PLP - 19

TOPIC 19—PROOF OF INDUCTION

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PROOF OF INDUCTION

WHY DOES INDUCTION WORK

THEOREM: MATHEMATICAL INDUCTION.

For all $n \in \mathbb{N}$ let $P(n)$ be a statement. Then if

- $P(1)$ is true, and
- $P(k) \implies P(k+1)$ is true for all $k \in \mathbb{N}$

then $P(n)$ is true for all $n \in \mathbb{N}$.

We won't give a rigorous proof, but will give a “proof sketch”

A GOOD SET AND A BAD SET

Assume, $P(1)$ is true, and $P(k) \implies P(k+1)$ is true.

Define two sets

- **Good set** — Let $G = \{n \text{ s.t. } P(n) \text{ is true}\}$. We know $1 \in G$.
- **Bad set** — Let $B = \{n \text{ s.t. } P(n) \text{ is false}\}$

Now either $B = \varnothing$ or $B \neq \varnothing$.

- If $B \neq \varnothing$, let q be the smallest element of B
- Then $P(q)$ is first number that makes $P(n)$ false.
- We must have $P(q-1)$ is true
- But *by assumption* $P(q-1) \implies P(q)$, so $P(q)$ is true.
- But then $q \notin B$.
- So there cannot be such an element q

So $B = \varnothing$, so $P(n)$ is true for all $n \in \mathbb{N}$

SNEAKY

We are doing two sneaky things here

- This is a **proof by contradiction** in disguise
- We are using the **well ordering principle** of $\mathbb{N}$

DEFINITION: (THE WELL ORDERING PRINCIPLE).

A set A is **well ordered** if every non-empty subset $B \subseteq A$ has a smallest element

Notice that $\mathbb{N}$ is well ordered, but $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are not.