PLP - 12

TOPIC 12 —PROOF BY CASES

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PROOF BY CASES

ANOTHER EQUIVALENCE

PROPOSITION:

$$(P \lor Q) \implies R \quad \equiv \quad (P \implies R) \land (Q \implies R)$$

You can prove this with a truth-table (tedious) or via equivalences (good exercise).

Useful because we can split the **hypothesis** into cases.

$$(n \in \mathbb{N}) \implies (n^2 + 5n - 7 \text{ is odd})$$
$$(n \text{ is even}) \lor (n \text{ is odd}) \implies (n^2 + 5n - 7 \text{ is odd})$$
$$\underbrace{(n \text{ is even}) \implies (n^2 + 5n - 7 \text{ is odd})} \text{ and } \underbrace{(n \text{ is odd}) \implies (n^2 + 5n - 7 \text{ is odd})}$$

We can prove each **case** in turn — **proof by cases**

PROOF BY CASES

PROPOSITION:

Let $n \in \mathbb{Z}$ then $n^2 + 5n - 7$ is odd.

PROOF.

Assume the hypothesis is true, so that $n \in \mathbb{Z}$. Hence n is even or odd.

- *Case 1:* Assume that n is even, so that $n = 2k$ for some $k \in \mathbb{Z}$. Hence $n^2 + 5n - 7 = 4k^2 + 10k - 7 = 2(2k^2 + 5k - 4) + 1$. Thus $n^2 + 5n - 7$ is odd.
- *Case 2:* Assume that n is odd, so that $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$. Hence $n^2 + 5n - 7 = 4\ell^2 + 4\ell + 1 + 10\ell + 5 - 7 = 2(2\ell^2 + 7\ell - 1) + 1$. Thus $n^2 + 5n - 7$ is odd.

Since $n^2 + 5n - 7$ is odd in *both cases*, the result holds.

WHAT CAN GO WRONG

Proof by cases can be tricky

- tell the **reader** that you are doing **case analysis**
- make sure you get *all* the cases — a common mistake
- cases are often very similar — be very careful of skipping steps.

  without loss of generality or **WLOG** is a good source of errors

Dangerous phrases in mathematics

- Without loss of generality…
- Clearly…
- Obviously…
- A quick calculation shows that…
- It is easy to show that…

ANOTHER EXAMPLE

PROPOSITION:

Let $n \in \mathbb{Z}$. If $3 \mid n^2$ then $3 \mid n$.

Scratch work —This smells of the contrapositive: $(3 \nmid n) \implies (3 \nmid n^2)$.

- Recall **Euclidean division** — every integer n can be written uniquely as

$$n = 3a \qquad n = 3a+1 \qquad n = 3a+2 \qquad \text{for some } a \in \mathbb{Z}$$

- If $3 \nmid n$ we must have either $n = 3a+1$ or $= 3a+2$ — our cases.
- If $n = 3a+1$ then $n^2 = 9a^2 + 6a + 1 = \dots$.
- If $n = 3a+2$ then $n^2 = 9a^2 + 12a + 4 = \dots$.

Time to write up.

WRITE IT UP NICELY

Let $n \in \mathbb{Z}$. If $3 \mid n^2$ then $3 \mid n$.

PROOF.

We prove the contrapositive, so assume that $3 \nmid n$. By Euclidean division, we know that $n = 3a + 1$ or $n = 3a + 2$.

- *Case 1:* Let $n = 3a + 1$, then $n^2 = 9a^2 + 6a + 1 = 3(3a^2 + 2a) + 1$ and so is not divisible by 3.
- *Case 2:* Let $n = 3a + 2$, then $n^2 = 9a^2 + 12a + 4 = 3(3a^2 + 4a + 1) + 1$ and so is not divisible by 3.

Since $3 \nmid n^2$ in both cases, the result holds.