PLP - 14

TOPIC 14—NEGATING QUANTIFIERS

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NEGATING QUANTIFIERS

NEGATING A QUANTIFIED STATEMENT

$\forall n \in \mathbb{N}, n^2 + 1$ is prime

- $n = 1$ gives $n^2 + 1 = 2$ is prime.
- $n = 2$ gives $n^2 + 1 = 5$ is prime.
- $n = 3$ gives $n^2 + 1 = 10 = 2 \times 5$ is *not* prime.

So statement is false. We actually proved that its **negation** is true:

$\exists n \in \mathbb{N}, n^2 + 1$ is not prime

NEGATING A QUANTIFIED STATEMENT

$$\exists n \in \mathbb{N} \text{ s.t. } n^2 < n$$

- $n = 1$ gives $n^2 = 1 = 1$
- $n = 2$ gives $n^2 = 4 > 2$
- $n = 3$ gives $n^2 = 9 > 3$

Looks false, but examples are not enough.

We need to show that $n^2 \geq n$ for every single $n \in \mathbb{N}$.

$$\forall n \in \mathbb{N}, n^2 \geq n$$

NEGATIONS

THEOREM:

Let $P(x)$ be an open sentence over the domain A, then

$$\begin{aligned} \sim(\forall x \in A, P(x)) &\equiv \exists x \in A \text{ s.t. } \sim(P(x)) \\ \sim(\exists x \in A \text{ s.t. } P(x)) &\equiv \forall x \in A, \sim(P(x)) \end{aligned}$$

Note that the negated statement still has the same **domain**.

Be careful of domains:

$$\sim(\forall x \in A, P(x)) \not\equiv \forall x \notin A, P(x)$$
$$\sim(\exists x \in A \text{ s.t. } P(x)) \not\equiv \exists x \notin A \text{ s.t. } P(x)$$

PROVE OR DISPROVE

$$\exists n \in \mathbb{N} \text{ s.t. } 4 \mid (n^2 + 1)$$

- To prove it true — show the **reader** value of n
- To prove it false — show that its negation is true

scratch work — explore some values

- $n = 1$ gives $n^2 + 1 = 2$ —nope
- $n = 2$ gives $n^2 + 1 = 5$ —nope
- $n = 3$ gives $n^2 + 1 = 10$ —nope
- $n = 4$ gives $n^2 + 1 = 17$ —nope

Smells false (esp for even n) — look at negation

$$\forall n \in \mathbb{N} \text{ s.t. } 4 \nmid (n^2 + 1)$$

PROVE OR DISPROVE — CONTINUED

$\forall n \in \mathbb{N}$ s.t. $4 \nmid (n^2 + 1)$

scratch work:

- If n is even, then $n = 2k$ so $n^2 + 1 = 4k^2 + 1$ — not divisible by 4.
- If n is odd then $n = 2\ell + 1$ so $n^2 + 1 = 4\ell^2 + 4\ell + 2$ — not divisible by 4.

PROOF.

We show the statement is false by proving its negation is true. Since $n \in \mathbb{N}$ it is either even or odd.

- Assume that n is even, so $n = 2k$ and thus $n^2 + 1 = 4k^2 + 1$
- Now assume that n is odd, so $n = 2\ell + 1$ and thus $n^2 + 1 = 4\ell^2 + 4\ell + 2$

In both cases, by Euclidean division, the result is not divisible by 4.