, b)$.
- Since $0 < r < |b|$ it follows that the value of r will be *strictly* smaller in each iteration. Further, since r is an integer, it follows that r will become zero in a finite number of iterations.

Hence the algorithm described will terminate in a finite number of steps and give the gcd. ■

Let's do another one:

$$\begin{aligned} \gcd(869, 442) &= \gcd(442, 427) && \text{since } 869 = 442 + 427 \\ &= \gcd(427, 15) && \text{since } 442 = 427 + 15 \\ &= \gcd(15, 7) && \text{since } 427 = 15 \cdot 28 + 7 \\ &= \gcd(7, 1) && \text{since } 15 = 2 \cdot 7 + 1 \end{aligned}$$

$$\begin{aligned} &= \gcd(1,0) && \text{since } 7 = 7 \cdot 1 + 0 \\ &= 1 \end{aligned}$$

Now notice that at each stage we take linear combinations of the arguments of the gcd to make new arguments.

$$\begin{aligned} 427 &= 869 - 442 \\ 15 &= 442 - 427 \\ 7 &= 427 - 28 \cdot 15 \\ 1 &= 15 - 2 \cdot 7 \end{aligned}$$

If we substitute these equations into each other, then we can write our result as some linear combination of our starting arguments. Let $a = 869, b = 442$, then:

$$\begin{aligned} 427 &= 869 - 442 = a - b \\ 15 &= 442 - 427 = b - \underbrace{(a-b)}_{=427} = 2b - a \\ 7 &= 427 - 28 \cdot 15 = \underbrace{(a-b)}_{=427} - 28\underbrace{(2b-a)}_{=15} = 29a - 57b \\ 1 &= 15 - 2 \cdot 7 = \underbrace{(2b-a)}_{=15} - 2\underbrace{(29a-75b)}_{=7} = 116b - 59a \end{aligned}$$

That is

$$\gcd(869, 442) = 1 = 116 \cdot 442 - 59 \cdot 869.$$

That one can write the gcd in this way is quite general and is known as Bézout's Lemma.

Lemma 9.5.7 Bézout's Lemma. *Let $a, b \in \mathbb{Z}$ so that at least one of a, b is non-zero. Then there exist $x, y \in \mathbb{Z}$ so that*

$$\gcd(a, b) = ax + by.$$

Proof. Essentially one follows the Euclidean algorithm backwards just as we have in the example above. However one can also prove this using an induction-like argument which is nearly the same thing.

Consider an execution of the Euclidean algorithm starting from a, b. It builds a finite sequence of equations of the form:

$$\begin{aligned} a &= q_1 b + r_1 \\ b &= q_2 r_1 + r_2 \\ r_1 &= q_3 r_2 + r_3 \\ r_2 &= q_4 r_3 + r_4 \\ \vdots &= \vdots \\ r_{n-1} &= q_{n+1} r_n + 0 \end{aligned}$$

where the gcd will be r_n. We now prove that for each i, there exist $x_i, y_i \in \mathbb{Z}$ so

that $r_i = ax_i + by_i$, using an induction-style proof.

Notice that when $i = 1$ we have

$$r_1 = a - q_1 b$$

so we have $x_1 = 1, y_1 = -q_1$. Similarly, when $i = 2$ we have

$$r_2 = b - q_2 r_1 = b - q_2(a - q_1 b) = (1 + q_1 q_2)b - q_2 a$$

Hence $x_2 = -q_2, y_2 = 1 + q_1 q_2$.

Now for general i we have that

$$r_{i+1} = r_{i-1} - q_{i+1} r_i$$

So if $r_i = x_i a + y_i b$ and $r_{i-1} = x_{i-1} a + y_{i-1} b$, then

$$r_{i+1} = (x_{i-1} a + y_{i-1} b) - q_{i+1}(x_i a + y_i b) = a(x_{i-1} - q_{i+1} x_i) + b(y_{i-1} - q_{i+1} y_i).$$

Hence

$$x_{i+1} = x_{i-1} - q_{i+1} x_i \qquad y_{i+1} = y_{i-1} - q_{i+1} y_i.$$

So using this recurrence and our starting values of $(x_1, y_1), (x_2, y_2)$ we can find integer $(x_3, y_3), (x_4, y_4)$ and so on up to (x_n, y_n). And hence we can write $gcd(a, b) = x_n a + y_n b$ with $x_n, y_n \in \mathbb{Z}$.

Notice that if we set

$$(x_0, y_0) = (0, 1) \qquad \text{and } (y_0, y_1) = (1, -q_1)$$

then our recurrence gives us

$$x_2 = 0 - q_2 \cdot 1 = -q_2 \qquad y_2 = 1 - q_2 \cdot (-q_1) = 1 + q_1 q_2$$

as required. ■

In the proof of the above lemma we give a construction of a recurrence that gives us the needed x, y to express $\gcd(a, b) = ax + by$. By combining that recurrence and the Euclidean algorithm we get the **extended Euclidean algorithm**.

Algorithm 9.5.8 The extended Euclidean algorithm. *Let $a, b \in \mathbb{Z}$ with $|a| \geq |b|$ and at least one of a, b non-zero. Then $\gcd(a, b)$ can be computed using the following steps:*

1. *If $b = 0$ then the gcd is a else go on to (2)*
2. *Compute the remainder of a divided by b, that is $a = qb + r$ with $0 \leq r < |b|$.*
3. *Go back to (1) with (a, b) replaced by (b, r).*

In the process of computing this we obtain a sequence of n quotients q_i and

remainders r_i. Define sequences x_i, y_i by the inital values

$$x_0 = 0, x_1 = 1 \qquad \textit{and} \qquad y_0 = 0, y_1 = -q_1$$

and then for $k \geq 1$:

$$x_{k+1} = x_{k-1} - q_{k+1}x_k$$
$$y_{k+1} = y_{k-1} - q_{k+1}y_k$$

Then

$$\gcd(a, b) = ax_n + by_n.$$

Proof. This follows quite directly from Proof 9.5.7.1 of Bézout's lemma above. ■

Returning to our example of $\gcd(869, 442)$ above, we had

$$q_1 = 1 \qquad q_2 = 1 \qquad q_3 = 28 \qquad q_4 = 2$$

which gives $x_0 = 0, x_1 = 1$ and $y_0 = 1, y_1 = -1$

$$\begin{array}{lll} x_2 = 0 - 1 = -1 & x_3 = 1 - 28 \cdot (-1) = 29 & x_4 = -1 - 2 \cdot 29 = -59 \\ y_2 = 1 - 1 \cdot (-1) = 2 & y_3 = -1 - 28 \cdot 2 = -57 & y_4 = 2 - 2 \cdot (-57) = 116 \end{array}$$

and we can verify that

$$-59 \cdot 869 + 116 \cdot 442 = -51271 + 51272 = 1$$

as required.

Bézout's turns out to be a useful result because it allows one to express $\gcd(a, b)$ as a simple equation in a, b which is then easier to manipulate. As an example of its utility, consider the following lemma about division which we attribute to Euclid. This feels like it should be "obvious", but it turns out to be not so easy to prove without invoking unique prime-factorisations of integers — the Fundamental Theorem of Arithmetic. Unfortunately that result is harder to prove than this one. We'll get around to proving the Fundamental Theorem of Arithmetic, but not just yet.

Lemma 9.5.9 Euclid's lemma. *Let $a, b, p \in \mathbb{Z}$ so that p is prime, and $p \mid ab$. If $p \nmid b$ then $p \mid a$.*

More generally let $a, b, d \in \mathbb{Z}$ so that $d \mid ab$. If $\gcd(d, b) = 1$ then $d \mid a$.

Via Bézout. Notice that the second statement implies the first statement, since by definition the only divisors of p are $1, p$, and so if $p \nmid b$ then $\gcd(p, b) = 1$.

By assumption, we know that $d \mid ab$, so we can write $ab = dk$ for some $k \in \mathbb{Z}$. Bézout's lemma allows us to write an equation linking b, d. In particular, it guarantees that there are $x, y \in \mathbb{Z}$ so that

$$1 = dx + by$$

Multiply this expression by a to get

$$a = adx + aby$$

But since $ab = dk$, we have

$$a = d(ax + ky)$$

and so $d \mid a$ as required. ■

Implicitly Bézout. Here is an alternative proof — it does not use Bézout's lemma explicitly, but some of the ideas are similar. Let $p \mid ab$ and assume that $p \nmid b$. Then form the set

$$S = \{x \in \mathbb{N} \mid p \mid xa\}$$

We know that S is non-empty since $b \in S$. Further, we also know that $S \subseteq \mathbb{N}$ and so obeys the well-ordering principle that we saw back in Chapter Chapter 7 on induction. It implies that S must have a smallest element, z. If we can show that $z = 1$ we are done because we have shown that $p \mid a$. We first that z divides all elements in S.

Let $x \in S$, then by Euclidean division we know that

$$x = qz + r \qquad \text{with } 0 \leq r < z$$

But we know that $p \mid xa$ and $xa = aqz + ra$. Rewrite this as

$$ra = xa - azq$$

Now by definition of S, we know that $p \mid xa$ and $p \mid az$, so $p \mid ra$. If $r \neq 0$ we have a problem because now $p \mid ra$ and $r < z$ which contradicts z being minimal. Hence we must have $r = 0$ and so $z \mid x$.

How do we now use this to show that $z = 1$? Since $p \mid pa$, we know that $p \in S$. Similarly, since $p \mid ba$, we know that $b \in S$. Hence $z \mid p$ and $z \mid b$. The only divisors of p are 1 and p and since $p \nmid b$, we must have $z = 1$. So finally, we know that $p \mid a$ as required. ■

Remark 9.5.10 Euclid without Bézout. Since Euclid lived around 300BC in ancient Greece and Egypt, and Bézout lived in 18th century France, it is pretty clear that there must be a way to prove the above without using Bézout's lemma. The interested reader should look at this excellent discussion of Euclid's proof[115] by David Pengelley and Fred Richman, and other good articles here[116] and here[117].

Closer to Euclid. The proof we give here is closer to Euclid's original proof. The key to that proof is the fact that if we have a fraction $\frac{a}{b}$ and then we find the smallest fraction equivalent to that, call if $\frac{x}{y}$ (ie where x is as small as possible), then there must be some n so that $a = nx$ and $b = ny$. Now this feels quite obvious. Obvious until you have to prove it.

[115] web.nmsu.edu/~davidp/euclid.pdf
[116] web.nmsu.edu/~davidp/fractions-monthly-final.pdf
[117] people.math.harvard.edu/~mazur/preprints/Eva.pdf

We follow the clever argument given in this excellent article[a] by Barry Mazur. Since the fractions $\frac{a}{b}$ and $\frac{x}{y}$ are equal, we know that

$$ay = bx.$$

If $a = x$ then $n = 1$ and we are done. So choose n to be the largest integer so that $a > nx$. Indeed we must have $a - nx \geq x$ (otherwise we would pick n to be larger). And since $x = \frac{y}{b}a$

$$nx = \frac{ny}{b}a$$

and so we must also have that $ny > b$.

Now consider the fraction $\dfrac{a - nx}{b - ny}$. One can quickly verify by cross-multiplication that

$$\frac{a - nx}{b - ny} = \frac{x}{y}.$$

So we have a new fraction that is also equal to $\frac{a}{b}$ and xy. But now, since $\frac{x}{y}$ was minimal, we must have $a - nx \geq x$, and at the same time, we choose n so that $a - nx \leq x$. Thus we must have $a - nx = x$. In other words

$$a = x(n + 1).$$

and so a is a multiple of x. Substituting this back into $ay = bx$ shows that

$$b = y(n + 1)$$

also.

This result implies that if we have

$$\frac{a}{b} = \frac{c}{d} = r$$

then there is a unique smallest fraction $\frac{x}{y} = r$ so that

$$\begin{aligned} a &= kx & b &= ky \\ c &= nx & d &= ny \end{aligned}$$

for some integers k, n. That is, the two fractions $\frac{a}{b}, \frac{c}{d}$ reduce to the same[b] fraction $\frac{x}{y}$.

Now let us go back and use this to prove that if $p \mid ab$ and $p \nmid b$ then $p \mid a$. Since $p \mid ab$, we know that $ab = pc$. Hence we have

$$\frac{a}{p} = \frac{c}{b}.$$

From the above, we know that there exist k, n, x, y so that

$$\begin{aligned} a &= kx & c &= nx \\ p &= ky & b &= ny \end{aligned}$$

But we know that p is prime, so either $(k, y) = (p, 1)$ or $(k, y) = (1, p)$. The first case implies that $p \mid a$, while the second implies that $p \mid b$. Since $p \nmid b$ by assumption, we know that $p \mid a$. And we are done. ■

[a] people.math.harvard.edu/~mazur/preprints/Eva.pdf

[b] This actually turns out to be a subtle point which is false in other contexts. In particular one can come up with perfectly well behaved sets of numbers, such as $\mathbb{Z}[\sqrt{-5}] = \left\{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\right\}$ where this result is no longer true. The interested reader should search-engine their way to discussions of "unique factorisation domains" and non-examples.

Here is another example of how Bézout's lemma can be very useful to prove a very useful[118] result linking the gcd and lcm.

Lemma 9.5.11 *The gcd and lcm satisfy the equation*

$$\gcd(a, b) \cdot \operatorname{lcm}(a, b) = |a \cdot b|.$$

Proof. Let $d = \gcd(a, b)$ and let $L = \frac{|ab|}{d}$. We need to show that both

- L is a common multiple of a, b, and
- any other common multiple of a, b is at least as large as L.

We know that, by definition, $d \mid a$ and $d \mid b$. Hence

$$a = dk \qquad \text{and } b = dn$$

which gives $|ab| = |d^2kn|$ and so

$$L = \frac{|ab|}{d} = |dkn| = |an| = |bk|.$$

Thus $a \mid L$ and $b \mid L$ as required.

Let M be a common multiple of a, b. It suffices to show that $L \mid M$ since this means that $|L| \leq |M|$. Since M is a common multiple of a, b we know that

$$M = ap = bq$$

for some integer p, q. Now since $dL = ab$, it is easier to compare dM and dL, and we do that via Bézout's lemma. Recall from Bézout's lemma that there are $x, y \in \mathbb{Z}$ so that:

$$d = ax + by$$

This means that

$$\begin{aligned} dM &= aMx + bMy \\ &= a(\underbrace{bq}_{=M})x + b(\underbrace{ap}_{=M})y \\ &= ab(qx + py) \end{aligned}$$

and thus $dL \mid dM$. Hence $L \mid M$ and so $|L| \leq |M|$. This means that L is the least common multiple of a, b. ■

[118] and obvious until you have to prove it!

9.6 Uniqueness of prime factorisation

Prime numbers follow very quickly when we learn about multiplication. Soon after that[119] we realise that every positive integer can be written as a product of primes.

$$30 = 5 \times 3 \times 2$$

While we can shuffle those numbers around

$$30 = 2 \times 5 \times 3 = 3 \times 2 \times 5 = \text{ etc}$$

those products will still contain the same number of each prime. Hence we can write it *uniquely* by putting the primes in order, smallest to largest:

$$30 = 2 \times 3 \times 5.$$

This **unique factorisation** of integers[120] into primes is called the "Fundamental theorem of arithmetic", and seems so obvious to us that it is hard to imagine that one even has to prove it! However, while it feels obvious, it needs a proof.

The history of the theorem is not quite as direct as one might think[121]. The result follows quite directly from the results of Euclid we discussed in the previous section, but does not actually appear explicitly in Euclid's Elements[122]. The 13th Century Persian mathematician al-Farisi[123] proved that numbers can be decomposed as products of primes, but not the uniqueness of that decomposition. Prestet, Euler and Lagrange[124] all seem to have been very close to writing it down, but it was actually Gauss[125] who first wrote it down explicitly in 1801.

Theorem 9.6.1 The fundamental theorem of arithmetic. *If $n > 1$ is an integer, then it can be written as a product of primes in exactly one way (up to permutations of the primes).*

[119] If this author is remembering back that far correctly…

[120] Of course we have to make exceptions for zero and one, which are not primes. We also have to agree that when we factor a negative integer we just put a "$(-1)\times$" and then factor the absolute value as usual.

[121] The interested reader can find out more about the history of this result using their favourite search-engine. A very good starting point is this article. It certainly surprises this author that the first explicit formal statement and proof of this fact comes a full century after Newton and Leibniz developed calculus.

[122] `wikipedia.org/wiki/Euclid's_Elements`

[123] Kamal al-Din al-Farisi (1265 – 1318) was a Persian mathematician and physicist who is also famous for his mathematically rigorous description of the formation of rainbows.

[124] Jean Prestet (1648–1690), Leonard Euler (1707 – 1783) and Joseph-Louis Lagrange (1736 –1813). The latter was born Giuseppe Ludovico De la Grange Tournier. Your favourite search engine will get you to much more information, especially about Euler and Lagrange.

[125] Johann Carl Friedrich Gauss (1777 – 1855) made an incredible number of contributions to mathematics and physics. Indeed, we have already come across him in this text back when we were busy computing $1 + 2 + \cdot + n$. His history is well worth a quick trip to your favourite search engine.

With our work in the previous section we are ready to prove this result. It follows quite directly from Euclid's lemma Lemma 9.5.9. Recall that this tells us that

$$(p \mid ab) \implies (p \mid a) \vee (p \mid b).$$

Proof. Let $n \in \mathbb{Z}$ so that $n \geq 2$. We use induction to prove that n can be decomposed as a product of primes. We then show that this decomposition is unique.

We proceed to show the existence of a prime factorisation by strong induction 7.2.2.

- Base case. Since $n = 2$ is prime, it is trivially written as a product of primes.
- Inductive step. Assume that the result holds for all integers $2 \leq k < n$. If n is prime the result holds since the factorisation is trivial. On the other hand, if n is not prime, then (by definition) it can be written as

  $$n = ab \qquad \text{with } 1 < a, b < n.$$

  Since $a, b < n$ we know, by assumption, that

  $$a = p_1 p_2 \cdots p_k \qquad \text{and} \qquad b = q_1 q_2 \cdots q_\ell$$

  where the p_i, q_j are all primes. Hence

  $$n = ab = p_1 p_2 \cdots p_k \cdot q_1 q_2 \cdots q_\ell$$

  is a product of primes as requried.

We must now show that the decomposition is unique. A very standard way to show the uniqueness of a mathematical object is to assume there are two of them and show that the two must be equal. So, let us assume that we can write (please forgive recycling of p's and q's from just above):

$$n = p_1 p_2 \cdots p_k = q_1 q_2 \cdots q_\ell$$

If there are any common prime factors between the sides, then divide through by them. Now either that cancels all the factors (in which case we are done), or we are left with an expression as above but with *no* common factors on each side. Well then take the smallest remaining prime on either side, call it z. Since z divides one side, it must, by Euclid's lemma, divide one of the factors on the other. However, since all the factors are prime, the factor z must be on both sides. But this cannot happen since we removed all the common factors. Thus the two factorisations of n must actually be the same. ■

9.7 Exercises

1. Let $A = \{1, 2, 3, 4, 5, 6\}$. Write out the relation R that expresses "$\nmid$" (does not divide) on A as a set of ordered pairs. That is, $(x, y) \in R$ if and only if $x \nmid y$. Is the relation reflexive? Symmetric? Transitive?

2. Define a relation on $\mathbb{R}$ as $x \mathcal{R} y$ if $|x - y| < 1$. Is R reflexive? Symmetric? Transitive?

3. For each of the following relations, determine whether or not they are reflexive, symmetric, and transitive.

(a) $R = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1)\} \subseteq \{1,2,3\} \times \{1,2,3\}$

(b) For $a, b \in \mathbb{N}$, $a \mathcal{R} b$ if and only if $a \mid b$.

(c) $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x - y \in \mathbb{Q}\}$

(d) For $A, B \subseteq \mathbb{R}$, $A \mathcal{R} B$ if and only if $A \cap B = \emptyset$.

(e) For functions $f, g : \mathbb{R} \to \mathbb{R}$, $f \mathcal{R} g$ if and only if $f - g$ is linear, that is, there are constants $m, b \in \mathbb{R}$ so that $f(x) - g(x) = mx + b$ for all $x \in \mathbb{R}$. The constants m, b depend on f and g, but not on x.

4. Determine whether the following relations are reflexive, symmetric and transitive. Prove your answers.

(a) On the set X of all functions $\mathbb{R} \to \mathbb{R}$, we define the relation:

$$f \mathcal{R} g \text{ if there exists } x \in \mathbb{R} \text{ such that } f(x) = g(x).$$

(b) Let $\mathcal{S}$ be a relation on $\mathbb{Z}$ defined by:

$$x \mathcal{S} y \text{ if } xy \equiv 0 \pmod 4.$$

5. For each of the following relations, show that they are equivalence relations, and determine their equivalence classes.

(a) $R = \left\{((x_1, y_1), (x_2, y_2)) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_1^2 + y_1^2 = x_2^2 + y_2^2\right\}$

(b) Let L be the set of all lines on the Euclidean plane, $\mathbb{R}^2$. For $\ell_1, \ell_2 \in L$, $\ell_1 \mathcal{R} \ell_2$ if and only if ℓ_1 and ℓ_2 have the same slope, or they are both vertical lines.

(c) Let R be a relation on $\mathbb{Z}^2$ defined by $x \mathcal{R} y$ if and only if $3 \mid x^2 - y^2$.

6. Define a relation on $\mathbb{Z}$ as

$$a \mathcal{R} b \quad \iff \quad 3 \mid (2a - 5b).$$

Is R an equivalence relation? Prove your answer.

7. Let E be a non-empty set and $q \in E$ be a fixed element of E. Consider the relation $\mathcal{R}$ on $\mathcal{P}(E)$ (power set of E) defined as

$$A \mathcal{R} B \iff (q \in A \cap B) \vee (q \in \overline{A} \cap \overline{B}),$$

where for any set $S \subseteq E$, we write $\overline{S} = E - S$ for the complement of S in E. Prove or disprove that $\mathcal{R}$ an equivalence relation.

8. Let R be a relation on $\mathbb{Z}$ defined by $a \mathcal{R} b$ if $7a^2 \equiv 2b^2 \pmod 5$. Prove that R is an equivalence relation. Determine its equivalence classes.

9. Let $n \in \mathbb{N}$ with $n > 1$ and let P be the set of polynomials with coefficients in $\mathbb{R}$. We define a relation, T, on P as follows:

Let $f, g \in P$. Then we say fTg if $f - g = c$ for some $c \in \mathbb{R}$. That is, if there exists a $c \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, $f(x) - g(x) = c$.

Show that T is an equivalence relation on P.

10. Prove or disprove the following statements:

(a) If R and S are two equivalence relations on a set A, then $R \cup S$ is also an equivalence relation on A.

(b) If R and S are two equivalence relations on a set A, then $R \cap S$ is also an equivalence relation on A.

11. Let R be a symmetric and transitive relation on a set A.

(a) Show that R is not necessarily reflexive.

(b) Suppose that for every $a \in A$, there exists $b \in A$ such that $a \mathcal{R} b$. Prove that R is reflexive.

12. Let R be a relation on a nonempty set A. Then $\overline{R} = (A \times A) - R$ is also a relation on A. Prove or disprove each of the following statements:

(a) If R is reflexive, then $\overline{R}$ is reflexive.

(b) If R is symmetric, then $\overline{R}$ is symmetric.

(c) If R is transitive, then $\overline{R}$ is transitive.

13. In this question we will call a relation $R \subset \mathbb{Z} \times \mathbb{Z}$ *sparse* if $(a, b) \in R$ implies that $(a, b + 1)$ and $(a + 1, b)$ are NOT elements of R.

(a) Prove that for all $n \in \mathbb{N}$ the equivalence relation $R = \{(a, b) \in \mathbb{Z} \times \mathbb{Z} : n \mid (a - b)\}$ is sparse if and only if $n \neq 1$.

(b) Prove or disprove that every equivalence relation R on $\mathbb{Z}$ is sparse.

14. Let A be a non-empty set and $P \subseteq \mathcal{P}(A)$ and $Q \subseteq \mathcal{P}(A)$ partitions of A. Prove that the set R defined as

$$R = \{S \cap T : S \in P,\ T \in Q\} - \{\emptyset\}$$

is also a partition of A.

15. Suppose that $n \in \mathbb{N}$ and $\mathbb{Z}_n$ is the set of equivalence class of congruent modulo n on $\mathbb{Z}$. In this question we will call an element $[u]_n$ *invertible* if it has a multiplicative inverse. That is,

$$[u]_n \text{ is invertible } \iff \text{ there exists } [v]_n \in \mathbb{Z}_n \text{ so that } [u]_n[v]_n = [1]_n.$$

Now, define a relation R on $\mathbb{Z}_n$ by

$$[x]_n \; R \; [y]_n \iff [x]_n[u]_n = [y]_n \text{ for some invertible } [u]_n \in \mathbb{Z}_n.$$

(a) Show that R is a equivalence relation.

(b) Compute the equivalence classes of this relation for $n = 6$.

16. Let $n \in \mathbb{Z}$ and $p \geq 5$ be prime.

(a) Prove, using Bézout's identity that if $3 \mid n$ and $8 \mid n$, then $24 \mid n$.

(b) Use the result in part (a) to show that $p^2 \equiv 1 \pmod{24}$.

17.

(a) Let p be a prime number, and suppose that $n \in \mathbb{Z}$ is such that $n \not\equiv 0 \pmod{p}$. Show that there is some $k \in \mathbb{Z}$ so that $nk \equiv 1 \pmod{p}$.

(b) Find an example to show that (a) may not be true if p is not prime. That is, find some composite number p and $n \in \mathbb{Z}$, $n \not\equiv 0 \pmod{p}$ such that $nk \not\equiv 1 \pmod{p}$ for all $k \in \mathbb{Z}$.

18. Let $a, b, d \in \mathbb{Z}$ such that $d \mid ab$. Show that the integer, $\frac{d}{\gcd(a,d)}$, divides b.

19. Let $a, b \in \mathbb{Z}$, at least one of which is non-zero.

(a) Suppose that d divides both a and b. Show that $d \mid \gcd(a, b)$.

(b) Let $m \in \mathbb{N}$. Show that $\gcd(ma, mb) = m \gcd(a, b)$.

(c) Let $c \in \mathbb{Z}$, $c \neq 0$. Show that the statement

$$\gcd(ac, b) = \gcd(a, b) \cdot \gcd(c, b)$$

does not hold in general.

20. A frequently used but false statement is

$$(x + y)^n = x^n + y^n$$

This is sometimes referred to by mathematicians as the "child's binomial theorem" (a quick trip to your search engine will turn up other names). One often sees examples of it such as

$$\sqrt{x + y} = \sqrt{x} + \sqrt{y} \qquad \text{and} \qquad (x + y)^2 = x^2 + y^2.$$

While it is definitely false, there is something here that can be rescued.

Notice that if we take $x, y \in \mathbb{Z}$ and let $n = 2$, then

$$(x+y)^2 = x^2 + 2xy + y^2$$

and so if we look at everything modulo 2 we get

$$(x+y)^2 \equiv (x^2 + 2xy + y^2) \equiv (x^2 + y^2) \pmod 2.$$

Similarly, with $n = 3$ we have

$$(x+y)^3 \equiv (x^3 + 3x^2y + 3xy^2 + y^3) \equiv (x^3 + y^3) \pmod 3.$$

Indeed, one can show that for any prime number p, and integers x, y we have

$$(x+y)^p \equiv x^p + y^p \pmod p.$$

Notice that this is not true for non-prime powers:

$$(1+3)^4 = 4^4 = 256 \equiv 0 \pmod 4$$

while

$$1^4 + 3^4 = 82 \equiv 2 \pmod 4.$$

(a) Use the recurrence in Exercise 7.3.21 (see Pascal's identity), together with the fact that $\binom{n}{0} = \binom{n}{n} = 1$, to prove that the binomial coefficients $\binom{n}{k}$ are integers.

(b) Prove that for prime p and integer $0 < k < p$ the binomial coefficient $\binom{p}{k}$ is a multiple of p, and so

$$\binom{p}{k} \equiv 0 \pmod p \qquad \text{for } 0 < k < p.$$

(c) Then using this and the Binomial Theorem (see Exercise 7.3.21) prove the result

$$(x+y)^p \equiv x^p + y^p \pmod p.$$

Chapter 10

Functions

One of the reasons to introduce relations is because they are nice intermediate mathematical object between sets (which have very little additional structure) and functions (which have quite a lot of structure). Indeed the idea of relations allows us to escape from the idea of a function as being a formula. Arguably, the usual high-school mathematics curriculum (especially the last couple of years of it) is really driving us towards being able to do calculus. And in calculus all the functions we look at are nice formulas that build up more complicated functions by doing arithmetic on simpler functions. At their core, these functions are really very algorithmic:

- Give me an input number x
- I do some arithmetic on x, and maybe look up some values (in a table or via a calculator or computer[126]) of things like sine, or logarithms.
- Then I return to you numerical result y.

Of course to be a function, this procedure has to be well defined — if you give me one input then I return to you one output. And if you give me the same input twice then I'd better return the same output each time.

10.1 Functions

So let's try to strip the idea of a function back as far as we can to make it both simpler and more general. First of all, we should escape from the idea that functions are restricted to have numbers are inputs and outputs; those of you with some programming experience will find this quite natural.

The following is a perfectly good function:

[126]Since doing complicated mathematical computations by hand can be very laborious, people who needed the results of those computations would hire people to do those computations for them. These people were called **computers**.

- Give me the name of a day of the week[127].
- I return to you the first letter of that word.

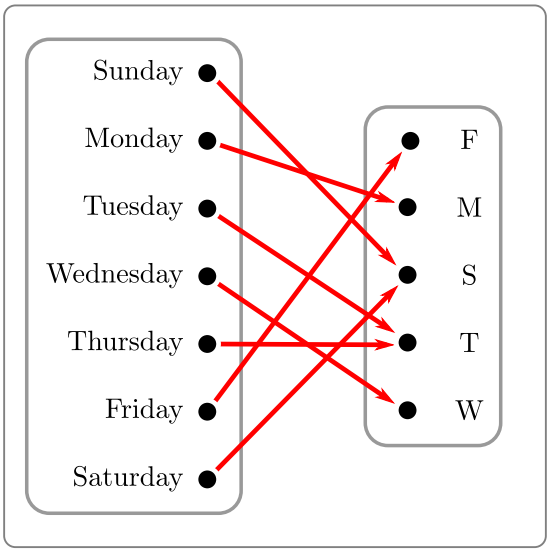
Now this is still quite algorithmic, but we have escaped from the tyranny of numbers — small steps first. The function takes inputs from the set

$$A = \{\text{Sunday}, \text{Monday}, \text{Tuesday}, \text{Wednesday}, \text{Thursday}, \text{Friday}, \text{Saturday}\}$$

and maps them to outputs in the set

$$B = \{F, M, S, T, W\}$$

We can summarise what is happening here by drawing a diagram that illustrates the inputs and outputs.



This is fine when the set of inputs is small, but will clearly become more and more painful as the set of inputs becomes larger. It is also still a quite algorithmic way of thinking about functions, but we can reduce the idea down further:

> Take an element of set A and do *something* to it to get an element of set B.

So we can summarise the function above by the pairs of inputs and outputs:

$$\Big\{(\text{Sunday}, S),(\text{Monday}, M), (\text{Tuesday}, T), (\text{Wednesday}, W), (\text{Thursday}, T), (\text{Friday}, F), (\text{Saturday}, S)\Big\}$$

So the set of ordered pairs of inputs and outputs is a subset of $A \times B$ — so it is a relation. However, its not *just any* relation — that is too general. We have extra conditions that a relation must satisfy in order to be a function.

[127] Written in correct — ie. Australian — English. The author might be biased in this assessment.

- We know that a function can only give one output for a given input. That is, if $f(a) = b_1$ and $f(a) = b_2$ then we must have $b_1 = b_2$. We can also express this in terms of relations:

$$((a, b_1) \in R \wedge (a, b_2) \in R) \implies b_1 = b_2$$

- Also everything in the input set has to have an output. That is

$$\forall a \in A, \exists b \in B \text{ so that } f(a) = b.$$

  Notice that this does *not* say that we have to reach *everything* in the output set — rather it just says that every input has to be "legal"; it results in some element in the output set.

Armed with these ideas we can define our new and more abstract idea of a function.

10.2 A more abstract definition

Definition 10.2.1 Let A, B be non-empty sets.

- A **function** from A to B, written $f : A \to B$ is a non-empty subset of $A \times B$ with two further properties
  - for every $a \in A$ there is some $b \in B$ so that $(a, b) \in f$.
  - if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$.
- In this context we call the set A the **domain** of f, and the set B is called the **co-domain**.
- If $(a, b) \in f$ then we write $f(a) = b$, and we call b the image of a. We also sometimes say that f maps a to b. With this notation the above two conditions are written as
  - for every $a \in A$ there is some $b \in B$ so that $f(a) = b$.
  - if $f(a) = b$ and $f(a) = c$ then $b = c$.
- We can further refine the co-domain to be exactly the set of elements of B that are mapped to by something in A; this set is called the **range**:

$$\text{rng} f = \{b \in B \mid \exists a \in A \text{ s.t. } f(a) = b\}$$

$\Diamond$

Some things to note

- The condition that if $f(a) = b$ and $f(a) = c$ then $b = c$ just means that the function is well defined. If we apply the function to a given element

then we only get 1 result. You might know this from high-school as the "vertical line test" — ie a graph-sketch corresponds to a function provided every vertical line intersects the graph once or not at all.

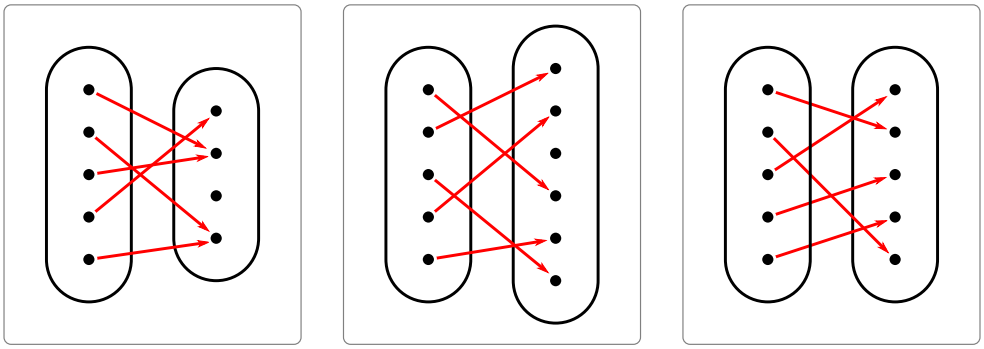
- Two functions $f, g : A \to B$ are equal when $f(a) = g(a)$ for all $a \in A$.
- The range is not (necessarily) the same as the co-domain. The range is always a subset of the co-domain. For example

$$f : \mathbb{R} \to \mathbb{R} \qquad\qquad f(x) = x^2$$

has domain $\mathbb{R}$, co-domain $\mathbb{R}$ and range $[0, \infty)$. We make the distinction between the two because sometimes it is really hard to write the range, but it is usually very easy to write the co-domain. We must be able to apply f to every single $a \in A$, but we don't have to arrive at every $b \in B$.

The term "function" only entered mathematics around the 17th century with Leibniz concurrent with the development of calculus and analytical geometry (such as the study of curves in the plane). Before this notions of dependent and independent variables that we are used to when studying $y = f(x)$ were not so well formalised. Arguably there was some work in this direction around the 12th-14th centuries by mathematicians such as Sharaf al-Din al-Tusi (who developed systematic method for numerical approximation of the roots of cubic polynomials) and Nicole Oresme (who first proved that the harmonic series diverges). Around the 19th century developments in mathematics required more general notions of functions and mathematicians, such as Dirichlet, Dedekind and Cantor, pushed away from the notion of a function as a formula and towards more general definitions such as the one above.

Once we have this more general idea of a function, we need ways to represent it. A very natural way (once Descartes introduced it — though it was also developed concurrently by Fermat and much earlier by Oresme) is to plot the function as a curve in the plane. Each point on the curve represents a pair of coordinates (x, y) so that $y = f(x)$. With this more general idea of a function we might try drawing something like we did above for the days-of-the-week example. That is, we might draw two sets of elements and edges between them to indicate that the function applied to that element in the first set gives the corresponding element in the second.



We could also just explicitly write out the mapping. So, for example, if the sets in the first of these functions are

$$A = \{1, 2, 3, 4, 5\} \qquad B = \{a, b, c, d\}$$

then the function can be written as

$$f = \{(1, b), (2, d), (3, b), (4, a), (5, d)\}$$

Both these approaches are reasonably practical when the domain and co-domain of the function are small, but is really not going to work as soon as they get a little bigger.

Example 10.2.2 Consider the sets

$$f = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 3x + 2y = 0\}$$
$$g = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid 3x + y = 0\}.$$

Both are subsets of $\mathbb{Z} \times \mathbb{Z}$ and so are (by definition) relations on $\mathbb{Z}$. Only one is a function from $\mathbb{Z}$ to $\mathbb{Z}$.

In order for f to be a function it has to satisfy two conditions:

- For every x in the domain, there must be a y in the co-domain so that $f(x) = y$, and
- If $f(x) = y$ and $f(x) = z$ then $y = z$.

The first of these fails, since if we set $x = 1$, then there is no integer y so that $3 + 2y = 0$. Indeed we require $y = -\frac{3}{2}$. The second condition is satisfied, because if $f(x) = y$ and $f(x) = z$, then we know that

$$3x + 2y = 0 \qquad \text{and} \qquad 3x + 2z = 0$$

then subtract one equation from the other to get

$$2y - 2z = 0$$

and so $y = z$.

The relation g satisfies both conditions:

- Let x be any integer, then we can set $y = -3x \in \mathbb{Z}$ and $3x + y = 0$ as required.
- The second condition is satisfied by the same argument we used for f.

The co-domain of g is the set of integers, but its range $\{n \in \mathbb{Z} \text{ so that } 3|n\}$. □

10.3 Images and preimages of sets

When we defined the function $f : A \to B$, we said that if $f(a) = b$ then we called b the image of a under f. This idea can be extended quite naturally to think of the image of a set of points. Also, given an element $b \in B$, we can ask for all the elements of A that map to it. This latter idea is not quite the inverse function, but it is getting close to it.

We should define these sets more precisely:

Definition 10.3.1 Image and preimage. Let $f : A \to B$ be a function, and let $C \subseteq A$ and let $D \subseteq B$.

- The set $f(C) = \{f(x) : x \in C\}$ is the **image of C in B**.
- The set $f^{-1}(D) = \{x \in A : f(x) \in D\}$ is the **preimage of D in A** or **f-inverse of D**.

$\Diamond$

Remark 10.3.2 Preimage of a single element. Note that the preimage of a set containing a single element of B is a (possibly) set of elements of A. For example, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then

$$f^{-1}(\{0\}) = \{0\} \qquad f^{-1}(\{1\}) = \{-1, 1\} \qquad f^{-1}(\{-1\}) = \varnothing.$$

This shows that the preimage of a set containing a single element is a set that may contain zero, one, two or even more elements. Indeed, it is not hard to construct an example in which the preimage contains infinitely many elements. When our function satisfies very specific conditions, we can ensure that the preimage of a set containing a single element is always set containing a single element. Understanding those conditions is one of the main aims of this chapter and we'll discuss it in detail in the next section. That, in turn, will help us to define the **inverse function**.

You will have noticed that in the preceding paragraph we have had to write "the preimage of a set containing a single element" several times. This becomes quite cumbersome. We will, from here, abuse the definition of the preimage a little to simplify our writing. In particular, we will often write "the preimage of an element x" to mean "the preimage of the set $\{x\}$". While this is modestly incorrect, it does make the writing and reading easier.

The notation for preimage, f^{-1}, is somewhat unfortunate in that we use the same notation to mean the inverse-function. Additionally, it is regularly confused with

$$(f(x))^{-1} = \frac{1}{f(x)}$$

ie the reciprocal. Alas, we are stuck with this notation and must be careful to understand its meaning by context.

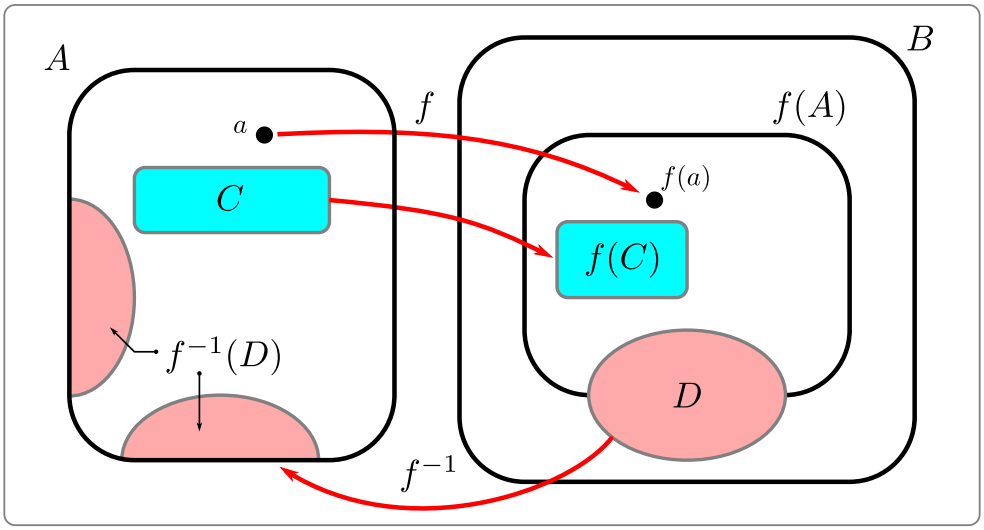
This is an important point, so we'll make it a formal warning:

Warning 10.3.3 Be careful with preimages:

- The preimage f^{-1} is *not* the inverse function.
- If certain special conditions are satisfied, then the inverse function exists and we use the same notation to denote that function.

Consequently, when you see f^{-1} you should think "preimage" and not "inverse function" unless we specifically know that the inverse exists.

After all those warnings and caveats, lets draw a schematic of images and preimages:



Notice in this figure that

- we have drawn $f(A)$ as a subset of B — in fact $f(A)$ is exactly the range of f and so must be a subset of B.
- we have drawn the preimage of D so that it looks like two copies of half of the set D — this is to emphasise the fact that not every element of B has to have a preimage in A. Further, a given point in B might have more than one preimage.

Our quick look at preimages of $f(x) = x^2$ above illustrated this second point. That was a little brief, but the following example looks at this in more detail.

Example 10.3.4 Images and preimages. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Find the following

(a) $f([0, 4])$

(b) $f([-3, -1] \cup [1, 2])$

(c) $f^{-1}(\{0\})$

(d) $f^{-1}(\{1\})$

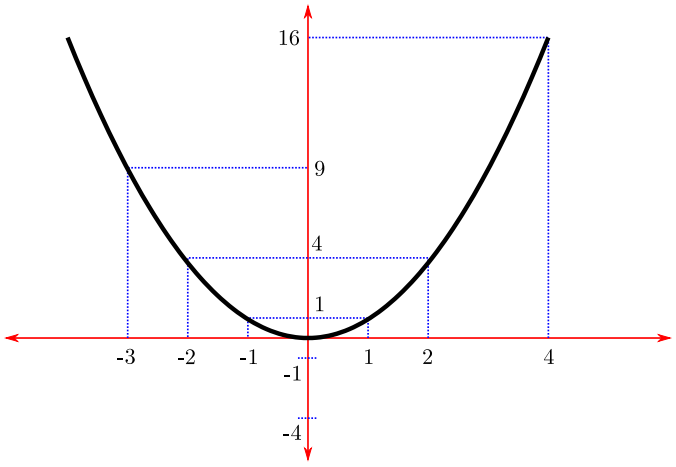
(e) $f^{-1}([0, 4]) = [-2, 2]$

(f) $f^{-1}([1,4]) = [-2,-1] \cup [1,2]$

(g) $f^{-1}(\{-1\})$

(h) $f^{-1}([-4,-1])$

Solution. It will help to make a quick sketch of $y = f(x) = x^2$ and think about the various intervals in the domain and co-domain. That $f(x)$ is decreasing for $x < 0$ and increasing for $x > 0$ does make this exercise a little easier.



(a) The interval $0 \le x \le 4$ maps to $0 \le x^2 \le 16$. So $f([0,4]) = [0,16]$

(b) The interval $-3 \le x \le -1$ maps to $1 \le x^2 \le 9$, and the interval $1 \le x \le 2$ maps to $1 \le x^2 \le 4$. Hence the points in the union $[-3,-1] \cup [1,2]$ map to the interval $[1,9] \cup [1,4] = [1,9]$. Hence $f([-3,-1] \cup [1,2]) = [1,9]$.

(c) To find the preimage of $\{0\}$, we need to solve $f(x) = x^2 = 0$. This only has a single solution, namely $x = 0$, and so $f^{-1}(\{0\}) = \{0\}$

(d) To find the preimage of $\{1\}$, we need to solve $f(x) = x^2 = 1$. This only two solutions, namely $x = \pm 1$, and so $f^{-1}(\{1\}) = \{-1, 1\}$

(e) The interval $0 \le x^2 \le 4$ is mapped to by any number in the interval $-2 \le x \le 2$. So $f^{-1}([0,4]) = [-2,2]$.

We can check this by looking at the above plot, but also by considering $f([-2,0] \cup [0,2])$. The interval $0 \le x \le 2$ maps to $0 \le x^2 \le 4$ and $-2 \le x \le 0$ maps to the same, $0 \le x^2 \le 4$.

(f) The interval $1 \le x^2 \le 4$ is mapped to by any number in the interval $1 \le x \le 2$ or any number in $-2 \le x \le -1$. So $f^{-1}([1,4]) = [-2,-1] \cup [1,2]$.

Again, we can check this by looking at the above plot, but also by considering $f([-2,-1] \cup [1,2])$. The interval $1 \le x \le 2$ mapes to $1 \le x^2 \le 4$, and $-2 \le x \le -1$ maps to the same.

(g) To find the preimage of $\{-1\}$, we need to solve $f(x) = x^2 = -1$. This has no solutions[128], so $f^{-1}(\{-1\}) = \varnothing$

(h) To find the preimage of $-4 \leq x^2 \leq -1$, we should recall that the square of any real number is non-negative. So there are no values of x that map into that interval. Thus $f^{-1}([-4, -1]) = \varnothing$.

□

This example shows that the preimage of a single element[129] in the co-domain can be empty, or can contain a single element, or can contain multiple elements. As noted above, we want to understand what conditions we can impose on a function so that the preimage of a single point[130] in the co-domain always contains exactly one point in the domain. This will allow us to properly define inverse functions — that is if $f(x) = y$ then how do we define a new function g so that $g(y) = x$.

Before we get to inverses we can do some more exploring of images and preimages. Since these are really operations on sets, we can (and should) ask ourselves how do these new things we can do to sets interact with the other things we can do to sets. So we now explore some of the relationships between subsets and their images and preimages, and also the interplay between functions, unions, intersections and differences.

For example — it is clearly [131] the case that if $C_1 \subseteq C_2 \subseteq A$ then $f(C_1) \subseteq f(C_2)$. Similarly if $D_1 \subseteq D_2 \subseteq B$ then $f^{-1}(D_1) \subseteq f^{-1}(D_2)$. While we've said "clearly", we should really state results carefully and make them a result and prove them. We'll follow this up with a more important result which we'll call a theorem.

Result 10.3.5 *Let $f : A \to B$, and let $C_1 \subseteq C_2 \subseteq A$ and $D_1 \subseteq D_2 \subseteq B$. Then*

$$f(C_1) \subseteq f(C_2) \qquad \text{and} \qquad f^{-1}(D_1) \subseteq f^{-1}(D_2).$$

Proof. We prove each inclusion in turn.

- Let $b \in f(C_1)$. Then (by the definition of image) there is some $a \in C_1$ so that $f(a) = b$. But since $C_1 \subseteq C_2$, we know that $a \in C_2$. Hence $b = f(a) \in f(C_2)$ as required.

- Let $a \in f^{-1}(D_1)$. Then (by definition of preimage) $f(a) \in D_1$. But since $D_1 \subseteq D_2$, we know $f(a) \in D_2$. Then, by the definition of preimage, we know that $a \in f^{-1}(D_2)$.

■

[130]To be more precise, it has no solutions over the reals, which is the domain of the function.

[129]Recall that we are abusing the definition of preimage here; we really mean "the preimage of a set containing a single element".

[130]Another similar abuse of the definition of preimage in order to keep the language flowing.

[131]This is always a dangerous word when writing mathematics, and the authors include it here as an example of what, perhaps, one should not do. One person's "clearly" is another persons "3 hours of confusion".

Now, while the above proof is not terribly technical, it does require us to know the definitions of image and preimage and to understand how to manipulate them. Even though the statement we have just proved is (arguably) obvious[132], its proof is not so trivial.

Theorem 10.3.6 *Let $f : A \to B$, and let $C \subseteq A$ and $D \subseteq B$. Further, let C_1, C_2 be subsets of A and let D_1, D_2 be subsets of B. The following are true*

(i) $C \subseteq f^{-1}(f(C))$

(ii) $f(f^{-1}(D)) \subseteq D$

(iii) $f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$ *— note: need not be equal*

(iv) $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$ *— note: are equal*

(v) $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$

(vi) $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$

So what does this theorem tell us? It says that preimages play very nicely with set operations — they are well-behaved:

- The preimage of the intersection is the intersection of the preimages
- The preimage of the union is the union of the preimages

It also tells us that images are *mostly* well-behaved:

- The image of the union is the union of the images
- The image of the intersection is *a subset of* the intersection of the images.

Of course, we should prove these results. We'll do some in the text and leave some of them as exercises. We'll prove (iii) first and then (vi) and leave (i) until last. In the authors' experience, people find (i) quite confusing, so we will tackle it after we've warmed up on the other two.

Proof.

- Proof of (iii):

  We need to show that if an element is in the set on the left then it is in the set on the right. Let $b \in f(C_1 \cap C_2)$. Hence there is $a \in C_1 \cap C_2$ such that $f(a) = b$. This means that $a \in C_1$ and $a \in C_2$. It follows that $f(a) = b \in f(C_1)$ and $f(a) = b \in f(C_2)$, and hence $b \in f(C_1) \cap f(C_2)$.

  The converse is false: $f(C_1) \cap f(C_2) \not\subseteq f(C_1 \cap C_2)$ — another good exercise

[132] "Obvious" is another dangerous word, and is a good example of **emotive conjugation**. "I found it obvious" but "they spent 4 days trying to work out what was going on". This sort of thing often arises in the descriptions that people give their own actions compared with the descriptions of others. The interested reader should search-engine their way to the BBC series "Yes Prime Minister" which has the following example: "I give confidential press briefings; you leak; he's being charged under section 2A of the Official Secrets Act."

for the reader.

- Proof of (vi):

  Let $a \in f^{-1}(D_1 \cup D_2)$ and so $f(a) \in D_1 \cup D_2$. This means that $f(a) \in D_1$ or $f(a) \in D_2$. If $f(a) \in D_1$ it follows that $a \in f^{-1}(D_1)$. Similarly if $f(a) \in D_2$ then $a \in f^{-1}(D_2)$. Since a lies in one of these two sets, it follows that $a \in f^{-1}(D_1) \cup f^{-1}(D_2)$.

  Let $a \in f^{-1}(D_1) \cup f^{-1}(D_2)$. Then a is an element of one of these two sets. If $a \in f^{-1}(D_1)$, then $f(a) \in D_1$. Similarly if $a \in f^{-1}(D_2)$ then $f(a) \in D_2$. In either case $f(a) \in D_1 \cup D_2$ and so $a \in f^{-1}(D_1 \cup D_2)$.

- Proof of (i):

  Let $x \in C$. We need to show that $x \in f^{-1}(f(C))$. So what is this set — by the definition it is $\{a \in A : f(a) \in f(C)\}$. Since $x \in C$ we have, by definition, $f(x) \in f(C)$. Since $f(x) \in f(C)$ it follows that $x \in f^{-1}(f(C))$.

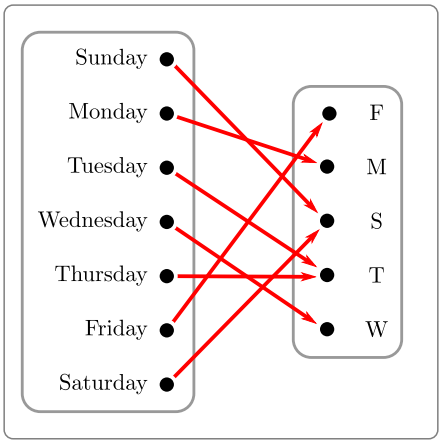
  Note that the converse of this statement is false: $f^{-1}(f(C)) \not\subseteq C$. This makes a good exercise. We also note that (ii) follows a similarly flavoured argument and is another good exercise for the reader.

■

10.4 Injective and surjective functions

We typically think of a function as taking objects from one set, A, doing "stuff" and turning them into elements from another set B. With this in mind, it is quite natural to ask whether or not we can reverse this process; take our result and turn it back into our original object. That is, "when is a given function invertible".

If you think back to our days-of-the-week example:



You can see that if we have arrived at the letter "M" then it is easy to determine that we started at "Monday" — so easy to undo the function. However, if we have arrived at "S" then things are not so simple — we could have started with

either "Sunday" or "Saturday". Obviously this is related to the preimage that we saw before. The good case of "M" worked because the preimage[133] had 1 element in it, "Monday". The bad case of "S" didn't work because the preimage[134] had 2 elements in it. More generally the preimage could have any number of elements in it — including zero.

Now, for us to be able to sensibly undo our function, we need the preimage of every element in our co-domain to have exactly one element in it. To be more precise, for any element y in our codomain, there must exist some x in the domain so that the preimage of $\{y\}$ is the set $\{x\}$. If you think about this a little, this means that the domain and co-domain must have the same size[135]. We'll discuss this more soon. But it should be clear from this discussion that not every function can be undone, and that those that are "undoable" have to satisfy special properties. This brings us to a couple of important definitions.

To get to this we need to define some simple properties that functions can have.

10.4.1 Injections and surjections

Definition 10.4.1 Let $a_1, a_2 \in A$ and let $f : A \to B$ be a function. We say that f is injective or one-to-one when

$$\text{if } a_1 \neq a_2 \text{ then } f(a_1) \neq f(a_2).$$

It is helpful to also write the contrapositive of this condition. We say that f is injective or one-to-one when

$$\text{if } f(a_1) = f(a_2) \text{ then } a_1 = a_2.$$

$\Diamond$

Things to note

- The term injective is, in this author's opinion, better to use than one-to-one. When we speak (and write) we are sometimes quite sloppy with our use of prepositions like "to" or "on" or "in" or "onto", so we might accidentally say one-onto-one, for example. The term injective is a nice latin-flavoured word that makes the speaker / writer sound more authoritative [136] .

[133] To be more precise, the preimage of the set {M} is the set {Monday}.

[134] Again, being more precise, the preimage of the set {S} is the set {Saturday, Sunday}.

[135] This is perhaps no so hard to see when everything is finite, but harder to see when things are infinite. Indeed, we'll see examples of just how weird this can be when we get to cardinality later in the text.

[136] Of course that doesn't mean the speaker knows what they are talking about, just that they sound like it. This is a variation of the "argument from authority" fallacy. Mind you there is the equally, or perhaps even more pernicious, fallacy that because a statement comes from an authority it should be mistrusted. Perhaps an example of an "appeal to common folk" fallacy. "Experts — bah! What would they know?!"

- A very common mistake made with this definition is to get the implications around the wrong way — to give the converse of what is required:
  - The right way — $f(a_1) = f(a_2) \implies a_1 = a_2$. Injective!
  - The wrong way — $a_1 = a_2 \implies f(a_1) = f(a_2)$ — this is just "same input implies same output" which just says the function is *well defined*.

  Be careful of this.
- So when a function is injective, different elements map to different elements.
- When a function is not injective there must be at least one pair $a_1, a_2 \in A$ so that $a_1 \neq a_2 \in A$ but $f(a_1) = f(a_2)$.

As a preview of what is to come when we reach the chapter on cardinality, think about what we can say about the sizes of *finite* sets A, B if there is an injective function between them $f : A \to B$. Each element $a \in A$ has to map to a *different* element $b \in B$. Consequently the set B must have at least as many elements as A. That is $|A| \leq |B|$. We note that when A, B are infinite, these sorts of questions become much less obvious.

Result 10.4.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 7x - 3$. Then f is injective.*

So — how do we prove this. We have two equivalent conditions

- $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$
- $f(a_1) = f(a_2) \implies a_1 = a_2$.

Arguably, the second is easier since we have all had much more practice manipulating equalities than manipulating inequalities. So lets run through the argument. Let $a_1, a_2 \in \mathbb{R}$, and assume that

$$\begin{aligned} f(a_1) &= f(a_2) && \text{then we must have} \\ 7a_1 - 3 &= 7a_2 - 3 && \text{and so} \\ 7a_1 &= 7a_2 && \text{and hence} \\ a_1 &= a_2. \end{aligned}$$

That wasn't too bad. Time to write it up as a proof.

Proof of Result 10.4.2. Let $a_1, a_2 \in \mathbb{R}$ and assume $f(a_1) = f(a_2)$. Then $7a_1 - 3 = 7a_2 - 3$ and so $7a_1 = 7a_2$ and thus $a_1 = a_2$. Hence f is injective. ■

Now what about an example that is not injective — we can recycle our x^2 example from above.

Result 10.4.3 *The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is not injective.*

Injective means

$$\forall a_1, a_2 \in A, f(a_1) = f(a_2) \implies a_1 = a_2$$

and hence non-injective is just the negation of this, namely

$$\exists a_1, a_2 \in A \text{ s.t. } f(a_1) = f(a_2) \wedge a_1 \neq a_2.$$

So we need a counter-example; there exists some pair of distinct $a_1, a_2 \in \mathbb{R}$ that map to the same value. Of course, we could have started with the equivalent contrapositive definition of injective:

$$\forall a_1, a_2 \in A, a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$$

The negation of that is

$$\exists a_1, a_2 \in A \text{ s.t. } a_2 \neq a_2 \implies f(a_1) = f(a_2)$$

which is the same as we found with the first definition of injective. Of course, this must be the case because the two definitions are equivalent.
Proof of Result 10.4.3. Since $-1, 1$ are in the domain of f and $f(-1) = f(1) = 1$, the function is not injective. ■

Remark 10.4.4 Preimages and injections. Consider an injection $f : A \to B$ and, for any given $b \in B$, the preimage of $\{b\}$:

$$f^{-1}(\{b\}) = \{a \in A \mid f(a) = b\}$$

This set is either empty or contains exactly 1 element. To see this consider $a, c \in f^{-1}(\{b\})$. By definition of the preimage, we know that $f(a) = b$ and $f(c) = b$. Since f is injective, this tells us that $a = c$. So the preimage of a point under an injection contains at most 1 element.

Another important class of functions are surjections.

Definition 10.4.5 Let $f : A \to B$ be a function. We say that f is surjective, or onto, when for every $b \in B$ there is some $a \in A$ such that $f(a) = b$. ♢

Things to note

- This simply means that every element in B is mapped to by some element of A.
- If the function is not surjective then there is some $b \in B$ such that for all $a \in A$, $f(a) \neq b$.
- Again, this author prefers the nice latin "surjective" over the term "onto" because it is less likely to confused with other prepositions.

Again, as a preview of cardinality, think about what we can say about the sizes of *finite* sets A, B if there is an surjection between them $g : A \to B$. Each element $b \in B$ must be mapped to by *at least one* element $a \in A$. Consequently the set A must have at least as many elements as B. That is $|A| \geq |B|$.

Result 10.4.6 *The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 7x - 3$ is surjective*

So we need to show that no matter which $y \in \mathbb{R}$ we can always find some $x \in \mathbb{R}$ such that $f(x) = 7x - 3 = y$. So we can just make x the subject giving $x = (y + 3)/7$.

Now that we have chosen x, we need to make sure it is actually an element of the domain of the function. In this case it is easy since $(y + 3)/7 \in \mathbb{R}$. However, if we consider a similarly function

$$g : \mathbb{Z} \to \mathbb{Z} \qquad g(x) = 7x - 3$$

we would also get $x = \frac{y+3}{7}$, but then x is not always in the domain. Time to write up.

Proof of Result 10.4.6. Let $y \in \mathbb{R}$. Choose $x = (y+3)/7$, then $f(x) = 7 \cdot \frac{y+3}{7} - 3 = y$. Hence f is surjective. ■

Notice that in the proof we *do not* have to explain to the reader how we found the choice of x. It is not necessary to work through solving $y = f(x)$ for x. All we need to do in the proof is tell the reader "Given y we choose this value of x" and then show that $f(x) = y$. This can often be a little frustrating for the reader who can be left thinking "How on earth did they get that?"; a good author might put in a little explanation in the text surrounding the proof, but it is not required for the proof to be valid.

Result 10.4.7 *The function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ is not surjective.*

Since surjective means

$$\forall b \in B, \exists a \in A \text{ s.t. } f(a) = b$$

its negation is

$$\exists b \in B \text{ s.t. } \forall a \in A, f(a) \neq b.$$

So in order to show that g is not surjective, we have to find (at least one) $b \in \mathbb{R}$ (in the co-domain) that is not the image of any $a \in \mathbb{R}$ (in the domain). Sometimes this can be tricky to prove, but for the example above we can use the fact that the square of a real is non-negative.

Proof of Result 10.4.7. Let $b = -1 \in \mathbb{R}$. Since the square of any real number is non-negative, we know that $f(x) = x^2 \geq 0$ for any $x \in \mathbb{R}$. Hence there is no $x \in \mathbb{R}$ so that $f(x) = -1$. Thus the function is not surjective. ■

Remark 10.4.8 Preimages and surjections. Consider a surjection $g : A \to B$ and, for any given $b \in B$, the preimage of $\{b\}$:

$$g^{-1}(\{b\}) = \{a \in A \mid g(a) = b\}$$

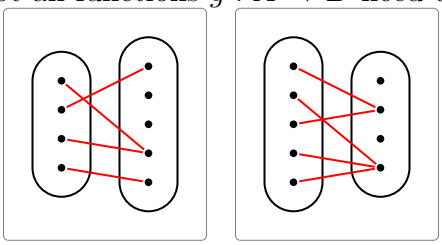
Since g is a surjection, for any given $b \in B$, there must be at least $a \in A$ so that $g(a) = b$. Hence the preimage must contain at least one element.

10.4.2 Bijective functions

Recall that we reasoned (but didn't really prove) that for *finite* sets A, B

- if there is an injection $f : A \to B$ then $|A| \leq |B|$, and
- if there is a surjection $g : A \to B$ then $|A| \geq |B|$

Warning 10.4.9 Be careful with your converses. Consider two finite sets A, B. If $|A| \leq |B|$ it does *not* mean that all functions $f : A \to B$ will be injections. Similarly if $|A| \geq |B|$ not all functions $g : A \to B$ need to surjections.



The function on the left is not injective (despite its domain being smaller than its co-domain). And the function on the right it not surjective (despite its domain being larger than its co-domain).

So given sets A, B if we can find such an injection and a surjection between them, then $|A| = |B|$. One way to do this is to find one function $h : A \to B$ that is *both* injective and surjective; these functions are called **bijections**. Finding a bijection between two sets is a good way to demonstrate that they have the same size — we'll do more on this in the chapter on cardinality.

Definition 10.4.10 Let $f : A \to B$ be a function. If f is injective and surjective then we say that f is **bijective**, or a **one-to-one correspondence**. ♢

The terms **injective**, **surjective** and **bijective** were coined by Nicholas Bourbaki. Bourbaki was not a person, but the pseudonym of a group of (mostly French) mathematicians who wrote a series of texts in the mid 20th century. The group still exists and published a book in 2016. The central aim of the group was to create a series of complete and self-contained texts on the core of mathematics. The texts strive to be extremely rigorous and very general in their treatment of the material and not without controversy. You can search engine your way to more information on this topic.

Example 10.4.11 Consider the following functions. Determine whether they are injective, surjective or bijective. They all have the same formula, but have different domains and co-domains (and so are different functions).

- $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$. Neither surjective, nor injective.
- $g : \mathbb{R} \to [0, \infty)$ given by $g(x) = x^2$. Surjective, but not injective.
- $h : [0, \infty) \to \mathbb{R}$ given by $h(x) = x^2$. Not surjective, but is injective.
- $\rho : [0, \infty) \to [0, \infty)$ given by $\rho(x) = x^2$. Is bijective.

Solution. All of these functions have the same formula, just different domains and co-domains.

- Think about how these functions can fail to be injective. We can verify that $f(-x) = (-x)^2 = x^2 = f(x)$. Hence these functions will fail to be injective if x and $-x$ are both within the domain. So neither f, g are injective since

$$f(-1) = f(1) = 1 \qquad \text{and} \qquad g(-1) = g(1) = 1$$

  We now prove that h is injective. So let $a_1, a_2 \geq 0$, so that $h(a_1) = h(a_2)$. Hence

$$a_1^2 = a_2^2$$

  Taking square-roots gives

$$a_1 = \pm a_2$$

  However, since neither a_1, a_2 are negative, we must have $a_1 = a_2$ and so h is an injection. The same argument works for ρ.

- Now think about how these functions might fail to be surjective. We know that the square of a real number is non-negative. That is $0 \leq x^2$. So if there is a negative number in the co-domain there is no real number that can map to it. Consequently, neither f nor h are surjections, since there is no $x \in \mathbb{R}$ so that

$$f(x) = -1 \qquad \text{or} \qquad h(x) = -1.$$

  We now prove that g is surjective using the square-root function. Given any y in the codomain of g, pick $x = \sqrt{y}$. Since $y \geq 0$, x is defined and non-negative and so in the domain of g. Further, we know that $g(x) = x^2 = (\sqrt{y})^2 = y$. Thus g is surjective. The argument for ρ is identical.

So to summarise

- f is neither injective, nor surjective,
- g is surjective but not injective,
- h is injective but not surjective, and
- ρ is both injective and surjective, and so bijective.

□

The simplest (useful) bijective function is the identity function.

Definition 10.4.12 Given a non-empty set A we define the identity function $i_A : A \to A$ by $i_A(a) = a$ for all $a \in A$. ◊

The authors are usually loath to use the word "clear", but we hope that it is clear that the identify function is surjective and injective and so bijective. We could prove it if we really had to. We will need the identity function to help us

define the inverse of a function.

We need a couple more examples.

Result 10.4.13 *The function $g : \mathbb{R} \to \mathbb{R}$, defined by $7x - 3$, is a bijection.*

Proof. We need to show that g is both injective and surjective.

- Injective: We proved this in Result 10.4.2.
- Surjective: We proved this in Result 10.4.6.

Since the function is both injective and surjective it is bijective. ■

A more interesting example. Let $a, b, c, d \in \mathbb{R}$ and define

$$h : \mathbb{R} - \left\{\frac{d}{c}\right\} \to \mathbb{R} - \left\{\frac{a}{c}\right\} \qquad h(x) = \frac{ax - b}{cx - d}$$

If the constants satisfy $ad \neq bc$, then this is a Möbius transformation[137] Notice that the denominator is zero when $cx = d$, and hence we have removed the point $x = \frac{d}{c}$ from the domain; the function is not defined there. Some similar reasoning can show that there is no $x \in \mathbb{R}$ so that $h(x) = \frac{a}{c}$, so we remove that point from the co-domain. Finally, notice that if $ad = bc$ then

$$\begin{aligned} h(x) &= \frac{ax - b}{cx - d} \\ &= \frac{cax - cb}{c^2x - cd} && \text{since } bc = ad \\ &= \frac{cax - ad}{c^2x - cd} = \frac{a(cx - d)}{c(cx - d)} \\ &= \frac{a}{c} \end{aligned}$$

and so is constant.

Möbius transforms are a good source of non-trivial bijective function examples for authors to give to students. So let us just do this in full generality.

Result 10.4.14 *Let $a, b, c, d \in \mathbb{R}$ with $ad \neq bc$. The function $h : \mathbb{R} - \left\{\frac{d}{c}\right\} \to \mathbb{R} - \left\{\frac{a}{c}\right\}$ defined by $h(x) = \frac{ax-b}{cx-d}$ is bijective.*

Scratch work.

- Injective. Let $x, y \in \mathbb{R} - \left\{\frac{d}{c}\right\}$ and assume $h(x) = h(y)$. Then

$$\frac{ax - b}{cx - d} = \frac{ay - b}{cy - d}$$

[137] Well, we should really take x over complex numbers, but the interested reader should search-engine their way to more on this topic. They are named for August Möbius who was 19th century German mathematician and astronomer. He is perhaps best known for discovering the Möbius strip which is an two-dimensional surface with only 1 side. The Möbius strip was actually discovered slightly earlier by Johann Listing; perhaps "Listing strip" doesn't have quite the same ring to it (sorry for the poor pun).

$$\begin{aligned} (cy-d)(ax-b) &= (ay-b)(cx-d) && \text{since denominator } \neq 0 \\ caxy - cyb - adx + db &= acxy - ady - bcx + bd \\ (ad-bc)y &= (ad-bc)x && \text{so we are done} \end{aligned}$$

- Surjective. Let $y = h(x)$, now find x

$$\begin{aligned} y &= \frac{ax-b}{cx-d} \\ cxy - dy &= ax - b \\ cxy - ax &= dy - b \\ x(cy-a) &= dy - b \\ x &= \frac{dy-b}{cy-a} \end{aligned}$$

  We now need to show that $x \neq \frac{d}{c}$ (and so is in the domain), and we can do so by considering

$$\begin{aligned} x - \frac{d}{c} &= \frac{dy-b}{cy-a} - \frac{d}{c} \\ &= \frac{dcy - bc - dcy + da}{c(cy-a)} \\ &= \frac{ad-bc}{c(cy-a)} \end{aligned}$$

  Now since $ad \neq bc$ and $y \neq \frac{a}{c}$, this is never zero. Hence this choice of x really does lie in the domain of the function.

Proof of Result 10.4.14. We need to show that h is both injective and surjective.

- Let $y \in \mathbb{R} - \frac{a}{c}$. Pick $x = \frac{dy-b}{cy-a} = \frac{d}{c} + \frac{ad-bc}{c(cy-a)}$. Since $\frac{ad-bc}{c(cy-a)} \neq 0$ for any $y \in \mathbb{R} - \left\{\frac{a}{c}\right\}$, this choice of x is in the domain of the function. Now

$$\begin{aligned} h(x) &= \frac{a\frac{dy-b}{cy-a} - b}{c\frac{dy-b}{cy-a} - d} \\ &= \frac{a(dy-b) - b(cy-a)}{c(dy-b) - d(cy-a)} \\ &= \frac{y(ad-bc)}{ad-bc} = y \end{aligned}$$

  Hence for any y in the co-domain, there is an x in the range such that $h(x) = y$. So the function is surjective.

- Now let x, y be two elements of the range such that $h(x) = h(y)$. Hence

$$\frac{ax-b}{cx-d} = \frac{ay-b}{cy-d}$$

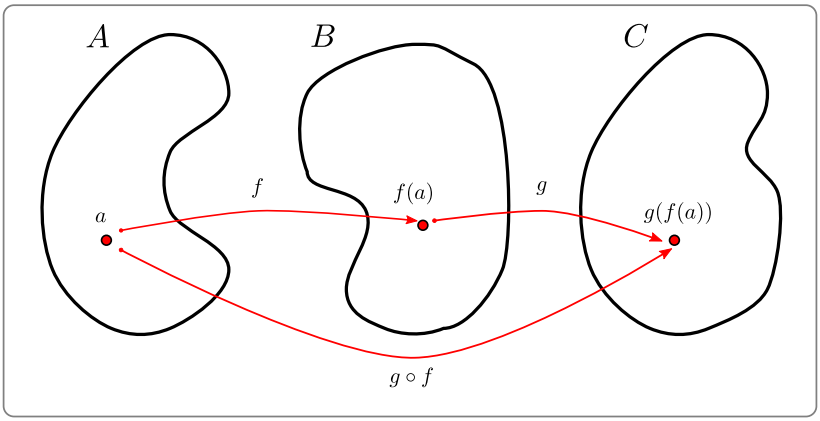
$$\begin{aligned}(cy-d)(ax-b) &= (ay-b)(cx-d) && \text{since denominator} \neq 0\\ cax-cby-dax+db &= acxy-day-bcx+db\\ (ad-bc)y &= (ad-bc)x\end{aligned}$$

Hence $x = y$. Thus if $h(x) = h(y)$ we must have $x = y$ and so h is injective.

Since the function is both injective and surjective, it is bijective as required. ■

10.5 Composition of functions

Our last step before defining inverse functions is to explain a generic way of combining functions. Now since we are dealing with general sets and not just sets of numbers, we can't add or subtract or divide or differentiate or any of the things we usually do with functions in calculus courses. Really the only thing we can with our abstract functions defined on abstract sets is **composition** — that is, take an element, apply the first function and then apply the second function.



Definition 10.5.1 Let $f : A \to B$ and $g : B \to C$ be two functions. The composition of f and g is denoted $g \circ f$ which we read "g of f". It defines a new function

$$g \circ f : A \to C \qquad (g \circ f)(a) = g(f(a)) \qquad \forall a \in A$$

◊

This standard notation for compositions looks a bit backwards — we read things from left to right, but when we actually evaluate the composition we do the rightmost function first.

$$(h \circ g \circ f)(x) = h(g(f(x)))$$

We start with x, apply f, then apply g to the result, and finally apply h to the result of that.

It is important to note that composition is not, in general, commutative:

$$g \circ f \neq f \circ g.$$

But it is associative

Theorem 10.5.2 *Let $f : A \to B$, $g : B \to C$ and $h : C \to D$ be functions. Then $h \circ (g \circ f) = (h \circ g) \circ f$.*

The proof of associativity is not difficult but is quite long; we leave it to the interested reader to work through it. Instead we'll look at the following useful (and nice) theorem. This tells us that composition of functions interacts nicely with injections, surjections and bijections.

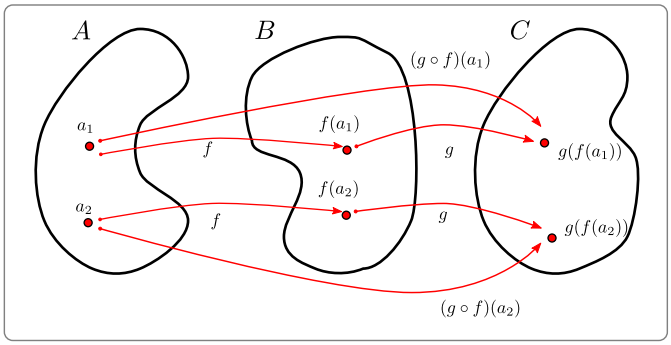
Theorem 10.5.3 *Let $f : A \to B$ and $g : B \to C$ be functions.*

- *If f and g are injective then so is $g \circ f$.*
- *If f and g are surjective then so is $g \circ f$.*

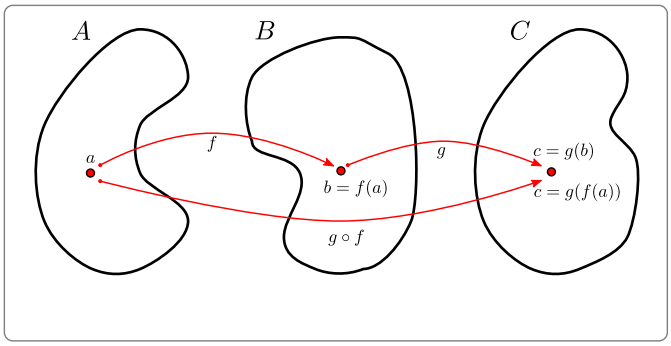
Consequently if f, g are bijective then so is $g \circ f$.

Scratch work:

- Injective: Assume both f and g are injective. So if $a_1 \neq a_2 \in A$ then $f(a_1) \neq f(a_2)$. Similarly if $b_1 \neq b_2$ then $g(b_1) \neq g(b_2)$. We can just put these together. If $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$ and so $g(f(a_1)) \neq g(f(a_2))$. Thus $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. The diagram below should help.



- Surjective: Assume both f and g are surjective. Then for each $c \in C$ there is some $b \in B$ such that $g(b) = c$. Similarly for each $b \in B$ there is some $a \in A$ such that $f(a) = b$. Thus $g(f(a)) = c$. Again, we refer the reader to the diagram below.



- Bijectiveness follows from surjectiveness and injectiveness. That is, if f, g are both bijective, then they are both injective and surjective. Hence their composition $g \circ f$ is injective and surjective, and so bijective.

We are ready to write things up nicely for the reader.
Proof of Theorem 10.5.3. It suffices to prove the first two points, since the final point follows immediately from them.

- Let f, g be injective functions. And let $a_1, a_2 \in A$ such that $a_1 \neq a_2$. Since f is injective, it follows that $f(a_1) \neq f(a_2)$. And since g is injective it followst that $g(f(a_1)) \neq g(f(a_2))$. Thus $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Thus $g \circ f$ is injective.

- Let f, g be surjective functions and let $c \in C$. Since g is surjective, $g(b) = c$ for some $b \in B$. Since f is surjective, there is some $a \in A$ such that $f(a) = b$. It follows that $g(f(a)) = c$ and so $g \circ f$ is surjective.

■

Here is another nice result about compositions, injections and surjections. In particular, if we know the composition $g \circ f$ is injective, then what can we say about f, g. Similarly, if $g \circ f$ is surjective, then what can we say about f, g. Similarly

Theorem 10.5.4 *Let $f : A \to B$ and $g : B \to C$ be functions. Then*

- *If $g \circ f$ is injective then f is injective*

- *If $g \circ f$ is surjective then g is surjective*

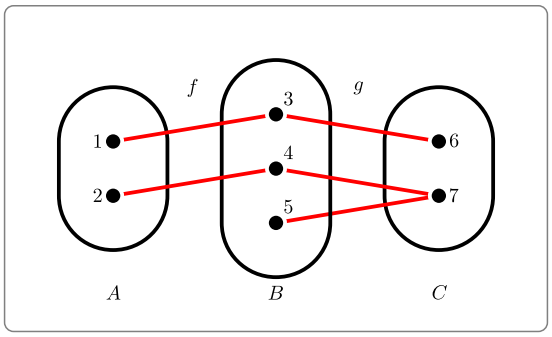
Proof. We prove each in turn.

- Assume that $g \circ f$ is injective, and let $a_1, a_2 \in A$ so that $f(a_1) = f(a_2)$. To show that f is injective it suffices to show that $a_1 = a_2$. Since $f(a_1) = f(a_2)$, we know that $g(f(a_1)) = g(f(a_2))$, and since $g \circ f$ is injective we have that $a_1 = a_2$.

- Now assume that $g \circ f$ is surjective and let $c \in C$. To prove that g is surjective it suffices to find $b \in B$ so that $g(b) = c$. Since $g \circ f$ is surjective, we know that there is $a \in A$ so that $g(f(a)) = c$. Now set $b = f(a)$. Then $g(b) = g(f(a)) = c$ as required.

■

Example 10.5.5 The above theorem proves that when $g \circ f$ is injective, we know that f is injective, the following example show that g need not be injective. Further, it also shows that just because $g \circ f$ and g are surjective, we need not have that f is surjective.



Let $A = \{1, 2\}, B = \{3, 4, 5\}, C = \{6, 7\}$ and consider the following functions

$$\begin{array}{ll} f : A \to B & f(1) = 3, \quad f(2) = 4 \\ g : B \to C & g(3) = 6, \quad g(4) = 7, \quad g(5) = 7 \\ g \circ f : A \to C & g(f(1)) = 6, \quad g(f(2)) = 7 \end{array}$$

Notice that f and $g \circ f$ are injective, but g is not injective since $g(4) = g(5)$. Additionally, g and $g \circ f$ are surjective, but f is not surjective. □

10.6 Inverse functions

We now have enough machinery to start working on inverse functions. Consider what an inverse *function* needs to do. If we start with $a \in A$ and apply a function $f : A \to B$ then we obtain some element $b \in B$. We want our inverse to "undo" this and take b back to a. We'll start by defining functions that are *nearly* inverses and give a few examples.

But before we do that, we note that some of the material below is a bit technical. The reader who is interested in these details should continue on. The reader who wishes to just get to the main results should instead skip ahead to Definition 10.6.6 and Theorem 10.6.8

Definition 10.6.1 Recall Definition 10.4.12, and let $f : A \to B$ and $g : B \to A$ be functions.

- If $g \circ f = i_A$ then we say that g is a **left-inverse** of f.
- Similarly, if $f \circ g = i_B$ then we say that g is a **right-inverse** of f.

◊

So notice that if we start at some point $a \in A$ and apply f to get $b \in B$, then a left-inverse g tells us how to get back from b to a:

$$g(b) = g(f(a)) = a$$

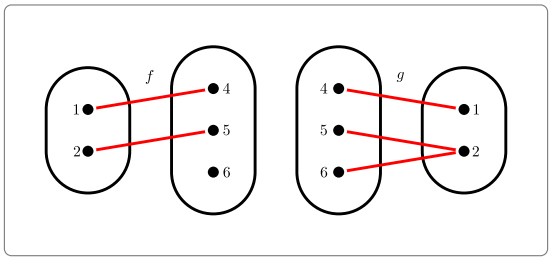
On the other hand, if are trying to get to a particular point $b \in B$ in the codomain using f, then the right-inverse tells you a possible starting point $a \in A$:

$$g(b) = a \qquad \text{so that} \qquad f(a) = f(g(b)) = b$$

Example 10.6.2 Let $A = \{1, 2\}$ and $B = \{4, 5, 6\}$ and define

$$\begin{array}{ll} f : A \to B & f(1) = 4, f(2) = 5 \\ g : B \to A & g(4) = 1, g(5) = 2, g(6) = 2 \end{array}$$

as depicted below.



Then notice that $g \circ f : A \to A$ satisfies

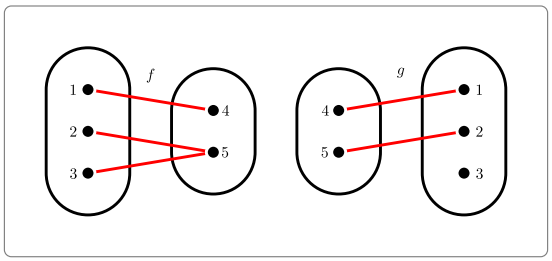
$$g(f(1)) = g(4) = 1 \qquad \text{and} \qquad g(f(2)) = g(5) = 2.$$

That is $g \circ f = i_A$ and so is a left-inverse of f. At the same time, $f \circ g : B \to B$ satisfies

$$f(g(4)) = f(1) = 4 \qquad f(g(5)) = f(2) = 5 \qquad \text{but} \qquad f(g(6)) = f(2) = 5$$

and so g is not a right-inverse of f.

By swapping the roles of f, g in the above we can construct a function that is a right-inverse but not a left-inverse. Consider the functions defined in the image below.



Then we see that

$$g(f(1)) = g(4) = 1 \qquad g(f(2)) = g(5) = 2 \qquad \text{but} \qquad g(f(3)) = g(5) = 2$$

and so g is not a left-inverse of f. And then

$$f(g(4)) = f(1) = 4 \qquad \text{and} \qquad f(g(5)) = f(2) = 5$$

making g a right-inverse of f. □

Notice in the above example, that the function that has a left-inverse is injective, while the function with the right-inverse is surjective. This is not a coincidence as the following two lemmas prove.

Lemma 10.6.3 *Let $f : A \to B$, then f has a left-inverse if and only if f is injective.*

Proof. We prove each implication in turn.

- Assume that f has a left-inverse, g. Now let $a_1, a_2 \in A$ so that $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$, but since g is the left-inverse of f, we know that $g(f(a_1)) = a_1$ and $g(f(a_2)) = a_2$. Thus $a_1 = a_2$ and so f is injective.

- Now let f be injective. We construct a left-inverse of f in two steps. First pick any $\alpha \in A$. Then consider the preimage of a given point $b \in B$. That preimage, $f^{-1}(\{b\})$, is empty or not.

  - If $f^{-1}(\{b\}) = \varnothing$, then define $g(b) = \alpha$
  - Now assume that $f^{-1}(\{b\}) \neq \varnothing$. Since f is injective, the preimage $f^{-1}(\{b\})$ contains exactly 1 element. To see why, consider a, c both in the preimage. We must have $f(c) = f(a) = b$ (since they are both in the preimage of b), but since f is an injection, we must have $c = a$. So define $g(b) = a$ the unique element in the preimage.

  To summarise

$$g(b) = \begin{cases} \alpha & \text{if preimage empty} \\ a & \text{otherwise take unique } a \text{ in the preimage} \end{cases}.$$

  Now let $a \in A$, under f it maps to some $b \in B$ with $b = f(a)$. Hence (as argued above) a is the unique element in the preimage of b, and so $g(f(a)) = g(b) = a$. Thus g is a left-inverse of f.

■

Lemma 10.6.4 *Let $f : A \to B$, then f has a right-inverse if and only if f is surjective.*

Proof. We prove each implication in turn.

- Assume that f has a right-inverse, g. Now let $b \in B$ and set $a = g(b)$. Then $f(a) = f(g(b)) = b$ and so f is surjective.
- Now let f be surjective and let $b \in B$. For the sake of this proof, let us denote the preimage of b as

$$A_b = \{a \in A \mid f(a) = b\}.$$

Since f is surjective, we know that $A_b \neq \varnothing$ for every $b \in B$. So now define $g(b)$ to be any[138] element of A_b.

Now let $b \in B$, then under g it maps to some $a \in A$ so that (by construction) $f(a) = b$. Hence $f(g(b)) = f(a) = b$ and thus g is a right-inverse as required.

■

These lemmas tell us that a function has both a left inverse and right inverse if and only if it is bijective. We can go further those one-sided inverses are actually the same function.

Lemma 10.6.5 *Let $f : A \to B$ have a left-inverse, $g : B \to A$ and a right inverse $h : B \to A$, then $g = h$. Further, the function f has a left and right inverse if and only if f is bijective.*

Proof. Let f, g, h be as stated. Then we know that

$$g \circ f = i_A \qquad \text{and} \qquad f \circ h = i_B$$

Now starting with g we can write:

$$\begin{aligned} g &= g \circ i_B \\ &= g \circ (f \circ h) = (g \circ f) \circ h \\ &= i_A \circ h = h \end{aligned}$$

and thus $g = h$ as required.

The last part of the lemma follows by combining the previous two lemmas. ■

So this lemma tells us conditions under which a function will have both a left- and right-inverse, and that those one-sided inverses are actually the same function. A function that is a left- and right-inverse is a (usual) inverse.

Definition 10.6.6 Let $f : A \to B$ and $g : B \to A$ be functions. If $g \circ f = i_A$ and $f \circ g = i_B$ then we say that g is an **inverse** of f.

Note that we will prove that the inverse is unique, and so we *will* be able to

[140] That one can do this is not as obvious as it might seem. In particular, if B is infinite we need the Axiom of Choice in order to make this selection. The interested reader should search engine their way to more information on this. Now in the event that our function f is injective, then A_b contains exactly one element and we don't need the Axiom of Choice to construct our function. Thankfully we apply this result when our function is bijective — phew.

say that g is *the* inverse of f and denote it f^{-1}.

Also note that if a function is an inverse then it is also a left- and right-inverse. ◊

Lemma 10.6.7 *If a function $f : A \to B$ has an inverse, then that inverse is unique.*

Proof. This proof is very similar to the proof of Lemma 10.6.5. Let $g : B \to A$ and $h : B \to A$ both be inverses to the function f. Then

$$\begin{aligned} g &= g \circ i_B \\ &= g \circ (f \circ h) = (g \circ f) \circ h \\ &= i_A \circ h = h \end{aligned}$$

and so $g = h$. Thus the inverse is unique. ■

We can now state our main theorem about inverse functions.

Theorem 10.6.8 *Let $f : A \to B$.*

- *The function f has an inverse function if and only if it is bijective*
- *The inverse of f, if it exists, is unique.*

Proof. We combine some of the lemmas above to prove this result.

- Assume that f has an inverse. Then that inverse is both a left-inverse and a right-inverse. Lemma 10.6.5 then implies that f is both injective and surjective, and so is bijective.

  Now assume that f is bijective. Then Lemma 10.6.5 tells us there exists a function g that is a left-inverse and right-inverse for f. Then, by definition g is an inverse for f.

- The uniqueness of the inverse is proven by Lemma 10.6.7.

■

Theorem 10.6.8 tells us under what circumstances a function has an inverse. However, it does not tell us if that inverse has a nice expression. If the original function is nice enough, then we may be able to state the inverse nicely. Here are a couple of such examples.

Example 10.6.9 The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 7x - 3$ is bijective and so has an inverse function. We proved this in Result 10.4.2 and Result 10.4.2. The inverse is $f^{-1} : \mathbb{R} \to \mathbb{R}$ and we can work out a formula for it by solving $y = f(x)$ for x in terms of y. Notice that we did exactly that when we proved that f was surjective. In particular, we found that

$$f^{-1} : \mathbb{R} \to \mathbb{R} \qquad \text{defined by} \qquad f^{-1}(y) = \frac{y+3}{7}.$$

To verify that this is correct it suffices to show that $f^{-1} \circ f$ is the identity function:

$$f^{-1} \circ f(x) = f^{-1}(7x - 3)$$

$$= \frac{(7x-3)+3}{y} = x$$

as required. □

Example 10.6.10 Möbius continued. In Result 10.4.14 above we saw that a Möbius transformation $h : \mathbb{R} - \left\{\frac{d}{c}\right\} \to \mathbb{R} - \left\{\frac{a}{c}\right\}$ defined by

$$h(x) = \frac{ax-b}{cx-d}$$

is bijective. The above result tells us that it has an inverse function. We can verify that the inverse is $h^{-1} : \mathbb{R} - \left\{\frac{a}{c}\right\} \to \mathbb{R} - \left\{\frac{d}{c}\right\}$ defined by

$$h^{-1}(y) = \frac{dy-b}{cy-a}.$$

(which we computed while proving the result). We simply need to show that $h^{-1} \circ h$ is the identity:

$$\begin{aligned} h^{-1}(h(x)) &= h^{-1}\left(\frac{ax-b}{cx-d}\right) \\ &= \frac{d\frac{ax-b}{cx-d} - b}{c\frac{ax-b}{cx-d} - a} \\ &= \frac{d(ax-b) - b(cx-d)}{c(ax-b) - a(cx-d)} \\ &= \frac{(ad-bc)x}{ad-bc} = x \end{aligned}$$

as required. □

10.7 (Optional) The axiom of choice

Consider the following, not terribly controversial, statement

> Given a non-empty set A, we can pick an element from it.

This is almost trivial. The fact that $A \neq \varnothing$ is equivalent to the statement $\exists a \in A$. So, we can simply take *that* element a, and we are done.

Let's turn up the complexity a little:

> Given two non-empty sets A, B, we can pick one element from each.

This is no harder, since A is non-empty, we can take $a \in A$. And since $B \neq \varnothing$ we can take $b \in B$. And, at this point, we need to start phrasing things a little differently so that we can be a little more formal and a little more careful.

> Given two sets A, B there exists a function $f : \{A, B\} \to A \cup B$

That is, our *choosing* of elements from A and B is really just a function that takes us from the collection $\{A, B\}$ to a specific elements $a, b \in A \cup B$. Similarly, if such a function exists, then we can use it to choose specific elements. We call such functions **choice functions**.

Definition 10.7.1 Choice function. Let $\mathcal{S}$ be a collection of non-empty sets. Then a **choice function** on $\mathcal{S}$ is a function

$$f : \mathcal{S} \to \bigcup_{X \in S} X$$

so that, for any $X \in \mathcal{S}$, $f(X) \in X$. That is, for any set X in our collection $\mathcal{S}$, the **choice function** f chooses an element $f(X) = x \in X$. ◊

This definition allows us to rephrase the above statements as

- A collection of a single set $\{A\}$ always has a choice function, and
- A collection of two sets $\{A, B\}$ always has a choice function.

This is quite easily extended to any finite collection of non-empty sets. Note that while the collection must be finite, the sets in the collection can be infinite.

Result 10.7.2 *Let $\mathcal{S} = \{A_i \text{ s.t. } i \in \{1, 2, \ldots, n\}\}$ be a finite collection of non-empty sets. Then there exists a choice function on $\mathcal{S}$.*
Proof. We prove this by induction.

- Let $\mathcal{S} = \{A\}$ consist of a single non-empty set A. Since A is non-empty, there is some $a \in A$. The function

  $$f : \{A\} \to A \qquad f(A) = a$$

  is a choice function on $\mathcal{S}$. Thus the statement is true for $|\mathcal{S}| = 1$.

- Now assume that the statement hold for all collections of k non-empty sets, and let $\mathcal{T} = \{A_i \text{ s.t. } i \in \{1, 2, \ldots, k+1\}\}$. Since $A_{k+1} \neq \varnothing$ there is some $q \in A_{k+1}$.

  Then, by assumption, the collection $\mathcal{S} = \{A_i \text{ s.t. } i \in \{1, 2, \ldots, k\}\}$ has a choice function f. We can then use this to define the required choice function g:

  $$g(A_i) = \begin{cases} f(A_i) & i = 1, 2, \ldots, k \\ q & i = k+1 \end{cases}$$

Then, by induction, the statement holds for any finite collection of non-empty sets. ■

The existence of such choice-functions is intuitively quite obvious. I can always grab an element out from a set, so I can grab an element out from each set in turn. Indeed, it is equivalent to the statement that the Cartesian product of a finite number of non-empty sets is also non-empty.

Result 10.7.3 *Let $A_1, A_2, \ldots, A_n$ be non-empty sets. The the collection $\{A_1, \ldots, A_n\}$ has a choice function if and only if the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ is non-empty.*

Proof. Let the A_i be as stated. We then prove each implication in turn.

Assume that the collection $\{A_1, \ldots, A_n\}$ has a choice function, f. We can use that to select $f(A_i) = a_i \in A_i$ for $i = 1, 2, \ldots, n$. Hence $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n$. Thus the product is non-empty.

Similarly, assume that the product $A_1 \times A_2 \times \cdots \times A_n \neq \varnothing$, and hence there is $(a_1, \ldots a_n) \in A_1 \times \cdots \times A_n$. We use that element to define

$$f : \{A_1, \ldots, A_n\} \to \cup A_i \qquad f(A_i) = a_i$$

giving the required choice function. ■

Where things become really very non-trivial, and the consequences very non-intuitive, is when we have infinite collections of sets. Our inductive argument above can't get us there — it cannot make the leap from large, but *finite* collections, to infinite collections.

Of course, if we the sets in our infinite collection are nice and have extra structure, then this might be easy. For example,

- A collection of subsets of $\mathbb{N}$ — just choose the smallest element of each subset (via the well-ordering principle).
- A collection of sets of English words — just choose the words that come first in alphabetic order[139].
- A collection of pairs of shoes — just choose the left shoe of each pair.

This last example is very famous and is due to Bertrand Russell[140]. The second half of the example points out that it is far from obvious how to do the same with an infinite collection of pairs of socks.

Indeed, for an infinite collection of non-empty sets, without extra structure, the existence of a choice function is taken as an axiom — the Axiom of Choice.

Axiom 10.7.4 Axiom of choice. *Let $\mathcal{S}$ be any collection of non-empty sets. Then there exists a choice function on $\mathcal{S}$.*

This statement feels so intuitively true, it was used quite implicitly until 1904 when Zermelo realised that the question of the existence of choice functions was not at all trivial. Indeed, it was subsequently shown that Axiom of Choice cannot be proved or disproved using usual set theory[141]. To be more precise, 1938 Kurt

[139] More formally, we pick the lexicographic least word from each subset. Lexicographic ordering is really useful and the interested reader should search-engine their way to more on this topic.

[140] No footnote can even begin to do justice to the many contributions of Russell to logic and mathematics and many other disciplines. The reader should search-engine their way to more information.

[141] By which we mean Zermelo-Fraenkel set theory, which is, roughly speaking, the formalisation of the usual notions of set theory, including those in this text.

Gödel proved that one can not disprove the existence of such a choice function using the standard axioms of set theory, while in 1963 Paul Cohen proved that the existence cannot be proven either. It is now accepted[142] as a standard part of set theory.

The Axiom of Choice 10.7.4 is equivalent to the statement that the Cartesian product of any collection of non-empty sets is itself non-empty — this seems reasonable and hardly controversial. However, the Axiom of Choice does have some very strange implications.

- Well-ordering theorem — every set can be well-ordered. That is, one can define an ordering of the elements of any set so that any subset has a first element! Cantor conjectured this result in 1883 but it was proved Zermelo in 1904. It was in that proof that the Axiom of Choice was first formalised. It is also known as Zermelo's theorem.

- Banach-Tarski paradox — it is possible to decompose a solid ball into finitely many pieces and reassemble them into two solid balls each having the same volume as the original!

- It allows you to predict the result of coin tosses — see this nice article[143] by Matt Baker. The interested reader should also examine a very nice (and somewhat provocative) paper[144] by Hardin and Taylor push this line of reasoning explaining how you can use the Axiom of Choice to predict future values of real-valued functions[145]. Here[146] is a nice and more accessible description of the result by Michael O'Connor.

The Axiom of Choice is an extremely interesting and complicated topic, and, well beyond the scope of this text; the interested reader should search-engine their way to more information[147].

10.8 Exercises

1. Is the set

$$\theta = \{((x, y), (5y, 4x, x + y)) : x, y \in \mathbb{R}\}$$

a function? If so, what is its domain and its range?

2. For which values of $a, b \in \mathbb{N}$ does the set $\phi = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : ax + by = 6\}$ define a function?

[142] By most mathematicians in most contexts. There is, however, an interesting body of research on what happens when the Axiom of Choice is false. Very strange things happen — the interested reader should search engine their way to more information.

[143] https://mattbaker.blog/2015/01/17/spooky-inference-and-the-axiom-of-choice/

[144] https://www.tandfonline.com/doi/abs/10.1080/00029890.2008.11920502

[145] Such as temperatures, stock-market prices, position of balls on roulette tables, etc.

[146] https://xorshammer.com/2008/08/23/set-theory-and-weather-prediction/

[147] There are many articles out there on this topic, but the authors found this blog to be a very nice discussion of the Axiom of Choice.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is defined by $f(x) = \dfrac{2x}{1+x^2}$. Show that $f(\mathbb{R}) = [-1, 1]$.

4. Consider the following functions and their images and preimages.

(a) Consider the function $f : \{1, 2, 3, 4, 5, 6, 7\} \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ given as

$$f = \{(1, 3), (2, 8), (3, 3), (4, 1), (5, 2), (6, 4), (7, 6)\}.$$

Find: $f\big(\{1, 2, 3\}\big)$, $f\big(\{4, 5, 6, 7\}\big)$, $f(\emptyset)$, $f^{-1}\big(\{0, 5, 9\}\big)$ and $f^{-1}\big(\{0, 3, 5, 9\}\big)$.

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 4x^2 - x - 3$.

Find: $g\left(\left\{\frac{1}{8}\right\}\right), g^{-1}(\{0\})$, $g((-1, 0) \cup [3, 4])$, and, $g^{-1}([-10, -5))$.

(c) Define $h : \mathbb{R} \to \mathbb{R}$ by $h(t) = \sin(2\pi t)$.

Find: $h(\mathbb{Z})$, $h\left(\left\{\frac{1}{4}, \frac{7}{2}, \frac{19}{4}, 22\right\}\right)$, $h^{-1}(\{1\})$, and $h^{-1}([0, 1))$.

5. Let A, B be sets and $f : A \to B$ be a function from $A$to B. Then prove that if E and F are subsets of B, then

$$f^{-1}(E - F) = f^{-1}(E) - f^{-1}(F).$$

Remember that since we do not know whether or not f is a bijection, f^{-1} denotes the preimage of f not its inverse.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + ax + b$, where $a, b \in \mathbb{R}$. Determine whether f is injective and/or surjective.

7. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ that are injective and such that for all $n \in \mathbb{N}$ we have $f(n) \leq n$.

8. Prove that $f : [3, \infty) \to [5, \infty)$, defined by $f(x) = x^2 - 6x + 14$ is a bijective function.

9. For $n \in \mathbb{N}$, let $A = \{a_1, a_2, a_3, \cdots, a_n\}$ be a fixed set where $a_j \neq a_i$ for any $i \neq j$, and let F be the set of all functions $f : A \to \{0, 1\}$.

What is $|F|$, the cardinality of F?

Now, for $\mathcal{P}(A)$, the power set of A, consider the function $g : F \to \mathcal{P}(A)$, defined as

$$g(f) = \{a \in A : f(a) = 1\}.$$

Is g injective? Is g surjective?

10. Let $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined by $f(n) = (2n + 1, n + 2)$. Check whether this function is injective and whether it is surjective. Prove your answer.

11. Let $f : E \to F$ be a function. We recall that for any $A \subseteq E$ the image of A by f, namely $f(A)$, is defined as

$$f(A) = \{f(x) : x \in A\}.$$

Show that f is surjective if and only if

$$\forall A \in \mathcal{P}(E),\ F - f(A) \subseteq f(E - A).$$

12. Let $f : C \to D$ be a function. Let $A, B \subset C$ be nonempty sets. Prove that if f is injective, then $f(A - B) = f(A) - f(B)$.

13. Consider the function

$$f : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \qquad \text{where} \qquad f(x, y) = 2^{x-1}(2y - 1).$$

Prove that f is bijective.

14. Let A, B be nonempty sets. Prove that if there is a bijection $f : A \to B$, then there is a bijection from $\mathcal{P}(A)$, the power set of A, to $\mathcal{P}(B)$, the power set of B.

15. Let $\mathcal{R}$ be the relation on $\mathbb{R}^2$ defined by

$$(x, y)\ \mathcal{R}\ (s, t) \text{ iff } x^2 + y^2 = s^2 + t^2$$

where $(x, y), (s, t) \in \mathbb{R}^2$.

(a) Show that $\mathcal{R}$ is an equivalence relation

(b) Let $\mathcal{S}$ be the set of equivalence classes of the relation $\mathcal{R}$ defined in part (a). Let f be defined by

$$f : \mathcal{S} \to [0, \infty) \qquad \text{with} \qquad f([(x, y)]) = \sqrt{x^2 + y^2}.$$

Prove that

- f is a function, and, further
- f is bijective.

16. Let $n \in \mathbb{N}$ with $n > 1$ and let $\mathbb{Z}_n$ be the set of equivalence classes modulo n. For any $x \in \mathbb{Z}$, let $[x]_n \in \mathbb{Z}_n$ denote its equivalence class modulo n.

Define the function $f : \mathbb{Z}_n \to \{0, 1, \ldots, n - 1\}$ by $f([x]_n) = r$, where r is the remainder of x upon division by n.

(a) Show that f is well-defined, meaning that f is defined on its whole domain and that f does not depend on the choice of representative for each equivalence class; i.e. $[x]_n = [y]_n \implies f([x]_n) = f([y]_n)$.

(b) Show that f is a bijection.

This question explains why when dealing with equivalence classes of integers modulo n, we often consider the set of representatives $\{0, 1, \ldots, n - 1\}$ instead.

17. Consider $f : A \to B$. Prove that f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$.

18. Consider $f : A \to B$. Prove that f is surjective if and only if $f(f^{-1}(Y)) = Y$ for all $Y \subseteq B$.

19. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if whenever $x_1 < x_2$ we have $f(x_1) < f(x_2)$. Similarly, a function $g : \mathbb{R} \to \mathbb{R}$ is *strictly decreasing* if whenever $x_1 < x_2$ we have $g(x_1) > g(x_2)$.

(a) Prove that the composition of two strictly increasing functions is strictly increasing.

(b) Prove that the composition of two strictly decreasing functions is strictly increasing.

20. Let f, g, and h be functions from $\mathbb{R} \to \mathbb{R}$. We define the following operations on functions:

- Function addition: $(f + g)(x) = f(x) + g(x)$
- Function division: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
- Function composition: $(f \circ g)(x) = f(g(x))$

Note that under this definition, $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$. This is NOT the inverse of $f(x)$.

Prove or give a counterexample to the following statements:

(a) $f \circ (g + h) = f \circ g + f \circ h$ for every $x \in \mathbb{R}$.

(b) $(g + h) \circ f = g \circ f + h \circ f$ for every $x \in \mathbb{R}$.

(c) $\frac{1}{f \circ g} = \frac{1}{f} \circ g$ for every $x \in \mathbb{R}$.

(d) $\frac{1}{f \circ g} = f \circ \frac{1}{g}$ for every $x \in \mathbb{R}$.

21. Find counterexamples to the following statements:

(a) Given a function $f : A \to B$ and subsets $W, X \subseteq A$, we have $f(W \cap X) = f(W) \cap f(X)$.

(b) Given a function $f : A \to B$ and a subset $Y \subseteq B$, we have $f(f^{-1}(Y)) = Y$.

Explain your answers.

22. Let A, B be nonempty sets. Let f, h be functions from A to B, and let g be a function from B to A.

(a) Suppose that g is injective. Prove that if $g \circ f = g \circ h$, then $f = h$.

(b) Suppose that g is surjective. Prove that if $f \circ g = h \circ g$, then $f = h$.

23. Consider the following functions and their compositions.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$. Does there exist a function

$g : \mathbb{R} \to \mathbb{R}$ such that $(f \circ g)(x) = (g \circ f)(x)$ for every $x \in \mathbb{R}$?

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = c$ for some $c \in \mathbb{R}$ (i.e. f is a constant function). Which functions $g : \mathbb{R} \to \mathbb{R}$ have the property $(f \circ g)(x) = (g \circ f)(x)$ for every $x \in \mathbb{R}$?

(c) Let $f : \mathbb{R} \to \mathbb{R}$. Suppose $(f \circ g)(x) = (g \circ f)(x)$ for every $x \in \mathbb{R}$ and for every function $g : \mathbb{R} \to \mathbb{R}$. Show that $f(x) = x$.

24. Let A be a nonempty set and $f : A \to A$ be a function.

(a) Prove that f is bijective if $f \circ f$ is bijective.

(b) Let

$$f : (0, \infty) \to (0, \infty) \qquad f(x) = \log\left(\frac{e^x + 1}{e^x - 1}\right)$$

where $\log(x)$ denotes the natural logarithm of x. Use part (a) to prove that this is a bijective function.

25. Prove that the function

$$f : \mathbb{R} - \{-2\} \to \mathbb{R} - \{1\}, \text{ defined by } f(x) = \frac{x+1}{x+2}$$

is bijective and find its inverse.

26. Let $f : \mathbb{Z} \to \mathbb{Z}$ defined so that $f(n) = -n + 3$ if n is even and $f(n) = n + 7$ if n is odd. Prove that f is bijective and find its inverse, f^{-1}.

27. The following question concerns the triple composition of a function.

(a) Let A be a non-empty set and let $g : A \to A$ be a function that satisfies $g \circ g \circ g = i_A$, where i_A is the identity function on the set A. Prove that g must be bijective.

(b) Let $A = \mathbb{R} - \{0, 1\}$ and let $f : A \to A$ be defined by $f(x) = 1 - \frac{1}{x}$. Show that $f \circ f \circ f = i_A$.

(c) Use part (a) to conclude that f is bijective and determine its inverse function f^{-1}.

Chapter 11

Proof by contradiction

Proof by contradiction is another general proof technique like direct proofs and the contrapositive proofs. When you first encounter proof by contradiction it can seem rather mysterious:

- Assume to be true something we know is false, then
- prove garbage, then
- from this deduce truth!

But after you get the hang of it, proof by contradiction becomes indispensable.

Warning 11.0.1 Not everything is a nail. One of the reasons that the authors have left this topic until quite late in the text is that we find that students try to use this method for *everything*. Remember, contradiction is just another method in our toolbox and just because we have a shiny new hammer, not every result is a nail[148]. Please do not forget the other proof techniques.

With that warning out of the way, what is proof by contradiction and how does it differ from other techniques? Well, roughly speaking, when we use proof by contradiction, we do not seek to prove a statement P to be true, but rather we prove that $(\sim P)$ is false.This might seem to be a delicate and pedantic distinction, but it does make the structure of the resulting proof very different from direct and contrapositive proofs.

Once we know that $(\sim P)$ is false we can deduce that P is true. To do so we rely on the **law of the excluded middle**[149] which states that a statement must either be true or its negation must be true:

$$(P \vee (\sim P)) \text{ is a tautology}$$

[150]Remember the Law of the Instrument (or look it up with your favourite search engine).

[149]Being middle child, one of the authors finds it difficult to not think of the law of the excluded middle as being some sort of analysis of birth-order and family dynamics. The interested reader should continue to the next section of the text in which we discuss it further. "It" being the law of the excluded middle and not anything to do with middle-child syndrome.

and also **modus tollens**[150]. Remember modus ponens is:

$$((P \implies Q) \wedge P) \implies Q$$

Modus tollens is (roughly speaking) applying modus ponens to the contrapositive:

$$((P \implies Q) \wedge \sim Q) \implies (\sim P)$$

So if we know that an implication $P \implies Q$ is true, and the conclusion Q is false, then the hypothesis P must be false[151].

So how do we put these things together to make a proof by contradiction?

11.1 Structure of a proof by contradiction

Say we wish to prove a statement P to be true. Since

$$(P \vee (\sim P)) \text{ is a tautology,}$$

either we must have P is true or $(\sim P)$ is true.

- Tell the reader something like "We will prove this by contradiction" otherwise the next step looks like a mistake.
- We assume that $(\sim P)$ is true, and then show that this leads to a falsehood — a contradiction — garbage.
- That is, we will construct a change of implications like:

$$\begin{aligned}
(\sim P) &\implies P_1 && \text{and}\\
P_1 &\implies P_2 && \text{and}\\
P_2 &\implies P_3 && \text{and}\\
&\vdots\\
P_{n-1} &\implies P_n && \text{and}\\
P_n &\implies \text{CONTRADICTION}
\end{aligned}$$

- The contradiction is a statement that is always false — for example

$$R \wedge (\sim R).$$

  But which contradiction do we aim for? Let's discuss that shortly.

- Since the contradiction is false, and all of those implications are true, we must have P_n is false (modus tollens).

[150] or "denying the consequent" as it is known in less latin moments.

[151] The skeptical reader should take a quick glimps at the truth table to see why this is so.

- Similarly, since P_n is false, we know P_{n-1} is false (again, modus tollens).
- Keep on moving back up the chain of implications, and we see that $(\sim P)$ must be false.
- Thus[152] P is true.

When when we write this up neatly for the reader we arrive at a proof that looks like the following:
Generic proof-by-contradiction proof. We prove this result by contradiction. So assume, to the contrary $(\sim P)$.

- A chain of implications showing that "$(\sim P) \implies$ contradiction".

Thus we conclude that $(\sim P)$ is false, and so P is true. ■

Warning 11.1.1 What contradiction should we aim for? As we warned earlier, once you are comfortable with the logic of proof by contradiction, it becomes tempting to use it everywhere. However, we caution the reader to use this method after first trying a direct or contrapositive proof. One very good reason for this caution is that a direct or contrapositive proof has a well defined starting point:

- the hypothesis is true,

and a well defined end point:

- the conclusion is true.

By exploring the conclusion in our scratch work we can work out how to make the proof work.

Proof by contradiction starts clearly enough

- the statement is false,

but in contrast with direct and contrapositive proofs, it is not clear what statement we need to generate the contradiction. We know we need *some* contradiction, but *which* contradiction we should reach? How do we know where to aim? This can make it much harder.

We can generate a contradiction for our proof by contradiction is to show that one of our assumptions is both true and false. For example, when we start a proof by contradiction, we assume that the result is actually false and this, in turn, requires us to make an assumption, say, Q. *One* way we can generate a contradiction is to reach a conclusion $(\sim Q)$. However this is not the *only* way to generate a contradiction.

Let us do a small example, in which the contradiction is quite easy to find.

Result 11.1.2 *There is no smallest positive real number.*

[152] By the law of the excluded middle

Proof by contradiction can work very nicely for results of this form "There is no smallest X" or "There is no largest X" (where X is some interesting mathematical object). We can construct the proof by

- assume that there is a smallest X, call it X_1, then
- use X_1 to construct an even smaller X, call it X_2

but then we have

- X_1 is the smallest possible X, and at the same time
- X_2 is smaller that X_1

This gives us a contradiction of the form

$$Q \wedge \sim Q.$$

Let's apply this approach to the above result.

Our scratch work should look something like this:

- We'll try a proof by contradiction
- The negation of the result is "There is a smallest positive real number".
- Hence we assume there is a smallest positive real number — call it r.
- But now the number $r/2$ is a positive real number and $r/2 < r$.
- Contradiction!

Proof of Result 11.1.2. We prove this result by contradiction. Assume the result is false, so there is some smallest positive real number r. But then $0 < r/2 < r$, making $r/2$ is a smaller positive real number. This contradicts our assumption that r was the smallest positive real number. Hence the result must be true. ■

11.2 Some examples

We'll start with a couple of good warm-up examples.

Example 11.2.1 No integer solutions. There are no integers a, b so that $2a + 4b = 1$.

Scratchwork. Now notice that this result hides some universal quantifiers:

$$\forall a, b \in \mathbb{Z}, 2a + 4b \neq 1$$

To prove this using a contradiction we need to assume the negation of this statement. That is, we will assume that

$$\exists a, b \in \mathbb{Z} \text{ s.t. } 2a + 4b = 1.$$

So let a, b be integers so that $2a + 4b = 1$. But from this we have (after a quick division by 2)

$$a + 2b = \frac{1}{2}$$

and this is sufficient to get a contradiction; $a \in \mathbb{Z}$ and $2b \in \mathbb{Z}$ so their sum must be an integer. We just need to write this up nicely[153].

Solution.

Proof. Assume, to the contrary, that there exist integers a, b so that $2a+4b = 1$. This implies that

$$a + 2b = \frac{1}{2}$$

which gives a contradiction because the sum of integers is also an integer. Consequently, there can be no such integer a, b and the result follows. ■

□

Example 11.2.2 Another no integer solutions. There are no integers a, b so that $a^2 - 4b = 3$.

Solution.

Proof. Assume, to the contrary that a, b are integers so that $a^2 - 4b = 3$. Hence $a^2 = 4b + 3$. Consequently a must be odd (otherwise the LHS is even while the RHS is odd). So we can write $a = 2k + 1$ for some integer k. Hence

$$(2k + 1)^2 = 4b + 3$$

and so

$$4k^2 + 4k + 1 = 4b + 3$$

which means that

$$4(k^2 + k - b) = 2.$$

And since $k^2 + k - b \in \mathbb{Z}$, this implies that 2 is divisible by 4. This is clearly false and so we have arrived at a contradiction. Thus there can be no such integers a, b and the result holds. ■

□

Time for something a bit more substantial — a result about irrational numbers. Recall the definition of rational and irrational numbers.

Definition 11.2.3 Let q be a real number. We say that q is rational if it can be written $q = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ with $b \neq 0$. If q is not rational we call it irrational. We will denote the set of irrational numbers as $\mathbb{I}$, and note that $\mathbb{I} = \mathbb{R} - \mathbb{Q}$. ◊

We note that the notation, $\mathbb{I}$, is not standard, and some authors will use $\mathbb{P}$ or $\mathbb{K}$. Unfortunately there is no widely accepted standard notation for the set of irrational numbers. When you do make use of a particular notation for irrational numbers[154], and you are unsure if your reader knows that notation, we

[155] and, of course, make sure we warn the reader that we are using proof by contradiction.

[154] and we do recommend using $\mathbb{I}$ for this course

recommend that you devote a quick short sentence clarifying to explain.

Thus a number q is irrational when (writing it with quantifiers)

$$\forall a, b \in \mathbb{Z}, \left(\frac{a}{b} \neq q\right).$$

So a good way to reach a contradiction when working with irrational numbers, is to show that a number that you have assumed to be irrational can actually be written as a ratio of integers. We use exactly this approach for the next result.

Result 11.2.4 *The sum of a rational number and an irrational number is irrational.*

We start by assuming that the result is actually false and then work our way to a contradiction — namely that the number we assumed to be irrational is actually rational. It pays, especially when starting out with proof by contradiction, to write statements carefully with with quantifiers, so that we can also write down the negation carefully. The original statement is

$$\forall x \in \mathbb{Q}, \forall y \in \mathbb{I}, x + y \in \mathbb{I}$$

and its negation is

$$\exists x \in \mathbb{Q} \text{ s.t. } \exists y \in \mathbb{I} \text{ s.t. } x + y \notin \mathbb{I}$$

We can simplify this a bit because we know that $x+y \in \mathbb{R}$, so if it is not irrational, it must be rational:

$$\exists x \in \mathbb{Q} \text{ s.t. } \exists y \in \mathbb{I} \text{ s.t. } x + y \in \mathbb{Q}.$$

Now we assume *this* statement is true. And hence we can find a rational number, $x = \frac{a}{b}$, and an irrational number, y, and their sum $x + y$ is rational. One way we could get the contradiction, is to leverage the facts we have (by assumption)

$$x \in \mathbb{Q} \qquad y \in \mathbb{I} \qquad \text{and} \qquad x + y \in \mathbb{Q}$$

to reach a contradiction; in this example we'll show that $y \in \mathbb{Q}$.

Since $x \in \mathbb{Q}$ we know that there are integers a, b so that

$$x = \frac{a}{b}.$$

similarly since $x + y \in \mathbb{Q}$ we know that there are integers c, d so that

$$x + y = \frac{c}{d}.$$

But now we can use these to obtain more information about y:

$$\begin{aligned} y &= (x + y) - x \\ &= \frac{c}{d} - \frac{a}{b} = \frac{cb - ad}{bd} \end{aligned}$$

But since all of $a, b, c, d \in \mathbb{Z}$, we've just shown that y is rational — contradiction!

Time to write it up. When we do so we should definitely be careful and show that those denominators are not zero.

Proof of Result 11.2.4. We prove the result by contradiction. Assume that the result is false, and so there is a rational number x and an irrational number y whose sum $z = x + y$ is rational. Since x and z are rational, $x = a/b$ and $z = c/d$ for some $a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$. But now

$$y = z - x = \frac{cb - ad}{bd}$$

with $bd \neq 0$ and so y must be rational. This contradicts our assumption that y was irrational. Hence the result is true. ■

When we use proof by contradiction to prove an implication, we just have to negate carefully. Say our statement P is of the form

$$P \equiv \Big(\forall x \in S, Q(x) \implies R(x)\Big)$$

Our proof by contradiction needs us to prove that the negation of this statement implies a contradiction. So negating carefully:

$$\begin{aligned}(\sim P) \equiv &\sim \Big(\forall x \in S, Q(x) \implies R(x)\Big) \\ &\equiv \exists x \in S \text{ s.t. } \sim \Big(Q(x) \implies R(x)\Big) \\ &\equiv \exists x \in S \text{ s.t. } \Big(Q(x) \wedge \sim R(x)\Big)\end{aligned}$$

So our proof will start by assuming the existence of some $x \in S$ such that $Q(x)$ is true and $R(x)$ is false. Of course, we should make sure that we alert the reader that we are using proof by contradiction. We might say "Suppose the statement is false" or something similar. Here is example of this in action:

Result 11.2.5 *Let $a, b \in \mathbb{Z}$ with $a \geq 2$. Then a does not divide b or a does not divide $b + 1$.*

This statement has an implication hiding inside and can be written as

$$\forall a, b \in \mathbb{Z}, \Big[(a \geq 2) \implies \big((a \nmid b) \vee (a \nmid (b+1))\big)\Big]$$

so when we negate it we obtain

$$\exists a, b \in \mathbb{Z} \text{ s.t. } \Big[(a \geq 2) \wedge \big((a \mid b) \wedge (a \mid (b+1))\big)\Big]$$

We'll start[155] our proof by assuming that we can find such integers a, b, so that $a \geq 2$ and $a \mid b$ and $a \mid (b + 1)$. From this we'll reach a contradiction.

Proof. Assume, to the contrary, that there is some $a, b \in \mathbb{Z}$ so that $a \geq 2$ and a divides both b and $b + 1$. This then implies that there exists $k, \ell \in \mathbb{Z}$ such that $b = ak$ and $(b + 1) = a\ell$.

But then $1 = (b + 1) - b = (\ell - k)a$. Since $\ell - k \in \mathbb{Z}$ this implies that a

[155] Of course, we'll warn the reader that it is a proof by contradiction before we get too far along. Be kind to your reader.

divides 1. This means that $a = 1$ or $a = -1$. This contradicts our assumption that $a \geq 2$. Hence the result is true. ■

This is not the only way to prove the above result and we could give a quick contrapositive proof which as some similarities with our proof above.

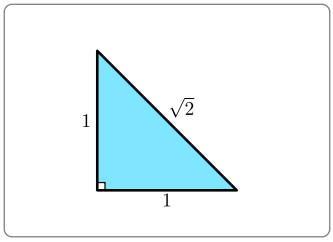
Proof. We prove the contrapositive. So let $a, b \in \mathbb{Z}$ and assume that $a \mid b$ and $a \mid (b+1)$. Then we know that $b = ak, (b+1) = a\ell$ for some $k, \ell \in \mathbb{Z}$. But then this tells us that

$$1 = (b+1) - b = a(k - \ell)$$

and so $a \mid 1$. However the only divisors of 1 are $1, -1$. And thus we know that $a < 2$ as required. ■

11.2.1 The irrationality of $\sqrt{2}$

The existence of a real quantity whose square is 2 follows directly from applying Pythagoras's Theorem to the following simple triangle.



It is, however, much less obvious that $\sqrt{2}$ cannot be expressed as the ratio of two integers; that result is one of the most famous in mathematics. Its proof, and so the proof of the existence of irrational numbers, is generally attributed to a member of the Pythagorean school in the 5th century BC, typically Hippasus of Metapontum. The evidence that exists linking Hippasus to the discovery of irrational numbers suggests that he was not praised for his work, but, rather, that he was expelled from his school. Some accounts indicate that he was even drowned as punishment! At the time the Pythagorean school thought that the positive integers numbers were somehow fundamental and beautiful and *natural*. The natural numbers were almost mystical objects and could be deployed to explain the universe. That such a simple and beautiful geometric object — the hypotenuse of a right-angle triangle — could not be expressed as the ratio of natural numbers was truly shocking. In some sense, it broke the link between number and the world.

Theorem 11.2.6 *The number $\sqrt{2}$ is not rational.*

We prove this result by finding a contradiction — that $\sqrt{2}$ is both rational and irrational. The same proof can be made to work (with minor adjustments) for any prime number. A key part of the proof is understanding that when we write a rational number

$$q = \frac{a}{b}$$

that we can insist that a, b do not have common factors. If we do have a representation $q = \frac{c}{d}$ where $\gcd(c, d) > 1$, then we can divide both numerator and denominator by that common factor and set

$$q = \frac{c}{d} = \frac{c/\gcd(c, d)}{d/gcd(c, d)} = \frac{a}{b}$$

where the new numerator and denominator, a, b have no common factors. In this way, the resulting a, b are the *smallest* integers whose ratio represents that rational number. Using this idea, our proof works by assuming that $\sqrt{2}$ is rational and so can be represented by a smallest a, b (ie with no common factors). We then obtain a contradiction by showing that the numerator and denominator do have a common factor. Along the way we will make use of a result we proved earlier[156].

Let $n \in \mathbb{Z}$. Then n^2 is even if and only if n is even.

Proof. We prove the result by contradiction, and so assume that $\sqrt{2}$ is rational. Thus we can write $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and a and b have *no common factors*. Hence $2 = \frac{a^2}{b^2}$, which can be rewritten as $2b^2 = a^2$.

This implies that a^2 is even and so (by the above fact) a must be even. Thus we can write $a = 2c$ for some $c \in \mathbb{Z}$.

Substituting $a = 2c$ into $2b^2 = a^2$ we find $2b^2 = 4c^2$, which implies $b^2 = 2c^2$.Hence b^2 is even and so b must be even.

Since both a and b are even, they must have a common factor of 2. This contradicts our assumption that a and b have no common factors. Hence the result is true and $\sqrt{2} \notin \mathbb{Q}$. ■

Here is another result with a similar proof.

Result 11.2.7 *Let a, b, c be odd integers. Then the polynomial $ax^2 + bx + c$ has no rational zeros.*

Proof. Assume, to the contrary, that there are odd a, b, c and rational x so that $ax^2 + bx + c = 0$. Since x is rational, we know that $x = \frac{k}{n}$ for some integers k, n with $n \neq 0$. Further, we can assume that k, n have no common factors; if they do have common factors, remove them. Then

$$a\left(\frac{k}{n}\right)^2 + b\left(\frac{k}{n}\right) + c = 0$$

and so (multiplying through by n^2)

$$ak^2 + bkn + cn^2 = 0$$

Now consider the parity of k and n. There are four possibilities

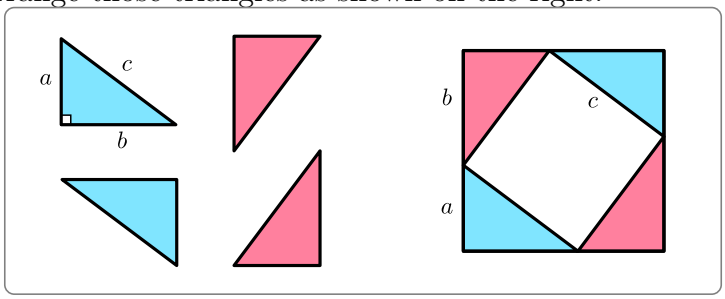
- If k, n are both odd, then since a, b, c are odd, the LHS is also odd, and so cannot equal zero.

[156] Lemma 5.1.5 is logically equivalent to the statement "The number n^2 is even iff n is even" (just take the contrapositive of both implications).

- If k is even and n is odd, then ak^2 and bkn are even, and cn^2 is odd. Hence the LHS is odd and so cannot equal zero.
- Similarly, if n is even and k is odd, then bkn and cn^2 are even and ak^2 is odd. Again this implies the LHS is odd and so cannot equal zero.
- Finally, if k, n are both even, then this contradicts our assumption that k, n have no common factors.

Thus there cannot exist such k, n and hence there is no such rational x. ■

Remark 11.2.8 Pythagoras' Theorem. There are a great many proofs of this famous theorem, but here is a pictorial argument that the authors particularly like (and we include here for completeness[157]). It is attributed to the 12th century mathematician and astronomer Bhaskara[158]. Construct a right-angle triangle with sides a, b, c, and then make another 3 copies, each rotated by 90 degrees, and then rearrange those triangles as shown on the right.



Now notice that the outer-square has area $(a+b)^2$, and the inner rotated square has area c^2. The four triangles have total area $2ab$. Equating the areas gives $(a+b)^2 = c^2 + 2ab$. But since $(a+b)^2 = a^2 + b^2 + 2ab$, subtracting $2ab$ from each side gives

$$c^2 = a^2 + b^2.$$

11.2.2 The infinitude of primes

Here is wonderful result about a fundamental property of numbers — that there are an infinite number of primes. The first recorded proof of this is due to Euclid in his Elements[159] from around 300 BC. The result does not rely on unique prime

[159] and because we can! We hope the reader will agree that it is pretty cool.

[160] Bhāskara, made many contributions to mathematics, including understanding some of the central ideas of differential calculus around 500 years before Newton and Leibniz. The interested reader should use their search engine to discover more. We note that Bhāskara is also known as Bhāskara II to distinguish him from Bhāskara I. Bhāskara I was a 7th Century mathematician and astronomer, who amongst other acheivements helped to develop the positional notation we use for representing numbers including a circle symbol for zero. He is also worth a trip to your favourite search engine. Both Bhāskara have satellites named after them by the Indian Space Research Organisation.

[159] This 13 book work on mathematics has been called the most famous and influential textbook in history. Indeed, it was *the* standard textbook for university students for many centuries. The interested reader should search-engine their way to more information.

factorisations, but it does require the fact that every integer greater than 1 has a prime divisor. We prove this first via strong induction — in fact we did this back in Example 7.2.20.

Result 11.2.9 *Let $n \in \mathbb{N}$ so that $n \geq 2$. Then n is divisible by a prime number.*
Proof. See Example 7.2.20. ■

Armed with this result, we can prove that the set of primes is not finite.

Theorem 11.2.10 Euclid — 300 BC. *There are infinitely many primes.*

We will prove this using a "proof by contradiction". We assume that there are only a finite number of primes and then deduce a contradiction. If there are a finite number of primes, we can list them out $\{p_1, p_2, \ldots p_r\}$ and we can form the new number $N = p_1 p_2 \ldots p_r$ — Now $N + 1$ is not on our list and it is not divisible by any of the primes. This will be the source of the contradiction (with a little more work).

Proof. Assume there are a finite number of primes. Since the primes are finite, we can write a finite list containing all of them — $p_1, p_2, \ldots, p_n$. Now let $N = (p_1 p_2 \ldots p_r)$ be a product of all the primes. Since the list of primes is finite, we know that N is finite. Now either $N + 1$ is prime or not.

- If $N + 1$ is prime then since $N + 1$ is bigger than all the primes on the list, $N + 1$ is not in our list of prime numbers. This gives a contradiction since our list was assumed to be *all* the primes.

- If $N+1$ is not prime then, by the above result, it must be divisible by one of the primes in our list — say $p_k \mid (N + 1)$. Hence we can write $N + 1 = p_k a$ for some $a \in \mathbb{N}$. Similarly we can write $N = p_k b$ for some $b \in \mathbb{N}$. But then

$$1 = (N + 1) - N = p_k(a - b).$$

  This implies that 1 is divisible by p_k which is clearly false.

  Thus $N + 1$ is not divisible by any prime on our list and so there must be some prime that is not contained in our list. Again this gives a contradiction since our list as assumed to contain *all* primes.

In both cases a contradiction is obtained and hence the result is true. ■

11.3 Exercises

1. Prove that there is no integer a that simulataneously satisfies

$$a \equiv 2 \pmod 6 \qquad \text{and} \qquad a \equiv 7 \pmod 9.$$

2. Let $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.

3. Let $n \in \mathbb{N}$. Suppose that $a \in \mathbb{Z}$ is such that $\gcd(a, n) > 1$. Show, by contradiction, that there is no $k \in \mathbb{Z}$ so that $ak \equiv 1 \pmod n$. This

statement implies that $[a]_n$ is not invertible, which is a concept defined in Exercise 9.7.17.

4. Let $n \in \mathbb{N}$, $n \geq 2$, and $a, b \in \mathbb{Z}$. Prove that if $ab \equiv 1 \pmod{n}$, then $\forall c \in \mathbb{Z}, c \not\equiv 0 \pmod{n}$ we have $ac \not\equiv 0 \pmod{n}$.

5. Prove that there do not exist $x, y \in \mathbb{Z}$ that satisfy the equation $5y^2 - 4x^2 = 7$.

6. Prove each of the following statements:

(a) There is no smallest positive rational number.

(b) There is no smallest positive irrational number.

7. Two irrationality proofs.

(a) Prove that $\sqrt{6}$ is an irrational number.

(b) Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

8. Prove that $\sqrt[3]{25}$ is irrational.

9. Prove that if k is a positive integer and $\sqrt{k}$ is not an integer, then $\sqrt{k}$ is irrational.

10. Let $r, x \in \mathbb{R}$, with $r \neq 0$. Prove by contradiction that if r is rational and x is irrational, then rx must be irrational.

11. Consider the following statements about preserving irrationality under addition and multiplication.

(a) Consider the following faulty proof of the statement, "If $x, y \in \mathbb{R}$ are irrational, then xy is irrational."

Faulty proof. Let $x, y \in \mathbb{R}$ be irrational. Then there are no $m, n, p, q \in \mathbb{Z}$ such that $x = m/n$ and $y = p/q$. Hence for any $m, n, p, q \in \mathbb{Z}$,

$$xy \neq \frac{mp}{nq}.$$

Since $mp, nq \in \mathbb{Z}$, we see that xy cannot be written as a fraction, and so xy is irrational. ■

Show by counterexample that the statement above is false.

(b) Prove or disprove the following statement: If $x, y \in \mathbb{R}$ are irrational, then $x + y$ is irrational.

12. Let $x \in \mathbb{R}$ satisfy $x^7 + 5x^2 - 3 = 0$. Then prove that x is irrational.

13. Consider the following questions about the irrationality of logarithmic values.

(a) Prove that 5^k is odd for all $k \in \mathbb{N}$.

(b) Prove that $\log_2(5)$ is irrational.

(c) Determine for which $n \in \mathbb{N}$ is $\log_2(n)$ irrational. Prove your answer. You may assume the following statement:

For any $n \in \mathbb{N}$, there is some $a \in \mathbb{Z}$, $a \geq 0$ and $b \in \mathbb{Z}$ that is odd, so that $n = 2^a b$.

For this question, you may assume the following properties about the logarithm:

- if $x > 1$, then $\log_2(x) > 0$;
- for any $x, y > 0$,

$$\log_2(xy) = \log_2(x) + \log_2(y);$$

14. Consider the subset of rational numbers

$$A = \left\{ x \in \mathbb{Q} \text{ s.t. } x \leq \sqrt{2} \right\}$$

Prove that it does not have a maximum.

See also Exercise 8.6.15 and Exercise 8.6.16.

15. Prove that there do not exist $a, n \in \mathbb{N}$ such that $a^2 + 35 = 7^n$.

16. Let $x \in (0, 1)$. Show that

$$\frac{1}{2x(1-x)} \geq 2$$

(a) by contradiction, and

(b) by a direct proof.

17. Consider the following statement: For all $x, y \in \mathbb{R}$ with $x, y > 0$ and $x \neq y$

$$\frac{x}{y} + \frac{y}{x} > 2.$$

(a) Prove the statement directly.

(b) Prove the statement by contradiction.

(c) How does the statement change if we remove the assumption $x \neq y$? That is: For all $x, y \in \mathbb{R}$ with $x, y > 0$, what can we say about $\frac{x}{y} + \frac{y}{x}$?

18. Let $a, b \in \mathbb{R}$ with $a, b > 0$. Show that

$$\frac{2}{a} + \frac{2}{b} \neq \frac{4}{a+b}$$

(a) using a contradiction, and

(b) using a direct proof.

19. Recall the *Intermediate Value Theorem:*

Let $g : U \to \mathbb{R}$ where $U \subseteq \mathbb{R}$. Suppose g is continuous on

$[a, b] \subseteq U$, and

$$f(a) \geq c \geq f(b) \quad \text{OR} \quad f(a) \leq c \leq f(b),$$

then there exists $x_0 \in [a, b]$ such that $g(x_0) = c$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, bijective function. Using the Intermediate Value Theorem, prove that f is strictly increasing or strictly decreasing. See Exercise 10.8.19 for the definitions of strictly increasing and decreasing.

Chapter 12

Cardinality

This last chapter of the text brings together many of the ideas and techniques that we have learned to make sense of cardinality — the size of a set. When a set is finite the cardinality is a very intuitive concept — just[160] count up the elements:

$$|\{1, 3, 7, 18, 53\}| = 5.$$

By carefully describing how we count elements in finite sets in terms of **bijections** we can extend our understanding of cardinality to infinite sets. This allows us to make sense of statements such as

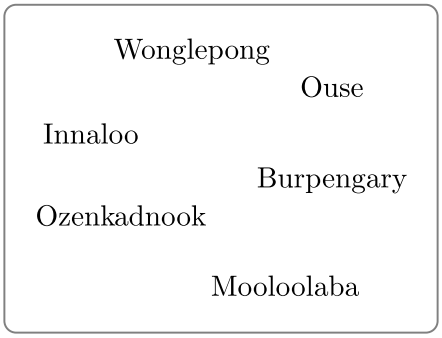
$$|\mathbb{Q}| = |\mathbb{Z}|$$

which are, to say the least, quite counter-intuitive. This also enables us to prove Cantor's Theorem 12.4.3. This result is arguably one of the most important pieces of mathematics that can be proven in undergraduate mathematics. It tells us something fundamental about the nature of infinity: not only are there different sorts of infinities, but there are an infinite number of different infinities!

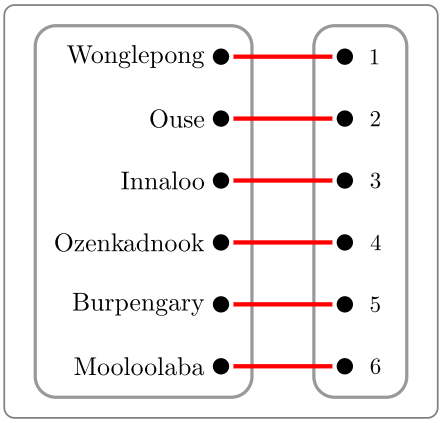
12.1 Finite sets

Consider the following set of place names from Australia:

[160]The reader is right to be a little skeptical of the use of the word "just" here; counting up the elements of a finite set can sometimes be quite difficult. Here we have listed out all the elements quite explicitly, but somes a set will be defined more *implicitly*, and then counting the elements can be quite challenging. The interested reader should search-engine their way to a description of enumerative combinatorics which is the mathematics of counting.



Think about what you do when you *count* the elements of that set. Now obviously[161] there are 6 elements in the set. However, think about how you count up those elements. Indeed, think about how we learn to count when we are very young. Typically we count by pointing at each element in turn (either physically pointing, or just pointing in our mind's eye), and counting off "one, two, three,…".



So what we are really doing here is constructing a function that takes us from the set of objects that we are counting to a subset $\{1, 2, 3, \ldots, n\}$. This function is both

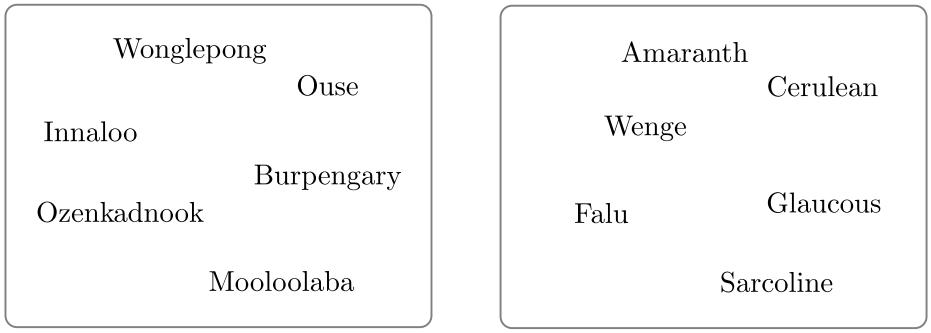
- injective — since different objects will be *counted* by different numbers, and
- surjective — since every number is used to *count* an object.

Hence when we count the objects in a finite set A we are really constructing a bijection from A to a subset of the natural numbers, $\{1, 2, 3, \ldots, n\}$.

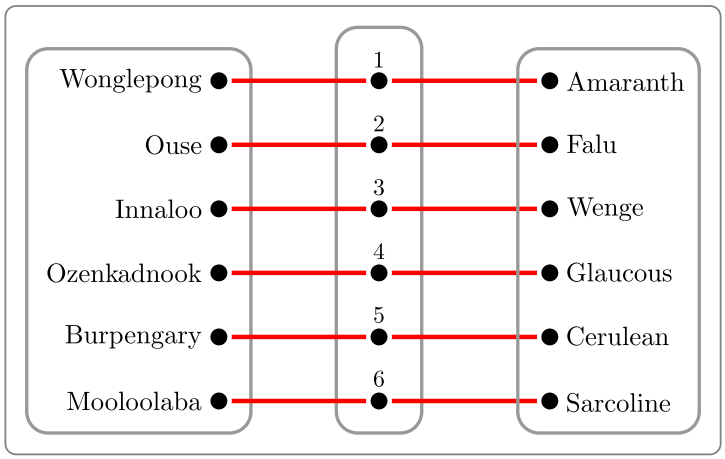
Now notice also that if we consider the following two sets[162]:

[161] While it is generally a good idea to avoid this word, it is probably safe to use it here. Clearly it is obviously safe to leave a decision on the use of this word to the reader who may edit their copy of the text accordingly.

[162] The same set of place names, and a set of "interesting" colour names.



Then we see that there are the same number of elements in each. We can do this in two ways, one is to count up as we did in the previous figure, but we could also simply establish a bijection between them.



These are actually equivalent. If there is a bijection, f from the set of place names to the set $\{1, 2, 3, 4, 5, 6\}$, and another g from the set of colours to the set $\{1, 2, 3, 4, 5, 6\}$, then g^{-1} is a bijection, and the composition $g^{-1} \circ f$ will be a bijection from the set of place names to the set of colours. In this way, two sets have the same size when there is a bijection between them.

In the above discussion we have tried to leverage the language of functions in order to compare the sizes of sets. Using functions allows us to extend the idea of "these sets are the same size" from finite sets to *infinite* sets. That is the main aim of this part of the text.

12.1.1 Equinumerous sets, bijections and pigeons

Oof! There are a lot of ideas above, so let's set them down slowly, carefully and rigorously.

Definition 12.1.1 Two sets A, B are said to have the same **cardinality** (or same **cardinal number**), written $|A| = |B|$, if either $A = B = \varnothing$ or there is a bijection from A to B. If A and B have the same cardinality then we say they are **equinumerous**[163]. Finally, if A and B are not equinumerous, we write $|A| \neq |B|$, and this is equivalent to saying that there is no bijection between them. ◊

This definition allows us to understand cardinality in terms of functions; this is critical for our understanding the size of infinite sets.

- When we write $|A| = n \in \mathbb{N}$, we are really stating that there is a bijection

$$f : A \to \{1, 2, 3, \ldots, n\}$$

  From our work on functions (see Theorem 10.6.8), we know that this also implies that there is a bijection back from $\{1, 2, \ldots, n\}$ to A — just the inverse function f^{-1}.

- And when $A = \varnothing$ we write $|A| = 0$. This is a special case that we have to treat separately; we cannot define a function with empty domain.

Example 12.1.2 Finite non-equinumerous sets. Say we have two finite sets A, B, so that they are not equal in size, $|A| \neq |B|$. Then there cannot be a bijection between them. We will shortly prove this in general, but suppose (for the sake of this discussion) that $|A| = 5$ and $|B| = 3$. So we can write our sets as

$$A = \{a_1, a_2, a_3, a_4, a_5\} \qquad B = \{b_1, b_2, b_3\}$$

First consider trying to construct a function $f : A \to B$. It is easy to construct a surjection:

$$f(a_1) = b_1, f(a_2) = b_2, f(a_3) = f(a_4) = f(a_5) = b_3.$$

However it is impossible to construct an injection. Once we have assigned $f(a_1), f(a_2), f(a_3)$ to three distinct values in B, it is impossible to assign $f(a_4)$ without repeating one of the values from B — making our function non-injective.

Now try construct a function $g : B \to A$. It is easy to construct an injection:

$$g(b_1) = a_1, g(b_2) = a_2, g(b_3) = a_3$$

However it is impossible to construct a surjection. Once we have assigned $g(b_1), g(b_2), g(b_3)$, there will still be two elements of A that have no preimage in B. □

The reasoning in the previous example is formalised by the **Pigeonhole Principle**[164].

[165] or "numerically equivalent", but "equinumerous" is a nicer word.

[164] While this is intuitively obvious, mathematicians like to make sure that obvious things are actually true. And if that obvious thing is really useful, then it should get a good name. Dirichlet was the first to formalise this idea in the 19th century and called it (in German) the

Theorem 12.1.3 Pigeonhole principle — Dirichlet's Schubfachprinzip. *If $n > 0$ objects are placed in $k > 0$ boxes then*

- *if $n < k$ then at least one box has zero objects in it, and*
- *if $n > k$ then at least one box has at least two objects in it.*

*In the second case, we can be more precise: at least one box contains at least $\lceil \frac{n}{k} \rceil$ objects, where $\lceil x \rceil$ is the **ceiling** of x and denotes the smallest integer larger or equal to x.*

Proof. We prove the contrapositive of the first statement, and then prove the second by contradiction.

- Assume each box has at least one element in it, then there must be at least k objects. Hence the number of objects is at least as large as the number of boxes.
- Assume, to the contrary, that every box contains at most $\lceil \frac{n}{k} \rceil - 1$ objects, then the total number of objects is

$$k \cdot \left(\left\lceil \frac{n}{k} \right\rceil - 1 \right) = k \left\lceil \frac{n}{k} \right\rceil - k < n,$$

where we have used the fact that $\lceil x \rceil - 1 < x \leq \lceil x \rceil$. This gives a contradiction. Hence at least one box contains at least $\lceil \frac{n}{k} \rceil$ objects.

■

An immediate corollary is the following result for *finite* sets that formalises the last example.

Corollary 12.1.4 *Let A, B be finite sets and let $f : A \to B$ be a function. Then*

- *If $|A| > |B|$ then f is not an injection.*
- *If $|A| < |B|$ then f is not a surjection.*

Proof. We prove each in turn. Assume that A, B are finite.

- Assume that $|A| > |B|$. Then when the images of elements of A are placed into B by the function, there must, by the pigeonhole principle, be at least one element of B which is the image of $\left\lceil \frac{|A|}{|B|} \right\rceil > 1$ elements of A. Hence f is not injective.
- Assume that $|A| < |B|$. Then when the images of elements of A are placed into B by the function, there must, by the pigeonhole principle, be at least one element of B which is not the image of any element of A. Hence f is not surjective.

---

drawer-principle — Schubfachprinzip. Since Dirichlet's father was a postmaster, it is perhaps an imperfect translation of his idea that led to "pigeonholes" in the sense of a small open drawer or shelf used to sorting or storing letters (slightly antiquated in these days of tweeting (pun!), email, and social media updates). This imperfect English translation has been imported into other languages, including back into German. No pigeons are involved.

■

We should notice the contrapositive of the results in the above corollary. Let A, B be finite sets and $f : A \to B$ be a function, then

- if f is an injection then $|A| \leq |B|$, and
- if f is a surjection then $|A| \geq |B|$.

Notice that one can do much more with the pigeonhole principle than just put objects in boxes — a little trip to your favourite search engine will turn up many many examples. We'll give a few examples, most quite standard, and one definitely not.

Example 12.1.5 Fix $n \in \mathbb{N}$ and let $S = \{1, 2, \cdots, 2n-1\}$ and let $A \subseteq S$ so that $|A| = n + 1$. Prove that there are two elements of A whose sum is $2n$.

Solution.

Proof. We can write $2n$ as the following sums of pairs from S:

$$2n = (2n-1) + 1 = (2n-2) + 2 = \cdots = (n+1) + (n-1)$$

Notice that the number n cannot be used in such a pair.

So split the set S up into 2 element subsets and $\{n\}$:

$$\{1, 2n-1\}, \{2, 2n-2\}, \ldots, \{n-1, n+1\}, \{n\}$$

There are n sets in this list. Consequently, when we choose n elements it is possible to choose one element from each of the above subsets, however when we choose one more, we must choose a second element from one of those two element subsets. The two elements from the two element-subset sum to $2n$ as required.

■

□

Example 12.1.6 Let $S = \{1, 2, \cdots, 20\}$ and let $A \subseteq S$ so that $|A| = 11$. That is, A contains 11 distinct integers from between 1 and 20 (inclusive). Then there exist $a, b \in A$ so that $a \mid b$.

Solution.

Proof. First notice that we can write any natural number as the product of an odd number and a power of 2.

$$\forall n \in \mathbb{N}, \exists k, \ell \in \mathbb{N} \text{ s.t. } n = (2k+1) \cdot 2^{\ell}$$

So we can take any $n \in A$ keep dividing by 2 until you get an odd number. When you do so the resulting odd number must be in the set $\{1, 3, 5, \cdots, 19\}$. Since this set contains 10 distinct elements, there must be two numbers in A that result the same odd number in $\{1, 3, 5, \cdots, 19\}$. We can write these numbers as

$$a = (2k+1)2^i \qquad \text{and} \qquad b = (2k+1)2^j$$

where $i \neq j$. If $i > j$ then $b \mid a$ and if $i < j$ then $a \mid b$. In either case there must be a pair of integers in A for which one divides the other. ■

□

Example 12.1.7 Spurious correlations. An excellent illustration of the pigeonhole principle is given by the database of "spurious correlations" — see the fantastic website and book by Tyler Vigen[165].

It goes roughly like this — consider how many simple (ie not-too wiggly) graphs you can draw whose horizontal axis is the last 100 years. There might be (say) a hundred such "simple" graphs. Now build a big database of any statistics you can think of — diary production in Quebec, deaths by lightning, number of twins born in a particular city, etc etc. There are thousands and thousands of such statistics.

Now — associate each of these statistics to the curve that best approximates it. Since there are only (say) a hundred such curves, and many thousands of statistics, there must — by the pigeonhole principle — be at least one curve which corresponds to many statistics. In practice, there are many statistics for each curve. This means that those statistics are correlated. In this way you can see that, say, the number of mathematics PhD's awarded is highly correlated with the quantity of uranium stored at US power plants. Is there any causal link — nope[166].

This can be a source of fun (well, mathematician fun), but it can also create problems. The idea of making use of "Big data" to solve problems is getting a lot attention. People even announcing that analysis of huge data sets will replace[167] the scientific method! However, given enough data, you will find correlations everywhere. It is just a matter of putting objects in boxes — the pigeonhole principle at work. □

12.1.2 Comparing with functions

In the previous section we started to link cardinality of sets A, B and the types of functions that can be constructed between them. For example, we saw that

- If $f : A \to B$ is an injection then $|A| \leq |B|$,
- If $g : A \to B$ is a surjection then $|A| \geq |B|$, and
- If $h : A \to B$ is a bijection then $|A| = |B|$

In this section we demonstrate that this way of comparing sizes of sets is well-defined. For example, we should have that equality of cardinality behaves just like equality of integers:

$$|A| = |A|$$

and

$$(|A| = |B|) \implies (|B| = |A|)$$

[167] tylervigen.com/spurious-correlations

[168] We hope not.

[169] This was a big headline article in Wired magazine in 2008.

and

$$(|A| = |B|) \wedge (|B| = |C|) \implies (|A| = |C|)$$

That is, we need to show that equinumerous is an equivalence relation.

Theorem 12.1.8 *Let A, B and C be sets. Then*

- $|A| = |A|$ *(reflexive).*
- *If* $|A| = |B|$ *then* $|B| = |A|$ *(symmetric).*
- *If* $|A| = |B|$ *and* $|B| = |C|$ *then* $|A| = |C|$ *(transitive).*

Proof.

- The identity function on A, $i_A : A \to A$ is a bijection and thus $|A| = |A|$.
- Assume $|A| = |B|$. Then there is a bijection $f : A \to B$. Since f is bijective the inverse function $f^{-1} : B \to A$ exists and is bijective. Thus there is a bijection from B to A and so $|B| = |A|$.
- Assume $|A| = |B|$ and $|B| = |C|$, so there are bijections $f : A \to B$ and $g : B \to C$. Since the composition of bijections is bijective, it follows that $h : A \to C$ defined by $h = g \circ f$ is bijective. Thus $|A| = |C|$.

■

This result is very useful. In order to show that two sets have the same size we can show they are equinumerous with a third set.

Remark 12.1.9 Cardinality inequalities. In addition to proving the above properties of equality of cardinalities, we should also prove analogous results for inequalities of cardinalities. In particular, we should also show that

$$|A| \leq |A|$$

and

$$(|A| \leq |B|) \wedge (|B| \leq |A|) \implies (|A| = |B|)$$

and

$$(|A| \leq |B|) \wedge (|B| \leq |C|) \implies (|A| \leq |C|)$$

Notice that the first follows immediately since the identity function on A is an injection. The last one follows because the composition of injections is itself an injection (this is exactly Theorem 10.5.3). The middle one is the Cantor-Schröder-Bernstein theorem and its proof is quite involved — see Section 12.5. It is reasonably easy to prove for finite sets, however. Maybe we'll set that as an exercise.

12.1.3 Infinite sets are strange.

When we deal with finite sets everything above is pretty clear cut — we can put things into a bijection with $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. We did exactly this

with our place-names and colours example at the start of this chapter. However, when we get to non-finite sets, things get much more interesting and weird.

Here is a very telling example. Consider the positive even numbers

$$E = \{2, 4, 6, 8, \dots\}$$

Now it is clear that E is a proper subset of $\mathbb{N}$; it only contains ever second number. So it should[168] be half the size. But we have to be careful; consider the following function

$$f : \mathbb{N} \to E \qquad\qquad f(n) = 2n$$

This function is a bijection from $\mathbb{N}$ to E:

- Injective because if $n_1 \neq n_2$ then $f(n_1) = n_1 \neq n_2 = f(n_2)$.
- Surjective because for any $b \in E$, we know that $b = 2k$ for $k \in \mathbb{N}$, and $f(k) = 2k = b$.

So the set of natural numbers is equinumerous with a proper subset of the natural numbers! No finite set can do this because of the pigeonhole principle, but an infinite set can. This is actually one way of defining when a set is infinite:

Definition 12.1.10 We say that a set A is **infinite** when there is a bijection between A and a proper subset of A. ♢

This definition was introduced by Dedekind in the late 19th century. Notice that this way of defining infinity does not rely on the natural numbers. When we first learn of the existence of infinity as children it is usually in the context of having learned to count

$$1, 2, 3, 4, \dots$$

and we ask ourselves (or our teacher, or our parent[169]) — what is the biggest number? And we realise that we can go on counting forever without stopping. This idea of infinity as being somehow intimately related to the natural numbers is hard to shake. But we will a little later in this chapter.

The apparent paradox[170] in Dedekind's definition of infinite sets goes back at least as far as Galileo, and is often called Galileo's paradox. In his "Two new sciences" Galileo imagines a conversation between two people, Simplicio and

[168] "Should" is another one of those dangerous words like "clearly" and "obviously" and "just". It should (ha!) set off alarm bells.

[169] Perhaps repeatedly?

[170] It is a veridical paradox. That is, a paradox that seems absurd but is actually true. Another good example, for the paradoxically inclined, is the Monte Hall problem. The interested reader can (and should) search-engine their way to that particular example.

Salviati[171] . Simplicio brings up the bijection[172] between the natural numbers at the squares:

$$f : \{1, 2, 3, \dots\} \mapsto \{1, 4, 9, \dots\} \qquad f(n) = n^2$$

and the apparent paradox of something larger being equinumerous with something smaller. After some discussion Salviati concludes that one cannot compare cardinalities of infinite sets. That restriction stayed until the 19th century work of Cantor and others.

Georg Cantor developed modern set theory (especially between about 1874 and 1884) — he developed the finer understanding of infinite sets we now have. Before him, there were finite sets, and a somewhat loosely defined idea of infinity. Cantor's work on infinity was not well received by the maths community of the 19th century and a number of big names in mathematics very publically criticised him — "corrupter of youth" was one of many insults[173] . It is worth search-engining your way to a biography of him and the intersection between his work on infinity and many ideas in the philosophy of mathematics and theology. Not light stuff to be summarised here in a quick paragraph.

12.2 Denumerable sets

As described above, our intuitive notion of an infinite set (or any infinite thing), is a process that keeps on going. We never get to the end because we can always take another step. What we are really doing when we think of infinity in this way is setting up a bijection between it and the set of natural numbers. There is some infinite process — we start at the beginning (the number 1) and then we take a step $n \mapsto n + 1$, and then another step, and another step and, ….. This is the first type of infinity that we will encounter — sets that are in bijection with $\mathbb{N}$. We call these sets denumerable because we can think of counting off the elements via the bijection.

[171] Salviati is named after one of Galileo's friends, while Simplicio is likely modelled on one of his philosophical opponents, or perhaps is intended to represent Galileo's beliefs early in his lief. Both characters appear in Galileo's work "Dialogue Concerning the Two Chief World Systems" which discusses the relative merits of the Copernican (heliocentric) and Ptolemaic (geocentric) models of the solar system. That book was subsequently banned by the Catholic Church for over 200 years.

[172] He doesn't call it a bijection — that term didn't come into mathematics until the middle of the 20th century with the work of Bourbaki.

[173] Henri Poincaré called his work on infinities a "grave disease" infecting the discipline of mathematics, while Leopold Kronecker described Cantor as a charlatan and a renegade as well as the famous epithet "corrupter of youth" and "I don't know what predominates in Cantor's theory — philosophy or theology, but I am sure that there is no mathematics there." Kronecker is reputed to have said that "God created the natural numbers; all else is the work of man." — making him (arguably) a proponent of mathematical finitism. It's an interesting topic and well worth a quick trip to your favourite search-engine.

Definition 12.2.1 Let A be a set.

- The set A is called **denumerable** if there is a bijection $f : \mathbb{N} \to A$.
- The cardinal number of a denumerable set is denoted $\aleph_0$ (read "aleph naught" or "aleph null").
- The set A is called **countable** if it is finite *or* denumerable.
- The set A is **uncountable** if it is not countable.

$\Diamond$

Notice that

- At this stage it is not clear that there is something out there that is "uncountable" — the existence of uncountable sets was highly controversial when Cantor proved it in 1874[174].
- There is a bijection from $\mathbb{N}$ to A if and only if there is a bijection from A to $\mathbb{N}$.
- Finally, a very nice property of denumerable sets is that even though they are infinite, we can still "list out the elements".

This last point is both very important and also is a little counter-intuitive, so we'll make it more precise. Consider a denumerable set A. By definition there is a bijection $f : \mathbb{N} \to A$. Now we can think of f as a relation

$$f = \{(1, f(1)), (2, f(2)), (3, f(3)), \dots\}$$

Hence we can write A as

$$A = \{f(1), f(2), f(3), \dots\} \qquad \text{or equivalently}$$
$$A = \{a_1, a_2, a_3, \dots\} \qquad \text{with } a_i = f(i).$$

Notice that this list has a couple of special properties.

- Since f is injective, our list does not repeat — different indices give different elements of A.
- Also, since f is surjective, any given element of A — say q — is mapped to by some integer $n_q \in \mathbb{N}$. Since n_q is finite, this means that any given element of A appears on the list in some finite position.

We can also go backwards — if we can list out the elements of some infinite set B, then B is denumerable. Say we have listed out our elements as $\{b_1, b_2, b_3, \dots\}$, so that

[174] It first appeared in an article called "On a Property of the Collection of All Real Algebraic Numbers". He did not use his famous diagonal-argument (we do it a little later in the text); that argument appeared 17 years later in a 1891 paper titled "On an elementary question of the theory of manifolds."

- the list does not repeat, and
- any given element of B appears at some finite position on the list.

Now we can define a function $g : \mathbb{N} \to B$ by $g(j) = b_j$. That is, just map any natural number to the element at that position in the list. The two conditions we have placed on the list then mean that g is injective and surjective, so g is a bijection. Hence B is denumerable.

Let's formalise this idea that denumerable means listable in a lemma. The proof is given by the argument above.

Lemma 12.2.2 Listing elements of denumerable sets. *Let A be a set. If A is denumerable then we can construct an infinite list, $a_1, a_2, a_3, \ldots$, of all its elements so that*

- *the list does not repeat any element of A, and*
- *any given element, $a \in A$ appears at some finite position in the list*

The converse of this statement is also true. Namely, if we can construct such a list of the elements of A, then A is denumerable.

Proof. Let A be denumerable. Then, by definition, there exists a bijection $f : \mathbb{N} \to A$. For any $n \in \mathbb{N}$, define the n^{th} element of our list to be $a_n = f(n)$. Then

- Since f is injective, we know that for $j \neq k$, $a_j = f(j) \neq f(k) = a_k$. So the items on our list are distinct.
- Since f is surjective, we know that for any $x \in A$ there exists $n \in \mathbb{N}$ so that $f(n) = x$. Thus the element $x = a_n$ is the n^{th} item on the list, and so appears at a finite position in the list.

Now assume such a list exists. Then for any $n \in \mathbb{N}$ define $g(n) = a_n \in A$. Since the list is infinite this defines a function $g : \mathbb{N} \to A$. Then

- Let $x \in A$. By assumption, the element x appears at some finite position in the list. So there is $n \in \mathbb{N}$ so that $x = a_n = g(n)$. Thus g is surjective.
- Since the list does not repeat we can take the items from the j^{th} and k^{th} positions on the list, namely a_j, a_k with $j \neq k$. Then we know that $a_j \neq a_k$ which in turn gives $g(j) \neq g(k)$. Thus g is injective.

Hence there is a bijection $g : \mathbb{N} \to A$ and so A is denumerable. ■

This infinite-but-listable way of looking at denumerable sets makes it much easier to work with them. Especially because, as we have seen with our even numbers example above, infinite sets are very counter-intuitive. Here is counter-intuitive result that is also a very important result.

Theorem 12.2.3 *The set of all integers is denumerable.*

Proof sketch. We will sketch out how to prove this result but leave the formal proof to the reader. It suffices to find a "nice way" to list out all the elements of $\mathbb{Z}$ (since this nice list is really a bijection). We can't just do $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ — which each integer appears on the list, it is not clear that any given integer appears at some finite position in the list. This sort of infinite list is typically called bi-infinite, since it extends to infinity in both direction.

On the other hand, if we write out the integers as

$$\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$$

then we can see that every integer will appear once, and that any given integer will appear at some finite position in the list. Indeed, this list, carefully described, is a proof of the result.

We can, in this case, make the bijection very explicit in this case and define a function

$$f = \begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & \cdots \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & 0 & 1 & -1 & 2 & -2 & \cdots \end{array}$$

With a bit of juggling this becomes

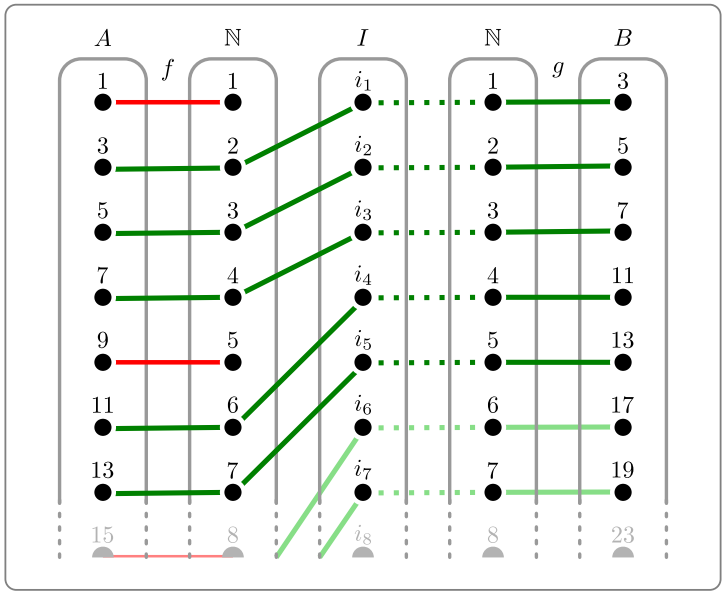
$$f(n) = \begin{cases} \frac{1-n}{2} & n \text{ is odd} \\ \frac{n}{2} & n \text{ is even} \end{cases} = \frac{1 + (-1)^n(2n-1)}{4}$$

It is not hard to show this is bijective — in fact, we'll make this an exercise. ■

So again we have produced an example of a strict subset having the same cardinality: $\mathbb{N} \subset \mathbb{Z}$ but $|\mathbb{N}| = |\mathbb{Z}|$. This is quite general; a subset of a denumerable set is either finite or itself denumerable.

Theorem 12.2.4 *Let A be a denumerable set and let $B \subseteq A$. Then B is countable.*

Now this proof is a little fiddly around the edges so we'll just give a proof sketch. Before we get started with the proof, consider the specific case of A being the set of positive odd numbers, and B being the set of odd prime numbers.



On the right-hand side of the figure we've drawn the set of odd primes, B, and the natural numbers and the bijection g between them. On the left-hand side we've given the set of odd numbers, A and next to it we've drawn the set of natural numbers $\mathbb{N}$ and the bijection f between them. Notice that we've drawn some edges in red and some in green. The red edges correspond to the elements of A that are not in B, while the green edges correspond to the elements in A that are in B. Notice that the red edges stop, while the green edges trace through a set I (described in the proof sketch below), and then another copy of $\mathbb{N}$ and then finally to the set B. Keep this picture in mind when you read the proof sketch below.

Actually just a sketch of a proof. Let B be a subset of a denumerable set A.

- If B is finite then we are done, because a finite set is countable.
- Assume B is infinite. There is a bijection $f : \mathbb{N} \to A$, and we can write $A = \{a_1, a_2, \dots\}$ and $f(n) = a_n$. Now define $I = \{n \in \mathbb{N} \mid a_n \in B\}$ — ie the indices of elements that are in B. Notice that $I \subseteq \mathbb{N}$ and so we can write the indices in order[a]

  Hence we can write the set of these indices as $I = \{i_1, i_2, i_3, \dots\}$. Now define a function $g : \mathbb{N} \to B$ by

  $$g(n) = a_{i_n}$$

  That is, for any input number n, look up the n^{th} index in I, and then return that element of A. By construction that is an element of B.

  We can trace this though our odd-primes figure above. In that case $g(3) = 7$ because $i_3 = 4$ and $f(4) = 7$. Similarly, $g(5) = 13$ since $i_5 = 7$ and $f(7) = 13$.
- Now that we have this function, we need to prove it is bijective.

- Injective — Assume that $g(n) = g(m)$, then $a_{i_n} = a_{i_m}$. Since the original function f is bijective, this implies that $i_n = i_m$. Hence $n = m$.
- Surjective — Let $b \in B$. Then since $B \subseteq A$, we know $b \in A$. Since f is surjective, there is some n so that $f(n) = a_n = b$. Hence $n \in I$. Hence there is some $1 \leq k \leq n$ so that $g(k) = a_n = b$.

- Thus there is a bijection from $\mathbb{N}$ to B, and so B is denumerable.

■

[a] This bit actually requires a bit of thought — one can use the well-ordering principle to find the smallest thing in the set I and call it i_1.Then remove that from the set and find the next smallest, call it i_2. etc etc.

Let's generalise our result above positive even numbers being in bijection with the naturals; we'll also do Galileo's paradox the same way. Using the above theorem we don't have to establish explicit bijections, we just need to show that they are infinite subsets of a denumerable set.

Result 12.2.5 *Let $k \in \mathbb{N}$, then the sets*

$$k\mathbb{Z} = \{kn \mid n \in \mathbb{Z}\} \qquad k\mathbb{N} = \{kn \mid n \in \mathbb{N}\} \qquad \left\{n^2 \mid n \in \mathbb{N}\right\}$$

are denumerable.
Proof. Let $k \in \mathbb{N}$. Then above sets all subsets of $\mathbb{Z}$. Hence by the previous theorem they are countable. All of the sets are infinite, so they must be denumerable. ■

One can show that the union of denumerable sets is denumerable and that the Cartesian product is too.

Result 12.2.6 *Let A and B be denumerable sets, then $A \times B$ is denumerable.*
Proof. It suffices to find a bijection from $\mathbb{N}$ to $A \times B$. Since A and B are denumerable we can write $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$. Construct the following (infinite) table

	b_1	b_2	b_3	$\dots$
a_1	(a_1, b_1)	(a_1, b_2)	(a_1, b_3)	$\dots$
a_2	(a_2, b_1)	(a_2, b_2)	(a_2, b_3)	$\dots$
a_3	(a_3, b_1)	(a_3, b_2)	(a_3, b_3)	$\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

We form the function $f : \mathbb{N} \to A \times B$ by sweeping successive diagonals $\swarrow \swarrow \cdots$.

$$\begin{array}{llll} f(1) = (a_1, b_1) & & & \\ f(2) = (a_1, b_2) & f(3) = (a_2, b_1) & & \\ f(4) = (a_1, b_3) & f(5) = (a_2, b_2) & f(6) = (a_3, b_1) & \\ f(7) = (a_1, b_4) & f(8) = (a_2, b_3) & f(9) = (a_3, b_2) & f(10) = (a_4, b_1) \end{array}$$

and so forth. Since any given ordered pair is reached eventually (ie after a finite

number of diagonal sweeps of finite length), the function is surjective. Since we never repeat an ordered pair, the function is injective. Thus f is bijective. ■

Notice that if we tried to list out the elements by reading out each column (or each row), then the resulting function would not be surjective — it would take an infinite time to reach the second column. Hence the preimage under that function of an element in the second column would not be a natural number.

Very similarly we arrive at the following very counter-intuitive result

Theorem 12.2.7 *The set of positive rational numbers $\mathbb{Q}^+$ is denumerable.*
Proof. Form the following table of positive rationals

	1	2	3	...
1	1/1	1/2	1/3	...
2	2/1	2/2	2/3	...
3	3/1	3/2	3/3	...
⋮	⋮	⋮	⋮	⋱

List things out in the same sweeping diagonal order $\swarrow\ \swarrow\ \cdots$ we used previously:

$$\begin{array}{lllll}
\frac{1}{1}, & & & & \\
\frac{1}{2}, & \frac{2}{2}, & & & \\
\frac{1}{3}, & \frac{2}{2}, & \frac{3}{1}, & & \\
\frac{1}{4}, & \frac{2}{3}, & \frac{3}{2}, & \frac{4}{1}, & \\
\frac{1}{5}, & \frac{2}{4}, & \frac{3}{3}, & \frac{4}{2}, & \frac{5}{1}, \\
\vdots & & & &
\end{array}$$

This list gives a surjective function, since any given positive rational number would appear at some finite positive position on the list. For example, we have $f(1) = \frac{1}{1}, f(2) = \frac{1}{2}, f(5) = \frac{2}{2}$, and so on. However, the list repeats rationals, so it is not injective.

Thankfully this is quite easy to fix, we can simply skip those rationals that have already appeared[a]:

$$\begin{array}{lllll}
\frac{1}{1}, & & & & \\
\frac{1}{2}, & \frac{2}{1}, & & & \\
\frac{1}{3}, & \cdot & \frac{3}{1}, & & \\
\frac{1}{4}, & \frac{2}{3}, & \frac{3}{2}, & \frac{4}{1}, & \\
\frac{1}{5}, & \cdot & \cdot & \cdot & \frac{5}{1},
\end{array}$$

where we have used a dot to indicate a fraction we have skipped. Define $f : \mathbb{N} \to \mathbb{Q}^+$ by $f(n)$ is the nth term of the list. So $f(1) = \frac{1}{1}, f(3) = \frac{2}{1}, f(5) = \frac{3}{1}, f(7) = \frac{2}{3}$ and so on.

Since every positive rational number appears on the list (at some finite position), f is surjective. Since no number is repeated, f is injective. Thus f is bijective and so $|\mathbb{N}| = \mathbb{Q}^+$. ■

[a]Notice that we do not have to be able to do this efficiently, we just need to be able to do it! Mind you, it is not too hard to work out that a given ratio $\frac{a}{b}$ has already appeared on the list when a, b have a common divisor.

An alternative proof. We can also prove this result using our previous theorem, by constructing a bijection from $\mathbb{Q}^+$ to an infinite subset of $\mathbb{N} \times \mathbb{N}$.

Recall that we can write any $q \in \mathbb{Q}^+$ uniquely as $\frac{a}{b}$ where a, b are natural numbers with no common divisors. Then we can define $f : \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$ by

$$f(q) = (a, b) \qquad \text{where } q = \frac{a}{b}$$

and $a, b \in \mathbb{N}$ with no common divisors. We claim that this function is an injection.

Let $p, q \in \mathbb{Q}^+$ so that $p \neq q$. Now if $f(p) = f(q) = (a, b)$ we must have $p = q = \frac{a}{b}$. Thus we must have $f(p) \neq f(q)$.

The injection f can be specialised to a bijection $\hat{f} : \mathbb{Q} \to \text{rng}(f)$ by reducing its codomain to exactly the range of f. Since $\text{rng}(f)$ is an infinite subset of $\mathbb{N} \times \mathbb{N}$, it must be denumerable. Hence $\mathbb{Q}^+$ is denumerable. ■

Now say that the above order gives

$$\begin{aligned} \mathbb{Q}^+ &= \{q_1, q_2, q_3, \dots\} \\ \mathbb{Q}^- &= \{-q_1, -q_2, -q_3, \dots\} \end{aligned}$$

Then we can write

$$\begin{aligned} \mathbb{Q} &= \{0\} \cup \mathbb{Q}^+ \cup \mathbb{Q}^- \\ &= \{0, q_1, -q_1, q_2, -q_2, \dots\} \end{aligned}$$

And so by using this sneaky way of listing the rationals, we get

Corollary 12.2.8 *The set of all rational numbers is denumerable.*

This is really weird. While the natural numbers and the integers feel very similar; there are nice discrete unit steps between numbers. The rational numbers, however, appear to be very different objects. Most striking being that they are dense — between any two rationals we can find another rational number. But despite that, the rationals are the same size as the integers!

The trick we used above to prove this corollary is a good one and can be extended to a more general result:

Result 12.2.9 *Let A, B be countable sets. Then the union $A \cup B$ and intersection $A \cap B$ are also countable.*
Proof.

- The intersection $A \cap B \subseteq A$, so if A is finite then so is the intersection. While if A is denumerable, then our previous theorem tells us that all its subsets are countable. So we are done.

- Now consider the union. Since A, B are countable, we can list out their elements as

$$A = \{a_1, a_2, a_3, \dots\} \qquad B = \{b_1, b_2, b_3, \dots\}$$

Assume, for the moment that $A \cap B = \varnothing$.

  - If both A, B are finite then their union is finite.
  - If both A, B are infinite, then write the union as

$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$$

    This listing of the elements of the union demonstrates that the union is denumerable.
  - If one of A, B is finite, but the other infinite, then we can write

$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots, a_n, b_n, a_{n+1}, a_{n+2}, a_{n+3}, \dots\} \quad \text{or}$$
$$A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots, a_\ell, b_\ell, b_{\ell+1}, b_{\ell+2}, b_{\ell+3}, \dots\}$$

    depending on which of A, B are finite. This demonstrates that the union is denumerable.

Thus the union of two disjoint countable sets is countable.

Now assume that $A \cap B \neq \varnothing$, then write $C = B - A$, so that

$$C \subseteq B \qquad A \cap C = \varnothing \qquad A \cup C = A \cup B.$$

These are quite straight-forward to prove:

  - If $x \in C$ then $x \in B - A$ and so $x \in B$.
  - Let $x \in A$. Then we must have that $x \notin B - A$, and thus $x \notin C$. Hence $A \cap C$ must be empty.
  - Since $C = B - A = B \cap \bar{A}$, we have

$$A \cup C = A \cup (B \cap \bar{A}) = (A \cup B) \cap (A \cup \bar{A}) = (A \cup B) \cap U = A \cup B$$

    where U denotes the universal set.

Now, since B is countable we know that C is countable. The reasoning used for disjoint sets above, then shows that $A \cup C$ is countable, and thus $A \cup B$ is countable.

■

12.3 Uncountable sets

We are now ready to move on to one of the nicest results of the course and arguably one of nicest in Mathematics. We will prove that there is no bijection between the natural numbers and the reals and so that the reals are uncountable. It is due to Georg Cantor. He proved the result first in 1873, and again by a simpler method in 1891. We will do the second version here, since it is easier. We give Cantor's first proof as an optional section 12.6 later in this chapter.

The proof works by a contradiction and also relies on some facts about decimal expansions of real numbers.

- Every rational number has a repeating decimal expansion
  - eg $1/3 = 0.333333333\ldots$
  - eg $2/11 = 0.1818181818181\ldots$
- Some rational numbers have two repeating expansions
  - eg $1/2 = 0.500000\ldots = 0.499999999\ldots$
  - eg $1/5 = 0.2\ldots = 0.1999999999\ldots$
- One can show that a rational number p/q (reduced) has two expansions if only if q is a product of powers of 2 and 5. In this case the expansions terminate either with 9's or 0's.
- Every irrational number has a unique (non-repeating) decimal expansion

None of the above is too hard to prove, but we won't do it here. We want to get on to this very important result.

Theorem 12.3.1 Cantor 1891. *The open interval* $(0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ *is uncountable.*

From this result we could prove that all of $\mathbb{R}$ is uncountable by constructing a suitable bijection (which we will do so at the end of this section). Alternatively, and perhaps more usefully, we can argue that because $(0,1)$ is a subset of $\mathbb{R}$, then $\mathbb{R}$ cannot be denumerable. By our previous theorem, if $\mathbb{R}$ were denumerable, then it could not have an uncountable subset. This observation is a pretty important one, so let us write it down as its own theorem.

Theorem 12.3.2 *Let A, B be sets with $A \subseteq B$. If A is uncountable then B is uncountable.*

Proof. Prove the contrapositive. If B is countable, then it is either finite or denumerable. If B is finite, then A must be finite. On the other hand if B is denumerable then all its subsets must be denumerable. In either case A must be countable. ■

So — how do we prove the uncountability of $(0,1)$. This comes down to showing that there cannot be a bijection from $\mathbb{N}$ to $(0,1)$. Showing the non-existence of an object is often most easily done by contradiction and that is the approach we will take (following in Cantor's footsteps).

So we assume that $(0,1)$ is denumerable meaning there is some bijection $f : \mathbb{N} \to (0,1)$. But this means that we can make a big list of all the numbers in $(0,1)$ (just like we did with the even numbers or the squares or … above). For example, we might have:

$$\begin{aligned} f(1) &= 0.78304492\ldots \\ f(2) &= 0.21892653\ldots \\ f(3) &= 0.15206327\ldots \\ &\vdots \end{aligned}$$

As we construct our list, if we come across a number in $(0,1)$ that has two expansions (like $1/2 = 0.49999 = 0.50000$) then we write down the expansion that ends in all zeros. To complete the proof we need to find a contradiction. We do so by finding a real number in $(0,1)$ that is not on our list. This implies that f is not a bijection and so — contradiction!

To build the number we use a very slick argument[175] argument. Consider the figure below.

$$\begin{array}{rcccccccccc} f(1) = & 0. & 7 & 8 & 3 & 0 & 4 & 4 & 9 & 2 & \cdots \\ f(2) = & 0. & 2 & 1 & 8 & 9 & 2 & 6 & 5 & 3 & \cdots \\ f(3) = & 0. & 1 & 5 & 2 & 0 & 6 & 3 & 2 & 7 & \cdots \\ f(4) = & 0. & 5 & 4 & 3 & 6 & 2 & 9 & 1 & 2 & \cdots \\ f(5) = & 0. & 8 & 9 & 7 & 5 & 1 & 7 & 5 & 9 & \cdots \\ f(6) = & 0. & 0 & 3 & 4 & 8 & 0 & 4 & 2 & 5 & \cdots \\ f(7) = & 0. & 7 & 4 & 3 & 7 & 5 & 8 & 1 & 2 & \cdots \\ f(8) = & 0. & 3 & 0 & 7 & 0 & 6 & 9 & 6 & 8 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ \\ \Delta = & 0. & 7 & 1 & 2 & 6 & 1 & 4 & 1 & 8 & \cdots \\ z = & 0. & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 & \cdots \end{array}$$

We have listed out the (hypothetical) values of each number in our list in order: $f(1), f(2), f(3)\ldots$. For each number we have carefully written out its decimal expansion so that all the digits are arranged neatly in an array. Now ignore the leading "0." and focus on those numbers down the diagonal.

[175] Cantor was very very clever. This argument is now called "Cantor's diagonal argument" and it has been used in many places in mathematics. One famous application of the idea is to show that the Halting Problem is not solvable. The interested reader should search-engine their way to more information.

From those digits down the diagonal, we can construct a new number Δ:

$$\Delta = 0.d_1d_2d_3\cdots$$

so that

- The first digit d_1 is first digit of $f(1)$.
- The second digit d_2 is second digit of $f(2)$.
- The third digit d_3 is third digit of $f(3)$.
- $\vdots$
- The n^{th} digit d_n is the n^{th} digit of $f(n)$.
- and so on.

Now because the digits of Δ agree with some of the digits of each of the $f(k)$, it is possible that Δ appears somewhere on our list — so this doesn't give us the contradiction. However, Cantor realised that from Δ, one can construct a number, z, that is definitely not on the list. In particular, construct z so that each digit of z is different from the corresponding digit of Δ.

This means that

- The first digit of z must be different from the first digit of $f(1)$ — so $z \neq f(1)$.
- The second digit of z must be different from the second digit of $f(2)$ — so $z \neq f(2)$.
- The third digit of z must be different from the third digit of $f(3)$ — so $z \neq f(3)$.
- $\vdots$
- The n^{th} digit of z must be different from the n^{th} digit of $f(n)$ — so $z \neq f(n)$.
- and so on.

So if we make the digits of z in this way, it follows that our number z cannot be any number of the list since it has a different expansion[176] . Contradiction!

Let us be more precise about this and make a proof.

Proof. Assume, to the contrary, that $(0, 1)$ is countable. Since it is finite, it must be denumerable and so there is a bijection $f : \mathbb{N} \to (0, 1)$. Hence we can write $(0, 1) = \{x_1, x_2, x_3, \dots\}$ where $x_i = f(i)$. Now each of these x's has an infinite

[176]We should be a little careful about 9's and 0's when we do this — and we will avoid those two digits when we do this in our proof.

decimal expansion, so we can write a big array as follows:

$$\begin{aligned} x_1 &= 0.a_{11}a_{12}a_{13}\cdots, \\ x_2 &= 0.a_{21}a_{22}a_{23}\cdots, \\ x_3 &= 0.a_{31}a_{32}a_{33}\cdots, \\ \vdots &= \vdots \end{aligned}$$

where each $a_{ij} \in \{0, 1, \ldots, 9\}$.

Now we construct a number z that is not on the list. Write $z = 0.b_1b_2b_3\cdots$. And set

$$b_n = \begin{cases} 1 & \text{if } a_{nn} \neq 1 \\ 2 & \text{if } a_{nn} = 1 \end{cases}$$

and since each digit[a] of z is either $1, 2$ we know that

$$\frac{1}{9} = 0.111111\ldots < z < 0.222222\cdots = \frac{2}{9}$$

and hence $z \in (0, 1)$.

Of course, since f is a bijection, our number z must appear somewhere on our list. So let us assume that it is the k^{th} number on the list, namely that $z = x_k$. But, by construction, the k^{th} digit of z is not the same as the k^{th} digit of x_k. So $z \neq x_k$. In this way, there is no k such that $z = x_k$ and z is not in the list — a contradiction since f must be a bijection.

Hence there is no bijection from $\mathbb{N}$ to the set $(0, 1)$, and thus $(0, 1)$ is uncountable. ■

[a]Note that we could have used any two digits except 9, in order to avoid the problem of representing the same number with two different expansions.

So this Theorem proves that there is more than one type of ∞ — something that is really not at all obvious[177] . Let us push through to all reals now.

Corollary 12.3.3 *The set $\mathbb{R}$ of real numbers is uncountable.*

Proof. If $\mathbb{R}$ were countable, then all its subsets must be denumerable. Since $(0, 1)$ is uncountable, it follows that $\mathbb{R}$ is uncountable. ■

Note that the cardinality of the reals is denoted c, for continuum. That is $|\mathbb{R}| = c$. Since $c \neq \aleph_0$ and $\mathbb{N} \subseteq \mathbb{R}$, we must have $\aleph_0 < c$. Of course, to give a concrete meaning to the symbol "$<$" in the context of cardinalities of infinite sets, we have to do some work. That is the subject of our next section.

[177]This is a really beautiful result about the nature of the infinite! The author still finds it incredible that one can prove such a deep statement about the universe in a one term course. Remember, we started out with truth tables, and sets and we've just demonstrated that there is not just one infinity! Amazing!

12.4 Comparing cardinalities

12.4.1 Extending to the infinite

So really all[178] we have been able to do so far is show that $|A| = |B|$ and, after borrowing some cunning from Cantor, that $|A| \neq |B|$. Now we will try to understand and give meaning to $|A| \leq |B|$ and $|A| < |B|$. As before, we will try to translate these statements whose meaning are quite simple for finite sets, into statements about functions. From there we can extend them to make sense of infinite sets. This, in turn, will enable us to prove that there is no biggest set and so there an infinite number of infinities!

Let us go back to finite sets for a moment. Consider the sets

$$A = \{a, b, c\} \qquad\qquad S = \{x, y, z, w\}$$

Of course we know $|A| < |B|$. How can we describe this in terms of functions.

- Is there an injection from A to B? — Yes, for example:

$$f(a) = x \qquad\qquad f(b) = y \qquad\qquad f(c) = z$$

- Is there a surjection from A to B? — No — by the pigeonhole principle.
- Since there is no surjection there cannot be a bijection.

In fact this was a corollary of the pigeonhole principle — Corollary 12.1.4.

In the same way that extended the equivalence between the existence of bijections and sets being equinumerous, from finite sets to infinite sets, we extend the above ideas from finite sets to all sets.

Definition 12.4.1 Let A and B be sets.

- We write $|A| \leq |B|$ if there is an injection from A to B.
- Further, we use $|A| < |B|$ to mean that $|A| \leq |B|$ and $|A| \neq |B|$.
- That is, we write $|A| < |B|$ and say A has a smaller cardinality than B if there is an injection from A to B but no bijection.

$\Diamond$

So with this definition we can now sensibly[179] state that

$$\aleph_0 = |\mathbb{N}| < |\mathbb{R}| = c$$

There are two different infinities — the infinity of the integers, and the larger infinity of the reals.

This inequality prompts two questions.

[178] Mind you we did quite a lot with "just" this.

[179] Well — perhaps "precisely" is a better word than "sensibly".

- Are there more[180] infinities?
- Is there some infinity that lies between α_0 and c?

This first question we can answer and will do so shortly. The second question is due to Cantor and remains unanswered; it now stated as a conjecture called the "continuum hypothesis".

Conjecture 12.4.2 The continuum hypothesis. *There is no set A such that $\aleph_0 < |A| < c$.*

This is generally believed to be true. It is still an active area of research and people have shown quite a bit about what might happen if this result is true and if it is false. In particular — it has been proved that one cannot disprove this using standard set theory (by Gödel 1931). It was then proved that one cannot prove it to be true either (by Cohen 1963).

12.4.2 Cantor's Theorem and infinite infinities

Back to the number of infinities. One relatively easy result shows that there is a set bigger than the reals. We do this by considering a set and its power set. This result, now known as Cantor's theorem, has very interesting consequences.

Theorem 12.4.3 Cantor's theorem. *Let A be a set. Then $|A| < |\mathcal{P}(A)|$.*

For finite sets this result is quite easy. One can prove (using induction, for example) that if $|A| = n$ then $|\mathcal{P}(A)| = 2^n$, and that $n < 2^n$ for any integer n. For infinite sets things are much less obvious.

We start by showing that there is an injection from A to $\mathcal{P}(A)$. An easy one is

$$f : A \to \mathcal{P}(A) \qquad \text{defined by} \qquad f(a) = \{a\}$$

another one is

$$h : A \to \mathcal{P}(A) \qquad \text{defined by} \qquad h(a) = A - \{a\}.$$

We can then show there is no bijection by showing that no function from A to its power set can be surjective. We do this using a proof by contradiction.

Before we do that, let us study these two functions, f and h, a little more. Consider $A = \{1, 2, 3\}$, then we have

$$\begin{array}{lll} f(1) = \{1\} & f(2) = \{2\} & f(3) = \{3\} \\ h(1) = \{2, 3\} & h(2) = \{1, 3\} & h(3) = \{1, 2\} \end{array}$$

[180] Once you show that something isn't unique, it is only reasonable to ask "Well - how many are there?". This was no small part of the controversy that Cantor's work generated. Some Christian theologians thought his results were challenging the notion that the Christian God is unique and absolute. You can certainly find much interesting reading on the implications of Cantor's work.

Notice that f, h take elements of A and turn them into subsets of A.

Look a little more closely at f and you see that $1 \in f(1)$, $2 \in f(2)$ and $3 \in f(3)$. That is

$$\forall a \in A, a \in f(a)$$

Looking at h we see that $1 \notin h(1), 2 \notin h(2)$ and $3 \notin h(3)$. That is

$$\forall a \in A, a \notin h(a)$$

More generally, if we have some function, g, that takes us from A to its power set, then we can try to understand which elements map into their image and which do not. In particular, for which $x \in A$ is $x \in g(x)$, and for which $y \in A$ is $y \notin g(y)$. Understanding those sets of elements will be key to the proof.

So, assume, to the contrary, that there is a surjection (call it g) from A to its power set. Hence the function g takes an element, $x \in A$ and maps it to a subset of $g(x) \subseteq A$. As we noted above, it is possible that when we apply g to an element x we'll get a subset of A that *contains* x. So now let us say that

- an element x is "good" if $x \in g(x)$, and
- an element x is "bad" if $x \notin g(x)$.

So we can now define the set of all good elements, G, and the set of all bad elements, B:

$$G = \{x \in A \mid x \in g(x)\}$$
$$B = \{x \in A \mid x \notin g(x)\}$$

Now both of these are subsets of A and so $G, B \in \mathcal{P}(A)$. Further we see that each element of A must be in exactly one of G, B, and thus

$$G \cap B = \varnothing \qquad \text{and} \qquad G \cup B = A$$

As is often the case, the set of good things is not as interesting as the set of bad things. So though we have defined G, we won't use it further. Instead concentrate on B. By assumption g is surjective, so there must be some element of A that maps to B. That is, there should be $q \in A$ so that

$$g(q) = B$$

We get the contradiction[181] by examining whether or not $q \in B$.

- If $q \in B$, then since $g(q) = B$ we must have (by definition of G) that $q \in G$. However, this is a contradiction — $q \in B$ and $q \notin B$.
- If $q \notin B$, then since $g(q) = B$ we must have (by definition of B) that $q \in B$. However, this is a contradiction — $q \notin B$ and $q \in B$.

In either case we get a contradiction, so no such surjection exists.

[181] This contradiction is reminiscent of Russell's paradox (due to Russell in 1901 and also by Zermelo in 1899). Consider the set of all sets that do not contain themselves. That is $R = \{X \mid X \notin X\}$. One realises the paradox when you try to decide whether $R \in R$ or not. If $R \in R$, then my definition of the set, it cannot be. While if $R \notin R$ then, by definition of the set, it must be. There is much of interest here for a reader armed with a good search-engine.

Proof. We split the proof into three steps.

- We show that the result holds when $A = \varnothing$
- We show that $|A| \leq |\mathcal{P}(A)|$ by giving an injection from A to $\mathcal{P}(A)$.
- Finally, we show that there cannot be a surjection from A to $\mathcal{P}(A)$.

Either A is empty or not. If $A = \varnothing$ then $0 = |A| < |\mathcal{P}(A)| = 1$. So in what follows we can assume $A \neq \varnothing$.

We now construct an injection $f : A \to \mathcal{P}(A)$. Define

$$f(x) = \{x\} \qquad \text{for all } x \in A.$$

To show that this function is injective, let $x_1, x_2 \in A$ and assume $f(x_1) = f(x_2)$. Then $\{x_1\} = \{x_2\}$ and thus $x_1 = x_2$. So f is injective. Thus $|A| \leq |\mathcal{P}(A)|$.

We prove there cannot be a bijection between A and $\mathcal{P}(A)$ by showing that there cannot be a surjection. We do this by contradiction. Assume, to the contrary, that there is a surjection $g : A \to \mathcal{P}(A)$. We then partition A into two subsets

$$G = \{x \in A \mid x \in g(x)\} \qquad \text{and} \qquad B = \{x \in A \mid x \notin g(x)\}.$$

Notice that $B \subseteq A$ and so $B \in \mathcal{P}(A)$.

Since g is, by assumption, a surjection, and B is an element of the codomain of g, there must be $q \in A$ so that $g(q) = B$. Now either $q \in B$ or $q \notin B$.

- If $q \in B$, then since $g(q) = B$ we have that $q \in g(q)$. But then definition of B implies that $q \notin B$, giving a contradiction.
- Similarly, if $q \notin B$, then since $g(q) = B$ we have that $q \notin g(q)$. But then definition of B implies that $q \in B$, again giving a contradiction.

In either case we get a contradiction, and so we must conclude that no surjection g can exist. Thus there is no surjection from $A \to \mathcal{P}(A)$, and thus there is no bijection from $A \to \mathcal{P}(A)$. ■

We can immediately apply Cantor's theorem to the natural numbers to see that

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$$

so we have 2 infinities! But, of course, we can do it again to get another infinity:

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))|$$

and again!

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))|$$

And we can keep going and going to get the following corollary.

Corollary 12.4.4 *There are an infinite number of different infinities.*

Proof. Starting with $\mathbb{R}$, we can form $\mathcal{P}(\mathbb{R})$, which is not equinumerous, by Cantor's theorem. But then we can take the power set of that to obtain a yet larger infinite set $\mathcal{P}(\mathcal{P}(\mathbb{R}))$. Continuing in this fashion gives an infinitely long sequence of infinite sets none of which are equinumerous. ■

So we have at least a denumerable number of infinites! In fact there are even more than that! But we'll let you search-engine your way to discussions of that result.

12.4.3 Congratulations are in order

At this point we invite the reader to take stock of what we — ie you! — have managed to do by following along the text. We started by looking at very basic ideas of sets, statements and truth-tables, and have now just proved something fundamental and highly-non-trivial about the nature of infinity! This is no small achievement!

12.4.4 One more question

There is one little question left in this section. From Theorem 12.3.1 we know that

$$|\mathbb{N}| < |\mathbb{R}|$$

and from Theorem 12.4.3 we know that

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$$

but we have not yet compared

$$|\mathcal{P}(\mathbb{N})| \stackrel{?}{=} |\mathbb{R}|$$

One can show that these sets are equinumerous, but constructing an explicit bijection between them is quite difficult. Instead one can prove that

$$|\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}| \qquad \text{and } |\mathbb{R}| \leq |\mathcal{P}(\mathbb{N})|$$

by finding injections between those sets. However in order to show that the existence of those injections implies the existence of a bijection, ie

$$(|A| \leq |B|) \wedge (|B| \leq |A|) \implies (|A| = |B|)$$

we need the Cantor-Schröder-Bernstein theorem.

The proof of the Cantor-Schröder-Bernstein theorem is quite involved, so *the proof* is not typically covered in a first course in proof. That being said, the result itself is useful and we do make use of it in some of the exercises for this chapter. Accordingly we encourage the reader to read about the result, skip over the proof, and see how Cantor-Schröder-Bernstein is used to show that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

The interested reader is, of course, encouraged to read the proof — it is a nice piece of mathematics.

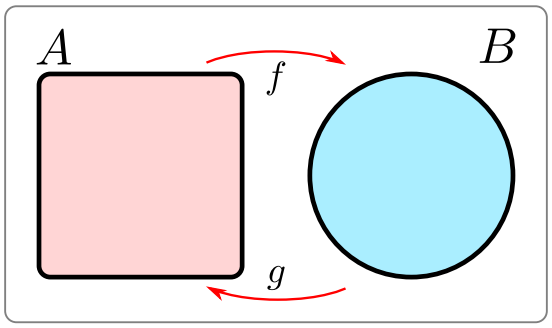
12.5 More comparisons of cardinalties

At this point, we have given extra meaning to the symbols "$\leq$" and '$<$' so that they can be used to handle cardinalities of infinite sets. In that context do they still behave the same way as they do in the context of (say) comparing real numbers? So, for example, does the following hold:

> if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$

This is equivalent to

> If there is an injection from A to B and an injection from B to A, then there is a bijection from A to B.

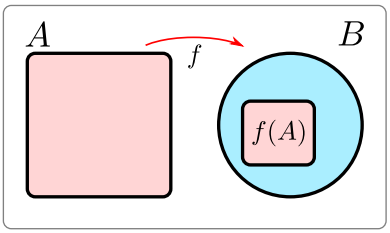


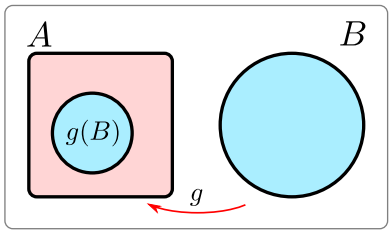
This result is the Cantor-Schröder-Bernstein[182] Theorem

12.5.1 The Cantor-Schröder-Bernstein theorem

Theorem 12.5.1 Cantor-Schröder-Bernstein (and also Dedekind). *Let A, B be sets. If there is an injection $f : A \to B$ and an injection $g : B \to A$, then there is a bijection $h : A \to B$. Equivalently, if $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.*

The proof is by no means trivial; we have some work to do. First up — let us assume that we have those injections $f : A \to B$ and $g : B \to A$.





[182] This was first published without proof by Cantor in 1887. In the same year it was proved by Dedekind, but not published. Schröder gave a proof sketch in 1896, but it was found to be incorrect. Then in 1897, almost simultaneously, Schröder and Bernstein gave correct proofs. At this time, Bernstein was 19 years old! Dedekin proved it again later that year. Why Dedekin doesn't get more credit for this theorem remains mysterious.

We can think of f injecting a copy of A into B — namely $f(A)$. It is not hard to show that by restricting the codomain we can build a function

$$\hat{f} : A \to f(A) \qquad \text{defined by } \hat{f}(a) = f(a)$$

is a bijection. So that looks like a good place to start making our bijection from A to B. We can similarly restrict the codomain of g to construct

$$\hat{g} : B \to g(B) \qquad \text{defined by } \hat{g}(b) = g(b)$$

Since this is also a bijection, its inverse is a bijection — giving us another bijection

$$\hat{g}^{-1} : g(B) \to B$$

This should give us some hope because by restricting codomains we have

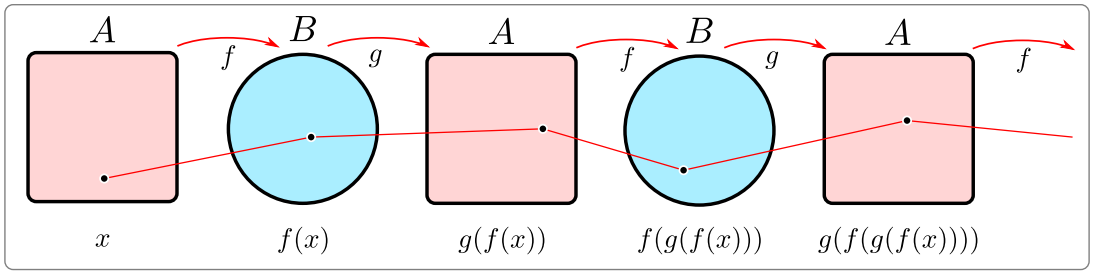
- a bijection, $\hat{f}$, from all of A to some (but not all) of B, and
- a bijection, $\hat{g}^{-1}$ from some (but not all) of A to all of B.

So perhaps we can build a bijection from all of A to all of B by carefully choosing between these two depending on our input.

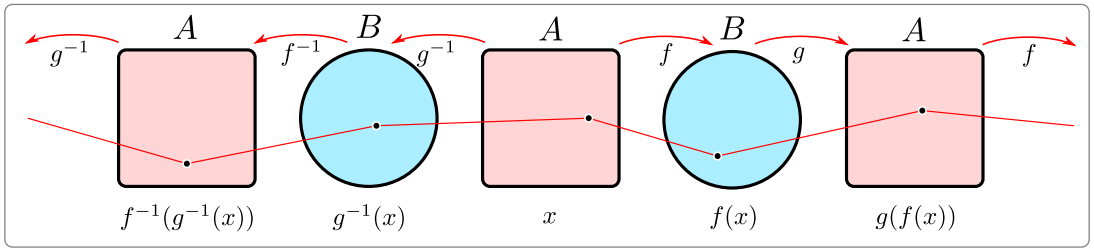
To investigate things a bit further, we should think about what happens when we compose f and g. Let $x \in A$ and consider the image of x under f — we get some element $f(x) \in B$. We cannot do much with this, but we can apply g to it, giving us $g(f(x))$. This, in turn, is some element of A, so we can apply f to it, etc etc. In this way, we can think of the trajectory of x under these two injections.

$$x \mapsto f(x) \mapsto g(f(x)) \mapsto f(g(f(x))) \mapsto \cdots$$

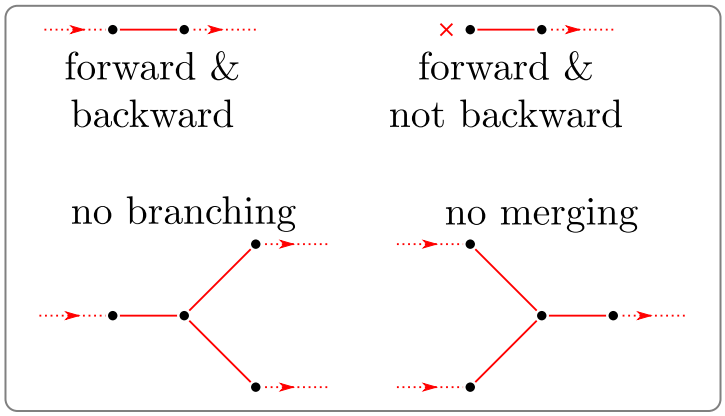
What we have here is the *forward* trajectory of x under f and g.



But we could also go backwards. For any $x \in A$, we can compute the preimage of $g^{-1}(\{x\})$. Since g is injective, we know that that this set is either empty or contains precisely 1 element (otherwise g would fail to be injective). We can make a similar argument about any $y \in B$, its preimage $f^{-1}(\{y\})$ is either empty or contains exactly 1 element. Hence we can extend this trajectory backwards via unique preimages — unless we get stuck at an element whose preimage is empty.

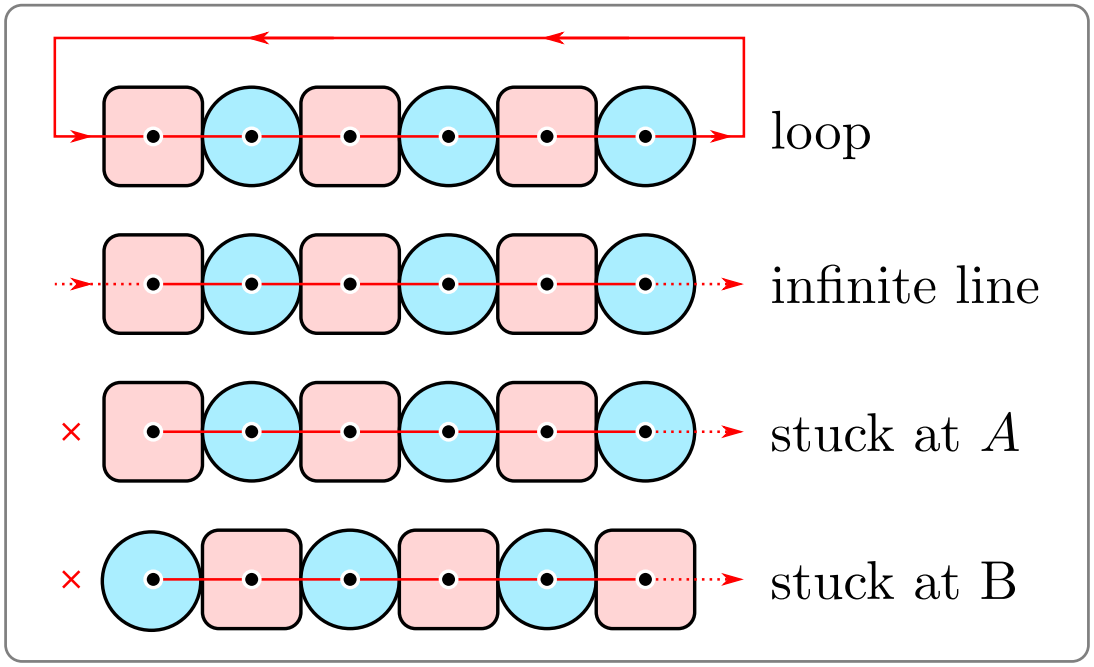


Each element has a unique forward step along its path by applying f or g as appropriate. Similarly, when we step back along the trajectory, we move to a unique element (due to injectivity) by application of f^{-1} or g^{-1}, or the preimage is empty and cannot move any further back. So two trajectories cannot merge, nor can a trajectory branch:



Because of this, the trajectory of any point x can be of 4 types:

- the trajectory forms an infinite line
- the trajectory is a loop
- the trajectory goes forward forever, but going backwards we get stuck at an element of A, or
- the trajectory goes forward forever, but going backwards we get stuck at an element of B.



We cannot have more complicated shapes. Let us call the first three types "good" and the last type — that gets stuck at B — "bad".

Now, define a function $h : A \to B$ by

$$h(x) = \begin{cases} f(x) & \text{if the trajectory of } x \text{ is good, and} \\ g^{-1}(x) & \text{if the trajectory of } x \text{ is bad.} \end{cases}$$

To see that this is actually function — take any element of $a \in A$. We can always form $f(a)$, since f is a function. If a lies on a bad trajectory, then we know we can go backwards until we get stuck at some element of B — thus the preimage of $g^{-1}(a)$ must contain exactly 1 element — which we take to be $h(a)$.

Oof. Now we "just" have to prove this is a bijection.

h is a surjection. Let $b \in B$. Examine the trajectory on which b lies — it is either "good" or "bad".

- If b lies on a good trajectory, then (by moving back one step along the trajectory) there is some $x \in A$ so that $f(x) = y$. This x must also lie on a good trajectory and so $h(x) = f(x) = y$.
- If b lies on a bad trajectory, then (by moving forward one step along the trajectory) we set $x = g(b)$. We know that x also lies on a bad trajectory, so $h(x) = g^{-1}(x) = g^{-1}(g(y)) = y$.

In either case, by moving forward or backward one step along the trajectory of y we find an $x \in A$ so that $h(x) = y$. ■

h is an injection. Let $a, c \in A$, and assume that $h(a) = h(c)$. Notice that by construction, $h(x)$ lies on the same trajectory as x. Thus $a, c, h(a)$ and $h(c)$ must all lie on the same trajectory. That trajectory is either good or bad.

- If the trajectory is good, then $h(a) = f(a)$ and $h(c) = f(c)$ and so $f(a) =$

$f(c)$. Since f is an injection, we know that $a = c$.

- If the trajectory is bad then $h(a) = g^{-1}(a)$ and $h(c) = g^{-1}(c)$. So $g^{-1}(a) = g^{-1}(c)$ — since g is an injection, that preimage contains exactly one element, so we must have $a = c$.

In either case, we have that $a = c$. ■

12.5.2 Applications

The Cantor-Schröder-Bernstein theorem makes proving the following results much easier.

Result 12.5.2 *The sets* $(0, 1), [0, 1), (0, 1]$ *and* $[0, 1]$ *are all equinumerous.*

Proof. By Cantor-Schröder-Bernstein it suffices to construct injections from $(0, 1)$ to $[0, 1)$ and vice-versa to show that they are equinumerous.

- Let $f : (0, 1) \to [0, 1)$ be defined by $f(x) = x$. This is an injection since if $x_1 \neq x_2$ then $f(x_1) = x_1 \neq x_2 = f(x_2)$.
- Let $g : [0, 1) \to (0, 1)$ be defined by $g(x) = 0.1(1+x)$ (there are many similar choices). Again, this is an injection since if $x_1 \neq x_2$ then $g(x_1) \neq g(x_2)$.

We can then show that the second and third sets are equinumerous via the explicit bijection

$$f : [0, 1) \to (0, 1] \qquad\qquad f(x) = 1 - x$$

This is injective since if $f(x_1) = f(x_2)$ then $1 - x_1 = 1 - x_2$ so $x_1 = x_2$. It is surjective since for any $y \in (0, 1]$ pick $x = 1 - y \in [0, 1)$. Then $f(x) = 1 - (1 - y) = y$ as required.

Finally we can prove that the first and last sets are equinumerous by very similar injections to those we used to prove that the first and second sets are equinumerous.

- Let $f : (0, 1) \to [0, 1]$ be defined by $f(x) = x$. It is (immediately) injective by the same argument used above.
- Let $g : [0, 1] \to (0, 1)$ be defined by $g(x) = 0.1(1 + x)$. It is injective by similar arguments.

So by CSB we have that $|(0, 1)| = |[0, 1]|$. ■

This can also be proved without CSB, but it is definitely more work — we'll demonstrate that the first two sets are equinumerous. The approach is to split the set $(0, 1)$ into the set $\left\{\frac{n}{n+1} \mid n \in \mathbb{N}\right\}$ and everything else. Then things of the form $\frac{n}{n+1}$ are mapped to $\frac{n-1}{n}$, so that

$$f(1/2) = 0 \qquad f(2/3) = 1/2 \qquad f(3/4) = 2/3 \qquad f(4/5) = 3/4$$

This way we can map something to 0 while not forgetting or missing any other element.

Proof. Let $f : (0,1) \to [0,1)$ be defined by

$$f(x) = \begin{cases} \frac{n-1}{n} & \text{if } x = \frac{n}{n+1}, n \in \mathbb{N} \\ x & \text{otherwise} \end{cases}$$

We then need to show that this is both injective and surjective.

- Injective — Let $a, b \in (0,1)$ and assume $f(a) = f(b)$. If $f(a) = f(b) = \frac{n-1}{n}$ for some $n \in \mathbb{N}$, then we must have $a = \frac{n}{n+1} = b$. On the other hand if $f(a) = f(b)$ is not of this form then $f(x) = x$, so $a = b$.
- Surjective — Let $y \in (0,1)$. If $y = \frac{n-1}{n}$ for some $n \in \mathbb{N}$, then set $x = \frac{n}{n+1}$. Otherwise set $x = y$. In either case $f(x) = y$ as required.

■

12.5.3 Proof that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$

From Cantor's theorem we saw that $|\mathbb{N}| < \mathcal{P}(\mathbb{N})|$, and by Cantor's diagonal argument we also saw that $|\mathbb{N}| < |\mathbb{R}|$. It is not unreasonable to ask to compare these two uncountable sets. The following really beautiful result shows that they are actually equinumerous.

Theorem 12.5.3 *The sets $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$ are equinumerous.*

Proof. By the Cantor-Schröder-Bernstein theorem it suffices to construct injections from $\mathcal{P}(\mathbb{N})$ to $\mathbb{R}$ and vice-versa.

- We define an injection $g : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$. Let $X \in \mathcal{P}(\mathbb{N})$, we can then construct $y = g(X)$ by constructing its decimal expansion. In particular, write $y = 0.y_1y_2y_3y_4\cdots$, and then

  $$y_n = \begin{cases} 1 & \text{if } n \in X \\ 0 & \text{if } n \notin X \end{cases}$$

  Now if we take $X_1, X_2 \in \mathcal{P}$ with $X_1 \neq X_2$, then

  - there is $n_1 \in X_1$ so that $n_1 \notin X_2$, or
  - there is $n_2 \in X_2$ so that $n_2 \notin X_1$, or both.

  In either case, this means that corresponding expansions differ at either y_{n_1} or y_{n_2}, and so $g(X_1) \neq g(X_2)$.

- It is easier to construct an injection from $[0,1)$ to $\mathcal{P}(\mathbb{N})$ rather than from $\mathbb{R}$ to $\mathcal{P}(\mathbb{N})$. However, since we have proved that $|[0,1)| = |(0,1)| = |\mathbb{R}|$, we know there is a bijection between $\mathbb{R}$ and $[0,1)$. So if we compose that

bijection with the injection we are about to construct, we get the required injection from $\mathbb{R}$ to $\mathcal{P}(\mathbb{N})$.

Given any $x \in [0,1)$ we write its decimal expansion as $x = 0.x_1x_2x_3\ldots$. As noted above we can make this expansion unique by avoiding any expansion that ends in an infinite sequence of 9's. Using this expansion we can define $h : [0,1) \to \mathcal{P}(\mathbb{N})$ by

$$h(x) = h(0.x_1x_2x_3\ldots) = \left\{x_1, 10x_2, 100x_3, \ldots, 10^{n-1}x_n, \ldots\right\} \subseteq \mathbb{N}$$

So, for example,

$$h\left(\frac{1}{4}\right) = h(0.25) = \{2, 50\}$$
$$h\left(\frac{1}{3}\right) = h(0.333\ldots) = \{3, 30, 300, 3000, \ldots\}$$
$$h\left(\frac{2}{11}\right) = h(0.181818\ldots) = \{1, 80, 100, 8000, \ldots\}$$

Now if $x \neq z$ then their decimal expansions must differ at least one digit. Consequently their images will be different subsets of $\mathbb{N}$. Hence the function h is an injection from $[0,1)$ to $\mathcal{P}(\mathbb{N})$, and so there is an injection from $\mathbb{R}$ to $\mathcal{P}(\mathbb{N})$ as required.

■

12.6 (Optional) Cantor's first proof of the uncountability of the reals

For completeness we include Cantor's first proof that $|\mathbb{N}| < |\mathbb{R}|$. This, proved in 1874, is more involved than his very famous diagonal argument which was published in 1891. Despite this it is still accessible to the tools developed in this text.

Like the diagonal argument, this first proof also is a proof by contradiction. We assume the existence of a bijection from $\mathbb{N}$ to an interval of the real-line and then show that this leads to a contradiction. However, rather than relying on decimal expansions of real numbers, it instead relies on the supremum — the least upper bound property. The reader should take a moment to examine Exercise 8.6.15 and Exercise 8.6.16, and also revise Section 6.4 before continuing.

Axiom 12.6.1 Least upper bound property of the reals. *Let $A \subseteq \mathbb{R}$ be bounded above. That is, there is some $M \in \mathbb{R}$ so that $a \leq M$ for all $a \in A$. Then the supremum of A exists and is a real number.*

This is the critical difference between the reals and the rationals — the rationals do not satisfy the least upper bound property. For example, one can

take the set of truncated decimal expansions of $\sqrt{2}$:

$$\begin{aligned}\{1.0, 1.4, 1.41, 1.414, 1.4142, \dots\} &= \left\{\frac{1}{1}, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \dots\right\} \\ &= \left\{\lfloor 10^n \sqrt{n} \rfloor \cdot 10^{-n} \text{ s.t. } n \in \mathbb{N}\right\} \\ &\subseteq \mathbb{Q}\end{aligned}$$

The least upper bound of this set is $\sqrt{2}$ which is not a rational number 11.2.6.

We can use the least upper bound property to prove[183] that increasing sequences of real numbers that are bounded above must converge to a real limit.

Lemma 12.6.2 *Let (a_n) be a sequence of real numbers that*

- *is bounded above by some real number M, and*
- *satisfies $a_k \leq a_{k+1}$ for all k.*

Then the sequence a_n converges as $n \to \infty$.
Proof. See Exercise 8.6.17. ■

Now we are ready to get into the proof. Let $[a, b]$ be an interval of the real line with $a < b$. Then assume to the contrary, that there is a bijection

$$g : \mathbb{N} \to [a, b]$$

This bijection defines an infinite sequence via $x_n = g(n)$ for any $n \in \mathbb{N}$. Notice that all the terms of this sequence must be distinct since g is injective.

The proof works by finding some $q \in [a, b]$ so that q is not part of the sequence. Hence there is no $n \in \mathbb{N}$ so that $g(n) = q$, and so g is not surjective and hence not bijective.

We construct two new sequences (a_n) and (b_n) from the sequence (x_n).

- Find the first two x_n lying strictly inside the interval, that is $a < x_n < b$. These two terms must be distinct and let a_1 be the smaller and b_1 be the larger.
- Similarly, find the first two terms of the sequence that lie inside (a_1, b_1) — call the smaller a_2 and the larger b_2.
- Keep repeating this to define (a_3, b_3), (a_4, b_4) and so on.

Notice that

- for any n we must have $a_n < b_n$, and
- the sequences of a's and b's satisfy

  $$a < a_1 < a_2 < a_3 < \ldots < b_3 < b_2 < b_1 < b$$

  and so the sequence (a_n) is increasing and bounded above by b, while the sequence (b_n) is decreasing and bounded below by a.

[183] This is a nice and intuitive result; in fact it is such a nice result we set it as an exercise 8.6.17.

- the sequences of a's and b's may or may not be infinite.
- for any n, the terms $x_1, x_2, \ldots, x_{2n}$ do not lie inside the interval (a_n, b_n).

This last point can be proved quite readily using induction on n.

Lemma 12.6.3 *The interval (a_n, b_n) does not contain the sequence terms $x_1, x_2, \ldots, x_{2n}$.*

Proof. We prove the result by induction on n.

- Notice that x_1, x_2 must belong to $\{a, b, a_1, b_1\}$ (by construction). Thus $x_1, x_2 \notin (a_1, b_1)$ since the open interval exludes the endpoints.
- Now assume that the result holds for $n = k$, and so $x_1, x_2, \ldots x_{2n}$ are lie outside the interval (a_n, b_n). Then either x_{2n+1}, x_{2n+2} lie outside this interval, or, if they lie inside the interval they are one or both of a_{n+1}, b_{n+1}. Consequently they lie outside (a_{n+1}, b_{n+1}).

Hence the result holds for all $n \in \mathbb{N}$. ■

The sequences $(a_n), (b_n)$ may or may not be infinite. So initially assume that they are finite. That is, the final interval is (a_N, b_N). Now notice that there cannot be two or more x_n inside the interval (a_N, b_N) because otherwise we would use those to define another interval (a_{N+1}, b_{N+1}). Thus there are either zero or one more term of the sequence (x_n) lying inside (a_N, b_N). But this means that any other number inside (a_N, b_N) is not part of the sequence (x_n). That is, there is some $q \in (a_N, b_N)$ so that $q \neq x_n = g(n)$ for any $n \in \mathbb{N}$.

Now instead, assume that the sequences of a's and b's are infinite. Since the sequence of a's is increasing and bounded above by b, Lemma 12.6.2 implies that it converges to a limit. The same argument (applied to $-b_n$) shows that the sequence of b's also converges to a limit. So define

$$\lim_{n\to\infty} a_n = \alpha \qquad \lim_{n\to\infty} b_n = \beta.$$

Notice that we cannot have $\beta < \alpha$ (otherwise we would have $a_k > b_k$ for some k), and so either $\alpha < \beta$ or $\alpha = \beta$.

- If $\alpha < \beta$, then every $q \in (\alpha, \beta)$ is not in the sequence of x's. If $q = x_n$ for some n, then this would contradict Lemma 12.6.3 above, since it would imply that $q = x_n \in (\alpha, \beta) \subseteq (a_n, b_n)$.
- On the other hand, if $\alpha = \beta$, then $q = \alpha$ is not in the sequence of x's. If $q = \alpha = x_n$ for some n, then again this would contradict Lemma 12.6.3 since it would imply that $q = x_n = \alpha \in (a_n, b_n)$.

Thus even when the sequences of a's and b's are infinite, there is always some $q \in [a, b]$ that is not in the sequence of x's. Hence the function $g : \mathbb{N} \to [a, b]$ is not a surjection, contradicting our initial assumption that it is bijective.

12.7 Exercises

1. Show that each of the following sets are denumerable, by construction a bijection between them and the natural numbers.

(a) $\mathbb{N} \cup \{0\}$

(b) $\{5, 6, 7, 8, \dots\}$

(c) $\{1, 3, 3^2, 3^3, \dots\}$

(d) $\mathbb{Z} \setminus \{0\}$

2. Suppose $A = \{(m, j) \in \mathbb{N} \times \mathbb{R} : j = \pi m\}$. Is it true that $|\mathbb{N}| = |A|$?

3. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be the function

$$f(m, n) = 2^{m-1}(2n - 1).$$

Can you use this f to conclude that $\mathbb{N} \times \mathbb{N}$ is denumerable?

4. Describe a partition of $\mathbb{N}$ that divides $\mathbb{N}$ into $\aleph_0$ countably infinite subsets. That is, partition $\mathbb{N}$ into an infinite number of subsets, each of which is itself infinite.

5. Theorem 12.2.3 states that $\mathbb{Z}$ is denumerable. We claimed that $f : \mathbb{N} \to \mathbb{Z}$ given by

$$f(n) = \begin{cases} \frac{1-n}{2} & n \text{ is odd} \\ \frac{n}{2} & n \text{ is even} \end{cases}$$

is a bijection. Prove this now.

6. Determine if each of the following sets is denumerable.

(a) The set of irrational numbers, $\mathbb{I}$

(b) $[0, 1] \cap \mathbb{Q}$

(c) $\{\pi + q : q \in \mathbb{Q}\}$

(d) $\{a + q : q \in \mathbb{Q}\}$ for some fixed $a \in \mathbb{R}$

(e) $\{\pi q : q \in \mathbb{Q}\}$

(f) $\{aq : q \in \mathbb{Q}\}$ for some fixed $a \in \mathbb{R}$

7. Prove that the set of all irrational numbers is uncountable. You may assume the fact that the set of real numbers is uncountable.

8. Prove that $(-\infty, -\sqrt{29})$ and $\mathbb{R}$ are equinumerous by constructing an explicit bijection between them.

9. Let S be the set of all functions $f : \mathbb{N} \to \{0, 1\}$. Prove that S is uncountable.

Notice that the codomain of these functions is the set that contains just

two elements, zero and one. It is not the set of all reals between 0 and 1.

10. Show that $\mathbb{R}$ and $(0, 1)$ are equinumerous by giving two different explicit bijections.

11. Show that the two given sets have equal cardinality by describing a bijection from one to the other. Describe your bijection with a formula (not as a table).

(a) $\mathbb{R}$ and $(\sqrt{2}, \infty)$

(b) The set of even integers and the set of odd integers

(c) $\mathbb{Z}$ and $S = \{x \in \mathbb{R} : \sin x = 1\}$

(d) $\{0, 1\} \times \mathbb{N}$ and $\mathbb{Z}$

12. Let A, B, C, D be any nonempty sets. Suppose that $A \cap C = \emptyset$ and $B \cap D = \emptyset$, and that $|A| = |B|$ and $|C| = |D|$. Show that $|A \cup C| = |B \cup D|$.

13. Construct an explicit bijection between the sets $(0, \infty)$ and $(0, \infty) - \{1\}$ to show that $|(0, \infty)| = |(0, \infty) - \{1\}\,|$.

You must prove that your function is a bijection.

14. Show that the following pairs of sets are equinumerous.

(a) $(0, 1) \times (0, 1)$ and $(0, 1)$

(b) $\mathbb{R}^2$ and $\mathbb{R}$

15. Let A and B be equinumerous sets. Show that $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

16. Prove or disprove: The set $\{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{Z}\}$ of infinite sequences of integers is countably infinite.

17. Let A and B be sets. Let P be a partition of A, and let Q be a partition of B. Suppose that we have a bijection between the partitions, $h\colon P \to Q$, with the additional property that $|X| = |h(X)|$ for every set $X \in P$.

Prove that the underlying sets, A and B, have the same cardinality.

18. Let A, B, and C be sets.

(a) Suppose that $|A| \leq |B|$ and $|B| \leq |C|$. Show that $|A| \leq |C|$.

(b) Show that the following statement is equivalent to the Cantor-Schröeder-Bernstein 12.5.1:

Suppose that $|A| \leq |B| \leq |C|$, and that $|A| = |C|$. Then $|A| = |B| = |C|$.

19. Let $F_n = \{X \subset \mathbb{N}\colon |X| = n\} \subseteq \mathcal{P}(\mathbb{N})$.

(a) Prove that for every $n \in \mathbb{N}$, $|F_n| = |\mathbb{N}|$.

(b) Also show that $\Big|\bigcup_{n\in\mathbb{N}} F_n\Big| = |\mathbb{N}|$.

(c) Does the result in part (b) contradict the fact that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$?

Explain why or why not (you do not need to give a formal proof).

20. Consider the following questions about countable unions of countable sets.

(a) Let $A_1, A_2, A_3, \ldots$ be denumerable sets, and suppose that $A_m \cap A_n = \emptyset$ whenever $m \neq n$. Show that

$$\bigcup_{n=1}^{\infty} A_n$$

is denumerable as well.

(b) Now suppose $A_1, A_2, A_3, \ldots$ are countable sets, and suppose that $A_m \cap A_n = \emptyset$ whenever $m \neq n$. Show that

$$\bigcup_{n=1}^{\infty} A_n$$

is countable as well.

(c) Redo part (b), but without the assumption that $A_m \cap A_n = \emptyset$ whenever $m \neq n$.

21. In the following exercises, you may use the result from Exercise 12.7.20 and the Fundamental Theorem of Algebra: a degree n polynomial has at most n real solutions.

(a) Let $m \in \mathbb{N}$. Define P_m to be the set of degree m polynomials with rational coefficients. That is,

$$P_m = \{a_0 + a_1 x + \cdots + a_m x^m \mid a_i \in \mathbb{Q} \text{ for all } i \in \{0, 1, \ldots, m\},\ a_m \neq 0\}.$$

Show that P_m is countable.

(b) Now, define P to be the set of all polynomials with rational coefficients. That is,

$$P = \{a_0 + a_1 x + \cdots + a_n x^n \mid n \in \mathbb{N} \text{ and } a_i \in \mathbb{Q} \text{ for all } i \in \{0, 1, \ldots, n\},\ a_n \neq 0\}.$$

Prove that P is countable.

(c) Define A to be the set of all real numbers that are the roots of a polynomial in P. That is,

$$A = \{x \in \mathbb{R} \mid \exists f \in P \setminus \{0\} \text{ s.t. } f(x) = 0\}.$$

Prove or disprove: $|A| = |P|$.

Appendix A

Hints for Exercises

1 · Sets
1.4 · Exercises

1.4.2.

(a) No hint.

(b) No hint.

(c) The elements of the set are close to other numbers that may be easier to find connection between.

(d) Can you see a connection between the numerators and the denominators? The previous question may give you a hint.

(e) What do the elements have in common?

(f) What do the elements have in common?

1.4.5. Pay close attention to notation! For part (h), this is an issue you would only come across if you are typing your math in a program such as LaTeX.

1.4.9. Try writing out the first few terms of each set.

2 · A little logic
2.7 · Exercises

2.7.9. Carefully determine what is the hypothesis and what is the conclusion of each implication, and then refer to the truth table of the implication.

2.7.14. What is the only situation in which an implication is false?

2.7.15. Is the conclusion of the implication true or false? What does that tell you?

2.7.16. You don't have to use truth tables to determine the answer (although you could).

3 · Direct proofs
3.5 · Exercises

3.5.1. Think about what does being even mean and how we can use it.

3.5.2. Think about what being odd means and how we can use it.

3.5.6. Think about what it means for n to divide a and b, and how we can use that information.

3.5.7. Think about the divisors of 1 and how you might use that.

3.5.8. Think about how we can get from 2 and 3 to 6.

3.5.9. Think about what it means for 3 to divide an integer and how we can use that information.

3.5.13. Try writing down different integer roots and check whether their product is also an integer root.

3.5.16. Try writing the inequality in a different way, the difference of squares might help.

4 · More logic
4.3 · Exercises

4.3.2. Refer back to Exercise 2.7.5 and use Theorem 4.2.3.

5 · More proofs
5.5 · Exercises

5.5.3. What is the contrapositive of this statement?

5.5.4. What is the contrapositive of the statement?

5.5.5. What is the contrapositive of the statement?

5.5.9. Think carefully about the parity of n.

5.5.10. The ideas used in the solution of Exercise 3.5.8 may be useful.

5.5.11. Try proving by cases.

5.5.12. Think carefully about the contrapositive.

5.5.13. What is the contrapositive of the statement?

5.5.15. If we assume the hypothesis is true, then what do we know about q?

5.5.16. Expand the cubes and simplify the sum

5.5.17. Modular arithmetic makes this much easier.

5.5.18. You may want to look at cases for $x \in \mathbb{R}$ to get rid of the absolute values.

5.5.21.

- Notice that the statement is equivalent to proving

$$-|x-y| \leq |x| - |y| \leq |x-y|.$$

- Use the triangle inequality!

5.5.22. Try sketching the function.

6 · Quantifiers
6.6 · Exercises

6.6.1. You can think about the division algorithm and cases for n. Also try factoring things.

6.6.2. Try to eliminate n. What does this tell you about k? What else do you know about k?

6.6.3. What can you say about a^2 modulo 4 for all $a \in \mathbb{Z}$?

6.6.6. Try playing around with some simple functions.

6.6.7. If you are unsure if the statement is true or not, then explore its negation — it might be easier to understand.

6.6.9. Which primes are even?

6.6.11. Think carefully about the truth table of the implication. Also, negate carefully. Finally, be careful with your inequalities.

6.6.12. Be careful of the order of quantifiers; make sure you pick variables in the correct order.

6.6.13. Start with a few simple functions you know well. Sketch them and try to decide if they are type A or type B or neither. Also, you should write out the negations of these definitions; what does it mean if a function is not type A? what does it mean if it is not type B?

6.6.19. Can you bound $\sin(1/x)$ by a simpler function?

6.6.20. Carefully negate the definition of convergence. Also, explore the first few terms of the sequence.

6.6.23. Split the sequence at some large N, so that when $n \geq N$, we know that a_n is really close to L. We can use this to bound $|a_n|$ when $n \geq N$. That leaves us to bound only finitely many of the terms $|a_n|$, the terms for which $n < N$. While we cannot take the maximum of an infinite set of values, we can find the maximum of a finite set of values and it will be finite. See Exercise 7.3.16 which explores this point.

6.6.24. For part (c), the contrapositive can help. Also, see Exercise 4.3.1 (c).

6.6.25. For part (b), try some of the sequences we have seen that converge with the Euclidean distance, and see whether they converge under this new distance? Choosing your ε satisfying $0 < \varepsilon < 1$, would be very useful in understanding the convergence of a sequence.

7 · Induction
7.3 · Exercises

7.3.2. Write out the statement for $n = k$ and $n = k + 1$ and try to work out how to make the left-hand side of the first statement look like the left-hand side of the second statement.

7.3.4. How can we get from z^{2n+1} to $z^{2(n+1)+1}$?

7.3.5. The even-ness of the exponent is required, otherwise the statement is simply not true:

$$3^3 - 1 = 27 - 1 = 26$$

and 8 definitely does not divide 26. Rephrase things to take advantage of what you know about even numbers.

7.3.10. The sum $\sum_{k=1}^{n} k$ is a very standard result and is in the main text. You can use that result without proof.

7.3.11. Be careful when you expand $(n+1)^3$ and $(n+1)^4$.

7.3.13.

- Take the first few derivatives and see if you can find a pattern.
- The factorial will be helpful. Recall that for any $n \in \mathbb{N}$,

  $$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

  and also the double-factorial $n!!$

  $$n!! = \begin{cases} n \cdot (n-2) \cdot (n-4) \cdots 2 & \text{when } n \text{ is even} \\ n \cdot (n-2) \cdot (n-4) \cdots 3 \cdot 1 & \text{when } n \text{ is odd} \end{cases}$$

7.3.14. Factoring a cubic can be painful. Perhaps write down the cubics you need and then expand them out. This might help with some of the arithmetic.

7.3.15. Adding to the end of the series will get you intro trouble. Try adding to the beginning instead. You might have to think about that a little.

7.3.17. For (b) try integration by parts.

7.3.19. Notice that every number is of the form

$$100\ \underbrace{1 \cdots 1}_{n\ \ 1s}\ 7$$

Also useful is the fact that if $d \mid a$ and $d \mid b$, then $d \mid (a + b)$.

7.3.20. The proof of Result 7.2.18 works because 5 is a Fibonacci number. See how you might generalise what is happening in the proof.

7.3.21.

- For Pascal's identity, rewrite the binomials as factorials and juggle carefully.
- For the binomial theorem, expand and group carefully. The following might

also be handy

$$\binom{n}{0} = \binom{n}{n} = 1.$$

Also, be careful around the edges of your expanded sums.

7.3.22. Integrate by parts!

7.3.23. As a base case, consider $n = 0$ and $n = 1$. This means that your inductive hypothesis can include both the $n - 1$ and n cases.

7.3.25. Shops will not be intimidated by excess purchasing power; even though you have lots of money, this fictional mathematical country does not allow you to overpay for an item.

7.3.27. Strong induction helps, as does the parity of the number.

7.3.28. You can simplify the analysis of the floor function by studying even and odd values separately.

7.3.29. Be careful as to how you can go from $k = n+1$ stones to $k \leq n$ stones in your inductive step and see whether different splittings change the calculations. Try playing this game with 5 or 6 stones to get a better understanding of how this works.

8 · Return to sets
8.6 · Exercises

8.6.3. Revise the definition of set-differences and then try to make a small example.

8.6.9. Remember that the power set is the set of subsets. That is

$$X \in \mathcal{P}(A) \iff X \subseteq A.$$

8.6.11. You may need to use mathematical induction on the size of A.

8.6.12. Try some small examples of sets A, B to gain some intuition.

8.6.13. Recall that

- $x \in \bigcup_{i \in I} A_i$ if and only if there is some $i \in I$ so that $x \in A_i$, and
- $x \in \bigcap_{i \in I} A_i$ if and only if for every $i \in I$ we have $x \in A_i$.

8.6.15. In order to prove that a is the supremum of a set S, it suffices to show that a is an upper bound for S, and that if $b < a$, then b is not an upper bound of the set. The latter statement may be rephrased as follows: if $b < a$, then there is some $s \in S$ with $s > b$.

If you want to show instead that S has no maximum, prove that any $s \in S$ is not an upper bound of the set. That is, for any $s \in S$, there is some $t \in S$ with $t > s$.

8.6.16. Suppose we are trying to prove that a is the least upper bound of a set S. Then we need prove that the two defining properties of the supremum hold for a. In order to prove the statement "if b is an upper bound for S, then $a \leq b$," it may be easier to show the contrapositive, "if $b < a$, then b is not an upper bound for S." In order to prove that contrapositive, we need to show that for any $b < a$, there is some $s \in S$ so that $s > b$. Then b will not be an upper bound for S, by definition.

8.6.17. For any ε, we know that $a - \varepsilon$ is not an upper bound of the set $\{a_n : n \in \mathbb{N}\}$.

9 · Relations

9.7 · Exercises

9.7.7. You can first look at the relation R for $E = \{1, 2, 3, 4\}$, and $q = 1$ to understand the relation better. Also, recall that $\overline{A} \cap \overline{B} = \overline{A \cup B}$.

9.7.10. Try looking at examples of relations on a small set, like $A = \{1, 2, 3\}$.

9.7.11. For part (a), try looking at examples of relations on a small set, like $A = \{1, 2, 3\}$.

9.7.12. Try some simple examples on small sets. Also, modus-tollens might help.

9.7.13. For the first part, the result in Exercise 3.5.7 will be useful.

9.7.14. Remember that every element of the set R is the intersection of two elements of P and Q. Also, read the definition of partitions carefully.

9.7.15. For the second part, start by finding the invertible elements in $\mathbb{Z}_6$

9.7.16.

(a) Bézout's identity tells you about the greatest common divisor of two numbers. What does it tell you about some of the numbers in the statement of the question? How can you use that information to get an equation for n?

(b) When a prime is at least 5 what do you know about its remainder when you divide by 3 or 8 or 24? And how can you turn the congruence we want to prove into a statement about divisibility? And how can we use (a) to reduce the number of cases we need to check?

9.7.17. Try using Bézout's identity.

9.7.18. Try modifying the proof of Euclid's lemma 9.5.9.

9.7.19.

- For part (a), try using Bézout's identity.
- For part (b), we can show that the two quantities are equal by showing that $m \gcd(a, b) \leq \gcd(ma, mb)$ and $\gcd(ma, mb) \leq m \gcd(a, b)$. Also, for any $d, e \in \mathbb{N}$, one way to show that $d \leq e$ is to prove that $d \mid e$.

- For part (c), some small numbers will help you build a counter-example.

9.7.20.

(a) The recurrence for the binomial coefficients is just adding integers.

(b) Think about prime factors.

(c) What coefficients in the sum are not divisible by p?

10 · Functions
10.8 · Exercises

10.8.1. The question asks for the range, not the codomain.

10.8.2. Remember that a function has to give a valid output for every valid input — why does this tell you about the y-values?

10.8.3. How do we show that two sets are equal?

10.8.5. Recall that

$$x \in f^{-1}(D) \iff f(x) \in D.$$

Also, modus tollens 2.5.2 can help you.

10.8.6. Completing squares may help with the answer (to be honest, completing squares will help you with almost everything quadratic functions related, cherish it).

Also, try choosing some small a, b values and sketch the function.

10.8.7. Can you determine $f(n)$ for small $n \in \mathbb{N}$?

10.8.8. This is a good example as to why when we want to determine whether a function is injective and/or surjective, we shouldn't only look at `what kind' of a function it is, but also consider the domain and the codomain of the function as well.

Remember, a function is not just a formula!

10.8.9.

- You can start the problem with a small set, say $A = \{1, 2, 3\}$, to understand the set F and the function g.
- To find $|F|$, think about how many different images there are there for each element in A.

10.8.11. For one side of the implication, think about how we can express the surjectivity in terms of the codomain and the range.

10.8.13. Think how an integer factors.

10.8.14.

- You need to do a lot of work mapping elements and subsets. We recommend that you use (say) x, y to denote elements of A and B, and (say) X, Y to denote subsets of A, B (which makes X, Y elements of $\mathcal{P}(A), \mathcal{P}(b)$).

- Try this with two small sets. For example, take $A = \{1, 2, 3\}$ and $B = \{10, 20, 30\}$. Construct a simple bijection from A to B and then use that to make a bijection from $\mathcal{P}(A)$ to $\mathcal{P}(B)$.

10.8.15.

- What do the equivalence classes of $\mathcal{R}$ look like?
- How can we show that f is, indeed, a function?

10.8.16. To show that an integer is zero, one can show that it is small and divisible by a bigger integer.

10.8.17. Be careful, the symbol f^{-1} denotes the preimage not the inverse-function. It only denotes the inverse-function when the function is bijective.

Also, see Theorem 10.3.6 and its proof.

10.8.18. Be careful, the symbol f^{-1} denotes the preimage not the inverse-function. It only denotes the inverse-function when the function is bijective.

Also, see Theorem 10.3.6 and its proof.

10.8.20. To show that two functions with same domains and codomains are different, it suffices to show that there is a single point at which they differ. To show that they are the same, you must show they are equal on every point of the domain.

10.8.21. Try to construct counterexamples on small sets (eg. $\{1, 2, 3\}$) rather than on $\mathbb{R}$.

10.8.22. Two functions are equal when they are equal at all points in their domains.

10.8.23. Your observations from part (b) can be helpful in solving part (c).

10.8.24.

- You may need to show that if $f \circ g$ is injective then g is injective, and then show that if $f \circ g$ is surjective then f is surjective.
- Combining part (a) with Theorem 10.5.3, we know that f is bijective if and only if $f \circ f$ is bijective.
- For part (b), it might help to compute $(f \circ f)(x)$.

10.8.26. Observe that f takes even numbers to odd numbers and odd numbers to even numbers. Considering that, what should the inverse of f look like?

Also - there are some very *handy* lemmas in Section 10.6.

10.8.27. Make good use of the fact that

$$g(g(g(x))) = (g \circ g)(g(x)) = i_A(x) = x.$$

11 · Proof by contradiction
11.3 · Exercises

11.3.1. If a were to satisfy both congruences, then we get 2 equations. Combining them helps.

11.3.3. Remember that 1 is not divisible by very many integers!

11.3.5. Modular arithmetic can really help with problems like this since they take all the infinite possible integers down to a small finite set of equivalence classes. Consider the equation modulo 4.

11.3.6. Try adapting the proof of Result 11.1.2.

11.3.7.

(a) Remember that $2 \mid 6$.

(b) The result from (a) can help you with (b). How can we manipulate $(\sqrt{2}+\sqrt{3})$ to somehow get an expression involving $\sqrt{6}$? Or, alternatively, how can we use that expression to say something about $\sqrt{2}$?

11.3.8. Please observe that $25 \mid 5^3$.
Euclid and the prime-ness of 5 also help.

11.3.9. Prove this by contradiction, but negate the statement carefully. Then, to get more information, Bézout's identity could be very useful.

11.3.12. What would the equation look like if x were a rational number?

11.3.14. If A were to have a maximum, what is the difference between $\sqrt{2}$ and that? It is not a rational number, but how can we use it to make a rational number that is still in A?

11.3.15.

- This may look like an induction question, but it's not!
- Euclid's lemma 9.5.9 should be useful, which tells us that if a prime p divides ab, then $p \mid a$ or $p \mid b$.
- What power of 7 divides 35?

11.3.19. What would happen if the function weren't strictly increasing or decreasing?

12 · Cardinality
12.7 · Exercises

12.7.4.

- Splitting $\mathbb{N}$ into even and odd doesn't work because it is a partition into only two parts.
- Similarly, splitting $\mathbb{N}$ as $\{\{n\} \,|n \in \mathbb{N}\}$ doesn't work because the parts are all finite.

12.7.6. Your answers for (d) or (f) may depend on (a).

12.7.8. Can you think of a bijection from $(0, \infty)$ to $\mathbb{R}$ How can we use that function in this question?

12.7.9. Question 13 in Section 10 may be useful.

12.7.10. Try sketching the graph of potential bijections from $(0, 1)$ to $\mathbb{R}$.

12.7.12. The sets do not have to be finite.

12.7.13. The explicit bijection in the proof of *{Result 12.5.2}* may be useful.

12.7.14. For part (a), try to define an injection from each set to the other; you can then use Cantor-Schröder-Bernstein 12.5.1 to infer there is a bijection between the sets. For the function from $(0, 1) \times (0, 1)$ to $(0, 1)$, consider the decimal representation of elements in $(0, 1)$.

12.7.15. Remember that the sets A and B may be infinite!

12.7.16.

- Any $x \in \mathbb{R}$ can be written as $x_0.x_1x_2x_3x_4\ldots$ where $x_0 \in \mathbb{Z}$ and $x_i \in \{0, 1, \ldots, 9\}$ for $i \geq 0$. For example, if $x = 3.141592\ldots$ then

$$\begin{array}{llll} x_0 = 3 & x_1 = 1 & x_2 = 4 & x_3 = 1 \\ x_4 = 5 & x_5 = 9 & x_6 = 2 & \cdots \end{array}$$

- We need to be careful! Notice that $0.25000000\cdots = 0.24999999\ldots$.

12.7.17.

- The function h is not a from A to B, it is a function from the partition P to the partition Q.
- What does it mean for X and $h(X)$ have the same cardinality?
- Corollary 9.3.7 may be useful.

12.7.20. Try using the diagonal sweeping argument given in the proof of Result 12.2.6. For part (c), try defining new sets B_n such that

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

and $B_n \cap B_m = \emptyset$ for $m \neq n$. Then apply part (b).

Appendix B

Scratchwork for Exercises

1 · Sets
1.4 · Exercises

1.4.1.

(a) A_1 is the set of all natural numbers whose square is less than 2. We see that there is only one natural number satisfying that condition, which is 1.

Thus, $A_1 = \{1\}$.

(b) We see that this set looks very similar to A_1. But we see that it is differrent in that we now are looking at the set of all *integers* whose square is less than 2, not natural numbers. In this case, we see that we have 3 numbers satisfying the condition, namely -1, 0, and 1.

Thus, $A_2 = \{-1, 0, 1\}$.

(c) A_3 is the set of all natural numbers that are multiples of 3 and also divisors of 216.

Therefore, $A_3 = \{3, 6, 9, 12, 18, 24, 27, 36, 54, 72, 108, 216\}$.

(d) A_4 is the set of all integers x such that if we add 2 and divide the result by 5 we still get an integer

So, we see that $A_4 = \{\ldots, -12, -7, -2, 3, 8, 13, \ldots\}$.

(e) We see that A_5 is the set of all elements a in B such that $6 \leq 4a + 1 < 1$. So, all we need to do is to check which elements of B satisfy this condition.

Then, we see $A_5 = \{2, 3\}$.

(f) We see that the set A_6 is the set of all elements of B whose product with at least one element of D is between 50 and 100, exclusive. So, to find the elements of A_6, we need to calculate the products of elements of B with elements of D.

Once we calculate that we see, $A_6 = \{7, 11, 13, 17, 19\}$.

(g) A_7 is the set of all integers n satisfying the inequality $n^2 - 5n - 16 \leq n$, or in other words, $n^2 - 6n - 16 \leq 0$. If we factor the expression on the left, we see that we want to find the integers satisfying $(n+2)(n-8) \leq 0$. Then, we see that for this inequality to be valid, we need to have $n \in [-2, 8]$.

Therefore, $A_7 = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}$.

1.4.2.

(a) We see that this set is the set of positive multiples of 5, or the set of all natural numbers that are muptiples of 5. Therefore we can write this set $A = \{5n \text{ s.t. } n \in \mathbb{N}\}$, or $A = \{n \in \mathbb{N} \text{ s.t. } n = 5m \text{ for some } m \in \mathbb{N}\}$.

We can also use different representations for this set (for any set really), but keeping our definition for the set accurate and concise makes it easier to understand and use.

(b) We see that B is the set of all natural numbers from 10 to 100, inclusive.

Therefore we can write this set as $B = \{n \in \mathbb{N} \text{ s.t. } 10 \leq n \leq 100\}$ or, equivalently, $B = \{n \in \mathbb{Z} \text{ s.t. } 10 \leq n \leq 100\}$.

(c) Even though it may seem like there is not a connection between the elements of this set. We see that the elements are all 1 away from the square of a natural number.

Therefore we can write this set as $C = \{n \in \mathbb{N} \text{ s.t. } n = m^2 - 1 \text{ for some } m \in \mathbb{N}\}$, or, equivalently, $C = \{n^2 - 1 \text{ s.t. } n \in \mathbb{N}\}$.

(d) We see that the elements of this set are fractions. So, to understand how we can write this set in set builder notation, we can try to find a connection between the numerator and the denominator of the elements of the set. Indeed, we see that $5 = 2^2 + 1$, $10 = 3^2 + 1$, $17 = 4^2 + 1$, etc..

Therefore, we can write the set as $D = \left\{\frac{m}{m^2+1} \text{ s.t. } m \in \mathbb{Z}\right\}$.

(e) In this set we see that we only have natural numbers. We also see that all the elements of the set are powers of 2 (even though the fact $65535 = 2^{16}$ may not be common knowledge, it is easy to check once we the `hunch' it may be). But, we also see that not all the powers of 2 are in the set. For example $32 = 2^5$ is not in the set. So, if we write the set as powers of 2, we see $E = \{2^1, 2^2, 2^4, 2^8, 2^{16}, \ldots\}$. This suggests that the powers themselves are also powers of 2.

Thus, we can write this set as $E = \left\{n \in \mathbb{N} \text{ s.t. } n = 2^{(2^k)} \text{ for some nonnegative integer } k\right\}$.

(f) In this example, we see that every element of the set is a natural number and also divisible by 2 or 3. This may suggest we write the set as $F = \{n \in \mathbb{N} \text{ s.t. } n = 2k \text{ or } n = 3k \text{ for some } k \in \mathbb{N}\}$. However, we see that this is not the right definition since we know that 2 divides 10, but 10 is not in

the set. So we realise that this is the set of all natural numbers that are only divisible by 2 or 3.

Therefore we can write this set as $F = \left\{2^a 3^b \text{ s.t. } (a, b \in \mathbb{Z}), (a, b \geq 0), \text{ and } (a + b \neq 0)\right\}$. Here, we are taking $a + b \neq 0$ so that a, b are not both 0, since $1 \notin F$.

2 · A little logic
2.7 · Exercises

2.7.9.

(a) We cannot determine whether or not it was raining on Monday.

(b) It was not raining on Tuesday.

(c) It was raining on Wednesday.

(d) We cannot determine whether or not it was raining on Thursday.

2.7.14. Recall that the only way an implication is false is if its conclusion is false while its hypothesis is true. So, $P \implies (Q \wedge R)$ being false means that $Q \wedge R$ is false while P is true. Note that $Q \wedge R$ being false means that at least one of Q and R are false

Consider the second implication, $\big((\sim Q) \wedge R\big) \implies (\sim P)$, which is true. Since P is true, the conclusion of this implication, $\sim P$ is false. But this means that the hypothesis, $(\sim Q) \wedge R$ is false, by modus tollens. More explicitly, if the hypothesis were true while the conclusion is false, then the implication is false, but if the hypothesis and conclusion are both false, then the implication is true. Therefore, at least one of $\sim Q$ and R are false.

We have determined that at least one of Q and R are false, and at least one of $\sim Q$ and R are false. In order for both of these conditions to be satisfied, R must be false. If R were true, then both Q and $\sim Q$ would necessarily be false, which is impossible!

We cannot determine the truth value of Q from the information given. While P must be true and R must be false, Q can be true or false. We show how you can verify this:

- Assume Q is false, then $Q \wedge R$ is false and P is true, so the first implication is false. Similarly, $(\sim Q) \wedge R$ is false, while $\sim P$ is false, so the second implication is true.
- Now assume Q is true, then $Q \wedge R$ is false and P is true, so the first implication is false. Similarly, $(\sim Q) \wedge R$ is false, while $\sim P$ is false, so the second implication is true.

Alternatively, we could determine the answer by analyzing the truth tables of the two implications.

P	Q	R	$P \implies (Q \wedge R)$	$\big((\sim Q) \wedge R\big) \implies (\sim P)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	F
T	F	F	F	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	T	T

The second and fourth rows are the only rows where the first implication is false, and the second true. In both these rows, P is true and R is false, so the truth values of these statements are determined. Since Q is true in the second row, but false in the fourth row, we cannot determine the truth value of Q. Either P is true, Q is true, and R is false, or P is true, Q is false, and R is false.

2.7.15. Since S is true, $\sim S$ is false, and consequently the conclusion of the implication, $Q \wedge (\sim S)$, is false. But since the implication is true, its hypothesis $R \vee (\sim P)$ must also be false. If the hypothesis were true while the conclusion were false, then the implication would be false.

Since $R \vee (\sim P)$ is false, both R and $\sim P$ are false. So R is false while P is true.

Finally, we use that $P \iff \big(Q \vee (\sim S)\big)$ is true. Since P is true, $Q \vee (\sim S)$ must also be true. But $\sim S$ is false, so Q must be true.

We have deduced that P and Q are true while R is false.

Alternatively, we could solve this question by analyzing the truth tables of the implication and biconditional. We only include the rows where S is true, since this is given in the question.

P	Q	R	S	$R \vee (\sim P)$	$\big(R \vee (\sim P)\big) \implies \big(Q \wedge (\sim S)\big)$	$P \iff \big(Q \vee (\sim S)\big)$
T	T	T	T	T	F	T
T	T	F	T	F	T	T
T	F	T	T	T	F	F
T	F	F	T	F	T	F
F	T	T	T	T	F	F
F	T	F	T	T	F	F
F	F	T	T	T	F	T
F	F	F	T	T	F	T

The only row in which the implication and biconditional are true is row 2, so we look at this row to discern the truth values of P, Q, and R: P is true, Q is true, and R is false.

2.7.16. We give a few different approaches.

- If A and B are statements, recall that $A \iff B$ is true only if both A and B are true, or both A and B are false. Applying this with $A = (P \vee Q) \implies R$ and $B = Q \wedge S$, and using the fact that $(P \vee Q) \implies R$ is false, we deduce that $B = Q \wedge S$ must be false. Now, since $Q \wedge S$ is false but S is true, Q must be false.

  Finally, we'll use the fact that $(P \vee Q) \implies R$ is false to determine the truth values of P and R. Recall that the only way that an implication is false is if its hypothesis is true, but its conclusion is false; in this case, that means the the hypothesis $P \vee Q$ is true, and the conclusion R is false. But remember that Q is false, so the only way $P \vee Q$ is true is if P is true.

  We have figured out that P is true, Q is false, and R is false.

- We can also start from the fact that the implication "$(P \vee Q) \implies R$" is false. An implication is only false when its hypothesis is true but its conclusion is false. This immediately tells us that R is false. It also tells us that $P \vee Q$ is true.

  Since the biconditional is true, but the clause on the left is false, we know that the clause on the right must also be false. That is $Q \wedge S$ must be false. Since we are told that S is true, we know that Q is false. Putting this together with the fact that $P \vee Q$ is true, we deduce that P is true.

  So again, we find that P is true, Q is false, and R is false.

- Alternatively, we could use truth tables to determine the answer. In the following, we build the truth table of $((P \vee Q) \implies R) \iff (Q \wedge S)$, but only include the rows where S is true, since this is given in the question.

P	Q	R	S	$(P \vee Q) \implies R$	$Q \wedge S$	$((P \vee Q) \implies R) \iff (Q \wedge S)$
T	T	T	T	T	T	T
T	T	F	T	F	T	F
T	F	T	T	T	F	F
T	F	F	T	F	F	T
F	T	T	T	T	T	T
F	T	F	T	F	T	F
F	F	T	T	T	F	F
F	F	F	T	T	F	F

  The fourth row is the only row where $(P \vee Q) \implies R$ is false but $((P \vee Q) \implies R) \iff (Q \wedge S)$ is true. Therefore we look to this row to determine the truth values of P, Q, and R: P is true, Q is false, and R is false.

Notice that the second solution is really just a reordering of the steps given in the first question. Since the initial deductions we made from the biconditional being true and the implication being false were independent of each other, we could make these deductions in either order. The last solution is different: rather than making logical deductions from the information given, we use the truth table to look at every possible case for the truth values of P, Q, and R, and look for the cases where the conditions given in the question are satisfied.

3 · Direct proofs
3.5 · Exercises

3.5.1. We see that this is a conditional statement, and to prove it, we are going to assume that the hypothesis is true and show that the conclusion follows. This means that we assume that n is an even number and show that $n^2 + 3n + 5$ is odd.

This means that, we have an assumption and we need to understand what this assumption means by using the definitions and extracting useful exressions, i.e. structures we can work with.

In this question, since n is even, we know that $n = 2k$ for some $k \in \mathbb{Z}$. This is a nice structure and we can build upon it. Now, since we want to show that $n^2 + 3n + 5$ is odd, we can try to see what this expression is equal to given the assumption, $n = 2k$.

We see that $n^2 + 3n + 5 = (2k)^2 + 3(2k) + 5 = 2(2k^2 + 3k + 2) + 1$. Then, using the definition of even and odd, we see that $n^2 + 3n + 5$ is an odd number since $2k^2 + 3k + 2 \in \mathbb{Z}$. This is what we wanted to show.

Of course, we don't need this much explanation in the actual proof. So, let's clear this up and turn this into a nice proof.

3.5.2. Even though this doesn't look like a condtional statement, we wee that we can rewrite it as:

$$\text{if } m, n \text{ are odd, then } mn \text{ is odd.}$$

Therefore, to prove it, we are going to assume that the hypothesis is true and show that the conclusion follows. This means that we assume that m, n are two odd numbers and show that mn is odd.

Here, we are going to rely on the definitions again to get as much information as we can from our assumption. Since m and n are odd, we know that $m = 2k+1$ for some $k \in \mathbb{Z}$ and $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$. Using this structure, we can get a nice expression for mn.

We see that $mn = (2k + 1)(2\ell + 1) = 2(2k\ell + k + \ell) + 1$. Then, using the definition of even and odd functions, we see that mn is an odd number since $2k\ell + k + \ell \in \mathbb{Z}$. This is what we wanted to show. Let's clean this up.

3.5.3. Let's consider the first case: the sum of two odd numbers. We're trying to determine the parity of $m + n$, where both m and n are odd. Since they're both odd, we know that $m = 2k + 1$ and $n = 2\ell + 1$ for some integers k, ℓ. Then

we can write

$$m + n = (2k + 1) + (2\ell + 1) = 2k + 2\ell + 2.$$

We want to write $m+n$ in either the form $2p$ or $2p+1$ for some integer p, which is possible depending on whether or not there is a factor of 2 in $m + n$, respectively. In this case, we can factor out a 2:

$$m + n = 2(k + \ell + 1)$$

and so $m + n$ is even. Note also that here we use the fact that $k + \ell + 1$ is an integer, since both k and ℓ are integers.

With this scratch work we can write a formal proof to the statement, "the sum of two odd numbers is even." The solutions to the remaining parts can be determined using the same logic: first, we'll suppose that m and n are two integers whose parity is given. We can then write out m in the form $2k$ if its even, or $2k + 1$ if its odd, and similarly for n (although we should be careful not to reuse the variable k in our expression for n). Using these expressions we can then obtain an expression for $m + n$ or mn. Then we manipulate this expression so that it's either in the form $2p$ or $2p + 1$, where p is an integer, to determine if it is even or odd.

3.5.6. This may look slightly harder then the previous questions since it involves 5 different variables and a conjunction (an "and") statement. But, since this is a conditional statement, our strategy is going to be the same. We are going to assume the hypothesis to be true and show the conclusion.

So, we assume that $n \mid a$ and $n \mid b$. This means that $a = nk$, and $b = n\ell$ for some $k, \ell \in \mathbb{Z}$. Once we have these expressions, we can bo back and try to see what we wanted to prove again.

We see that we want to prove that $n \mid (ax + by)$. In other words, we want to prove $ax + by = nm$ for some $m \in \mathbb{Z}$. Since we know that $a = nk$, and $b = n\ell$, we can write $ax + by = nkx + n\ell y = n(kx + \ell y)$. This is what we wanted to show. Let's write this in a nice proof.

3.5.7. The hypothesis tells us that $a = nk$ and $a + 1 = n\ell$. This means we can write $1 = (a + 1) - a = n\ell - nk = n(\ell - k)$. But this means $n \mid 1$. This is the basis of our proof.

3.5.8. This is a conditional statement. To prove this, we need to assume that the hypothesis is true and show that the conclusion follows. This means that we assume $3 \mid a$ and $2 \mid a$ and show that $6 \mid a$.

This is "one of those questions" where our previous knowledge may be misleading. We may want to say that since $6 = 3 \cdot 2$, then if a is divisible by 2 and 3, it should be divisible by 6. But this would be a wrong chain of implications. Even though the result is true in this instance, it may be wrong if we change the numbers a little. For example, if we change 3 by 4, that is, "If $4 \mid a$ and $2 \mid a$, then $8 \mid a$", even though the previous chain of implications doesn't change, the result doesn't follow anymore (for an example you can see the statement is false for $a = 4$).

It is always preferable to use the definitions and the results we have proven to prove these statements. Let's see what we can do.

Assume $3 \mid a$ and $2 \mid a$, that is, $a = 2k$ and $a = 3m$ for some $k, m \in \mathbb{Z}$. Since we want to show that $6 \mid a$, we can multiply both sides of the equation $a = 2k$ by 3 and both sides of the equation $a = 3m$ by 2 and get $3a = 6k$ and $2a = 6m$. Then, subtracting the second form the first, we get $a = 3a - 2a = 6k - 6m = 6(k - m)$, which is the desired result. Now, we can write this nicely in a proof.

Alternatively, assuming $a = 2k$ and $a = 3m$ for some $k, m \in \mathbb{Z}$, we can try to show that $m = 2\ell$ for some $\ell \in \mathbb{Z}$. We can do this by writing, $a = 2k = 3m$, and hence, $2k - 2m = 2(k - m) = m$. Therefore $a = 3m = 6(k - m)$, and the result follows.

3.5.9. To prove this conditional statement, we are going to assume the hypothesis to be true and show that the conclusion follows. That means, we assume that $3 \mid (n - 4)$ and show that $3 \mid (n^2 - 1)$.

We see that assuming $3 \mid (n - 4)$ means that $n - 4 = 3k$ for some $k \in \mathbb{Z}$, and under this assumption, we want to show that $3 \mid (n^2 - 1)$, that is, $n^2 - 1 = 3m$ for some $m \in \mathbb{Z}$. This suggests that if we have an expression on n we can simply calculate $n^2 - 1$ and get the desired result (alternatively, we could try to find a connection between $(n - 4)$ and $(n^2 - 1)$).

Since we know that $n - 4 = 3k$ for some $k \in \mathbb{Z}$, we see that $n = 3k + 4$. Thus, we get $n^2 - 1 = (3k + 4)^2 - 1 = (9k^2 + 24k + 16) - 1 = 3(3k^2 + 8k + 5)$, which is what we wanted to show. Of course, in a proof we don't need this much explanation. Indeed, we can write this explanation nicely in a proof.

3.5.13. Before we get into the proof we see that every integer ℓ is an integer root since we can take $k = 1$ and $m = \ell$ as given in the definition. But these numbers also include non-integer numbers such as $\sqrt[3]{7}$, since we can take $k = 3$ and $m = 7$ or $\sqrt{5}$ since we can teke $k = 2$ and $m = 5$ as given in the definition. We also see that the statement also says that the product of $\sqrt[3]{7}$ and $\sqrt{5}$ should also be an integer root. If we check carefully, we see that

$$\sqrt[3]{7} \cdot \sqrt{5} = 7^{1/3}5^{1/2} = (7^2)^{1/6}(5^3)^{1/6} = (7^2 5^3)^{1/6}.$$

Indeed, the product is an integer root.

Now, assume that a and b are integer roots. This means that there are natural numbers k, ℓ and integers m, n such that $a^k = n$ and $b^\ell = m$. Now, we are going to do the same trick we did in the example above to make the powers "same" so that we can multiply the bases. We see $a^{k\ell} = n^\ell$ and $b^{\ell k} = m^k$. Thus, $(ab)^{k\ell} = n^\ell m^k$, which is the desired result, since $n^\ell m^k \in \mathbb{Z}$. Now, we can write this in a proof.

3.5.16. It is a common (and easy to make) mistake to apply any operation on an inequality and implicitly assuming that the operation keeps the order of the inequality. Namely, if $f(x)$ is a function and we know $x < y$, we may want to say $f(x) < f(y)$. Unfortunately this is not always true, and we have to be careful. This reasoning would only be true if the function f is an increasing function

in the given domain; and this question tells us that $f(x) = \sqrt{x}$ is, indeed, an increasing function.

We see that what we want to prove in this question is a conditional statement. So, we can assume the hypothesis and try to show the conclusion. So, we assume $x > y$ and try to show that $\sqrt{x} > \sqrt{y}$. Since we cannot simply take the square roots of both sides (because of the above argument), we need to find a different way to prove the conclusion.

We see that we can rewrite the hypothesis as $x - y > 0$. We also know that

$$\underbrace{x - y}_{>0} = (\sqrt{x})^2 - \sqrt{y})^2 = (\sqrt{x} - \sqrt{y})\underbrace{(\sqrt{x} + \sqrt{y})}_{>0},$$

where we have factored the difference of squares: $a^2 - b^2 = (a - b)(a + b)$. This tells us that $\sqrt{x} - \sqrt{y} > 0$, which is what we wanted to show. Now, we can write this nicely in a proof.

3.5.18. For scratchwork, we'll begin with the inequality

$$\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}$$

Note that this is what we actually want to show though! When we write up the formal proof, we'll have to reverse the order of our logic. We'll try to derive something true from this inequality, which will be the starting point of our actual proof. Starting with

$$\sqrt{x + y} \leq \sqrt{x} + \sqrt{y},$$

note that $0 \leq \sqrt{x + y}$, and so we can square both sides to obtain

$$x + y \leq x + 2\sqrt{x}\sqrt{y} + y.$$

Here we are using that $(\sqrt{x})^2 = x$ since $x \geq 0$, and similarly for y. Subtracting $x + y$ from both sides of this inequality, we obtain

$$0 \leq \sqrt{x}\sqrt{y}$$

and this inequality is true since the square root function is never negative. So, for the actual proof of the statement, we should start with the inequality

$$0 \leq \sqrt{x}\sqrt{y},$$

and reverse the argument above to show the desired conclusion,

$$\sqrt{x + y} \leq \sqrt{x} + \sqrt{y}.$$

5 · More proofs
5.5 · Exercises

5.5.1. We see that this is a conditional statement and as we did in the previous chapters, we can try direct proof where we assume the hypothesis and show the conclusion. That means, we assume that $n^2 + 4n + 5$ is odd and show that n is even.

This means that we assume $n^2 + 4n + 5 = 2k + 1$ for some $k \in \mathbb{Z}$ and show that $n = 2m$ for some $m \in \mathbb{Z}$. However, to get from $n^2 + 4n + 5 = 2k + 1$ to showing that n is even will not an easy task. It would require us to take a square root, and so we would also need to understand the effect of taking square root on the parity of the number. Oof! That would be a whole other exercise.

So, we may need to look at a different proof method. Since this is a conditional statement, we can look at its contrapositive and see whether it simplifies the problem. First, let $n \in \mathbb{Z}$, then the contrapositive is:

$$n \text{ is not odd} \implies n^2 + 4n + 5 \text{ is not odd.}$$

But since n is an integer, we know that $n^2 + 4n + 5$ is an integer. So if n and $n^2 + 4n + 5$ are not odd, then they must be even. Hence we can write the contrapositive as

$$n \text{ is odd} \implies n^2 + 4n + 5 \text{ is even.}$$

This immediately looks like a simpler statement since the hypothesis in on n, which is the simpler term, and the conclusion is on $n^2 + 4n + 5$, which is the more complex term. From here, the rest of the proof is straightforward. We assume $n = 2k + 1$ for some $k \in \mathbb{Z}$ and then we write the expression for $n^2 + 4n + 5$ and show that it is even. Namely,

5.5.2. Trying to prove this statement directly doesn't make sense, and instead we're going to prove this statement by its contrapositive; that way, we will end up with a statement about what happens when 5 does divide an integer, and we can directly use the definition of that.

The contrapositive of this statement is: If $5 \mid n$, then $5 \mid n^2$. We've had plenty of practice proving statements like this.

5.5.3. In this question we see that we have a conditional statement. Hence, we can assume the hypothesis and try to show the conclusion. In other words, we can assume $5 \nmid n$ or $2 \nmid n$ and try to show that $10 \nmid n$. But, we we have seen before, whenever we have a hypothesis that is an "or"-statement, we may need to look at cases. Moreover, since each part of the hypothesis is of the form $a \nmid b$, we may need to look at cases for those as well. This means that if we want to prove this statement using direct proof and cases, it may get very long. Also, when we have lots of cases, the chances of making a mistake go up.

However, we can also look at the contrapositive of the statement and see whether it may be easier to prove. For $n \in \mathbb{Z}$, the contrapositive of the statement of the statement is:

$$\text{if } 10 \mid n, \text{ then } 5 \mid n \text{ and } 2 \mid n,$$

which looks much more promising, since if a number is divisible by 10, then it should be divisible by 5 and 2. Let's see how we can put that in a proof.

5.5.4. We see that this is a conditional statement. First let $n, m \in \mathbb{N}$. So, to prove the statement, we can assume the hypothesis, i.e. $n \neq 1$ and $n \neq 2$ and show that $n \nmid m$ or $n \nmid (m+2)$. However, the assumption doesn't give us much to work with. We can try to look at cases to create more structure to work with, but unfortunately, it is also not clear what those cases should be.

Changing our strategy, we can also look at the contrapositive of the statement:

$$\text{If } n \mid m \text{ and } n \mid (m+2), \text{ then } n = 1 \text{ or } n = 2.$$

Here we see that we assume $n \mid m$ and $n \mid (m+2)$. Then we can write $m = xn$ and $m + 2 = yn$. Now since the result says something about $n, 1$ and 2, it makes sense to try to use this equations to say something about 2. Taking the difference of these gives us

$$2 = xn - yn = n(x - y)$$

which implies that n should be able to divide 2 as well. But we know only natural number divisors of 2 are 1 and 2, which is just what we wanted to show. Now, let's write this up.

5.5.5. We see that this is a conditional statement. To prove it, we can assume that $n^2 + m^2$ is even and try to show that then n, m have the same parity, that is, n, m are both odd or both even. We know that $n^2 + m^2$ being even means that $n^2 + m^2 = 2k$ for some $k \in \mathbb{Z}$. We see that it may not be trivial to get from this assumption to the conclusion on n, m. It would require us to take square-roots or solve a quadratic or something like that. Urgh.

What we may try instead, is to look at the contrapositive of the statement we want to prove:

$$\text{if } n, m \text{ have different parities, then } n^2 + m^2 \text{ is not even,}$$

and since we know that $n^2 + m^2 \in \mathbb{Z}$, the conclusion also means that $n^2 + m^2$ is odd.

Here, if we assume the hypothesis of the contrapositive, we see that we have 2 cases: n is even and m is odd, and m is even and n is odd.

- If n is even and m is odd, then we have $n = 2a$ and $m = 2b + 1$ for some $a, b \in \mathbb{Z}$. This means that $n+2+m^2 = (2a)^2+(2b+1)^2 = 4a^2+4b^2+4b+1 = 2(2a^2 + 2b^2 + 2b) + 1$, which is odd.

- If n is odd and m is even, then we have $n = 2a + 1$ and $m = 2b$ for some $a, b \in \mathbb{Z}$. This means that $n+2+m^2 = (2a+1)^2+(2b)^2 = 4a^2+4a+1+4b^2 = 2(2a^2 + 2b^2 + 2a) + 1$, which is also odd.

Here, we also see that these cases are almost the same since the conclusion is symmetric with respect to n, m. So, practically, we will only need to prove one of case, and the other will be proven very similarly. So, we can use "WLOG" and

prove only one of the cases. But, whenever you want to use "WLOG", you should always convince yourself that the cases are similar by doing both possibilities and seeing just how similar they are. Let's see how we can make this work in a proof.

5.5.6. If we were to provide a direct proof of this statement, we'd have to start with the inequality $x^3 + 5x \geq x^2 + 1$ which would be difficult, since we have a cubic term. Instead we're going to prove this statement by its contrapositive; this will allow us to assume that $x \leq 0$, and from that we will know the sign of powers of x.

The contrapositive of this statement is: If $x \leq 0$, then $x^3 + 5x < x^2 + 1$. Notice that the right-hand side is positive, since $x^2 \geq 0$. At the same time, since $x \leq 0$, we know that $x^3 \leq 0$ and $5x \leq 0$. So the left-hand side is non-positive. Thus we can write $x^3 + 5x \leq 0 < x^2 + 1$. And now we have everything we need for the proof.

5.5.7. Proving things directly looks difficult, so we look at the contrapositive: *If a and b are consecutive, then $a + b$ is odd.* Now we can use the definition of consecutive to prove things.

5.5.8. We see that this is a conditional statement. This means that we can assume the hypothesis and then we try to show the conclusion. Hence, we assume that $n = 2k$ for some $k \in \mathbb{Z}$, and show that $n = 4m$ or $n = 4m + 2$ for some $m \in \mathbb{Z}$. However, when we wrote $n = 2k$ for some $k \in \mathbb{Z}$, we have extracted all the information from our assumption and we still need more to finish the proof.

Proof by cases is a very useful tool in stuations like these since it creates extra structure for us to work with.

In this question, if we pay a little closer attention to the conclusion of the statement, we see that we want to show is $n = 4m = 2(2m)$ or $n = 2(2m + 1)$ for some $m \in \mathbb{Z}$. Moreover, since $k \in \mathbb{Z}$, we know that $k = 2m$ or $k = 2m + 1$ for some $m \in \mathbb{Z}$, which suggests that we can use cases to finish the proof.

5.5.9. Since this is a biconditional statement, we need to prove the implication in both direction. That is we must prove

$$\big(2 \mid (n^4 - 7)\big) \implies \big(4 \mid (n^2 + 3)\big). \quad \text{and} \quad \big(4 \mid (n^2 + 3)\big) \implies \big(2 \mid (n^4 - 7)\big)$$

We'll prove each implication in turn.

5.5.10. We see that this is a biconditional statement and thus we need to prove both implications in both direction. That is, we need to prove

$$(3 \mid 5a) \implies (3 \mid a) \qquad \text{and} \qquad (3 \mid a) \implies (3 \mid 5a).$$

We can see that the second statement should be quite straightforward, as if a number is a multiple of 3 then 5 times that number will also be a multiple of 3.

However, the first statement may be a little more involved. We can try to look at the contrapositive of the statement:

$$(3 \nmid a) \implies (3 \nmid 5a).$$

So that we have an assumption on the simple term a. But, this also suggests that we use cases: $a = 3k + 1$ or $a = 3k + 2$ for some $k \in \mathbb{Z}$. Then, we can calculate $5a$ and show that it is also not divisible by 3.

We can also try to prove it directly by assuming $3 \mid 5a$, that is, $5a = 3k$ for some $k \in \mathbb{Z}$ and then manipulating this equation to show $a = 3n$ for some $n \in \mathbb{Z}$. Starting from $5a = 3k$, add a to both sides to get $6a = 3k + a$. Now subtract $3k$ from both sides to get $6a - 3k = a$. That is $a = 3(2a - k)$. Since $2a - k \in \mathbb{Z}$, we have shown that $3 \mid a$. Let's put everything together.

5.5.11. In order to show that $(n^2 - 1)(n^2 + 2n)$ is divisible by 4, we need to show that $(n^2 - 1)(n^2 + 2n) = 4k$ for some $k \in \mathbb{Z}$. This looks complicated, so we should split it into cases!

Factoring the expression, we have

$$(n^2 - 1)(n^2 + 2n) = (n - 1)(n + 1)n(n + 2)$$

This is a product of four consecutive terms! Using this factorization, we could do the proof by looking at four cases $n = 4k$, $4k + 1$, $4k + 2$ and $4k + 3$. That will work, but can we complete the proof in fewer cases?

Since the polynomial is the product of two factors, it would be sufficient to show that both are even, or that one of them is divisible by 4. This suggests that we might get away with considering the parity of n — what happens when n is even, and when n is odd. Lets try that first.

- In order to show that $(n^2 - 1)(n^2 + 2n)$ is divisible by 4, we need only show one of these factors is divisible by 4.
- When n is even n^2 is even, so $n^2 - 1$ is odd, and not divisible by 4. We will need to show that $n^2 + 2n$ is divisible by 4 in this case. That isn't too hard to show.
- When n is odd n^2 is odd, and so $n^2 + 2n$ is odd, and not divisible by 4. We will need to show that $n^2 - 1$ is divisible by 4 in this case. Since n is odd, we know $n = 2k + 1$, so $n^2 - 1 = 4k^2 + 4k + 1 - 1$, so this isn't too hard.

5.5.12. We would like to prove the contrapositive, but first we need to figure out what the contrapositive statement is. Following the technical definition, we know that the contrapositive is *If* $\sim$*(x or y is odd, but not both), then* $\sim$*(x + y is odd).* But we would like to translate this into a statement that is easier to comprehend.

There are four possibilities for the parities of x and y:

(a) x and y are both even.

(b) x is even and y is odd.

(c) x is odd and y is even.

(d) x and y are both odd.

Options 2 and 3 fit the condition *x or y is odd, but not both.* Therefore, *$\sim$(x or y is odd, but not both)* is satisfied by conditions 1 and 4. We can summarize these two conditions as "x and y have the same parity." Moreover, we can rewrite the statement *$\sim$($x + y$ is odd)* as "$x + y$ is even."

Putting these together, we get the contrapositive: *If x and y have the same parity, then $x + y$ is even.*

5.5.13. We see that this is a conditional statement. So, to prove that we can assume the hypothesis and show the conclusion. That is, we assume that $3 \mid (n^2+4n+1)$ and try to show that $n \equiv 1 \pmod 3$. We see that our assumption means that $n^2 + 4n + 1 = 3m$ for some $m \in \mathbb{Z}$. From this assumption, we need to get to our conclusion on n. Even though it is not impossible, it will require factoring or taking the square root, and we may need to prove more statements about how those operations affect divisibility by 3.

Instead, we can look at the contrapositive of the statement and try to have an assumption on the simpler term n. First let $n \in \mathbb{Z}$. Then, we see that the contrapositive of the statement is

$$\text{if } n \not\equiv 1 \pmod 3 \text{ then } 3 \nmid (n^2 + 4n + 1).$$

Therefore we assume that $n \not\equiv 1 \pmod 3$. This means that we have 2 cases, $n \equiv 0 \pmod 3$ or $n \equiv 2 \pmod 3$. We see that both of these cases give us nice structure to work on and say something about $n^2 + 4n + 1$. Let's work this out in a proof.

5.5.14. We see that this is a conditional statement. As we did before we can try to prove it directly by assuming the hypothesis and proving the conclusion. This means that we assume $5 \nmid m$ and show that $m^2 \equiv 1 \pmod 5$ or $m^2 \equiv -1 \pmod 5$.

Since the hypothesis is $5 \nmid m$, this suggests we may need to use cases to prove the result. Once we divide the proof into cases, the rest becomes quite straightforward.

We could also think about looking at the contrapositive of this statement. This may make sense since the conclusion of the statement is an "or"-statement. We see that the contrapositive of the statement is:

$$\text{if } m^2 \not\equiv 1 \pmod 5 \text{ and } m^2 \not\equiv -1 \pmod 5, \text{ then } 5 \mid m.$$

This still suggests that we use case, and, on top of that, we would have assumptions on the more conplex term, m^2, instead of the simpler term, m. Therefore, using direct proof with cases may be easier in this question. Let's write the proof using cases.

5.5.15. Before attempting the question, we should think about what method of proof will be easiest to show. Based on the content of the chapter, it might be wise to consider a proof by contrapositive or a direct proof with cases.

The contrapositive statement is: *If $q^2 \not\equiv 1 \pmod 3$, then $3 \mid q$.* This means that either $q^2 \equiv 0 \pmod 3$ or $q^2 \equiv 2 \pmod 3$. Working through this problem

would be difficult because we start with cases on q^2 and we need to get information about q from there. For this reason, we look into another method of proof.

Now, we consider what a direct proof with cases might look like. If $3 \nmid q$, then when we divide q by 3, we will have a remainder of 1 or 2. This gives two very clear cases. It is also easier in general to start with information about q and work towards information about q^2. Because of this, we proceed directly using cases.

5.5.18. Questions of this type are a good examples of how useful the triangle inequality can be. We see that if we wanted to use triangle inequality we could just say:

$$|x+4| + |x-3| = |x+4| + |3-x| \geq |(x+4)+(3-x)| = |7| = 7.$$

However, we are not allowed to use triangle inequality in this question. This means that we need to find a way to simplify the absolute values in the question. For that, we need to understand when the expressions $(x+4)$ and $(x-3)$ change signs. We see that these expressions change signs at $x = -4$ and $x = 3$, respectively.

It is quite common to think here to look at cases $x < -4$, $x \geq 4$, $x < 3$, and $x \geq 3$. However, this is not the most efficient way to divide this hypothesis into cases since even when we look at $x \geq -4$, we need to consider the sign of $x - 3$. This means that within some of the cases, we may have extra cases to consider.

Instead, we can divide these into 3 cases: $x < -4$, $-4 \leq x \leq 3$, and $x > 3$. This is more efficient since the expressions $(x+4)$ and $(x-3)$ do not change signs in the intervals $(-\infty, -4)$, $[-4, 3]$, and $(3, \infty)$.

Let's see how this works in the proof.

5.5.19. We assume the hypothesis is true and then try to reach the conclusion. So we assume $|x-1| < 1$ and then we have to bound $|x^2 - 1|$. In order to bound $|x^2 - 1|$, we can factor the expression, and try to bound each term.

$$|x^2 - 1| = |(x-1)(x+1)| = |x-1| \cdot |x+1|$$

The factor $|x-1| < 1$, by assumption. But how can we bound the factor $|x+1|$? We'll need to use the inequality $|x-1| < 1$ again.

Recall that $|x-1| < 1$ is equivalent to the inequality

$$-1 < x - 1 < 1.$$

Adding 2 to everything, we end up with

$$1 < x + 1 < 3.$$

In particular, this implies that $|x+1| < 3$.

There is a slicker way to obtain this inequality: we'll use a common trick, where we "add zero in a fancy way," in order to introduce the term $x - 1$. Then, we'll use the triangle inequality and the bound $|x-1| < 1$.

$$|x+1| = |(x-1)+2| \leq |x-1| + |2| < 1 + 2 = 3.$$

With this scrap work, we're ready to write up the proof of the statement. Remember to be careful with the logic of the proof! We need to start by assuming the hypothesis, $|x-1|<1$, and show the conclusion is true, that $|x^2-1|<3$.

5.5.20. In order to bound $|2x^2-3x-2|$, we can factor the expression, and try to bound each term.

$$|2x^2-3x-2| = |(2x+1)(x-2)| = |2x+1|\cdot|x-2|$$

The factor $|x-2|<1$, by assumption. But how can we bound the factor $|2x+1|$? We'll need to use the inequality $|x-2|<1$ again.

Recall that $|x-2|<1$ is equivalent to the inequality

$$-1 < x-2 < 1.$$

Adding 2 to everything, we end up with

$$1 < x < 3$$

which implies that $|x|<3$. Now we can use the triangle inequality to bound $|2x+1|$:

$$|2x+1| \leq |2x|+1 = 2|x|+1 < 2\cdot 3+1 = 7.$$

With this scrap work, we're ready to write up the proof of the statement. But remember, we have to make sure that our logic flows in the correct direction. We must make sure our proof starts from the hypothesis, $|x-2|<1$, and ends at the conclusion, $|2x^2-3x-2|<7$.

5.5.21. We start with some scratch work. We are asked to prove an inequality involving absolute values. Often, the triangle inequality can be helpful in these types of situations. Recall that the triangle inequality states: For any $a, b \in \mathbb{R}$, $|a+b| \leq |a|+|b|$. Following the hint, we would like to show that $-|x-y| \leq |x|-|y|$ and $|x|-|y| \leq |x-y|$.

We rearrange the triangle inequality and attempt to get the formula $|x|-|y| \leq |x-y|$ (the other formula can be obtained similarly).

First, we want to rearrange the inequality to have a minus sign on the left, and only one set of absolute values on the right:

$$|a+b|-|b| \leq |a|.$$

Then we can try to write a and b in terms of x and y to make the equation above look more like the equation we want to prove. Notice that we currently have $|a|$ on the right hand side, whereas the expression we want to obtain has $|x-y|$. Let's try setting $a = x-y$:

$$|x-y+b|-|b| \leq |x-y|.$$

Now, the left hand side has $|x-y+b|-|b|$ where we want $|x|-|y|$. We set $b=y$ to obtain the desired expression:

$$|x-y+y|-|y| = |x|-|y| \leq |x-y|.$$

All that's left to do is to write this up as a proof!

5.5.22. The statement as written is false, which we can see by looking at the value of the function for a negative point in the domain, say $x = -1$, and then at a positive point in the domain, say $y = 1$. Then $x \leq y$ but $f(-1) = -1 < 1 = f(1)$. So f is not decreasing on all of $\mathbb{R} - \{0\}$. We will not run into this issue if we instead restrict the domain of the function to $(0, \infty)$.

Let's rewrite the statement as this: Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = 1/x$. Then f is decreasing.

We are going to take $0 < x < y$ and we want to show that $1/x > 1/y$. To do this start from $x < y$ and then divide both sides by xy (which is strictly positive). This gives $1/y < 1/x$ as required. Notice that this argument fails precisely when xy is negative, and that happens when x and y have opposite signs - exactly as we saw in our counter-example above.

6 · Quantifiers
6.6 · Exercises

6.6.1. We see that this statement is a universal statement. Since we don't have any information on n other than being an integer, we see that we don't really have a useful assumption on n. For example, if n were even, then we could say $n = 2m$ for some $m \in \mathbb{Z}$. But in this question, n is just an integer, which is a broad assumption without much structure.

One strategy to create such structure is to divide our assumption into cases. This may create extra steps, but eventually may help us have "something" that we can work with. In this example, since we are trying to prove a statement on divisibility by 3, it would make sense to look at cases that makes use of division algorithm and divisibility by 3.

6.6.4. Even though this result looks counter-intuitive, it is indeed true.

We are going to use proof by contrapositive. We see that the expression $\forall a, b \in \mathbb{Z}$ is not a part of the conditional statement, and therefore we see that the contrapositive of the statement is:

$$\forall a, b \in \mathbb{Z}, \text{ if } 3 \nmid a \text{ or } 3 \nmid b, \text{ then } \forall a, b \in \mathbb{Z}, \text{ we have } 3 \nmid (a^2 + b^2).$$

Looking at the contrapositive still doesn't sound like the right thing to do since we are trading a "3 divides (blah)" type of statement as the hypothesis of the conditional statement, with two "3 doesn't divide (blah)" type of statement that are connected with "or". This may seem like it will make our assumption more complicated. But, by looking at the contrapositive, we now have an assumption on simpler building blocks, a and b, instead of the their complicated, complex, counterpart $a^2 + b^2$. Since we now have "$3 \nmid a$ or $3 \nmid b$", it also suggests that we look at cases.

Since these two cases, $3 \nmid a$ and $3 \nmid b$, are very similar we will will make the proof a little shorter by using "without loss of generality". To make it shorter still, we will contract "without loss of generality" to "WLOG". Note that one should always be careful using WLOG since it is easy to introduce errors by

assuming two cases are very similar when, in fact, there are subtle differences between them.

Once we assume, WLOG that $3 \nmid a$, this will introduce 2 new cases. Then, within those cases, we may still need to have some extra cases for the choices of b. We want to mention that this is quite standard to have cases within cases, within cases, within cases, within…

6.6.5. The converse of the statement is:

> For any $n, a, b \in \mathbb{Z}$, if $n \mid ab$, then $n \mid a$ or $n \mid b$.

While the negation of this statement is:

> There are some $n, a, b \in \mathbb{Z}$ such that $n \mid ab$ but $n \nmid a$ and $n \nmid b$.

We can test some values of n, a, and b to see if the converse or its negation is true. Suppose $n = 4$. We can factor $n = 2 \cdot 2$. This means is we take $a = b = 2$, then $n = ab$ and so $n \mid ab$. But $4 \nmid 2$, so $n \nmid a$ and $n \nmid b$. Therefore we see that the converse of the original statement is false. Now we write up a formal disproof.

6.6.6. We'll try to construct a pair of function f, g so that $f + g$ is odd, but one of them, say f, is even. We can test the result by look at some simple odd and even functions. An example of an odd function is $y = x$, and an example of an even function is $y = x^2$. Let's suppose that $f(x) = x^2$, which is even. We want to choose g so that $f + g = x$, an odd function. Then we can take $g(x) = x - f(x) = x - x^2$. Also, we can show that g is neither even nor odd; $g(2) = -2$ but $g(-2) = -6$.

6.6.16. Suppose that $\varepsilon > 0$ is given. We need to find some $\delta > 0$ so that

$$0 < |x - 4| < \delta \implies |-3x + 5 - (-7)| < \varepsilon.$$

We can simplify the inequality on the right-hand side as

$$|-3x + 12| < \varepsilon.$$

Moreover, we can factor out $|-3|$ from the absolute value, and we'll end up with a factor of $|x - 4|$:

$$3|x - 4| < \varepsilon.$$

So in order to satisfy the initial implication, we need to find $\delta > 0$ so that

$$0 < |x - 4| < \delta \implies 3|x - 4| < \varepsilon.$$

From this, we see that this implication will be satisfied when $\delta \leq \varepsilon/3$. So let's just take $\delta = \varepsilon/3$.

Now we can write up the formal proof.

6.6.17. We need to show the following:

For any $\varepsilon > 0$, there is some $\delta > 0$ so that whenever $0 < |x - 1| < \delta$, we have

$|x^2 - 1| < \varepsilon$.

In order to bound $|x^2 - 1|$, we can factor the expression, and try to bound each term.

$$|x^2 - 1| = |(x - 1)(x + 1)| = |x - 1| \cdot |x + 1|$$

The factor $|x - 1| < \delta$, by assumption. But how can we bound the factor $|x + 1|$? The idea is this: as long as x is close to 1, say $|x - 1| \leq 1$, then $|x + 1|$ can't be too large. When we decide how to choose delta, we don't have to make our choice immediately; we can impose several conditions on delta, and then when we have worked out all those conditions, we can make our final choice. So, we are going to require that $\delta \leq 1$, to ensure that $|1 + x|$ is not too large. To see this assume that $|x - 1| \leq 1$. Then

$$-1 \leq x - 1 \leq 1.$$

Adding 2 to everything, we end up with

$$1 \leq x + 1 \leq 3.$$

In particular, this implies that $|x + 1| \leq 3$.

So let's assume that $|x - 1| < \delta$ and $|x - 1| \leq 1$, the latter implying that $|x + 1| \leq 3$. Returning back to the factorization of $|x^2 - 1|$, we have

$$|x^2 - 1| = |x + 1| \cdot |x - 1| < 3\delta$$

So we want to choose delta so that

$$3\delta \leq \varepsilon$$

and so we are going to place another condition on delta, namely that

$$\delta \leq \frac{\varepsilon}{3}.$$

Now we don't immediately set $\delta = \varepsilon/3$, since we also required (above) that $\delta \leq 1$. We can satisfy both of these requirements by setting $\delta = \min\{1, \varepsilon/3\}$.

With this scratchwork, we're ready to write up the proof.

6.6.18. Given $\varepsilon > 0$, we need to find a number N so that whenever $n > N$, we have

$$\left| \frac{1}{n^2} - 0 \right| = \frac{1}{n^2} < \varepsilon.$$

Rearranging this gives (and using the fact that ε is positive)

$$\frac{1}{\varepsilon} < n^2$$

Taking the square-root of both sides then tells us that we need

$$n > \frac{1}{\sqrt{\varepsilon}}$$

Therefore, we could take N to be any number greater than $1/\sqrt{\varepsilon}$. In particular, we could take N to be the smallest integer greater than $1/\sqrt{\varepsilon}$, which we denote by

$$\lceil 1/\sqrt{\varepsilon} \rceil .$$

6.6.19. For a given $\varepsilon > 0$, we want to find $\delta > 0$ so that the following holds:

$$0 < |x - 0| < \delta \implies |f(x) - 0| < \varepsilon.$$

Since $x \neq 0$, the latter inequality is equivalent to

$$\left| 6x \sin\left(\frac{1}{x}\right) \right| < \varepsilon.$$

We can simplify the inequality we're working with by using the fact that $|\sin(\theta)|$ is always bounded by 1. This means that

$$\left| 6x \sin\left(\frac{1}{x}\right) \right| \leq |6x|.$$

You may have seen something like this argument given when discussing the "Squeeze theorem" or "Sandwich theorem" in a calculus course.

Then, if we have $|6x| < \varepsilon$, we'd also have

$$\left| 6x \sin\left(\frac{1}{x}\right) \right| < \varepsilon.$$

But $|6x| < \varepsilon$ is satisfied when

$$|x| < \frac{\varepsilon}{6}.$$

Therefore, we need $\delta \leq \varepsilon/6$.

Now we can write up the formal proof.

6.6.20. When the sequence $(x_n)_{n\in\mathbb{N}} = \left((-1)^n + \frac{1}{n}\right)_{n\in\mathbb{N}}$ does not converge to 0 it means that (x_n) satisfies the negation of the definition of convergence with $L = 0$. That is (x_n) does not converge to 0 is equivalent to the statement

$$\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n > N, \left| \left((-1)^n + \frac{1}{n}\right) - 0 \right| \geq \varepsilon.$$

Observe that because of the $(-1)^n$ term in the sequence, it behaves slightly differently when n is even or odd.

- For all even $k \in \mathbb{N}$, we have $x_k = (-1)^k + \frac{1}{k} = 1 + \frac{1}{k} > 1$.
- While for all odd $k \in \mathbb{N}$ with $k \geq 3$, $x_k = (-1)^k + \frac{1}{k} = -1 + \frac{1}{k} \leq -\frac{2}{3}$.

Thus, for any integer $k \geq 2$, we know that $|x_k - 0| = |x_k| \geq \frac{2}{3}$.

So take $\varepsilon = \dfrac{1}{2}$, and then for any $N \in \mathbb{N}$, so that $n > N$, set $n = \max\{N, 2\} + 1$ and we can make the proof work.

6.6.21. Recall from the definition of a limit that the proof should go like this: We're given some $\varepsilon > 0$. We need to find some $N \in \mathbb{N}$ so that whenever $n > N$, we have

$$\left|1 - \frac{2}{n^2} - \frac{3}{n^3} - 1\right| < \varepsilon.$$

Let's unravel this inequality to see how large n needs to be. We want to rewrite this inequality so that's in the form n greater than *something*; this *something* will tell us how large we need to take N so that the inequality above is satisfied.

Now since,

$$\left|\left(1 - \frac{2}{n^2} - \frac{3}{n^3}\right) - 1\right| = \left|-\frac{2}{n^2} - \frac{3}{n^3}\right| = \frac{2}{n^2} + \frac{3}{n^3}$$

the inequality can be simplified to

$$\frac{2}{n^2} + \frac{3}{n^3} < \varepsilon.$$

Because we have two different terms with n in them, it may be difficult to put the inequality in the form we'd like, that is, n greater than something. What we can do instead is consider which of the factors $1/n^2$ or $1/n^3$ is larger - and then only working with the larger one, by bounding the smaller one by it.

Since $n \in \mathbb{N}$ we know that $n \geq 1$. Dividing this inequality by n^3 (which is positive), then gives us

$$\frac{1}{n^2} \geq \frac{1}{n^3}$$

This means that

$$\frac{2}{n^2} + \frac{3}{n^3} \leq \frac{2}{n^2} + \frac{3}{n^2} = \frac{5}{n^2}.$$

Therefore, if we show that

$$\frac{5}{n^2} < \varepsilon,$$

we'll also have that

$$\frac{2}{n^2} + \frac{3}{n^3} < \varepsilon.$$

Now, we can take the square root of both sides of the inequality $5/n^2 < \varepsilon$, and use the reverse implication in the fact given in the question to say that

$$\frac{\sqrt{5}}{\sqrt{\varepsilon}} < n.$$

We can actually reverse all of these steps (so we use the forward implication in the given fact), to get the following implication:

$$\frac{\sqrt{5}}{\sqrt{\varepsilon}} < n \implies \left|1 - \frac{2}{n^2} - \frac{3}{n^3} - 1\right| < \varepsilon.$$

This tells us that we can take N to be any natural number that is larger than $\sqrt{5}/\sqrt{\varepsilon}$.

So we can now write up the proof.

But we have actually missed an opportunity to simplify things further here. Recall that by dividing the inequality $n \geq 1$ by n^3 we showed that $\frac{1}{n^2} \geq \frac{1}{n^3}$ and used that to simplify the inequalities that we needed for our proof. We can simplify things further still by noting that by dividing the inequality $n \geq 1$ by n^2 we get

$$\frac{1}{n} \geq \frac{1}{n^2}$$

So we can use that to show

$$\frac{2}{n^2} + \frac{3}{n^3} \leq \frac{2}{n^2} + \frac{3}{n^2} = \frac{5}{n^2} \leq \frac{5}{n}.$$

Then it suffices to find n big enough so that

$$\frac{5}{n} \leq \varepsilon$$

which is quite a bit easier than the mucking around with square-roots we had to do before.

6.6.22.

- (c): Let $M > 0$. We need to find some $N \in \mathbb{N}$ so that whenever $n \geq N$, we have $\sqrt{n} \geq M$. But we know that $\sqrt{n} \geq M$ if $n \geq M^2$, by Exercise 3.5.16. Therefore we can take $N = M^2$.

- (d): We need to show that there is some $M > 0$, so that for any $N \in \mathbb{N}$, there's some $n \geq N$ such that $(-1)^n\sqrt{n} < M$. But if n is odd, then $(-1)^n\sqrt{n}$ is negative, so $(-1)^n\sqrt{n} < M$. Thus we could actually take M to be any positive number, but for the purposes of the proof, we just need to fix a particular value of M. For example, we can take $M = 1$.

- (e): Let $M > 0$. We need to find some $N \in \mathbb{N}$ so that whenever $n \geq N$, we have $n^2 - 100n \geq M$. Factoring the expression on the lefthand side of the inequality, we have

  $$n(n - 100) \geq M.$$

  Notice that if $n - 100 \geq 1$, then

  $$n(n - 100) \geq n \cdot 1 = n.$$

So if $n \geq 101$ and $n \geq M$ we would have

$$n^2 - 100n = n(n-100) \geq n \geq M,$$

as desired. So we need M to satisfy both $n \geq M$ and $n \geq 101$; therefore we can take $N = \max\{M, 101\}$.

6.6.23.

(a) By definition of the limit, we know that for any $\varepsilon > 0$ there is some $N_\varepsilon \in \mathbb{N}$ such that

$$n \geq N_\varepsilon \implies |a_n - L| < \varepsilon.$$

We would like to use the inequality $|a_n - L| < \varepsilon$ to bound $|a_n|$ from above. We can do this by writing $|a_n| = |L + (a_n - L)|$ and applying the triangle inequality. Indeed, we have

$$|a_n| = |L + (a_n - L)| \leq |L| + |a_n - L| < |L| + \varepsilon.$$

This inequality is true for all $n \geq N_\varepsilon$, where N_ε depends on our choice of $\varepsilon > 0$. We are free to choose ε, so let's take $\varepsilon = 1$. So for $n \geq N_1$, we have

$$|a_n - L| < 1 \implies |a_n| < |L| + 1.$$

Now, we only have to bound

$$|a_1|, |a_2|, \ldots, |a_{N_1 - 1}|.$$

But there are only finitely many values here, so

$$M_0 = \max\{|a_1|, |a_2|, \ldots, |a_{N_1-1}|\}$$

exists, and is a real number. So for any value of n, we know that $|a_n| \leq M_0$ if $1 \leq n < N_1$, or $|a_n| \leq |L| + 1$ if $n \geq N_1$. Therefore we may take $M = \max\{M_0, |L| + 1\}$.

(b) We need to find a sequence $\{a_n\}_{n \in \mathbb{N}}$ that is bounded, but does not converge. We can try a sequence that oscillates between two fixed numbers, say the sequence $a_n = (-1)^n$. This sequence is bounded, as $|a_n| \leq 1$ for all n. Intuitively, this sequence shouldn't converge since it doesn't get arbitrarily close to a single number. Indeed, in Example 6.4.6, we saw that this sequence doesn't converge to 1. But we need to show a bit more than that for the purposes of this question; we need to show that for any $L \in \mathbb{R}$, the sequence doesn't converge to L.

To this end, we'll fix $L \in \mathbb{R}$. To show that a_n doesn't converge to L, we need to find some ε so that the following is true:

$$\forall N \in \mathbb{N}\ \exists n \geq N \text{ such that } |a_n - L| \geq \varepsilon.$$

In Example 6.4.6, we saw that for $L = 1$ we could take $\varepsilon = 1$, because then for any odd n, $|a_n - 1| = |-1 - 1| > 1$.

6.6.24. We will prove part (c) by contrapositive. The contrapositive of the statement given in the question is

$$\lim_{n\to\infty} a_n \neq +\infty \implies (a_n)_{n\in\mathbb{N}} \text{ is bounded, or } (a_n)_{n\in\mathbb{N}} \text{ is not increasing.}$$

So this statement is of the form $P \implies (R \vee \sim Q)$. Now Q is either true or false

- If Q is false, then $R \vee \sim Q$ is false, and so $P \implies F$ is true. Easy!
- While if Q is true, then $R \vee \sim Q$ is true only when R is true, and so we have to show that $P \implies R$ is true.

Also notice that this statement is of the same form as Exercise 4.3.1 (c), and so is logically equivalent to

$$\lim_{n\to\infty} a_n \neq +\infty \implies ((a_n)_{n\in\mathbb{N}} \text{ is increasing} \implies a_n \text{ is bounded.})$$

and so is of the form $P \implies (Q \implies R)$. Now, either Q is true or false:

- If Q is false, then $F \implies R$ is true, and so $P \implies T$ is true. Easy!
- On the other hand, if Q is true, then $T \implies R$ is only true when R is true, so we really have to show that $P \implies R$ is true.

We know one of the following is true: either $(a_n)_{n\in\mathbb{N}}$ is not increasing, or it is increasing. We will use these two cases to prove the contrapositive statement. In the first case, when $(a_n)_{n\in\mathbb{N}}$ is not increasing, the conclusion (that $(a_n)_{n\in\mathbb{N}}$ is bounded or not increasing) is automatically satisfied. Therefore we may focus on the second case, when $(a_n)_{n\in\mathbb{N}}$ is increasing. We need to show that the sequence is bounded (then the conclusion, $(a_n)_{n\in\mathbb{N}}$ is bounded or not increasing, will be true).

By assumption, we know that $\lim_{n\to\infty} a_n \neq +\infty$. By definition, this means there is some $C > 0$ such that for any $n \in \mathbb{N}$, there's some $m \geq n$ with $a_m < C$. But since $(a_n)_{n\in\mathbb{N}}$ is increasing, we know that $a_n \leq a_m < C$. While it's not necessarily true that $a_n \leq a_m$ for all n, since m depends on n, we do know that $a_n < C$ for all n. Moreover, we also know that $a_n \geq a_1$ for all $n \in \mathbb{N}$, as a_n is increasing. Combining these inequalities to get $a_1 \leq a_n < C$, we can show that $|a_n|$ is bounded, as desired.

6.6.25. This particular distance function is known as the "discrete distance" or the "discrete metric". Proving (a) is straight-forward by case analysis with either $x = z$ or $x \neq z$.

Now, (b) looks like a complicated question since the distance that is given in the definition is not the standard distance function (where the distance between two points is defined as the absolute value of their difference). It also forces the reader to parse a new definition carefully and rework their understanding of the topic at hand.

Now, let's try to see how we can prove this statememt.

First, we see that this is a conditional statement. This means that we can start by assuming the hypothesis and trying to show the conclusion. That is, assume that $(x_n)_{n\in\mathbb{N}}$ converges to L, and try to show that the set $\{n \in \mathbb{N} : x_n \neq L\}$ is finite.

Since $(x_n)_{n\in\mathbb{N}}$ converges to L, we know that

$$\forall \varepsilon > 0, \exists N_0 \in \mathbb{N} \text{ such that } \forall n \geq N_0, D(x_n, L) < \varepsilon.$$

Hence, since this is true for all $\forall \varepsilon > 0$, we know that this should also work for $\varepsilon = 1/2$ (any ε strictly between 0 and 1 would work here). So, for $\varepsilon = 1/2$, we should be able to find an $N_\varepsilon \in \mathbb{N}$, such that $\forall n > N_\varepsilon$, we have $D(x_n, L) < \varepsilon = 1/2$.

Now, if we recall that the distance function D only takes values 0 or 1, this means that $D(x_n, L) = 0 < \varepsilon = 1/2 < 1$.

We see that this is what we needed to show, since this means that $\forall n > N_\varepsilon$, we have $x_n = L$. So the only $x_n \neq L$ must occur for $n \leq N_\varepsilon$, and that is a finite set.

Let's clean this up and write in a proof.

7 · Induction
7.3 · Exercises

7.3.2. When $n = k$ the statement is

$$k! \leq k^k$$

and we will need to prove that when $n = k + 1$

$$(k+1)! \leq (k+1)^(k+1)$$

Notice that $(k+1)! = k! \cdot (k+1)$, and so we can make the first statement look like the second by multiplying by $(k+1)$.

7.3.3. Since we want to show that the inequality $n! > 3^n$ is true for all $n \geq 7$, we can try to understand how it behaves for any natural number. We can do that by starting to look at the statement for different values of $k \in \mathbb{N}$, and try to develop a pattern.

We see that for $n = 1$, the inequality becomes $1 > 3$, which is not true. Moreover, if we check couple more numbers, we get,

$n = 2$	$2 > 9$	False,
$n = 3$	$6 > 27$	False,
$n = 4$	$24 > 81$	False,
$n = 5$	$120 > 243$	False,
$n = 6$	$720 > 729$	False,
$n = 7$	$5040 > 2187$	True,
$n = 8$	$40320 > 6561$	True,

$$n = 9 \qquad 362880 > 19683 \qquad \text{True.}$$

After $n = 7$, we see that the left hand side grows much faster than the right hand side of the inequlity. This suggests that the statement shoul be true. Now, we can try to show it using induction.

7.3.5. We want to prove this statement by induction. As written, we want to prove something about all $m \in \mathbb{N}$, with m even. In order to prove this by induction, however, we need reformulate the statement into something that we want to prove for all natural numbers instead. We can do this by writing $m = 2n$ for $n \in \mathbb{N}$. Then we need to prove that 8 divides $3^{2n} - 1$. We can prove this by induction on n.

- Base case: Take $n = 1$. Then $3^{2\cdot n} - 1 = 9 - 1 = 8$,, so the result holds.
- Inductive step: Assume that
$$8 \mid 3^{2k} - 1.$$
We want to show that
$$8 \mid 3^{2(k+1)} - 1.$$
Our assumption implies that there is some ℓ so that
$$3^{2k} - 1 = 8\ell$$
Rephrasing this we have $3^{2k} = 8\ell + 1$. Notice that this implies that
$$3^{2k+2} = 72\ell + 9$$
and thus
$$3^{2k+2} - 1 = 72\ell + 8 = 8(9\ell + 1)$$
giving us the divisibilty result we need.
- Inductive step — another way: Another way to approach the inductive step is to consider the difference between
$$(3^{2k} - 1) \qquad \text{and} \qquad (3^{2k+2} - 1).$$
If that difference is a multiple of 8 then we can infer the result we need. So again, we assume that $8 \mid 3^{2k} - 1$, and then consider the difference:
$$(3^{2k+2} - 1) - (3^{2k} - 1) = 3^{2k+2} - 3^{2k} = 9 \cdot 3^{2k} - 3^{2k} = 8 \cdot 3^{2k}.$$
Then since $8 \mid 3^{2k} - 1$ we can write it as 8ℓ for some integer ℓ, and so
$$(3^{2k+2} - 1) - \underbrace{(3^{2k} - 1)}_{=8\ell} = 8 \cdot 3^{2k}$$
and so, with a little rearranging:
$$(3^{2k+2} - 1) = 8\ell + 8 \cdot 3^{2k}.$$
So we see that $3^{2(k+1)} - 1$ is divisible by 8.

Now we can write the formal proof up:

7.3.6. We want to prove this statement by induction on n.

- Base case: Take $n = 2$. Then the distributive law gives the result in this case.
- Inductive step: Assume that the result holds for $n = k$, and let $a, b_1, \ldots, b_k, b_{k+1}$ be real numbers. The inductive step tells us that

$$a \cdot (b_1 + b_2 + \cdots + b_k) = a \cdot b_1 + a \cdot b_2 + \cdots + a \cdot b_k,$$

which we want to use to show that

$$a \cdot (b_1 + b_2 + \cdots + b_k + b_{k+1}) = a \cdot b_1 + a \cdot b_2 + \cdots + a \cdot b_k + a \cdot b_{k+1}.$$

We want to consider the sum $b_1 + b_2 + \cdots + b_k + b_{k+1}$ as the sum of two elements, $b_1 + b_2 + \cdots + b_k$ and b_{k+1}. Then the distributive law tells us that

$$a \cdot ((b_1 + b_2 + \cdots + b_k) + b_{k+1}) = a \cdot (b_1 + b_2 + \cdots + b_k) + a \cdot b_{k+1}.$$

Then we can apply the inductive hypothesis to obtain the desired result.

Now we can write the formal proof up:

7.3.7. We need to prove the statement by induction. Let's take a look at the base case and the inductive step.

- Base case: $n = 1$. Then $2^n = 2 = 2n$, so $2^n \geq 2n$.
- Inductive step: Suppose that the inequality holds for $n = k$, so that $2^k \geq 2k$. We need to show that $2^{k+1} \geq 2(k + 1)$.

Note that if we multiply $2^k \geq 2k$ by 2, we'll end up with 2^{k+1} on the right-hand side, which is the right-hand side of the inequality we want to show. So, let's do that to obtain $2^{k+1} \geq 4k$.

Now, we just need to show that $4k \geq 2(k + 1)$, and then we'll have $2^{k+1} \geq 2(k+1)$, as desired. Let's try to simplify the inequality $4k \geq 2(k+1)$ to see why it may be true. Rewrite the inequality as

$$4k \geq 2k + 2.$$

Subtracting $2k$ from both sides of the inequality, we have $2k \geq 2$, which after dividing by 2, gives $k \geq 1$. But we know that $k \geq 1$, since $k \in \mathbb{N}$.

We can reverse all of these steps to show that $k \geq 1$ implies $4k \geq 2(k + 1)$. When we write up the formal proof, it's important to start with what we know, namely, that $k \geq 1$, in order to show the desired inequality, that $4k \geq 2(k + 1)$.

Now we're ready to write up the proof.

7.3.8. We want to prove this statement by induction on n.

- Base case: Take $n = 0$. Then $2^{2n+1} + 3^{2n+1} = 2 + 3 = 5$,, so the result holds.
- Inductive step: Assume that

$$5 \mid \left(2^{2k+1} + 3^{2k+1}\right).$$

We want to show that

$$5 \mid \left(2^{2(k+1)+1} + 3^{2(k+1)+1}\right).$$

The assumption tells us that there is some $\ell \in \mathbb{Z}$ such that

$$2^{2k+1} + 3^{2k+1} = 5\ell.$$

Note that

$$2^{2(k+1)+1} + 3^{2(k+1)+1} = 4 \cdot 2^{2k+1} + 9 \cdot 3^{2k+1} = 4(2^{2k+1} + 3^{2k+1}) + 5 \cdot 3^{2k+1}$$

and so by the inductive hypothesis

$$2^{2(k+1)+1} + 3^{2(k+1)+1} = 4 \cdot 5\ell + 5 \cdot 3^{2k+1} = 5(4\ell + 3^{2k+1}).$$

Therefore $2^{2(k+1)+1} + 3^{2(k+1)+1}$ is divisible by 5.

- Another way to prove the inductive step is to consider the difference

$$2^{2(k+1)+1} + 3^{2(k+1)+1} - (2^{2k+1} + 3^{2k+1})$$

and show that it is divisible by 5. To this end, let's simplify that difference:

$$\begin{aligned} 2^{2(k+1)+1} + 3^{2(k+1)+1} - (2^{2k+1} + 3^{2k+1}) &= 3^{2(k+1)+1} - 3^{2k+1} + 2^{2(k+1)+1} - 2^{2k+1} \\ &= 3^2 \cdot 3^{2k+1} - 3^{2k+1} + 2^2 \cdot 2^{2k+1} - 2^{2k+1} \\ &= 8 \cdot 3^{2k+1} + 3 \cdot 2^{2k+1} \end{aligned}$$

We want to rewrite this expression in terms of $2^{2k+1} + 3^{2k+1}$, so that we can use the inductive hypothesis. We rewrite the expression above as

$$8 \cdot 3^{2k+1} + 3 \cdot 2^{2k+1} = 8 \cdot (3^{2k+1} + 2^{2k+1}) - 8 \cdot 2^{2k+1} + 3 \cdot 2^{2k+1}$$

This is equal to

$$8 \cdot (3^{2k+1} + 2^{2k+1}) - 5 \cdot 2^{2k+1}$$

which is divisibly by 5, since the second term is divisible by 5, and the first is by the inductive hypothesis.

Now we can write the formal proof.

7.3.9. The base case of our induction will be $n = 2$. For this part, we'll need to show that the sum of two rational numbers is rational. For the inductive step, we'll suppose that any sum of k rational numbers is rational, and we need to show that if $x_1, x_2, \ldots, x_{k+1} \in \mathbb{Q}$, then their sum lies in $\mathbb{Q}$. The inductive hypothesis tells us that $x_1 + x_2 + \cdots + x_k \in \mathbb{Q}$. But then $x_1 + x_2 + \cdots + x_k + x_{k+1}$ is really the sum of two rational numbers, and hence rational itself, as we already proved this in the base case.

7.3.10. Here we see that we have a statement we want to show is true for all $n \in \mathbb{N}$. In the previous chapters we have seen that to show such statements we assume that $n \in \mathbb{N}$ and try to show the conclusion. Moreover, since the hypothesis is broad, we would probably need to look at cases. But, in this question, it is not clear what the cases should be. Instead, we can also apply mathematical induction. We see that this equation doesn't seem to give us a formula, but instead gives us two expressions. However, we know what the right hand side is equal to.

We know that $\displaystyle\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$ (see Result 7.1.8). This means that what we wat to show now becomes

$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}.$$

Now, we see that the how we can use induction becomes more clear.

7.3.11. In this question we want to prove that an inequality is true for all $n \in \mathbb{N}$. This suggests that induction may be a good method. However, in general, proving inequalities using induction may be a little harder than proving equalities or formulas. The reason for that is the fact that to prove the inductive step, we may need to manipulate the inequality, add or subtract terms, etc. We should also be very careful to make sure that our logic flows correctly in the proof of the inductive step; start from the assumption and reach the conclusion.

If we look at this question, and try to understand how we can handle the inductive step, we would assume that

$$\sum_{j=1}^{n} j^3 > \frac{1}{4}n^4,$$

and try to show

$$\sum_{j=1}^{n+1} j^3 > \frac{1}{4}(n+1)^4.$$

We see that (based on our assumption)

$$\sum_{j=1}^{n+1} j^3 = \sum_{j=1}^{n} j^3 + (n+1)^3 > \frac{1}{4}n^4 + (n+1)^3.$$

But, $\frac{1}{4}n^4 + (n+1)^3 \neq \frac{1}{4}(n+1)^4$.

To complete the proof, we need to show that

$$\frac{1}{4}n^4 + (n+1)^3 > \frac{1}{4}(n+1)^4 = \frac{1}{4}n^4 + n^3 + \frac{3}{2}n^2 + n + \frac{1}{4},$$

which requires just a little careful algebra.

7.3.12. Let $r \neq 1$, and then we need to prove the result

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$$

for all $n \in \{0, 1, 2, \dots\}$. Notice that since $r \neq 1$ the expression is okay; we do not divide by zero. For the base case, we need to show that the result holds for $n = 0$, while for the inductive step, we need to assume that the result is true for $n = k$, and then show that the result also holds for $n = k + 1$.

- The base case is true, since

$$\sum_{i=0}^{0} r^i = r^0 = 1 = \frac{1 - r}{1 - r} = \frac{1 - r^{0+1}}{1 - r}.$$

- For the inductive step, we assume that

$$\sum_{i=0}^{k} r^i = \frac{1 - r^{k+1}}{1 - r}$$

and we need to show that

$$\sum_{i=0}^{k+1} r^i = \frac{1 - r^{(k+1)+1}}{1 - r}.$$

Notice that we can obtain the left-hand side of the equation above by adding r^{k+1} to the left-hand side of the inductive hypothesis. Then

$$\sum_{i=0}^{k+1} r^i = \sum_{i=0}^{k} r^i + r^{k+1} = \frac{1 - r^{k+1}}{1 - r} + r^{k+1},$$

by the inductive hypothesis. Now we just need to simplify the right-hand side of this equation:

$$\frac{1 - r^{k+1}}{1 - r} + r^{k+1} = \frac{1 - r^{k+1} + r^{k+1}(1 - r)}{1 - r} = \frac{1 - r^{k+2}}{1 - r},$$

and we end up with

$$\sum_{i=0}^{k+1} r^i = \frac{1 - r^{(k+1)+1}}{1 - r}$$

which is what we wanted to show.

7.3.13.

(a) $f(x) = x^n$ for $n \in \mathbb{N}$ with $0 \leq k \leq n$.

We compute the first few derivatives:

$$\begin{aligned} f^{(1)}(x) &= nx^{n-1} \\ f^{(2)}(x) &= n(n-1)x^{n-2} \\ f^{(3)}(x) &= n(n-1)(n-2)x^{n-3} \\ f^{(4)}(x) &= n(n-1)(n-2)(n-3)x^{n-4} \end{aligned}$$

A reasonable guess for the k^{th} derivative would be

$$f^{(k)}(x) = n(n-1)\dots(n-(k-1))x^{n-k} = \frac{n!}{(n-k)!}x^{n-k}$$

(b) $g(x) = x^{-n}$ for $n \in \mathbb{N}$.

$$\begin{aligned} g^{(1)}(x) &= -nx^{-n-1} \\ g^{(2)}(x) &= -n(-n-1)x^{-n-2} \\ g^{(3)}(x) &= -n(-n-1)(-n-2)x^{-n-3} \\ g^{(4)}(x) &= -n(-n-1)(-n-2)(-n-3)x^{-n-4} \end{aligned}$$

A reasonable guess for the k^{th} derivative would be

$$g^{(k)}(x) = (-1)^k n(n+1)\dots(n+(k-1))x^{n-k} = \frac{(-1)^k(n+(k-1))!}{(n-1)!}x^{-n-k}$$

(c) $h(x) = \frac{1}{\sqrt{9-2x}} = (9-2x)^{-1/2}$. This one is a little harder:

$$\begin{aligned} h^{(1)}(x) &= -\frac{1}{2}(-2)(9-2x)^{-3/2} = (9-2x)^{-(2(1)+1)/2} \\ h^{(2)}(x) &= \frac{-3}{2}(-2)(9-2x)^{-5/2} = 3(9-2x)^{-(2(2)+1)/2} \\ h^{(3)}(x) &= (3)\frac{-5}{2}(-2)(9-2x)^{-7/2} = 3\cdot 5(9-2x)^{-(2(3)+1)/2} \\ h^{(4)}(x) &= (3\cdot 5)\frac{-7}{2}(-2)(9-2x)^{-9/2} = 3\cdot 5\cdot 7(9-2x)^{-(2(4)+1)/2} \end{aligned}$$

A reasonable guess for the k^{th} derivative would be

$$h^{(k)}(x) = (3)(5)\dots(2(k-1)+1))(9-2x)^{-(2k+1)/2} = (2k-1)!!(9-2x)^{-(2k+1)/2}$$

where we have used the double factorial $\ell!!$

$$\ell!! = \begin{cases} \ell\cdot(\ell-2)\cdots 2 & \text{when } \ell \text{ is even} \\ \ell\cdot(\ell-2)\cdots 3\cdot 1 & \text{when } \ell \text{ is odd} \end{cases}.$$

7.3.14. As scratch work for this problem, we work through the computation we will have to do after we adopt the inductive hypothesis. That is, we want to work through the $n = j + 1$ case after assuming that the $n = j$ case is true.

$$\sum_{k=1}^{j+1} k^2 = (j+1)^2 + \sum_{k=1}^{j} k^2$$

By the inductive hypothesis,

$$= (j+1)^2 + \frac{j(j+1)(2j+1)}{6}.$$

Expanding the expression and giving both terms a common denominator yields

$$= \frac{(6j^2 + 12j + 6) + (2j^3 + 3j^2 + j)}{6}.$$

Grouping like terms,

$$= \frac{2j^3 + 9j^2 + 13j + 6}{6}$$

We now factor the top of the expression to get

$$= \frac{(j+1)(j+2)(2j+3)}{6}$$

Finally, we rearrange into the desired form

$$= \frac{(j+1)((j+1)+1)(2(j+1)+1)}{6}.$$

The above method works, but, in line (2), we could also notice that $j + 1$ is a common factor, rather than expanding the cubic. That computation looks like:

$$\sum_{k=1}^{j+1} k^2 = (j+1)^2 + \sum_{k=1}^{j} k^2$$

By the inductive hypothesis,

$$= (j+1)^2 + \frac{j(j+1)(2j+1)}{6}.$$

Factoring out $j + 1$ and giving both terms a common denominator yields

$$= \frac{(j+1)(6(j+1) + j(2j+1))}{6}.$$

Simplifying,

$$= \frac{(j+1)(2j^2 + 7j + 6)}{6}$$

We now factor the top of the expression to get

$$= \frac{(j+1)(j+2)(2j+3)}{6}$$

Finally, we rearrange into the desired form

$$= \frac{(j+1)((j+1)+1)(2(j+1)+1)}{6}.$$

7.3.17.

(a) We'll prove this by induction on n.

- Base case: take $n = 0$. Then

$$\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} e^{-x} = 0.$$

- Inductive step: Assuming

$$\lim_{x\to\infty} x^k e^{-x} = 0,$$

we want to show that

$$\lim_{x\to\infty} x^{k+1} e^{-x} = 0.$$

Rewrite the limit as

$$\lim_{x\to\infty} \frac{x^{k+1}}{e^x}$$

Then both the numerator and the denominator go to infinity as $x \to \infty$. Then we can use l'Hôpital's rule to obtain

$$\lim_{x\to\infty} \frac{x^{k+1}}{e^x} = \lim_{x\to\infty} \frac{(k+1)x^k}{e^x}.$$

Now we have a limit that almost looks like the one from the inductive hypothesis; we just need to pull out the constant factor of $k+1$, and then apply the inductive hypothesis:

$$\lim_{x\to\infty} \frac{x^{k+1}}{e^x} = (k+1) \lim_{x\to\infty} \frac{x^k}{e^x} = (k+1) \cdot 0 = 0.$$

(b) We'll prove this by induction on n.

- Base case: take $n = 0$. Then

$$\int_0^\infty x^n e^{-x} \mathrm{d}x = \int_0^\infty e^{-x} \mathrm{d}x$$

Evaluating this integral yields

$$\int_0^\infty e^{-x} \mathrm{d}x = \lim_{t\to\infty} (-e^{-x}) \Big]_0^t = 1.$$

And since $1 = 0!$ the base case holds.

- Inductive step: Suppose that

$$k! = \int_0^\infty x^k e^{-x} \mathrm{d}x.$$

We want to use this equation in order to show that

$$(k+1)! = \int_0^\infty x^{k+1} e^{-x} \mathrm{d}x.$$

We can connect the integral in the equation for $(k+1)!$ to the integral in the equation for $k!$ by using integration by parts, because $\frac{d}{\mathrm{d}x} x^{k+1} = (k+1)x^k$. Indeed, by integration by parts,

$$\begin{aligned}\int_0^\infty x^{k+1} e^{-x} \mathrm{d}x &= \left(-x^{k+1} e^{-x}\right)\Big]_0^\infty + (k+1) \int_0^\infty x^k e^{-x} \mathrm{d}x \\ &= \lim_{t\to\infty} \left(-t^{k+1} e^{-t}\right) + 0^{k+1} e^{-0} + (k+1) \int_0^\infty x^k e^{-x} \mathrm{d}x\end{aligned}$$

We then use our result from (a) to evaluate that limit, giving

$$= 0 + (k+1) \int_0^\infty x^k e^{-x} \mathrm{d}x$$

Now we're left with $(k+1)$ times the integral that we've assumed is equal to $k!$ in the inductive hypothesis. But this gives exactly the equation that we wanted to show!

7.3.20. Let's try making this work with $F_7 = 13$. That is, we want to show that $F_7 \mid F_7 n$. Se we'll assume that $F_7 \mid F_{7k}$ and then we'll try to show that this implies that $F_7 \mid F_{7k+7}$. Now, from the definition of the Fibonacci numbers, we can write F_{7k+7} in terms of F_{7k+6} and F_{7k+5}:

$$F_{7k+7} = F_{7k+6} + F_{7k+5}$$

and then similarly expand $F_{7k+6} = F_{7k+5} + F_{7k+4}$, so:

$$F_{7k+7} = 2F_{7k+5} + F_{7k+4}$$

and keep going:

$$\begin{aligned}F_{7k+7} &= 2F_{7k+5} + F_{7k+4} && \text{expand } F_{7k+5} \\ &= 3F_{7k+4} + 2F_{7k+3} && \text{expand } F_{7k+4} \\ &= 5F_{7k+3} + 3F_{7k+2} && \text{expand } F_{7k+3} \\ &= 8F_{7k+2} + 5F_{7k+1} && \text{expand } F_{7k+2} \\ &= 13F_{7k+1} + 8F_{7k}\end{aligned}$$

Now since, by assumption $13 = F_7 \mid F_{7k}$, we are done.

Notice that each time we expand and regroup the terms we get another Fibonacci number. So, let us try to write this more explicitly in terms of Fibonacci numbers:

$$F_n = 1 \cdot F_{n-1} + 1 \cdot F_{n-2}$$

$$\begin{aligned}
&= F_2 \cdot F_{n-1} + F_1 \cdot F_{n-2} \\
&= F_3 \cdot F_{n-2} + F_2 \cdot F_{n-3} \\
&= F_4 \cdot F_{n-3} + F_3 \cdot F_{n-4} \\
&= F_5 \cdot F_{n-4} + F_4 \cdot F_{n-5}
\end{aligned}$$

So we might try to show, more generally, that

$$F_n = F_\ell \cdot F_{n-\ell+1} + F_{\ell-1} \cdot F_{n-\ell}$$

or equivalently (after substituting $n \mapsto n + \ell$)

$$F_{n+\ell} = F_\ell \cdot F_{n+1} + F_{\ell-1} \cdot F_n.$$

Setting $n = 7k, \ell = 7$ in this gives us

$$F_{7k+7} = F_7 \cdot F_{7k+1} + F_6 \cdot F_{7k}$$

which is precisely what we need.

7.3.23. As scratch work, we want to try to break the $n+1$ case into something involving the previous cases.

Rather than trying to factor $\alpha^{n+1} + \frac{1}{\alpha^{n+1}}$, we attempt to get to it by multiplying the previous cases. We try to multiply the $n = 1$ and n cases because this will give a power of $n + 1$

$$\left(\alpha^n + \frac{1}{\alpha^n}\right)\left(\alpha + \frac{1}{\alpha}\right) = \alpha^{n+1} + \frac{1}{\alpha^{n+1}} + \alpha^{n-1} + \frac{1}{\alpha^{n-1}}$$

We now have the term we want in right hand side! Rearranging, we see

$$\alpha^{n+1} + \frac{1}{\alpha^{n+1}} = \left(\alpha^n + \frac{1}{\alpha^n}\right)\left(\alpha + \frac{1}{\alpha}\right) - \left(\alpha^{n-1} + \frac{1}{\alpha^{n-1}}\right).$$

Since we can write the $(n+1)^{\text{st}}$ case in terms of the n^{th} and $(n-1)^{\text{st}}$ cases, we should make sure to have two consecutive cases (e.g. $n = 0, 1$) in the base case.

Notice that we are able to use an induction argument with two parts to the base case, since this allows us to have two parts to the inductive hypothesis. Another way of proving this is to notice that for the $n+1$ case we need to reference more than just the previous case, so we can use proof by strong induction.

7.3.24. This question tell us that from a natural number n, we can divide out the the biggest possible power of 3, say 3^m, so that $3 \nmid \left(\frac{n}{3^m}\right)$. Since this is a statement that should hold for all natural numbers, induction may be useful method to use. But, because of the nature of the problem, i.e. since we are trying to factor out as many powers of 3 as possible, we see that it may be hard to get from $n = k$ to $n = k + 1$, but it may be easier to get from $n = \frac{(k+1)}{3}$ to $n = k + 1$ assuming that

$\frac{(k+1)}{3} \in \mathbb{Z}$. This suggests that we may need to use strong induction rather than regular induction.

7.3.26. We want to prove this statement by induction on n. Since a_0 and a_1 are not given by the recurrence relation, we'll prove that the formula is satisfied for $n = 0$ and $n = 1$ in the base case. When $n \geq 2$, we have to use the recurrence relation $a_n = a_{n-1} + 6a_{n-2}$ to prove that the formula holds. We can do this by plugging in the formula for a_{n-1} and a_{n-2}. Consequently, for the inductive hypothesis, we need to rely on the statement for $n-1$ and $n-2$, not just $n-1$. Therefore we need to use strong induction.

7.3.27. You may have already seen how to write a natural number in binary:

- We take the natural number,
- find the biggest power of 2 that is smaller than that number,
- subtract that power of 2 from the number,
- and keep doing this till you end up with 0 or 1.

Here we see that, even though this method is practically useful, this doesn't give us the proof. Since there are some gaps in the argument. For example, how can we pick that "biggest factor of 2", or what does it mean to "keep doing till we end up with 0 or 1"?

This is a good use of induction methods since induction takes care of that "keep going till..." portion of the process.

Now, we may want to use regular induction in this question. If we think about how we can prove the inductive step, to get from n to $n+1$, we would assume $n = c_m 2^m + c_{m-1}2^{m-1} + \cdots + c_1 2 + c_0$ for some $m \geq 0$ and constants $c_0, c_1, c_2, \ldots, c_m \in \{0, 1\}$. Then, we see $n+1 = c_m 2^m + c_{m-1}2^{m-1} + \cdots + c_1 2 + c_0 + 1$. But then, we have to have $2m$ different cases depending on whether or not c_i is 0 or not, for $0 \leq i \leq m$. We see that this has a very similar "keep going till..." type of problem. Instead, we can use strong induction and the fact that every natural number is even or odd.

Notice that in our argument we are implicitly making use of the distributive law

$$2 \cdot (a_0 + a_1 + a_2 + \cdots + a_n) = 2 \cdot a_0 + 2 \cdot a_1 + 2 \cdot a_2 + \cdots + 2 \cdot a_n$$

Now normally, we state the distributive law as

$$a \cdot (b + c) = a \cdot b + a \cdot c,$$

that is, we show how we distribute the produce over two summands, rather than n summands. So we should really prove that it extends in this way. That is precisely Exercise 7.3.6.

8 · Return to sets

8.6 · Exercises

8.6.2. The sets F and G may look like regular sets in $\mathbb{R}^2$, but we see that in both cases, the second item of each ordered pair is given as a function of the first coordinate. This suggests that F and G, indeed, can be seen as representing the graphs of the functions $f(x) = x^2 - 3x + 2$ and $g(x) = x + 2$ respectively. So, finding the intersection of these sets is the same as finding the intersection of these two graphs. Let's see how that works in the proof.

Any element in F is of the form $(x, x^2 - 3x + 2)$ and any element of G is of the form $(z, z + 2)$. So if an element is in the intersection we must have that

$$(x, x^2 - 3x + 2) = (z, z + 2)$$

and so (since these are ordered pairs) $x = z$ and $x^2 - 3x + 2 = z + 2$. Combining these we get the equation

$$x^2 - 3x + 2 = x + 2$$

and so

$$x^2 - 4x = 0$$

which implies that $x = 0, 4$. So the points in the intersection are $\{(0, 2), (4, 6)\}$.

But, we still have to prove that

$$F \cap G = \{(0, 2), (4, 6)\}$$

We do this in two ways, first we prove a sequence of set-equalities, and then second we show that each side is a subset of the other.

8.6.3. At a first glance, this statement looks like a true statement. It is because it sounds like it is saying: "if I remove elements of C from B and then add them back again, I get back to B". But unfortunately it is not exactly what it says. What it says is: "if I remove all elements of C from B and add *all* the elements of C to it, then you get back to B". Now, we see that this doesn't have to be true since the set C may contain element that were not in B in the first place. So those elements wouldn't have been removed when we took the difference, and would be added on top when we took the union. So, as long as the set C has an element that is not in B, we will have a counterexample. Let's see how we can create such a counterexample for this statement.

8.6.4. We want to prove that $\{x^a \text{ s.t. } a \in \mathbb{Q}\} = \{y^a \text{ s.t. } a \in \mathbb{Q}\}$ given that $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$ satisfying, $x, y > 0$ and $x^k = y$. At first glance it may look like a difficult question since it has many variables. But, we can simplify this statement and have a better intuition by choosing certain values of x and y.

For example, we can take $x = \sqrt{3}$ and $y = 3$, i.e. $k = 2$. Then we see that the statement becomes:

$$\left\{\left(\sqrt{3}\right)^a \text{ s.t. } a \in \mathbb{Q}\right\} = \left\{\left(3^{\frac{1}{2}}\right)^a \text{ s.t. } a \in \mathbb{Q}\right\} = \left\{3^{\frac{a}{2}} \text{ s.t. } a \in \mathbb{Q}\right\} = \left\{3^b \text{ s.t. } b \in \mathbb{Q}\right\}.$$

We see that the set on the left and the one on the right are the same sets since for every $a \in \mathbb{Q}$, we have $b \in \mathbb{Q}$ such that $b = \frac{a}{2}$ and for every $b \in \mathbb{Q}$, we have $a \in \mathbb{Q}$ such that $a = 2b$. This definitely does not work if we were forced to take integer a, b.

Now if we replace $\sqrt{3}$ and 3 with x and y with $x, y > 0$ and $x^k = y$, same argument would still hold, but instead of $k = 2$, we would have some other k. Namely,

$$\{x^a \text{ s.t. } a \in \mathbb{Q}\} = \left\{\left(y^{\frac{1}{k}}\right)^a \text{ s.t. } a \in \mathbb{Q}\right\} = \left\{y^{\frac{a}{k}} \text{ s.t. } a \in \mathbb{Q}\right\} = \left\{y^b \text{ s.t. } b \in \mathbb{Q}\right\}.$$

We can now write this neatly up as a proof.

8.6.5. We see that this is a "prove or disprove" question. This means that we need to figure out whether it is true or not.

The statement claims that every element that is both in the sets $\{x \in \mathbb{Z} : m \mid x\}$ and $\{x \in \mathbb{Z} : n \mid x\}$ must be in $\{x \in \mathbb{Z} : mn \mid x\}$ as well. In other words, if a number is a multiple of m and also is a multiple of n, then it is also a multiple of mn. This statement doesn't really sound right since m can itself be a multiple of n, and thus, the hypothesis wouldn't give us any more information other than the number being a multiple of m (since it automatically becomes a multiple of n in that situation). This also suggests that we find a counterexample to this statement. Moreover, this suggest what kind of a counterexample would work. Let's see how we can create a counterexample.

8.6.6. We see that this is a `prove or disprove' question. This means that we need to figure out whether it is true or not.

The statement claims that every element of $\{x \in \mathbb{Z} : mn \mid x\}$ is an element of $\{x \in \mathbb{Z} : m \mid x\}$ and $\{x \in \mathbb{Z} : n \mid x\}$. In other words, every multiple of mn is a multiple of m and also a multiple of n. We can easily tell that this statement is true. So, we need to prove it.

Of course, we should be extra careful around statements that we can "easily" decide are true. Sometimes they are very deceptive. As a general rule, we should avoid using words like "easily" when we do mathematics.

8.6.9. We see that since we are dealing with power sets in this question, we may not be able to use Venn diagrams to determine the set equality. This means that to see whether we want to prove or disprove the statement, we need to understand what the statement claims.

We can see that the statement says: "intersection of the power sets of A and B is the power set of their intersection". In other words, it says that every set that is both a subset of A and B must be a subset of their intersection AND a subset of their intersection must be a subset of both A and B. This makes more sense. We can tell now that this statement is true. So, we can try to prove it.

8.6.10. We can see that the statement says: "the union of the power sets of A and B is the power set of their union". In other words, it claims that every set that is a subset of A or B must be a subset of their union *and* every subset

of the union must be a subset of A or B. We can tell that the first part of the statement is indeed true, that is, every set that is a subset of A or B must be a subset of their union. But, second part of the statement doesn't sound right, since the union of two sets can be "bigger" than both sets. This would create a problem since we can simply take the union of these two sets as the "subset" of the union, i.e. the element of the power set of the union. This suggests that we should be able to create a counterexample as long as $A \cup B \neq A$ and $A \cup B \neq B$. Let's see how we can create a counterexample.

8.6.11. Let A be a set. We can check the statement for some numbers to see whether it works for, at least, sets with small sizes. This may also give us some ideas as to how to prove it for any size set.

We see that if $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$. This means that $|\mathcal{P}(A)| = 1 = 2^0$.

Now, assume that $|A| = 1$, e.g. $A = \{1\}$. Then, we see that $\mathcal{P}(A) = \{\emptyset, \{1\}\}$. Thus, $|\mathcal{P}(A)| = 2 = 2^1$.

Similarly, assume $|A| = 2$, e.g. $A = \{1, 2\}$. Then, we see that $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Thus, $|\mathcal{P}(A)| = 4 = 2^2$.

Finally, if $|A| = 3$, e.g. $A = \{1, 2, 3\}$. Then, we see that

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Thus, $|\mathcal{P}(A)| = 8 = 2^3$.

This suggests that the stamement may, indeed, be true, and since it is a statement on the size of A, induction may be a useful tool.

To see how induction may work, we may need to understand how we can count the size of the power set of a set of size $n + 1$, when we know the size for a set of size n. For that, we observe that

$$\begin{aligned}\mathcal{P}(\{1, 2, 3\}) &= \mathcal{P}(\{1, 2\}) \cup \{\{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\} \\ &= \mathcal{P}(\{1, 2\}) \cup \{\emptyset \cup \{3\}, \{1\} \cup \{3\}, \{2\} \cup \{3\}, \{1, 2\} \cup \{3\}\}.\end{aligned}$$

This suggests that when we add an element to a set, we double its power set's size, since every subset that contains 3 is a subset, B, of $\{1, 2\}$ union $\{3\}$.

8.6.12. Both statements are false. In the first case, notice that the empty set is an element of every power set. This means that the empty set is in the left-hand side, but not in the right-hand side.

The second statement is a little harder to disprove. Think about what happens when A and B have some, but not all, elements in common. For example, $A = \{1, 2\}$ and $B = \{2, 3\}$, so that $A - B = \{1\}$. Then $\{1, 2\}$ is in the left-hand side, but not in the right-hand side. We can simplify this counter-example a little further, since we don't actually need "3" to make it work.

8.6.13.

(a) In order to show that the two sets are equal, we need to show that each is a subset of the other. First, we need to show that

$$x \in \bigcup_{n=3}^{\infty} \left(\frac{1}{n}, 1 - \frac{1}{n}\right) \implies x \in (0, 1).$$

This direction is more straightforward to prove, as $(1/n, 1-1/n) \subset (0,1)$ for all $n \in \mathbb{N}$, $n \geq 3$. Then, we need to show that

$$y \in (0,1) \implies y \in \bigcup_{n=3}^{\infty} \left(\frac{1}{n}, 1 - \frac{1}{n} \right).$$

This direction is trickier. For $y \in (0,1)$, we need to find some $N \in \mathbb{N}$ so that

$$\frac{1}{N} \leq y \leq 1 - \frac{1}{N}.$$

The inequality $1/N \leq y$ implies that we need $N \geq 1/y$. The inequality $y \leq 1 - 1/N$ implies that we need $N \geq 1/(1-y)$. So we need to take N bigger than both $1/y$ and $1/(1-y)$; for example, we can set $N = \max\left\{ \left\lceil \frac{1}{y} \right\rceil, \left\lceil \frac{1}{1-y} \right\rceil \right\} + 1$

With this scratchwork, we can write up the proof.

(b) Similarly as the previous part, we need to show each of the sets is a subset of the other. First, we need to show that

$$x \in [0,1] \implies x \in \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n} \right)$$

This direction is fairly straightforward, since $[0,1] \subseteq (-1/n, 1+1/n)$ for all $n \in \mathbb{N}$. The trickier part is showing that

$$y \in \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n} \right) \implies y \in [0,1].$$

We can prove this by contrapositive, that is, instead prove

$$y \notin [0,1] \implies y \notin \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1 + \frac{1}{n} \right).$$

If $y \notin [0,1]$, then either $y < 0$ or $y > 1$. If $y < 0$, then we could find some $N \in \mathbb{N}$ so that $y < -1/N$. But then $y \notin (-1/N, 1+1/N)$, which means that $y \notin \cap_{n=1}^{\infty}(-1/n, 1+1/n)$. If $y > 1$, then we could find some $N \in \mathbb{N}$ so that $1 + 1/N < y$. Indeed, by rearranging, we see that this inequality will hold as long as $N > 1/(y-1)$. From here, we can similarly show the desired result.

Now we can write the proof.

8.6.15.

(a) Any real number that is at least 3 is larger than or equal to every element of the set $[1,3]$. That means that any real number that is at least 3 is an

upper bound of $[1, 3]$. Since 3 is an element of $[1, 3]$, and any number less than 3 is not an upper bound for $[1, 3]$, we see that 3 is the least upper bound and the maximum of $[1, 3]$.

(b) We claim that $(1, 3)$ has no maximum. In order to show this, we need to show that any $x \in (1, 3)$ is not an upper bound of the set.

We claim that 3 is the supremum of $(1, 3)$. For any $x \in (1, 3)$, $x < 3$, and therefore 3 is an upper bound of $(1, 3)$. Now we need to show that it is the *least* upper bound. That is, we need to show that if a is an upper bound of $(1, 3)$, then $3 \leq a$. We will instead prove the converse of this statement: if $a < 3$, then a is not an upper bound of $(1, 3)$.

(c) It is important to realise that this is a subset of integers, not reals. Then we can simplify the set before we determine its maximum or supremum. The inequality $|2(m - 4)| \leq 15$ holds if and only if

$$-15 \leq 2(m - 4) \leq 15.$$

Dividing by 2, we then have

$$-\frac{15}{2} \leq m - 4 \leq \frac{15}{2}$$

and then adding 4,

$$-\frac{7}{2} \leq m \leq \frac{23}{2}.$$

Since each of these steps can be reversed, we see that this inequality is equivalent to the original one. Hence

$$\{m \in \mathbb{Z} : |2(m - 4)| \leq 15\} = \left\{m \in \mathbb{Z} : -\frac{7}{2} \leq m \leq \frac{23}{2}\right\}.$$

In this form, it's easier to see that the maximum and supremum of the set is 11: indeed, any integer that is at most $\frac{23}{2} = 11 + \frac{1}{2}$ must be at most 11, so 11 is an upper bound for the set. Since 11 is an element of the set, it is the maximum. Since any element less than 11 cannot be an upper bound for the set, we see that 11 is also the supremum.

(d) Let

$$S = \left\{2 - \frac{1}{n} : n \in \mathbb{N}\right\}.$$

We claim that the set has no maximum. We need to show that for any $x \in S$, there is some other element of S that is greater than x; then x will not be an upper bound for S, and hence not its maximum.

We claim that the supremum of S is 2. First, we'll need to show that 2 is an upper bound for S. Then, we'll need to show that for any $a < 2$, a is

not an upper bound of S. In order to do that, we need to find some $n \in \mathbb{N}$ so that $a < 2 - 1/n$. Rearranging this, we have $1/n < 2 - a$. Since $a < 2$, we have $2 - a > 0$, and so the inequality $1/n < 2 - a$ may be rearranged to $1/(2 - a) < n$. So the choice of $n = \lceil 1/(2 - a) \rceil + 1$ will work.

(e) Recall that $\cos(\theta) = 1$ holds if and only if $\theta = 2\pi k$ for some $k \in \mathbb{Z}$. Hence, if $\cos(2x) = 1$, we must have $x = \pi k$ for some $k \in \mathbb{Z}$. So

$$\{x \in \mathbb{R} : \cos(2x) = 1\} = \{\pi k : k \in \mathbb{Z}\}.$$

We claim that this set has no upper bound; this means that the set cannot have a maximum or a supremum, since both the maximum and supremum would be upper bounds of the set.

8.6.16.

(a) We need to show that $a = \max\{s, t\}$ satisfies the following two properties:

- a is an upper bound for $S \cup T$, so $x \leq a$ for all $x \in S \cup T$, and
- $a \leq b$ for any upper bound b of $S \cup T$.

(b) Let's look at some examples for S and T to see if we can find any patterns.

- Say $S = (0, 2)$ and $T = (0, 1)$, so that $s = 2$ and $t = 1$. Then $S \cap T = T$, and so $\sup(S \cap T) = 1 = t$.
- Suppose $S = (0, 2)$ and $T = (1, 3)$, so that $s = 2$ and $t = 3$. Then $S \cap T = (1, 2)$, and so $\sup(S \cap T) = 2 = s$.

From these examples, we may believe that $\sup(S \cap T) = \min(S, T)$. However, we haven't yet looked at an example where S and T are disjoint.

- Say $S = (0, 1)$ and $T = (1, 3)$, in which case $s = 1$ and $t = 3$. Then $S \cap T = \emptyset$. What's the least upper bound of the empty set? We claim that any real number is an upper bound for the empty set. Indeed, suppose $a \in \mathbb{R}$. The statement, if $s \in \emptyset$ then $s \leq a$, is true, since the hypothesis is always false! Since every real number of $\emptyset$ is an upper bound, it can't have a least upper bound.

(c) We need to show that $a = s + t$ satisfies the following two properties:

- a is an upper bound for $S + T$.
- If b is an upper bound of $S + T$, then $a \leq b$.

The second part is trickier to prove, and we'll try to prove the contrapositive instead:

If $b < a$, then b is not an upper bound of $S + T$.

This means that we need to find some $x \in S+T$ such that $x > b$. We want to write b as a sum of two elements, say u and v, that are less than s and t respectively. If we have such an u and v, then they will not be upper bounds for S and T, respectively. Then, by definition, there will be $c \in S$ and $d \in T$ so that $c > u$ and $d > v$. But then $c+d > u+v = b$, so b would not be an upper bound for $S+T$.

How can we find these elements u and v? There are many choices, but we'll set

$$u = s - \varepsilon \qquad \text{and} \qquad v = t - \varepsilon$$

where $\varepsilon = \frac{a-b}{2}$. Then

$$u + v = s + t - 2\varepsilon = a - (b - a) = b.$$

8.6.17. By definition, we need to show that given any $\varepsilon > 0$, there is some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \varepsilon$. Unravelling the inequality, we need to show

$$-\varepsilon < a_n - a < \varepsilon \iff a - \varepsilon < a_n < a + \varepsilon.$$

Since a is the supremum of $\{a_n : n \in \mathbb{N}\}$, we know that $a_n \leq a$ for all $n \in \mathbb{N}$. So the inequality $a_n < a + \varepsilon$ is satisfied for all $n \in \mathbb{N}$. Therefore, we can focus on satisfying the inequality

$$a - \varepsilon < a_n.$$

To establish this inequality for sufficiently large n, we need to use the fact that a is the *least* upper bound of $\{a_n : n \in \mathbb{N}\}$. Since $\varepsilon > 0$, we know that $a - \varepsilon < a$. Since $a - \varepsilon < a$ and a is the least upper bound, we know that $a - \varepsilon$ is not an upper bound of the set $\{a_n : n \in \mathbb{N}\}$. Therefore, there is some $N \in \mathbb{N}$ such that $a - \varepsilon < a_N$. Finally, we just need to use the fact that a_n is increasing to show that $a - \varepsilon < a_n$ for all $n \geq N$.

9 · Relations
9.7 · Exercises

9.7.4.

(a) For the first relation, we see that the definition of the relation is symmetric with respect to f and g. This suggests that the relation is symmetric as well. Also, we can see that it is reflexive, since a function is equal to itself at every point. However we see that it may not be transitive since for two functions to be related, they have to be same at a point, and if we have $f \mathcal{R} g$ and $g \mathcal{R} h$, we may have two different points $x \neq y \in \mathbb{R}$ such that $f(x) = g(x)$ and $g(y) = h(y)$, but this does not mean that f and h are equal anywhere. But of course, we will need to find a counterexample to show that the relation is not transitive when we are writing the proof. Simply saying that we can find f, g, h and $x, y \in \mathbb{R}$ satisfying the condition above is not enough.

(b) For the second relation, we see that it is symmetric since the definition of

the relation is symmetric with respect to x and y. But, we can also see that is is not reflexive since $(1,1) \notin \mathcal{S}$ and also not transitive since every integer is related to 0, that is, we can take $(1,0) \in \mathcal{S}$, $(0,1) \in \mathcal{S}$, but we see that $(1,1) \notin \mathcal{S}$ as mentioned.

9.7.7. The hint suggests that this relation says for the nonempty set E, $q \in E$, and $A, B \in \mathcal{P}(E)$, $A \mathcal{R} B$ if q is in both of them or in neither of them. This observation immediately suggests that the relation is reflexive and symmetric. Moreover, we see that if $A \mathcal{R} B$ and $B \mathcal{R} C$, then we see that q is in all three sets A, B, and C; or in none of them. In either case, we see that $A \mathcal{R} C$. Thus, the relation should be transitive too.

9.7.13. For the first part, we first realize the relation $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : n \mid (a-b)\}$ is nothing but the congruence modulo n relation. Moreover, we see that if R is not sparse than there are $a, b \in \mathbb{Z}$ such that $a \equiv b \pmod{n}$ and either $a \equiv (b+1) \pmod{n}$ or $(a+1) \equiv b \pmod{n}$. Since these cases are symetric with respect to a and b, WLOG, we can assume that $a \equiv (b+1) \pmod{n}$. Then, we see that $n \mid (b-a)$ and $n \mid ((b+1)-a)$, that is $n \mid ((b-a)+1)$. Therefore, as the hint suggested, we get $n = 1$ (since $n \in \mathbb{N}$). We can also see that if $n = 1$, then the congruence relation cannot be sparse, since every element will be related to every nonzero element.

This final observation is also going to be useful in the second part, since we know that the relation $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : 1 \mid (a-b)\} = \mathbb{Z} \times \mathbb{Z}$ is an equivalence relation, and it is not sparse.

Now, we can write this nicely as a proof.

9.7.17.

(a) We want to show that $nk \equiv 1 \pmod{p}$. Then by definition, we want to show that $p \mid nk - 1$, that is, there's some $\ell \in \mathbb{Z}$ such that $p\ell = nk - 1$. Let's rewrite this equation as $1 = kn + (-\ell)p$. This equation may look familiar from Bézout's identity, which tells us that if $a, b \in \mathbb{Z}$, with at least one of them non-zero, then there are some $x, y \in \mathbb{Z}$ so that $ax + by = \gcd(a,b)$. In order to show that there is some $\ell \in \mathbb{Z}$ so that $1 = kn + (-\ell)p$, it suffices, by Bézout's identity, to show that $\gcd(n,p) = 1$.

To show that $\gcd(n,p) = 1$, we'll use the hypothesis that $n \not\equiv 0 \pmod{p}$. By definition, this means that $p \nmid (n-0)$, so $p \nmid n$. Since p is prime, its only divisors are 1 and p. Since p isn't a divisor of n, we must have $\gcd(n,p) = 1$.

Now we can write the proof.

(b) Let's try to find such an n when p is not prime; we'll use $p = 4$ since it is a small composite number. We need to choose n so that $\gcd(n,p) \neq 1$; if n is such that $\gcd(n,p) = 1$, then we could use Bézout's identity as in part (a) to find some k with $nk \equiv 1 \pmod{p}$. So let's try $n = 2$. We need to check that $nk \not\equiv 1 \pmod{4}$ for each $k \in \{0,1,2,3\}$, since these represent all equivalence classes mod 4. We have

$$2{\cdot}0 \equiv 0 \pmod{4}, \quad 2{\cdot}1 \equiv 2 \pmod{4}, \quad 2{\cdot}2 \equiv 0 \pmod{4}, \quad 2{\cdot}3 \equiv 2 \pmod{4},$$

so there is no $k \in \mathbb{Z}$ with $2k \equiv 1 \pmod 4$.

We chose $n = 2$ because then nk is always an even number, however in order to have $nk \equiv 1 \pmod 4$, we need nk to be odd. Since a number cannot be even and odd at the same time, that cannot happen.

9.7.18. Lemma 9.5.9 proves the statement in the special case that $\gcd(a, d) = 1$. We'll try to modify the proof of that lemma. Bézout's identity tells us that there are $x, y \in \mathbb{Z}$ so that

$$\gcd(a, d) = xa + yd.$$

We can rewrite this as

$$1 = x \cdot \frac{a}{\gcd(a, d)} + y \cdot \frac{d}{\gcd(a, d)}$$

since $\gcd(a, d)$ divides both a and d. Now multiplying the entire equation by b, we have

$$b = x \cdot \frac{ab}{\gcd(a, d)} + yb \cdot \frac{d}{\gcd(a, d)}.$$

The last step is to use that $d \mid ab$ in order to show that this equation implies that $d/\gcd(a, d)$ divides b.

9.7.19.

(a) Bézout's identity tells us that there are $x, y \in \mathbb{Z}$ so that $\gcd(a, b) = ax + by$. This equation implies that any divisor of both a and b must divide $\gcd(a, b)$.

(b) We will try to show that $m \gcd(a, b) \leq \gcd(ma, mb)$ and $\gcd(ma, mb) \leq m \gcd(a, b)$.

We claim that $m \gcd(a, b) \mid \gcd(ma, mb)$. We know that $d = \gcd(a, b)$ divides both a and b; from this, we will be able to show that md divides both ma and mb.Then $md \leq \gcd(a, b)$.

Now let's consider the converse, $\gcd(ma, mb) \leq m \gcd(a, b)$. Now m divides both ma and mb, so by part (a), $m \mid \gcd(ma, mb)$. Then there is some $e \in \mathbb{Z}$ so that $me = \gcd(ma, mb)$. To prove that $\gcd(ma, mb) \leq m \gcd(a, b)$, we can show that $e \leq \gcd(a, b)$. Let's try to prove that e divides both a and b. By definition of $\gcd(ma, mb)$ there are $u, v \in \mathbb{Z}$ so that $meu = \gcd(ma, mb)u = ma$ and $mev = \gcd(ma, mb)v = ma$; then we can cancel the factors of m to get that $e \mid a$ and $e \mid b$.

(c) Take $a = b = c = 2$. Then $gcd(a, b) = \gcd(c, b) = \gcd(ac, b) = 2$, and so $\gcd(ac, b) \neq \gcd(a, b) \cdot \gcd(c, b)$.

10 · Functions
10.8 · Exercises

10.8.2. Let us start by rearranging things a little. We must have $ax + by = 6$

which means that

$$y = \frac{6 - ax}{b}.$$

So for any given x there is only a single value of y.

But we also require that $y \in \mathbb{Z}$, which means that we must have

$$b \mid (6 - ax)$$

for every $x \in \mathbb{Z}$. Setting $x = 0$ immediately tells us that $b \mid 6$, and so $b \in \{1, 2, 3, 6\}$.

- When $b = 1$, then $y = 6 - ax \in \mathbb{Z}$ for any $a \in \mathbb{N}$.
- When $b = 2$, then $y = \frac{6-ax}{2} = 3 - \frac{ax}{2}$ will be an integer for all x provided $2 \mid a$.
- When $b = 3$, then $y = \frac{6-ax}{3} = 2 - \frac{ax}{3}$ will be an integer for all x provided $3 \mid a$.
- And finally, when $b = 3$, then $y = \frac{6-ax}{6} = 1 - \frac{ax}{6}$ will be an integer for all x provided $6 \mid a$.

That is, ϕ is a function provided $b \in \{1, 2, 3, 6\}$ (that is $b \mid 6$) and $b \mid a$.

Now, strictly speaking we have shown that if ϕ is a function, then we have $b \mid 6, b \mid a$. We can also readily show that when $b \mid 6, b \mid a$ then ϕ is a function.

So now assume that $b \mid 6, b \mid a$, and sot $6 = kb, a = \ell b$ for integers k, ℓ. Hence

$$y = \frac{6 - ax}{b} = \frac{kb - \ell bx}{b} = k - \ell x$$

and thus if $x \in \mathbb{Z}$ then $y \in \mathbb{Z}$, and so we have a function.

10.8.3. As part of our proof we need to prove that $f(x) \in [-1, 1]$. To do this let us start with the inequality we wish to prove, namely

$$-1 \leq \frac{2x}{1 + x^2} \leq 1$$

Now since $x \in \mathbb{R}$ we know that $x^2 \geq 0$ and so $1 + x^2 \geq 1$. Hence, we have

$$-1 - x^2 \leq 2x \leq 1 + x^2$$

Subtracting $2x$ from everything gives

$$-1 - 2x - x^2 \leq 0 \leq 1 - 2x + x^2$$
$$-(1 + x)^2 \leq 0 \leq (1 - x)^2$$

which we know to be true. In our proof we will use this chain of reasoning, but in reverse; starting from what we know is true and going back to the inequality we need.

We will also need to show that given $y \in [-1, 1]$ that $y \in f(\mathbb{R})$. That is, there is some $x \in \mathbb{R}$ so that $f(x) = y$. Our proof doesn't have to explain how we found that x-value, just that it works. Let us find it now:

$$\begin{aligned} y = f(x) &= \frac{2x}{1+x^2} \\ yx^2 - 2x + y &= 0 \\ x &= \frac{2 \pm \sqrt{4-4y^2}}{2y} = \frac{2 \pm 2\sqrt{1-y^2}}{2y} \end{aligned}$$

This looks like it should work since $-1 \leq y \leq 1$ and so $0 \leq y^2 \leq 1$, but we might have a problem when $y = 0$. However, when $y = 0$ we can just take $x = 0$.

10.8.6. We see that for any values of a and b this function $f(x) = x^2 + ax + b$ is a quadratic function. This means that we wouldn't expect the function to be injective or surjective. Moreover, we see that we can complete the square and get $f(x) = \left(x + \frac{a}{2}\right)^2 + \left(b - \frac{a^2}{4}\right)$. This tells us the minimum of the function (which we can use to show that the function is not surjective) and also the vertex of the graph of the function (which we can use to show that the function is not injective).

10.8.7. We see that the condition in the question tells us that $f(n) \leq n$ for all $n \in \mathbb{N}$ and that f is an injective function. If we want to understand the function, we see that for large $n \in \mathbb{N}$, say $n = 1000000$, we have $f(1000000) \leq 1000000$. This tells us that $f(1000000)$ has 1000000 different options. If we also want to consider that f is an injective function, it may not be clear as to which of those options $f(1000000)$ can take.

Instead, we can try to understand what happens to f when $n \in \mathbb{N}$ is small. This will give us fewer options for $f(n)$ and maybe we can even figure out some values of f.

Now, if we check $n = 1$, we see $f(1) \leq 1$. Since $f : \mathbb{N} \to \mathbb{N}$, this tells us that $f(1) = 1$. Similarly, if we check $n = 2$, we see that $f(2) \leq 2$. This means that $f(2) = 1$ or $f(2) = 2$. But, since f is injective, we see that $f(2) \neq 1$. Thus $f(2) = 2$.

This suggests that if we keep going like this, we will end up with $f(n) = n$. But, we also realize that if we want to "keep going like this", we need to apply induction. Moreover, since we want to know something about the values of the function at any point to be able to determine the next one since we want to make sure that the function is injective, we may need strong induction.

10.8.8. To prove that the function is injective we start by assuming $f(a) = f(b)$ and then prove that this implies that $a = b$. This requires a little juggling and factoring of polynomials. To prove that it is surjective we take some $y \in [5, \infty)$ and then show that there is an $x \in [3, \infty)$ so that $y = f(x)$. Now, the proof doesn't have to show how we came up with that x, only that it works. We find it

by solving $y = f(x)$:

$$\begin{aligned} x^2 - 6x + 16 &= y && \text{complete the square} \\ (x-3)^2 &= y - 5 && \text{careful with square-roots} \\ (x-3) &= \pm\sqrt{y-5} \\ x &= 3 \pm \sqrt{y-5} \end{aligned} \qquad (x^2 - 6x + 9) + 5 = y$$

Now since $y \geq 5$, the square root is defined. And since we want $x \geq 3$, we take the positive branch. That is, given a y-value in the codomain, set $x = 3 + \sqrt{y-5}$.

10.8.15. When we look at the relation defined in the question, we can easily see that it is reflexive (since equality is reflexive), and symmetric (since equality is symmetric). Moreover, if $(x, y)\ \mathcal{R}\ (s, t)$ and $(s, t)\ \mathcal{R}\ (a, b)$, we see that $x^2 + y^2 = s^2 + t^2 = a^2 + b^2$. Therefore it also is a transitive relation. As we can see, trying to show that this relation is an equivalence relation is not too hard to show.

Now, let's try to understand the equivalence classes of this relation. Let $(s, t) \in \mathbb{R}^2$. then we see

$$\begin{aligned} [(s,t)] &= \left\{(x,y) \in \mathbb{R}^2 \colon (s,t)\ \mathcal{R}\ (x,y)\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \colon x^2 + y^2 = s^2 + t^2\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \colon \sqrt{x^2+y^2} = \sqrt{s^2+t^2}\right\}. \end{aligned}$$

This means that $[(s, t)]$ is the set of all points on the plane that have the same distance from the origin, that is, the points on the circle, centred at origin of radius $\sqrt{s^2 + t^2}$.

This is an important observation in understanding whether f is a function. We know that for f to be a function, we need to show that

- f is defined on all its domain — this is easy since for any x, y we know that $x^2 + y^2 \geq 0$ and so $\sqrt{x^2 + y^2} \in \mathbb{R}$ as required.
- if we input different representatives of the same equivalence class, we get the same result, namely

  $$\Big([(x,y)] = [(s,t)]\Big) \implies f\big([(x,y)]\big) = f\big([(s,t)]\big).$$

  We also know that if $[(x, y)] = [(s, t)]$, then the points (x, y) and (s, t) are on the same circle centred at the origin. Then, since f takes an equivalence class (which is a circle) and sends it to the radius of the circle, we see that it shouldn't matter which points on the circle we pick as the representative of the equivalence class, we get the same output for f.

This means that f defines a function.

Once we see that f is a function we can also easily see that it is surjective since for all $z \in [0, \infty)$, we can take $[(x, y)] = [(0, z)]$, and see that

$$f\big([(x, y)]\big) = f\big([(0, z)]\big) = \sqrt{z^2} = z.$$

To show the injectivity we remember, again, that f takes the circles centred at the origin and sends them to their radius. This suggests that if

$$f\big([(x, y)]\big) = f\big([(s, t)]\big),$$

then the radius of the circle defined by $[(x, y)]$ and $[(s, t)]$ must be the same. Since these circles are centred at the origin, we see that they must be the same, that is,

$$[(x, y)] = [(s, t)],$$

which means that f is injective.

10.8.16.

(a) Let $[z]_n \in \mathbb{Z}_n$. We need to show that $f([z]_n) \in \{0, 1, \ldots, n-1\}$. We know that $z \in [z]_n$, and then by Euclidean division, we know that there exist $q, r \in \mathbb{Z}$ so that $z = qn + r$ and $r \in \{0, 1, \ldots, n-1\}$. Hence $f([z]_n) = r \in \{0, 1, \ldots, n-1\}$ as required.

Now suppose that $[x]_n = [y]_n$. We need to show that $f([x]_n) = f([y]_n)$. Untangling the definitions, we know that n divides $x-y$, and we need to show that x and y have the same remainder upon division by n. By Euclidean division, we can write $x = q_x n + r_x$ and $y = q_y n + r_y$ for some integers q_x, q_y, r_x, r_y with $0 \leq r_x, r_y \leq n-1$. Then $x - y = (q_x - q_y)n + (r_x - r_y)$. Since n divides $x - y$ and $(q_x - q_y)n$, we can show from this equation that n divides $r_x - r_y$. Now we can use the restrictions on the size of the remainders, namely that $0 \leq r_x, r_y \leq n-1$, combined with the fact that $n \mid r_x - r_y$ to show that $r_x - r_y = 0$.

(b) In order to show that f is a bijection, we need to show that f is both injective and surjective.

For injectivity, we assume that $f([x]_n) = f([y]_n)$ for some $[x]_n, [y]_n \in \mathbb{Z}_n$, and we need to show that $[x]_n = [y]_n$. Note that $f([x]_n) = f([y]_n)$ implies that x and y have the same remainder when divided by n. From this observation, we can show that n divides $x - y$, and consequently, x and y are congruent modulo n.

For surjectivity, we need to show that for any $r \in \{0, 1, \ldots, n-1\}$, the codomain of the function, there is some $[x]_n \in \mathbb{Z}_n$ so that $f([x]_n) = r$. This is the same as saying that there is some $x \in \mathbb{Z}$ so that its remainder upon division by n is r.

10.8.20. We do some rough computation here to try to figure out which statements are true or false. When we write up the proof, we will include more

detail in these computations.

(a) Fix $x \in \mathbb{R}$. We compute, starting with the left hand side. The computation below uses the definition of function composition, followed by the definition of function addition.

$$\begin{aligned}(f \circ (g+h))(x) &= f((g+h)(x)) \\ &= f(g(x)+h(x))\end{aligned}$$

Now, for the right hand side, we compute using the definitions of function addition and composition

$$\begin{aligned}(f \circ g + f \circ h)(x) &= (f \circ g)(x) + (f \circ h)(x) \\ &= f(g(x)) + f(h(x))\end{aligned}$$

Most functions f do not have the property that

$$f(g(x)+h(x)) = f(g(x)) + f(h(x)).$$

This is a property of linear functions. Therefore, we can create a counterexample using a nonlinear function such as $f(x) = x^2$.

(b) Fix $x \in \mathbb{R}$. We compute, starting from the left hand side. By the definition of function composition,

$$((g+h) \circ f)(x) = (g+h)(f(x))$$

By the definition of function addition

$$= g(f(x)) + h(f(x))$$

By the definition of composition

$$= (g \circ f)(x) + (h \circ f)(x)$$

By the definition of function addition

$$= (g \circ f + h \circ f)(x)$$

This computation did not rely on our choice of x, so it holds for all $x \in \mathbb{R}$. Thus, we can show

$$(g+h) \circ f = g \circ f + h \circ f.$$

(c) We first use the definition of function composition to see that

$$\left(\frac{1}{f} \circ g\right)(x) = \left(\frac{1}{f}\right)(g(x)).$$

By the definition of function division, we have

$$= \frac{1}{f(g(x))}$$

By the definition of function composition, we have

$$= \frac{1}{(f \circ g)(x)}$$

Using the definition of function division yields

$$= \left(\frac{1}{f \circ g}\right)(x)$$

This computation did not rely on our choice of x, so it holds for all $x \in \mathbb{R}$. Thus, we can show

$$\left(\frac{1}{f}\right) \circ g = \frac{1}{f \circ g}.$$

(d) Fix $x \in \mathbb{R}$. We start with the left hand side, and use the definition of function division to compute,

$$\left(\frac{1}{f \circ g}\right)(x) = \frac{1}{(f \circ g)(x)}$$

By the definition of function composition

$$= \frac{1}{f(g(x))}$$

On the other hand, we start from the right hand side. Using the definition of function composition to compute

$$\left(f \circ \frac{1}{g}\right)(x) = f\left(\left(\frac{1}{g}\right)(x)\right)$$

By the definition of function division

$$= f\left(\frac{1}{g(x)}\right)$$

In order for

$$\frac{1}{f(g(x))} = f\left(\frac{1}{g(x)}\right),$$

f must have the property that

$$\frac{1}{f(x)} = f\left(\frac{1}{x}\right).$$

This is not true for every function $f : \mathbb{R} \to \mathbb{R}$. For example, if $f(x) = x + 2$, we have

$$\frac{1}{f(x)} = \frac{1}{x+2} \neq \frac{1}{x} + 2 = f\left(\frac{1}{x}\right).$$

10.8.21.

(a) We will try to construct a function f and sets X, W such that $f(W \cap X)$ has few elements, but $f(W)$ and $f(X)$ have many elements in common. One way of accomplishing this is to make $W \cap X$ small (for example, having only one element in common), but designing the function f to produce the same output for distinct elements of X and W.

As a counterexample, take $f : \{-1, 0, 1\} \to \{0, 1\}$ defined as $f(x) = |x|$ and consider the subsets $W = \{-1, 0\}$ and $X\,\{0, 1\}$. Then we see $f(W \cap X) = f(\{0\}) = \{0\}$ and $f(W) = f(X) = \{0, 1\}$. Hence $f(W \cap X) = \{0\} \neq \{0, 1\} = f(W) \cap f(X)$.

(b) Recall from Theorem 10.3.6(ii), $f(f^{-1}(Y)) \subseteq Y$ for any $Y \subseteq B$. Therefore we need to find an example where Y is not a subset of $f(f^{-1}(Y))$. This means that we need to find a function that is not surjective, i.e. not every element of Y is in the range of the function.

As a counterexample, we take $f : \{1, 2\} \to \{1, 2, 3\}$, defined as $f(x) = x$ and let $Y = \{2, 3\}$. Notice that there is no element in the domain that maps to 3, so f fails to be surjective.

10.8.25. We have to show the function is both injective and surjective.

- Injective: Assume that $f(x) = f(z)$ and show that $x = z$; this requires some simple manipulations of polynomials.

- Surjective: Given some y in the codomain we have to find an x in the domain so that $f(x) = y$. Our proof does not have to contain how we found that x-value, just that it is in the domain and that it satisfies $f(x) = y$. To find the x we solve

$$\begin{aligned} y &= \frac{x+1}{x+2} \\ (x+2)y &= x+1 \\ xy - x &= 1 - 2y \\ x &= \frac{1-2y}{y-1} = -2 - \frac{1}{y-1} \end{aligned}$$

where we have done a little partial fractions at the last step.

Notice that since $y \neq 1$ we know that $x \in \mathbb{R}$, and since $\frac{1}{y-1} \neq 0$, we know that $x \neq -2$.

This gives us the result we wanted. Now, we can write the proof.

10.8.26. In this question, we can try to show that f is an injective and surjective function, and conclude that f has an inverse, and calculate the inverse afterwards.

However, if we know what the inverse of f should look like, call it a new function g, we can check whether g is a right and also a left inverse of f. If so,

using Lemma 10.6.3 and Lemma 10.6.4, we can conlude that f is bijective and moreover g is its inverse.

We can find what the inverse of f should look like quite easily in this question. We see that f takes even numbers to odd numbers and the odd numbers to even numbers. For example, $f(5) = 5 + 7 = 12$. This means that if we want to find, g, the the inverse of f we should have $g(12) = 5 = 12 - 7$ (since we added 7 to 5 to get 12), that is $g(n) = n - 7$ for n even. Similarly for n even, we see that $f(n) = -n + 3$, so we must have $g(-n + 3) = n = -((-n + 3) - 3)$, that is, $g(n) = -n + 3$ for n odd. Now, we can check whether this function g is the right- and the left-inverse of f.

11 · Proof by contradiction

11.3 · Exercises

11.3.1. Proof by contradiction can be a useful tool for proving statements where we want to show the *non-existence* of an object satisfying certain conditions. This is because when we assume for a contradiction that such an object exists, then we would have some structure we can work with and, hopefully, get a contradiction with our prior assumptions or knowledge.

11.3.2. We'll try to prove the statement by contradiction, by assuming that both a and b are odd. Using the definition of a and b being odd, we can prove that $a^2 + b^2 = 4n + 2$ for some $n \in \mathbb{Z}$. Since $c^2 = a^2 + b^2$, this implies that c^2 is even, and so c is even. (The contrapositive of this statement is given in Result 3.2.10) But then 4 will divide c^2. This contradicts the equation $c^2 = a^2 + b^2 = 4n + 2$, which implies that c^2 is not divisible by 4.

11.3.3. In order to prove the statement by contradiction, we need to assume that there is some $k \in \mathbb{Z}$ so that $ak \equiv 1 \pmod{n}$, and then show that this leads to a contradiction, implying that the statement must be true. From the equation $ak \equiv 1 \pmod{n}$, we know that $n \mid (ak - 1)$. From here we can show that $\gcd(a, n) \mid 1$. This contradicts that $\gcd(a, n) > 1$.

11.3.5. Since we are trying to show that there are no integers satisfying , it may be useful to use proof by contradiction.

So, assume $\exists x, y \in \mathbb{Z}$ such that $5y^2 - 4x^2 = 7$. This may not look too helpful since we have 2 unknowns and one equation, and it is not obvious as to how x and y would interact with each other to produce 7. One thing we can do in this question is to try to find a way to reduce the number of the variables. This may look a little unconventional, but if we look at the equation modulo 4, we see that it becomes

$$\begin{aligned} 5y^2 - 4x^2 &\equiv 7 \pmod{4} \\ 1y^2 - 0x^2 &\equiv 3 \pmod{4} \\ y^2 &\equiv 3 \pmod{4} \end{aligned}$$

which eliminates x completely!

Then, if we can show that there is no integer y satisfying $y^2 \equiv 3 \pmod{4}$, we would get our contradiction. Moreover, we see that if y is even, then $y^2 \equiv 0$

(mod 4) and if y is odd, then $y^2 \equiv 1 \pmod 4$. Therefore, we have our contradiction.

Notice that we could also consider the four classes of $y \pmod 4$, but it will give the same result, just with more work.

We can also eliminate y instead of x by considering the equation modulo 5. This gives

$$\begin{aligned} 5y^2 - 4x^2 &\equiv 7 \pmod 5 \\ 0y^2 + 1x^2 &\equiv 2 \pmod 5 \end{aligned}$$

So we need to find $x^2 \equiv 2 \pmod 5$. Writing the square modulo 5 gives the following table:

x	0	1	2	3	4
x^2	0	1	4	4	1

So there is no x so that $x^2 \equiv 2 \pmod 5$.

11.3.6.

(a) We'll prove the statement by contradiction, and assume that there is some smallest positive rational number, say r. We'll try to find a positive rational number that's strictly less than r. Let's consider $r/2$, which we know satisfies the inequality $0 < r/2 < r$. This inequality isn't enough though; we also need to show that $r/2$ is a rational number, and then we'll have reached a contradiction.

(b) The proof is very similar to part (a). We'll assume that we have a smallest positive irrational number, say r. We'll try to show that $r/2$ is irrational. We can prove this statement by using contradiction as well: assume that $r/2$ is rational, and show this contradicts the assumption that r is irrational.

11.3.8. This is a good proof-by-contradiction question. We'll assume that $\sqrt[3]{25}$ is rational, write it as a simple fraction, and then show that we get a contradiction.

11.3.10. Let's assume the conclusion is false, that is, that rx is rational. Then we can write $rx = p/q$ and $r = m/n$ for integers p, q, m, n. Using this, we can show that x is rational, which is a contradiction. However, when we write the proof up, we need to be careful to always show that our denominators are non-zero

11.3.11.

(a) We have to be careful when dealing with the not equal signs, because the property of being not equal is not transitive. It can be very easy to reach faulty conclusions. For example it is certainly *not* the case that $2 \neq -2$ implies that $(2)^2 \neq (-2)^2$.

In the same way, the statement that $x \neq m/n$ for all $m, n \in \mathbb{Z}$ and $y \neq p/q$ for all $p, q \in \mathbb{Z}$ does not imply that $xy \neq (mp)/(nq)$ for all $m, p, n, q \in \mathbb{Z}$. This statement isn't true because we can factor an integer into irrational

components; for example, we can write $2 = \sqrt{2} \cdot \sqrt{2}$.

11.3.13.

(a) We know that the product of two odd numbers is odd, from Exercise 3.5.2. From this, we know that if 5^k is odd, then 5^{k+1} will be too. So we can prove the statement for all k by using induction.

(b) We'll prove the statement by contradiction, by assuming that the statement is false, that is, $\log_2(5)$ is rational. Since $\log_2(5) = m/n$ we know that $5 = 2^{m/n}$ and so $5^n = 2^m$. But this means that 5^n is even which gives a contradiction. We need to be a little careful with the signs of m, n.

(c) Let $n \in \mathbb{N}$. As given in the question, there is some $a \in \mathbb{Z}$, $a \geq 0$ and $b \in \mathbb{Z}$ that is odd, so that $n = 2^a b$. We know that

$$\log_2(n) = \log_2(2^a b) = \log_2(2^a) + \log_2(b) = a + \log_2(b).$$

Since a is an integer, and so rational, the rationality or irrationality of $\log_2(n)$ will depend on that of $\log_2(b)$. Indeed, we know that

- the sum of two rational numbers is also rational (see, for example, Exercise 7.3.9);
- the sum of a rational and an irrational number is irrational (by Result 11.2.4).

This implies, respectively, that

- $\log_2(n)$ is rational if $\log_2(b)$ is rational;
- $\log_2(n)$ is irrational if $\log_2(b)$ is irrational.

We can try to adapt the argument from part (b) to show that $\log_2(b)$ is irrational, since (b) is a special case of this, with $b = 5$. However, in part (b), we used the fact that $\log_2(5) > 0$, which is only true for $\log_2(b)$ if $b \neq 1$. We can try to prove that $\log_2(b)$ is irrational when $b \neq 1$ and b is odd.

The last case to analyze is if $b = 1$. Then $\log_2(b) = 0$, and hence it is rational. To summarize, we have shown that

- if $b = 1$, then $\log_2(b)$ is rational, and so $\log_2(n)$ is rational;
- if $b \neq 1$, then $\log_2(b)$ is irrational, and so $\log_2(n)$ is irrational.

These two statements imply that for $n \in \mathbb{N}$, $\log_2(n)$ is irrational if and only if n is not a power of 2.

11.3.15. We'll prove this by contradiction by first assuming that there are some $a, n \in \mathbb{N}$ satisfying the equation. Writing $a^2 = 7^n - 35$, we can show that a^2 is divisible by 7. By Euclid's lemma, this implies that $7 \mid a$. Then $7^2 \mid a^2$. Notice that $7 \mid 35$, but $7^2 \nmid 35$. If we also had $7^2 \mid 7^n$, then that would imply 7^2

divides $35 = 7^n - a^2$, which is a contradiction. So we have a contradiction in the case $n \geq 2$. Now we just need to show that the case $n = 1$ also leads to a contradiction. In this case, we have the equation $a^2 + 35 = 7$. This is imposible, since $a^2 + 35 \geq 35$.

11.3.16. Let's explore the inequality a little bit before we leap into proving it. Notice that because $x \in (0, 1)$ we know that the denominator is positive. Therefore we can rearrange the inequality to give

$$1 \geq 4x - 4x^2,$$

which we can rearrange further to give us

$$(2x - 1)^2 = 4x^2 - 4x - 1 \geq 0.$$

This looks good because it tells us that a square is non-negative and we know that squares are non-negative. That smells like the direct proof, what about contradiction?

We need to have x so that the opposite inequality holds. That is

$$\frac{1}{2x(1-x)} < 2,$$

but this gives us (by similar reasoning) that

$$1 < 4x - 4x^2$$

and so

$$(2x - 1)^2 = 4x^2 - 4x + 1 < 0.$$

This smells like a contradiction since it tells us a square is negative.

11.3.17.

(a) Let's try to rearrange the inequality in the statement to achieve something more obvious.

$$\frac{x}{y} + \frac{y}{x} > 2.$$

We give the left hand side a common denominator

$$\frac{x^2 + y^2}{xy} > 2.$$

Now we multiply both sides by xy. This won't change the sign of our inequality because $x, y > 0$, so $xy > 0$

$$x^2 + y^2 > 2xy.$$

We now subtract $2xy$ from both sides of the equation and see if the resulting quadratic equation has any nice properties

$$x^2 - 2xy + y^2 > 0.$$

We factor the left hand side

$$(x-y)^2 > 0.$$

Since we assumed that $x \neq y$, we know that $x - y \neq 0$, and the square of any non-zero real must be positive. Therefore we've reached an obvious fact! To write the direct proof, we start with the obvious fact and work backwards through the steps above until we reach the desired result.

(b)

(c) We check what happens when $x = y$. When $x = y$, $\frac{x}{y} = \frac{y}{x} = 1$. Thus,

$$\frac{x}{y} + \frac{y}{x} = 2.$$

We'll alter the statement to read:

For all $x, y \in \mathbb{R}$ with $x, y > 0$

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

11.3.18. Consider the expression needed for a proof by contradiction. Let $a, b > 0$. What would it mean for us to have

$$\frac{2}{a} + \frac{2}{b} = \frac{4}{a+b}?$$

Multiply both sides by $ab(a+b)$ to clear the denominators

$$2a^2 + 2b^2 + 4ab = 4ab,$$

and thus

$$a^2 + b^2 = 0.$$

But since $a, b > 0$, $a^2 + b^2 > 0$ and we've obtained a contradiction!

We can leverage this reasoning into a direct proof too. We leave that to the reader.

12 · Cardinality
12.7 · Exercises

12.7.3. We know that we can use this function to show that $\mathbb{N} \times \mathbb{N}$ is denumerable, *if* it is a bijective function. That is, if the function is bijetive, then we can conclude that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$, which implies that $\mathbb{N} \times \mathbb{N}$ is denumerable. But, if the function is *not* bijective, this wouldn't give us any more information, since just because f is not bijective wouldn't imply $|\mathbb{N}| \neq |\mathbb{N} \times \mathbb{N}|$.

We actually saw that if we have two denumerable sets A, B, then $A \times B$ is also denumerable. We proved it using a diagonalization argument (see Result 12.2.6). Applying this result to $A = B = \mathbb{N}$, we can immediately see that $\mathbb{N} \times \mathbb{N}$ is, indeed,

denumerable.

So, assuming that we can show f is bijective, this question suggests that we could also show that $\mathbb{N} \times \mathbb{N}$ is denumerable directly by finding a bijective function from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$.

12.7.6.

(a) Since we're trying to figure out whether or not $\mathbb{I}$ is denumerable, let's assume that it is, and see what kind of conclusion we may reach. That is, let's see if under this assumption we reach a contradiction.

(b) We know that
$$[0,1] \cap \mathbb{Q} \subset \mathbb{Q},$$
and that $\mathbb{Q}$ is denumerable. Since any subset of a denumerable set is countable, and the subset is infinite, it must be denumerable.

(c) The set
$$\{\pi + q : q \in \mathbb{Q}\}$$
shifts the set of rational numbers by π. Intuitively, we think that shifting shouldn't change the size of a set, but of course, we need to prove this by showing that the set is denumerable, like $\mathbb{Q}$. Indeed, we can define a bijection between $\{\pi + q : q \in \mathbb{Q}\}$ and $\mathbb{Q}$. Since $\mathbb{Q}$ is denumerable, the other is too.

(d) Part (c) is a special case of this, with $a = \pi$. We need to check that the argument given in part (c) works when a is an arbitrary real number.

(e) The set
$$\{\pi q : q \in \mathbb{Q}\}$$
scales the set of rational numbers by π. In comparison to part (c), it may not be as intuitive that this set has the same size as the rational numbers. However, we can define the bijection $g(q) = \pi q$ from the rational numbers to $\{\pi q : q \in \mathbb{Q}\}$.

(f) Part (e) is a special case of this, with $a = \pi$. We need to check that the argument given in part (e) works when a is an arbitrary real number.

12.7.8. The hints suggests that we think about a bijection from $(0, \infty)$ to $\mathbb{R}$. We see that the easiest of such function is $f(x) = \log(x)$. But, we also realize that the set given to us is not $(0, \infty)$, but $(-\infty, -\sqrt{29})$. This may look like a big problem, but it is a problem we can easily overcome. First observe that we can take $g(x) = \log(-x)$, instead of f, it becomes a bijective function (at least intuitively, we need to show that such changes preserve bijectivity) from $(-\infty, 0)$ to $\mathbb{R}$. Then, we can finalize our construction by shifting g by $\sqrt{29}$ to the left, i.e. by defining $h(x) = \log(-x - \sqrt{29})$. This function, h, should be a bijection from $(-\infty, -\sqrt{29})$ to $\mathbb{R}$.

Now, of course, we need to show that h is indeed a bijection. We can do this

either by showing that it is injective and surjective, or by showing that it has both a left and a right inverse. Since we know how to find the inverse of the log function, the second way may be easier.

12.7.9. Question 13 in Section 10 suggests that we should be able to construct a bijective function $g : S \to \mathcal{P}(N)$ (as done in that question) implying that S is uncountable.

12.7.10.

- *First solution*: Let's try to find a bijection, say h, from $(0, 1)$ to $\mathbb{R}$. We'll need h to blow up to infinity at some point, and negative infinity at another point. Let's try to create h so that

$$\lim_{x \to 0^+} h(x) = +\infty, \quad \lim_{x \to 1^-} h(x) = -\infty.$$

  We know that the function $1/x$ satisfies the first condition, and that the function $1/(x-1)$ satisfies the second condition.

  We can define h to be a piecewise function involving rational functions like these. Suppose we tried to define

$$h(x) = \begin{cases} \frac{1}{x} & \text{if } 0 < x \leq \frac{1}{2} \\ \frac{1}{x-1} & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

  This function doesn't quite work, since it jumps from 2 to -2 at $x = 1/2$, and is thus not surjective. Let's fix this by defining

$$h(x) = \begin{cases} \frac{1}{x} - 2 & \text{if } 0 < x \leq \frac{1}{2} \\ \frac{1}{x-1} + 2 & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

  We now need to show that this is a bijection.

- *Second solution:* Instead of working with the piecewise function, we could also try the function $y : (0, 1) \to \mathbb{R}$ defined by

$$y(x) = \frac{1}{x} - \frac{1}{1-x} = \frac{1-2x}{x(1-x)}$$

  which also blows up to positive infinity as x approaches 0 from the right, and blows up to negative infinity as x approaches 1 from the left. We need to prove that it is bijective.

- *Third solution*: We can show that both sets are equinumerous by showing they are both equinumerous to the interval $(1, \infty)$ via simpler bijections.

12.7.12. In this question, we intuitively know that the result must be true. This is because we realize that if all the sets are finite, the result is more or less trivial. When the sets are finite we know that the we can equate the cardinality of a set

with its "size", i.e. the number of elements it has. This means that if $|A| = |B|$ and $|C| = |D|$, we can conclude that A and B have the same number of elements and C and D have the same number of elements. Thus, since $A \cap C = \emptyset$ and $B \cap D = \emptyset$, we get

$$|A \cup C| = |A| + |C| = |B| + |D| = |B \cup D|,$$

and the result follows.

Unfortunately, this argument fails when at least one of the sets is infinite, since we cannot talk about the "sizes" of infinite sets. So what we need to, instead, is to use the actual definition of two sets having equal cardinalities. That is, we know that $|A| = |B|$ and $|C| = |D|$ means that there exist two bijective functions $f : A \to B$ and $g : C \to D$.

Now, we need to be able to find a bijective function

$$h : A \cup C \to B \cup D.$$

For this, we realize that if $x \in A \cup C$, then $x \in A$ or $x \in C$. This means that for each case, we can use either f or g to send $x \in A \cup C$ to $B \cup D$, which will define our function h. Then all we will have to do is to show that this new function is indeed bijective.

12.7.13. We see that if we only want to show that these two sets are equinumerous, our best choice would be to use The Cantor-Schröder-Bernstein theorem. We can do that by fiinding two injective functions:

$$g_1 : (0, \infty) \to (0, \infty) - \{1\},$$

and

$$g_2 : (0, \infty) - \{1\} \to (0, \infty).$$

In this case, we see that we can take g_1 and g_2 defined as $g_1(x) = x + 1$, and $g_2(x) = x$. We see that these functions are injective which imlplies that $|(0, \infty)| = |(0, \infty) - \{1\}|$.

But, in this question, we want to find an explicit bijection from $(0, \infty)$ to $(0, \infty) - \{1\}$. The problem with such bijections is that it generally feels almost trivial since the two sets are almost identical. This gives the impression that we should be just take the identity function, or a tiny variation of it to get the bijection.

However, this idea fails when we realize that our bijection should send 1 to an element in $(0, \infty) - \{1\}$, call it y_1. But, then, se see $y_1 \in (0, \infty)$ and thus should go to another element in the codomain (different than itself), call it y_2. Then, this y_2 should go to another element, and that should go to another, and so on. This means that we cannot take a simple variation of the identity function here, we may need to make a more drastic change to account for this "snowball" effect.

The simplest way to handle this situation is by defining a function that sends 1 to 2, 2 to 3, 3 to 4, and so on, very similar to "Hilbert's Hotel Problem". As

for elements that are not natural numbers, we can just send them to themselves.

12.7.14.

(a) Cantor-Schröder-Bernstein 12.5.1 tells us that it is enough to find an injection from $(0,1) \to (0,1)^2$ and then another injection back. The first injection is easy, we can take, say, the function $f : (0,1) \to (0,1)^2$ defined by $f(x) = (x, 1/2)$.

The injection back is harder; we need to make a pair of reals (a, b) to a single real c. A nice "trick" uses the decimal expansions of a, b. In particular, we can write $a = 0.a_1a_2a_3\ldots, b = 0.b_1b_2b_3\ldots$. We can then interleave these expansions to create a new number $c = 0.a_1b_1a_2b_2a_3b_3\ldots$. Some care is required to ensure that this mapping is injective.

(b) We already know that $|(0,1)| = |\mathbb{R}|$ and so we know that there is a bijection between these sets. We can then leverage that bijection to construct a bijection between $(0,1) \times (0,1)$ and $\mathbb{R}^2$. Then part (a) tells us that $|(0,1) \times (0,1)| = |(0,1)|$. And then again we can use $|(0,1)| = |R|$. Putting all these together gives us the required result.

12.7.15. Since A and B are equinumerous, there is a bijection $f : A \to B$. We can use f to create a bijection from the power set of A to the power set of B. Let $F : \mathcal{P}(A) \to \mathcal{P}(B)$ be defined by

$$F(C) = \{f(a) : a \in C\}.$$

Notice that this function F takes as input a subset of A, and maps it to a subset of B. We need to show that F is surjective and injective.

12.7.17. We saw in Exercise 12.7.12 that if A_1, A_2, B_1, B_2 are any nonempty sets satisfying $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$, and that $|A_1| = |B_1|$ and $|A_2| = |B_2|$, then we have $|A_1 \cup A_2| = |B_1 \cup B_2|$. The idea in the proof relied on the fact that since $|A_1| = |B_1|$ and $|A_2| = |B_2|$, there are bijections $f : A_1 \to B_1$ and $g : A_2 \to B_2$. Using these functions, we could construct a piecewise function $h : A_1 \cup A_2 \to B_1 \cup B_2$ that turned out to be a bijection as well.

We can see this result as a specific example to this question, where our partitions for the sets $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$ are given as

$$P = \{A_1, A_2\},$$

and

$$Q = \{B_1, B_2\}.$$

Then, we see that $F : P \to Q$, defined as $F(A_1) = B_1$, and $F(A_2) = B_2$ is a bijection from P to Q, and moreover $|A_1| = |B_1|$ and $|A_2| = |B_2|$.

This suggests that if we have partitions with more sets in them, we should be able to construct a bijection in a very similar piecewise manner. Also, since sets in a partition are mutually disjoint, and that their union gives the entire set, such a construction would still work.

12.7.18.

(a) By definition, we know that there is an injection from A to B, and from B to C. We need to use these to create an injection from A to C.

(b) We need to prove that the statement given in the question implies the Cantor-Schröeder-Bernstein Theorem, and conversely, that the Cantor-Schröeder-Bernstein Theorem implies the given statement.

To prove the first implication, we assume that the statement given in the question is true; that is, if $|A| \leq |B| \leq |C|$ and $|A| = |C|$, then $|A| = |B| = |C|$. When we take $A = C$, we have the statement of the Cantor-Schroeder-Bernstein Theorem.

To prove the converse, we need to use the Cantor-Schröeder-Bernstein Theorem to establish the given statement:

$$\text{If } |A| \leq |B| \leq |C| \text{ and } |A| = |C|, \text{ then } |A| = |B| = |C|.$$

We'll try to prove that $|B| = |C|$. By Theorem 12.5.1, we can do this by showing that $|B| \leq |C|$ and $|C| \leq |B|$. The first is given by the statement of the problem. The second we will have to deduce from the assumption that $|A| = |C|$.

12.7.20.

(a) Since each A_n is denumerable, we can list the elements of each set, like

$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \dots\}.$$

We can then construct an infinite table, where the n^{th} row of the table lists the elements of A_n. Then, just as in the proof of Result 12.2.6, we can construction a bijection from the natural numbers to the elements of the table by sweeping over successive diagonals.

(b) This part differs to part (a), as the sets A_n may be finite or denumerable. In particular, it's possible that some of the A_n are the empty set.

Initially, suppose that only finitely many of the sets A_n are non-empty. We'll return to the other possibility shortly. By relabeling if necessary, we may assume that the first N sets, $A_1, \dots, A_N$ are non-empty, and $A_n = \emptyset$ for $n > N$. Then

$$\bigcup_{n=1}^{N} A_n = \bigcup_{n=1}^{\infty} A_n.$$

We are now only taking the union over a finite number of sets, and we need to show that it is countable. Result 12.2.9 tells us that the union of two countable sets is again countable. To get that the union of N countable sets is also countable, we'll need to apply induction.

(c) We would like to define sets B_n such that

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

and $B_m \cap B_n = \emptyset$ for $m \neq n$. That way we may apply part (b) to get the desired result.

Let $B_1 = A_1$. Then, let B_2 consist of all elements of A_2 that are not also elements of A_1. That way $A_1 \cup A_2 = B_1 \cup B_2$, and $B_1 \cap B_2 = \emptyset$, by construction. Next, we would define B_3 to be all elements of A_3 that are not elements of A_1 or A_2. Another way of writing this is

$$B_3 = A_3 \setminus (A_1 \cup A_2).$$

We proceed in this manner to define B_n for all $n \in \mathbb{N}$. We need to show that the union of all the sets B_n is equal to the union of all the sets A_n, and that the sets B_n satisfy the conditions of part (b).

Appendix C

Solutions to Exercises

1 · Sets
1.4 · Exercises

1.4.1.

$$
\begin{aligned}
A_1 &= \{1\} \\
A_2 &= \{-1, 0, 1\} \\
A_3 &= \{3, 6, 9, 12, 18, 24, 27, 36, 54, 72, 108, 216\} \\
A_4 &= \{\ldots, -12, -7, -2, 3, 8, 13, \ldots\} \\
A_5 &= \{2, 3\} \\
A_6 &= \{7, 11, 13, 17, 19\} \\
A_7 &= \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8\}
\end{aligned}
$$

1.4.2.

(a) $A = \{5n \text{ s.t. } n \in \mathbb{N}\}$

(b) $B = \{n \in \mathbb{N} \text{ s.t. } 10 \leq n \leq 100\}$

(c) $C = \left\{n^2 - 1 \text{ s.t. } n \in \mathbb{N}\right\}$

(d) $D = \left\{\dfrac{m}{m^2+1} \text{ s.t. } m \in \mathbb{Z}\right\}$

(e) $E = \left\{n \in \mathbb{N} \text{ s.t. } n = 2^{(2^k)} \text{ for some nonnegative integer } k\right\}$

(f) $F = \left\{2^a 3^b \text{ s.t. } (a, b \in \mathbb{Z}), (a, b \geq 0), \text{ and } (a + b \neq 0)\right\}$

1.4.3.

(a) We can write A in set builder notation in the following two equivalent ways:

$$A = \{m \text{ s.t. } m \in \mathbb{Z},\ 0 \leq m \leq 100,\ m \text{ is even}\} = \{2k \text{ s.t. } k \in \mathbb{Z},\ 0 \leq k \leq 50\}$$

We could also describe it in words by saying that A is the set of all even

non-negative integers that are at most 100.

(b) We can write B in set builder notation as follows:

$$B = \{3^n \text{ s.t. } n \in \mathbb{N}\}$$

Also, B is the set of all positive integer powers of three.

(c) The set C is given in set builder notation. Listing out its elements,

$$C = \{-3, -2, -1, 0, 1, 2, 3\}$$

Also, C is the set of all integers that are at most three in absolute value.

(d) Listing out the elements of D, we have

$$D = \{\ldots, -11, -7, -3, 1, 5, 9, 13, \ldots\}$$

Since the elements of D "go off" to both positive and negative infinity, we have ellipsis at the beginning and end of the set. We can describe the set in words by saying that D is the set of all integers that are one more than a (possibly negative) multiple of four. Alternatively, we could describe D as the set of all integers that have a remainder of one when divided by four.

(e) The set E is described in words. The set E can be given by listing elements, and using set builder notation, as follows:

$$E = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\} = \left\{\frac{1}{n} \text{ s.t. } n \in \mathbb{N}\right\}$$

(f) The set F can be given by listing elements, and using set builder notation, as follows:

$$F = \{\ldots, -18, -13, -8, -3, 2, 7, 12, 17, 22, \ldots\} = \{5k + 2 \text{ s.t. } k \in \mathbb{Z}\}$$

1.4.4. The set is ill-defined because we haven't written out enough elements to determine the set's "rule". We could, for example, interpret this set as the set of all positive even integers,

$$\{2, 4, 6, 8, \ldots\},$$

or the the set of all positive integer powers of two,

$$\{2, 4, 8, 16, \ldots\}.$$

1.4.5.

(a) We have not specified what the variable k is, so this set is ambiguous. For example, one person might assume that k is a constant, while another person assumes $k \in \mathbb{R}$. Neither of these gives the desired set.

(b) We have used the letter N rather than the symbol $\mathbb{N}$, which indicates the natural numbers.

(c) We use the letter c to represent this set. It is more conventional to use capital letters to denote sets.

(d) We have used the letter k to write the set elements $2k + 1$, and then we have stated that another variable n is a natural number. Because of this, the variable k is not defined, and the set is ambiguous.

(e) We have used the Greek letter epsilon (ϵ) rather than the "element of" symbol $\in$.

(f) We have used capital letters to denote elements of the set. It is more conventional to use lowercase letters for set elements when possible.

(g) We have used the Greek letter epsilon (ε) rather than the "element of" symbol $\in$. Note that this is the same issue as part (e), but is included twice because of the two representations of epsilon.

(h) This is an issue you would encounter if you are typing up math using a program such as LaTeX. We have written this set using normal text, rather than in math mode. In LaTeX, one uses dollar signs (e.g. "x") or backslash with parentheses (e.g. "\(x\)")around their math. This produces italicized math characters such as "x" instead of "x." Notice that in the set, none of the letters are italicized, and the word "in" is used, rather than the symbol "$\in$". This is very sloppy, and makes it more difficult for the reader to follow. When typing, you should always be careful about mixing up text and mathematics. Make sure you type characters deliberately to avoid confusion.

1.4.6. The goal of this question is to highlight that the empty set is not the same as the zero set, nor is it same as the set containing the empty set. When in doubt, refer the empty bag analogy provided in Section 1.2.

(a) False. The empty set contains no elements, but the set $\{0\}$ contains 1 element, namely the number 0.

(b) False. This is for the same reason as above. $\{\varnothing\}$ is a set with 1 element; it contains the empty set.

(c) True. There are 0 elements in the empty set.

(d) False. The set $\{\varnothing\}$ is a set, containing a set with no elements (this is like a bag containing another empty bag), whereas the set $\{\{\varnothing\}\}$ is a set containing a set containing a set with no elements (like a bag containing a bag containing a third empty bag).

(e) True. These are both sets containing a set with no elements.

1.4.7.

- The only integer that belongs to A is -1, so all of the numbers listed are not elements of A.
- The numbers 2 and 8 do not belong to B because their squares do not exceed 100; -12 is not an element of B because it is negative.
- The elements of C are the set containing 2, the set containing 8, and the set containing -12. C itself does not actually have the elements 2, 8, and -12.
- The number 2 does not belong to D because it is not a multiple of 4, 8 does not belong to D because it is not an *odd* multiple of 4, and -12 does not belong to D because it is not a *positive* multiple of 4.

1.4.8.

(a) $\mathbb{Z} \neq \{a : a \in \mathbb{N} \text{ or } -a \in \mathbb{N}\}$. This is because $0 \in \mathbb{Z}$, but $0 \notin \{a : a \in \mathbb{N} \text{ or } -a \in \mathbb{N}\}$.

(b) $\{1, 2, 2, 3, 3, 3, 2, 2, 1\} = \{1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 3, 3, 3, 3, 3\}$. The number of occurrences of an element of a set does not matter. Both of these sets are equal to the set $\{1, 2, 3\}$.

(c) $\{d : d \text{ is a day of the week with 40 hours}\} = \{w : w \text{ is a week with 6 days}\}$. Since there are no days with 40 hours and no weeks with 6 days, both of these are equal to the empty set.

(d) $\{p : p \text{ is prime}, p < 42\} \neq \{1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41\}$. The number 1 is not a prime, thus there is an element in the set on the right hand side that is not present in the left hand side.

1.4.9. The sets B, C, E, and F equal S. The sets A and D do not.

- $A \neq S$ because $1 \notin A$ but $1 \in S$.
- $B = S$ since the elements of B are just the reciprocal of the absolute value of non-zero integers. Hence $-1, 1$ both give $1/1$, $-2, +2$ both give $1/2$ and so on.
- $C = S$ since the elements of C can be reduced to the form $1/n$, $n \in \mathbb{N}$. Indeed, $1/n = 2/k$ when $k = 2n$, and in the definition of C, we require that k is even.
- $D \neq S$ because $3/2 \in D$ but $3/2 \notin S$. Since D is the set of rational numbers, there are many elements in D that are not in S.
- $E = S$ because we compensate for writing its elements as $\frac{1}{n-1}$ by restricting to $n \in \mathbb{N}$ that are greater than one. Then $n = 2$ gives $1/1$, $n = 3$ gives $1/2$, and so on.

- $F = S$ since the requirement that $m \in \mathbb{Z},\ m > 0$ is the same as saying $m \in \mathbb{N}$.

2 · A little logic

2.7 · Exercises

2.7.1.

(a) This is a statement, and it is an implication. Since the hypothesis of the implication is true, but the conclusion is false, the full statement is false.

(b) This is a statement. Since the hypothesis of the implication is false, the full statement is automatically true.

(c) This is an open sentence, since it is true or false depending on the function f.

(d) This is a statement, and a conjunction of the statement "13 is prime" and the statement "6 is prime." Since the first is true, but the second is false, the conjunction is false.

(e) This is a statement, and a disjunction of the statement "13 is prime" and the statement "6 is prime." Since the first is true, but the second is false, the disjunction is true.

(f) This is an open sentence, since it is true or false depending on the circle.

2.7.2.

(a) True. A weekend consists of the days Saturday and Sunday, hence, if it is Saturday, we can conclude that it is the weekend.

(b) False. Since a weekend consists of Saturday and Sunday, knowing that it is a weekend is not enough to conclude that it is Saturday.

(c) True. The hypothesis is false, so we automatically conclude that the implication is true.

2.7.3.

(a) False. Recall that we can also read this statement as "whenever x is an even number, it is also twice a natural number ." Notice that the set in the conclusion is the set of positive even numbers, so taking a negative even number, such as $x = -2$, we see that there is no $n \in \mathbb{N}$ such that $x = 2n$.

(b) False. The statement can also be read as "every prime number is odd," but 2 is an even prime number, so the statement must be false.

(c) False. This statement can be read as "every multiple of 3 is a multiple of 6." The number 3 is a multiple of 3 (in particular, $3 = 3 \cdot 1$), but it is not a multiple of 6.

(d) True. This statement can be read as "every multiple of 6 is a multiple of

3." Here is a brief proof of our claim:

Proof. Suppose that $x \in \{6k : k \in \mathbb{Z}\}$. By definition, we may write $x = 6k$ for some $k \in \mathbb{Z}$. Factoring the 6 yields $x = 3(2k)$. Since $2k$ is an integer, we have shown that we can write x as a multiple of 3. Hence we may conclude that $x \in \{3k : k \in \mathbb{Z}\}$, as desired. ■

We will do many more proofs like this in Chapter 3.

2.7.4.

(a) False. The statement "3 is even" is false, so the conjunction is false.

(b) True. The statement "3 is prime" is true, so the disjunction is true.

(c) True. Both of the statements "$x^2 > x$ when $x > 1$" and "18 is composite" are true, so the conjunction is true.

(d) True. Both of the statements "$x^2 > x$ when $x > 1$" and "18 is composite" are true, so the disjunction is also true.

2.7.5.

(a) This sentence can be written as $P \wedge Q$ for

$$\begin{aligned} P: &\quad 8 \text{ is even }, \\ Q: &\quad 5 \text{ is prime.} \end{aligned}$$

This is a statement.

(b) This sentence can be written as $(P \wedge Q) \implies R$ for

$$\begin{aligned} P: &\quad n \text{ is a multiple of 4}, \\ Q: &\quad n \text{ is a multple of 6}, \\ R: &\quad n \text{ is a multiple of 24}. \end{aligned}$$

Even though P, Q and R are open sentences, this sentence is a conditional statement.

(c) This sentence can be written as $\sim P \implies \big(Q \wedge (\sim R)\big)$ for

$$\begin{aligned} P: &\quad n \text{ is a multiple of 10}, \\ Q: &\quad n \text{ is a multiple of 2}, \\ R: &\quad n \text{ is a multiple of 5}. \end{aligned}$$

This is a statement.

(d) This expression can be written as $P \wedge Q$ for

$$\begin{aligned} P: &\quad x \geq 3, \\ Q: &\quad x \leq 6. \end{aligned}$$

This is an open sentence.

(e) This sentence can be written as $R \implies (P \vee Q)$ for

$$\begin{aligned} P: &\quad x \text{ is less than } -2, \\ Q: &\quad x \text{ is greater than } 2, \\ R: &\quad x^2 \text{ is greater than } 4. \end{aligned}$$

This is a statement.

(f) This sentence can be written as $P \implies (Q \implies R)$ for

$$\begin{aligned} P: &\quad f(x) \text{ is differentiable everywhere}, \\ Q: &\quad x \text{ is a local maximum of } f(x), \\ R: &\quad f'(x) = 0. \end{aligned}$$

This is a statement.

2.7.6. We list a few possible English statements:

(a)
- If x is a real number, then x^2 is also a real number and x^2 is non-negative.
- If x is a real number, then x^2 is a non-negative real number.
- The square of a real number is non-negative and real.

(b)
- 4 is an element of the set of positive, even numbers.
- We can write $4 = 2\ell$ for some $\ell \in \mathbb{N}$.
- 4 is a positive even number.

(c)
- For a natural number x, it is not the case that $x^2 = 0$.
- When x is a natural number, x^2 is never 0.
- The square of a natural number is non-zero.

(d)
- If x is an integer, then x is an even number or x is an odd number.
- If x is an integer, then it is either even or odd.
- Every integer is even or odd.

2.7.7.

(a) We can refer to the truth table for the implication, except this time the hypothesis is $\sim P$ rather than P.

P	Q	$\sim P$	$(\sim P) \implies Q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

(b) We can use the truth table from part (a) to help figure out the new truth table.

P	Q	$(\sim P) \implies Q$	$P \wedge Q$	$(P \wedge Q) \vee \big((\sim P) \implies Q\big)$
T	T	T	T	T
T	F	T	F	T
F	T	T	F	T
F	F	F	F	F

(c) From the truth table, we see that $P \wedge (\sim P)$ is always false.

P	$\sim P$	$P \wedge (\sim P)$
T	F	F
F	T	F

(d) From the truth table, we see that $P \vee (\sim P)$ is always true.

P	$\sim P$	$P \vee (\sim P)$
T	F	T
F	T	T

(e) From the truth table, we see that $(P \implies Q) \iff (Q \implies P)$ is true only when P and Q have the same truth value.

P	Q	$P \implies Q$	$Q \implies P$	$(P \implies Q) \iff (Q \implies P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

2.7.8. We start with the truth table for $P \implies Q$ given in Definition 2.4.1. The next column (fourth column of the table) gives the truth table for $\sim (P \implies Q)$. We compare this column to the last column of the table, which is the truth table for $P \wedge \sim Q$.

P	Q	$P \implies Q$	$\sim (P \implies Q)$	$P \wedge \sim Q$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

2.7.9. Recall the truth table of the implication:

P	Q	$P \implies Q$
T	T	T
T	F	F
F	T	T
F	F	T

(a) We cannot determine whether or not it was raining on Monday. The conclusion of the implication (Q = "I brought an umbrella to work") is true, which means that the hypothesis (P = "it's raining") may be true or false, while the implication remains true. See the first and third rows of the truth table.

(b) It was not raining on Tuesday. Because the conclusion of the implication if false, the hypothesis must also be false. If the hypothesis were true, then the implication would be false. See the second and fourth rows of the truth table.

(c) It was raining on Wednesday. Because the hypothesis of the implication (P = "I am late for work") is true, the conclusion (Q = "it's raining") must also be true. If the conclusion were false, then the implication would also be false. See the first and second rows of the truth table.

(d) We cannot determine whether or not it was raining on Thursday. The hypothesis of the implication is false, which means that the conclusion may be true or false, while the implication remains true. See the third and fourth rows of the truth table.

2.7.10.

(a) We cannot conclude anything. We have not satisfied the hypothesis of the conditional statement.

(b) We conclude that sailors should take warning by modus ponens.

(c) We cannot conclude anything. If we did, we would be "affirming the consequent," and this is a false deduction. There are many other things that could cause sailors to take warning, such as pirates or a group of hungry sharks.

(d) We cannot conclude anything. If we did, we would be "denying the antecedent," and this is a false deduction.

(e) We conclude that it is not true that (the sky is red and it is morning) by modus tollens.

2.7.11.

(a) If n is not a multiple of 24, then it is not a multiple of 4 or 6.

(b) We give two ways of writing this; they are quite similar.

- If n is not a multiple of 2, but it is a multiple of 5, then n is a multiple of 10.
- If n is a multiple of 5 and not a multiple of 2, then n is a multiple of 10.

(c) First we rearrange the original statement to be in a format where we can more easily identify the hypothesis and the conclusion:

- If x is a real number whose square is greater than 4, then x is less than -2 or greater than 2.

We should also make sure we are careful with the inequalities. In the given statement these are strict, $x < -2$, $x > 2$, so when we form the contrapositive they will be inclusive, $x \geq -2$, $x \leq 2$. So the contrapositive is

- If x is between -2 and 2 (inclusive), then the square of x is less than or equal to 4.

(d) $(x^2 \notin \mathbb{R}) \vee (x^2 < 0) \implies x \notin \mathbb{R}$.

(e) $x^2 = 0 \implies x \notin \mathbb{N}$.

(f) $x \notin \{6k : k \in \mathbb{Z}\} \implies x \notin \{3k : k \in \mathbb{Z}\}$.

2.7.12. Let $P =$ "m^2 is even," $Q =$ "m is even," and $R =$ "m^2 is divisible by 4."

The statement "If m is odd, then m^2 is odd" converts to $(\sim Q) \implies (\sim P)$. Its contrapositive is $\sim (\sim P) \implies \sim (\sim Q)$, which is the same as $P \implies Q$. In words, the contrapositive is "If m^2 is even, then m is even." Since an implication and its contrapositive have the same truth table, and $(\sim Q) \implies (\sim P)$ is true, $P \implies Q$ is also true.

The statement "If m is even, then m^2 is divisible by 4" converts to $Q \implies R$. Its contrapositive is $(\sim R) \implies (\sim Q)$, or "If m^2 is not divisible by 4, then m is odd."

We have the four distinct implications, $P \implies Q$, $(\sim Q) \implies (\sim P)$, $Q \implies R$, and $(\sim R) \implies (\sim Q)$.

Chaining the implications $P \implies Q$ and $Q \implies R$, we end up with the implication $P \implies R$, or

"If m^2 is even, then m^2 is divisible by 4."

Chaining the implications $(\sim R) \implies (\sim Q)$ and $(\sim Q) \implies (\sim P)$, we end up with the implication $(\sim R) \implies (\sim P)$, or

"If m^2 is not divisible by 4, then m^2 is odd."

Notice that this implication is the contrapositive of the other implication we formed.

2.7.13. Let $P =$ "p is prime" and $Q =$ "$\sqrt{p}$ is irrational."

- The contrapositive of $P \implies Q$ is $\sim Q \implies \sim P$, or "If $\sqrt{p}$ is rational, then p is not prime."
- The converse of $P \implies Q$ is $Q \implies P$, or "If $\sqrt{p}$ is irrational, then p is prime."
- The inverse of $P \implies Q$ is $\sim P \implies \sim Q$, "If p is not prime, then $\sqrt{p}$ is rational."

Since the truth tables of the implication and its contrapositive are the same, the contrapositive is also a true statement. Since the truth tables of the converse and inverse differ from that of the original implication, we cannot determine whether or not the converse and inverse are true statements from the information provided (although they are indeed false, as evidenced by the fact that $\sqrt{6}$ is irrational; this is something you could prove using the tools developed later on in this book).

2.7.14. P is true and R is false. The truth value of Q cannot be determined from the given information.

2.7.15. P is true, Q is true, and R is false.

2.7.16. P is true, Q is false, and R is false.

3 · Direct proofs

3.5 · Exercises

3.5.1.

Proof. Assume n is even. Then we know $n = 2k$ for some $k \in \mathbb{Z}$. Hence, $n^2 + 3n + 5 = (2k)^2 + 3(2k) + 5 = 2(2k^2 + 3k + 2) + 1$. Since $2k^2 + 3k + 2 \in \mathbb{Z}$ for $k \in \mathbb{Z}$, we see $n^2 + 3n + 5$ is odd. ■

3.5.2.

Proof. Assume n, m are odd integers. Then we know $n = 2k+1$ and $m = 2\ell+1$ for some $k, \ell \in \mathbb{Z}$. Thus, $nm = (2k + 1)(2\ell + 1) = 2(2k\ell + k + \ell) + 1$. Now since $k, \ell \in \mathbb{Z}$ we know that $2k\ell + k + \ell \in \mathbb{Z}$, and so nm is odd. ■

3.5.3.

(a) The sum of two odd numbers is even.

Proof. Let m, n be two odd numbers. Then there are some $k, \ell \in \mathbb{Z}$ such that $m = 2k + 1$ and $n = 2\ell + 1$, and so

$$m + n = (2k + 1) + (2\ell + 1) = 2(k + \ell + 1).$$

Since $k, \ell \in \mathbb{Z}$, we also have that $k + \ell + 1$ is an integer, and thus $m + n$ is even. ■

(b) The sum of two even numbers is even.

Proof. Let m, n be two even numbers. Then there are some $k, \ell \in \mathbb{Z}$ such that $m = 2k$ and $n = 2\ell$, and so

$$m + n = (2k) + (2\ell) = 2(k + \ell).$$

Since $k, \ell \in \mathbb{Z}$, we also have that $k + \ell$ is an integer, and thus $m + n$ is even. ■

(c) The sum of an even number and an odd number is odd.

Proof. Let m be an even number and n be an odd number. Then there are some $k, \ell \in \mathbb{Z}$ such that $m = 2k$ and $n = 2\ell + 1$, and so

$$m + n = (2k) + (2\ell + 1) = 2(k + \ell) + 1.$$

Since $k, \ell \in \mathbb{Z}$, we also have that $k + \ell$ is an integer, and thus $m + n$ is odd. ■

(d) The product of two even numbers is even.

Proof. Let m, n be two even numbers. Then there are some $k, \ell \in \mathbb{Z}$ such that $m = 2k$ and $n = 2\ell$, and so

$$mn = (2k)(2\ell) = 2(2k\ell).$$

Since $k, \ell \in \mathbb{Z}$, we also have that $2k\ell$ is an integer, and thus mn is even. ■

(e) The product of an even number and an odd number is even.

Proof. Let m be an even number and n be an odd number. Then there are some $k, \ell \in \mathbb{Z}$ such that $m = 2k$ and $n = 2\ell + 1$, and so

$$mn = (2k)(2\ell + 1) = 2(2k\ell + k)).$$

Since $k, \ell \in \mathbb{Z}$, we also have that $2k\ell + k)$ is an integer, and thus mn is even. ■

3.5.4. We have accidentally proven the converse rather than the desired statement. We started by assuming the conclusion, and ended up showing that the hypothesis is true. A correct proof can be given by using cases or proving the contrapositive Chapter 5.

3.5.5. The main issue with proof as written is that the variables are undefined. The variables a, b, k, and ℓ came out of nowhere! Presumably, a and b are odd integers, and k and ℓ are integers, but the proof didn't define them as such. Moreover, note that saying $k + \ell + 1 \in \mathbb{Z}$ does not guarantee that $k, \ell \in \mathbb{Z}$; for example, taking $k = \ell = 1/2$ shows why.

So while the ideas of the proof are correct, it is, at best, very sloppy. You should not assume that the reader will "just get it" - you need to explain what

"it" is!

Let's rewrite the proof with these issues resolved:

Proof. Let a and b be odd integers. Then there are $k, \ell \in \mathbb{Z}$ such that $a = 2k+1$ and $b = 2\ell + 1$. Hence

$$a + b = (2k + 1) + (2\ell + 1) = 2(k + \ell + 1).$$

Since k, ℓ are integers, $k + \ell + 1 \in \mathbb{Z}$, and so $a + b$ is even. Thus the sum of any two odd integers is even. ■

3.5.6.

Proof. Assume $n, a, b, x, y \in \mathbb{Z}$, $n \mid a$, and $n \mid b$. This implies that $a = nk$ and $b = n\ell$ for some $k, \ell \in \mathbb{Z}$. Thus, $ax + by = nkx + n\ell y = n(kx + \ell y)$. Since $kx + \ell y \in \mathbb{Z}$ for $k, x, \ell, y \in \mathbb{Z}$, then we see $n \mid (ax + by)$. ■

3.5.7.

Proof. Let $n, a \in \mathbb{Z}$. Assume that $n \mid a$ and $n \mid (a+1)$. Then we see that $a = nk$ for some $k \in \mathbb{Z}$ and $(a+1) = n\ell$ for some $\ell \in \mathbb{Z}$. Thus, $1 = (a+1) - a = n(\ell - k)$ and since $(\ell - k) \in \mathbb{Z}$, we see $n \mid 1$. Therefore, since n is an integer, the only possibilities are that $n = 1$ or $n = -1$. ■

3.5.8.

Proof. Let $a \in \mathbb{Z}$. Assume that $2 \mid a$ and $3 \mid a$. This implies $a = 2k$ some some $k \in \mathbb{Z}$ and $a = 3m$ for some $m \in \mathbb{Z}$. Thus, we see $3a = 6k$ and $2a = 6m$. Hence, $a = 3a - 2a = 6k - 6m = 6(k - m)$.Therefore, since $(k - m) \in \mathbb{Z}$ for $k, m \in \mathbb{Z}$, we see $6 \mid a$. ■

Proof. Let $a \in \mathbb{Z}$. Assume that $2 \mid a$ and $3 \mid a$. This implies $a = 2k$ some some $k \in \mathbb{Z}$ and $a = 3m$ for some $m \in \mathbb{Z}$. Thus, we see $a = 2k = 3m$, and hence, $2k - 2m = 2(k - m) = m$. This implies $a = 3m = 6(k - m)$. Therefore, since $(k - m) \in \mathbb{Z}$ for $k, m \in \mathbb{Z}$, we see $6 \mid a$. ■

3.5.9.

Proof. Let $n \in \mathbb{Z}$ and assume that $3 \mid (n - 4)$. Then we see that $(n - 4) = 3k$ for some $k \in \mathbb{Z}$. Thus, $n = 3k + 4$, which implies $n^2 - 1 = (3k + 4)^2 - 1 = (9k^2 + 24k + 16) - 1 = 3(3k^2 + 8k + 5)$. Therefore, since $k \in \mathbb{Z}$, we know that $3k^2 + 8k + 5 \in \mathbb{Z}$, and so we see $3 \mid (n^2 - 1)$ as required. ■

3.5.10. The issue with this proof is that we used the variable k in two different situations. In general, the integer multiple that b is of a would be different than the integer multiple that c is of b. While the logic of the proof is mostly correct, we end up with the statement $c = k^2 a$ (that c is a square multiple of a), which is incorrect. For example, if a is positive while c is negative, then we definitely can't find an integer k satisfying $c = k^2 a$.

Let's rewrite the proof with these issues resolved:

Proof. Assume a, b, and c are integers such that $a \mid b$ and $b \mid c$. Since a divides

b, we have that $b = ka$ for some $k \in \mathbb{Z}$. Moreover, since b divides c, we have that $c = \ell b$ for some $\ell \in \mathbb{Z}$. But then

$$c = k(\ell a) = (k\ell)a,$$

Since k and ℓ are integers, $k\ell \in \mathbb{Z}$, and it follows that a divides c. ■

3.5.11. First, this solution is quite terse, and it could benefit from more explanation. Explaining the steps taken in each line could make the error more clear:

Assume that $x = y$. Then

$$\begin{aligned} x^2 &= xy && \text{multiplying by } x \\ \Rightarrow x^2 - y^2 &= xy - y^2 && \text{Subtracting } y^2 \\ \Rightarrow (x+y)(x-y) &= y(x-y) && \text{Factoring} \\ \Rightarrow x + y &= y && \text{Dividing by } (x-y) \\ \Rightarrow 2y &= y && \text{Using } x = y \end{aligned}$$

Letting $x = y = 1$, we have shown that $2 = 1$.

The above proof is now more clear to read, and we notice that in the fourth line, we divide by $x - y$. We should always take care when we divide; we cannot divide by zero. Recall that we previously assumed that $x = y$, thus we are dividing by 0 in the fourth line. This is our logical error, and we can be reassured that $2 \neq 1$.

In this proof, a careless division of an equation caused all the problems. Similar problems can arise when we work with inequalities. For example, multiplication or division of an inequality by a negative number will change the sign of the inequality, so we must take extra care to check if quantities are negative or not.

3.5.12.

Proof. We prove each implication in turn.

- Assume that $\lfloor x \rfloor = x$. Then, since the floor function, by definition, returns an integer, we see that $x = \lfloor x \rfloor \in \mathbb{Z}$ as required.
- Now assume that $x \in \mathbb{Z}$. Then, since the floor function returns the greatest integer less than or equal to x, we know that $\lfloor x \rfloor \leq x$. Additionally, since $x \in \mathbb{Z}$ and $x \leq x$, we know that x is an integer less than or equal to x. Now $\lfloor x \rfloor$ is the greatest integer less than or equal to x, so we must have that $x \leq \lfloor x \rfloor$. Because $x \leq \lfloor x \rfloor$ and $x \geq \lfloor x \rfloor$, we know $x = \lfloor x \rfloor$.

■

What we are really doing here is proving a "biconditional statement". We will see much more on this in the next couple of chapters.

3.5.13.

Proof. Assume that a, b are integer roots, then, by definition, there are natural numbers k, ℓ and integers m, n such that $a^k = n$ and $b^\ell = m$. Therefore we see that $(ab)^{k\ell} = n^\ell m^k$. Since $k\ell \in \mathbb{N}$ and $n^\ell m^k \in \mathbb{Z}$, we see that ab is also an integer root. ■

3.5.14. The issue with the proof is that it is written in the wrong order — we start with the conclusion of the implication, and end up with the hypothesis of the implication. Essentially, the proof is backwards. Often, when doing scratch work for proofs involving inequalities, you will end up with a backwards proof. However, when you're writing up the proof formally, always remember to start off with the hypothesis.

Here's a correct proof of the statement:

Proof. Let x be positive. Assume that $x < 1$. Then $2x < 2$, and so $5x < 3x + 2$. Since $x > 0$, we can divide this inequality by $5x$ to obtain

$$1 < \frac{3x + 2}{5x}$$

which is the desired inequality. ■

3.5.15. There are a number of issues with this proof. First, it is written in the wrong order — we start with the conclusion of the implication, and end up with the hypothesis of the implication. Also, there are two times that we divide by a negative number but fail to flip the sign of the inequality. This happens when we divide by $3x - 5$, which is negative since $x < 0$, and when we divide by -3. Notice that since this happens twice, we end up with the right inequality (by luck!), but the proof is still incorrect since we took the wrong path to this inequality.

Here's a correct proof of the statement:

Proof. Let $x < 0$. Multiplying by -3, we have $-3x > 0$, and so $-3x + 5 > 5$. Writing this as

$$-(3x - 5) > 5$$

we divide through by $3x - 5$, which is negative as $x < 0$, and obtain

$$-1 < \frac{5}{3x - 5}.$$

■

3.5.16.

Proof. Assume that $0 < y < x$. This means that $x - y > 0$. Then, since $x, y > 0$, we can factor the expression $x - y$ and get

$$(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) > 0$$

and since the square root function is nonnegative, and $x, y > 0$, we also know

that $\sqrt{y} + \sqrt{x} > 0$. So dividing both sides of the inequality by $\sqrt{x} + \sqrt{y}$ gives

$$\sqrt{x} - \sqrt{y} > 0$$

Hence we conclude that $\sqrt{x} > \sqrt{y}$ as required. ■

It is a good extension of this exercise to think how you might extend this to the slightly more general result:

$$x \geq y \geq 0 \implies \sqrt{x} \geq \sqrt{y}.$$

3.5.17. There are a few issues with this proof. First of all, the flow of logic in this proof is backwards. The computation presented in the question could be used as scratch work, but is not a proof. A proof should start with a basic fact (here: the square of any nonzero real number is positive), and work towards the desired conclusion. Furthermore, this solution lacks explanation. A good proof should explain the steps taken to reach the conclusion. Our proofs should not only be correct but they should also be easy on the reader. Often, we forget to include extra explanation because our brains are filling in those extra details as we write. Think about the reader - they don't have the same brain as you to fill in those gaps! How can you make it easier for them to understand?

Proof. Let $a, b \in \mathbb{R}$ and $0 < a < b$. Then $a - b \neq 0$, and we know that $(a-b)^2 > 0$. We expand and complete the square to get $(a+b)^2$,

$$\begin{aligned} 0 &< (a-b)^2 \\ &= a^2 - 2ab + b^2 \\ &= a^2 - 2ab + b^2 + 4ab - 4ab \\ &= (a^2 + 2ab + b^2) - 4ab \\ &= (a+b)^2 - 4ab \end{aligned}$$

Therefore $4ab < (a+b)^2$.

We would now like to take square roots. From Exercise 3.5.16, the function $f(x) = \sqrt{x}$ is a positive, increasing function when $x > 0$ (these two conditions are necessary to ensure that the inequality will not change when we take the square root of each side). Now, taking square roots, we see

$$2\sqrt{ab} < (a+b).$$

Dividing by two yields $\sqrt{ab} < \frac{a+b}{2}$, as desired. ■

3.5.18.

Proof. Let $x, y \in \mathbb{R}$ such that $x, y \geq 0$. Since the square root function is never negative, $\sqrt{x}\sqrt{y} \geq 0$. Then multiplying by 2 and adding $x + y$ to both sides, we have

$$x + y \leq x + 2\sqrt{x}\sqrt{y} + y.$$

Since $x, y \geq 0$, we have that $(\sqrt{x})^2 = x$ and $(\sqrt{y})^2 = y$, and so we may factor the

righthand side of the above inequality to obtain

$$x + y \leq (\sqrt{x} + \sqrt{y})^2.$$

Since $x + y \geq 0$, we then have

$$\sqrt{x+y} \leq \sqrt{(\sqrt{x} + \sqrt{y})^2}$$

But

$$\sqrt{(\sqrt{x} + \sqrt{y})^2} = \sqrt{x} + \sqrt{y},$$

since $\sqrt{x} + \sqrt{y} \geq 0$. Therefore

$$\sqrt{x+y} \leq \sqrt{x} + \sqrt{y},$$

as required. ■

4 · More logic
4.3 · Exercises
4.3.1.

(a) $(\sim P) \vee Q$ and $P \Rightarrow Q$.

P	Q	$(\sim P)$	$(\sim P) \vee Q$	$P \implies Q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We see that the last two columns are the same. Therefore, these are logically equivalent.

(b) $P \Leftrightarrow Q$ and $(\sim P) \Leftrightarrow (\sim Q)$.

P	Q	$\sim P$	$\sim Q$	$P \Leftrightarrow Q$	$(\sim P) \Leftrightarrow (\sim Q)$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

We see that the last two columns are the same. Therefore, these are logically equivalent.

(c) $P \Rightarrow (Q \vee R)$ and $P \Rightarrow ((\sim Q) \Rightarrow R)$.

P	Q	R	$Q \vee R$	$(\sim Q) \Rightarrow R$	$P \Rightarrow (Q \vee R)$	$P \Rightarrow ((\sim Q) \Rightarrow R)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
F	T	T	T	T	T	T
T	F	F	F	F	F	F
F	T	F	T	T	T	T
F	F	T	T	T	T	T
F	F	F	F	F	T	T

We see that the last two columns are the same. Therefore, these are logically equivalent.

(d) $(P \vee Q) \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$.

P	Q	R	$P \vee Q$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \vee Q) \Rightarrow R$	$(P \Rightarrow R) \wedge (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
F	T	T	T	T	T	T	T
T	F	F	T	F	F	F	F
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

We see that the last two columns are the same. Therefore, these are logically equivalent.

(e) $P \Rightarrow (Q \vee R)$ and $(Q \wedge R) \Rightarrow P$.

P	Q	R	$Q \vee R$	$Q \wedge R$	$P \Rightarrow (Q \vee R)$	$(Q \wedge R) \Rightarrow P$
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	F	T	T	F	T	T
F	T	T	T	T	T	F
T	F	F	F	F	F	T
F	T	F	T	F	T	T
F	F	T	T	F	T	T
F	F	F	F	F	T	T

We see that the last two columns are different. Therefore, these are not logically equivalent. Indeed, we see that when P is false and Q, R are true (i.e. the fourth row), we see that $P \Rightarrow (Q \vee R)$ is true, whereas $(Q \wedge R) \Rightarrow P$ is false.

Even though questions like this are very easy to show and may feel quite unnecessary, they play an important role in understanding what logical equivalences we can think of when we are proving statements.

For example, part (c) tells us that if we have a conditional statement where the conclusion is an `or'-statement, we can indeed turn that into a double implication statement which may be easier to prove.

Similarly part (d) is going to play an important role in proving statements using cases in the upcoming chapter.

So, it is important to think about which statements are logically equivalent and how we can use that to our advantage when we want to prove different statements.

4.3.2. In this question, first we are going to write the sentences in symbolic logic, then negate them using the Theorem 4.2.3 and finally we will rewrite these negations in English.

(a) This sentence can be written as $P \wedge Q$ for

$$\begin{aligned} P: &\quad 8 \text{ is even} \\ Q: &\quad 5 \text{ is prime.} \end{aligned}$$

So, its negation is:

$$\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q)$$

Here, we used DeMorgan's law. Then, the negation can be rewritten as:

$$8 \text{ is not even or } 5 \text{ is not prime.}$$

(b) This sentence can be written as $(P \wedge Q) \implies R$ for

$$\begin{aligned} P: &\quad n \text{ is a multiple of 4,} \\ Q: &\quad n \text{ is a multple of 6,} \\ R: &\quad n \text{ is a multiple of 24.} \end{aligned}$$

Then its negation is:

$$\sim((P \wedge Q) \Rightarrow R) \equiv \sim(\sim(P \wedge Q) \vee R) \equiv (P \wedge Q) \wedge \sim R.$$

Here, we used the equivalence of implication, DeMorgan's law and double negation. Then, the negation can be rewritten as:

$$n \text{ is a multiple of 4 and 6, but it is not a multiple of 24.}$$

(c) This sentence can be written as $(\sim P) \implies \big(Q \wedge (\sim R)\big)$ for

$$\begin{aligned} P: &\quad n \text{ is a multiple of 10,} \\ Q: &\quad n \text{ is a multiple of 2,} \\ R: &\quad n \text{ is a multiple of 5.} \end{aligned}$$

Then, we see that the negation is:

$$\begin{aligned}\sim\big(\sim P \implies \big(Q \wedge (\sim R)\big)\big) &\equiv \sim\big(\sim(\sim P) \vee \big(Q \wedge (\sim R)\big)\big)\\ &\equiv (\sim P) \wedge \sim\big(Q \wedge (\sim R)\big)\\ &\equiv (\sim P) \wedge \big((\sim Q) \vee R\big)\\ &\equiv \big((\sim P) \wedge (\sim Q)\big) \vee \big((\sim P) \wedge R\big).\end{aligned}$$

Here, first we used the equivalence of implication. Then DeMorgan's law and double negation, and finally distribution law. This means that we can rewrite the negation as:

> n is not a multiple of 10 and n is not a multiple of 2, or
> n is not a multiple of 10 and n is a multiple of 5.

(d) This expression can be written as $P \wedge Q$ for

$$\begin{aligned}P:&\quad x \geq 3,\\ Q:&\quad x \leq 6.\end{aligned}$$

Then its negation is:

$$\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q).$$

Here we used DeMorgan's law. Therefore, the negation can be rewritten as:

$$x < 3 \text{ or } x > 6.$$

(e) This sentence can be written as $R \implies (P \vee Q)$ for

$$\begin{aligned}P:&\quad x \text{ is less than -2,}\\ Q:&\quad x \text{ greater than 2,}\\ R:&\quad x^2 \text{ is greater than 4..}\end{aligned}$$

Then its negation is:

$$\sim(R \Rightarrow (Q \vee P)) \equiv \sim(\sim R \vee (Q \vee P)) \equiv R \wedge ((\sim Q) \wedge (\sim P)).$$

Here, first we used the equivalence of implication. Then DeMorgan's law and double negation. Thus, the negation can be rewritten as:

> The square of a real number x is greater than 4 and
> x is greater than or equal to -2, and less than or equal to 2.

(f) This sentence can be written as $P \implies (Q \implies R)$ for

$$P:\quad f \text{ is differentiable everywhere,}$$

$$\begin{aligned} Q: &\quad x \text{ is a local maximum of } f, \\ R: &\quad f'(x) = 0. \end{aligned}$$

Here, first we used the equivalence of implication. Then DeMorgan's law and double negation. Then its negation is:

$$\sim(P \Rightarrow (Q \Rightarrow R)) \equiv \sim(\sim P \vee (\sim Q \vee R)) \equiv P \wedge (Q \wedge (\sim R)).$$

Here, first we used the equivalence of implication twice. Then DeMorgan's law and double negation. Hence, the negation can be rewritten as:

A function f is differentiable everywhere and $x \in \mathbb{R}$ is a local maximum of f, but $f'(x) \neq 0$.

4.3.3.

(a) Using the biconditional equivalence, we have

$$(P \iff Q) \equiv ((P \implies Q) \wedge (Q \implies P)).$$

Then using the contrapositive equivalence, we have

$$((P \implies Q) \wedge (Q \implies P)) \equiv \big(((\sim Q) \implies (\sim P)) \wedge ((\sim P) \implies (\sim Q))\big)$$

and by commutativity, this is logically equivalent to

$$((\sim P) \implies (\sim Q)) \wedge ((\sim Q) \implies (\sim P)).$$

Finally, by the biconditional equivalence, this is logically equivalent to

$$(\sim P) \iff (\sim Q)$$

(b) By the implication equivalence and the double negation equivalence, we have

$$(P \implies (\sim Q \implies R)) \equiv \big(P \implies ((\sim(\sim Q)) \vee R)\big) \equiv (P \implies (Q \vee R)).$$

(c) By the implication equivalence and DeMorgan's law,

$$((P \vee Q) \implies R) \equiv ((\sim(P \vee Q)) \vee R) \equiv \big(((\sim P) \wedge (\sim Q)) \vee R\big).$$

Then using commutativity and then the distribution law, this is logically equivalent to

$$\big(R \vee ((\sim P) \wedge (\sim Q))\big) \equiv \big((R \vee (\sim P)) \wedge (R \vee (\sim Q))\big)$$

Then by commutativity and then the implication equivalence, this is logically equivalent to

$$\big(((\sim P) \vee R) \wedge ((\sim Q) \vee R)\big) \equiv ((P \implies R) \wedge (Q \implies R)).$$

5 · More proofs
5.5 · Exercises

5.5.1.

Proof. (Proof by contrapositive) Assume that n is odd. Then we see that $n = 2k + 1$ for some $k \in \mathbb{Z}$. Thus, $n^2 + 4n + 5 = (2k + 1)^2 + 4(2k + 1) + 5 = 2(2k^2 + 6k + 5)$. Since $(2k^2 + 6k + 5) \in \mathbb{Z}$, we see that $n^2 + 4n + 5$ is even. ■

Notice that we have started our proof with an indication to the reader that we are using the contrapositive. Always make things easier for your reader.

5.5.2.

Proof. Let $n \in \mathbb{Z}$, and suppose $5 \mid n$. Then $n = 5k$ for some $k \in \mathbb{Z}$, and so

$$n^2 = (5k)^2 = 5(5k^2).$$

Since $k \in \mathbb{Z}$, we also have $5k^2 \in \mathbb{Z}$, and so we see that $5 \mid n^2$. ■

5.5.3.

Proof. We are going to use contrapositive proof. First, let $n \in \mathbb{Z}$. Then, we see that the contrapositive is:

$$\text{if } 10 \mid n, \text{ then } 5 \mid n \text{ and } 2 \mid n,$$

where we used the definition of the contrapositive and DeMorgan's Law.

Now, assume that $10 \mid n$. Then, we see that $n = 10k$ for some $k \in \mathbb{Z}$. This implies that $n = 5(2k) = 2(5k)$. Since both $2k, 5k \in \mathbb{Z}$, we conclude $5 \mid n$ and $2 \mid n$. So, the result follows. ■

5.5.4.

Proof. We prove the contrapositive. Let $n, m \in \mathbb{N}$ and suppose that $n|m$ and $n|(m + 2)$. Then there exists $x, y \in \mathbb{N}$ so that $m = xn$ and $m + 2 = yn$. Combining these we have $xn + 2 = yn$, and rearranging gives $n(y - x) = 2$ so that $n|2$. However the only natural numbers dividing 2 are 1 and 2 so that either $n = 1$ or $n = 2$. ■

5.5.5.

Proof. (Proof by contrapositive) Assume n, m have opposite parities. The we have two cases; either n is even and m is odd, or m is even and n is odd. Since the statement and the cases are symmetric with respect to n and m, WLOG we can assume n is even and m is odd. In this case, we can write $n = 2a$ and $m = 2b + 1$ for some $a, b \in \mathbb{Z}$. Hence, $n^2 + m^2 = (2a)^2 + (2b + 1)^2 = 4a^2 + 4b^2 + 4b + 1 = 2(2a^2 + 2b^2 + 2b) + 1$. Since $(2a^2 + 2b^2 + 2b) \in \mathbb{Z}$, we see that $n^2 + m^2$ is odd. ■

Notice that we have used the standard contraction "WLOG" to mean "without loss of generality". If you are unsure if your reader knows such an abbreviation, then you should explain it to them, say with a quick comment in brackets or a footnote.

5.5.6.

Proof. We prove the contrapositive. Let $x \in \mathbb{R}$, and suppose that $x \leq 0$. Then

$$x^2 + 1 > 0$$

and both $x^3 \leq 0$ and $5x \leq 0$. Putting this all together, we have

$$x^3 + 5x \leq 0 < x^2 + 1$$

and hence $x^3 + 5x < x^2 + 1$. ■

5.5.7.

Proof. We proceed by contrapositive. Without loss of generality, assume that $b > a$. Assume that a and b are consecutive, so that $b = a + 1$. We compute,

$$a + b = a + (a + 1) = 2a + 1.$$

Since $a \in \mathbb{Z}$, we see that $a + b$ fits the definition of an odd integer, as desired. ■

5.5.8.

Proof. Assume n is an even integer. Then we see $n = 2a$ for some $a \in \mathbb{Z}$. Since $a \in \mathbb{Z}$ we see that a is either even or odd.

- *Case 1:* Assume a is even. So in this case, we see that $a = 2m$ for some $m \in \mathbb{Z}$. Thus, $n = 2a = 2(2m) = 4m$ for some integer m.
- *Case 2:* Now assume that a is odd. We see that $a = 2t + 1$ for some $t \in \mathbb{Z}$.Thus, $n = 2a = 2(2t + 1) = 4t + 2$ for some integer t.

Therefore, if n is an even integer than $n = 4k$ or $n = 4k + 2$ for some integer k. ■

5.5.9.

Proof. Let $n \in \mathbb{Z}$. We prove each implication in turn.

- ($\Rightarrow$): Assume that $2 \mid (n^4 - 7)$. Then we see that $n^4 - 7 = 2k$ for some $k \in \mathbb{Z}$. Thus, we see that $n^4 = 2(k + 3) + 1$. Since $(k + 3) \in \mathbb{Z}$, we see that n^4 is odd. Since the square of an even number is even and $n^4 = n^2 n^2$, we see that n^2 is odd. Similarly, we see that n is odd. Thus $n = 2m + 1$ for some $m \in \mathbb{Z}$. Hence, $n^2 + 3 = (4m^2 + 4m + 1) + 3 = 4(m^2 + m + 1)$ and since $(m^2 + m + 1) \in \mathbb{Z}$, we see that $4 \mid (n^2 + 3)$.
- ($\Leftarrow$): Assume that $4 \mid (n^2 + 3)$. Then we see that $n^2 + 3 = 4k$ for some $k \in \mathbb{Z}$. Thus, we see that $n^2 = 2(2k - 2) + 1$. Since $(2k - 2) \in \mathbb{Z}$, we see that n^2 is odd. Moreover, since the product of two odd numbers is odd, we see that $n^4 = n^2 n^2$ is odd. Thus $n^4 = 2m + 1$ for some $m \in \mathbb{Z}$. Hence, $n^4 - 7 = 2m - 6 = 2(m - 3)$ and since $(m - 3) \in \mathbb{Z}$, we see that $2 \mid (n^4 - 7)$.

Since both implications are true, the biconditional holds. ■

5.5.10.

Proof. This is a biconditional statement. Hence, we need to prove both implications:

$$(3 \mid 5a) \implies (3 \mid a) \qquad \text{and } (3 \mid a) \implies (3 \mid 5a).$$

- ($\Leftarrow$): Assume that $3 \mid a$. Then we see that $a = 3k$ for some $k \in \mathbb{Z}$. Hence, $5a = 5(3k) = 3(5k)$. Since $5k \in \mathbb{Z}$, we get $3 \mid 5a$.
- ($\Rightarrow$): Assume that $3 \mid 5a$. This implies $5a = 3m$ for some $m \in \mathbb{Z}$. Therefore we see that by adding a to both sides, we get $6a = 3m + a$. Thus, $a = 6a - 3m = 3(2a - m)$. Since $(2a - m) \in \mathbb{Z}$, we see $3 \mid a$.

Therefore the result follows. ■

Notice that we have indicated to the reader which implication we are proving by starting with ($\Leftarrow$) or ($\Rightarrow$). This is a very simple useful way to help tell the reader what is going on. Always make life easier for your reader.

5.5.11.

Proof. Let $n \in \mathbb{Z}$. We consider the cases that n is odd and n is even. First suppose that n is even, so that $n = 2k$ for some $k \in \mathbb{N}$. Then

$$n^2 + 2n = (2k)^2 + 2(2k) = 4k^2 + 4k = 4(k^2 + k)$$

and so

$$(n^2 - 1)(n^2 + 2n) = 4(k^2 + k)(n^2 - 1).$$

Since $k, n \in \mathbb{Z}$, $(k^2 + k)(n^2 - 1) \in \mathbb{Z}$, and thus $(n^2 - 1)(n^2 + 2n)$ is divisible by 4.

Next suppose that n is odd, so that $n = 2k + 1$ for some $k \in \mathbb{N}$. Then

$$n^2 - 1 = (2k + 1)^2 - 1 = 4k^2 + 4k + 1 - 1 = 4(k^2 + k)$$

and so

$$(n^2 - 1)(n^2 + 2n) = 4(k^2 + k)(n^2 + 2n).$$

Since $k, n \in \mathbb{Z}$, $(k^2 + k)(n^2 + 2n) \in \mathbb{Z}$, and thus $(n^2 - 1)(n^2 + 2n)$ is divisible by 4.

We have proved the statement in all cases, and so the statement holds for all $n \in \mathbb{N}$. ■

5.5.12.

Proof. We proceed by contrapositive. We see that thee contrapositive of the statement is: If x and y have the same parity, then $x + y$ is even. Now assume that x and y are both even or both odd. Then we have 2 cases:

- Case 1: Assume that both x and y are even. Then $x = 2n$ and $y = 2m$ for some $n, m \in \mathbb{Z}$. Therefore,

$$x + y = 2n + 2m = 2(n + m).$$

Since $n, m \in \mathbb{Z}$ and the sum of integers is also an integer, we see that $n + m \in \mathbb{Z}$, so that $x + y$ fits the definition of an even number, as required.

- Case 2: Assume that x and y are both odd. Then $x = 2j+1$ and $y = 2k+1$ for some $j, k \in \mathbb{Z}$. Then

$$x + y = 2j + 1 + 2k + 1 = 2(j + k + 1).$$

Since $j, k, 1 \in \mathbb{Z}$ and the sum of integers is an integer, $j + k + 1 \in \mathbb{Z}$, and we see that $x + y$ is even, as desired.

■

5.5.13.

Proof. We prove the contrapositive. Let $n \not\equiv 1 \pmod 3$. Then we see that we have 2 cases: $n = 3k$ or $n = 3k + 2$ for some $k \in \mathbb{Z}$.

- *Case 1:* $n = 3k$ for some $k \in \mathbb{Z}$. In this case, we see that

$$n^2 + 4n + 1 = (3k)^2 + 4(3k) + 1 = 9k^2 + 12k + 1 = 3(3k^2 + 4k) + 1.$$

Hence, since $3k^2 + 4k \in \mathbb{Z}$, we see that $3 \nmid (n^2 + 4n + 1)$.

- *Case 2:* $n = 3k + 2$ for some $k \in \mathbb{Z}$. In this case, we see that

$$n^2+4n+1 = (3k+2)^2+4(3k+2)+1 = 9k^2+12k+4+12k+8+1 = 3(3k^2+8k+4)+1.$$

Since $3k^2 + 8k + 4 \in \mathbb{Z}$, this implies $3 \nmid (n^2 + 4n + 1)$.

Hence the contrapositive holds, and therefore, we see that if $3 \mid (n^2 + 4n + 1)$, then $n \equiv 1 \pmod 3$. ■

5.5.14.

Proof. Assume that $5 \nmid m$. By the division algorithm, there are four cases for m: $m = 5k + 1$, $m = 5k + 2$, $m = 5k + 3$, or $m = 5k + 4$ for some $k \in \mathbb{Z}$.

- *Case 1*: $m = 5k+1$ for some $k \in \mathbb{Z}$. In this case, we have $m^2 = 25k^2+10k+1$. Thus, we see $m^2-1 = 5(5k^2+2k)$. Since $(5k^2+2k) \in \mathbb{Z}$, we see $5 \mid (m^2-1)$, that is $m^2 \equiv 1 \pmod 5$.

- *Case 2*: $m = 5k+2$ for some $k \in \mathbb{Z}$. In this case, we have $m^2 = 25k^2+20k+4$. Thus, we see $m^2 + 1 = 5(5k^2 + 2k + 1)$. Since $(5k^2 + 2k + 1) \in \mathbb{Z}$, we see $5 \mid (m^2 + 1)$, that is $m^2 \equiv -1 \pmod 5$.

- *Case 3*: $m = 5k+3$ for some $k \in \mathbb{Z}$. In this case, we have $m^2 = 25k^2+30k+9$. Thus, we see $m^2 + 1 = 5(5k^2 + 2k + 2)$. Since $(5k^2 + 2k + 2) \in \mathbb{Z}$, we see $5 \mid (m^2 + 1)$, that is $m^2 \equiv -1 \pmod 5$.

- *Case 4*: $m = 5k + 4$ for some $k \in \mathbb{Z}$. In this case, we have $m^2 = 25k^2 +$

$40k + 16$. Thus, we see $m^2 - 1 = 5(5k^2 + 2k + 3)$. Since $(5k^2 + 2k + 3) \in \mathbb{Z}$, we see $5 \mid (m^2 - 1)$, that is $m^2 \equiv 1 \pmod 5$.

Therefore, if $5 \nmid m$, then $m^2 \equiv 1 \pmod 5$ or $m^2 \equiv -1 \pmod 5$. ■

5.5.15.

Proof. Assume that $3 \nmid q$. Then we have two possible cases, $q \equiv 1 \pmod 3$ or $q \equiv 2 \pmod 3$.

- Case 1: $q \equiv 1 \pmod 3$. Then by definition, $q = 3k + 1$ for some $k \in \mathbb{Z}$. Squaring q, we see that

$$q^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1.$$

  Because $3k^2 + 2k \in \mathbb{Z}$, we conclude that $q^2 \equiv 1 \pmod 3$.

- Case 2: $q \equiv 2 \pmod 3$. Then by definition, $q = 3k + 2$ for some $k \in \mathbb{Z}$. Squaring q, we see that

$$q^2 = (3k+2)^2 = 9k^2 + 12k + 4 = (9k^2 + 12k + 3) + 1 = 3(3k^2 + 4k + 1) + 1.$$

  Because $3k^2 + 4k + 1 \in \mathbb{Z}$, we conclude that $q^2 \equiv 1 \pmod 3$.

Since we have verified both possible cases, we deduce that if $3 \nmid q$, then $q^2 \equiv 1 \pmod 3$. ■

5.5.16. We see that this is a conditional statement, so we can assume the hypothesis and try to show the conclusion. This means that we assume that $n \in \mathbb{Z}$ and show that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9. Since the hypothesis is very broad, this suggests that we use proof by cases; and since the result is about divisibility by 9, this suggests that we use 9 cases using division algorithm. Oof! That sounds ugly.

Instead, let's try to rewrite what we want to show and see if it can be simplified. Notice that

$$\begin{aligned} n^3 + (n+1)^3 + (n+2)^3 &= n^3 + (n^3 + 3n^2 + 3n + 1) + (n^3 + 6n^2 + 12n + 8) \\ &= 3n^3 + 9n^2 + 15n + 9 \\ &= 3(n^3 + 5n) + 9(n^2 + 1) \\ &= 3n(n^2 + 5) + 9(n^2 + 1). \end{aligned}$$

This means that all we need to do is show that $3n(n^2 + 5) + 9(n^2 + 1) = 9k$ for some $k \in \mathbb{Z}$. In fact since one of the terms on the LHS is already a multiple of 9, we only need to show that $3n(n^2 + 5) = 9t$ for some $t \in \mathbb{Z}$, i.e. $3 \mid n(n^2 + 5)$. Even though this still requires cases, we now only need 3. Let's see how we can make use of this in the proof.

Proof. Let $n \in \mathbb{Z}$. Then, we have

$$n^3 + (n+1)^3 + (n+2)^3 = n^3 + (n^3 + 3n^2 + 3n + 1) + (n^3 + 6n^2 + 12n + 8)$$

$$\begin{aligned} &= 3n^3 + 9n^2 + 15n + 9 \\ &= 3(n^3 + 5n) + 9(n^2 + 1) \\ &= 3n(n^2 + 5) + 9(n^2 + 1). \end{aligned}$$

We see that $9 \mid 9(n^2 + 1)$. Hence, to show $9 \mid \left(n^3 + (n + 1)^3 + (n + 2)^3\right)$, it is enough to show $9 \mid 3n(n^2 + 5)$, or equivalently $3 \mid n(n^2 + 5)$. To prove that we use cases.

- Case 1: $n \equiv 0 \pmod 3$: In this case, we see that $n(n^2 + 5) \equiv 0 \pmod 3$, which implies that $3 \mid n(n^2 + 5)$.
- Case 2: $n \equiv 1 \pmod 3$: In this case, we see that $n^2 \equiv 1 \pmod 3$, which implies that $(n^2 + 5) \equiv 0 \pmod 3$. Thus, $3 \mid n(n^2 + 5)$.
- Case 3: $n \equiv 2 \pmod 3$: In this case, we see that $n^2 \equiv 4 \equiv 1 \pmod 3$, which implies that $(n^2 + 5) \equiv 0 \pmod 3$. Thus, $3 \mid n(n^2 + 5)$.

Therefore, we see that $3 \mid n(n^2+5)$ for all n, and thus, for every integer $n \geq 0$, the sum $n^3 + (n + 1)^3 + (n + 2)^3$ is divisible by 9. ■

In our proof here we have proved a small additional result to help us prove the main result. We could have, instead, made that small result a separate lemma and proved it separately. Both approaches are common practice.

5.5.17.

Proof. Let $a \in \mathbb{Z}$. By Euclidean division, the number a can be written uniquely as $a = 5k + r$ with $r \in \{0, 1, 2, 3, 4\}$. This means that

$$a \equiv r \pmod 5 \qquad \text{with } r \in \{0, 1, 2, 3, 4\}\,.$$

This means that we can consider five cases — one for each value of r:

- $a \equiv 0 \pmod 5$: In this case, we see that $a^5 \equiv 0^5 \equiv 0 \equiv a \pmod 5$.
- $a \equiv 1 \pmod 5$: In this case, we see that $a^5 \equiv 1^5 \equiv 1 \equiv a \pmod 5$.
- $a \equiv 2 \pmod 5$: In this case, we see that $a^5 \equiv 2^5 \equiv 32 \equiv 2 \equiv a \pmod 5$.
- $a \equiv 3 \pmod 5$: In this case, we see that $a^5 \equiv 3^5 \equiv 243 \equiv 3 \equiv a \pmod 5$.
- $a \equiv 4 \pmod 5$: In this case, we see that $a^5 \equiv 4^5 \equiv 1024 \equiv 4 \equiv a \pmod 5$.

In each case $a^5 \equiv a \pmod 5$ as required. ■

This result generalises to any prime number, p:

$$a^p \equiv a \pmod p$$

and is called Fermat's little theorem. With a little work it can be proved using induction. One can also, with a little work, turn this into a very good test of whether or not a number is prime — the interested reader should search-engine their way to the Miller-Rabin test.

5.5.18.

Proof. We see that the statement involves the expressions $|x+4|$ and $|x-3|$. Thus, we need to understand when the expressions $(x+4)$ and $(x-3)$ change signs. To do that, we need to consider three cases: $x < -4$, $-4 \leq x \leq 3$, and $x > 3$.

- *Case 1*: Let $x < -4$. In this case, we see that $|x+4| = -x-4$ and $|x-3| = 3-x$. Therefore, $|x+4|+|x-3| = (-x-4)+(3-x) = -2x-1$. Moreover, since $x < -4$, we see $-2x-1 \geq 7$ which implies $|x+4|+|x-3| \geq 7$.
- *Case 2*: Let $-4 \leq x \leq 3$. In this case, we see that $|x+4| = x+4$ and $|x-3| = 3-x$. Therefore, $|x+4|+|x-3| = (x+4)+(3-x) = 7$. Hence, $|x+4|+|x-3| \geq 7$.
- *Case 3*: Finally, assume that $x > 3$. In this case, we see that $|x+4| = x+4$ and $|x-3| = x-3$. Therefore, $|x+4|+|x-3| = (x+4)+(x-3) = 2x+1$. Moreover, since $x > 3$, we see $2x+1 \geq 7$ which implies $|x+4|+|x-3| \geq 7$.

Therefore if $x \in \mathbb{R}$, then $|x+4|+|x-3| \geq 7$. ■

5.5.19.

Proof. Let $x \in \mathbb{R}$, and suppose that $|x-1| < 1$. Then

$$|x^2-1| = |(x-1)(x+1)| = |x-1| \cdot |x+1|.$$

Using the inequality $|x-1| < 1$, we therefore have

$$|x^2-1| < 1 \cdot |x+1|.$$

However,using the triangle inequality, and again the bound $|x-1| < 1$, we have

$$|x+1| = |(x-1)+2| \leq |x-1|+|2| < 1+2 = 3.$$

Putting everything together, we see that $|x^2-1| < 3$, as required. ■

5.5.20.

Proof. Let $x \in \mathbb{R}$, and suppose that $|x-2| < 1$. Note that this inequality implies that

$$-1 < x-2 < 1.$$

Adding 2 to everything, we end up with

$$1 < x < 3$$

which implies that $|x| < 3$.

Furthermore, note that

$$|2x^2-3x-2| = |(2x+1)(x-2)| = |2x+1| \cdot |x-2|.$$

Using the inequality $|x-2|<1$, we therefore have

$$|2x^2-3x-2|<|2x+1|\cdot 1.$$

However, using the triangle inequality, and the bound $|x|<3$ we already established, we have

$$|2x+1|\leq |2x|+1=2|x|+1<2\cdot 3+1=7.$$

Thus

$$|2x^2-3x-2|<|2x+1|\cdot 1<7\cdot 1=7.$$

■

5.5.21.

Proof. Assume that $x,y\in\mathbb{R}$. Recall the triangle inequality, which states: For any $a,b\in\mathbb{R}$, $|a+b|\leq |a|+|b|$. We set $a=x-y$ and $b=y$. Since $x,y\in\mathbb{R}$, $x-y\in\mathbb{R}$, so a and b satisfy the hypothesis of the triangle inequality. Plugging our values of a and b into the triangle inequality yields,

$$|x-y+y|\leq |x-y|+|y|.$$

Subtracting $|y|$ from both sides gives

$$|x|-|y|\leq |x-y|. \tag{1}$$

We use the triangle inequality again. This time setting $a=x$ and $b=y-x$. Again, since x and $y-x$ are real numbers, the hypothesis of the triangle inequality is satisfied, so we obtain

$$|x+y-x|\leq |x|+|y-x|.$$

Subtracting $|x|$ from each side, we see $|y|-|x|\leq |y-x|$. Dividing both sides by -1 yields

$$|x|-|y|\geq -|y-x|=-|x-y|. \tag{2}$$

Combining equations (1) and (2), we obtain the desired result

$$-|x-y|\leq |x|-|y|\leq |x-y|$$

■

5.5.22.

Proof. Let $x,y\in(0,\infty)$ and suppose that $x\leq y$. Since both x and y and positive, we can divide this equation by xy to obtain

$$f(y)=\frac{1}{y}\leq \frac{1}{x}=f(x).$$

Therefore f is decreasing. ■

We could have instead restricted f to the domain (a, ∞) for any $a > 0$, or to the domain $(-\infty, b)$ for any $b < 0$, and also obtained a decreasing function. So even though the function is decreasing on $(-\infty, 0)$ and also on $(0, \infty)$, it is not decreasing; in order to show that a function is decreasing, we have to look at the function and its domain as a whole.

6 · Quantifiers
6.6 · Exercises
6.6.1.

Proof. Let $n \in \mathbb{Z}$. Notice that $n^3 - n = n(n-1)(n+1)$, so it is sufficient to show that $3 \mid n(n-1)(n+1)$. By division algorithm we see that we have 3 cases: $n = 3k$, $n = 3k+1$ or $n = 3k+2$ for some $k \in \mathbb{Z}$.

- Case 1: $n = 3k$. In this case we have $n^3 - n = n(n-1)(n+1) = 3k(3k-1)(3k+1)$. Then, since $k(3k-1)(3k+1) \in \mathbb{Z}$ we can conclude $3 \mid (n^3 - n)$.
- Case 2: $n = 3k+1$. Now we have $n^3 - n = n(n-1)(n+1) = (3k+1)(3k)(3k+2)$. Then, since $k(3k+1)(3k+2) \in \mathbb{Z}$ we get $3 \mid (n^3 - n)$.
- Case 3: $n = 3k+2$. Finally $n^3 - n = n(n-1)(n+1) = (3k+2)(3k+1)(3k+3) = 3(3k+2)(3k+1)(k+1)$. Then, since $(3k+2)(3k+1)(k+1) \in \mathbb{Z}$ we have $3 \mid (n^3 - n)$.

Hence, for all $n \in \mathbb{Z}$, $3 \mid (n^3 - n)$. ■

6.6.2.

Proof. Let $n, k \in \mathbb{Z}$. Assume $k \mid (2n+1)$ and $k \mid (4n^2+1)$. This implies that $2n+1 = ka$ and $4n^2+1 = kb$ for some $a, b \in \mathbb{Z}$. Combining these equations gives

$$\begin{aligned} k^2a^2 &= (2n+1)^2 = 4n^2 + 4n + 1 \\ &= kb + 4n = kb + 2(ka - 1) \qquad \text{since } 2n = ka - 1. \end{aligned}$$

Hence, if we group all the k terms together, we get

$$2 = kb + 2ka - k^2a^2 = k(b + 2a - ka^2).$$

But this tells us that $k \mid 2$. The only divisors of 2 are $\pm 1, \pm 2$.

However, since $k \mid 2n+1$, we know that k must be odd (if it were even, then $2n+1$ would have to be even). Therefore k must be an odd divisor of 2, that is $k = \pm 1$. ■

6.6.3.

Proof. We are going to use proof by contrapositive. We see that the expression $\forall n \in \mathbb{Z}$, comes before the conditional statement starts, and hence is not a part of it. Thus, we see that the contrapositive of the statement is:

$$\forall n \in \mathbb{Z}, \text{ if } n \equiv 3 \pmod 4, \text{ then } \forall a, b \in \mathbb{Z}, \text{ we have } a^2 + b^2 \neq n.$$

Now, let $n \in \mathbb{Z}$ and assume that $n \equiv 3 \pmod 4$. To prove the statement, we are going to use 3 cases: a and b are both even, a and b are both odd, and only one of a or b is even.

- *Case 1: a and b are both even:* In this case, we know that $a = 2k$ and $b = 2\ell$ for some $k, \ell \in \mathbb{Z}$. Hence, we see that $a^2 = 4k^2$ and $b^2 = 4\ell^2$. Thus, $a^2 + b^2 = 4k^2 + 4\ell^2 = 4(k^2 + \ell^2)$. Since $k^2 + \ell^2 \in \mathbb{Z}$ we see $4 \mid (a^2 + b^2)$. Therefore $(a^2 + b^2) \equiv 0 \pmod 4$.

- *Case 2: a and b are both odd:* Then, we know that $a = 2k + 1$ and $b = 2\ell + 1$ for some $k, \ell \in \mathbb{Z}$. In this case, we see that $a^2 = 4k^2 4k + 1$ and $b^2 = 4\ell^2 + 4\ell + 1$. Thus, $a^2 + b^2 = 4k^2 + 4k + 1 + 4\ell^2 + 4\ell + 1 = 4(k^2 + k + \ell^2 + \ell) + 2$. Since $k^2 + k + \ell^2 + \ell \in \mathbb{Z}$ we see $4 \mid (a^2 + b^2 - 2)$. Therefore $(a^2 + b^2) \equiv 2 \pmod 4$.

- *Case 3: only one of a or b is even:* For this case, WLOG, we can assume that a is odd and b is even. Then, we know that $a = 2k + 1$ and $b = 2\ell$ for some $k, \ell \in \mathbb{Z}$. This implies that $a^2 = 4k^2 4k + 1$ and $b^2 = 4\ell^2$. Thus, $a^2 + b^2 = 4k^2 + 4k + 1 + 4\ell^2 = 4(k^2 + k + \ell^2) + 1$. Since $k^2 + k + \ell^2 + \in \mathbb{Z}$ we see $4 \mid (a^2 + b^2 - 1)$. Therefore $(a^2 + b^2) \equiv 1 \pmod 4$.

Hence, there does not exist $a, b \in \mathbb{Z}$, where $(a^2 + b^2) \equiv 3 \pmod 4$. Therefore there does not exist $a, b \in \mathbb{Z}$, where $a^2 + b^2 = n$. ■

6.6.4.

Proof. We are going to use proof by contrapositive. Let $a, b \in \mathbb{Z}$. Assume that $3 \nmid a$ or $3 \nmid b$. WLOG we can assume $3 \nmid a$. Then we see that we have 2 cases $a = 3k + 1$ or $a = 3k + 2$ for some $k \in \mathbb{Z}$.

- *Case 1:* Let $a = 3k+1$ for some $k \in \mathbb{Z}$, then we see that $a^2 = 9k^2+6k+1 = 3(3k^2 + 2k) + 1$.

- *Case 2:* Now let $a = 3k + 2$ for some $k \in \mathbb{Z}$. Then $a^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$.

Thus, we see that since $3 \nmid a$, we have $a^2 = 3m + 1$ for some $m \in \mathbb{Z}$.

Now, either $3 \mid b$ or $3 \nmid b$.

- When $3 \mid b$, $b = 3\ell$ for some $\ell \in \mathbb{Z}$, and so $a^2 + b^2 = 3m + 9\ell^2 + 1 = 3(m + 3\ell^2) + 1$.

- On the other hand, when $3 \nmid b$, then, by the reasoning above, we get $b^2 = 3\ell + 1$ for some $\ell \in \mathbb{Z}$. And so $a^2 + b^2 = 3(m + \ell) + 2$.

In both cases, we see that $3 \nmid (a^2 + b^2)$ as required. ■

6.6.5.

Proof. Let $n = 4$, $a = 2$, and $b = 2$. Since $4 \mid 4$,

$$n \mid a \cdot b$$

but $4 \nmid 2$ and so $n \nmid a$ and $n \nmid b$. Therefore we see that the statement, if $n \mid ab$, then $n \mid a$ or $n \mid b$, does not hold for all $n, a, b, \in \mathbb{Z}$. ■

6.6.6.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, and let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x - x^2$. Then for any $x \in \mathbb{R}$,

$$f(x) = x^2 = (-x)^2 = f(-x)$$

so f is even, not odd. Moreover, $(f + g)(x) = x^2 + (x - x^2) = x$. Then

$$(f + g)(x) = x = -(f + g)(-x)$$

and so $f + g$ is odd. Therefore just because the sum of two functions is an odd function, it does not follow that those two functions are odd functions. ■

6.6.7.

(a) True. We just need to find one example of x, y so that $x + y = 3$. So take $x = 0, y = 3$.

(b) False. We show the negation is true; the negation is $\forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}$ s.t. $x + y \neq 3$. We must be careful to choose x before we choose y. So, let x be any integer, and then set $y = -x$. Then $x + y = 0 \neq 3$. Since the negation is true, the original is false.

(c) True. Let x be any integer and then pick $y = 3 - x$. Then $x + y = 3$.

(d) False. We show the negation is true; the negation is $\exists x \in \mathbb{Z}$ s.t. $\exists y \in \mathbb{Z}$ s.t. $x + y \neq 3$. So we only need one example of x, y so that $x + y \neq 3$. Pick $x = 3, y = 7$ then $x + y \neq 3$. Since the negation is true, the original is false.

6.6.8.

(a) True. We only need an example of x, y so that $x^2 < y$. Take $x = 0, y = 1$.

(b) False. We show the negation is true; the negation is $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $x^2 \geq y$. Let x be any real number, then we know $x^2 \geq 0$, so set $y = 0$. Then we must have $x^2 \geq 0 = y$ as required. Since negation is true, original is false.

(c) True. Let x be any real number and then pick $y = x^2 + 1$ (remember we must pick y after we pick x. Then we have $x^2 < x^2 + 1 = y$ as required.

(d) False. We show the negation is true; the negation is $\exists x \in \mathbb{R}$ s.t. $\exists y \in \mathbb{R}$ s.t. $x^2 \geq y$. So we only need an example of x, y. Pick $x = 1, y = 0$ then

$x^2 = 1 > 0 = y$. Since the negation is true, the original is false.

6.6.9. The fact that 2 is the only even prime really helps to solve this problem.

(a) *Disproof*: This statement is false. The negation of the statement is

$$\exists x \in P \text{ s.t. } \exists y \in P \text{ s.t. } x + y \notin P.$$

So we to prove the negation is true we simply need an example of x, y. That example is then a counterexample to the original. Take $x = y = 3 \in P$. Then we see that $x + y = 6 \notin P$. Notice that there are many such counter examples; one can take any pair of odd primes.

(b) *Disproof*: This statement is false. The negation of the statement is

$$\exists x \in P \text{ s.t. } \forall y \in P \text{ s.t. } x + y \notin P.$$

Take $x = 7 \in P$, and let $y \in P$. Either y is even or y is odd.

- When y is even, it must be 2, but then $x + y = 9 \notin P$.
- When y is odd, $x + y > 2$ is even, and so not prime.

In either case the negation is true, and so the original statement is false.

(c) *Disproof*: This statement is false. The negation of the statement is

$$\forall x \in P, \exists y \in P \text{ s.t. } x + y \notin P.$$

We can use a similar proof to the previous statement. Let $x \in P$, then either $x = 2$ or x is odd.

- *Case 1:* $x = 2$: In this case, we can pick $y = 7 \in P$, and get $x + y = 9 \notin P$.
- *Case 2: x is odd*: In this case, we can take $y = x$. This implies that $x + y > 2$ is even and thus, $x + y \notin P$.

Therefore the negation of the statement is true, and hence, the original statement is false.

(d) *Proof*: This statement is true. As an example, we can take any $x = 2 \in P$, and $y = 3 \in P$. Then we see that $x + y = 5 \in P$.

6.6.10.

(a) $\forall a \in A$, $\exists b \in B$ such that $a + b \in C$.

This statement is true.

Let $a \in A$. Then we see that $a = 3k$ for some $k \in \mathbb{Z}$. Thus, since $k = 2n$ or $k = 2n+1$ for some $n \in \mathbb{Z}$, we see that $a = 6n$ or $a = 6n+3$ for some $n \in \mathbb{Z}$. Therefore if we pick $b = 1 \in B$, we get, $a + b = 6n + 1$ or $a + b = 6n + 4$.

Hence, $6 \nmid (a+b)$.

(b) $\exists a \in A$ such that $\forall b \in B$, $a+b \in C$.

This statement is false.

It helps to write out the negation: $\forall a \in A, \exists b \in B$ such that $a+b \notin C$. Let a be any element of A, then, as we saw in part (a), we know that $a = 6n$ or $a = 6n+3$ for some $n \in \mathbb{Z}$.

- If $a = 6n$, then we can pick $b = 6 \in B$, and we get $a+b = 6n+6 = 6(n+1)$. Hence, $6 \mid (a+b)$, that is, $(a+b) \notin C$.
- Similarly, if $a = 6n+3$, we can pick $b = 3 \in B$, and get $a+b = 6n+6 = 6(n+1)$. Hence, $6 \mid (a+b)$, that is, $(a+b) \notin C$.

So in either case, we can pick $b \in B$ so that $a+b \notin C$. Since the negation is true, the original statement is false.

(c) $\forall a \in A$, $\forall b \in B$, $a+b \in C$.

This statement is false.

Again, it helps to write out the negation: $\exists a \in A$ such that $\exists b \in B$ such that $a+b \notin C$. So it suffices to find an example of a, b so that $a+b \notin C$. We can take $a = 3 \in A$ and $b = 3 \in B$; and get $a+b = 6$. This means $6 \mid (a+b)$. Thus, $(a+b) \notin C$. Since the negation is true, the original statement is false.

(d) $\exists a \in A$ and $\exists b \in B$ such that $a+b \in C$.

This statement is true.

Since both quantifiers are "there exists", we only need an example. We can take $a = 0 \in A$ and $b = 1 \in B$, and get $a+b = 1$ and hence, $6 \nmid (a+b)$. Thus, $(a+b) \in C$.

6.6.11.

(a) True. We only need an example, so pick $x = y = 1$. Then $xy = 1 > 0$ and $x+y = 2 > 0$. Since both the hypothesis and conclusion are true, the implication is true.

(b) True. Note that when the hypothesis is false, the implication is always true. By picking $x = 0$, then no matter what y is chosen, $xy = 0$, making the hypothesis false and the implication true.

(c) True. We can use a similar "trick" to the previous statement. Let x be any real number, and then pick $y = 0$. Then $xy = 0$, making the hypothesis false and so the implcation holds.

(d) False. The negation is $\exists x \in \mathbb{R}$ s.t. $\exists y \in \mathbb{R}$ s.t. $(xy > 0) \wedge (x+y \leq 0)$, so we only need an example of x, y. Pick $x = y = -1$. Then $xy = 1 > 0$ and

$x + y = -2 \leq 0$. Since the negation is true, the original is false.

(e) True. Let $x = 1$, and then let $y \in \mathbb{R}$. When $y < 0$, the hypothesis is false, since $xy = y < 0$, so the implication is true. While if $y \geq 0$, the hypothesis is true, and the conclusion is true, since $xy = y \geq 0$ and $x + y = 1 + y \geq 1 > 0$. Thus the statement holds.

(f) True. Let x be any real number and then set $y = 1$. When $x < 0$, the hypothesis is false since $xy = x < 0$, and thus the implication is true. On the other hand, when $x \geq 0$, the hypothesis and conclusion are both true since $xy = x \geq 0$ and $x + y = x + 1 \geq 1 > 0$. So the statement is true.

6.6.12.

(a) True. Let y, z be any real numbers, and then pick $x = z - y$ (which is a real). Then $x + y = z - y + y = z$ as required.

(b) False. The negation of the statement is $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ s.t. $\exists z \in \mathbb{R}$ s.t. $x + y \neq z$. So let x be any real number and then set $y = -x$ and $z = 1$. Then $x + y = x - y = 0 \neq 1 = z$. Since the negation is true, the original is false.

(c) True. Recall that when the hypothesis is false the implication is true. So let x be any real number and set $y = 1, z = 0$. Then $z = 0 < 1 = y$, so the hypothesis is false, making the implication true.

(d) False. The negation is $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}, \exists z \in \mathbb{R}$ s.t. $(z > y) \wedge (z \leq x + y)$. So pick $x = 2$ and then let y be any real number. Now set $z = y + 1$, then $z = y + 1 > y$ and $z = y + 1 \leq y + 2 = x + y$. Since the negation is true, the original is false.

6.6.13. Notice that to prove either (a) or (b) to be true, we must show that they work for any functions that satisfy the hypotheses. But to show they are false, it is sufficient to find counter-examples.

(a) Disproof: This statement is false. Consider the function $f(x) = \sin(x)$. We show that this is type A but not type B.

- To see that f is type A, let x be any real number. Since sine is a periodic function with period 2π and that $\sin(\pi/2) = 1$, we know that there is some point $y \in (x, x + 2\pi)$ so that $f(y) = 1$. Hence sine is a type A function.
- To see that f is not type B, first consider the negation of the definition. A function g is not type B when

$$\forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } (y \geq x) \wedge (|g(y)| < 1).$$

So let x be any real number, and then (using similar reasoning to the above), we know that there will be $y \in (x, x + 2\pi)$ so that $f(y) = \sin(y) = 0$ (just pick the smallest integer multiple of π that is larger than x). Thus sine is not type B.

(b) Proof. We show that any function that is type B is also type A. So assume that the function h is type B. Hence, there is some $t \in \mathbb{R}$ so that when $y \geq t$ then $|h(y)| \geq 1$. Now, let $x \in \mathbb{R}$, and then set $y = \max\{x, t\}$. Since h is type B and $y \geq t$, it follows that $|h(y)| \geq 1$. Thus h is type A.

6.6.14.

(a) $\exists x \in \mathbb{Z}$ such that $x > 84$ and $x \equiv 75 \pmod{84}$.

The negation of this sentence is:

"$\forall x \in \mathbb{Z}$ such that $x > 84$, $x \not\equiv 75 \pmod{84}$".

Or, equivalently,

"$\forall x \in \mathbb{Z}$, $x \leq 84$ or $x \not\equiv 75 \pmod{84}$".

We see that the original statement is true. For an example, we can take $x = 159$ (since $159 = 84 \times 1 + 75$).

(b) $\exists x, y \in \mathbb{Z}$ such that if $1 \geq x^2 \geq y^2$, then $x \geq y$.

The negation of this sentence is:

"$\forall x, y \in \mathbb{Z}$, $1 \geq x^2 \geq y^2$ and $y > x$".

We see that the original statement is true. For an example, we can take $x = y = 0$.

(c) $\forall z \in \mathbb{N}$, $\exists x, y \in \mathbb{Z}$ such that $z = x^2 + y^2$.

The negation of this sentence is:

"$\exists z \in \mathbb{N}$, $\forall x, y \in \mathbb{Z}$, $z \neq x^2 + y^2$".

We see that the negation of the statement is true. For an example, we can take $z = 3$. Then, there doesn't exist $x, y \in \mathbb{Z}$ such that $x^2 + y^2 = 3$. Since the negation is true, the original statement is false.

If we wanted to give a more formal solution to this problem, we could use the fact that the only way we can have $x^2 \leq 3$ is when $x \in \{-1, 0, 1\}$. Thus, in order for $x^2 + y^2 \leq 3$ we have only 9 possibilities to check, namely $x, y \in \{-1, 0, 1\}$, and none of them work.

(d) $\exists a \in \mathbb{R}$ such that $a > 0$ and $\forall x \in \mathbb{R}$, if $x \geq a$, then $2^{-x} < \dfrac{1}{100}$.

The negation of this sentence is:

"$\forall a \in \mathbb{R}$ such that $a > 0$, $\exists x \in \mathbb{R}$, $x \geq a$ and $2^{-x} \geq \dfrac{1}{100}$".

Or

"$\forall a \in \mathbb{R}$, $a \leq 0$; or $\exists x \in \mathbb{R}$, $x \geq a$ and $2^{-x} \geq \dfrac{1}{100}$".

We see that the original statement is true. For an example, we can take $a = 7$. Then we see that $\frac{1}{2^7} = \frac{1}{128} < \frac{1}{100}$, and whenever $x \geq 7$, we get $\frac{1}{2^x} \leq \frac{1}{2^7} < \frac{1}{100}$.

Notice that we are implicitly using the fact that the function 2^x is an increasing function of x, and so 2^{-x} is a decreasing function of x. This is not hard to prove. If we assume that $a < b$, then $\Delta = 2^b - 2^a = 2^a\left(2^{b-a} - 1\right)$. And since $b - a > 0$, we know that $2^{b-a} > 1$. Thus $\Delta > 0$, and so $2^b > 2^a$.

(e) $\forall n \in \mathbb{R}$, n is even if and only if n^2 is even.

The negation of this sentence is:

"$\exists n \in \mathbb{R}$ such that n is even but n^2 is not even; or n^2 is even but n is not even."

We see that the negation of the statement is true. For an example, we can take $n = \sqrt{2}$. Then, we see $n^2 = 2$ is even whereas n is not even since it is not an integer.

6.6.15.

(a) *Proof*: Let $a, b \in \mathbb{N}$. Assume that $b < a$. Thus, since $b^2 \geq 0$, we get $b - b^2 \leq b < a$.

(b) *Disproof*: Consider the negation:

$$\exists p \in \mathbb{N} \text{ such that } \exists q \in \mathbb{N} \text{ such that } \sqrt{\frac{p}{q}} \in \mathbb{N} \text{ and } \left(\sqrt{p} \notin \mathbb{N} \text{ or } \sqrt{q} \notin \mathbb{N}\right).$$

So we can prove that this is true using an example — that is a counter-example to the original statement. Take $p = 8$ and $q = 2$. Then we see that $\sqrt{\frac{p}{q}} = \sqrt{4} = 2 \in \mathbb{N}$, whereas $\sqrt{p} = \sqrt{8} \notin \mathbb{N}$ and $\sqrt{q} = \sqrt{2} \notin \mathbb{N}$.

Note that We haven't proved that $\sqrt{2}, \sqrt{8}$ are not integers, but we do so later in the book 11. For the moment, it is safe to assume this.

(c) *Proof*: Let $a, b \in \mathbb{R}$, then we can pick $c = a$ and $d = b$ and get $ab = cd$ and moreover, since $a = c$ and $b = d$, we see that the statement is true.

(d) *Disproof*: Consider the negation:

$$\exists a, b \in \mathbb{N} \text{ such that } \left(\exists x, y \in \mathbb{Z} \text{ and } \exists k \in \mathbb{N} \text{ such that } ax + by = k\right) \text{ and } (k \nmid a \vee k \nmid b).$$

Since all of these are existential quantifiers we can prove the negation is true by an example — this is just a counter-example to the original statement. Take $a = 2$ and $b = 3$ so that both are prime numbers. At the same time, taking $x = y = 1$ and $k = 5$ and get $ax + by = k$, but $5 \nmid 2$ and $5 \nmid 3$.

6.6.16.

Proof. Let $\varepsilon > 0$. Take $\delta = \varepsilon/3$, and suppose that $x \in \mathbb{R}$ so that $0 < |x-4| < \delta$. Then

$$|-3x+5-(-7)| = |-3x+12| = |-3(x-4)| = |-3||x-4| = 3|x-4|.$$

Using the inequality, $|x-4| < \delta$, we then have

$$|-3x+5-(-7)| = 3|x-4| < 3 \cdot \delta = 3 \cdot \frac{\varepsilon}{3} = \varepsilon$$

as required. ■

6.6.17.

Proof. Let $\varepsilon > 0$ be given. Take $\delta = \min\{1, \varepsilon/3\}$. Suppose that $x \in \mathbb{R}$ and $0 < |x-1| < \delta$. In particular, $|x-1| \leq 1$, and so

$$-1 \leq x - 1 \leq 1.$$

Adding 2 to everything, we end up with

$$1 \leq x + 1 \leq 3.$$

In particular, this implies that $|x+1| \leq 3$. Then using this inequality, and the inequality $|x-1| < \delta \leq \varepsilon/3$, we have

$$|x^2 - 1| = |x+1| \cdot |x-1| \leq 3\delta \leq 3 \cdot \frac{\varepsilon}{3} = \varepsilon.$$

■

6.6.18.

Proof. Let $\varepsilon > 0$. Take

$$N = \lceil 1/\sqrt{\varepsilon} \rceil$$

Then for $n > N$, we have

$$\left| \frac{1}{n^2} - 0 \right| < \frac{1}{N^2} < \varepsilon.$$

Hence

$$\lim_{n\to\infty} 1/n^2 = 0.$$

■

6.6.19.

Proof. Let $\varepsilon > 0$ be given. Take $\delta = \varepsilon/6$. Let $x \in \mathbb{R}$ such that $0 < |x-0| < \delta$. Since $|\sin(\theta)| \leq 1$ for all $\theta \in \mathbb{R}$, we have

$$|f(x) - 0| = \left| 6x \sin\left(\frac{1}{x}\right) \right| = |6x| \left| \sin\left(\frac{1}{x}\right) \right| \leq |6x|.$$

But then by choice of δ,

$$|f(x) - 0| \leq 6|x| = 6|x - 0| < 6\delta = 6 \cdot \frac{\varepsilon}{6} = \varepsilon.$$

Hence the function f converges to 0 as $x \to 0$. ■

6.6.20.

Proof. Let $\varepsilon = \frac{1}{2}$, then for any any $N \in \mathbb{N}$, so that $n > N$, set $n = \max\{N, 2\}+1$. Then

- If n is even, then $x_n = (-1)^n + \frac{1}{n} = 1 + \frac{1}{n} > 1$.
- While, if n is odd then $x_n = (-1)^n + \frac{1}{n} = -1 + \frac{1}{n} \leq -\frac{2}{3}$

In either case $|x_n - 0| = |x_n| > \frac{2}{3} > \frac{1}{2} = \varepsilon$. Therefore, the sequence $(x_n)_{n\in\mathbb{N}} = \left((-1)^n + \frac{1}{n}\right)_{n\in\mathbb{N}}$ does not converge to 0. ■

6.6.21.

Proof. Let $\varepsilon > 0$ be given. Let $N = \left\lceil \sqrt{\frac{5}{\varepsilon}} \right\rceil$. Note this implies that $N^2 \geq 5/\varepsilon$. Now suppose that $n \in \mathbb{N}$, $n > N$. Since $n = \sqrt{n^2}$ and $N = \sqrt{N^2}$ we can use the given fact to say that

$$n^2 > N^2 \geq \frac{5}{\varepsilon},$$

and so $5/n^2 < \varepsilon$. Moreover, since $n \geq 1$, we have $n^3 \geq n^2$, and so $1/n^3 \leq 1/n^2$. Using these two inequalities, we therefore have

$$\left|1 - \frac{2}{n^2} - \frac{3}{n^3} - 1\right| = \frac{2}{n^2} + \frac{3}{n^3} \leq \frac{5}{n^2} < \varepsilon.$$

And thus the sequence converges to 1 as required. ■

Proof. Let $\varepsilon > 0$ be given. Let $N = \left\lceil \frac{5}{\varepsilon} \right\rceil$. Notice that when $n \geq N$, we have that

$$\frac{5}{n} < \frac{5}{N} \leq \varepsilon.$$

Since $n \geq 1$, we know that

$$\frac{1}{n} > \frac{1}{n^2} \geq \frac{1}{n^3}$$

Hence

$$\left|1 - \frac{2}{n^2} - \frac{3}{n^3} - 1\right| = \frac{2}{n^2} + \frac{3}{n^3} \leq \frac{5}{n} < \varepsilon.$$

Thus the sequence converges to 1. ■

We shouldn't feel bad that we made a more complicated proof first; It is a normal part of doing mathematics. Our first proof of a result is often "bettered" by our second proof.

6.6.22.

(a) The symbolic statement

$$\forall M > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \in \mathbb{N}, n \geq N \implies s_n \geq M$$

means that for all $M > 0$, there is some $N \in \mathbb{N}$ so that for all $n \in \mathbb{N}$, if $n \geq N$, then we have $s_n \geq M$. So, s_n can be made arbitrarily large (larger than M, where M is given), by taking n large enough (larger than N, where the choice of N depends on M).

(b) The negation of the statement is

$$\exists M > 0 \text{ such that } \forall N \in \mathbb{N}, \exists n \in \mathbb{N} \text{ such that } n \geq N \text{ and } s_n < M.$$

This means that there is some $M > 0$ so that no matter how large we take N, there's some $n \geq N$ with $s_n < M$. This doesn't necessarily mean that all the terms are bounded by some fixed number, but rather, that we can always find some `small' value.

Proof. Let $M > 0$. Take $N = M^2$. Let $n \in \mathbb{N}$, and suppose $n \geq N$. Then $n \geq M^2$, and so $\sqrt{n} \geq M$. Therefore

$$\lim_{n\to\infty} \sqrt{n} = +\infty.$$

■

Proof. Take $M = 1$, and let $N \in \mathbb{N}$. If N is odd, then

$$(-1)^N \sqrt{N} < 0 < M.$$

If N is even, then $N + 1$ is odd, and

$$(-1)^{N+1} \sqrt{N+1} < 0 < M.$$

Thus in any case, there is some $n \geq N$ with $(-1)^n \sqrt{n} < M$. Therefore

$$\lim_{n\to\infty} (-1)^n \sqrt{n} \neq +\infty.$$

■

Proof. Let $M > 0$. Take $N = \max\{M, 101\}$. Let $n \in \mathbb{N}$, and suppose $n \geq N$. Since $n \geq N \geq 101$, we have $n - 100 \geq 1$. Thus

$$n^2 - 100n = n(n - 100) \geq n \geq N \geq M.$$

Therefore

$$\lim_{n\to\infty} (n^2 - 100n) = +\infty.$$

■

6.6.23.

Proof. Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence that converges to $L \in \mathbb{R}$. Then by definition, there is some $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $|a_n - L| < 1$. Therefore for $n \geq N_1$,

$$|a_n| = |L + (a_n - L)| \leq |L| + |a_n - L| < |L| + \varepsilon.$$

Let

$$M = \max\{|a_1|, |a_2|, \ldots, |a_{N_1-1}|, |L| + \varepsilon\},$$

which exists, since we are taking the maximum of only finitely many real numbers. But then for all $n \in \mathbb{N}$, $|a_n| \leq M$, so $\{a_n\}$ is bounded. ■

Proof. Let $L \in \mathbb{R}$, and take $\varepsilon = \frac{1}{2}\max\{|1 - L|, |-1 - L|\}$. Note that $\varepsilon > 0$. Let $N \in \mathbb{N}$. If $\varepsilon = \frac{1}{2}|1 - L|$, take $n = 2N$. Then we have $n \geq N$ but

$$|a_n - L| = |(-1)^{2N} - L| = |1 - L| = 2\varepsilon > \varepsilon.$$

If $\varepsilon = \frac{1}{2}|-1 - L|$, take $n = 2N + 1$. Then we have $n \geq N$ but

$$|a_n - L| = |(-1)^{2N+1} - L| = |-1 - L| = 2\varepsilon > \varepsilon.$$

In both cases, we have some $n \geq N$ such that $|a_n - L| > \varepsilon$, and so a_n does not converge to L. As L was an arbitrary real number, a_n does not converge. ■

6.6.24.

(a) We saw in part (d) of Exercise 6.6.22 that $\lim_{n\to\infty}(-1)^n\sqrt{n} \neq +\infty$, and $((-1)^n\sqrt{n})_{n\in\mathbb{N}}$ is an unbounded sequence. Indeed, given $M \geq 0$, we choose N be an integer strictly greater than M^2. Using the ceiling function, we set $N = \lceil M^2 \rceil + 1$. Then $N > M^2$ implies $|a_N| = \sqrt{N} > M$.

(b) We need to find a sequence that increases, and does not go to infinity. Take $a_n = -1/n$. Then $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$. Moreover, given any $M \geq 0$, we have that $a_n < M$ for all n. This implies that $\lim\limits_{n\to\infty} a_n \neq +\infty$.

Proof. Suppose that $\lim\limits_{n\to\infty} a_n \neq +\infty$, and that a_n is increasing. We show that $(a_n)_{n\in\mathbb{N}}$ is bounded. Since $\lim\limits_{n\to\infty} a_n \neq +\infty$, there is some $C > 0$ such that for any $n \in \mathbb{N}$, there's some $m \geq n$ with $a_m < C$. Fix $n \in \mathbb{N}$, and let $m \geq n$ be such that $a_m < C$. Since $(a_n)_{n\in\mathbb{N}}$ is increasing and $m \geq n$, we have

$$a_n \leq a_m < C.$$

But as $(a_n)_{n\in\mathbb{N}}$ is increasing, we also know that $a_1 \leq a_n$. Let $M = \max\{|a_1|, |C|\}$. Note that

$$-|a_1| \leq a_1 \leq a_n < C \leq |C|$$

But by choice of M, $|C| \leq M$, and $-M \leq -|a_1|$. Therefore

$$-M \leq a_n \leq M,$$

and so $|a_n| \leq M$. This holds for all n, and so a_n is bounded. Thus if $\lim_{n\to\infty} a_n \neq +\infty$, either $(a_n)_{n\in\mathbb{N}}$ is not increasing or $(a_n)_{n\in\mathbb{N}}$ is bounded. ■

6.6.25.

Proof. We are going to show that D is a distance, by showing that it satisfies triangle inequality. Since the function D is defined piecewise, we see that we would need to prove this statement using cases.

Let $x, y, z \in \mathbb{R}$. Now either $x = z$ or $x \neq z$, so we have two cases to check.

- *Case 1:* If $x = z$ then $D(x, z) = 0$. Since $D(x, y), D(y, z) \geq 0 = D(x, z)$, we know that $D(x, y) \leq D(x, y) + D(y, z)$.
- *Case 2:* If $x \neq z$ then $D(x, z) = 1$. So we must show that $D(x, y) + D(y, z) \geq 1$. Now either $x = y$ or $x \neq y$; this gives us two sub-cases to check.
  - *Case 2a:* If $x = y$ then we must have $y \neq z$ since $x \neq z$. But then $D(x, y) + D(y, z) = 0 + 1 = 1 = D(x, z)$.
  - *Case 2b:* If $x \neq y$ then $D(x, y) = 1$. Now we don't know if $y = z$ or $y \neq z$, but it doesn't matter, we know that $D(y, z) \geq 0$. Putting this together we have that $D(x, y) + D(y, z) = 1 + D(y, z) \geq 1 = D(x, z)$.

  So in both sub-cases, $D(x, z) \leq D(x, y) + D(y, z)$ as required.

So D satisfies the triangle inequality and therefore it is a distance. ■

Proof. Assume that $x_n \to L$. Let $\varepsilon = 1/2$. Then, by definition there is some $N \in \mathbb{N}$ s.t. for all $n > N$ we must have $D(x_n, L) < 1/2$. By definition of $D(x, y)$ it follows that we must have $D(x_n, L) = 0$ and so $x_n = L$. Hence when $x_n \neq L$ we have $n < N$. Thus, the set $\{n \in \mathbb{N} : x_n \neq L\}$ is finite. ■

7 · Induction
7.3 · Exercises

7.3.1. We have seen this statement before where we used cases to prove it. In this question, we are asked to use induction instead. This tells us that we can prove statements like this in more than one way.

Proof. We are going to use mathematical induction.

- *Base Case*: We see that for $n = 1$, the stament is "$3 \mid (1^3 - 1) = 0$". Hence, we see that this statement is true for $n = 1$.
- *Inductive Step*: Let $k \geq 1$, and assume that the statement is true for $n = k$, that is, $3 \mid (k^3 - k)$. Then we see that $k^3 - k = 3m$ for some $m \in \mathbb{Z}$. Thus,

$$(k+1)^3-(k+1) = k^3+3k^2+3k+1-k-1 = (k^3-k)+3(k^2+k) = 3(m+k^2+k),$$

and since $(m + k^2 + k) \in \mathbb{Z}$, we see $3 \mid \big((k+1)^3 - (k+1)\big)$.

Therefore the statement is true for $n = k + 1$, and hence, by induction, it is true for all $n \in \mathbb{N}$. ■

7.3.2.

Proof. We prove the result by induction.

- Base case: when $n = 2$ we have $n! = 2 \cdot 1 = 2$ and $n^n = 2^2 = 4$. Since $2 \leq 4$, the result holds when $n = 2$.
- Inductive step: Assume the result holds when $n = k \geq 2$. Then we have

$$\begin{aligned} k! &\leq k^k && \text{multiply both sides by } k+1 > 0 \\ (k+1)! &\leq (k+1) \cdot k^k && \text{since } k < k+1 \\ &\leq (k+1) \cdot (k+1)^k \\ &= (k+1)^{k+1} \end{aligned}$$

  That is $(k+1)! \leq (k+1)^{k+1}$, so the result holds when $n = k + 1$.

Since the base case and inductive step both hold, the result follows by induction. ■

7.3.3.

Proof. We are going to use mathematical induction.

- *Base Case*: We see that for $n = 7$, the stament is "$7! > 3^7$". We also know that $7! = 5040$ and $3^7 = 2187$. Thus, the statement is true for $n = 7$
- *Inductive Step*: Let $k \geq 7$, and assume that the statement is true for $n = k$, that is, $k! > 3^k$. Then we see that

$$(k+1)! = (k+1)k! > (k+1)3^k > (3)3^k = 3^{k+1},$$

  since $(k + 1) > 3$.

Therefore the statement is true for $n = k + 1$, and hence, by induction, it is true for all $n \in \mathbb{N}$. ■

We should also note that the inductive step doesn't really require the assumption $n \geq 7$. For the inductive step to be true, all we need is $n \geq 3$. But, as we have seen in our scratch work, the statement isn't true for $n < 7$, so the corresponding base cases would be false. This means that we can only prove this statement for $n \geq 7$.

7.3.4. The question wants us to use mathematical induction. However, we also see that this is not a standard induction question in the sense that for all $n \in \mathbb{N}$, we actually want to prove an existential statement, that is, the existence of x, y,

and z satisfying $x^2 + y^2 = z^{2n+1}$. Let's see how we can do that in the proof.

Proof. We are going to use mathematical induction.

- *Base Case*: We see that for $n = 1$, the stament is "$\exists x, y, z \in \mathbb{Z}$ such that $x \geq 2$, $y \geq 2$, and $z \geq 2$ and satisfy $x^2 + y^2 = z^3$". We see that for $x = 2$, $y = 2$, and $z = 2$, this statement is true.
- *Inductive Step*: Let $k \geq 1$, and assume that the statement is true for $n = k$, that is, $\exists x, y, z \in \mathbb{Z}$ such that $x \geq 2$, $y \geq 2$, and $z \geq 2$ and satisfy $x^2 + y^2 = z^{2k+1}$. Then we see that if we multiply the equation by z^2, we get $(xz)^2 + (yz)^2 = z^{2k+3} = z^{2(k+1)+1}$. Moreover, we see that $(xz), (xy) \geq 2$ since $x, z, y \geq 2$.

Therefore the statement is true for $n = k + 1$, and hence, by induction, it is true for all $n \in \mathbb{N}$. ■

7.3.5.

Proof. Let $n \in \mathbb{N}$. We proceed by induction on n. When $n = 1$, we have

$$3^{2\cdot 1} - 1 = 9 - 1 = 8,$$

which is divisible by 8. Now suppose that the result holds for $n = k$, so

$$8 \mid 3^{2k} - 1.$$

So we can write $3^{2k} - 1 = 8\ell$ for some $\ell \in \mathbb{Z}$. Now, notice that

$$\begin{aligned} 3^{2k+2} - 1 &= 9 \cdot 3^{2k} - 1 \\ &= 8 \cdot 3^{2k} + (3^{2k} - 1) \\ &= 8 \cdot 3^{2k} + 8\ell \end{aligned}$$

Thus 8 divides $3^{2(k+1)}$ as required.

Therefore, by induction, 8 divides $3^{2n} - 1$ for all $n \in \mathbb{N}$. Hence, equivalently, 8 divides $3^m - 1$ for any even $m \in \mathbb{N}$. ■

7.3.6.

Proof. Let $n \in \mathbb{N}$. We proceed by induction on n. When $n = 2$, the result holds by the distributive law. Now assume the result holds for $n = k$, and suppose that $a, b_1, b_2, \ldots, b_k, b_{k+1}$ are real numbers. Then

$$a\cdot(b_1+b_2+\cdots+b_k+b_{k+1}) = a\cdot((b_1+b_2+\cdots+b_k)+b_{k+1}) = a\cdot(b_1+b_2+\cdots+b_k)+a\cdot b_{k+1}$$

by the distributive law. But by inductive hypothesis,

$$a \cdot (b_1 + b_2 + \cdots + b_k) + a \cdot b_{k+1} = a \cdot b_1 + a \cdot b_2 + \cdots + a \cdot b_k + a \cdot b_{k+1}.$$

and so

$$a \cdot (b_1 + b_2 + \cdots + b_k + b_{k+1}) = a \cdot b_1 + a \cdot b_2 + \cdots + a \cdot b_k + a \cdot b_{k+1}.$$

Therefore, the result holds for any $n \in \mathbb{N}$, $n \geq 2$. ■

This proof gives us a general strategy on how we may use induction to extend a result between two objects to a result between any finite number of objects.

7.3.7.

Proof. Let $n \in \mathbb{N}$. We proceed with induction on n. For the base case, suppose $n = 1$. Since $2^1 = 2 \cdot 1$, we have $2^n \geq 2n$.

Now suppose that the statement is true for $n = k$; that is, $2^k \geq 2k$. Multiplying both sides of the inequality by 2, we have $2^{k+1} \geq 4k$. Rewrite this inequality as $2^{k+1} \geq 2k + 2k$. Since $k \geq 1$, we have $2k \geq 2$, and so

$$2^{k+1} \geq 2k + 2k \geq 2k + 2 = 2(k + 1)$$

and thus the result is established for $n = k + 1$. By induction, $2^n \geq 2n$ holds for any $n \in \mathbb{N}$. ■

7.3.8.

Proof. Let $n \in \mathbb{N}$. We proceed by induction on n. When $n = 0$, we have

$$2^{2n+1} + 3^{2n+1} = 2 + 3 = 5,$$

which is divisible by 5. Now suppose that the result holds for $n = k$, so

$$5 \mid \left(2^{2k+1} + 3^{2k+1}\right).$$

Then there is some $\ell \in \mathbb{Z}$ such that

$$2^{2k+1} + 3^{2k+1} = 5\ell.$$

Note that

$$2^{2(k+1)+1} + 3^{2(k+1)+1} = 4 \cdot 2^{2k+1} + 9 \cdot 3^{2k+1} = 4(2^{2k+1} + 3^{2k+1}) + 5 \cdot 3^{2k+1}.$$

Then by the inductive hypothesis

$$2^{2(k+1)+1} + 3^{2(k+1)+1} = 4 \cdot 5\ell + 5 \cdot 3^{2k+1} = 5(4\ell + 3^{2k+1})$$

and so 5 divides $2^{2(k+1)+1} + 3^{2(k+1)+1}$. Therefore, by induction, 5 divides $2^{2n+1} + 3^{2n+1}$ for all $n \in \mathbb{N}$. ■

7.3.9.

Proof. Let $n \in \mathbb{N}$. Suppose $n = 2$, and that $x_1, x_2 \in \mathbb{Q}$. Then there are $a_1, a_2, b_1, b_2 \in \mathbb{Z}$ with $b_1 \neq 0$ and $b_2 \neq 0$ so that $x_1 = a_1/b_1$ and $x_2 = a_2/b_2$. Then

$$x_1 + x_2 = \frac{b_2 a_1 + b_1 a_2}{b_1 b_2}$$

where $b_2 a_1 + b_1 a_2, b_1 b_2 \in \mathbb{Z}$ with $b_1 b_2 \neq 0$. Thus $x_1 + x_2 \in \mathbb{Q}$.

Now assume the result holds for $n = k$ for some $k \geq 2$, and suppose that

$x_1, x_2, \ldots, x_{k+1} \in \mathbb{Q}$. By the inductive hypothesis, $x_1 + x_2 + \cdots + x_k \in \mathbb{Q}$. Moreover, by the base case, the sum of two rationals is rational, and therefore

$$x_1 + x_2 + \cdots + x_{k+1} = (x_1 + x_2 + \cdots + x_k) + x_{k+1} \in \mathbb{Q}.$$

By induction, the result holds for any $n \geq 2$. ■

Note that the proof by induction only gives the result for a sum of finitely many terms. We cannot conclude that the sum of an infinite number of rational numbers is rational; indeed that statement is false. To see this, suppose that $x_0 = 1$, $x_1 = 4$, $x_3 = 1$, and in general, x_n is the n^{th} digit of $\sqrt{2}$. Then $x_n 10^{-n} \in \mathbb{Q}$ for all $n = 0, 1, 2, 3, \ldots$. But

$$\sqrt{2} = \sum_{n=0}^{\infty} x_n 10^{-n} \notin \mathbb{Q}.$$

We'll take the statement that $\sqrt{2} = \sum_{n=0}^{\infty} x_n 10^{-n}$ to be a fact, but this does require a proof, since we're dealing with an infinite number of terms. Really, we'd need to prove that

$$\sqrt{2} = \lim_{N \to \infty} \sum_{n=0}^{N} x_n 10^{-n}$$

which is beyond the scope of what we are trying to do in this chapter.

7.3.10.

Proof. We are going to use mathematical induction.

- *Base Case*: We see that for $n = 1$, the statement is "$\sum_{k=1}^{1} k^3 = 1 = \left(\sum_{k=1}^{1} k\right)^2$". Thus, the statement is true for $n = 1$.
- *Inductive Step*: Let $m \geq 1$ and assume that the statement is true for $n = m$, that is, assume that $\sum_{k=1}^{m} k^3 = \left(\sum_{k=1}^{m} k\right)^2$. Then,

$$\sum_{k=1}^{m+1} k^3 = \sum_{k=1}^{m} k^3 + (m+1)^3 = \left(\sum_{k=1}^{m} k\right)^2 + (m+1)^3.$$

Moreover, we know that $\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$. Hence,

$$\sum_{k=1}^{m+1} k^3 = \frac{m^2(m+1)^2}{4} + (m+1)^3 = \frac{(m+1)^2(m^2+4m+1)}{4} = \frac{(m+1)^2(m+2)^2}{4} = \left(\sum_{k=1}^{m+1} k\right)^2.$$

Therefore the statement is true for $n = m + 1$, and hence, by induction, it is true for all $n \in \mathbb{N}$. ■

7.3.11.

Proof. We are going to prove this statement using induction.

- *Base case*: We see that $\sum_{j=1}^{1} j^3 = 1 > \frac{1}{4} = \frac{1}{4}n^4$. Thus, the statement is true for $n = 1$.
- *Inductive step*: Assume that the statement is true for $n = k$ for some $k \geq 1$, that is,

  $\sum_{j=1}^{k} j^3 > \frac{1}{4}k^4$. Then, we see,

$$\begin{aligned}
\sum_{j=1}^{k+1} j^3 &= \sum_{j=1}^{k} j^3 + (k+1)^3 \\
&> \frac{1}{4}k^4 + (k+1)^3 && \text{by assumption} \\
&= \frac{1}{4}k^4 + k^3 + 3k^2 + 3k + 1 \\
&= \frac{1}{4}(k+1)^4 + \left(\frac{3}{2}k^2 + 2k + \frac{3}{4}\right) \\
&> \frac{1}{4}(k+1)^4,
\end{aligned}$$

  since $\frac{3}{2}k^2 + 2k + \frac{3}{4} > 0$ for all $k \in \mathbb{N}$. This means that the statement is true for $n = k + 1$.

Therefore, by mathematical induction, we see that the statement is true for all $n \in \mathbb{N}$. ■

7.3.12.

Proof. Let $r \in \mathbb{R}$, $r \neq 1$. We proceed with induction on n. For the base case, let $n = 0$. Then

$$\sum_{i=0}^{n} r^i = r^0 = 1 = \frac{1-r}{1-r} = \frac{1-r^{n+1}}{1-r}.$$

Now suppose that

$$\sum_{i=0}^{k} r^i = \frac{1-r^{k+1}}{1-r}$$

for $n = k$, where $k \in \mathbb{Z}$ and $k \geq 0$. Then

$$\sum_{i=0}^{k+1} r^i = \left(\sum_{i=0}^{k} r^i\right) + r^{k+1} = \frac{1-r^{k+1}}{1-r} + r^{k+1},$$

by the inductive hypothesis. Some arithmetic gives

$$\frac{1-r^{k+1}}{1-r} + r^{k+1} = \frac{1-r^{k+1}+r^{k+1}(1-r)}{1-r} = \frac{1-r^{k+2}}{1-r}.$$

Putting everything together, we have

$$\sum_{i=0}^{k+1} r^i = \frac{1-r^{k+2}}{1-r} = \frac{1-r^{(k+1)+1}}{1-r}$$

and so the result holds for $n = k+1$. Then, by induction, the result holds for any $n \in \{0, 1, 2, \dots\}$. ■

7.3.13.

(a) *Claim:*

$$f^{(k)}(x) = \frac{n!}{(n-k)!}x^{n-k}$$

Proof. We proceed by induction on k. As a base case, take $k = 1$. Computing the derivative of f, we see $f'(x) = nx^{n-1} = \frac{n!}{(n-1)!}x^{n-1}$, as desired.

Now assume that the derivative holds for an arbitrary $k \in \mathbb{N}$:

$$f^{(k)}(x) = \frac{n!}{(n-k)!}x^{n-k}.$$

We would like to show that the claim holds for $k+1$. We compute $f^{(k+1)(x)}$ by taking one derivative of $f^{(k)}(x)$.

$$f^{(k+1)}(x) = \frac{d}{dx}f^{(k)}(x)$$

By the inductive hypothesis,

$$= \frac{d}{dx}\frac{n!}{(n-k)!}x^{n-k}.$$

We compute the derivative

$$= \frac{n!}{(n-k)!}(n-k)x^{(n-k)-1}.$$

Finally, we rearrange into the desired form

$$= \frac{n!}{(n-(k+1))!}x^{n-(k+1)}.$$

By the inductive principle, our claim is true. ■

(b) *Claim:*

$$g^{(k)}(x) = \frac{(-1)^k(n+(k-1))!}{(n-1)!}x^{-n-k}$$

Proof. We proceed by induction on k. As a base case, take $k = 1$. Taking derivatives, we see that $g'(x) = -nx^{-n-1} = \frac{(-1)n!}{(n-1)!}x^{-n-1}$. Since this is the desired form, the base case is true.

Now, we assume that the derivative holds for an arbitrary $k \in \mathbb{N}$:

$$g^{(k)}(x) = \frac{(-1)^k(n+(k-1))!}{(n-1)!}x^{-n-k}.$$

We would like to show that the result holds for $k+1$. We compute

$$g^{(k+1)(x)} = \frac{d}{dx}g^{(k)}(x).$$

By the inductive hypothesis:

$$= \frac{d}{dx}\frac{(-1)^k(n+(k-1))!}{(n-1)!}x^{-n-k}.$$

We compute the derivative

$$= \frac{(-1)^k(n+(k-1))!}{(n-1)!}(-n-k)x^{-n-k-1},$$

and rearrange to obtain the desired form

$$= \frac{(-1)^{k+1}(n+k)!}{(n-1)!}x^{-n-(k+1)}.$$

By induction, our claim holds. ■

(c) *Claim:*

$$h^{(k)}(x) = (2k-1)!!(9-2x)^{-(2k+1)/2}$$

Proof. We prove our claim by induction on k. As a base case, let $k = 1$. By the power rule and the chain rule, we see that $h'(x) = \frac{-1}{2}(9-2x)^{-3/2}(-2) = (9-2x)^{-3/2}$. Hence, our base case holds.

Now, we assume that the derivative holds for an arbitrary $k \in \mathbb{N}$:

$$h^{(k)}(x) = (2k-1)!!(9-2x)^{-(2k+1)/2}$$

We would like to show that the result holds for $k+1$. We compute,

$$h^{(k+1)}(x) = \frac{d}{dx}h^{(k)}(x).$$

By the inductive hypothesis,

$$= \frac{d}{dx}(2k-1)!!(9-2x)^{-(2k+1)/2}.$$

We take the derivative

$$= (2k-1)!!\frac{-(2k+1)}{2}(-2)(9-2x)^{-(2k+1)/2-1}$$

Finally, we rearrange

$$= (2k+1)!!(9-2x)^{-(2k+1)/2-1}$$

This is the desired form, so our claim holds by the principle of induction. ■

7.3.14.

Proof. As a base case, we take $n = 1$. Then we see that

$$\sum_{k=1}^{1} k^2 = 1^2 = \frac{1(2)(3)}{6}.$$

We assume that the statement is true for $n = j \in \mathbb{N}$, that is

$$\sum_{k=1}^{j} k^2 = \frac{j(j+1)(2j+1)}{6},$$

and show that the statement is true for $n = j + 1$.

$$\sum_{k=1}^{j+1} k^2 = (j+1)^2 + \sum_{k=1}^{j} k^2$$

By the inductive hypothesis,

$$= (j+1)^2 + \frac{j(j+1)(2j+1)}{6}.$$

Factoring out $j + 1$ and giving both terms a common denominator yields

$$= \frac{(j+1)(2j^2+7j+6)}{6}$$

We now factor the numerator to get

$$= \frac{(j+1)(j+2)(2j+3)}{6}$$

Finally, we rearrange into the desired form

$$= \frac{(j+1)((j+1)+1)(2(j+1)+1)}{6}.$$

And so the statement holds for $n = j + 1$, as required.

Since the base case and inductive step are true, by the principle of induction, the statement holds true for each $n \in \mathbb{N}$. ■

7.3.15.

Proof. As a base case, we take $n = 1$. The base case is true since $\frac{1}{2} < 1$.

Now assume that the statement is true for an arbitrary value $n = j \in \mathbb{N}$:

$$\sum_{k=1}^{j} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^j} < 1.$$

Now we consider the desired sum

$$\sum_{k=1}^{j+1} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^j} + \frac{1}{2^{j+1}}$$

We factor $\frac{1}{2}$ from all but the first term to obtain

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^j}\right)$$

By the inductive hypothesis, $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^j} < 1$, therefore

$$\begin{aligned} &< \frac{1}{2} + \frac{1}{2}(1) \\ &= 1. \end{aligned}$$

So the inductive hypothesis is true.

By the principle of mathematical induction, we have our desired result. ■

7.3.16. When we prove a statement, say $P(n)$, by induction, we prove the statement for all $n \in \mathbb{N}$. Even though n can be as large as we like, it must always be a finite number. We can't actually set n to infinity.

The proposed proof actually shows that if $A \subseteq \mathbb{N}$ and $|A| = n$ for some $n \in \mathbb{N}$, then A has a maximum element. So, we have shown that if A is a non-empty subset of $\mathbb{N}$ with a finite number of elements, then A has a maximum element. This is not true if A has infinitely many elements. For example, if $A = \mathbb{N}$, then A does not have a maximum.

7.3.17.

Proof. Let $n \in \mathbb{Z}$, $n \geq 0$. We proceed by induction on n. First assume $n = 0$. Then

$$\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} e^{-x} = 0,$$

and so the result holds. Next suppose the result holds for $n = k$, so that

$$\lim_{x\to\infty} x^k e^{-x} = 0.$$

By l'Hôpital's rule,

$$\lim_{x\to\infty} x^{k+1} e^{-x} = \lim_{x\to\infty} \frac{x^{k+1}}{e^x} = \lim_{x\to\infty} \frac{(k+1)x^k}{e^x}.$$

Pulling out the constant, and applying the inductive hypothesis, we have

$$\lim_{x\to\infty}\frac{x^{k+1}}{e^x} = (k+1)\lim_{x\to\infty}\frac{x^k}{e^x} = (k+1)\cdot 0 = 0.$$

Therefore the result holds for $n = k+1$, and by induction, for any n. ■

Proof. Let $n \in \mathbb{Z}$, $n \geq 0$. We proceed by induction on n. First assume $n = 0$. Then

$$\int_0^\infty x^n e^{-x}\mathrm{d}x = \int_0^\infty e^{-x}\mathrm{d}x = \lim_{t\to\infty}(-e^{-x})\Big]_0^t = 1 = 0!.$$

Next suppose that the result holds for $n = k$; that is,

$$k! = \int_0^\infty x^k e^{-x}\mathrm{d}x.$$

Then, by integration by parts,

$$\begin{aligned}\int_0^\infty x^{k+1}e^{-x}\mathrm{d}x &= (-x^{k+1}e^{-x})\Big]_0^\infty + (k+1)\int_0^\infty x^k e^{-x}\mathrm{d}x \\ &= \lim_{t\to\infty}\left(-t^{k+1}e^{-t}\right) + 0^{k+1}e^{-0} + (k+1)\int_0^\infty x^k e^{-x}\mathrm{d}x \\ &= 0 + (k+1)\int_0^\infty x^k e^{-x}\mathrm{d}x\end{aligned}$$

where we have used the fact that

$$\lim_{x\to\infty} x^k e^{-x} = 0.$$

which we proved in (a). Now using the induction hypothesis, we therefore have

$$\int_0^\infty x^{k+1}e^{-x}\mathrm{d}x = (k+1)\int_0^\infty x^k e^{-x}\mathrm{d}x = (k+1)k! = (k+1)!.$$

Thus, by induction, the result holds for all n. ■

7.3.18. This is a generalization of the triangle inequality to arbitrary sets of finitely many real numbers. This proof is a standard way of generalizing statements that are stated for only two elements to arbitrary number of elements. This will also come up in later chapters when we discuss the union and intersection of multiple sets.

Proof. We are going to prove this statement using induction.

- *Base case:* We see that for $n = 1$ we have the statement $a_1 \leq a_1$, which is true for all real numbers. Moreover for $n = 2$ the statement is:

  "Let a_1, a_2 be real numbers. Then we have $\left|\sum_{k=1}^2 a_k\right| = |a_1 + a_2| \leq |a_1| + |a_2| = \sum_{k=1}^2 |a_k|$."

This is the triangle inequality (see Theorem 5.4.6) which we proved in Section 5.4.

- *Inductive step:* Assume that this statement is true for $m = n \geq 2$. This means that whenever $a_1, a_2, \ldots a_n \in \mathbb{R}$, then we have

$$\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|.$$

Now assume that we have some set of $n+1$ real numbers, $b_1, b_2, \ldots b_n, b_{n+1} \in \mathbb{R}$. Then we see

$$\begin{aligned}
\left|\sum_{k=1}^{n+1} b_k\right| &= \left|\left(\sum_{k=1}^{n} b_k\right) + b_{n+1}\right| \\
&\leq \left|\left(\sum_{k=1}^{n} b_k\right)\right| + |b_{n+1}| && \text{by Triangle Inequality} \\
&\leq \sum_{k=1}^{n} |b_k| + |b_{n+1}| && \text{by induction hypothesis} \\
&\leq \sum_{k=1}^{n+1} |b_k|.
\end{aligned}$$

Hence the statement is true for $m = n + 1$.

Therefore, by induction, the statement is true for all $n \in \mathbb{N}$. ■

7.3.19.

Proof. Let n be defined as the number of 1's between 100 and 7. For example, in the $n = 0$ case, the number is 1007 and in the $n = 4$ case, the number we are working with is 10011117. We proceed by induction on n.

For our base case, take $n = 0$. That is, we wish to show that 1007 is divisible by 53. Since $1007 = 19 \cdot 53$, our base case is true.

We assume that the statement holds true for $n = k$ and wish to prove it for the $n = k + 1$ case. Notice that we can write

$$100\underbrace{1\ldots1}_{\substack{k+1\\ \text{times}}}7 = 100\underbrace{1\ldots1}_{\substack{k\\ \text{times}}}7 + 901\underbrace{0\ldots0}_{\substack{n\\ \text{times}}}.$$

Since $901 = 17 \cdot 53$, we see that $901\underbrace{0\ldots0}_{k} = 17 \cdot 53 \cdot 10^k$. Therefore, $53 \mid 901\underbrace{0\ldots0}_{k}$. By the inductive hypothesis, $53 \mid 100\underbrace{1\ldots1}_{k}7$. Finally, applying the hint tells us that 53 divides $100\underbrace{1\ldots1}_{k+1}7$, as required.

Since the base case and inductive hypothesis are both true, the result follows. ■

7.3.20.

Proof. Let $n \in \mathbb{N}$. We prove, using strong induction, that for all $\ell \in \mathbb{N}$

$$F_{n+\ell} = F_\ell \cdot F_{n+1} + F_{\ell-1} \cdot F_n.$$

and then use this to prove the result.

- Base case: When $\ell = 1$ we have

  $$F_{n+1} = F_1 \cdot F_{n+1} + F_0 \cdot F_n$$

  and since $F_0 = 0, = F_1 = 1$, this follows.

- Inductive step: Now assume that

  $$F_{n+k} = F_k \cdot F_{n+1} + F_{k-1} \cdot F_n.$$

  holds for all $k \leq \ell$. Then

  $$\begin{aligned} F_{n+\ell+1} &= F_{n+\ell} + F_{n+\ell-1} \\ &= (F_\ell \cdot F_{n+1} + F_{\ell-1} \cdot F_n) + (F_{\ell-1} \cdot F_n + F_{\ell-2} \cdot F_{n-1}) \\ &= (F_\ell + F_{\ell-1}) \cdot F_n + (F_{\ell-1} + F_{\ell-2}) \cdot F_{n-1} \\ &= F_{\ell+1} \cdot F_n + F_\ell \cdot F_{n-1} \end{aligned}$$

  as required.

Thus

$$F_{n+\ell} = F_\ell \cdot F_{n+1} + F_{\ell-1} \cdot F_n.$$

holds for all $\ell \in \mathbb{N}$.

Now we prove the main result by induction.

- Base case: Since $F_q \mid F_q$ the case case is true.

- Inductive step: Assume that

  $$F_q \mid F_{qn}$$

  and consider F_{qn+q}. By the inductive result we proved above (with $n \mapsto qn$ and $\ell \mapsto q$) we know that

  $$F_{qn+q} = F_{q+1} \cdot F_{qn} + F_q \cdot F_{qn-1}$$

  Now since $F_q \mid F_{qn}$ by assumption, this shows that $F_q \mid F_{qn+q}$ as required.

Thus $F_q \mid F_{qn}$ for all $n \in \mathbb{N}$. ■

7.3.21. We prove each result in turn. First, Pascal's identity:

Proof. Let n, r be as given, then

$$\begin{aligned}\binom{n}{r} + \binom{n}{r-1} &= \frac{n!}{(n-r)!r!} + \frac{n!}{(n-r+1)!(r-1)!} \\ &= \frac{n!(n-r+1)}{(n-r+1)!r!} + \frac{n! \cdot r}{(n-r+1)!r!} \\ &= \frac{n!}{(n-r+1)!r!}(n-r+1+r) \\ &= \frac{n!(n+1)}{(n+1-r)!r!} \\ &= \frac{(n+1)!}{((n+1)-r)!r!} \\ &= \binom{n+1}{r}\end{aligned}$$

as required. ■

Next, the Binomial Theorem:

Proof. Let a, b be as stated, and then we prove the result by induction on n.

- When $n = 1$ we have that
$$(a+b)^1 = a + b = \binom{1}{0}a + \binom{1}{1}b$$
and so the base case holds.
- Now assume that the result holds for some $n = \ell \in \mathbb{N}$ and we show that it holds for $n = \ell + 1$. Start with
$$(a+b)^\ell = \sum_{k=0}^{\ell} \binom{\ell}{k} a^{\ell-k}b^k.$$
Multiplying both sides by $(a+b)$ gives:
$$\begin{aligned}(a+b)^{\ell+1} &= (a+b)\sum_{k=0}^{\ell} \binom{\ell}{k} a^{\ell-k}b^k \\ &= \sum_{k=0}^{\ell} \binom{\ell}{k} a^{\ell-k}b^k(a+b) \\ &= \sum_{k=0}^{\ell} \binom{\ell}{k} a^{\ell-k+1}b^k + \sum_{k=0}^{\ell} \binom{\ell}{k} a^{\ell-k}b^{k+1}\end{aligned}$$
Extract out the very first and very last terms
$$= \binom{\ell}{0}a^{\ell+1} + \sum_{k=1}^{\ell} \binom{\ell}{k} a^{\ell-k+1}b^k + \sum_{k=0}^{\ell-1} \binom{\ell}{k} a^{\ell-k}b^{k+1} + \binom{\ell}{\ell}a^0b^{\ell+1}$$

re-index the second sum to make it start from 0 by replacing k with $k-1$

$$= a^{\ell+1} + \sum_{k=1}^{\ell} \binom{\ell}{k} a^{\ell-k+1}b^k + \sum_{k=1}^{\ell} \binom{\ell}{k-1} a^{\ell-k+1}b^k + a^0 b^{\ell+1}$$

If we group-together terms with similar powers of a, b, we get

$$= a^{\ell+1} + \sum_{k=1}^{\ell} \left[\binom{\ell}{k} + \binom{\ell}{k-1} \right] a^{\ell-k+1}b^{k+1} + b^{\ell+1}$$

then, Pascal's identity gives

$$= a^{\ell+1} + \sum_{k=1}^{\ell} \binom{\ell+1}{k} a^{\ell+1-k}b^k + b^{\ell+1}$$

Finally, since $\binom{\ell+1}{0} = \binom{\ell+1}{\ell+1} = 1$, we have

$$= \binom{\ell+1}{0} a^{\ell+1}b^0 + \sum_{k=1}^{\ell} \binom{\ell+1}{k} a^{\ell+1-k}b^k + \binom{\ell+1}{\ell+1} a^0 b^{\ell+1}$$
$$= \sum_{k=0}^{\ell+1} \binom{\ell+1}{k} a^{\ell+1-k}b^k$$

as required.

Since the base-case and inductive step hold, the result follows by induction.

■

7.3.22.

Proof. We proceed by induction on k.

As a base case, let $k = 1$. We show that

$$\mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

using integration by parts. Using the definition of $\mathcal{L}\{f'\}(x)$, we compute

$$\mathcal{L}\{f'\}(s) = \int_0^\infty f'(x)e^{-sx}\mathrm{d}x$$

Integrating by parts, we see,

$$= f(x)e^{-sx}\Big|_0^\infty - \int_0^\infty f(x)(-se^{-sx})\mathrm{d}x$$
$$= \lim_{t\to\infty} f(t)e^{-st} - f(0) + s\int_0^\infty f(x)e^{-sx}\mathrm{d}x$$

By assumption 2. the first term is 0, so we have

$$= -f(0) + s\mathcal{L}\{f\}(s)$$

This proves the base case.

Now for some $k \in \mathbb{N}$, we assume that

$$\mathcal{L}\left\{f^{(k)}\right\}(s) = s^k\mathcal{L}\{f\}(x) - \sum_{i=0}^{k-1} s^{k-1-i}f^{(i)}(0)$$

We wish to show that the result holds for $k+1$. The computation is quite similar to the base case above.

We use integration by parts to compute

$$\begin{aligned}\mathcal{L}\left\{f^{(k+1)}\right\}(s) &= \int_0^\infty f^{(k+1)}(x)e^{-sx}\mathrm{d}x \\ &= f^{(k)}(x)e^{-sx}\Big|_0^\infty - \int_0^\infty f^{(k)}(x)(-se^{-sx})\mathrm{d}x \\ &= \lim_{t\to\infty} f^{(k)}(t)e^{-st} - f^{(k)}(0) + s\mathcal{L}\{f^{(k)}\}(s)\end{aligned}$$

By assumption 3. the first term is 0, so we have

$$= -f^{(k)}(0) + s\mathcal{L}\{f^{(k)}\}(s).$$

By the inductive hypothesis,

$$= -f^{(k)}(0) + s\left(s^k\mathcal{L}\{f\}(x) - \sum_{i=0}^{k-1} s^{k-1-i}f^{(i)}(0)\right)$$

Expand the bracketed terms

$$= -f^{(k)}(0) + s^{k+1}\mathcal{L}\{f\}(x) - \sum_{i=0}^{k-1} s^{k-i}f^{(i)}(0)$$

Finally, we absorb $f^{(k)}(0)$ into the sum

$$= s^{k+1}\mathcal{L}\{f\}(x) - \sum_{i=0}^{k} s^{k-i}f^{(i)}(0)$$

So the inductive step holds.

Since both the base case and inductive step are true, the result follows by induction. ■

7.3.23. We offer two proofs of the statement. We first use mathematical induction with $n = 0$, $n = 1$ as the base case.

Proof. Let α be as stated and then we proceed by strong induction.

As a base case, we take both $n = 0$ and $n = 1$. When $n = 0$, we have $\alpha^0 + \frac{1}{\alpha^0} = 2 \in \mathbb{Z}$, and the $n = 1$ case follows directly by assumption.

For our inductive hypothesis, we assume that for an arbitrary $n = k \in \mathbb{N}$,

$$\alpha^{k-1} + \frac{1}{\alpha^{k-1}} \in \mathbb{Z} \qquad \text{and} \qquad \alpha^k + \frac{1}{\alpha^k} \in \mathbb{Z}$$

We wish to show that the statement holds true for $n = k + 1$.

$$\alpha^{k+1} + \frac{1}{\alpha^{k+1}} = \left(\alpha^k + \frac{1}{\alpha^k}\right)\left(\alpha + \frac{1}{\alpha}\right) - \left(\alpha^{k-1} + \frac{1}{\alpha^{k-1}}\right)$$

By assumption, $\alpha + \frac{1}{\alpha} \in \mathbb{Z}$. Since the product and difference of integers is an integer, we conclude that $\alpha^{k+1} + \frac{1}{\alpha^{k+1}} \in \mathbb{Z}$.

Since both the base case and inductive step are true, the result follows. ■

We now appeal to the principle of strong mathematical induction for an alternate proof.

Proof. We proceed by strong induction. As a base case, we take $n = 0$. Then we have $\alpha^0 + \frac{1}{\alpha^0} = 2 \in \mathbb{Z}$.

Fix $n \in \mathbb{N}$. For our inductive hypothesis, we assume that for any $m \in \mathbb{N} \cup \{0\}$ such that $m \leq n$,

$$\alpha^m + \frac{1}{\alpha^m} \in \mathbb{Z}$$

We wish to show that the statement holds true for $n + 1$.

$$\alpha^{n+1} + \frac{1}{\alpha^{n+1}} = \left(\alpha^n + \frac{1}{\alpha^n}\right)\left(\alpha + \frac{1}{\alpha}\right) - \left(\alpha^{n-1} + \frac{1}{\alpha^{n-1}}\right)$$

By assumption, $\alpha + \frac{1}{\alpha} \in \mathbb{Z}$. By the inductive hypothesis,

$$\alpha^n + \frac{1}{\alpha^n} \in \mathbb{Z} \quad \text{and} \quad \alpha^{n-1} + \frac{1}{\alpha^{n-1}} \in \mathbb{Z}.$$

Since the product and difference of integers is an integer, we conclude that

$$\alpha^{n+1} + \frac{1}{\alpha^{n+1}} \in \mathbb{Z}.$$

By the principle of strong mathematical induction, the result is proven. ■

7.3.24.

Proof. We are going to use strong induction on a.

- *Base case:* We see that this statement is true for $n = 1$ since we can take $m = 0$ and $3 \nmid 1$.
- *Inductive step:* Assume that the statement is true for all $1 \leq \ell \leq k$.

  Then for $n = k + 1$ we have two cases.

- *Case 1:* $3 \nmid n$. In this case, we can take $m = 0, a = n$, that is $n = 3^0 n$. Thus, the statement is true.
- *Case 2:* $3 \mid n$. Then, we know that $n = 3b$ for some $b \in \mathbb{N}$. Moreover, we see that $b < k + 1$, and so $b \leq k$. Thus, by inductive hypothesis, we see that

$$b = 3^m a,$$

for some $m, a \in \mathbb{N}$ with $3 \nmid a$. Hence

$$n = 3b = 3^{m+1} a$$

And since $3 \nmid a$, the statement is true for $n = k + s1$.

So the inductive hypothesis holds for all $n \in \mathbb{N}$.

Therefore the statement is true for all $n \in \mathbb{N}$. ■

7.3.25. *Claim:* You can buy any souvenir you want!

Proof. We proceed by strong induction. As a base case, we take $p = 8$, $p = 9$, and $p = 10$. Since $p = 3 + 5$, we can pay for an \$8 item with one 3 dollar bill and one 5 dollar bill. Similarly, $9 = 3 \cdot 3$, so we can use three \$3 bills to purchase a \$9 item, and $10 = 2 \cdot 5$, so we can use two \$5 bills to purchase a \$10 item.

Let $n \in \mathbb{N}$ with $n > 10$. As our inductive hypothesis, we assume that we can pay for items with price $p = 8$, $p = 9, \ldots, p = n$ using only 3 and 5 dollar bills. We show that we can purchase an $\$(n + 1)$ item.

By the inductive hypothesis, we can purchase an $\$(n - 2)$ souvenir. That is, there exist natural numbers a, b such that $3a + 5b = n - 2$. Adding 3 to each side, we see $3(a + 1) + 5b = n + 1$. Therefore, we can purchase an $\$(n + 1)$ souvenir using $(a + 1)$ \$3 bills and b \$5 bills.

By the principle of strong mathematical induction, the result follows. ■

7.3.26.

Proof. Let $n \in \mathbb{N}$. We proceed with strong induction on n. The result holds for $n = 0$ and $n = 1$, since

$$a_0 = 2 = (-2)^0 + 3^0, \quad a_1 = 1 = (-2)^1 + 3^1.$$

Now suppose that $k \geq 1$ and for all $n \leq k$, we have $a_n = (-2)^n + 3^n$. Then by the recurrence for the sequence, and the inductive hypothesis, we have

$$a_{k+1} = a_k + 6a_{k-1} = (-2)^k + 3^k + 6\left((-2)^{k-1} + 3^{k-1}\right)$$

We can rewrite this as

$$a_{k+1} = (-2)^k + 3^k - 3(-2)^k + 2 \cdot 3^k = -2 \cdot (-2)^k + 3 \cdot 3^k = (-2)^{k+1} + 3^{k+1}$$

Therefore the result holds for $n = k+1$, and so, by strong induction, all $n \in \mathbb{N} \cup \{0\}$. ■

7.3.27.

Proof. We are going to use strong induction on n.

- *Base case:* We see that the statement is true for $n = 1$ since we can take $m = 0$ and $c_0 = 1$.
- *Inductive step:* Assume that the statement is true for all $1 \leq \ell \leq k$ Then for $n = k + 1$ we have two cases.
  - *Case 1:* n is even. So $n = 2a$ for some $a \in \mathbb{N}$. Since $a < n$, we know $a \leq k$ and so the induction hypothesis implies that there exists $m \in \mathbb{Z}$, $m \geq 0$ and constants $c_0, c_1, c_2, \ldots, c_m \in \{0, 1\}$ such that

    $$a = c_m \cdot 2^m + c_{m-1} \cdot 2^{m-1} + \cdots + c_1 \cdot 2 + c_0.$$

    This implies that

    $$n = 2a = c_m \cdot 2^{m+1} + c_{m-1} \cdot 2^m + \cdots + c_1 \cdot 2^2 + c_0 \cdot 2,$$

    Hence, the statement is true for $n = k + 1$.
  - *Case 2:* n is odd. So, $n = 2b + 1$ for some $b \in \mathbb{N}$. Again, since $b < n$, we know that $b \leq k$ and so the induction hypothesis tells us that there exists $m \in \mathbb{Z}$, $m \geq 0$ and constants $c_0, c_1, c_2, \ldots, c_m \in \{0, 1\}$ such that

    $$b = c_m \cdot 2^m + c_{m-1} \cdot 2^{m-1} + \cdots + c_1 \cdot 2 + c_0.$$

    This, then, implies

    $$2b = c_m \cdot 2^{m+1} + c_{m-1} \cdot 2^m + \cdots + c_1 \cdot 2^2 + c_0 \cdot 2.$$

    and so

    $$n = 2b + 1 = c_m \cdot 2^{m+1} + c_{m-1} \cdot 2^m + \cdots + c_1 \cdot 2^2 + c_0 \cdot 2 + 1.$$

    Hence, the statement is true for $k = n + 1$.

  Since both cases are true, the inductive step holds.

Therefore the statement is true for all $n \in \mathbb{N}$. ■

7.3.28.

Proof. We prove the result by strong induction.

- When $n = 1$ we have $T(1) = 1$ and since $1 \log_2 1 + 1 = 1$, the result holds.
- Now assume that the result holds for all $1 \leq k \leq \ell$ and we wish to show that it holds for $k = \ell + 1$. Since the recurrence is slightly different for even or odd $k = \ell + 1$, we split into two cases.

- Assume that $k = \ell + 1 = 2j$, then

$$\begin{aligned} T(k) &= 2T(j) + k \\ &\leq 2\left(j \log_2(j) + j\right) + k \\ &= 2j \log_2(j) + 2j + k \\ &= k \log_2\left(\frac{2j}{2}\right) + 2k \\ &= k\left(\log_2(k) - \log_2(2)\right) + 2k \\ &= k \log_2(k) + k, \end{aligned}$$

since $\log_2(2) = 1$.

- Now assume that $k = \ell + 1 = 2j + 1$, then

$$\begin{aligned} T(k) &= 2T(j) + k \\ &\leq 2\left(j \log_2(j) + j\right) + k \\ &= 2j \log_2(j) + 2j + k \\ &= k \log_2\left(\frac{2j}{2}\right) + 2k - 1 \\ &= k \log_2(k) + k - 1 < k \log_2(k) + k \end{aligned}$$

where again we have used $\log_2(2) = 1$.

In both cases the inductive step holds.

Since both the base case and inductive step are true, the result follows by induction. ■

7.3.29.

Proof. We are going to use mathematical induction.

- *Base Case*: We see that for $n = 2$, we have only one way of splitting them, by splitting them into two piles of one stone each. Thus, our number becomes $1 \times 1 = 1$ and we also have $\frac{2(2-1)}{2} = 1$. Hence, the statement is true for $n = 2$.

- *Inductive Step*: Let $m \geq 2$, and assume that the statement is true for all $k \leq m$. Now, assume we have $m + 1$ stones. Then for the first splitting, we have two cases.

  - *Case 1: Splitting into two piles of* 1 *and* m *stones*: In this case, we have our first number to be $m \times 1 = m$. Moreover, from the inductive hypothesis, we see that if we keep splitting the pile of m stones we get the number $\frac{m(m-1)}{2}$. Thus, our final number is

$$m + \frac{m(m-1)}{2} = \frac{(m+1)m}{2}.$$

Therefore, in this case, the statement is true for $n = m + 1$.

- *Case 2: Splitting into two piles of p and q stones, $p, q > 1$:* In this case, we have our first number to be pq. Moreover, since $p, q > 1$, we have $p, q < m$. Thus, by inductive hypothesis, the number we should get by splitting the two piles into smaller piles are $\frac{p(p-1)}{2}$ and $\frac{q(q-1)}{2}$. We also know that $q = (m+1) - p$, and hence, our final number is

$$\frac{p(p-1)}{2} + \frac{q(q-1)}{2} + pq = \frac{(p^2 - p + q + 2 - q)}{2} + pq = \frac{p^2 + q^2 - (p+q) + 2pq}{2},$$

which implies

$$\frac{(p+q)^2 - (p+q)}{2} = \frac{(m+1)^2 - (m+1)}{2} = \frac{(m+1)m}{2}.$$

Therefore, in this case, the statement is true for $n = m + 1$.

Since both cases hold, the inductive step is true.

Hence, we conclude that the statement is true for all $n \in \mathbb{N}$. ■

8 · Return to sets
8.6 · Exercises

8.6.1.

(a) $A_2 - A_3 = \{6, 9, 12, 24, 27, 36, 54, 72, 216\}$.

(b) $A_5 \cap A_6 = \emptyset$.

(c) $\mathcal{P}(A_1) = \{\emptyset, \{-1\}, \{0\}, \{1\}, \{-1, 1\}, \{-1, 0\}, \{0, 1\}, \{-1, 0, 1\}\}$. Notice that since $|A_1| = 3$ we have $|\mathcal{P}(A_1)| = 2^3 = 8$.

(d) $\mathcal{P}(\mathcal{P}(A_1 - \{-1\}))$. We see that $A_1 - \{-1\} = \{0, 1\}$. Then we get, $\mathcal{P}(A_1 - \{-1\})) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. Thus we see,

$$\begin{aligned}\mathcal{P}(\mathcal{P}(A_1 - \{-1\})) = \Big\{&\emptyset, \{\emptyset\}, \{\{0\}\}, \{\{1\}\}, \{\{0, 1\}\}, \\ &\{\emptyset, \{0\}\}, \{\emptyset, \{1\}\}, \{\emptyset, \{0, 1\}\}\, \{\{0\}, \{1\}\}, \{\{0\}, \{0, 1\}\}\, \{\{1\}, \{0, 1\}\}, \\ &\{\emptyset, \{0\}, \{1\}\}, \{\emptyset, \{0\}, \{0, 1\}\}, \{\emptyset, \{1\}, \{0, 1\}\}, \{\{0\}, \{1\}, \{0, 1\}\}, \\ &\{\emptyset, \{0\}, \{1\}, \{0, 1\}\}\Big\}.\end{aligned}$$

Since $|\mathcal{P}(A_1 - \{-1\})| = 4$, the power set of that has $2^4 = 16$ elements. We have arranged our answer here so that the single subset with 0 elements is followed by the four subsets with 1 element, then the 6 with two elements,

the four with 3 elements and finally the single subset with 4 elements. Of course, the order doesn't matter, as long as all 16 are there.

(e) $A_3 \cap A_4 = \{3\}$,

(f) $A_2 - A_7 = \emptyset$.

(g) $A_5 \cup A_2 = \{3, 7, 6, 9, 11, 12, 13, 1718, 19, 24, 27, 36, 54, 72, 108, 216\}$.

(h) $A_2 \cap A_7 = \{3, 6, 9, 12, 18, 24, 27, 36, 54, 72, 108, 216\} = A_2$.

(i) $A_5 \cap \overline{A_2}$, given the universal set $U = \mathbb{R}$. We have $A_5 \cap \overline{A_2} = \{7, 11, 13, 17, 19\} = A_5$.

(j) Verify whether $(A_4 \times B) \cap (B \times A_4) = A_4 \times A_4$ and $(A_4 \times B) \cup (B \times A_4) = B \times B$.

We see

$$A_4 \times B = \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\},$$

and

$$B \times A_4 = \{(1,2), (2,2), (3,2), (4,2), (5,2), (6,2), (1,3), (2,3), (3,3), (4,3), (5,3), (6,3)\}.$$

Then,

$$(A_4 \times B) \cap (B \times A_4) = \{(2,2), (2,3), (3,2), (3,3)\} = A_4 \times A_4.$$

We also see,

$$\begin{aligned}(A_4 \times B) \cup (B \times A_4) &= \{(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6),\\ &\quad (1,2), (4,2), (5,2), (6,2), (1,3), (4,3), (5,3), (6,3)\} \neq B \times B,\end{aligned}$$

since $(1,1) \in B \times B$ but $(1,1) \notin (A_4 \times B) \cup (B \times A_4)$.

8.6.2.

Proof. We see that $F \cap G = \{(a,b) \in \mathbb{R}^2 \colon (a,b) \in F \text{ and } (a,b) \in G\}$

$$\begin{aligned}F \cap G &= \{(a,b) \in \mathbb{R}^2 \colon (a,b) \in F \text{ and } (a,b) \in G\}\\ &= \{(a,b) \colon a, b \in \mathbb{R}, b = a^2 - 3a + 2 \text{ and } b = a + 2\}\\ &= \{(a,b) \colon a, b \in \mathbb{R}, b = a^2 - 3a + 2 = a + 2\}\\ &= \{(a,b) \colon a, b \in \mathbb{R}, a^2 - 4a = 0 \text{ and } b = a + 2\}\\ &= \{(a,b) \colon a, b \in \mathbb{R}, (a = 0 \text{ and } b = a + 2 = 2), \text{ or } (a = 4 \text{ and } b = a + 2 = 6)\}\\ &= \{(0,2), (4,6)\}\end{aligned}$$

■

Proof. We prove that

$$F \cap G \subseteq \{(0,2),(4,6)\} \qquad \text{and} \qquad \{(0,2),(4,6)\} \subseteq F \cap G$$

- Let $(x,y) \in F \cap G$. So we know that $(x,y) \in F$ and $(x,y) \in G$. Since $(x,y) \in F$ we know that $x, y \in \mathbb{R}$ so that $y = x^2 - 3x + 2$. Then since $x, y \in G$, we know that $y = x + 2$. Combining these two equations we see that we must have $x + 2 = x^2 - 3x + 2$ or, equivalently $x^2 - 4x = 0$. This, in turn, implies that $x = 0, 4$. And thus $(x,y) = (0,2)$ or $(x,y) = (4,6)$. In both cases $(x,y) \in \{(0,2),(4,6)\}$ as required.
- Now let $(x,y) \in \{(0,2),(4,6)\}$. Then either $(x,y) = (0,2)$ or $(x,y) = (4,6)$.
  - Let $(x,y) = (0,2)$. Then since $2 = 0^2 - 3 \cdot 0 + 2$, it follows that $(x,y) \in F$. Then, since $2 = 0 + 2$, we know that $(x,y) \in G$.
  - Very similarly, let $(x,y) = (4,6)$. Then since $6 = 4^2 - 3 \cdot 4 + 2$, we know that $(x,y) \in F$. And since $6 = 4 + 2$ we also know that $(x,y) \in G$.

  So in both cases, we have that $(x,y) \in F$ and $(x,y) \in G$, and thus $(x,y) \in F \cap G$ as required.

Since both inclusions hold, the sets are equal. ■

8.6.3.

Proof. This statement is false. To see this let $A = B = \{1\}$ and $C = \{2\}$. Then notice that $A = B - C$, but $A \cup C = \{1,2\} \neq \{1\} = B$. ■

8.6.4.

Proof. This is a set equality, so we need to prove $\{x^a \text{ s.t. } a \in \mathbb{Q}\} \subseteq \{y^a \text{ s.t. } a \in \mathbb{Q}\}$ and also the reverse inclusion.

- ($\subseteq$): Let $z \in \{x^a : a \in \mathbb{Q}\}$. Then we know that $z = x^a$ for some $a \in \mathbb{Q}$. Since $x^k = y$, we know that $z = y^{a/k}$. Then since $a \in \mathbb{Q}$ and $k \in \mathbb{N}$, $\frac{a}{k} \in \mathbb{Q}$. Thus, we see that $z \in \{y^b : b \in \mathbb{Q}\}$.
- ($\supseteq$): Let $u \in \{y^b : b \in \mathbb{Q}\}$. Then we see that $u = y^b$ for some $b \in \mathbb{Q}$. Since $y = x^k$, we know that $u = y^{bk}$. Because $b \in \mathbb{Q}$ and $k \in \mathbb{N}$, we know that $bk \in \mathbb{Q}$. Thus, we see that $z \in \{x^a : a \in \mathbb{Q}\}$.

Since both inclusions hold, the sets are equal. ■

8.6.5.

Proof. This statement is false. We see that if we take $m = n = 2$, then $\{x \in \mathbb{Z} \colon mn \mid x\} = \{x \in \mathbb{Z} \colon 4 \mid x\}$, and $\{x \in \mathbb{Z} \colon m \mid x\} \cap \{x \in \mathbb{Z} \colon n \mid x\} = \{x \in \mathbb{Z} \colon 2 \mid x\}$. Thus, since $2 \in \{x \in \mathbb{Z} \colon 2 \mid x\}$, but $2 \notin \{x \in \mathbb{Z} \colon 4 \mid x\}$, we see that the statement is false. ■

8.6.6.

Proof. Let $m, n \in \mathbb{Z}$. Assume that $s \in \{x \in \mathbb{Z} : mn \mid x\}$. Then we see that $mn \mid s$ and thus, $s = mnk$ for some $k \in \mathbb{Z}$.

- Now since $s = m(nk)$ and $nk \in \mathbb{Z}$, we know that $m \mid s$.
- Similarly, since $s = n(mk)$ and $mk \in \mathbb{Z}$, we also know that $n \mid s$.

Consequently $s \in \{x \in \mathbb{Z} : n \mid x\}$ and $s \in \{x \in \mathbb{Z} : m \mid x\}$. Since s is an element of both of those sets, it is an element of their intersection. Thus

$$\{x \in \mathbb{Z} : mn \mid x\} \subseteq \{x \in \mathbb{Z} : m \mid x\} \cap \{x \in \mathbb{Z} : n \mid x\}$$

as required. ■

8.6.7.

(a) *Claim:* The statement is true.

Proof. By the definition of the Cartesian product, an element of $A \times \emptyset$ takes the form (a, b) where $a \in A$ and $b \in \emptyset$. But by the definition of the empty set, no such b exists. Therefore the ordered pair (a, b) cannot exist, and we see that there are no elements in $A \times \emptyset$. Therefore, $A \times \emptyset = \emptyset$. Hence, $A \times \emptyset \subseteq A$. ■

(b) *Claim:* The statement is false for general sets A. It is, however, true when A$= \emptyset$.

Proof. We saw in part (a), that $A \times \emptyset = \emptyset$. Therefore $A \subseteq A \times \emptyset = \emptyset$ if and only if $A = \emptyset$. Thus, $A \times \emptyset = A$ if and only if $A = \emptyset$. ■

8.6.8.

Proof. We require two cases.

- *Case 1:* $C = \emptyset$. By the definition of the Cartesian product, an element of $A \times C$ takes the form (a, c) where $a \in A$ and $c \in C = \emptyset$. But by the definition of the empty set, no such c exists. Therefore the ordered pair (a, c) cannot exist, and we see that there are no elements in $A \times C$. Therefore, $A \times C = \emptyset$. We can similarly argue that $B \times C = \emptyset$. We conclude that

$$\emptyset = A \times C \subseteq B \times C = \emptyset.$$

- *Case 2:* $C \neq \emptyset$. Suppose that $x \in A \times C$. Then by the definition of the Cartesian product, $x = (a, c)$ for some elements $a \in A$ and $c \in C$. Since $A \subseteq B$, we also have $a \in B$. Then by the definition of the Cartesian product, $x = (a, c) \in B \times C$. We conclude that $A \times C \subseteq B \times C$.

■

In order for (b) to be true, we claim that we require that $C \neq \emptyset$.

Proof. First, notice that if $C = \emptyset$, then by *Case 1* of the proof of part (a), we would have $A \times C = \emptyset = B \times C$. Thus, we must require that $C \neq \emptyset$.

Now, we repeat the argument of *Case 2* from part (a).

Suppose that $x \in A \times C$. Then by the definition of the Cartesian product, $x = (a, c)$ for some elements $a \in A$ and $c \in C$. Since $A \subset B$, we also have $a \in B$. Then by the definition of the Cartesian product, $x = (a, c) \in B \times C$. Therefore $A \times C \subseteq B \times C$.

It remains to show that $B \times C \nsubseteq A \times C$. Since $A \subset B$, there exists an element $b \in B$ such that $b \notin A$. Let $c \in C$, then by the definition of the Cartesian product, $(b, c) \in B \times C$. However, since $b \notin A$, $(b, c) \notin A \times C$. We conclude that $B \times C \nsubseteq A \times C$. Therefore,

$$A \times C \subset B \times C,$$

as required. ■

8.6.9.

Proof. This is a set equality, so we have to prove $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$ and the reverse inclusion, $\mathcal{P}(A) \cap \mathcal{P}(B) \supseteq \mathcal{P}(A \cap B)$.

- $\subseteq$: Let $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then we see that $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$. This implies that $X \subseteq A$ and $X \subseteq B$. Then we see that if $z \in X$, then $z \in A$ and $z \in B$, that is $z \in A \cap B$. Hence, $X \subseteq A \cap B$. Then we see that $X \in \mathcal{P}(A \cap B)$.

- $\supseteq$: Let $Y \in \mathcal{P}(A \cap B)$. Then we see that $Y \subseteq A \cap B$. Thus, if $t \in Y$. then $t \in A \cap B$. This implies $t \in A$ and $t \in B$. Thus, we see that $Y \subseteq A$ and $Y \subseteq B$. This means that $Y \in \mathcal{P}(A)$ and $Y \in \mathcal{P}(B)$. Therefore $Y \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

Hence, the result follows. ■

8.6.10.

Proof. This statement is false. For a counterexample, we can take $A = \{1\}$ and $B = \{2\}$. Then

$$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}\} \quad \text{but} \quad \mathcal{P}(A \cup B) \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$$

Notice that $\{1, 2\} \in \mathcal{P}(A \cup B)$ but $\{1, 2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$. Consequently the inclusion does not hold. ■

8.6.11.

Proof. Let A be a finite set. We will prove this statement using induction on the size of A, where $|A| = n$, for $n \in \mathbb{N} \cup \{0\}$.

- *Base case*: We see that if $|A| = 0$, then $A = \emptyset$. Therefore $\mathcal{P}(A) = \{\emptyset\}$. This means that $|\mathcal{P}(A)| = 1 = 2^0$. Hence, the statement holds for $n = 0$.

Moreover, we see that if $|A| = 1$, then $A = \{a\}$ for some a. Thus, we see that $\mathcal{P}(A) = \{\emptyset, \{a\}\}$. Thus, $|\mathcal{P}(A)| = 2 = 2^1$. This means that the statement is true for $n = 1$ as well.

- *Inductive step*: Assume that the statement is true for $k = n$, that is, if $|A| = n$, then we have $|\mathcal{P}(A)| = 2^n$.

  Now, let B be a set with $|B| = n+1$. Then, we see $B = \{b_1, b_2, b_3, \ldots, b_{n+1}\}$. Therefore, we see that we can write

  $$B = \{b_1, b_2, b_3, \ldots, b_n\} \cup \{b_{n+1}\} = \tilde{B} \cup \{b_{n+1}\},$$

  where $|\tilde{B}| = n$.

  Therefore we know that $|\mathcal{P}\left(\tilde{B}\right)| = 2^n$.

  Now we can write $\mathcal{P}(B)$ as a union of all subsets that contain the element b_{n+1}, call $\mathcal{P}$, and subsets that don't contain the element b_{n+1}, call $\mathcal{Q}$. Moreover, we see that the set of all subsets that don't contain b_{n+1} is the set of all subsets of $\tilde{B}$, i.e. $\mathcal{P} = \mathcal{P}\left(\tilde{B}\right)$.

  We also see that a subset, D, of B contains b_{n+1} if and only if $D - \{b_{n+1}\}$ is a subset of $\tilde{B}$, that is

  $$|\mathcal{Q}| = |\mathcal{P}| = |\mathcal{P}\left(\tilde{B}\right)|.$$

  Thus, since $\mathcal{P} \cap \mathcal{Q} = \emptyset$, we get

  $$\begin{aligned} |\mathcal{P}(B)| &= |\mathcal{P}| + |\mathcal{Q}| \\ &|\mathcal{P}\left(\tilde{B}\right)| + |\mathcal{Q}| \\ &|\mathcal{P}\left(\tilde{B}\right)| + |\mathcal{P}\left(\tilde{B}\right)| = 2|\mathcal{P}\left(\tilde{B}\right)| \\ &= 2.2^n = 2^{n+1}. \end{aligned}$$

  Hence, the statement is true for $k = n + 1$.

Therefore, by mathematical induction, we conclude that if A is a finite set with $|A| = n$. Prove that $|\mathcal{P}(A)| = 2^n$. ■

8.6.12. We give a disproof of both statements.

Proof. Both statements are false. We disprove each in turn.

- Let $A = \{1\}$ and $B = \{2\}$. Then $\emptyset$ is an element of $\mathcal{P}(A), \mathcal{P}(B)$ and $\mathcal{P}(A - B)$. But this means that $\emptyset \in \mathcal{P}(A - B)$ but $\emptyset \notin \mathcal{P}(A) - \mathcal{P}(B)$. Hence the first inclusion does not hold.

- Now let $A = \{1, 2\}, B = \{2\}$. Then

$$\mathcal{P}(A) - \mathcal{P}(B) = \{\emptyset, \{1\}, \{1, 2\}\} \quad \text{but} \quad \mathcal{P}(A - B) = \{\emptyset, \{1\}\}.$$

  Hence $\{1, 2\} \in LHS$ but $\{1, 2\} \notin RHS$. So the second inclusion does not hold.

■

8.6.13.

Proof. We show that each side is included in the other.

- Suppose $x \in \cup_{n=3}^{\infty}(1/n, 1 - 1/n)$. Then $x \in (1/n, 1 - 1/n)$ for some $n \in \mathbb{N}$. Since $0 < 1/n$ and $1 - 1/n < 1$, we also have $x \in (0, 1)$. Therefore the LHS is a subset of $(0, 1)$.

- Suppose $y \in (0, 1)$. Let $N_1 = \left\lceil \frac{1}{y} \right\rceil + 1$ and $N_1 = \left\lceil \frac{1}{1-y} \right\rceil + 1$. Take $N = \max\{N_1, N_2\}$. Then

$$\frac{1}{1 - y} < N_2 \leq N.$$

  Multiplying the inequality through by $1 - y$, which is positive as $y < 1$, and dividing by N, we have $1/N < 1 - y$, from which we may obtain $y < 1 - 1/N$. Moreover, $N \geq N_1$ implies that

$$\frac{1}{N} \leq \frac{1}{N_1} < y.$$

  Thus $y \in (1/N, 1 - 1/N)$ and so y is an element of the LHS. This gives us the reverse inclusion as well.

Since both inclusions hold, we can conclude

$$\bigcup_{n=3}^{\infty} \left(\frac{1}{n}, 1 - \frac{1}{n} \right) = (0, 1)$$

as required. ■

Proof. We prove each side is included in the other.

- Suppose $x \in [0, 1]$. Then for any $n \in \mathbb{N}$,

$$-\frac{1}{n} < 0 \leq x \leq 1 < 1 + \frac{1}{n}.$$

  So $x \in \cap_{n=1}^{\infty}(-1/n, 1 + 1/n)$, and therefore $[0, 1]$ is a subset of the LHS.

- Conversely, suppose $y \notin [0,1]$. If $y < 0$, then $N = \left\lceil \frac{1}{|y|} \right\rceil + 1$ is a natural number, and $y < -1/N$. But then $y \notin (-1/N, 1 + 1/N)$. Therefore y is not an element of the LHS.

  If $y > 1$, then $N = \left\lceil \frac{1}{y-1} \right\rceil + 1$ is a natural number, and we'd have $N > 1/(y-1)$. Multiplying through by $y - 1$, which is positive as $y > 1$, and dividing by N, we'd obtain $y - 1 > 1/N$. But then $y > 1 + 1/N$, and so $y \notin (-1/N, 1 + 1/N)$. So in either case we have that y is not an element of the LHS. Thus, by contrapositive, we have

$$\bigcap_{n=1}^{\infty} (-1/n, 1 + 1/n) \subseteq [0,1].$$

As we have proved the reverse inclusion as well, the two sets are equal. ■

8.6.14.

(a) We claim that

$$\bigcup_{n \in \mathbb{N}} [-n, n] = \mathbb{R}.$$

Proof. For any $n \in \mathbb{N}$, we have $[-n, n] \subseteq \mathbb{R}$, and so

$$\bigcup_{n \in \mathbb{N}} [-n, n] \subseteq \mathbb{R}.$$

We need to show that the reverse inclusion holds. To this end, let $x \in \mathbb{R}$. If $x = 0$,

$$0 \in [-1, 1] \subseteq \bigcup_{n \in \mathbb{N}} [-n, n].$$

If $x \neq 0$, let $N = \lceil |x| \rceil$. Recall that $f(y) = \lceil y \rceil$ is the ceiling function, defined to be the smallest integer m such that $y \leq m$. We have $N \in \mathbb{N}$, and $|x| \leq N$. Hence

$$x \in [-N, N] \subseteq \bigcup_{n \in \mathbb{N}} [-n, n]$$

for this case as well. Thus

$$\mathbb{R} \subseteq \bigcup_{n \in \mathbb{N}} [-n, n]$$

holds as well, establishing that the sets are equal. ■

(b) For a fixed $r \in \mathbb{R}$, $r > 0$, the set

$$B_r = \{(x, y) : x^2 + y^2 < r\}$$

represents the inside of a disk in $\mathbb{R}^2$ with radius r. We claim that

$$B_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 < r\} = \mathbb{R} \times \mathbb{R}.$$

Proof. For any $r \in \mathbb{R}$, $r > 0$, we have by definition that

$$B_r = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 < r\} \subseteq \mathbb{R} \times \mathbb{R}.$$

Therefore

$$\bigcup_{r\in\mathbb{R},r>0} B_r \subseteq \mathbb{R} \times \mathbb{R}.$$

We need to show that the reverse inclusion holds. To this end, let $(x, y) \in \mathbb{R} \times \mathbb{R}$. Let $r = x^2 + y^2 + 1$. Then $r > 0$, and $x^2 + y^2 < r$. Hence

$$(x, y) \in B_r \subseteq \bigcup_{r\in\mathbb{R},r>0} B_r.$$

As (x, y) was arbitrary, we have

$$\mathbb{R} \times \mathbb{R} \subseteq \bigcup_{r\in\mathbb{R},r>0} B_r$$

and so the sets are in fact equal. ■

8.6.15.

Proof. If $x \in [1, 3]$, then $x \leq 3$, and so 3 is an upper bound of the set. Since $3 \in [1, 3]$, it is also the maximum of the set. Suppose a is an upper bound for $[1, 3]$. Then $y \leq a$ for all $y \in [1, 3]$; in particular, $3 \leq a$. Since 3 is an upper bound for $[1, 3]$, and any upper bound of the set must be at least 3, we have that the supremum of $[1, 3]$ is 3. ■

Proof. Let $x \in (1, 3)$. Then $x < 3$. Set $y = x + \frac{3-x}{2} = \frac{x+3}{2}$, which is the number half-way between x and 3. Then notice that

$$x + x < 2y = x + 3 < 3 + 3$$

and so $x < y < 3$. Thus $y \in (1, 3)$ and larger than x. Thus x cannot be an upper bound of $(1, 3)$. Since this is true for any $x \in (1, 3)$, this set has no maximum.

Note that we could choose other y-values, but this choice makes the argument quite clean. ■

Proof. Since this is a finite set of integers we can list out its elements explicitly and find the maximum and supremum from that. The set is equivalent to $\left\{n \in \mathbb{Z} : -\frac{7}{2} \leq n \leq \frac{23}{2}\right\}$. Hence it is the set $\{-3, -2, \ldots, 11\}$. Thus the element 11 is the maximum. It is also the supremum, since there is no greater element, and any $x < 11$ is not an upper bound. ■

Proof. We first prove that the supremum is 3 and then prove that there is no maximum.

Since $x < 3$ for all $x \in (1,3)$, 3 is an upper bound of $(1,3)$. Now let $a < 3$. If $a \leq 1$, then it is not an upper bound of $(1,3)$, since $2 \in (1,3)$ but $a < 2$. Now suppose $a > 1$. Then set y to be the number half-way between a and 3, namely set $y = a + \frac{3-a}{2} = \frac{a+3}{2}$. Then, as argued above,

$$2 < a + a < 2y = a + 3 < 3 + 3$$

Then $y \in (1,3)$ but $y > a$, so a is not an upper bound of $(1,3)$. Thus 3 is the supremum of $(1,3)$, as claimed.

Now let $x \in S$. Then $x = 2 - 1/n$ for some $n \in \mathbb{N}$. But then

$$\frac{1}{n+1} < \frac{1}{n},$$

and so

$$x = 2 - \frac{1}{n} < 2 - \frac{1}{n+1} = y.$$

Since $y \in S$ and $y > x$, x is not the maximum of S. As x was an arbitrary element of S, the set has no maximum. ■

Proof. Let $a \in \mathbb{R}$. We show that a is not an upper bound of the set. Let $k = \lceil a/\pi \rceil + 1$. Then

$$\pi k > \pi \cdot \frac{a}{\pi} = a.$$

Since πk is an element of $\{x \in \mathbb{R} : \cos(2x) = 1\}$, a is not an upper bound of the set. Since a was arbitrary, the set has no upper bound, and hence no maximum or supremum. ■

After completing this question, you may be wondering when a set has a supremum or a maximum. Every non-empty subset of the real numbers that has an upper bound also has a supremum; this property is called *completeness of the real numbers*. This may fail if the set has no upper bound, as we saw in part (e).

However, even if a non-empty subset of the real numbers has an upper bound, it may not have a maximum. For example, in part (b), we saw that $(1,3)$ has an upper bound (for example, 3), but no maximum.

Note that rational numbers do not share this property; they are not complete. For example, the set

$$A = \left\{ r \in \mathbb{Q} \ : \ 0 < r \leq \sqrt{2} \right\}$$

is the set of positive rational numbers that are at most $\sqrt{2}$. This set has a supremum in the real numbers; it is $\sqrt{2}$. We will show in Chapter 11 that $\sqrt{2} \notin \mathbb{Q}$. We could consider A to be a subset of the rational numbers, rather than the real numbers. Then A does not have a maximum nor a supremum in the rational numbers, even though it has an upper bound in the rational numbers (for example, 2). With some more work, one can show that for any $\frac{a}{b}$ in this set, you can construct another $\frac{c}{d}$ in the set which is larger. In fact, we'll make that an exercise in Chapter 11. This statement implies that $\sqrt{2}$ is the set's supremum in the real numbers, and that the set has no supremum in the rational numbers.

8.6.16.

Proof. Let $S, T \subset \mathbb{R}$, and $s = \sup(S)$, $t = \sup(T)$. Suppose $a = \max\{s, t\}$. Then for any $x \in S \cup T$ we know that $x \in S$ or $x \in T$.

- If $x \in S$, then $x \leq s \leq a$.
- Similarly, when $x \in T$, then $x \leq t \leq a$.

Hence a is an upper bound for $S \cup T$.

Now let b be any upper bound for $S \cup T$. By definition, this means that for all $x \in S \cup T$, $x \leq b$. Since all elements of S and T are also elements of $S \cup T$, it follows that b is an upper bound for S and T. Now, since s and t are the least upper bounds of S and T respectively, we must have $s \leq b$ and $t \leq b$. Hence $a = \max\{s, t\} \leq b$. Thus a is smaller than any upper bound of $S \cup T$, and so is the least upper bound of $S \cup T$, as required. ■

For (b), one cannot determine the supremum of the intersection $S \cap T$ from s, t. One requires more detailed information about the sets. See the scratchwork above.

Proof. Let $S, T \subset \mathbb{R}$, and $s = \sup(S)$, $t = \sup(T)$. Let $a = s + t$.

Then a is an upper bound for $S \cup T$; indeed, for any $x \in S + T$, we may write $x = y + z$ for some $y \in S$, $z \in T$. But by definition of the supremum, $y \leq s$ and $z \leq t$. Thus $x = y + z \leq s + t = a$.

Next, suppose that $b < a$. Then $\varepsilon = \frac{a-b}{2}$ is a positive real number. Hence $s - \varepsilon$ and $t - \varepsilon$ are not upper bounds for S and T respectively, by definition of the supremum. Consequently, there are some $c \in S$, $d \in T$ such that $c > s - \varepsilon$ and $d > t - \varepsilon$. But then

$$c + d > (s - \varepsilon) + (t - \varepsilon) = a - 2\varepsilon = a - (a - b) = b,$$

and since $c + d \in S + T$, we see that b is not an upper bound for $S + T$.

Thus a is the least upper bound of $S + T$, as required. ■

8.6.17.

Proof. Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence such that $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, and such that

$$a = \sup\{a_n : n \in \mathbb{N}\}$$

exists as a real number. Let $\varepsilon > 0$ be given. Then $a - \varepsilon < a$, so $a - \varepsilon$ is not an upper bound of the set $\{a_n : n \in \mathbb{N}\}$. Consequently, there is some $N \in \mathbb{N}$ such that $a - \varepsilon > a_N$. But then for any $n \geq N$, we have

$$a - \varepsilon < a_N \leq a_n.$$

Moreover, since a is an upper bound of the set $\{a_n : n \in \mathbb{N}\}$, we have

$$a_n \leq a < a + \varepsilon$$

for all $n \in \mathbb{N}$, and particularly, all $n \geq N$. Adding a to both sides of these inequalities, and putting these two inequalities together, we have

$$-\varepsilon < a_n - a < \varepsilon$$

for all $n \geq N$. Equivalently, $|a_n - a| < \varepsilon$ for all $n \geq N$. ■

9 · Relations
9.7 · Exercises

9.7.1. The relation is

$$R = \big\{(2,1),(2,3),(2,5),(3,1),(3,2),(3,4),(3,5),(4,1),(4,2),(4,3),(4,5),\\ (4,6),(5,1),(5,2),(5,3),(5,4),(5,6),(6,1),(6,2),(6,3),(6,4),(6,5)\big\}.$$

The relation is not reflexive: $(1,1) \notin R$. The relation is not symmetric: $(2,1) \in R$, but $(1,2) \notin R$. The relation is not transitive: $(2,3) \in R$ and $(3,2) \in R$, but $(2,2) \notin R$.

9.7.2.

Proof. We show that the relation is reflexive and symmetric but not transitive.

- Reflexive. For all $x \in \mathbb{R}$ we have $|x - x| = 0 < 1$. Thus, $x \mathcal{R} x$ for any $x \in \mathbb{R}$.
- Symmetric. Since $|x-y| = |y-x|$, we see that if $x \mathcal{R} y$, that is, $|x-y| < 1$, then $|y - x| < 1$, which implies $y \mathcal{R} x$.
- Not transitive. Consider $x = 0$, $y = 3/4$, and $z = 3/2$. Then we see that $|x - y| < 1$ and $|y - z| < 1$, but $|x - z| = 3/2 > 1$. Thus, $x \mathcal{R} y$, $y \mathcal{R} z$, but x is not related to z. Therefore the relation is not transitive.

■

9.7.3.

(a) The relation is a subset of $S \times S$, where $S = \{1,2,3\}$. Since $(i,i) \in R$ for all $i \in S$, the relation is reflexive. We can also see that the relation is symmetric, since $(i,j) \in R$ whenever $(j,i) \in R$. The relation is not transitive: for example, we have $(2,1) \in R$ and $(1,3) \in R$, but $(2,3) \notin R$.

(b) The relation is reflexive, since $a \mid a$ for any $a \in \mathbb{N}$. The relation is not symmetric; for example, $2 \mathcal{R} 4$, since $2 \mid 4$, but the reverse does not hold, since $4 \nmid 2$. The relation is transitive: for any $a, b, c \in \mathbb{N}$, if $a \mid b$ and $b \mid c$, then $a \mid c$, as shown in Exercise 3.5.10.

(c) For any $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Q}$, and so the relation is reflexive. For any $x, y \in \mathbb{R}$ such that $x - y \in \mathbb{Q}$, we have $y - x = -(x - y)$ is in $\mathbb{Q}$ as well. So the relation is symmetric. Finally, if $x, y, z \in \mathbb{R}$ so that $x - y \in \mathbb{Q}$ and $y - z \in \mathbb{Q}$, then $x - z = (x - y) - (y - z)$ is also in $\mathbb{Q}$, since the sum of

two rational numbers is also rational. Thus the relation is transitive.

(d) The relation is not reflexive since, for example, if $A = \{1\}$ then $A \cap A \neq \emptyset$. Indeed, it fails for any non-empty set A. The relation is symmetric, since $A \cap B = \emptyset$ if and only if $B \cap A = \emptyset$. The relation is not transitive. For example, if we take $A = \{1, 2\}$, $B = \{3\}$, and $C = \{1\}$, then $A \cap B = \emptyset$ and $B \cap C = \emptyset$, but $1 \in A \cap C$.

(e) Let $f : \mathbb{R} \to \mathbb{R}$. Then $f(x) - f(x) = 0$ for all $x \in \mathbb{R}$, and so $f - f$ is a linear function. Thus the relation is reflexive. Now let $g : \mathbb{R} \to \mathbb{R}$ also. If $f(x) - g(x) = mx + b$ for all $x \in \mathbb{R}$, then $g(x) - f(x) = -mx - b$ for all $x \in \mathbb{R}$. Then $g - f$ is also a linear function, so the relation is symmetric. Finally let $h : \mathbb{R} \to \mathbb{R}$. If $f(x) - g(x) = m_1x + b_1$ and $g(x) - h(x) = m_2x + b_2$ for all $x \in \mathbb{R}$, then

$$f(x) - h(x) = (f(x) - g(x)) - (g(x) - h(x)) = (m_1 - m_2)x + (b_1 - b_2)$$

for all $x \in \mathbb{R}$. So $f - h$ is also linear, and thus the relation is transitive.

9.7.4.

(a) Let $\mathcal{R}$ be a relation on the set X of all functions $\mathbb{R} \to \mathbb{R}$, defined as:

$$f \mathcal{R} g \text{ if there exists } x \in \mathbb{R} \text{ such that } f(x) = g(x).$$

- We see that by definition of the relation, it is reflexive.
- Moreover, if $f \mathcal{R} g$, then we know that $\exists x \in \mathbb{R}$ such that $f(x) = g(x)$. But, this also means that $g(x) = f(x)$. Hence the relation is also symmetric.
- But, the relation is not transitive. For a counterexample, let f, g and h such that $f(x) = 0$, $g(x) = x$ and $h(x) = 1$. We have $f \mathcal{R} g$ and $g \mathcal{R} h$ but it is not true that $f \mathcal{R} h$.

(b) Let $\mathcal{S}$ be a relation on $\mathbb{Z}$ defined by:

$$x \mathcal{S} y \text{ if } xy \equiv 0 \pmod 4.$$

- We see that $(1, 1) \notin \mathcal{S}$, since $1 \cdot 1 = 1 \not\equiv 0 \pmod 4$. Therefore, the relation is not reflexive.
- This relation is symmetric since if $xy \equiv 0 \pmod 4$, then $yx = xy \equiv 0 \pmod 4$, that is, if $(x, y) \in \mathcal{S}$, then $(y, x) \in \mathcal{S}$.
- This relation is not transitive. For a counterexample, we can take, $a = 1, b = 0$, and $c = 1$. Then, we see that $(1, 0), (0, 1) \in \mathcal{S}$, whereas, $(1, 1) \notin \mathcal{S}$.

9.7.5.

(a) *Proof.* We show that the relation is reflexive, symmetric, and transitive,

and hence an equivalence relation.

- It is reflexive, since the equation $x_1^2 + y_1^2 = x_2^2 + y_2^2$ holds when $(x_1, y_1) = (x_2, y_2)$.
- The relation is symmetric, as if $(x_1, y_1)\ \mathcal{R}\ (x_2, y_2)$, then we have $x_1^2 + y_1^2 = x_2^2 + y_2^2$, in which case $x_2^2 + y_2^2 = x_1^2 + y_1^2$.
- Finally, the relation is transitive, as $x_1^2 + y_1^2 = x_2^2 + y_2^2$ and $x_2^2 + y_2^2 = x_3^2 + y_3^2$ gives $x_1^2 + y_1^2 = x_3^2 + y_3^2$.

■

Now we need to determine the equivalence classes. Suppose $(a, b) \in \mathbb{R}^2$. We want to find the equivalence classe that (a, b) belongs to, namely, $[(a, b)]$. Note that $(x, y) \in [(a, b)]$ if and only if $x^2 + y^2 = a^2 + b^2$. If $a^2 + b^2 = 0$, then this forces $x = y = 0$. So $(0, 0)$ is the only member of its equivalence class. If $a^2+b^2 > 0$, then the equivalence class of $[(a, b)]$ consists of the points lying on the circle centered around the origin with radius $\sqrt{a^2 + b^2}$. This also implies that $[(a, b)] = [(0, \sqrt{a^2 + b^2})] = [(\sqrt{a^2 + b^2}, 0)]$, since $(a, b)\ \mathcal{R}\ (0, \sqrt{a^2 + b^2})$ and $(a, b)\ \mathcal{R}\ (0, \sqrt{a^2 + b^2})$. Therefore we can conclude that all equivalence classes can be written as $\{[(0, r)] : r > 0\}$, or $\{[(r, 0)] : r > 0\}$.

(b) *Proof.* The relation is an equivalence relation:

- A line is either vertical or has the same slope as itself, so the relation is reflexive.
- The statement that two lines have the same slope or are both vertical is symmetric, so the relation is symmetric.
- Suppose $\ell_1\ \mathcal{R}\ \ell_2$ and $\ell_2\ \mathcal{R}\ \ell_3$. Then either both ℓ_1 and ℓ_2 are vertical, or they have the same slope. The same is true for the pair ℓ_2 and ℓ_3. If ℓ_2 is vertical, then both ℓ_1 and ℓ_3 are, and $\ell_1\ \mathcal{R}\ \ell_3$. If ℓ_2 has slope m, then so do ℓ_1 and ℓ_3, giving $\ell_1\ \mathcal{R}\ \ell_3$. So the relation is transitive.

■

Let $\ell \in L$, and suppose the slope of ℓ is m. Then its equivalence class $[\ell]$ consists of all lines with slope m. For any $m \in \mathbb{R}$, $m \geq 0$, let ℓ_m be the line through the origin with slope m. Moreover, let ℓ_∞ be the vertical line passing through the origin. Then the set of equivalence classes is $\{[\ell_m] : m \in \mathbb{R}, m \geq 0\} \cup \{[\ell_\infty]\}$.

(c) *Proof.* The relation is an equivalence relation:

- The relation is reflexive, since $3 \mid (x^2 - x^2)$.
- The relation is symmetric, since if $3 \mid (x^2 - y^2)$, then $3 \mid (y^2 - x^2)$.

- The relation is also transitive: suppose $3 \mid (x^2 - y^2)$ and $3 \mid (y^2 - z^2)$. Then there are $k, \ell \in \mathbb{Z}$ so that $x^2 - y^2 = 3k$ and $y^2 - z^2 = 3\ell$, and so

$$x^2 - z^2 = (x^2 - y^2) + (y^2 - z^2) = 3(k + \ell).$$

  Since $k + \ell \in \mathbb{Z}$, $3 \mid (x^2 - z^2)$.

■

Let $x \in \mathbb{Z}$. Then by the Division Algorithm, $x = 3q + r$ for some $q \in \mathbb{Z}$, and for $r \in \{0, 1, 2\}$. Since the relation depends on divisibility by 3, it makes sense to examine what happens for the different remainders $x = 3q$, $x = 3q + 1$, and $x = 3q + 2$. In the first case, $x^2 = (3q)^2 = 9q^2$, which is divisible by 3. In the second case,

$$x^2 = (3q + 1)^2 = 9q^2 + 6q + 1 = 3(3q^2 + 2q) + 1,$$

so x^2 has a remainder of 1 when divided by 3. In the third case,

$$x^2 = (3q + 2)^2 = 9q^2 + 6q + 4 = 3(3q^2 + 2q + 1) + 1$$

so again x^2 has a remainder of 1 when divided by 3. Similar conclusions can be drawn for any other $y \in \mathbb{Z}$. Thus, if $3 \mid x^2 - y^2$, then either both x and y are divisible by 3 (in which case x^2 and y^2 are divisible by 3), or both x and y are not divisible by 3 (in which case x^2 and y^2 are both congruent to 1 mod 3). So the equivalence relation only has two equivalence classes, namely [0], which consists of all integers that are congruent to 0 mod 3, and [1], which consists of all integers that are congruent to 1 or 2 mod 3.

9.7.6. We prove that R is indeed an equivalence relation.

Proof. We must show that R is reflexive, symmetric, and transitive.

- Reflexive:For any $a \in \mathbb{Z}$, we have $(2a - 5a) = 3(-a)$, which implies $3 \mid (2a - 5a)$. Thus $a \mathcal{R} a$.
- Symmetric: Let $a, b \in \mathbb{Z}$ and assume $a \mathcal{R} b$. Then we see $3 \mid (2a - 5b)$, and so $2a - 5b = 3k$ for some $k \in \mathbb{Z}$. Then

$$2b - 5a = (5b - 2a) - 3b - 3a = -3k - 3b - 3a = 3(-k - b - a).$$

  Since $(-b - a - k) \in \mathbb{Z}$ we see that $3 \mid (2b - 5a)$. Therefore R is symmetric.
- Transitive: Let $a, b, c \in \mathbb{Z}$ and assume $a \mathcal{R} b$ and $b \mathcal{R} c$. Then we see $3 \mid (2a - 5b)$ and $3 \mid (2b - 5c)$, so that $2a - 5b = 3k$ and $2b - 5c = 3\ell$ for some $k, \ell \in \mathbb{Z}$. Then

$$2a - 5c = (2a - 5b) + 3b + (2b - 5c) = 3(k + b + \ell)$$

  Since $(k + b + \ell) \in \mathbb{Z}$ we see that $3 \mid (2a - 5c)$. Therefore R is transitive.

Thus R is an equivalence relation. ■

9.7.7.

Proof. Let us prove that $\mathcal{R}$ is an equivalence relation.

- Reflexivity: Let $A \in \mathcal{P}(E)$. Then $(q \in A) \vee (q \in \overline{A})$ which we can rewrite as $(q \in A \cap A) \vee (q \in \overline{A} \cap \overline{A})$. Hence, $A \mathcal{R} A$.
- Symmetry: The symmetry is immediate from the symmetry of the intersection of sets.
- Transitivity: Let $A, B, C \in \mathcal{P}(E)$ and assume that $A \mathcal{R} B$ and $B \mathcal{R} C$ so that

$$\big((q \in A \cap B) \vee (q \in \overline{A} \cap \overline{B})\big) \wedge \big((q \in B \cap C) \vee (q \in \overline{B} \cap \overline{C})\big) .$$

  Now we can study 4 cases in turn:

  - Case 1: $(q \in A \cap B) \wedge (q \in B \cap C)$. Then $q \in A$ and $q \in C$ so $q \in A \cap C$, which implies $A \mathcal{R} C$.
  - Case 2: $(q \in A \cap B) \wedge (q \in \overline{B} \cap \overline{C})$, which entails that $q \in B \cap \overline{B}$ so this case never happens.
  - Case 3: $(q \in \overline{A} \cap \overline{B}) \wedge (q \in B \cap C)$. This case does not happen for the same reason as above.
  - Case 4: $\big(q \in \overline{A} \cap \overline{B})\big) \wedge (q \in \overline{B} \cap \overline{C})$. From there $q \in \overline{A} \cap \overline{C}$ and so $A \mathcal{R} C$.

  Therefore we see that the relation is an equivalence relation.

■

9.7.8.

Proof. To prove that this is an equivalence relation, we need to show that R is reflexive, symmetric, and transitive.

- R is reflexive: Let $a \in \mathbb{Z}$. Notice that $7a^2 - 2a^2 = 5a^2$ and $5 \mid 5a^2$. Thus, $a \mathcal{R} a$, and so R is relexive.
- R is symmetric: Let $a, b \in \mathbb{Z}$ such that $a \mathcal{R} b$. This means that $7a^2 \equiv 2b^2 \pmod 5$, that is $5 \mid (7a^2 - 2b^2)$, and so $7a^2 - 2b^2 = 5\ell$ for some $\ell \in \mathbb{Z}$. Then notice that we can write

$$7b^2 - 2a^2 = 5a^2 + 5b^2 - (7a^2 - 2b^2) = 5(a^2 + b^2 - \ell)$$

  Thus $5 \mid (7b^2 - 2a^2)$, which implies $7b^2 \equiv 2a^2 \pmod 5$ Thus, $b \mathcal{R} a$ and so R is symmetric.
- R is transitive: Let $a, b, c \in \mathbb{Z}$ such that $a \mathcal{R} b$ and $b \mathcal{R} c$. This means that $7a^2 \equiv 2b^2 \pmod 5$, and $7b^2 \equiv 2c^2 \pmod 5$. So for some $k, \ell \in \mathbb{Z}$ we have

$$7a^2 - 2b^2 = 5k \qquad \text{and} \qquad 7b^2 - 2c^2 = 5\ell$$

Then

$$7a^2 - 2c^2 = (7a^2 - 2b^2) + (7b^2 - 2c^2) - 5b^2 = 5(k + \ell - b^2)$$

Hence, $5 \mid (7a^2 - 2c^2)$, which implies $7a^2 \equiv 2c^2 \pmod 5$ Thus, $a \mathcal{R} c$, and so R is symmetric.

■

Now let's consider the equivalence classes of R. Let $a, b \in \mathbb{Z}$ and suppose that $a \mathcal{R} b$. We know that $7a^2 \equiv 2b^2 \pmod 5$. It makes sense to consider the integers modulo 5, and by the division algorithm there are unique integers q_a, q_a, r_a, r_b with $0 \leq r_a, r_b < 5$ so that

$$a = 5q_a + r_a \qquad\qquad b = 5q_b + r_b$$

Then

$$\begin{aligned} 7a^2 &= 7(5q_a + r_a)^2 = 5(35q_a^2 + 14q_a r_a + r_a^2) + 2ra^2 \\ &\equiv 2r_a^2 \pmod 5 \qquad\qquad \text{and similarly} \\ 2b^2 &\equiv 2r_b^2 \pmod 5 \end{aligned}$$

Hence

$$a \mathcal{R} b \qquad \Longleftrightarrow \qquad 2r_a^2 \equiv 2r_b^2$$

So, for example, to determine all the numbers equivalent to $a = 0$, which has $r_a = 0$, we need to find all the remainders that give $2r_b^2 \equiv 0 \pmod 5$. Rather than do the specific case, we compute $2r^2 \pmod 5$ for $r = 0, 1, 2, 3, 4$ and then compare the results:

$2 \cdot 0^2 \equiv 0 \pmod 5$, $\quad 2 \cdot 1^2 \equiv 2 \pmod 5$ $\quad 2 \cdot 2^2 = 8 \equiv 3 \pmod 5$,
$2 \cdot 3^2 = 18 \equiv 3 \pmod 5$ $\quad 2 \cdot 4^2 = 32 \equiv 2 \pmod 5$

Hence the integers equivalent to 0 must have $r = 0$. That is

$$[0] = \{5n \text{ s.t. } n \in \mathbb{Z}\}$$

The integers equivalent to 1 must have $r = 1, 4$, since they both result in $2r^2 \equiv 2 \pmod 5$. That is

$$[1] = \{5n + 1 \text{ s.t. } n \in \mathbb{Z}\} \cup \{5n + 4 \text{ s.t. } n \in \mathbb{Z}\}$$

Notice that since $4 \equiv -1 \pmod 5$ we can also write this as

$$[1] = \{5n \pm 1 \text{ s.t. } n \in \mathbb{Z}\}$$

The integers equivalent to 2 must have $r = 2, 3$ since they both result in $2r^2 \equiv 3 \pmod 5$. And since $3 \equiv -2 \pmod 5$ we can write

$$[2] = \{5n \pm 2 \text{ s.t. } n \in \mathbb{Z}\}$$

Since every integer has remainder $0, 1, 2, 3, 4$ modulo 5, the above are all the equivalence classes.

9.7.9.

Proof. We need to check that the relation is reflexive, symmetric and transitive.

Reflexivity: Let $f \in P$. Then, since $f - f = 0$ and $0 \in \mathbb{R}$, we see that fTf. Hence T is reflexive.

Symmetry: Let $f, g \in P$ and assume that fTg. Then, by definition, we see that $f - g = c$ for some $c \in \mathbb{R}$. Hence, we see that $g - f = -c$, and since $-c \in \mathbb{R}$, we have gTf, that is, T is symmetric.

Transitive:. Let $f, g, h \in P$ and assume that fTg and gTh. This means $f - g = c$ for some $c \in \mathbb{R}$ and $g - h = d$ for some $d \in \mathbb{R}$. Hence, we see that

$$f - h = (f - g) + (g - h) = c + d.$$

Thus, since $c + d \in \mathbb{R}$, we see fTh. Therefore T is transitive.

Hence the relation T is an equivalence relation. ■

9.7.10.

(a) This statement is false.

Proof. As a counterexample, take $A = \{1, 2, 3\}$ and the relations

$$R = \{(1,1), (2,2), (3,3), (1,2), (2,1)\} \text{ and } S = \{(1,1), (2,2), (3,3), (1,3), (3,1)\}.$$

Note that R and S are equivalence relations. However,

$$R \cup S = \{(1,1), (2,2), (3,3), (1,2), (2,1), (1,3), (3,1)\}$$

is not an equivalence relation since it is not transitive; $(2,1), (1,3) \in R \cup S$, but $(2,3) \notin R \cup S$. ■

(b) This statement is true.

Proof. We need to show that $R \cap S$ is reflexive, symmetric, and transitive.

- Reflexive: Let $a \in A$. Since R and S are reflexive, $(a,a) \in R$ and $(a,a) \in S$. But then $(a,a) \in R \cap S$, so $R \cap S$ is reflexive.
- Symmetric: Suppose $(a,b) \in R \cap S$. Then $(a,b) \in R$ and $(a,b) \in S$ by definition of the intersection. Since R and S are symmetric, $(b,a) \in R$ and $(b,a) \in S$. Thus $(b,a) \in R \cap S$, and so $R \cap S$ is symmetric.
- Transitive: Suppose $(a,b) \in R \cap S$ and $(b,c) \in R \cap S$. Then $(a,b), (b,c) \in R$ and $(a,b), (b,c) \in S$, by definition of the intersection. Since R and S are transitive, $(a,c) \in R$ and $(a,c) \in S$. Hence $(a,c) \in R \cap S$, and $R \cap S$ is transitive.

■

9.7.11.

(a) Take $A = \{1, 2, 3\}$ and take the relation $R = \{(1,1), (1,2), (2,1), (2,2)\}$.

Then R is a symmetric and transitive relation, but it's not reflexive, since $3 \in A$ but $(3,3) \notin R$.

Proof. Assume that for all $a \in A$, there exists $b \in A$ such that $a \mathcal{R} b$. Let $x \in A$ be arbitrary. Then, by the assumption, there is some $y \in A$ such that $x \mathcal{R} y$. Since R is symmetric, we see that $y \mathcal{R} x$. Moreover, since R is transitive and $x \mathcal{R} y$ and $y \mathcal{R} x$, we see $x \mathcal{R} x$. As x was arbitrary, R is reflexive. ■

9.7.12.

(a) This statement is false.

Proof. Let $a \in A$. Then, we see that if R is reflexive, then $(a,a) \in R$. Thus, $(a,a) \notin \overline{R}$. Therefore $\overline{R}$ is not reflexive. ■

One can make a more explicit counter-example by considering the set $A = \{1\}$ and $R = \{(1,1)\}$. Then $\bar{R} = \emptyset$. While $\bar{R}$ is a relation it is not reflexive since $(1,1) \notin \bar{R}$.

(b) This statement is true.

Proof. Let $a, b \in A$. Assume R is symmetric and assume that $(a,b) \in \overline{R}$. By definition, we see that $(a,b) \notin R$. Since R is symmetric we know that if $(b,a) \in R$, then $(a,b) \in R$. The contrapositive of this statement implies that if $(a,b) \notin R$, then $(b,a) \notin R$. Hence, $(b,a) \in \overline{R}$. Therefore $\overline{R}$ is symmetric. ■

Notice that we are using modus-tollens in this proof. We know that $(a,b) \in R \implies (b,a) \in R$, and that $(b,a) \notin R$, so we conclude $(a,b) \notin R$.

(c) This statement is false.

Proof. For a counterexample, we can take $A = \{1,2\}$ and $R = \{(1,1),(2,2)\}$. Notice that R is the "equality" relation, and it is transitive. Then the complementary relation $\overline{R} = \{(1,2),(2,1)\}$. Since $(1,2) \in \overline{R}$ and $(2,1) \in \overline{R}$ but $(1,1) \notin \overline{R}$, we see that $\overline{R}$ is not transitive. ■

9.7.13.

Proof. We prove each implication in turn.

First, suppose that R is sparse, and observe that since $n \mid 0$ for all $n \in \mathbb{N}$, $(a,a) \in \mathbb{R}$ for all $a \in \mathbb{Z}$. Then since R is sparse, $(a, a+1) \notin R$ and hence $n \nmid (a - a - 1) = -1$. In particular, $n \neq 1$, as desired.

For the second implication we will prove the contrapositive. Suppose that R is not sparse. Then we have two cases: either there exist $a, b \in \mathbb{Z}$ so that (a,b) and $(a, b+1)$ are in R, or there exist $a, b \in \mathbb{Z}$ so that (a,b) and $(a+1, b)$ are in R.

- In the first case, we have that $n|(a-b)$ and $n|(a-(b+1))$, and hence n divides their difference. That is, $n|1$. Since $n \in \mathbb{N}$, this implies $n = 1$.

- The second case is very similar. We have that $n|(a-b)$ and $n|(a+1-b)$, and hence $n|1$. In particular $n=1$.

Note that these are the modular equivalence relations. ■

The statement in (b) is false.

Proof. Consider $R = \mathbb{Z} \times \mathbb{Z}$. This is an equivalence relation: all three conditions are trivially satisfied since R contains all pairs of integers. However, both $(0,0)$ and $(0,1)$ are elements of R, so R is not sparse. ■

Note that the counter-example in this disproof is just the case $n=1$ in the previous part of the question.

9.7.14.

Proof. Let A, P, Q, R be as given in the statement of the problem. Then to show that R is a partition of A, we need to show that

$$\bigcup_{X \in R} X = A$$

and

$$(U_1, U_2 \in R, U_1 \neq U_2) \implies (U_1 \cap U_2 = \emptyset).$$

We prove each of these in turn.

- To prove the set equality we need to show that each side is included in the other. One of these inclusions is straight-forward, since if $X \in R$, then, by definition of R, $X \subseteq A$. Therefore, we see that $\bigcup_{X \in R} X \subseteq A$.

  Thus, we just need to show that $A \subseteq \bigcup_{X \in R} X$. Let $x \in A$. Since P and Q are partitions of A, there exists $S \in P$ and $T \in Q$ such that $x \in S$ and $x \in T$. This entails that $x \in S \cap T$ and $S \cap T \in R$. Therefore $x \in \bigcup_{X \in R} X$.

- To show the second property of partitions let $U_1, U_2 \in R$. By definition $U_1 = S_1 \cap T_1$ for some $S_1 \in P$ and $T_1 \in Q$. Similarly, $U_2 = S_2 \cap T_2$ for some $S_2 \in P$ and $T_2 \in Q$.

  We can prove the contrapositive of the implication. Assume that $U_1 \cap U_2 \neq \emptyset$. Hence there is some $x \in U_1 \cap U_2$. Since $x \in U_1$ this implies that $x \in S_1 \cap T_1$, and since $x \in U_2$ we know that $x \in S_2 \cap T_2$. Thus x is in all of S_1, S_2, T_1, T_2.

  - Since $x \in S_1$ and $x \in S_2$, it follows that $x \in S_1 \cap S_2$. Thus, since S_1, S_2 are parts of a partition, we have $S_1 = S_2$.
  - Similarly, since $x \in T_1$ and $x \in T_2$, we know that $T_1 = T_2$.

  Thus we must have $U_1 = S_1 \cap T_1 = S_2 \cap T_2 = U_2$. Hence when $U_1 \cap U_2 \neq \emptyset$ it follows that $U_1 = U_2$.

In the end, R is a partition of A. ■

9.7.15. We first prove that R is an equivalence relation.

Proof. We have to show that R is reflexive, symmetric and transitive.

- (reflexive) We have $[x]_n \S_n$ since $[x]_n[1]_n = [x]_n$ for all $[x]_n \in \mathbb{Z}_n$.
- (symmetric) Suppose $[x]_n \dagger_n$. Then by definition of R, we know that $[x]_n[u]_n = [y]_n$ for some *invertible* $[u]_n$. Thus, we know that $[u]_n$ has an inverse in $\mathbb{Z}_n$, and call $[v]_n \in \mathbb{Z}_n$ to be a inverse of $[u]_n$, i.e. $[u]_n[v]_n = [1]_n$. Then since $[y]_n = [x]_n[u]_n$, we know that $[y]_n[v]_n = [x]_n[u]_n[v]_n = [x]_n[1]_n = [x]_n$ and thus $[y]_n \S_n$.
- (transitive) Suppose $[x]_n \dagger_n$ and $[y]_n \ddagger_n$, that is $[x]_n[u]_n = [y]_n$ and $[y]_n[v]_n = [z]_n$ with $[u]_n, [v]_n$ both admitting multiplicative inverse.

  Then we have $[x]_n[u]_n[v]_n = [y]_n[v]_n = [z]_n$. Denoting the multiplicative inverses of $[u]_n$ and $[v]_n$ by $[\bar{u}]_n$ and $[\bar{v}]_n$, respectively, we see that

$$[u]_n[v]_n[\bar{v}]_n[\bar{u}]_n = [u]_n[1]_n[\bar{u}]_n = [u]_n[\bar{u}]_n = [1]_n.$$

  This means that $[u]_n[v]_n$ admits a multiplicative inverse.

  Therefore, we have $[x]_n \ddagger_n$.

■

For the second part, we first note that the set of elements in $\mathbb{Z}_6$ with a multiplicative inverses are $U = \{[1]_6, [5]_6\}$. We can see this by looking at the multiplication table for $\mathbb{Z}_6$:

	$[0]_6$	$[1]_6$	$[2]_6$	$[3]_6$	$[4]_6$	$[5]_6$
$[0]_6$	$[0]_6$	$[0]_6$	$[0]_6$	$[0]_6$	$[0]_6$	$[0]_6$
$[1]_6$	$[0]_6$	$[1]_6$	$[2]_6$	$[3]_6$	$[4]_6$	$[5]_6$
$[2]_6$	$[0]_6$	$[2]_6$	$[4]_6$	$[0]_6$	$[2]_6$	$[4]_6$
$[3]_6$	$[0]_6$	$[3]_6$	$[0]_6$	$[3]_6$	$[0]_6$	$[3]_6$
$[4]_6$	$[0]_6$	$[4]_6$	$[2]_6$	$[0]_6$	$[4]_6$	$[2]_6$
$[5]_6$	$[0]_6$	$[5]_6$	$[4]_6$	$[3]_6$	$[2]_6$	$[1]_6$

Notice that the only products that are equal to $[1]_6$ are $[1]_6[1]_6 = [1]_6$ and $[5]_6[5]_6 = [1]_6$.

Thus we may list the equivalence classes defined by R:

- We see that

$$\begin{aligned}[[0]_6] &= \{[y]_6 \in \mathbb{Z}_6 \colon [y]_6 = [0]_6[u]_6 \text{ for some invertible } [u]_6 \in \mathbb{Z}_6\} \\ &= \{[0]_6 u : u \in U\} = \{[0]_6\}.\end{aligned}$$

  Then similarly,

- $[[1]_6] = \{[1]_6 u : u \in U\} = \{[1]_6, [5]_6\}$.

- $[[2]_6] = \{[2]_6 u : u \in U\} = \{[2]_6, [4]_6\}$.
- $[[3]_6] = \{[3]_6 u : u \in U\} = \{[3]_6\}$.

Since these equivalence classes contain all the elements in $\mathbb{Z}_6$, we see that there are no other equivalence classes.

9.7.16.

Proof. Let $n \in \mathbb{Z}$. Assume that $3 \mid n$ and $8 \mid n$. This means that $n = 3k$ and $n = 8l$ for some $k, l \in \mathbb{Z}$. Thus, we see $8n = 24k$ and $3n = 24l$. Moreover, since $\gcd(3, 8) = 1$, by Bézout's identity, we see that $\exists x, y \in \mathbb{Z}$ such that $3x + 8y = 1$ (we can take x=3, y=-1). Using the x, y in the Bézout's identity, we get $3xn = 24xl$ and $8yn = 24yk$. Thus, adding two equations together, we get $(3x + 8y)n = n = 24(xl + yk)$. Since $(xl + yk) \in \mathbb{Z}$, we see $24 \mid n$ as desired. ■

We use the result of (a) to prove (b).

Proof. Assume p is prime. We see that proving $p^2 \equiv 1 \pmod{24}$ is equivalent to showing $24 \mid (p^2 - 1)$ and thus by part (a), it is enough to show $8 \mid (p^2 - 1)$ and $3 \mid (p^2 - 1)$.

Proof that $3 \mid (p^2 - 1)$: We know, since p is prime, that $3 \nmid p$. Then we see that we have 2 cases $p \equiv 1 \pmod 3$ or $p \equiv 2 \pmod 3$.

- *Case 1:* $p \equiv 1 \pmod 3$*:* Then we see that $p^2 \equiv 1 \pmod 3$, that is, $3 \mid (p^2 - 1)$.
- *Case 2:* $p \equiv 2 \pmod 3$*:* Then we see that $p^2 \equiv 4 \equiv 1 \pmod 3$, which again proves $3 \mid (p^2 - 1)$.

Proof that $8 \mid (p^2 - 1)$: Since p is prime and $p \geq 5$ we know that p is odd. Thus, $p = 2k + 1$ for some $k \in \mathbb{Z}$. This means $p^2 - 1 = 4k(k + 1)$. Moreover, we see that for k we have two cases: $k = 2n$ or $k = 2n + 1$ for some $n \in \mathbb{Z}$.

- *Case 1:* $k = 2n$: In this case we see that $p^2 - 1 = 4k(k + 1) = 8n(k + 1)$; and since $n(k + 1) \in \mathbb{Z}$, we see that $8 \mid (p^2 - 1)$.
- *Case 2:* $k = 2n + 1$: In this case we see that $p^2 - 1 = 4k(k + 1) = 4(2n + 1)(2n + 2) = 8(n + 1)(2n + 1)$; and since $(n + 1)(2n + 1) \in \mathbb{Z}$, we see that $8 \mid (p^2 - 1)$.

Therefore, $3 \mid (p^2 - 1)$ and $8 \mid (p^2 - 1)$ which implies $24 \mid (p^2 - 1)$. ■

9.7.17.

Proof. Let p be a prime number, and suppose that $n \in \mathbb{Z}$ is such that $n \not\equiv 0 \pmod p$. Then $p \nmid (n - 0)$, so $p \nmid n$. Since p is prime, its only divisors are 1 and p, and as p isn't a divisor of n, we have $\gcd(n, p) = 1$. Then, by Bézout's identity, there are some $k, \ell \in \mathbb{Z}$ so that $nk + \ell p = 1$. Rearranging, we have $\ell p = 1 - nk$, and so $p \mid (1 - nk)$. Then, by definition, $nk \equiv 1 \pmod p$. ■

For (b), take $p = 4$ and set $n = 2$. Then for any k, we know that nk is

even, and so $nk-1$ is odd. Since $nk-1$ is odd it is not divisible by 4. Thus $nk \not\equiv 1 \pmod 4$.

9.7.18.

Proof. Let $a, b, d \in \mathbb{Z}$ such that $d \mid ab$. By Bézout's lemma, there are some $x, y \in \mathbb{Z}$ such that $\gcd(a,d) = xa + yd$. Multiplying the equation by b, we have $b\gcd(a,d) = xab + ydb$. Since $d \mid ab$, there is some $k \in \mathbb{Z}$ so that $ab = kd$. Hence

$$b\gcd(a,d) = xab + ydb = xkd + ydb = d(xk + yb).$$

Now $\gcd(a,d) \mid d$, and so there is some $\ell \in \mathbb{Z}$ so that $\ell\gcd(a,d) = d$. But then

$$b\gcd(a,d) = \ell\gcd(a,d)(xk + yb)$$

and as $\gcd(a,d) \neq 0$,

$$b = \ell(xk + yb).$$

Since $xk + yb \in \mathbb{Z}$, we have $\ell \mid b$. But $\ell = d/\gcd(a,d)$, and so $d/\gcd(a,d)$ divides b, as required. ■

9.7.19.

Proof. Let $a, b \in \mathbb{Z}$, at least one of which is non-zero. Moreover, suppose $d \in \mathbb{Z}$ divides both a and b. Then there are $k, \ell \in \mathbb{Z}$ such that $a = kd$ and $b = \ell d$. By Bézout's identity, there are $x, y \in \mathbb{Z}$ such that

$$\gcd(a,b) = ax + by = kdx + \ell dy = d(kx + \ell y).$$

Since $kx + \ell y \in \mathbb{Z}$, $d \mid \gcd(a,b)$. ■

Proof. Let $a, b \in \mathbb{Z}$, at least one of which is non-zero, and let $m \in \mathbb{N}$. Let $d = \gcd(a,b)$. Then $d \mid a$ and $d \mid b$, and so there are $k, \ell \in \mathbb{Z}$ so that $a = kd$ and $b = \ell d$. Multiplying both equations through by m, we have $ma = k(md)$ and $mb = \ell(md)$. Therefore $md \mid ma$ and $md \mid mb$. Hence $md \leq \gcd(ma, mb)$.

Note that m divides both ma and mb, so by part (a), $m \mid \gcd(ma, mb)$. Suppose that $e \in \mathbb{Z}$ is such that $me = \gcd(ma, mb)$. Since $me \mid ma$ and $me \mid mb$, there are $u, v \in \mathbb{Z}$ so that $ma = meu$ and $mb = mev$. But $m \neq 0$, implying that $a = ue$ and $b = ve$, so e is a common divisor of a and b. Therefore $e \leq \gcd(a,b)$. Since $m > 0$, we have $\gcd(ma, mb) = me \leq m\gcd(a,b)$. As we have already shown the reverse inequality, we conclude that $\gcd(ma, mb) = m\gcd(a,b)$, as required. ■

For (c) set $a = b = c = 2$. Then $gcd(a,b) = \gcd(c,b) = \gcd(ac,b) = 2$, and so $\gcd(ac,b) \neq \gcd(a,b) \cdot \gcd(c,b)$.

9.7.20.

Proof. Pascal's identity tells us that every binomial coefficient $\binom{n}{r}$ is either a boundary term $\binom{n}{n} = 1$, $\binom{n}{0} = 1$ or can be written as the sum of two earlier terms $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$. We can use this as the basis of an induction argument.

- Base case: When $n = 0$ we have $\binom{n}{n} = 1$.
- Inductive step: Assume that $\binom{k}{r} \in \mathbb{Z}$ for all $0 \leq r \leq k$. Then consider $\binom{k+1}{s}$. If $s = 0, k+1$ then, by the boundary conditions, we know that the coefficient is an integer. Otherwise, we can write $\binom{k+1}{s} = \binom{k}{s-1} + \binom{k}{s}$ which is, by hypothesis, the sum of two integers, and so is itself an integer.

Thus, by mathematical induction, the binomial coefficients are integers. ■

Proof. Let p be prime and let $0 < k < p$ be an integer. Then the binomial coefficient

$$\binom{p}{k} = \frac{p!}{(p-k)!k!}.$$

The numerator contains a factor of p. However, since $0 < k < p$ the denominator is a product of positive integers that are strictly smaller than p. Thus there is no term in the denominator that can cancel the *prime* p in the numerator. Thus the binomial coefficient is an integer that is divisible by p. ■

Proof. Let x, y and p be as given. Then, by the Binomial Theorem

$$(x+y)^p = x^p + y^p + \sum_{k=1}^{p-1} \binom{p}{k} x^{p-k} y^k$$

Hence it suffices to show that

$$\binom{p}{k} \equiv 0 \pmod{p} \qquad \text{when } 0 < k < p$$

This is precisely what we proved above. From this, the result follows since every term in the expansion of $(x+y)^p$

$$\binom{p}{k} x^{p-k} y^k$$

is a multiple of p, excepting x^p and y^p. ■

10 · Functions
10.8 · Exercises

10.8.1. The set is a function from $\mathbb{R}^2$ to $\mathbb{R}^3$.

Proof. We see that for every pair $(x, y) \in \mathbb{R}^2$, there is a unique ordered pair $((x, y), (5y, 4x, x+y)) \in \mathbb{R}^2 \times \mathbb{R}^3$. Therefore, θ defines a function from $\mathbb{R}^2$ to $\mathbb{R}^3$. Moreover, this implies that the domain of θ is $\mathbb{R}^2$.

We know that the range of θ is the set $Y \subseteq \mathbb{R}^3$ satisfying:

$$\begin{aligned} Y &= \left\{(a,b,c) \in \mathbb{R}^3 : (a,b,c) = (5y, 4x, x+y) \text{ for some } (x,y) \in \mathbb{R}^2\right\} \\ &= \left\{(a,b,c) \in \mathbb{R}^3 : a = 5y, b = 4x, c = x+y \text{ for some } (x,y) \in \mathbb{R}^2\right\} \end{aligned}$$

Writing c in terms of a and b,

$$= \left\{(a,b,c) \in \mathbb{R}^3 : a = 5y, b = 4x, c = a/5 + b/4 \text{ for some } (x,y) \in \mathbb{R}^2\right\}$$
$$= \left\{(a,b,a/5 + b/4) \in \mathbb{R}^3 : a = 5y, b = 4x \text{ for some } (x,y) \in \mathbb{R}^2\right\}$$

Using the fact that $\{(5y, 4x) : (x,y) \in \mathbb{R}^2\} = \mathbb{R}^2$,

$$= \left\{(a,b,a/5 + b/4) \in \mathbb{R}^3 : a, b \in \mathbb{R}\right\}.$$

Hence the range of the function is the plane $20z - 4a - 5b = 0$ in $\mathbb{R}^3$. ■

10.8.2. *Claim:* ϕ is a function for all $a, b \in \mathbb{N}$ satisfying $b \mid 6$ and $b \mid a$.

Proof. For ϕ to define a function $\forall x \in \mathbb{Z}$, there has to be a unique point $y \in \mathbb{Z}$ such that $(x,y) \in \phi$. Let $x \in \mathbb{Z}$. Then, by definition of ϕ, we see that $y = \dfrac{6 - ax}{b}$, which is uniquely defined for any $x \in \mathbb{Z}$. Then, we see that for ϕ to be a function, $\dfrac{6 - ax}{b}$ must be an integer.

Since this has to be true for any $x \in \mathbb{Z}$, we see that it has to be true for $x = 0$, so $b \mid 6$. Thus, $b \in \{1, 2, 3, 6\}$.

This also means that for ϕ for be a function $\dfrac{ax}{b}$ must also be an integer, that is $b \mid ax$. Setting $x = 1$ tells us that a must be a multiple of b as claimed.

Now assume that $b \mid 6$ and $b \mid a$, so that $6 = bk, a = b\ell$ for some $k, \ell \in \mathbb{Z}$. Notice that $k \neq 0$. Then the condition that $ax + by = 6$ becomes

$$\ell bx + by = bk$$

and dividing through by k we have

$$\ell x + y = k$$

or $y = k - \ell x$. Then for any $x \in \mathbb{Z}$ we know that $y \in \mathbb{Z}$ and so ϕ is a function. ■

10.8.3.

Proof. Let f be defined as above. To show that

$$f(\mathbb{R}) = [-1, 1]$$

we prove each side is included in the other.

- $f(\mathbb{R}) \subseteq [-1,1]$: Let $y \in f(\mathbb{R})$. Then, by definition, we know that there is a $x \in \mathbb{R}$, such that $y = f(x) = \dfrac{2x}{1 + x^2}$.

  Now, for all $x \in \mathbb{R}$ we know that

  $$(1 + 2x + x^2) = (1 + x)^2 \geq 0 \quad \text{and} \quad (1 - 2x + x^2) = (1 - x)^2 \geq 0$$

We can rearrange this to give

$$1 + x^2 \geq -2x \qquad \text{and} \qquad 1 + x^2 \geq 2x$$

Combining these gives us

$$-(1 + x^2) \leq 2x \leq 1 + x^2$$

Since $1 + x^2 \geq 1$ we can divide both sides of this inequality by $1 + x^2$ to get

$$-1 \leq \frac{2x}{1 + x^2} \leq 1$$

which is precisely the result we require. Thus $f(\mathbb{R}) \subseteq [-1, 1]$.

- $[-1, 1] \subseteq f(\mathbb{R})$: Let $y \in [-1, 1]$. We need to show that $f \in f(\mathbb{R})$. That is, we need to find $x \in \mathbb{R}$, such that $f(x) = \dfrac{2x}{1 + x^2} = y$. Now, either $y = 0$ or $y \neq 0$.
  - When $y = 0$, set $x = 0$ and then note that $f(0) = 0$ as required.
  - When $y \neq 0$ set

$$x = \frac{1 + \sqrt{1 - y^2}}{y}.$$

Since $y \neq 0$ and $y \in [-1, 1]$, the above is a real number. Then we verify that

$$\begin{aligned}
f(x) &= \frac{\frac{2+2\sqrt{1-y^2}}{y}}{\left(\frac{1+\sqrt{1-y^2}}{y}\right)^2 + 1} && \text{multiply by } y^2 \\
&= \frac{2y + 2y\sqrt{1 - y^2}}{\left(1 + \sqrt{1 - y^2}\right)^2 + y^2} && \text{expand} \\
&= \frac{2y + 2y\sqrt{1 - y^2}}{\left(1 + 2\sqrt{1 - y^2} + 1 - y^2\right) + y^2} \\
&= \frac{2y + 2y\sqrt{1 - y^2}}{2 + 2\sqrt{1 - y^2}} \\
&= y
\end{aligned}$$

Thus $y \in f(\mathbb{R})$ and so $[-1, 1] \subseteq f(\mathbb{R})$.

Thus, we see that $f(\mathbb{R}) = [-1, 1]$. ■

10.8.4.

(a) We see that the function can be written as $f : \{1,2,3,4,5,6,7\} \to \{0,1,2,3,4,5,6,7,8,9\}$ defined by

$$\begin{aligned} f(1) &= 3 & f(2) &= 8 \\ f(3) &= 3 & f(4) &= 1 \\ f(5) &= 2 & f(6) &= 4 \\ f(7) &= 6. \end{aligned}$$

Therefore, we compute

$$\begin{aligned} f\big(\{1,2,3\}\big) &= \{3,8\}, \\ f\big(\{4,5,6,7\}\big) &= \{1,2,4,6\}, \\ f(\emptyset) &= \emptyset \\ f^{-1}\big(\{0,5,9\}\big) &= \emptyset, \text{ and} \\ f^{-1}\big(\{0,3,5,9\}\big) &= \{1,3\}. \end{aligned}$$

(b) We begin by writing the function g in a few ways. Below, we have written the quadratic in standard form, factored form, and vertex form, respectively.

$$g(x) = 4x^2 - x - 3 = (4x+3)(x-1) = \frac{1}{16}(8x-1)^2 - \frac{49}{16}.$$

From the vertex form of $g(x)$, we see that the function is symmetric around $x = \frac{1}{8}$. Then plugging in $\frac{1}{8}$ to $g(x)$, we find

$$g\left(\left\{\frac{1}{8}\right\}\right) = \left\{-\frac{49}{16}\right\}.$$

Moreover, from the factored form of $g(x)$, we see that

$$g^{-1}(\{0\}) = \left\{-\frac{3}{4}, 1\right\}.$$

In particular, the previous two computations tell us that $g(x)$ has a minimum when $x = \frac{1}{8}$. Therefore, $g(x)$ is decreasing on the interval $\left(-\infty, \frac{1}{8}\right)$ and increasing on $\left(\frac{1}{8}, \infty\right)$. With this in mind, we compute

$$\begin{aligned} g((-1,0) \cup [3,4]) &= g((-1,0)) \cup g([3,4]) \\ &= (g(0), g(-1)) \cup [g(3), g(4)] \\ &= (-3,2) \cup [30,57]. \end{aligned}$$

Finally, since $g(x)$ has a minimum when $x = \frac{1}{8}$, and $g\left(\frac{1}{8}\right) = -\frac{49}{16}$, we find

$$g^{-1}([-10,-5)) = \emptyset.$$

(c) First, we notice that $h(t)$ is 1-periodic, that is the function satisfies $h(t) = h(t+1)$. We use this fact to compute

$$\begin{aligned} h(\mathbb{Z}) = h(\{0\}) = \{0\} \text{ and} \\ h\left(\left\{\frac{1}{4}, \frac{7}{2}, \frac{19}{4}, 22\right\}\right) &= h\left(\left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\}\right) \\ &= \left\{h\left(\frac{1}{4}\right), h\left(\frac{1}{2}\right), h\left(\frac{3}{4}\right), h(1)\right\} = \{-1, 0, 1\}. \end{aligned}$$

To find the desired preimages, we first find the preimage of h for $t \in [0,1)$, and then use 1-periodicity to find the preimage for $t \in \mathbb{R}$.

For $t \in [0,1)$, $h^{-1}(\{1\}) = \left\{\frac{1}{4}\right\}$. Therefore for $t \in \mathbb{R}$,

$$h^{-1}(\{1\}) = \left\{\frac{1}{4} + k : k \in \mathbb{Z}\right\}.$$

Finally, for $t \in [0,1)$, $h^{-1}([0,1)) = \left[0, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \frac{1}{2}\right]$. Therefore for $t \in \mathbb{R}$,

$$h^{-1}([0,1)) = \bigcup_{k \in \mathbb{Z}} \left(\left[k, k + \frac{1}{4}\right) \cup \left(k + \frac{1}{4}, k + \frac{1}{2}\right]\right).$$

10.8.5.

Proof. We have to prove that $LHS \subseteq RHS$ and $RHS \subseteq LHS$.

- Let $x \in LHS$. Then, by definition of the preimage, we know that $f(x) \in E - F$. Since $x \in E$, we know that $x \in f^{-1}(E)$. Similarly since we know (via modus tollens 2.5.2) that $f(x) \notin f(F)$ implies $x \notin f^{-1}(F)$. Hence $x \in f^{-1}(E) - f^{-1}(F) = RHS$.
- Now let $x \in RHS$, so that $x \in f^{-1}(E)$ and $x \notin f^{-1}(F)$. Since $x \in f^{-1}(E)$, the definition of the preimage implies that $f(x) \in E$. Similarly (again via modus tollens 2.5.2) since $x \notin f^{-1}(F)$, it follows that $f(x) \notin F$. So $f(x) \in E - F$ and the definition of preimage implies that $x \in f^{-1}(E - F) = LHS$.

We now conclude: $f^{-1}(E) - f^{-1}(F) = f^{-1}(E - F)$ ∎

10.8.6.

Proof. We will show that this function is neither injective nor surjective. First, observe that, by completing the square, we can rewrite the function as

$$f(x) = \left(x + \frac{a}{2}\right)^2 + (b - \frac{a^2}{4}).$$

- f is not injective. Since f can be rewritten as above we see that

$$f\left(-1 - \frac{a}{2}\right) = \left(-1 - \frac{a}{2} + \frac{a}{2}\right)^2 + \left(b - \frac{a^2}{4}\right)$$

$$= 1 + \left(b - \frac{a^2}{4}\right)$$

and also

$$f\left(1 - \frac{a}{2}\right) = \left(1 - \frac{a}{2} + \frac{a}{2}\right)^2 + \left(b - \frac{a^2}{4}\right)$$
$$= 1 + \left(b - \frac{a^2}{4}\right) = f\left(-1 - \frac{a}{2}\right)$$

Since $-1 - \frac{a}{2} \neq 1 - \frac{a}{2}$, we see that f is not injective.

- f is not surjective. Since $f(x) = \left(x + \frac{a}{2}\right)^2 + \left(b - \frac{a^2}{4}\right)$ and since $\left(x + \frac{a}{2}\right)^2 \geq 0$ for all $x \in \mathbb{R}$, we can conclude that $f(x) \geq b - \frac{a^2}{4}$ for all $x \in \mathbb{R}$. So if we take $y = b - a^2 - 1 < b - a^2 \leq b - \frac{a^2}{4} \leq f(x)$ for all $x \in \mathbb{R}$. Thus there is no $x \in \mathbb{R}$ such that $f(x) = y$, and hence f is not surjective.

■

10.8.7.

Proof. First we see that the function defined by $g(n) = n$ satisfies the properties above.

Then, let us prove that any function f that satisfies the above has to be equal to g. Let us prove this using strong induction on the property $P(n) : f(n) = n$.

- Base case: $n = 1$. Since $f(1) \leq 1$ and $f(1) \in \mathbb{N}$, then $f(1) = 1$.
- Induction step: assume that $f(k) = k$ for all $k \leq n$. By assumption, $f(n+1) \in \{1, 2, 3, \ldots, n+1\}$. We cannot have $f(n+1) = k$ for any $k \leq n$ by injectivity of f, because we already have $f(k) = k$ and $k \neq n+1$. Hence the only possible value is $f(n+1) = n+1$.

In the end, we conclude by strong induction that for all $n \in \mathbb{N}$ we have $f(n) = n$.

■

10.8.8.

Proof. Let $f : [3, \infty) \to [5, \infty)$, defined by $f(x) = x^2 - 6x + 14$. We want to show that f is a bijective function. To do that, we need to show that f is both injective and surjective.

- Injective: Let $a, b \in [3, \infty)$ and $f(a) = f(b)$. This means that $a^2 - 6a + 14 = b^2 - 6b + 14$. Cancelling 14 from both sides and collecting terms together, we see $a^2 - b^2 - 6(a - b) = (a + b - 6)(a - b) = 0$. Then, we see that $a + b - 6 = 0$ or $a - b = 0$. If $a - b = 0$ then we know that $a = b$. On the other hand, when $a + b = 6$, since $a, b \geq 3$, we see that $a = 3 = b$. Thus, we see that in all the cases, $a = b$.

  Hence, f is injective.

- Surjective: Let $y \in [5, \infty)$. Then set $x = 3 + \sqrt{y-5}$. Since $y \geq 5$, we know that $\sqrt{y-5}$ is a non-negative real number, and thus $x \geq 3$ lies in the domain of the function. We now verify that $f(x) = y$:

$$\begin{aligned} f(x) &= (3 + \sqrt{y-5})^2 - 6(3 + \sqrt{y-5}) + 14 \\ &= 9 + 6\sqrt{y-5} + (y-5) - 18 - 6\sqrt{y-5} + 14 \\ &= y \end{aligned}$$

  as required. Hence f is surjective.

Therefore f is a bijective function. ■

10.8.9.

Proof. We see that for $f : A \to \{0, 1\}$ to be a function, every element in A has to go to a unique element in $\{0, 1\}$. Then we see that for every element a_i, $i \in \{1, 2, \cdots, n\}$, there are two different options for the value of $f(a_i)$, $f(a_i) = 0$ or $f(a_i) = 1$. Thus, there are 2^n diffferent ways we can construct a function from A to $\{0, 1\}$. Therefore, $|F| = 2^n$. ■

Proof. Now, let $g : F \to \mathcal{P}(A)$ be defined as $g(f) = \{a \in A : f(a) = 1\}$. We prove show that this function is both surjective and injective.

- Surjective: Let $B \in \mathcal{P}(A)$. Then, we see that $B \subseteq A$. Thus, we can define $f_B : A \to \{0, 1\}$, defined as $f_B(x) = 1$ if $x \in B$ and $f_B(x) = 0$ if $x \notin B$. Then we see that f_B is well-defined and moreover, we have $g(f_B) = B$. Therefore, g is surjective.

- Injective: Let $f_1, f_2 \in F$ and assume that $g(f_1) = g(f_2)$. This implies,

$$\{a \in A : f_1(a) = 1\} = \{a \in A : f_2(a) = 1\}.$$

  Call this set B. To show that $f_1 = f_2$, we show that the functions agree at all inputs. Let $x \in A$, then either $x \in B$ or $x \notin B$. When $x = \in B$, we have, by definition of B, $f_1(x) = 1 = f_2(x)$. Then when $x \notin B$ we have $f_1(x) = 0 = f_2(x)$.Therefore $\forall x \in A$, we have $f_1(x) = f_2(x)$, which implies $f_1 = f_2$. Therefore g is injective.

■

Notice that we can also prove the injectiveness of g as follows:

Proof. Let $f_1, f_2 \in F$ so that $f_1 \neq f_2$. This means that there is some $x \in A$ so that $f_1(x) \neq f_2(x)$. Now, either $f_1(x) = 0$, or $f_1(x) = 1$.

- Assume $f_1(x) = 1$ and so $f_2(x) = 0$. Then $x \in g(f_1)$ but $x \notin g(f_2)$, and so $g(f_1) \neq g(f_2)$.

- Similarly, when $f_1(x) = 0$ we know $f_2(x) = 1$. Hence $x \notin g(f_1)$ but $x \in g(f_2)$, and again $g(f_1) \neq g(f_2)$

Thus when $f_1 \neq f_2$ we know that $g(f_1) \neq g(f_2)$ and thus g is injective. ■

10.8.10. *Claim:* The function is injective, but not surjective.

Proof. To prove that the function is injective, let $m, n \in \mathbb{Z}$ so that $f(n) = f(m)$. Then $(2n+1, n+2) = (2m+1, m+2)$. This gives us two equations

$$2n+1 = 2m+1 \qquad \text{and} \qquad n+2 = m+2$$

Both of these give $n = m$ as required.

Now, to show that the function is not surjective, we find a point in $(y_1, y_2) \in \mathbb{Z} \times \mathbb{Z}$ which is not the image of any $x \in \mathbb{Z}$. Consider $(2, 0) \in \mathbb{Z} \times \mathbb{Z}$. We see that $(2n+1, n+2) \neq (2, 0)$ for any $n \in \mathbb{Z}$ since 2 is even and $2n+1$ is always odd, that is $2 \neq 2n+1$ for any $n \in \mathbb{Z}$. ■

Notice that we can also prove that the function is an injection as follows:

Proof. Let $n, m \in \mathbb{Z}$ and assume $n \neq m$. Then we see that $n+2 \neq m+2$. Hence, we see $(2n+1, n+2) \neq (2m+1, m+2)$, that is $f(n) \neq f(m)$. Therefore the function is injective. ■

10.8.11.

Proof. Let us prove the two implications in turn.

- First, assume that f is surjective. Let $A \in \mathcal{P}(E)$, and let $y \in F - f(A)$, so $y \notin f(A)$. We need to prove that $y \in f(E - A)$.

  Since f is surjective $y = f(x)$ for some $x \in F$. Now, by definition of the image, if $x \in A$ then $f(x) \in f(A)$. Since we know, by assumption that $f(x) = y \notin f(A)$, it follows (by modus-tollens 2.5.2) that $x \notin A$.

  Hence $x \in E \setminus A$, so that $y \in f(E \setminus A)$. Therefore, $F \setminus f(A) \subseteq f(E \setminus A)$ so the first implication is proved.

- Now, assume that $\forall A \in \mathcal{P}(E),\ F \setminus f(A) \subseteq f(E \setminus A)$. We apply the above with $A = \emptyset$ to get that

  $$F \setminus f(\emptyset) \subseteq f(E \setminus \emptyset) \quad \text{so that} \quad F \subseteq f(E).$$

  This entails that f is surjective. Indeed, let $y \in F$. By the inclusion above, we have $y \in f(E)$ and so $y = f(x)$ for some $x \in E$.

■

10.8.12.

Proof. Assume f is injective. Then we need to show $f(A - B) \subseteq f(A) - f(B)$ and $f(A) - f(B) \subseteq f(A - B)$. Let us start with the second inclusion.

- Let $y \in f(A - B)$. Then, by definition, there exists $x \in A - B$ such that $f(x) = y$. Then, we see that $y \in f(A)$. Now we must show that $y \notin f(B)$. Let z be any element of B. Then we must have that $x \neq z$, and so, since f is injective, we know that $f(x) \neq f(z)$. This means that y cannot be the image of any $z \in B$, and so $y \notin f(B)$. Hence $y \in f(A) - f(B)$, and so

$f(A-B) \subseteq f(A) - f(B)$ as required.

We note that the contrapositive of this argument is illuminating. If $y \in f(B)$ then there is some $z \in B$ so that $f(z) = y$. Now, since $y = f(x)$ this means $f(x) = f(z)$. But since $x \notin B$ and $z \in B$, we know $x \neq z$ and thus f is not injective.

- Now, let $y \in f(A) - f(B)$. This means that $y \in f(A)$ and so there is some $x \in A$ so that $y = f(x)$. At the same time, $y \notin f(B)$, which means that $y \neq f(z)$ for any $z \in B$. Hence it cannot be the case that $x \in B$. Thus $x \in A - B$ and so $y = f(x) \in f(A - B)$, and therefore $f(A) - f(B) \subseteq f(A - B)$ as required.

■

Notice that the proof that $f(A) - f(B) \subseteq f(A - B)$ does not actually require that f is surjective. It holds for all functions. Also notice that when we prove first inclusion we are actually proving that

$$f \text{ is injective } \implies (LHS \subseteq RHS)$$

So we could prove the contrapositive of this instead. Namely

$$(LHS \not\subseteq RHS) \implies f \text{ is not injective.}$$

Proof. Assume that $f(A-B) \not\subseteq f(A)-f(B)$. Hence there is some $y \in f(A-B)$ so that $y \notin f(A)-f(B)$. So we know that there is some $x \in A-B$ with $f(x) = y$. Notice that since $x \in A$, we know that $y = f(x) \in f(A)$.

Now since $y \notin f(A) - f(B)$ and $y \in f(A)$ it must be the case that $y \in f(B)$. However, this means that there is some $z \in B$ so that $f(z) = y$. This implies that $f(z) = y = f(x)$ and at the same time $x \neq z$ and thus f is not injective. ■

10.8.13.

Proof. We will show that f is injective and then that it is surjective.

To prove that f is injective, let $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$ and assume that $f(a, b) = f(c, d)$. Without loss of generality, let us further assume that $a \geq c$; the case that $a < c$ is very similar.

Using the definition of the function, we see that $2^{a-1}(2b - 1) = 2^{c-1}(2d - 1)$, which implies

$$2^{a-c}(2b - 1) = (2d - 1).$$

Notice that the right hand side of this expression, $(2d - 1)$, is odd. On the other hand, the left hand side of the expression has a factor of 2^{a-c} and $a - c \geq 0$. Therefore, we must have $a - c = 0$ (so that $2^{a-c} = 1$), otherwise the left hand side will be even. Thus, $a = c$. Hence, we get $(2b - 1) = (2d - 1)$, that is, $b = d$. Hence $(a, b) = (c, d)$, and we see that f is injective.

To prove that f is surjective, let $n \in \mathbb{N}$. Then, we know that n has a unique prime factorisation. We can write the factorisation as $n = 2^{a_1}3^{a_2}5^{a_3}7^{a_4} \cdots p_m^{a_m}$ for

some $m \in \mathbb{N}$ where $a_i \in \mathbb{N} \cup \{0\}$ for all $i \in \{1, 2, 3, \cdots, m\}$.

Since the product of odd numbers is odd, we see that $3^{a_2}5^{a_3}7^{a_4} \cdots p_m^{a_m}$ is odd. Thus, there exists $\ell \in \mathbb{Z}$ such that $3^{a_2}5^{a_3}7^{a_4} \cdots p_m^{a_m} = 2\ell - 1$. Note that since that product of primes is positive and odd, we know that $\ell \geq 1$ and so $\ell \in \mathbb{N}$. Similarly, since $a_1 \geq 0$, we know that $k = a_1 + 1 \in \mathbb{N}$ and so we can write $n = 2^{k-1}(2\ell - 1) = f(k, \ell)$ with $k, \ell \in \mathbb{N}$. Therefore f is surjective. ■

10.8.14.

Proof. Let A, B be nonempty sets and assume that there is a bijection $f : A \to B$. Then, since f is a function, we can define a new function $g : \mathcal{P}(A) \to \mathcal{P}(B)$ as follows:

$$g(X) = \{f(x) : x \in X\} \in \mathcal{P}(B).$$

That is, our new function g takes each element of a subset to its image under f. We need to check that g is injective and surjective.

- Surjective: Let $Y \in \mathcal{P}(B)$. Then we know that $Y \subseteq B$. We also know that f is surjective. Then, we see that $\forall y \in Y$ there is an $x \in A$ such that $f(x) = y$. Consider the set $X = \{x \in A : f(x) \in Y\}$. We would like to show that $g(X) = Y$.

  First, let $b \in g(X)$. Then by the definition of g, $b = f(a)$ for some $a \in X$. In particular, X is the set of $x \in A$ such that $f(x) \in Y$, so $a \in X$ implies $a = f(b) \in Y$. Thus, $g(X) \subseteq Y$.

  Now, let $b \in Y$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Thus, by the definition of X we have $a \in X$, and by the definition of g, $f(a) = b \in g(X)$.

  Therefore we see $g(X) = Y$. Hence g is surjective.

- Injective: We proceed by contrapositive. Let $Z, W \in \mathcal{P}(A)$. We will show that if $g(Z) = g(W)$ then $Z = W$. Assume $g(Z) = g(W)$.

  Let $z \in Z$. By the definition of g, $f(z) \in g(Z) = g(W)$. Thus, there is some $w \in W$ so that $f(w) = f(z)$. By the injectivity of f, $w = z$, so $z \in W$, and we see $Z \subseteq W$.

  Now, let $w \in W$. By the definition of g, $f(w) \in g(W) = g(Z)$. Thus, there is some $z \in Z$ so that $f(w) = f(z)$. By the injectivity of f, $w = z$, so $w \in Z$, and we see $W \subseteq Z$. Therefore, $Z = W$, and we see that g is injective.

  We conclude that g is a bijection.

■

10.8.15.

Proof. Let $\mathcal{R}$ be the relation on $\mathbb{R}^2$ defined as

$$(x, y)\ \mathcal{R}\ (s, t) \text{ if } x^2 + y^2 = s^2 + t^2.$$

(a) We need to show that this relation is reflexive, symmetric and transitive.

- {Reflexivity:} Let $(x, y) \in \mathbb{R}^2$. Then, we see that since $x^2+y^2 = x^2+y^2$, we get $(x, y)\ \mathcal{R}\ (x, y)$. Hence $\mathcal{R}$ is reflexive.
- {Symmetry:} Let $(x, y), (s, t) \in \mathbb{R}^2$ and assume that $(x, y)\ \mathcal{R}\ (s, t)$. This means $x^2 + y^2 = s^2 + t^2$. Then, we see that $s^2 + t^2 = x^2 + y^2$. Hence, we see that $(s, t)\ \mathcal{R}\ (x, y)$, that is, $\mathcal{R}$ is symmetric.
- {Transitive:}. Let $(x, y), (s, t), (a, b) \in \mathbb{R}^2$ and assume that $(x, y)\ \mathcal{R}\ (s, t)$ and $(s, t\ \mathcal{R}\ (a, b))$. This means $x^2 + y^2 = s^2 + t^2$ and $s^2 + t^2 = a^2 + b^2$. Then, we see that $x^2 + y^2 = a^2 + b^2$. Hence, we see that $(x, y)\ \mathcal{R}\ (a, b)$, that is, $\mathcal{R}$ is transitive.

Hence the relation $\mathcal{R}$ is an equivalence relation.

(b) Now, let $\mathcal{S}$ be the set of equivalence classes of the relation $\mathcal{R}$ defined in part (a) and define $f : \mathcal{S} \to (0, \infty)$, defined by $f([(x, y)]) = \sqrt{x^2 + y^2}$.

- *Show f is a function:* Let $[(x, y)]$ be an equivalence class. Then since $x, y \in \mathbb{R}$ we know that $x^2 + y^2 \geq 0$ and so $\sqrt{x^2 + y^2} \geq 0$. Hence f is defined everywhere on its domain. Now let $[(x, y)] = [(s, t)]$. This means that $x^2 + y^2 = s^2 + t^2$. Thus, we get

$$f\big([(x, y)]\big) = \sqrt{x^2 + y^2} = \sqrt{s^2 + t^2} = f\big([(s, t)]\big).$$

  Therefore, $f : \int \to (0, \infty)$ defines a function.
- *Show f is bijective:* We need to show that f is injective and surjective.
  - *Injective:* Let $(x, y), (s, t) \in \mathbb{R}^2$, and assume that

$$f\big([(x, y)]\big) = f\big([(s, t)]\big).$$

    This means that $\sqrt{x^2 + y^2} = \sqrt{s^2 + t^2}$, which in turn implies that

$$x^2 + y^2 = s^2 + t^2.$$

    Thus, we get $(x, y)\ \mathcal{R}\ (s, t)$, that is,

$$[(x, y)] = [(s, t)].$$

    Hence f is injective.
  - *Surjective:* Let $z \in [0, \infty)$. Then, we can take $[(x, y)] = [(0, z)] \in \mathcal{S}$, and see that

$$f\big([(x, y)]\big) = f\big([(0, z)]\big) = \sqrt{z^2} = z.$$

    Hence, f is surjective.

  Therefore, we see that $f : \mathcal{S} \to [0, \infty)$ is bijective.

■

10.8.16.

Proof. Let $[z]_n \in \mathbb{Z}_n$, then we know that $z \in [z]_n$. By Euclidean division we know that $z = qn + r$ with $q \in \mathbb{Z}$ and $r \in \{0, 1, \ldots, n-1\}$. Hence $f([x]_n) \in \{0, 1, \ldots, n-1\}$ as required.

Now let $x, y \in \mathbb{Z}$, and suppose that $[x]_n = [y]_n$. Since $x, y \in \mathbb{Z}$, by Euclidean division, there are integers q_x, q_y and $r_x, r_y \in \{0, 1, \ldots, n-1\}$

$$x = q_x n + r_x \qquad \text{and} \qquad y = q_y n + r_y.$$

So, by our definition of f, $f([x]_n) = r_x$ and $f([y_n]) = r_y$.

However, since $[x]_n = [y]_n$ we need to show that $f([x]_n) = f([y]_n)$. That is, we need to show that $r_x = r_y$. Since $[x]_n = [y]_n$, we must have that x and y are congruent modulo n, so $n = x - y$. That is, there is some $k \in \mathbb{Z}$ so that $(x - y) = kn$.

Hence the equation $(x - y) = kn$ gives

$$q_x n + r_x - (q_y n + r_y) = kn$$

and rearranging, we have

$$r_x - r_y = n(k + q_y - q_x).$$

Thus $n \mid r_x - r_y$. Notice that since $0 \leq r_x, r_y \leq n - 1$,

$$-(n-1) \leq r_x - r_y \leq n - 1.$$

Equivalently, we can say $|r_x - r_y| < n$. Since $n \mid (r_x - r_y)$ and $|r_x - r_y| < n$, it must be the case that $r_x - r_y = 0$. That is, $r_x = r_y$, and hence $f([x]_n) = f([y]_n)$. ■

Proof. We prove that f is both injective and surjective, and so, bijective.

First we prove that f is injective. Suppose that for $[x]_n, [y]_n \in \mathbb{Z}_n$, we have $f([x]_n) = f([y]_n)$. Then there is some $r \in \{0, 1, \ldots, n-1\}$ so that $f([x]_n) = f([y]_n) = r$. By definition of f, we know that r is the remainder when x or y is divided by n. By Euclidean division, there are integers p and q so that $x = pn + r$ and $y = qn + r$. Thus

$$x - y = (pn + r) - (qn + r) = (p - q)n$$

and since $p - q \in \mathbb{Z}$, we have that n divides $x - y$. Therefore x and y are congruent mod n, and so $[x]_n = [y]_n$. Hence f is injective.

Now we prove that f is surjective. Let $r \in \{0, 1, \ldots, n-1\}$. Since r is an integer and $0 \leq r \leq n - 1$, r is itself its remainder when r is divided by n. Therefore $f([r]_n) = r$, and so f is surjective. ■

10.8.17.

Proof. This is a biconditional statement, so we have to prove both implications

in turn.

- Assume that f is injective. Then we need to show $X \subseteq f^{-1}(f(X))$ and $f^{-1}(f(X)) \subseteq X$.

  First, let $a \in X$. Then, by definition of image, we see $f(a) \in f(X)$, which also implies $a \in f^{-1}(f(X))$. Therefore $X \subseteq f^{-1}(f(X))$.

  Next, let $b \in f^{-1}(f(X))$. Then, by definition of preimage, we see that $f(b) \in f(X)$. Thus, we see that there exists $c \in X$ such that $f(c) = f(b)$. Moreover, since f is injective, we see that $b = c$. In particular, $b \in X$. Therefore $f^{-1}(f(X)) \subseteq X$.

  Hence we can conclude that if f is injective, then $X = f^{-1}(f(X))$.

- We are going to use proof by contrapositive:

  $$f \text{ not injective} \implies X \neq f^{-1}(f(X)) \text{ for some } X \subseteq A.$$

  Assume f is not injective. Then we see that there are $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$. Then we see that for the set $Y = \{y\}$, we get $f^{-1}(f(Y)) = f^{-1}(\{f(y)\}) = \{x, y\} \neq Y$.

  Since the contrapositive is true, we have shown that if $X = f^{-1}(f(X))$ for all $X \subseteq A$ then f is injective.

■

10.8.18.

Proof. This is a biconditional statement, so we have to prove both implications in turn.

- Assume that f is surjective. Then we need to show $Y \subseteq f(f^{-1}(Y))$ and $f(f^{-1}(Y)) \subseteq Y$.

  First, let $b \in Y$. Then, since f is surjective, we see that there is $a \in A$ such that $f(a) = b$. Thus, we see that $a \in f^{-1}(Y)$, which implies $f(a) = b \in f(f^{-1}(Y))$. Therefore $Y \subseteq f(f^{-1}(Y))$.

  Next, let $b \in f(f^{-1}(Y))$. Then, by definition, we see that $b = f(a)$ for some $a \in f^{-1}(Y)$. Thus, by definition, we see that $f(a) = b \in Y$. Therefore $f(f^{-1}(Y)) \subseteq Y$.

  Hence we can conclude that if f is surjective, then $Y = f(f^{-1}(Y))$.

- We prove the contrapositive, so assume that f is not surjective. Then we see that there exists $b \in B$ such that there does not exist $a \in A$ with $f(a) = b$. Hence for the set $Y = \{b\}$, we get $f(f^{-1}(Y)) = f(\emptyset) = \emptyset \neq \{b\} = Y$. Therefore if $Y = f(f^{-1}(Y))$, then f is surjective.

■

10.8.19.

Proof. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two strictly increasing functions. Fix $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$. Then because g is strictly increasing,

$$z_1 = g(x_1) < g(x_2) = z_2.$$

Now, using the fact that f is strictly increasing and that $z_1 < z_2$, we see that

$$(f \circ g)(x_1) = f(g(x_1)) = f(z_1) < f(z_2) = f(g(x_2)) = (f \circ g)(x_2).$$

Therefore $f \circ g$ is a strictly increasing function. ■

Proof. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two strictly decreasing functions. Fix $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$. Then because g is strictly decreasing,

$$z_1 = g(x_1) > g(x_2) = z_2.$$

Now, using the fact that f is strictly decreasing and $z_2 < z_1$, we see that

$$(f \circ g)(x_2) = f(g(x_2)) = f(z_2) > f(z_1) = f(g(x_1)) = (f \circ g)(x_1)$$

Therefore $f \circ g$ is a strictly increasing function. ■

10.8.20.

(a) *Claim:* This statement is false.

Proof. Consider $f(x) = x^2$ and $g(x) = h(x) = x$. Then

$$\begin{aligned} f \circ (g + h)(x) &= f((g + h)(x)) \\ &= f(g(x) + h(x)) \end{aligned}$$

Using the choice of $g(x)$ and $h(x)$,

$$= f(2x)$$

Using the choice of $f(x)$,

$$= 4x^2$$

On the other hand, we compute

$$(f \circ g + f \circ h)(x) = f(g(x)) + f(h(x))$$

Using the definitions of $g(x)$ and $h(x)$,

$$= f(x) + f(x)$$

Using the definitions of $f(x)$,

$$= 2x^2$$

Consider $x = 1$. Since

$$f \circ (g + h)(1) = (4)1^2 = 4 \neq 2 = (2)1^2 = (f \circ g + f \circ h)(1),$$

we have found a value of x such that the functions are not equal. Therefore the statement is false. ■

(b) *Claim:* The statement is true.

Proof. Fix $x \in \mathbb{R}$. We compute, using the definition of function composition

$$(g + h) \circ f(x) = (g + h)(f(x)).$$

Using the definition of function addition, we obtain

$$= g(f(x)) + h(f(x)).$$

We now use the definition of function composition to rewrite the statement as

$$= (g \circ f)(x) + (h \circ f)(x).$$

Finally, we use the definition of function addition to obtain

$$= (g \circ f + h \circ f)(x).$$

Since x was chosen arbitrarily, we have that

$$(g + h) \circ f(x) = (g \circ f + h \circ f)(x)$$

for every x. We conclude that

$$(g + h) \circ f = g \circ f + h \circ f$$

■

(c) *Claim:* The statement is true.

Proof. Fix $x \in \mathbb{R}$. First, we use the definition of function composition to see that

$$\left(\frac{1}{f} \circ g\right)(x) = \left(\frac{1}{f}\right)(g(x)).$$

We regard the numerator as the constant function $h(x) = 1$. This lets us rewrite the expression as

$$= \left(\frac{h}{f}\right)(g(x))$$

By the definition of function division, we have

$$= \frac{h(g(x))}{f(g(x))}$$

Since h is a constant function, we have $h(g(x)) = 1 = h(x)$, thus

$$= \frac{h(x)}{f(g(x))}$$

By the definition of function composition, we have

$$= \frac{h(x)}{(f \circ g)(x)}$$

Using the definition of function division yields

$$= \left(\frac{h}{f \circ g}\right)(x)$$

Finally, we use the definition of h to conclude,

$$= \left(\frac{1}{f \circ g}\right)(x).$$

Since x was chosen arbitrarily, we see that

$$\left(\frac{1}{f} \circ g\right)(x) = \left(\frac{1}{f \circ g}\right)(x)$$

for every $x \in \mathbb{R}$. We conclude that

$$\frac{1}{f} \circ g = \frac{1}{f \circ g}.$$

■

(d) *Claim:* The statement is false.

Proof. Consider $f(x) = x + 2$ and $g(x) = x^2$. We compute

$$\frac{1}{f \circ g}(x) = \frac{1}{f(g(x))}$$

Using the definition of $g(x)$,

$$= \frac{1}{f(x^2)}$$

Using the definition of $f(x)$,

$$= \frac{1}{x^2 + 2}.$$

On the other hand,

$$\left(f \circ \frac{1}{g}\right)(x) = f\left(\frac{1}{g(x)}\right)$$

Using the definition of $g(x)$,

$$= f\left(\frac{1}{x^2}\right)$$

Using the definition of $f(x)$,

$$= \frac{1}{x^2} + 2.$$

Consider $x = 1$. Since

$$\frac{1}{f \circ g}(1) = \frac{1}{1^2 + 2} = \frac{1}{3} \neq 3 = \frac{1}{1^2} + 2 = \left(f \circ \frac{1}{g}\right)(x),$$

we have found a value of x such that the functions are not equal. Therefore the statement is false. ■

10.8.21.

Proof. As a counterexample, take $f : \{-1, 0, 1\} \to \{0, 1\}$ defined as $f(x) = |x|$ and consider the subsets $W = \{-1, 0\}$ and $X\,\{0, 1\}$. Then we see $f(W \cap X) = f(\{0\}) = \{0\}$ and $f(W) = f(X) = \{0, 1\}$. Hence $f(W \cap X) = \{0\} \neq \{0, 1\} = f(W) \cap f(X)$. Thus the statement is false. ■

Proof. As a counterexample, take $f : \{1, 2\} \to \{1, 2, 3\}$, defined as $f(x) = x$ and let $Y = \{2, 3\}$. Then we see that $f^{-1}(Y) = \{2\}$, which implies $f(f^{-1}(Y)) = \{2\} \neq \{2, 3\} = Y$. Hence the statement is false. ■

10.8.22.

Proof. Assume that g is injective and $g \circ f = g \circ h$. Then, we know that $\forall x \in A$ we have $g \circ f(x) = g \circ h(x)$. Since $f(x), h(x) \in B$, we have, by the definition of function composition, $g(f(x)) = g(h(x))$. Moreover, since g is injective, we conclude $f(x) = h(x)$. Since this is true for all $x \in A$, we conclude that $f = h$. ■

Proof. Assume that g is surjective and $f \circ g = h \circ g$. Let $x \in A$. Since g is a surjective function, we see that $\exists y \in B$ such that $g(y) = x$. Hence, $f \circ g(y) = h \circ g(y)$. This implies $f(x) = f(g(y)) = h(g(y)) = h(x)$. Since this is true for all $x \in A$, we conclude that $f = h$. ■

10.8.23.

Proof. *Claim:* There exists a suitable function g

Consider $g(x) = x$. Then

$$(f \circ g)(x) = f(g(x)) = f(x) = x + 1.$$

And on the other hand,

$$(g \circ f(x)) = g(f(x)) = g(x + 1) = x + 1.$$

Thus, $f \circ g = g \circ f$, as desired. ■

Proof. *Claim:* g can be any function such that $g(c) = c$.

Notice that $(f \circ g)(x) = f(g(x)) = c$, since f is a constant function. Then in order for $(f \circ g)(x) = (g \circ f)(x)$ for every $x \in \mathbb{R}$,

$$c = (g \circ f)(x) = g(f(x)) = g(c).$$

Hence, we require that $g(c) = c$. There are no other conditions on the function g. ■

Proof. Since the statement

$$(f \circ g)(x) = (g \circ f)(x) \tag{1}$$

must be true for any $x \in \mathbb{R}$, and any function $g : \mathbb{R} \to \mathbb{R}$, we begin by examining the restrictions placed on f when g is a constant function.

Fix $c \in \mathbb{R}$ and define the function $g_c : \mathbb{R} \to \mathbb{R}$ by $g_c(x) = c$. By assumption, the statement $(f \circ g_c)(x) = (g_c \circ f)(x)$ must hold. Applying part (b), we see that we must have $f(c) = c$.

Moreover, since the choice of $c \in \mathbb{R}$ was arbitrary and $(f \circ g_c)(x) = (g_c \circ f)(x)$ must hold for every $c \in \mathbb{R}$, we know that $f(c) = c$ for every $c \in \mathbb{R}$. This tells us that the function $f(x) = x$ is the only function which could possibly satisfy (1).

It remains to check that (1) holds for the function $f(x) = x$. Working with the left hand side, we see

$$(f \circ g)(x) = f(g(x)) = g(x)$$

Where $f(g(x)) = g(x)$ by the definition of f. On the other hand,

$$(g \circ f)(x) = g(f(x)) = g(x)$$

Where we obtain the second equality using the fact that $f(x) = x$. Therefore,

$$(f \circ g)(x) = g(x) = (g \circ f)(x),$$

and we conclude that $f(x) = x$ is the only solution to (1). ■

10.8.24.

Proof. Assume $f \circ f$ is bijective. Let $x, y \in A$ and assume that $f(x) = f(y)$.

Then, since f is a function, we get $(f \circ f)(x) = (f \circ f)(y)$. Moreover, since $f \circ f$ is injective we see $x = y$. Therefore f is injective.

Moreover, if $b \in A$, then we see that since $f \circ f$ is surjective, there exists $a \in A$ such that $(f \circ f)(a) = b$. Thus, we see that $f(f(a)) = b$, which means that for $c = f(a) \in A$, we have $f(c) = b$. Therefore f is surjective.

Hence, f is bijective. ■

Proof. We see that if $f(x) = \log\left(\dfrac{e^x+1}{e^x-1}\right)$, then

$$f \circ f(x) = f(f(x))$$

Using the definition of f for the "inner" function,

$$= f\left(\log\left(\frac{e^x+1}{e^x-1}\right)\right)$$

Using the definition of f for the "outer" function,

$$= \log\left(\frac{e^{\log\left(\frac{e^x+1}{e^x-1}\right)}+1}{e^{\log\left(\frac{e^x+1}{e^x-1}\right)}-1}\right)$$

We now use the fact that e and log are inverse functions to obtain,

$$= \log\left(\frac{\left(\frac{e^x+1}{e^x-1}\right)+1}{\left(\frac{e^x+1}{e^x-1}\right)-1}\right)$$

Simplifying, we see

$$= \log\left(\frac{\left(\frac{2e^x}{e^x-1}\right)}{\left(\frac{2}{e^x-1}\right)}\right)$$
$$= \log(e^x)$$
$$= x.$$

Therefore, we see that $f \circ f$ is the identity function, $i_{(0,\infty)}$. And then by part (a), since $i_{(0,\infty)}$ is bijective, f is bijective. ■

10.8.25.

Proof. Let $f : \mathbb{R} - \{-2\} \to \mathbb{R} = \{1\}$, be defined by $f(x) = \dfrac{x+1}{x+2}$. Then we

want to show that f is both injetive and surjective.

- Injective: Assume that $a, z \in R - \{-2\}$, and $f(x) = f(z)$. Then

$$\begin{aligned} \frac{x+1}{x+2} &= \frac{z+1}{z+2} && \text{since } x, z \neq -2 \\ (z+2)(x+1) &= (x+2)(z+1) && \text{expand} \\ xz + z + 2x + 2 &= xz + x + 2x + 2 && \text{cancel} \\ x &= z \end{aligned}$$

  Hence $x = z$ and so f is injective.

- Surjective: Let $y \in \mathbb{R} - \{1\}$. Then set $x = -2 + \frac{1}{1-y}$. Since $y \neq 1$ we know that $x \in \mathbb{R}$. And since $\frac{1}{1-y} \neq 0$, we know that $x \neq -2$. Hence x lies in the domain of the function. Then we verify that $f(x) = y$ as follows:

$$\begin{aligned} f(x) &= \frac{x+1}{x+2} \\ &= \frac{-1 + \frac{1}{1-y}}{\frac{1}{1-y}} \\ &= \frac{y - 1 + 1}{1} = y \end{aligned}$$

  as required. Hence, f is surjective.

Therefore f is bijective.

Since it is bijective it has an inverse, denoted f^{-1}, such that $f^{-1} : \mathbb{R} - \{1\} \to \mathbb{R} - \{-2\}$. Our proof of surjectiveness shows that for $x = \dfrac{2y-1}{1-y}$, we have $f(x) = f\left(\dfrac{2y-1}{1-y}\right) = y$, which in turn implies that $f^{-1}(y) = \dfrac{2y-1}{1-y}$. ■

10.8.26.

Proof. Let $g : \mathbb{Z} \to \mathbb{Z}$ be defined as

$$g(n) = \begin{cases} 3 - n & \text{if } n \text{ is odd} \\ n - 7 & \text{if } n \text{ is even} \end{cases}$$

We prove that g is a left and right inverse of f, which implies, via Lemma 10.6.3 and Lemma 10.6.4, that f is bijective and that g is its inverse.

- *g is the left inverse of f*: Let $n \in \mathbb{Z}$ and consider $g(f(n))$. Then n is either even or odd.
  - *n is even*: In this case, $g(f(n)) = g(-n + 3) = -(-n + 3) + 3 = n$ since $-n + 3$ is odd.

- n *is odd*: Then, we see that $g(f(n)) = g(n+7) = (n+7) - 7 = n$ since $n+7$ is even.

Hence $g \circ f = i_{\mathbb{Z}}$, where $i_{\mathbb{Z}}$ is the identity function on $\mathbb{Z}$, and so g is a left-inverse of f.

- *g is the right inverse of f*: Let $n \in \mathbb{Z}$ and consider $f(g(n))$. Then, again, we have two cases, n is either even or odd;

  - n *is even*: Then, we have $f(g(n)) = f(n-7) = (n-7)+7 = n$ since $n-7$ is odd.
  - n *is odd* : Finally, in this case we have $f(g(n)) = f(-n+3) = -(-n+3)+3 = n$ since $-n+3$ is even.

Hence $f \circ g = i_{\mathbb{Z}}$ and so g is a right-inverse of f.

Therefore f is bijective and g is the inverse of f . ■

10.8.27.

Proof. Let B a set and assume that $g : A \to A$ satisfies the condition $g \circ g \circ g = i_A$. We need to show that g is injective and surjective.

- *Injective:* Let $x, y \in A$ so that $g(x) = g(y)$. Since g is a function, this implies that $g(g(x)) = g(g(y))$. Applying g again gives $g(g(g(x))) = g(g(g(y)))$, that is $(g \circ g \circ g)(x) = (g \circ g \circ g)(y)$. But, since $g \circ g \circ g = i_A$, the above implies that $i_A(x) = i_A(y)$ and so $x = y$. Therefore g is injective.
- *Surjective*: Let $b \in A$. The,n since $g \circ g \circ g = i_A$, we get $g \circ g \circ g(b) = b$. Therefore, if we set $a = (g \circ g)(b) \in A$, we have $g(a) = b$. Hence, g is surjective.

Therefore, we see that g is bijective. ■

Proof. This requires some careful algebraic manipulations. Start by computing and simplifying $(f \circ f)(x)$

$$\begin{aligned} f(f(x)) &= 1 - \frac{1}{1 - \frac{1}{x}} \\ &= 1 - \frac{x}{x-1} = \frac{x-1-x}{x-1} \\ &= \frac{-1}{x-1} = \frac{1}{1-x} \end{aligned}$$

Then use this to compute $(f \circ f \circ f)(x)$:

$$\begin{aligned} (f \circ f \circ f)(x) = f((f \circ f)(x)) &= 1 - \frac{1}{\frac{1}{x-1}} \\ &= 1 - \frac{x-1}{1} = x \end{aligned}$$

as required. Thus $f \circ f \circ f = i_A$ as required. ■

Proof. The result from (a), implies that f is bijective. We can compute the inverse in two ways. First, observe that since $f \circ f \circ f$ is the identity function, it follows that $\forall x \in A$, we have $(f \circ f)(x) = f^{-1}(x) = \frac{1}{1-x}$ as calculated in part (b).

We could also solve for f^{-1} directly. Namely, if $f^{-1}(x) = y$, then we have

$$\begin{aligned} x = f(y) &= 1 - \frac{1}{y} \\ xy &= y - 1 \\ xy - y &= -1 \end{aligned}$$

from which we can conclude that

$$y = f^{-1}(x) = \frac{1}{1-x}.$$

■

11 · Proof by contradiction
11.3 · Exercises

11.3.1.

Proof. Assume for a contradiction that such an integer a exists. Then the first congruence implies that $a = 6k + 2$ for some integer k. Similarly, the second congruence implies that $a = 9\ell + 7$ for some integer ℓ. Thus we have

$$6k + 2 = 9\ell + 7.$$

We can rewrite this as

$$6k - 9\ell = 3(2k - 3\ell) = 5.$$

Therefore, since $2k + 3\ell \in \mathbb{Z}$, we conclude that $3 \mid 5$, which is a contradiction.

Hence, there is no integer a so that $a \equiv 2 \pmod 6$ and $a \equiv 7 \pmod 9$. ■

11.3.2.

Proof. Let $a, b, c \in \mathbb{Z}$ be such that $a^2 + b^2 = c^2$. Assume for a contradiction that and a and b are both odd. Then, there are $k, \ell \in \mathbb{Z}$ such that $a = 2k + 1$ and $b = 2\ell + 1$. Then we see that

$$c^2 = a^2+b^2 = (2k+1)^2+(2\ell+1)^2 = 4k^2+4k+4\ell^2+4\ell+2 = 2(2k^2+2\ell^2+2k+2\ell+1).$$

Since $(2k^2 + 2\ell^2 + 2k + 2\ell + 1) \in \mathbb{Z}$, we see that c^2 is even, which implies c is even. Thus, $c = 2m$ for some $m \in \mathbb{Z}$. Substituting this into the equation

$$4k^2 + 4k + 4\ell^2 + 4\ell + 2 = c^2$$

we obtain

$$4k^2 + 4k + 4\ell^2 + 4\ell + 2 = 4m^2.$$

But then we may write

$$2 = 4m^2 - (4k^2 + 4k + 4\ell^2 + 4\ell) = 4(m^2 - k^2 - k - \ell^2 - \ell)$$

and as $m^2 - k^2 - k - \ell^2 - \ell \in \mathbb{Z}$, we can conclude that $4 \mid 2$, a contradiction. Hence our original assumption was false, and so a or b must be even. ■

11.3.3.

Proof. Let $n \in \mathbb{N}$ and let $a \in \mathbb{Z}$ be such that $d = \gcd(a, n) > 1$. Assume, toward a contradiction, that there is some integer $k \in \mathbb{Z}$ such that $ka \equiv 1 \pmod{n}$. By definition, we have $n \mid (ka - 1)$. Then there is some integer $\ell \in \mathbb{Z}$ so that $\ell n = ka - 1$. Rearranging, we have $1 = ka - \ell n$. Since $d = \gcd(a, n)$ divides both a and n, there are integers u, v such that $a = ud$ and $n = vd$. Then

$$1 = kud + \ell vd = (ku + \ell v)d$$

and as $ku + \ell v \in \mathbb{Z}$, d divides 1. But this is a contradiction, since $d > 1$. Hence there is no integer $k \in \mathbb{Z}$ such that $ka \equiv 1 \pmod{n}$. ■

11.3.4.

Proof. Let $a, b \in \mathbb{Z}$ and $n \geq 2$. By way of contradiction, suppose that there exists an integer $c \not\equiv 0 \pmod{n}$ such that we have both

$$ab \equiv 1 \pmod{n} \quad \text{and} \tag{1}$$
$$ac \equiv 0 \pmod{n}. \tag{2}$$

We consider $abc \pmod{n}$. On the one hand, we use statement (1) to see that

$$(ab)c \equiv c \pmod{n}. \tag{3}$$

On the other hand, we use expression (2) to see

$$(ac)b \equiv 0(b) \equiv 0 \pmod{n}. \tag{4}$$

Therefore, combining statements (3) and (4) we obtain

$$c \equiv abc \equiv 0 \pmod{n}.$$

This contradicts the assumption that $c \not\equiv 0 \pmod{n}$. ■

11.3.5.

Proof. Suppose for a contradiction that there are integers x and y satisfying $5y^2 - 4x^2 = 7$. Then we see that $5y^2 - 4x^2 \equiv 7 \pmod{4}$. This implies

$$\begin{aligned} 1y^2 - 0x^2 &\equiv 3 \pmod{4} \\ y^2 &\equiv 3 \pmod{4}. \end{aligned}$$

However, we have 2 cases for y, y is even or y is odd.

- *y is even*: In this case, we see that $y = 2k$ for some $k \in \mathbb{Z}$. Therefore $y^2 = 4k^2 \equiv 0 \pmod 4$, which contradicts with $y^2 \equiv 3 \pmod 4$.
- *y is odd*: Then, we see that $y = 2k + 1$ for some $k \in \mathbb{Z}$. Thus, we get $y^2 = 4k^2 + 4k + 1 \equiv 1 \pmod 4$, which, again, contradicts with $y^2 \equiv 3 \pmod 4$.

Hence we can conclude that there doesn't exist $x, y \in \mathbb{Z}$ satisfying the equation $5y^2 - 4x^2 = 7$. ■

It is worth noting that this approach to showing the non-existence of integer solutions to equations can work very effectively (as it does above). However, consider what happens in the above if we choose a "bad" modulus. Consider the equation modulo 3:

$$\begin{aligned} 5y^2 - 4x^2 &\equiv 7 \pmod 3 \\ 2y^2 + 2y^2 &\equiv 1 \pmod 3 \end{aligned}$$

This equation *does* have a solution $x, y = 1 \pmod 3$ since

$$2 \cdot 1 + 2 \cdot 1 = 4 \equiv 1 \pmod 3.$$

This does not mean the equation has a solution over the integers. Let us be careful with the flow of logic:

- If the equation has a solution, $x = a, y = b$ over the integers, then
- those numbers satisfy $5b^2 - 4a^2 = 7$, so then
- taking that equation modulo n gives the equation $5b^2 - 4a^2 \equiv 7 \pmod n$

That chain of reasoning has to be true for all modulii n. So when we find a moulus with no solutions, we can use modus tollens 2.5.2 to tell us that the equation has no solution. Unfortunately, when we do find a solution for a particular modulus, we cannot infer that there has to be a solution over the integers; that would be an example of affirming the consequent 2.5.3.

11.3.6.

Proof. Assume, for a contradiction, that $r \in \mathbb{Q}$ is the smallest positive rational number. Then there are $a, b \in \mathbb{Z}$, $b \neq 0$, such that $r = a/b$. Thus

$$\frac{r}{2} = \frac{a}{2b}$$

and as $a, 2b \in \mathbb{Z}$ with $2b \neq 0$, we have that $r/2$ is rational. But as $r > 0$,

$$0 < \frac{r}{2} < r,$$

so $r/2$ is a positive rational number smaller than r. This contradicts our assumption on r, and so there is no smallest positive rational number. ■

Proof. Assume, for a contradiction, that $r \in \mathbb{R}$ is the smallest positive irrational number. We claim that $r/2$ is also irrational. If this were not the case, then there would be $a, b \in \mathbb{Z}$ with $b \neq 0$ such that $r/2 = a/b$. But then

$$r = \frac{2a}{b}$$

and as $2a, b \in \mathbb{Z}$ with $b \neq 0$, we have that r is rational. This is a contradiction, and so $r/2$ must be irrational. Since $r > 0$,

$$0 < \frac{r}{2} < r,$$

so $r/2$ is a positive irrational number smaller than r. This contradicts our assumption on r, and so there is no smallest positive irrational number. ■

11.3.7.

Proof. Assume, to the contrary that $\sqrt{6}$ is rational. Then we can write $\sqrt{6} = \frac{a}{b}$ with $a \in \mathbb{Z}, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Squaring both sides gives us $6 = \frac{a^2}{b^2}$ and so $a^2 = 6b^2$. Thus $6 \mid a^2$. This means that a^2 is even, and so a is even.

Since a is even we can write $a = 2k$ for some $k \in \mathbb{Z}$. But then

$$a^2 = 4k^2 = 6b^2$$

This implies that $2k^2 = 3b^2$. This, in turn, implies that $3b^2$ is even, since 3 is odd, we must have that b^2 is even and so b is even. Now we have a contradiction since 2 divides both a and b but we assumed that $\gcd(a, b) = 1$.

Hence $\sqrt{6}$ is irrational. ■

Notice that we did not actually need, as part of the contradiction, to show that $6 \mid a \implies 6 \mid b$. We only needed that $\gcd(a, b) \neq 1$ and it was sufficient to show that both were even.

We give two proofs of (b).

Proof. Assume, to the contrary that $\sqrt{2} + \sqrt{3}$ is rational. Then we can write

$$\sqrt{2} + \sqrt{3} = \frac{a}{b}$$

where $a \in \mathbb{Z}, b \in \mathbb{N}$ and $\gcd(a, b) = 1$. Squaring both sides of this gives

$$\begin{aligned} \frac{a^2}{b^2} &= \left(\sqrt{2} + \sqrt{3}\right)^2 \\ &= 2 + 2\sqrt{2} \cdot \sqrt{3} + 3 \\ &= 5 + 2\sqrt{6} \end{aligned}$$

Rearranging this further, to isolate $\sqrt{6}$, gives

$$2\sqrt{6} = \frac{a^2}{b^2} - 5$$

$$\sqrt{6} = \frac{a^2 - 5b^2}{2b^2}$$

which implies that $\sqrt{6} \in \mathbb{Q}$. This contradicts our result from (a). Thus it follows that $\sqrt{2} + \sqrt{3}$ is irrational as required. ■

Proof. Assume, to the contrary, that $\sqrt{2} + \sqrt{3}$ is rational. Hence we can write $\sqrt{2} + \sqrt{3} = \frac{a}{b}$ with $a \in \mathbb{Z}, b \in \mathbb{N}$. But now rearrange this as

$$\sqrt{3} = \frac{a}{b} - \sqrt{2}.$$

Square both sides to get

$$3 = \frac{a^2}{b^2} - \frac{2a}{b}\sqrt{2} + 2$$

Now isolate $\sqrt{2}$:

$$\frac{2a}{b}\sqrt{2} = \frac{a^2}{b^2} - 1 = \frac{a^2 - b^2}{b^2}$$
$$\sqrt{2} = \frac{a^2 - b^2}{2ab}$$

This contradicts the fact that $\sqrt{2}$ is irrational. Thus $\sqrt{2} + \sqrt{3}$ is irrational.

Notice that our manipulations require that $a \neq 0$, but since $\sqrt{2} + \sqrt{3} \neq 0$, this is easily verified. ■

11.3.8.

Proof. Assume for a contradiction that $\sqrt[3]{25} \in \mathbb{Q}$. Then we can write $\sqrt[3]{25} = a/b$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{N}$, with $\gcd(a, b) = 1$. This implies that $25 = a^3/b^3$, which in turn implies

$$25b^3 = a^3.$$

Therefore $25 \mid a^3$, so $5 \mid a^3$. Thus, since 5 is prime, we have $5 \mid a$, by Euclid's lemma.

This means we can write $a = 5n$ for some $n \in \mathbb{Z}$. Then we can substitute this into our equation above:

$$25b^3 = a^3 = 125n^3.$$

This, then implies that

$$b^3 = 5n^3,$$

so $5 \mid b^3$. Again, since 5 is prime, Euclid's lemma implies that $5 \mid b$. This gives a contradiction since $5 \mid a$ and $5 \mid b$, but we assumed that $\gcd(a, b) = 1$.

Therefore, $\sqrt[3]{25}$ is irrational. ■

11.3.9.

Proof. Assume for a contradiction that k is a positive integer, $\sqrt{k}$ is not an integer, but that $\sqrt{k}$ is rational. Then we see that $\sqrt{k} = \frac{a}{b}$ for some $a \in \mathbb{Z}$ and

$b \in \mathbb{N}$, where $\gcd(a, b) = 1$. This gives us two equations:

- By rearranging the expression for $\sqrt{k}$ we have
$$kb^2 = a^2.$$
- At the same time, since $\gcd(a, b) = 1$, Bézout's identity implies that there exists integers x, y so that
$$ax + by = 1.$$

We can link these two equations together by multiplying the the second by a and then substituing in the first gives

$$kb^2x + aby = a.$$

Factoring this then gives

$$b(kbx + ay) = a$$

and since $kbx + ay \in \mathbb{Z}$, this implies that $b \mid a$.

We know that $\gcd(a, b) = 1$ and so if $b \mid a$, we must have that $b = 1$. This, in turn, implies that we can write $\sqrt{k} = \frac{a}{b} = a \in \mathbb{Z}$. This contradicts our assumption that k is not an integer.

Therefore the statement is true. ■

11.3.10.

Proof. Let $r, x \in \mathbb{R}$ with $r \neq 0$. Suppose that r is rational and x is irrational. Assume, toward a contradiction, that $rx \in \mathbb{Q}$. Then there are integers p, q, with q non-zero, such that $rx = p/q$. As $r \neq 0$ and rational, there are non-zero integers m, n such that $r = m/n$. Then

$$\frac{p}{q} = rx = \frac{m}{n}x,$$

and since $m \neq 0$ we have

$$x = \frac{pn}{qm}.$$

But $pn, qm \in \mathbb{Z}$ and $qm \neq 0$, which contradicts the fact that x is irrational. Thus $x \notin \mathbb{Q}$, as required. ■

11.3.11.

Proof. As a counter example take $x = y = \sqrt{2}$, which is irrational, so $x \neq m/n$ and $y \neq p/q$ for any integers m, n, p, q. But $xy = 2$, and we can write 2 in the form $(mp)/(nq)$, by taking, for example $m = 2$ and $n = p = q = 1$. ■

The statement in (b) is false.

Proof. For a counterexample, we know that $\sqrt{2}$ is irrational. Then $-\sqrt{2}$ is irrational as well. Indeed, if this were not the case, we could write $-\sqrt{2} = m/n$

for some $m, n \in \mathbb{Z}$, $n \neq 0$. But then $\sqrt{2} = (-m)/n$ for $-m, n \in \mathbb{Z}$, $n \neq 0$, contradicting that $\sqrt{2}$ is irrational. But $\sqrt{2} + (-\sqrt{2}) = 0$ is rational. ■

11.3.12.

Proof. Assume for a contradiction that x is a rational number. This means that we can write $x = \frac{m}{n}$, where $m \in \mathbb{Z}$, $n \in \mathbb{N}$, where $\gcd(m, n) = 1$. Then, plugging this into the equation, we get

$$\frac{m^7}{n^7} + 5\frac{m^2}{n^2} - 3 = 0.$$

Then multiplying both sides by n^7 we get

$$m^7 + 5m^2n^5 - 3n^7 = 0.$$

Since, by assumption, this is satisfied by m, n, it must also be satisfied modulo 2. It becomes:

$$m^7 + m^2n^5 + n^7 \equiv 0 \pmod 2$$

There are four possibilities depending on the parity of m, n: both even, both odd, or different parities,

- both even: This implies that $2 \mid m$ and $2 \mid n$ which contradicts our assumption that $\gcd(m, n) = 1$. Thus this cannot happen.
- both odd: in this case $m \equiv 1 \pmod 2, n \equiv 1 \pmod 2$, but then

  $$m^7 + m^2n^5 + n^7 \equiv 3 \equiv 1 \pmod 2$$

  But this cannot happen since $m^7 + m^2n^5 + n^7 \equiv 0 \pmod 2$, and $1 \not\equiv 0 \pmod 2$.
- m even and n odd: In this case $m \equiv 0 \pmod 2$ and $n \equiv 1 \pmod 2$ and so

  $$m^7 + m^2n^5 + n^7 \equiv 1 \pmod 2$$

  Again, this cannot happen since $m^7 + m^2n^5 + n^7 \equiv 0 \pmod 2$.
- m odd and n even: Now $m \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 2$ and so

  $$m^7 + m^2n^5 + n^7 \equiv 1 \pmod 2$$

  This cannot happen since $m^7 + m^2n^5 + n^7 \equiv 0 \pmod 2$.

This gives us a contradiction since there is no integer solution to this equation modulo 2.

Therefore, any real solution of the equation $x^7 + 5x^2 - 3 = 0$ is irrational. ■

11.3.13.

Proof. Let $k \in \mathbb{N}$. If $k = 1$, then $5^k = 5$ is odd, establishing the base case. Assume that 5^k is odd for $k = m$. Then $5^{m+1} = 5^m \cdot 5$ is the product of two odd numbers, and so odd. Thus, by induction 5^k is odd for all $k \in \mathbb{N}$. ■

Proof. Assume, towards contradiction, that $\log_2(5)$ is rational. Then there are integers m, n with $n \neq 0$ such that

$$\log_2(5) = \frac{m}{n}.$$

Moreover, since $\log_2(5) > 0$, we may assume that m and n are positive. This equation implies that $2^{m/n} = 5$, and so $2^m = 5^n$. Since $m > 0$, we have that $2 \mid 5^n$. However, this contradicts part (a), which tells us that 5^n is odd. Hence our initial assumption is false, and so $\log_2(5)$ is irrational. ■

Proof. Since $2^0 = 1$ we know that $\log_2(1) = 0$ is rational.

Let $b \in \mathbb{N}$, $b \neq 1$, be odd. Assume, towards contradiction, that $\log_2(b)$ is rational. Then there are integers m, n with $n \neq 0$ such that

$$\log_2(b) = \frac{m}{n}.$$

Moreover, since $\log_2(b) > 0$, we may assume that m and n are positive. This equation implies that $2^{m/n} = b$, and so $2^m = b^n$. Since $m > 0$, we have that $2 \mid b^n$.

However, we claim that b^k is odd for all $k \in \mathbb{N}$, and so for $k = n$. Indeed, the statement holds for $k = 1$, since b is odd. Assume that b^k is odd. Then $b^{k+1} = b^k \cdot b$ is the product of two odd numbers, and so odd itself. By induction, b^k is odd for all $k \in \mathbb{N}$.

Thus we have shown that simultaneously $2 \mid b^n$ and that b^n is odd, a contradiction. Thus $\log_2(b)$ is irrational.

Thus $\log_2(b)$ with b odd is rational when $b = 1$ and irrational for $b > 1$.

Finally, consider any natural number $n = 2^a b$ with b odd. Then if $b = 1$, so $n = 2^a$ then we know that $\log_2(n) = a$ is rational. While if b is odd and greater than 1, then $\log_2(n) = a + \log b$, where $\log_2(b)$ is irrational. Since the sum of a rational and irrational is irrational, we know that $\log_2(n)$ is irrational. ■

11.3.14.

Proof. Assume, to the contrary, that this set does have a maxium. Then we can write that maximum as $\frac{a}{b} \in \mathbb{Q}$. Since $\sqrt{2} \notin \mathbb{Q}$, we must have that

$$\sqrt{2} - \frac{a}{b} = \delta > 0.$$

Now let $n = \left\lceil \frac{1}{\delta} \right\rceil$. Then

$$0 < \frac{1}{n} < \delta = \sqrt{2} - \frac{a}{b}$$

and so

$$\frac{a}{b} < \frac{a}{b} + \frac{1}{n} < \sqrt{2}$$

Thus $\frac{a}{b} + \frac{1}{n}$ is a rational number larger than $\frac{a}{b}$ and smaller than $\sqrt{2}$, which contradicts our assumption that the maximum of A was $\frac{a}{b}$.

Hence the set has no maximum. ■

11.3.15.

Proof. Assume for a contradiction that there exist $a, n \in \mathbb{N}$ such that $a^2 + 35 = 7^n$. Then we can write $a^2 = 7(7^{n-1} - 5)$, and as $n \in \mathbb{N}$, we have that $7^{n-1} - 5 \in \mathbb{Z}$; thus $7 \mid a^2$. Since 7 is prime, by Euclid's lemma, we have $7 \mid a$. Thus, $a = 7m$ for some $m \in \mathbb{Z}$. Plugging this into the original equation, we obtain

$$7^2m^2 + 35 = 7^n.$$

Now, we have two cases for n, either $n = 1$ or $n \geq 2$.

- If $n = 1$, then the equation becomes $7^2m^2 + 35 = 7$. But $7^2m^2 + 35 \geq 35$, so this is a contradiction.
- Now assume that $n \geq 2$. Rewrite the equation as

  $$7^2(7^{n-2} - m^2) = 7^n - 7m^2 = 35.$$

  Since $n - 2 \geq 0$, we have $7^{n-2} - m^2 \in \mathbb{Z}$, implying that $7^2 \mid 35$. This is a contradiction.

Either case leads to a contradiction, and thus are no $a, n \in \mathbb{N}$ such that $a^2 + 35 = 7^n$. ■

11.3.16.

Proof. By way of contradiction, suppose there exists a value $x \in (0, 1)$ such that

$$\frac{1}{2x(1-x)} < 2.$$

Note that $2x(1 - x) > 0$ because $x \in (0, 1)$. Therefore multiplying both sides by $2x(1 - x)$ gives

$$1 < 4x(1 - x) = 4x - 4x^2. \tag{1}$$

Now, subtracting $4x - 4x^2$ from both sides of (1) yields

$$4x^2 - 4x + 1 = (2x - 1)^2 < 0.$$

This gives a contradiction because the square of a real number is non-negative. ■

Proof. Assume $x \in (0, 1)$. Then $2x - 1 \in \mathbb{R}$ as well. We use the fact that the square of a real number is non-negative to see

$$(2x - 1)^2 \geq 0.$$

Expanding, we have

$$4x^2 - 4x + 1 \geq 0.$$

Now, we subtract $4x^2 + 4x$ from each side to obtain

$$1 \geq 4x - 4x^2 = 4x(1 - x).$$

Finally, by assumption, we chose $x \in (0, 1)$. This means $x(1 - x) > 0$. Thus, dividing both sides by $2x(1 - x)$, we have

$$\frac{1}{2x(1-x)} \geq 2,$$

as desired. ■

11.3.17.

Proof. Recall that the square of any non-zero real number is positive. Since $x \neq y$, we know $x - y \neq 0$, so we have

$$(x - y)^2 > 0.$$

We expand the left hand side to achieve

$$x^2 - 2xy + y^2 > 0,$$

and add $2xy$ to each side, giving

$$x^2 + y^2 > 2xy.$$

Now, by assumption $x, y > 0$, hence $xy > 0$ and we may divide both sides by xy without changing the sign of the inequality

$$\frac{x^2}{xy} + \frac{y^2}{xy} > 2.$$

Simplifying gives the desired result

$$\frac{x}{y} + \frac{y}{x} > 2.$$

■

Proof. By way of contradiction, suppose that there exist values $x, y \in \mathbb{R}$ with $x, y > 0$ and $x \neq y$ such that

$$\frac{x}{y} + \frac{y}{x} \leq 2.$$

Since $x, y > 0$, we have $xy > 0$. Therefore, multiplying both sides by xy, we see

$$x^2 + y^2 \leq 2xy.$$

Subtracting $2xy$ from each side gives

$$x^2 - 2xy + y^2 = (x - y)^2 \leq 0 \tag{1}$$

On the other hand, $x - y \in \mathbb{R}$ implies that $(x-y)^2 \geq 0$. Combined with (1), we see

$$(x-y)^2 = 0,$$

and thus we have $x = y$. This yields a contradiction since we assumed $x \neq y$. ■

Proof. *Claim:* For all $x, y \in \mathbb{R}$ with $x, y > 0$

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

We prove the claim using two cases.
Case 1: $x = y$.
If $x = y$, then $\frac{x}{y} = \frac{y}{x} = 1$. Therefore

$$\frac{x}{y} + \frac{y}{x} = 1 + 1 = 2.$$

Case 2: $x \neq y$.
By either of the previous parts, we have

$$\frac{x}{y} + \frac{y}{x} > 2.$$

Combining cases 1 and 2, we conclude that for all $x, y \in \mathbb{R}$ with $x, y > 0$

$$\frac{x}{y} + \frac{y}{x} \geq 2.$$

■

11.3.18.

Proof. By way of contradiction, suppose there exist $a, b \in \mathbb{R}$ with $a, b > 0$ such that

$$\frac{2}{a} + \frac{2}{b} = \frac{4}{a+b}.$$

We rearrange the expression, multiplying both sides by ab to achieve

$$2b + 2a = \frac{4ab}{a+b}.$$

Multiplying both sides by $a + b$, we have

$$4ab + 2b^2 + 2a^2 = 4ab.$$

Subtracting $4ab$ from each side and dividing by 2 yields,

$$a^2 + b^2 = 0.$$

This gives a contradiction since $a, b \in \mathbb{R} \setminus \{0\}$ implies $a^2 + b^2 > 0$. ■

Proof. Recall that the square of a positive real number is positive. Therefore, $a^2 > 0$ and $b^2 > 0$, and we must have

$$2a^2 + 2b^2 > 0.$$

We add $4ab$ to each side of our expression

$$2a^2 + 2b^2 + 4ab > 4ab$$

and factor the left hand side to obtain

$$(2a + 2b)(a + b) > 4ab.$$

Since $a, b > 0$, $a + b > 0$ and we may divide both sides by $a + b$ to achieve

$$2a + 2b > \frac{4ab}{a + b}$$

Finally, since $a, b > 0$, we may divide both sides by ab to obtain

$$\frac{2}{b} + \frac{2}{a} > \frac{4}{a + b}.$$

In particular,

$$\frac{2}{b} + \frac{2}{a} \neq \frac{4}{a + b}.$$

■

11.3.19.

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, bijective function.

By way of contradiction, assume that f is neither strictly increasing, nor strictly decreasing.

Then there exists $x_1, x_2, x_3 \in \mathbb{R}$ with $x_1 < x_2 < x_3$ such that

(1) $f(x_1) \leq f(x_2)$ and $f(x_3) \leq f(x_2)$ (the function increases and then decreases). Or,

(2) $f(x_1) \geq f(x_2)$ and $f(x_3) \geq f(x_2)$ (the function decreases and then increases).

First, we consider case (1). In this case, the function increases and then decreases, but we do not have information about how $f(x_1)$ compares to $f(x_3)$. This gives us two further sub-cases.

(i) Suppose $f(x_1) \leq f(x_3)$. We combine this with the inequalities from (1) to find

$$f(x_1) \leq f(x_3) \leq f(x_2).$$

We can regard $f(x_3)$ as the value c in the statement of the Intermediate Value Theorem. Since f is continuous, by the Intermediate Value Theorem,

there exists $x_0 \in [x_1, x_2]$ such that $f(x_0) = f(x_3)$. Because $x_0 \leq x_2 < x_3$, $x_0 \neq x_3$. This contradicts the assumption that f is injective.

(ii) Suppose $f(x_3) \leq f(x_1)$. Combining this with the inequalities from (1) yields

$$f(x_3) \leq f(x_1) \leq f(x_2).$$

We can now regard $f(x_1)$ as the value c in the statement of the Intermediate Value Theorem. Since f is continuous, the Intermediate Value Theorem tells us that there exists $x_0 \in [x_2, x_3]$ such that $f(x_0) = f(x_1)$. Because $x_1 < x_2 \leq x_0$, $x_1 \neq x_0$ and we find that f is not injective.

The argument is similar for case (2). We break into two further sub-cases which compares the value of $f(x_1)$ with $f(x_3)$.

(i) Suppose $f(x_1) \leq f(x_3)$. Then the inequalities from (2) tell us

$$f(x_2) \leq f(x_1) \leq f(x_3).$$

We regard $f(x_1)$ as the value c in the statement of the Intermediate Value Theorem. Since f is continuous, the Intermediate Value Theorem says there exists $x_0 \in [x_2, x_3]$ such that $f(x_0) = f(x_1)$. Notice that $x_1 < x_2 \leq x_0$, so $x_1 \neq x_0$. We conclude that f is not injective.

(ii) Finally, suppose $f(x_3) \leq f(x_1)$. f is continuous, and from (2) we know that

$$f(x_2) \leq f(x_3) \leq f(x_1).$$

We regard $f(x_3)$ as the value c in the statement of the Intermediate Value Theorem. By the Intermediate Value Theorem, there exists $x_0 \in [x_1, x_2]$ such that $f(x_0) = f(x_3)$. Because $x_3 > x_2 \geq x_0$, we see that $x_3 \neq x_0$, so f is not injective.

In each of the four cases, we reach the conclusion that f is not injective, contradicting our assumption that f is bijective.

We conclude that f must be strictly increasing or strictly decreasing ■

12 · Cardinality
12.7 · Exercises

12.7.1.

(a) We can list the elements of this set as $\{0, 1, 2, 3, \dots\}$. The function corresponding to this list is $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ defined by $f(n) = n - 1$. This function is well-defined, since $n - 1 \in \mathbb{N} \cup \{0\}$ whenever $n \in \mathbb{N}$. We need to show that this function is a bijection.

- Surjective: Let $m \in \mathbb{N} \cup \{0\}$. Then $m + 1 \in \mathbb{N}$, and $f(m + 1) = m$.
- Injective: If $f(m) = f(n)$, then $m - 1 = n - 1$. Hence $m = n$.

Alternatively, we could show that f is a bijection by showing that it has an inverse function. Indeed, its inverse is given by $g : \mathbb{N} \cup \{0\} \to \mathbb{N}$ with $g(n) = n + 1$.

(b) Let $f : \mathbb{N} \to \{5, 6, 7, 8, \dots\}$ be defined by $f(n) = n + 4$. This function is well-defined, since $n + 4 \in \{5, 6, 7, 8, \dots\}$ whenever $n \in \mathbb{N}$. We need to show that this function is a bijection.

- Surjective: Let $m \in \{5, 6, 7, 8, \dots\}$. Then $m - 4 \in \mathbb{N}$, and
$$f(m - 4) = (m - 4) + 4 = m.$$
- Injective: If $f(m) = f(n)$, then $m + 4 = n + 4$. Hence $m = n$.

(c) Let $f : \mathbb{N} \to \{1, 3, 3^2, 3^3, \dots\}$ be given by $f(n) = 3^{n-1}$. We need to show that this function is a bijection.

- Surjective: Let $a \in \{1, 3, 3^2, 3^3, \dots\}$. Then $a = 3^k$ for some $k \in \{0, 1, 2, \dots\}$. But then $k = n - 1$ for some $n \in \mathbb{N}$, and $f(n) = 3^{n-1} = 3^k = a$.
- Injective: If $f(m) = f(n)$, then $3^m = 3^n$. By taking logarithm with base 3 of each side, we have $m = n$.

(d) We can construct a bijection in a similar fashion as in Theorem 12.2.3, where we showed that $\mathbb{Z}$ is denumerable. We can alternate between the positive and negative integers, listing out the elements of $\mathbb{Z} \setminus \{0\}$ as
$$1, -1, 2, -2, 3, -3, \dots$$
So, the odd natural numbers will be set to the positive integers, and the even natural numbers will be sent to the negative integers. Define the function $f : \mathbb{N} \to \mathbb{Z} \setminus \{0\}$ by
$$f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ -\frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$
We show that f is a bijection:

- Surjective: Let $m \in \mathbb{Z} \setminus \{0\}$. First suppose that $m \geq 1$. Then $2m - 1$ is an integer, and $2m - 1 \geq 1$, so $2m - 1 \in \mathbb{N}$. Moreover, $2m - 1$ is odd, and hence
$$f(2m - 1) = \frac{(2m - 1) + 1}{2} = \frac{2m}{2} = m.$$
Next suppose that $m \leq -1$. Then $-2m \in \mathbb{N}$, and $-2m$ is even. Hence
$$f(-2m) = -\frac{-2m}{2} = m.$$

- Injective: Suppose $f(m) = f(n)$. First note that this implies both m and n are even, or both m and n are odd. Indeed, if this were not the case, than the sign of $f(m)$ and $f(n)$ would be opposite, meaning they could not be equal. If both m and n are even, then

$$\frac{m+1}{2} = f(m) = f(n) = \frac{n+1}{2}$$

implying $m = n$. If both m and n are odd, then

$$-\frac{m}{2} = f(m) = f(n) = -\frac{n}{2}$$

and so $m = n$.

12.7.2. *Claim:* It is true that $|A| = |\mathbb{N}|$.

Proof. We define a function $f : \mathbb{N} \to A$ by $f(k) = (k, k\pi)$.

Suppose $f(n) = f(\ell)$ for some $n, \ell \in \mathbb{N}$. Then $(n, n\pi) = (\ell, \ell\pi)$, and in particular, the first entries in the pair must match, so $n = \ell$. Thus, f is injective.

Now, consider $a \in A$. By the definition of A, this element takes the form $a = (m, \pi m)$ for some $m \in \mathbb{N}$. Therefore $f(m) = a$, and we see that f is surjective.

We conclude that $|A| = |\mathbb{N}|$. ■

12.7.3.

Proof. We show that the function

$$f(m, n) = 2^{m-1}(2n - 1),$$

is a bijection, and so show that $\mathbb{N} \times \mathbb{N}$ is denumerable.

We first show that the function is injective. So assume that $f(m_1, n_1) = f(m_2, n_2)$, then $2^{m_1-1}(2n_1 - 1) = 2^{m_2-1}(2n_2 - 1)$. WLOG, assume $m_1 \geq m_2$. Then $2^{m_1-m_2}(2n_1 - 1) = 2n_2 - 1$. If $m_1 \neq m_2$, then the left side is even but this contradicts the right side is odd. Thus, $m_1 = m_2$. Then $2n_1 - 1 = 2n_2 - 1$, hence $n_1 = n_2$. We conclude that f is injective.

Now we show that f is also surjective. Let $k \in \mathbb{N}$. Then we can write k as a nonnegative power of 2 times an odd number, namely, $k = 2^n(2q - 1)$ where $n \geq 0$ is an integer and $q \in \mathbb{N}$. To see that this expression for k is unique, assume $k = 2^{n_1}(2q_1 - 1) = 2^{n_2}(2q_2 - 1)$. WLOG, assume $n_1 \geq n_2$. Then $2^{n_1-n_2}(2q_1 - 1) = 2q_2 - 1$. If $n_1 > n_2$ then the left side is even but the right side is odd, a contradiction. Thus $n_1 = n_2$ and it follows $q_1 = q_2$. Now we have $n + 1, q \in \mathbb{N}$ $f(n + 1, q) = 2^n(2q - 1)$. So f is surjective.

Hence, since f is a bijection, we see that $\mathbb{N} \times \mathbb{N}$ is denumerable. ■

12.7.4.

Proof. Here we can use the fact that the set of prime numbers is countably infinite. Let P be the set of prime numbers. We have seen that P is countably infinite. The we can list the elements of P as $\{p_1, p_2, p_3, \ldots\}$. then $\forall i \in \mathbb{N}$ we

can define the sets

$$A_i = \{p_i^n : n \in \mathbb{N}\}.$$

We see by definition that $|A_i| = |\mathbb{N}|$.

Moreover consider the set:

$A_0 = \{1\} \cup \{n \in \mathbb{N} : \exists p_i, p_j \in P, \text{ s.t. } p_i \neq p_j, \text{ and } p_i \mid n \text{ and } p_j \mid n\}$. then we see that A_0 is also coutably infinite (since, say, it contains all the multiples 6 and still a subset of the set of natural numbers).

We also observe that there are countable infinitely many A_k's, which are all countably infinite themselves and $\bigcup_{n\geq 0} A_n = \mathbb{N}$. ■

Proof. *Another solution:* We know that we have a bijection, $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, defined as $f(n, m) = 2^{n-1}(2m - 1)$. Then, $\forall n \in \mathbb{N}$, we can define the set $A_n = \{2^{n-1}z : z \text{ is odd}\}$. Then we see that A_n is countably infinite for all n, there are countable many of such A_n's, where $A_k \cap A_l = \emptyset$ if $k \neq l$, and $\bigcup_{n\geq 0} A_n = \mathbb{N}$. ■

12.7.5.

Proof. First we should check that f is well-defined; in particular, if $n \in \mathbb{N}$, then $f(n) \in \mathbb{Z}$. If n is odd, then 2 divides $1 - n$, and so

$$f(n) = \frac{1-n}{2} \in \mathbb{Z}.$$

If n is even, then 2 divides n, and so

$$f(n) = \frac{n}{2} \in \mathbb{Z}.$$

Now we need to check that f is a bijection.

- Surjective: Let $m \in \mathbb{Z}$. First suppose that $m \leq 0$. Since m is an integer, $1 - 2m \in \mathbb{Z}$. Moreover, since $-2m \geq 0$, we have $1 - 2m \geq 1$. Thus $1 - 2m \in \mathbb{N}$. Furthermore,$1 - 2m = 2(-m) + 1$ is odd, and hence

$$f(1-2m) = \frac{1-(1-2m)}{2} = \frac{2m}{2} = m.$$

  Next suppose that $m \geq 1$. Then $2m \in \mathbb{N}$, and $2m$ is even. Hence

$$f(2m) = \frac{2m}{2} = m.$$

- Injective: Suppose $f(m) = f(k)$. If $f(m) = f(k)$ is non-positive, then both m and k are odd, since $f(\ell) > 0$ for even ℓ. In this case, we have

$$\frac{1-m}{2} = f(m) = f(k) = \frac{1-k}{2}$$

  implying $m = k$.

If $f(m) = f(k)$ is positive, then both m and k are even, since $f(\ell) \leq 0$ for odd ℓ. In this case,

$$\frac{m}{2} = f(m) = f(k) = \frac{k}{2}$$

and so $m = k$. We have reached this conclusion in all cases, so f is injective.

■

12.7.6.

Proof. Assume, towards contradiction, that $\mathbb{I}$ is denumerable. Then both $\mathbb{I}$ and $\mathbb{Q}$ are countable, and by Result 12.2.9,

$$\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$$

is countable as well. But this is a contradiction, since the real numbers are uncountable. Hence $\mathbb{I}$ is uncountable. ■

Proof. We know that

$$[0, 1] \cap \mathbb{Q} \subset \mathbb{Q},$$

and that $\mathbb{Q}$ is denumerable. Hence, by Theorem 12.2.4, $[0, 1] \cap \mathbb{Q}$ is countable. Moreover, $[0, 1] \cap \mathbb{Q}$ is infinite, since it contains the set $\{1/n : n \in \mathbb{N}\}$. Since the set is countable and infinite, it must be denumerable. ■

Proof. Let $f : \mathbb{Q} \to \{\pi + q : q \in \mathbb{Q}\}$ be given by $f(q) = \pi + q$. This function is surjective: if $x \in \{\pi + q : q \in \mathbb{Q}\}$, then $x = \pi + q$ for some $q \in \mathbb{Q}$. But then $f(q) = x$. The function is injective: if $f(q) = f(r)$, then $\pi + q = \pi + r$, and so $q = r$. Therefore f is a bijection, and

$$|\{\pi + q : q \in \mathbb{Q}\}| = |\mathbb{Q}|$$

and as $\mathbb{Q}$ is denumerable, $\{\pi + q : q \in \mathbb{Q}\}$ is as well. ■

Proof. Let $f : \mathbb{Q} \to \{a + q : q \in \mathbb{Q}\}$ be given by $f(q) = a + q$. This function is surjective: if $x \in \{a + q : q \in \mathbb{Q}\}$, then $x = a + q$ for some $q \in \mathbb{Q}$. But then $f(q) = x$. The function is injective: if $f(q) = f(r)$, then $a + q = a + r$, and so $q = r$. Therefore f is a bijection, and

$$|\{a + q : q \in \mathbb{Q}\}| = |\mathbb{Q}|$$

and as $\mathbb{Q}$ is denumerable, $\{a + q : q \in \mathbb{Q}\}$ is as well. ■

Proof. Let $g : \mathbb{Q} \to \{\pi q : q \in \mathbb{Q}\}$ be given by $g(q) = \pi q$. This function is surjective: if $x \in \{\pi q : q \in \mathbb{Q}\}$, then $x = \pi q$ for some $q \in \mathbb{Q}$. But then $g(q) = x$. The function is injective: if $g(q) = g(r)$, then $\pi q = \pi r$, and so $q = r$. Therefore g is a bijection, and

$$|\{\pi q : q \in \mathbb{Q}\}| = |\mathbb{Q}|$$

and as $\mathbb{Q}$ is denumerable, $\{\pi q : q \in \mathbb{Q}\}$ is as well. ■

Proof. As in the proof of (e), we can define the function $g : \mathbb{Q} \to \{aq : q \in \mathbb{Q}\}$ by $g(q) = aq$.

The argument for surjectivity is similar as part (e): if $x \in \{aq : q \in \mathbb{Q}\}$, then $x = aq$ for some $q \in \mathbb{Q}$, and $g(q) = x$. However, we have to be careful with injectivity. Suppose $g(q) = g(r)$, in which case $aq = ar$. This implies that $q = r$ as long as $a \neq 0$.

Thus, we see that when $a \neq 0$, g is a bijection and $\{aq : q \in \mathbb{Q}\}$ is denumerable. If $a = 0$, then $\{aq : q \in \mathbb{Q}\} = \{0\}$, and so the set is finite, not denumerable. ■

12.7.7.

Proof. Let $\mathbb{I}$ denote the set of irrational numbers. By way of contradiction, assume that $\mathbb{I}$ is countable. Then we see that $\mathbb{R} = \mathbb{Q} \cup \mathbb{I}$. Since $\mathbb{Q}$ is countable and $\mathbb{I}$ is countable, and the union of countable sets is countable, we conclude that $\mathbb{R}$ is countable, which is a contradiction. Therefore the set of irrational numbers is uncountable. ■

12.7.8.

Proof. Let $f(x) = \log(-x - \sqrt{29})$. We then construct a function $g(y) = -e^y - \sqrt{29}$. Note that f has domain $(-\infty, -\sqrt{29})$ and g has domain $\mathbb{R}$. Now

$$\forall y \in \mathbb{R}, \quad f(g(y)) = \log(e^y + \sqrt{29} - \sqrt{29}) = y,$$

and

$$\forall x \in (-\infty, -\sqrt{29}), \quad g(f(x)) = -e^{\log(-x-\sqrt{29})} - \sqrt{29} = -(-x-\sqrt{29}) - \sqrt{29} = x.$$

We conclude that $f \circ g$ and $g \circ f$ are both identity functions (on $\mathbb{R}$ and $(-\infty, -\sqrt{29})$, respectively), and so f has a two-sided inverse, which means that it is a bijection. Therefore, $(-\infty, -\sqrt{29})$ and $\mathbb{R}$ are equinumerous. ■

12.7.9.

Proof. Consider the function $G : S \to \mathcal{P}(\mathbb{N})$ defined by $G(f) = \{n : f(n) = 1\}$. We claim that G is a bijection. This implies that $|S| = |\mathcal{P}(\mathbb{N})|$ and thus is uncountable.

To prove the claim, it suffices to show that G is injective and surjective.

- (Injectivity) Suppose that $f, h \in S$ such that $G(f) = G(h)$. Then we have
$$\{n : f(n) = 1\} = \{n : h(n) = 1\},$$
So the functions f, h take the value 1 at the same inputs. At any other value of n the functions must be zero. Hence they take the value 1 at the same times and the value 0 at the same time. So $f = h$.

- (Surjectivity) Suppose that $X \in \mathcal{P}(\mathbb{N})$ and so $X \subseteq \mathbb{N}$. Define $f : \mathbb{N} \to \{0, 1\}$ by
$$f(n) = \begin{cases} 1 & \text{if } n \in X \\ 0 & \text{otherwise.} \end{cases}$$
Then we have $G(f) = X$ by definition.

■

12.7.10.

Proof. Let $h : (0,1) \to \mathbb{R}$ be defined by

$$h(x) = \begin{cases} \frac{1}{x} - 2 & \text{if } 0 < x \leq \frac{1}{2} \\ \frac{1}{x-1} + 2 & \text{if } \frac{1}{2} < x < 1 \end{cases}$$

In order to prove that this is bijective, we show that it has an inverse function, namely

$$h^{-1}(x) = \begin{cases} \frac{1}{x+2} & \text{if } x \geq 0 \\ \frac{1}{x-2} + 1 & \text{if } x < 0 \end{cases}$$

It is then a straightforward (and slightly tedious) computation to show that $h \circ h^{-1} = id$ and $h^{-1} \circ h = id$. ■

Proof. Let $y : (0,1) \to \mathbb{R}$ defined by

$$y(x) = \frac{1}{x} - \frac{1}{1-x} = \frac{1-2x}{x(1-x)}$$

We prove that this is a bijection.

- *Injective:* Suppose that $y(a) = y(b)$, with $a, b \in (0,1)$. Then

  $$\frac{1-2a}{a(1-a)} = \frac{1-2b}{b(1-b)} \implies (1-2a)b(1-b) = (1-2b)a(1-a)$$

  Expanding and simplifying, we have

  $$b - b^2 + 2ab^2 = a - a^2 + 2a^2b.$$

  We want to show that $a = b$, so we'd like to rewrite this expression with a factor of $a - b$. We write

  $$\begin{aligned} 0 &= a - b - (a^2 - b^2) + 2a^2b - 2ab^2 \\ 0 &= a - b - (a-b)(a+b) + 2ab(a-b) \\ 0 &= (a-b)(1 - (a+b) + 2ab). \end{aligned}$$

  Now, we need to show that $1 - (a+b) + 2ab \neq 0$, as then $a - b = 0$. To this end, we'll show that

  $$1 + 2ab > a + b.$$

  We will split the argument into two cases, $a + b \leq 1$ and $a + b > 1$, and use the assumption that $0 < a, b < 1$. If $a + b \leq 1$, then $a + b < 1 + 2ab$, as a and b are positive. If $a + b > 1$, then either $a > 1/2$ or $b > 1/2$. Without loss of generality, assume $a > 1/2$, as we may relabel if necessary. Then $2a > 1$, and as $b > 0$, we also have $2ab > b$. Therefore

  $$1 + 2ab > 1 + b > a + b$$

  with the last two inequalitiy following from $1 > a$. In all cases, we have $1 + 2ab > a + b$, and so $1 - (a+b) + 2ab \neq 0$. As mentioned earlier, this implies that $a = b$, and so y is injective.

- *Surjective:* Let $d \in \mathbb{R}$. We need to find $c \in (0,1)$ such that $y(c) = d$. That is, we need to show that there is such a c that solves the equation

$$d = \frac{1-2x}{x(1-x)}.$$

This equation simplifies to

$$dx^2 - (d+2)x + 1 = 0.$$

If $d = 0$, then we have the solution $x = 1/2$. If $d \neq 0$, by the quadratic formula, the equation has solutions

$$x = \frac{d+2 \pm \sqrt{(d+2)^2 - 4d}}{2d} = \frac{d+2 \pm \sqrt{d^2+4}}{2d}.$$

By Exercise 3.5.16, we know that

$$\sqrt{d^2+2} \geq \sqrt{d^2} = |d|$$

Suppose $d > 0$. Then

$$d + \sqrt{d^2+4} \geq d + d = 2d,$$

implying that

$$\frac{d+2+\sqrt{d^2+4}}{2d} \geq \frac{2d+2}{2d} > 1,$$

so this candidate for c doesn't work, as this solution lies outside the interval $(0,1)$. Therefore, we aim to show that

$$0 < \frac{d+2-\sqrt{d^2+4}}{2d} < 1. \qquad (*)$$

We'll split the proof of this into the cases $d > 0$ and $d < 0$.

First assume $d > 0$. Then $(*)$ is equivalent to the inequality

$$0 < d + 2 - \sqrt{d^2+4} < 2d. \qquad (**)$$

By Exercise 3.5.18, we know that

$$\sqrt{d^2+4} < \sqrt{d^2} + \sqrt{4} = d + 2.$$

Therefore

$$d + 2 - \sqrt{d^2+4} > d + 2 - (d+2) = 0.$$

Moreover, by Exercise 3.5.16, we know that $d^2 + 4 \geq 4$ implies

$$\sqrt{d^2+4} \geq \sqrt{4} = 2.$$

Thus

$$d + 2 - \sqrt{d^2 + 4} \leq d + 2 - 2 < 2d.$$

So we have established $(**)$, and so $(*)$, for the case $d > 0$.

Now assume $d < 0$. Then $(*)$ is equivalent to the inequality

$$0 > d + 2 - \sqrt{d^2 + 4} > 2d. \tag{$\dagger$}$$

By Exercise 3.5.18, we know that

$$\sqrt{d^2 + 4} < \sqrt{d^2} + \sqrt{4} = |d| + 2 = -d + 2.$$

Therefore

$$d + 2 - \sqrt{d^2 + 4} > d + 2 - (-d + 2) = 2d.$$

Moreover, by Exercise 3.5.16, we know that $d^2 + 4 \geq 4$ implies

$$\sqrt{d^2 + 4} \geq \sqrt{4} = 2 > 2 + d,$$

with the last inequality holding as $d < 0$. Thus

$$d + 2 - \sqrt{d^2 + 4} < d + 2 - (2 + d) = 0.$$

So we have established $(\dagger)$, and so $(*)$, for the case $d < 0$.

We have found a solution $c \in (0, 1)$ to the equation $y(c) = d$ in all cases, and so y is surjective.

■

Proof. We show that both sets are equinumerous to the interval $(1, \infty)$.

- First we find a bijection $f : \mathbb{R} \to (1, \infty)$. A function that almost works is the exponential function, 2^x; the issue is its range is $(0, \infty)$ rather than $(1, \infty)$. We can resolve this by taking $f(x) = 2^x + 1$. Moreover, we know that f has an inverse, $f^{-1} : (1, \infty) \to \mathbb{R}$, given by $f^{-1}(x) = \log_2(x - 1)$. Then by Theorem 10.6.8, f is bijective. Thus $\mathbb{R}$ and $(1, \infty)$ are equinumerous.
- Next, we want to find a bijection $g : (1, \infty) \to (0, 1)$. Take $g(x) = 1/x$. This function is well-defined, as for $x > 1$, we have $0 < 1/x < \infty$. Moreover, its inverse is given by $g^{-1} : (0, 1) \to (1, \infty)$, with $g^{-1}(x) = 1/x$. Again by Theorem 10.6.8, g is bijective. Thus $(1, \infty)$ and $(0, 1)$ are equinumerous.

Combining the two statements we proved, and appealing to Theorem 12.1.8, we can conclude that $\mathbb{R}$ and $(0, 1)$ are equinumerous, as desired. Indeed, the function $g(f(x)) = 1/(2^x + 1)$ is a bijection from $\mathbb{R}$ to $(0, 1)$. ■

12.7.11.

Proof. We know that the function $g : \mathbb{R} \to (0, \infty)$ defined by $g(x) = e^x$ is a

bijective function. Using this function we can define a bijection $f : \mathbb{R} \to (\sqrt{2}, \infty)$ as $f(x) = g(x) + \sqrt{2}$.

To see that f is injective, consider $x, y \in \mathbb{R}$ such that $f(x) = f(y)$. Then by the definition of f, $g(x) + \sqrt{2} = g(y) + \sqrt{2}$. Subtracting $\sqrt{2}$ from each side, we see that we must have $g(x) = g(y)$. By the injectivity of g, we see that $x = y$, and conclude that f is injective.

Moreover, consider $z \in (\sqrt{2}, \infty)$. Then $z - \sqrt{2} \in (0, \infty)$. Since g is bijective, there exists a value $x \in \mathbb{R}$ such that $g(x) = z - \sqrt{2}$. Adding $\sqrt{2}$ to each side, we see that $z = g(x) + \sqrt{2} = f(x)$, so f is surjective.

We conclude that $|\mathbb{R}| = |(\sqrt{2}, \infty)|$. ■

Proof. Let O denote the set of odd integers and E denote the set of even integers. Then we can define the function $f : O \to E$ as $f(x) = x + 1$.

To see that f is injective, consider $x, y \in O$ such that $f(x) = f(y)$. Then $x + 1 = y + 1$. Subtracting 1 from each side, we see that $x = y$. Also, for any even integer y, we see $y - 1$ is odd and $f(y - 1) = y$, which implies that f is surjective. Therefore f is bijective, and we see that $|O| = |E|$ ■

Proof. We see that $S = \{x \in \mathbb{R} : \sin x = 1\} = \{x \in \mathbb{R} : x = \pi/2 + 2\pi n \text{ for some } n \in \mathbb{Z}\}$. Thus we can define a bijection $f : \mathbb{Z} \to S$, where $f(k) = \frac{\pi}{2} + 2\pi k$. Consider $i, j \in \mathbb{Z}$ such that $f(i) = f(j)$. Then by the definition of f, $\frac{\pi}{2} + 2\pi i = \frac{\pi}{2} + 2\pi j$. Subtracting $\frac{\pi}{2}$ from each side gives $2\pi i = 2\pi j$. Finally, dividing by 2π, we see that $i = j$, so f is injective.

Now, consider $s \in S$. As we noticed above, $s = \frac{\pi}{2} + 2\pi k$ for some $k \in \mathbb{Z}$. Thus, $f(k) = s$, and we see that f is a surjection.

Hence f is bijective, and we conclude that $|\mathbb{Z}| = |S|$. ■

Proof. Define a function $f : \mathbb{Z} \to \{0, 1\} \times \mathbb{N}$, by

$$f(k) = \begin{cases} (1, k+1) & \text{if } k \geq 0 \\ (0, -k) & \text{if } k \leq -1 \end{cases}$$

Then we see that for $k, m \in \mathbb{Z}$, if $f(k) = f(m)$, then either $k, m \leq -1$ or $k, m \geq 0$. If $k, m \geq 0$ we have $(1, k + 1) = (1, m + 1)$ in which case $k = m$. If $k, m \leq -1$ we have $(0, -k) = (0, -m)$, in which case $k = m$ too. We also see that these two are the only two cases since if $k \geq 0$ and $m \leq -1$ or $m \geq 0$ and $k \leq -1$ we see $f(k) \neq f(m)$.

We also see that f is surjective. If $(a, n) \in (0, 1) \times \mathbb{N}$ we have two cases: $a = 0$ or $a = 1$. If $a = 0$, we can define $k = -n$ so that $f(-k) = (0, n)$ and if $a = 1$, we ca define $k = n - 1$ so that $f(k) = (1, n)$. We conclude that f is a bijection, and therefore $|\{0, 1\} \times \mathbb{N}| = |\mathbb{Z}|$. ■

12.7.12.

Proof. Since $|A| = |B|$ and $|C| = |D|$, there are bijections $f : A \to B$ and $g : C \to D$. Since $A \cap C = \emptyset$, any $x \in A \cup C$, either $x \in A$ or $x \in C$ but not in

both. Define $F : A \cup C \to B \cup D$ by

$$F(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in C. \end{cases}$$

And since $A \cap C = \emptyset$ this is well defined.

We now show F is bijective. To see F is injective: assume $F(x) = F(y)$.

(a) $x, y \in A$. So $f(x) = f(y)$ hence $x = y$ as f is injective.

(b) $x, y \in C$. So $g(x) = g(y)$ hence $x = y$ as g is injective.

(c) $x \in A$ and $y \in C$ (or similarly $x \in C$ and $y \in A$), $F(x) = f(x) \in B$ and $F(y) = g(y) \in D$. But $B \cap D = \emptyset$, which gives us a contradiction. So this case cannot happen.

Since we always have $x = y$, we conclude F is injective.

To see F is surjective: For any $y \in B \cup D$, %since $B \cap D = \emptyset$, $y \in B$ or $y \in D$ but not in both.

(a) $y \in B$. There is $x \in A$ s.t. $f(x) = y$ since f is surjective. So $F(x) = f(x) = y$.

(b) $y \in D$. There is $x \in C$ s.t. $g(x) = y$ since g is surjective. So $F(x) = g(x) = y$.

We conclude F is surjective. Hence F is bijective and $|A \cup C| = |B \cup D|$. ■

12.7.13.

Proof. We can define such a bijection, $f : (0, \infty) \to (0, \infty) - \{1\}$, as follows:

$$f(x) = \begin{cases} x + 1 & x \in \mathbb{N} \\ x & \text{otherwise.} \end{cases}$$

Now, we need to show that f is indeed a bijective function.

Show f is surjective: Let $y \in (0, \infty) - \{1\} = (0, 1) \cup (1, \infty)$. Then we have 2 cases.

Case 1: $y \in \mathbb{N}$, $y \neq 1$, that is, $y \geq 2$. Then, we see that we can take $x = y - 1$ and get $f(x) = x + 1 = y$.

Case 2: $y \notin \mathbb{N}$. Then, we see that we can take $x = y$ and get $f(x) = x = y$.

Show f is injective: Let $x, y \in (0.\infty)$, such that $f(x) = f(y)$. Then we know that if $f(x) \in \mathbb{N}$, then $f(y) \in \mathbb{N}$, which implies that $x, y \in \mathbb{N}$ and since for $x, y \notin \mathbb{N}$, we see that $f(x) = x \notin \mathbb{N}$ and $f(y) = y \notin \mathbb{N}$.

Hence, we see that $x = f(x) - 1 = f(y) - 1 = y$.

Moreover, we see that if $f(x), f(y) \notin \mathbb{N}$, we see that $x = f(x) = f(y) = y$.

Therefore f is injective, which in turn implies that f is bijective. ■

12.7.14.

Proof. We will prove this statement using Cantor-Schröder-Bernstein 12.5.1, defining an injection from $(0,1)$ to $(0,1)\times(0,1)$, and another injection in the reverse direction. This is easier than trying to define an explicit bijection from one set to the other; as we will see, the fact that some numbers have two distinct decimal representations causes trouble if we tried that approach instead.

First let's define an injection $f:(0,1)\to(0,1)\times(0,1)$. We can take

$$f(x)=(x,1/2),$$

so that f maps the interval $(0,1)$ onto a strip contained in the square $(0,1)\times(0,1)$.

Next, we try to define an injection $g:(0,1)\times(0,1)\to(0,1)$. This direction is trickier. In Section 12.3, we saw that any element $a\in(0,1)$ can be written as a decimal expansion,

$$a=0.a_1a_2a_3\ldots \quad a_i\in\{0,1,\ldots,9\}.$$

Moreover, we discussed that this representation is unique, with the exception of numbers whose digits are eventually all zero or nine, such as

$$1/2=0.5000\cdots=0.4999\ldots$$

In these cases, we choose the representation ending in all zeros. Therefore, none of the decimal expansions will end in all repeating nines.

Let $(a,b)\in(0,1)\times(0,1)$, which we write as

$$(a,b)=(0.a_1a_2a_3\ldots,0.b_1b_2b_3\ldots).$$

We want to map this to a single element in $(0,1)$. We'll identify this element by its own decimal representation, which will be defined in terms of the digits a_i and b_i. Since there are infinitely digits in the expansions of a and b, we'll need to combine their digits by going back and forth between the a_i and b_i. Define

$$g((a,b))=0.a_1b_1a_2b_2a_3b_3\ldots$$

We need to check that this map is injective.

Injectivity will follow in most cases by the uniqueness of the decimal representation. However, we need to be careful for numbers that have two representations, like $1/2$. In such a case, one of these representations ends in repeating nines. In order for $0.a_1b_1a_2b_2a_3b_3\ldots$ to end in repeating nines, the digit expansions chosen for a and b would also need to end in repeating nines. But this can't happen, since we already made the choice to not use any decimal expansions ending in all nines, opting for the expansion ending in zeros instead.

Notice that the function g isn't a surjective. For example, the element

$$(0.5,0.1119191919\ldots)$$

is not in the range of g; indeed, if it were, then we would need $b = 0.1999\ldots$, but for this number we use the representation $0.200\ldots$ instead. ■

Proof. From Exercise 12.7.10, we know that $|(0,1)| = |\mathbb{R}|$. So there is a bijection $h : (0,1) \to \mathbb{R}$. We can use h to form a bijection H from $(0,1) \times (0,1)$ to $\mathbb{R} \times \mathbb{R}$. Indeed, let

$$H((x,y)) = (h(x), h(y)).$$

Since h is invertible, H is as well; namely,

$$H^{-1}((x,y)) = (h^{-1}(x), h^{-1}(y)).$$

Thus H is indeed bijective. Therefore

$$|(0,1) \times (0,1)| = |\mathbb{R} \times \mathbb{R}|.$$

From part (a), we know that

$$|(0,1)| = |(0,1) \times (0,1)|$$

and we also already know that $|(0,1)| = |\mathbb{R}|$. Putting all these cardinality equalities together, we have $|\mathbb{R}^2| = |\mathbb{R}|$, as desired. ■

12.7.15.

Proof. Since A and B are equinumerous, there is a bijection $f : A \to B$. Use this to define $F : \mathcal{P}(A) \to \mathcal{P}(B)$ where

$$F(C) = \{f(a) : a \in C\}.$$

We show that F is surjective and injective.

- Surjective: Let $D \in \mathcal{P}(B)$, so that $D \subseteq B$. Since f is bijective, it has an inverse $f^{-1} : B \to A$, and we may define $C = \{f^{-1}(b) : b \in D\}$. Notice that $C \in \mathcal{P}(A)$, since $f^{-1}(b) \in A$ for all $b \in D$, meaning that C is a subset of A.

  We claim that $F(C) = D$. This is an equality of sets, so we need to show that $F(C) \subseteq D$ and $D \subseteq F(C)$. Let $c \in F(C)$. By definition, $c = f(a)$ for some $a \in C$. But since $a \in C$, there is some $b \in D$ such that $a = f^{-1}(b)$. Thus

  $$c = f(a) = f(f^{-1}(b)) = b$$

  giving $c = b \in D$. So $F(C) \subseteq D$.

  Conversely, let $d \in D$. We need to show that $d \in F(C)$, or equivalently, that there is some $a \in C$ such that $f(a) = d$. Using the definition of C, we need some $b \in D$ such that $f(f^{-1}(b)) = d$. Of course, $b = d$ will work. Indeed,

  $$d = f(f^{-1}(d)) \in F(C).$$

  Thus $D \subseteq F(C)$.

- Injective: Suppose that $F(C) = F(E)$ for some $C, E \in \mathcal{P}(A)$. We need to show that $C = E$, which we do by showing each set is a subset of the other.

  Suppose $c \in C$. Since $f(c) \in F(C)$ and $F(C) = F(E)$, we have $f(c) \in F(E)$. Hence, there is some $e \in E$ such that $f(c) = f(e)$. But f is injective, so $c = e$. Thus $c \in E$, meaning $C \subseteq E$. A similar argument gives $E \subseteq C$.

■

12.7.16. *Claim:* The statement is not true.

Proof. Let $\mathcal{A} = \{(a_1, a_2, a_3, \dots) : a_i \in \mathbb{Z}\}$, and consider

$$\mathcal{B} = \{(b_1, b_2, b_3, \dots) : b_i \in \{0, 1, \dots, 9\} \text{ and } \nexists k \in \mathbb{N} \text{ s.t. } b_n = 9 \,\forall\, n > k\} \subset \mathcal{A}.$$

That is, infinite sequences of natural numbers from 0 to 9 which do not end in infinitely many consecutive 9's. We will first show that $\mathcal{B}$ is uncountable by constructing a bijection from $\mathcal{B}$ to $[0, 1)$. Define $f : \mathcal{B} \to [0, 1)$ by

$$f((b_1, b_2, b_3, b_4, b_5, \dots)) = 0.b_1b_2b_3b_4b_5\dots.$$

This functions takes the i^{th} coordinate of $(b_1, b_2, b_3, b_4, b_5, \dots)$ and sends it to the i^{th} decimal place following 0. For example

$$f((1, 2, 3, 4, 5, 6, \dots)) = 0.123456\dots.$$

To see that f is injective, let $x = (x_1, x_2, x_3, \dots)$ and $y = (y_1, y_2, y_3, \dots)$ be elements of $\mathcal{B}$ and suppose that $f(x) = f(y)$. Then

$$0.x_1x_2x_3\cdots = 0.y_1y_2y_3\dots.$$

Since our sequence does not terminate in 9's, each real number in $[0, 1)$ has a unique decimal representation, and we must have

$$x_1 = y_1 \qquad x_2 = y_2 \qquad x_3 = y_3 \qquad \cdots$$

Therefore $x = y$ and f is injective. It remains to show that f is surjective. Suppose $s \in [0, 1)$. Then s takes the form

$$0.s_1s_2s_3s_4s_5\dots$$

where $s_j \in \{0, 1, \dots, 9\}$. Since

$$f((s_1, s_2, s_3, s_4, s_5, \dots)) = 0.s_1s_2s_3s_4s_5\dots$$

we have found an element of $\mathcal{B}$ mapping to s, so f is surjective. Therefore, $|\mathcal{B}| = |[0, 1)|$, so $\mathcal{B}$ is uncountable. Since $\mathcal{B} \subset \mathcal{A}$, $\mathcal{A}$ is uncountable as well. ■

Proof. We are going to prove this set is uncountable by showing that it has an uncountable subset. Let $\mathcal{A} = \{(a_1, a_2, a_3, \dots) : a_i \in \mathbb{Z}\}$, and let $\mathcal{C} =$

$\{(c_1, c_2, c_3, \ldots) : c_i \in \{0, 1\}\}$. Then, we see that $\mathcal{C} \subset \mathcal{A}$. Now, we can show that $\mathcal{C}$ is uncountable by finding a bijection from $\mathcal{C}$ to $\mathcal{P}(\mathbb{N})$.

We can define the bijection f as follows:

$$f : \mathcal{C} \to \mathcal{P}(\mathbb{N})$$
$$f(c_1, c_2, c_3, \ldots) = \{k \in \mathbb{N} : c_k = 1\}.$$

for example, $f(0, 1, 0, 1, 0, 0, 0, 0, \ldots) = \{2, 4\}$.

We see that f is surjective since for any $X \subseteq \mathbb{N}$, we can construct the sequence $(x_1, x_2, x_3, \ldots)$, where $x_k = 1$ if $k \in X$ and 0 otherwise. Then, $f(x_1, x_2, x_3, \ldots) = X$.

We also see that f is injective since if $(a_1, a_2, a_3, \ldots) \neq (b_1, b_2, b_3, \ldots)$, then $\exists m \in \mathbb{N}$ such that $a_m = 1$ and $b_m = 0$, in which case $m \in f(a_1, a_2, a_3, \ldots)$ and $m \notin f(b_1, b_2, b_3, \ldots)$, or $a_m = 0$ and $b_m = 1$, in which case $m \notin f(a_1, a_2, a_3, \ldots)$ and $m \in f(b_1, b_2, b_3, \ldots)$. In both cases, we see $f(a_1, a_2, a_3, \ldots) \neq f(b_1, b_2, b_3, \ldots)$. We conclude that f is a bijection. Therefore we see that $|\mathcal{C}| = |\mathcal{P}(\mathbb{N})|$, which means $\mathcal{C}$ is uncountable. Thus, $\mathcal{A}$ is uncountable too. ■

Note: We can also use a diagonal argument directly to show that the set $\{(a_1, a_2, a_3, \ldots) : a_i \in \mathbb{Z}\}$ is uncountable.

12.7.17.

Proof. For each $X \in P$, since X and $h(X)$ have the same cardinality, there exists a bijection $g_X \colon X \to h(X)$. Define f to be the union of all these g_X:

$$f = \{(a, b) \colon \text{there exists } X \in P \text{ such that } a \in X \text{ and } b = g_X(a)\}.$$

Of course, it is not immediately obvious that this is a function, let alone that it is a bijection. We must prove that.

We show that f is a function from A to B, that it is surjective, and that it is injective.

- For any $(a, b) \in f$, choose $X \in P$ such that $a \in X$ and $b = g_X(a)$. Since $X \in P$ and P is a partition of A, we know that $X \subset A$, and therefore $a \in A$. Similarly, $g_X(a) \in h(X)$ and $h(X) \in Q$; since Q is a partition of B, we see that $h(X) \subset B$, and so $b = g_X(a) \in B$. This shows that $f \subset A \times B$.

  Let $a \in A$ be arbitrary. Since P is a partition of A, there is a unique $X \in P$ with $a \in X$. Since g_X is a function from X to $h(X)$, there is a unique b such that $g_X(a) = b$. This shows that there is a unique $b \in B$ with $f(a) = b$, and so f is a function.

- Let $b \in B$ be arbitrary. Since Q is a partition of B, there exists $Y \in Q$ with $b \in Y$. Since h is surjective, there exists $X \in P$ with $h(X) = Y$, and therefore g_X is a function from X to Y. Since g_X is surjective, there exists $a \in X$ such that $g_X(a) = b$. By definition, we see that $f(a) = b$. Therefore f is surjective.

- Let $a_1, a_2 \in A$, and suppose that $f(a_1) = f(a_2)$. Since Q is a partition of B, there is a unique $Y \in Q$ with $f(a_1) = f(a_2) \in Y$. Since $h\colon P \to Q$ is injective, there is a unique $X \in P$ with $h(X) = Y$. This X corresponds to the only function g_X whose codomain is Y, and so we must have $a_1, a_2 \in X$ and $f(a_1) = g_X(a_1)$ and $f(a_2) = g_X(a_2)$. Since g_X is injective, we conclude that $a_1 = a_2$. Therefore f is injective.

■

12.7.18.

Proof. Let A, B and C be sets such that $|A| \leq |B|$ and $|B| \leq |C|$. Then, by definition, there are injections $f : A \to B$ and $g : B \to C$. By Theorem 10.5.3, the function $g \circ f : A \to C$ is an injection. Thus, by definition, $|A| \leq |C|$. ■

Proof. First assume that whenever sets A, B, and C satisfy $|A| \leq |B| \leq |C|$ and $|A| = |C|$, then we also have $|A| = |B| = |C|$. Let S and T be set such that $|S| \leq |T|$ and $|T| \leq |S|$. Then applying the previous statement with $A = C = S$ and $B = T$, we see that $|S| = |T|$. This is precisely the statement of Cantor-Schröeder-Bernstein 12.5.1.

Conversely, let A, B and C be sets such that $|A| \leq |B|$ and $|B| \leq |C|$, and $|A| = |C|$. Since $|A| = |C|$, there is a bijection, say h, from C to A. But h is also an injection, so $|C| \leq |A|$. Combining this with the inequality $|A| \leq |B|$, we have $|C| \leq |B|$, by part (a). Therefore $|B| \leq |C|$ and $|C| \leq |B|$, implying by Cantor-Schröeder-Bernstein 12.5.1 that $|B| = |C|$. ■

12.7.19.

Proof. Let $n \in \mathbb{N}$. We see that the set F_n is the set of all subsets of $\mathbb{N}$ of size n. We see that F_n is an infinite set. This means that we need to show that F_n is countable. As F_n is infinite, we know that $|F_n| \geq |\mathbb{N}|$. We must show that $|F_n| \leq |\mathbb{N}|$, so it is enough to show that there is an injection from F_n to a countable set. We can define such injection as follows:

$$f : F_n \to \mathbb{N}^n,$$
$$f(\{a_1, a_2, a_3, \cdots, a_n\}) = (a_{k_1}, a_{k_2}, a_{k_3}, \cdots, a_{k_n}),$$

where $\{a_1, a_2, a_3, \cdots, a_n\} = \{a_{k_1}, a_{k_2}, a_{k_3}, \cdots, a_{k_n}\}$ and $a_{k_1} < a_{k_2} < a_{k_3} < \cdots < a_{k_n}$. We see that f is a well-defined function since if the sets $\{a_1, a_2, a_3, \cdots, a_n\} = \{b_1, b_2, b_3, \cdots, b_n\}$, then $(a_{k_1}, a_{k_2}, a_{k_3}, \cdots, a_{k_n}) = (b_{k_1}, b_{k_2}, b_{k_3}, \cdots, b_{k_n})$, that is, when the a_i's and b_j's are arranged in ascending order, the corresponding n-tuples are equal.

Moreover, if $f(\{a_1, a_2, a_3, \cdots, a_n\}) = f(\{b_1, b_2, b_3, \cdots, b_n\})$, then $(a_{k_1}, a_{k_2}, a_{k_3}, \cdots, a_{k_n}) = (b_{k_1}, b_{k_2}, b_{k_3}, \cdots, b_{k_n})$ and since $\{a_1, a_2, a_3, \cdots, a_n\} = \{a_{k_1}, a_{k_2}, a_{k_3}, \cdots, a_{k_n}\}$ and $\{b_1, b_2, b_3, \cdots, b_n\} = \{b_{k_1}, b_{k_2}, b_{k_3}, \cdots, b_{k_n}\}$ we see $\{a_1, a_2, a_3, \cdots, a_n\} = \{b_1, b_2, b_3, \cdots, b_n\}$, meaning that f is injective.

We have also seen that the finite cartesian product of countable sets is countable (via Result 12.2.6). Thus, we see $\mathbb{N}^n$ is countable. Therefore since f is an injection from F_n to $\mathbb{N}^n$ and since $\mathbb{N}^n$ is countable, we see F_n is countable. ■

Proof. Since we are taking the union over denumerably many denumerable sets, by Exercise 12.7.20, we see that $\left| \bigcup_{n\in\mathbb{N}} F_n \right| = |\mathbb{N}|$. ■

(c) This result says that the set of all finite subsets of $\mathbb{N}$ is countable. This is not really a contradiction with $\mathcal{P}(\mathbb{N})$ being uncountable, since we can show that the set of infinite subsets of $\mathbb{N}$ is indeed uncountable.

12.7.20.

Proof. Let $A_1, A_2, A_3, \ldots$ be denumerable sets, and suppose that $A_m \cap A_n = \emptyset$ whenever $m \neq n$. Since each A_n is denumerable, we have

$$A_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\}.$$

We can construct the following infinite table:

$$\begin{array}{c|cccc} A_1 & a_{11}, & a_{12}, & a_{13}, & \cdots \\ A_2 & a_{21}, & a_{22}, & a_{23}, & \cdots \\ A_3 & a_{31}, & a_{32}, & a_{33}, & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Define the function $f : \mathbb{N} \to \cup_{n=1}^{\infty} A_n$ as

$$\begin{array}{llll} f(1) = a_{11} & & & \\ f(2) = a_{12} & f(3) = a_{21} & & \\ f(3) = a_{13} & f(4) = a_{22} & f(5) = a_{31} & \\ f(6) = a_{14} & f(7) = a_{23} & f(8) = a_{32} & f(9) = a_{41} \end{array}$$

and so on. Since every element a_{nk} is eventually reached (it will be reached after a finite number of diagonal sweeps, and each of these diagonal sweeps has finite length), f is surjective. The assumption that $A_m \cap A_n = \emptyset$ whenever $m \neq n$ implies that there are no repeated elements in the table. The diagonal sweeping counts every element in the table only once, and so f is injective. ■

Proof. We claim that $A_1 \cup \cdots \cup A_k$ is countable for each $k \in \mathbb{N}$. This is the case for $k = 1$, by assumption. Suppose that it is true for $k = n$. Then

$$A_1 \cup \cdots \cup A_n \cup A_{n+1} = (A_1 \cup \cdots \cup A_n) \cup A_{n+1}$$

is the union of two countable sets, and so countable itself, by Result 12.2.9. Thus the result holds for $k = n + 1$. Thus by induction, the result holds for all $k \in \mathbb{N}$, and in particular, for $k = N$.

Now assume that there are infinitely many sets that are non-empty. By removing the empty sets, and relabeling if necessary, we may assume that $A_n \neq \emptyset$ for all $n \in \mathbb{N}$. The argument to prove that the union of these sets is countable is similar to part (a). We can construct a table with the n^{th} row listing the elements of A_n. There is at least one element in every row, but some rows may have only finitely many elements.

We can construct a bijection $f : \mathbb{N} \to \cup_{n=1}^{\infty} A_n$ by sweeping over diagonals of the table. However, some positions in the table may be empty. For example, suppose that $|A_1| = 2$ and $|A_2| = 1$, and the table looks like

$$\begin{array}{c|cccc} A_1 & a_{11}, & a_{12} & & \\ A_2 & a_{21} & & & \\ A_3 & a_{31}, & a_{32}, & a_{33}, & \cdots \\ A_4 & a_{41}, & a_{42}, & a_{43}, & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Then f would be given by

$$\begin{array}{ll} f(1) = a_{11} & \\ f(2) = a_{12} & f(3) = a_{21} \\ f(4) = a_{13} & \\ f(5) = a_{32} & f(6) = a_{41} \end{array}$$

and so on and so on. Notice that the first column of our table contains no empty spaces, because each set A_n is non-empty. We need to make sure that we don't pass over infinitely many empty spaces in the table one after another. If this were the case, $f(n)$ would be defined for only finitely many $n \in \mathbb{N}$, and we wouldn't reach every element in the table. Indeed this cannot happen. Each diagonal sweep starts from a (possibly empty) spot in the topmost row and moves to the leftmost column in a finite number of steps. Since the leftmost column contains no empty spaces, each diagonal sweep reaches a non-empty space in the table after a finite number of steps. ■

Proof. Let $A_1, A_2, A_3, \ldots$ be countable sets. Let $B_1 = A_1$. For $n > 1$, let

$$B_n = A_n \setminus \left(\cup_{m=1}^{n-1} A_m\right).$$

We claim that

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$$

and that $B_n \cap B_m = \emptyset$ for $n \neq m$. First we show (1). Suppose that

$$x \in \bigcup_{n=1}^{\infty} A_n.$$

Let

$$N = \min\{n : x \in A_n\}.$$

Then N exists, since $\{n : x \in A_n\}$ is non-empty (as x is an element of the union, so must be in some A_n), and a subset of the natural numbers (so it has a minimum). Then

$$x \notin \bigcup_{m=1}^{N-1} A_m$$

as $x \notin A_m$ for $1 \leq m \leq N-1$ by choice of N. Combining this with the fact that $x \in A_N$, we have

$$x \in B_N = A_N \setminus \left(\cup_{m=1}^{N-1} A_m\right).$$

Therefore

$$x \in B_N \subset \bigcup_{n=1}^{\infty} B_n.$$

Conversely, suppose

$$y \in \bigcup_{n=1}^{\infty} B_n.$$

Then there is some n such that

$$y \in B_n = A_n \setminus \left(\cup_{m=1}^{n-1} A_m\right)$$

implying that $y \in A_n$. Thus (1) is established.

Now we show that the sets B_n are disjoint. Let $m \neq n$. Without loss of generality, we may assume that $m < n$, as we may relabel if necessary. Then

$$B_m \subset A_m \subset \bigcup_{k=1}^{n-1} A_k.$$

So, when choosing elements of A_n to include in B_n, we leave out any elements that appear in B_m. That is,

$$B_m \cap B_n = B_m \cap \left(A_n \setminus \left(\cup_{k=1}^{n-1} A_k\right)\right)$$

is empty.

Finally, for all $n \in \mathbb{N}$, $B_n \subset A_n$, and so $|B_n| \leq |A_n|$. Since A_n is countable, B_n is countable as well.

By part (b), $\cup_{n=1}^{\infty} B_n$ is countable, and hence by (1), $\bigcup_{n=1}^{\infty} A_n$ is countable as well. ■

12.7.21.

Proof. We will show this by constructing a bijection. Let $f : P_m \to (\mathbb{Q}^m \times (\mathbb{Q} \setminus \{0\}))$ be defined by

$$f(a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m) = \big((a_0, a_1, a_2, \cdots, a_{m-1}), a_m\big).$$

Consider $((b_0, \ldots, b_{m-1}), b_m) \in (\mathbb{Q}^m \times (\mathbb{Q} \setminus \{0\}))$. By construction,

$$f(b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m) = ((b_0, \ldots, b_{m-1}), b_m),$$

so we see f is surjective.

Moreover, let $(a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m), (b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m) \in P_m$ such that $f(a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m) = f(b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m)$. Then $\big((a_0, a_1, a_2, \cdots, a_{m-1}), a_m\big) = \big((b_0, b_1, b_2, \cdots, b_{m-1}), b_m\big)$, and we see that

$a_i = b_i$ for all $i \in \{0, 1, 2, \ldots, m\}$. Thus, $a_0 + a_1x + a_2x^2 + \cdots + a_mx^m = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$, which implies that the function f is injective.

Therefore $|P_m| = |(\mathbb{Q}^m \times (\mathbb{Q} \setminus \{0\}))|$. Since the Cartesian product of finitely many countable sets is countable, we have $|P_m| = |(\mathbb{Q}^m \times (\mathbb{Q} \setminus \{0\}))| = |\mathbb{N}|$. ■

Proof. By part (a), $|P_m| = |\mathbb{N}|$. Notice that $P = \bigcup_{m \in \mathbb{N}} P_m$. By exercise Exercise 12.7.20, the union of countably many countable sets is countable. Therefore, $|P| = |\mathbb{N}|$. ■

Proof. *Claim:* $|A| = |P|$.

Let $m \in \mathbb{N}$. By part (a), the polynomials of degree m are countable. Therefore, they are in bijection with $\mathbb{N}$, and we may enumerate them as $p_{m,1}, p_{m,2}, p_{m,3}, \ldots$. That is, any polynomial in P_m can be described as $p_{m,k}$ for $k \in \mathbb{N}$, where m refers to the degree of the polynomial and k comes from the enumeration of P_m. We define the set

$$R_{p_{m,k}} = \{x \in \mathbb{R} : p_{m,k}(x) = 0\}$$

to be the set of real roots of the polynomial $p_{m,k}$. By the Fundamental Theorem of Algebra, we know that every polynomial of order m has at most m distinct real roots. Therefore for any $k \in \mathbb{N}$, $|R_{p_{m,k}}| \leq m$, in particular, $R_{p_{m,k}}$ is countable.

Thus, we see that $A_m = \bigcup_{k \in \mathbb{N}} R_{p_{m,k}}$ gives us the set of all possible roots of polynomials of order m. Notice that A_m is the countable union of countable set, and thus, by Exercise 12.7.20, is countable.

Again, using Exercise 12.7.20, since $A = \bigcup_{m \in \mathbb{N}} A_m$, we see that A is also countable union of countable sets, and hence is countable.

Moreover, since any $q \in \mathbb{Q}$ is a zero of a polynomial with rational coefficients, e.g. $f(x) = x - q$, we see that $\mathbb{Q} \subseteq A$. This implies that A is infinite. Thus, we see that A is countably infinite.

Therefore $|A| = |P|$. ■

(This is a very important result. These numbers that are solutions to rational (or equivalently integer) polynomials are called algebraic numbers. As an example we see $\sqrt{2}$ is irrational, but is the solution to the equation $x^2 - 2 = 0$. Thus, since the set of algebraic numbers is countable and the set of real numbers is uncountable, we see that there has to be uncountable many real numbers that cannot be written as a solution to a polynomial with rational coefficients. We call such numbers *transcendental* numbers. For example, π is a transcendental number-as one can guess, showing a number is transcendental is generally a very hard question.)

Colophon

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