PLP - 18

TOPIC 18—INDUCTION

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INDUCTION

INDUCTION

Mathematical induction is a specialised technique for proving

$$\forall n \in \mathbb{N}, P(n)$$

It breaks the proof into 2 simpler steps

base case

prove that $P(1)$ is true

inductive step

show that $P(k) \implies P(k+1)$

- **base case** $=$ step onto bottom rung of ladder
- **inductive step** $=$ from the current rung you can reach the next rung
- so you can climb the ladder as high as you want

AN EXAMPLE

PROPOSITION:

For all $n \in \mathbb{N}$, $n^2 + 5n - 7$ is odd

scratch work

- **base case** when $n = 1$ we have $1 + 5 - 7 = -1$ which is odd
- **inductive step** need to prove

$$(k^2 + 5k - 7 \text{ is odd}) \implies ((k+1)^2 + 5(k+1) - 7 \text{ is odd})$$

- The inductive step is a sub-proof inside our proof

INDUCTIVE STEP "SUB PROOF"

$$\forall k \in \mathbb{N}, (k^2 + 5k - 7 \text{ is odd}) \implies ((k+1)^2 + 5(k+1) - 7 \text{ is odd})$$

scratch work

- so we assume $k^2 + 5k - 7 = 2\ell + 1$
- need to show $(k^2 + 2k + 1) + 5(k+1) - 7 = \underbrace{(k^2 + 5k - 7)}_{2\ell+1} + (2k + 6)$ is odd.
- Since $k^2 + 5k - 7 = 2\ell + 1$ we know

$$(k+1)^2 + 5(k+1) - 7 = 2(\ell + k + 3) + 1$$

  and since $\ell + k + 3 \in \mathbb{Z}$ we are done.

Of course we still need to put the two parts of the proof together.

PRINCIPLE OF MATHEMATICAL INDUCTION

THEOREM: MATHEMATICAL INDUCTION.

For all $n \in \mathbb{N}$ let $P(n)$ be a statement. Then if

- $P(1)$ is true, and
- $P(k) \implies P(k+1)$ is true for all $k \in \mathbb{N}$

then $P(n)$ is true for all $n \in \mathbb{N}$.

Warnings

- Induction *is not* "adding the next term to both sides"
- Induction *does not* prove all statements — **the law of the instrument**
- Tell your **reader** if you use induction in your proof

COMPLETING OUR PROOF

For all $n \in \mathbb{N}$, $n^2 + 5n - 7$ is odd

PROOF.

We prove the result by induction.

- Base case: When $n = 1$ we have $1 + 5 - 7 = -1$ which is odd.
- Inductive step: Assume that $k^2 + 5k - 7$ is odd, so we can write

$$\begin{aligned} k^2 + 5k - 7 &= 2\ell + 1 \text{ for some } \ell \in \mathbb{Z} \text{ and so} \\ (k+1)^2 + 5(k+1) - 7 &= 2(\ell + k + 3) + 1 \end{aligned}$$

and since $\ell + k + 3 \in \mathbb{Z}$, it follows that $(k+1)^2 + 5(k+1) - 7$ is odd.

Since the base case and inductive step hold, the result follows *by induction*.

ANOTHER EXAMPLE

PROPOSITION:

For every natural number n, $3 \mid (4^n - 1)$

Scratch work

- When $n = 1$, easy $3 \mid (4 - 1)$
- Assume $3 \mid (4^k - 1)$, so $4^k - 1 = 3\ell$.
- Writing $4^k = 3\ell + 1$ shows

$$4^{k+1} = 12\ell + 4 \qquad \text{so} \qquad 4^{k+1} - 1 = 12\ell + 3$$

done!

WRITE IT UP NICELY

For every natural number n, $3 \mid (4^n - 1)$

PROOF.

We prove the result by induction.

- Base case: When $n = 1$ we have $3 \mid (4 - 1)$, so the result holds.
- Inductive step: Assume that $3 \mid (4^k - 1)$, so $4^k = 3\ell + 1$ for some $\ell \in \mathbb{Z}$. Then

$$4^{k+1} - 1 = 4(3\ell + 1) - 1 = 3(4\ell + 1)$$

  and so $3 \mid (4^{k+1} - 1)$ as required.

Since the base case and inductive step hold, the result follows by induction.