PLP - 1

TOPIC 1 — INTRODUCTION TO SETS

Demirbaş & Rechnitzer

INTRODUCTION TO SETS

GETTING STARTED WITH SETS

We need tools to understand collections of objects

DEFINITION: (A NOT SO FORMAL DEFINITION OF SETS).

A **set** is a collection of objects.

The objects are referred to as **elements** or **members** of the set.

Informal because

- is simple and intuitive
- rigorous definition is way harder than we need
- we just want to get on with things

QUICK ASIDE ON CONVENTIONS AND NOTATION

Mathematics has many conventions / traditions

- not firm rules, but
- make it easier for the **reader** to understand

Use

- Capitals for sets: A, B, C, X, Y
- Lowercase for elements: a, b, c, x, y

Similarly, use

- i, j, k, ℓ, m, n to denote integers
- x, y, z, w to denote real numbers

A SET ANSWERS ONLY ONE QUESTION

- There is only one question we can ask a set:

  > *"Is this object in the set"*

  and the set will answer

  > *"yes" or "no"*

- Example: let E be the set of all positive even numbers.

  > *The number 4 is in E, but 3 is not in E.*

- The set E does no know anything else about $3, 4$.

SOME NOTATION

Mathematicians use a lot of shorthand and notation — please use standard notation.

DEFINITION: (SOME SHORTHAND NOTATION).

- If a is an element of the set A we write $a \in A$.
- If b is not an element of the set B we write $b \notin B$

Note "ε" is *not* the same as "$\in$"

- Back to E. We know that

$$4 \in E \qquad \text{and} \qquad 3 \notin E$$

but "$4\varepsilon E$" is the product of 4, ε and E — *garbage!*

DESCRIBING A SET

When we define a set it must be very clear.

- "Let A be the set of even integers between 1 and 13." — nice and clear.
- "Let B be the set of tall people at UBC." — not clear.

If only a few elements — just list them all inside *braces*

$$\text{Let } C = \{1, 2, 3, 4\}.$$

- Since a set only cares about membership, the order does not matter:

$$C = \{1, 2, 3, 4\} = \{2, 3, 1, 4\} = \{4, 1, 2, 3\}$$

- Repetition does not matter

$$\{1, 2, 3, 4\} = \{1, 2, 2, 3, 4, 1, 4, 2, 2, 1\}$$

- Be nice to the **reader** — ordered and no repeats.

DESCRIBING A SLIGHTLY LARGER SET

Use "$\ldots$" as shorthand for the skipped elements:

- $C = \{1, 2, 3, \ldots, 40\}$ the set of all integers between 1 and 40 (inclusive).
- $A = \{1, 4, 9, 16, \ldots\}$ the set of all positive square integers

Be careful — $B = \{3, 5, 7, \ldots\}$ is what set?

- all odd primes, or
- all odd numbers bigger than 1, or
- primes that are 1 less than a power of 2, or $\ldots$???

Definitions must be precise

- If the definition is vague then it is not a set
- Help your **reader** — don't assume that "they get what I mean"

THE MOST FUNDAMENTAL SET CONTAINS NOTHING

DEFINITION: (EMPTY SET).

- The **empty set** is the set which contains no elements.
- It is denoted $\varnothing$ and $\varnothing = \{\}$
- For any object x we know $x \notin \varnothing$

- The empty set is like an empty bag — it is not nothing.
- The set $A = \{\varnothing\}$ is not empty, it contains 1 element.
- The set $B = \{\varnothing, \{\varnothing\}\}$ contains 2 elements.

DEFINING SETS BY A RULE

For larger, complicated sets easier to define them using a rule.

- All even numbers $E = \{x \text{ so that } x \text{ is even}\}$
- “so that” shortened $\{x \text{ s.t. } x \text{ is even}\} = \{x \mid x \text{ is even}\} = \{x : x \text{ is even}\}$

This is **set builder** notation

$$S = \{\text{expression} : \text{rule}\}$$

- $A = \{n^2 \mid n \text{ is a whole number}\} = \{0, 1, 4, 9, 36, \ldots\}$
- $B = \{a \in A \mid a < 12\} = \{0, 1, 4, 9\}$
- $C = \{a \in A \mid \text{ and } a + 1 \text{ is prime}\} = \{1, 4, 16, 36, 100, \ldots\}$

It is important that the rule is clear — help your **reader**.

A TOUR OF OTHER IMPORTANT SETS

DEFINITION: (IMPORTANT SETS OF NUMBERS).

- **Natural numbers** (or positive integers) $\mathbb{N} = \{1, 2, 3, \ldots\}$ – note $0 \notin \mathbb{N}$
- **Integers** $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
- **Rational numbers** (fractions) $\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z}, b \in \mathbb{N} \right\}$
- **Real numbers** $\mathbb{R}$

All are denoted using **blackboard bold** letters

Please use correct notation: $N \neq \mathbb{N}$, $Z \neq \mathbb{Z}$.

We might also see

- **Irrational numbers** $\mathbb{I}$ = real numbers that are not rational.

We will prove that $\sqrt{2} \in \mathbb{I}$.

CARDINALITY

DEFINITION:

- We write $|S|$ to denote the **cardinality** of S.
- For a finite set S, the cardinality is the number of elements in S.

Examples

$$|\varnothing| = 0 \qquad |\{1,2,3\}| = 3 \qquad |\{\varnothing, \{1,2\}\}| = 2$$

For infinite sets things become very strange.

- $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$
- $|\mathbb{Z}| < |\mathbb{R}|$

We will prove these.