PLP - 16

TOPIC 16—EXISTENCE PROOFS

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EXISTENCE PROOFS

AN EXAMPLE

There exist integers x, y so that $x^3 - y^2 = 13$.

PROOF.

Consider $x = 17$ and $y = 70$. Since $17^3 - 70^2 = 4913 - 4900 = 13$ we are done.

Why is this sufficient?

- The statement is “$\exists x, y \in \mathbb{N}$ **s.t.** $x^3 - y^2 = 13$”
- So to prove it true we only need to give at least one instance that makes it true
- We *do not* have to explain how we found that example

This is a **constructive** proof.

ANOTHER EXAMPLE

There exists $x \in \left[0, \frac{\pi}{2}\right]$ *so that* $\cos(x) = x$

PROOF.

Let $f(x) = \cos(x) - x$. Note that $f(0) = 1 > 0$ and $f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2} < 0$ and that $f(x)$ is a continuous function. Then by the *Intermediate Value Theorem*, we know that there exists a point $c \in \left(0, \frac{\pi}{2}\right)$ so that $f(c) = 0$. From this we know that $\cos(c) = c$ as required.

Why is this sufficient?

- To prove this we only need to *infer* that an example exists
- We *do not* have to give the example explicitly

This is a **non-constructive** proof.

EXISTENCE PROOFS

Proofs of existence results fall into 2 broad categories

- **constructive proofs**
  - in which a specific example is given explicitly and verified
  - an explanation of *how* the example was found is *not required*
- **non-constructive proofs**
  - in which the existence is *inferred* but an example is not explicitly stated

UNIQUENESS PROOF

After demonstrating that a required object exists, one often also wants uniqueness

> *There exists a unique* x *so that* $P(x)$

A simple way to approach such proofs is

- Let x, y be objects so that $P(x)$ and $P(y)$ are true
- Do *stuff* to show that $x = y$

The *fun* is in working out what *stuff* is.

AN EXAMPLE

The equation $ax = b$ with $a, b \in \mathbb{R}$ and $a \neq 0$ has a unique real solution.

PROOF.

First note that since $a \neq 0$, we can solve the equation by choosing $x = \frac{b}{a} \in \mathbb{R}$. Thus a solution exists.

Now assume that numbers r, s both satisfy the equation. Hence

$$
\begin{aligned}
ar &= b \qquad as = b && \text{and so} \\
ar &= as && \text{and since } a \neq 0 \\
r &= s
\end{aligned}
$$

So both solutions are in fact equal and the solution must be unique.