PLP - 9

TOPIC 9 —MORE PROOF EXAMPLES

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MORE EXAMPLES

A DIVISIBILITY EXAMPLE

PROPOSITION:

Let $a, b, c \in \mathbb{Z}$. If $a \mid b$ and $b \mid c$ then $a \mid c$.

We start with **scratch work** and *assume the hypothesis is true*.

- So $a \mid b$ and $b \mid c$
- *By definition of divisibility*, $b = ka$ and $c = \ell b$ for some $k, \ell \in \mathbb{Z}$
- We *want to show that* $a \mid c$. That is $c = na$ for some $n \in \mathbb{Z}$.
- Since $c = \ell b$ and $b = ka$ we know $c = \ell ka$
- Since $k, \ell \in \mathbb{Z}$ we know that $k\ell \in \mathbb{Z}$ so we are done!

After **scratch work** we have to write the proof *nice and neat* for our **reader**

CLEANING IT UP

We need to clean up our **scratch work**

- make sure logic flows correctly
- no dead-ends, no scribbles, keep presentation neat and tidy
- skip *very* obvious steps — only if *very* obvious to the **reader** (not you)
- make the text easy to read — we add “hence”, “we know that”, “it follows that”, etc
- dot-point form is okay when you are learning how to write proofs

PROOF.

We start by assuming the hypothesis to be true.

- Assume that $a \mid b$ and $b \mid c$, so that $b = ka$ and $c = \ell b$ for some $k, \ell \in \mathbb{Z}$.
- It follows that $c = k\ell a$
- Since $k\ell \in \mathbb{Z}$, we know that $a \mid c$ as required.

AN INEQUALITY

PROPOSITION:

Let $x, y \in \mathbb{R}$ then $x^2 + y^2 \geq 2xy$.

Scratch work:

- The implication hiding here is $(x, y \in \mathbb{R}) \implies (x^2 + y^2 \geq 2xy)$
- We don't know much about inequalities, except $(x \in \mathbb{R}) \implies (x^2 \geq 0)$.
- Rearrange inequality to make it look like a square?

$$x^2 + y^2 - 2xy \geq 0 \qquad \text{so} \qquad (x - y)^2 \geq 0$$

- This is what we want. The square of something is non-negative.

BE CAREFUL OF FLOW OF LOGIC

Logic flow in scratch work doesn't always match logic needed for proof.

- We started from **conclusion** $x^2 + y^2 \geq 2xy$
- Reached the **fact** that $(x - y)^2 \geq 0$

This is backwards — very common for proofs of inequalities.

PROOF.

- Assume that $x, y \in \mathbb{R}$
- Since the square of any real is non-negative, we know that $(x - y)^2 \geq 0$
- This implies that $x^2 - 2xy + y^2 \geq 0$
- From this $x^2 + y^2 \geq 2xy$ as required.