PLP - 13

TOPIC 13—QUANTIFIERS

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QUANTIFIERS

BACK TO OPEN SENTENCES

> *The number x^2 is non-negative*

This is an *open sentence*. One can prove this to be true for *all real x*.

> *For every $x \in \mathbb{R}$, the number x^2 is non-negative.*

- The extra text “*For every $x \in \mathbb{R}$*” adds scope.
- It is an example of a **quantifier**
- Adding quantifiers to an **open sentence** turns it into a **statement**

AN EXAMPLE

Consider the open sentence "$P(x) : x^2 - 5x + 4 = 0$" — when is it true?

- $P(0)$ is false
- $P(1)$ is true
- $P(2)$ is false, and so on

To decide truth values we need to decide *from what set do we take x*?

Eg. consider the truth values of $P(x)$ over the set $S = \{0, 1, 2, 3, 4\}$.

$$P(1), P(4) \text{ are true, but } P(0), P(2), P(3) \text{ are false}$$

Summarise this as

- $P(x)$ is true *for some* $x \in S$
- $P(x)$ is not true *for all* $x \in S$

the extra text "*for some*" and "*for all*" are **quantifiers**.

UNIVERSAL AND EXISTENTIAL

DEFINITION:

- The **universal quantifier** is denoted $\forall$ and is read as "for all" or "for every".

  "$\forall x \in A, P(x)$" is true provided $P(x)$ is true for *every* $x \in A$ and otherwise false.
- The **existential quantifier** is denoted $\exists$ and is read as "there exists".

  "$\exists x \in A$ so that $P(x)$" is true when *at least one* $x \in A$ makes $P(x)$ true, and otherwise false.

So our example becomes

$$\begin{aligned} \exists x \in S \text{ s.t. } x^2 - 5x + 4 = 0 & \qquad \text{true} \\ \forall x \in S, x^2 - 5x + 4 = 0 & \qquad \text{false} \end{aligned}$$

The "*s.t.*" and "*,*" separate the quantifier and open sentence — helps the **reader**

EXAMPLES — TRUE

- $\exists n \in \mathbb{Z}$ **s.t.** $\frac{7n-6}{3} \in \mathbb{Z}$

  True — set $n = 3$. Then $n \in \mathbb{Z}$, $\frac{7n-6}{3} = \frac{21-6}{3} = 5 \in \mathbb{Z}$.

  To show $\exists$ is true we just need 1 value that makes it true.

- $\forall n \in \mathbb{Z}, n^2 + 1 \in \mathbb{N}$

  True. Let n be *any* integer. Hence $n^2 + 1 \in \mathbb{Z}$. Since n is real, we know $n^2 + 1 \geq 1$. Hence $n^2 + 1 \in \mathbb{N}$.

  To show $\forall$ is true we must show it is true *generically*.

You cannot prove universal quantifiers using examples

EXAMPLES — FALSE

- $\forall n \in \mathbb{Z}$ s.t. $\frac{7n-6}{3} \in \mathbb{Z}$

  False — set $n = 1$. Then $n \in \mathbb{Z}$, but $\frac{7n-6}{3} = \frac{1}{3} = \notin \mathbb{Z}$.

  To show $\forall$ is false we just need 1 value that makes it false.

- $\exists n \in \mathbb{Z}$ s.t. $-n^2 \in \mathbb{N}$

  False. Let n be *any* integer. Then since $n \in \mathbb{R}$ we know that $n^2 \geq 0$, so $-n^2 \leq 0$. Hence $-n^2 \notin \mathbb{N}$.

  To show $\exists$ is false we must show it is false *generically*.

You cannot disprove existential quantifiers using examples

READING QUANTIFIERS

$\exists x \in A, P(x)$

- There exists x in A so that $P(x)$ is true.
- There is at least one x in A so that $P(x)$ is true.
- $P(x)$ is true for at least one value of x from A
- We can find an x in A so that $P(x)$ is true.
- $\dots$

$\forall x \in A, P(x)$

- For all x in A, $P(x)$ is true.
- For every x in A, $P(x)$ is true.
- No matter which x we choose from A, $P(x)$ is true.
- Every choice of x from A makes $P(x)$ true.
- $(x \in A) \implies P(x)$
- $\dots$