# PLP - 25

## TOPIC 25—MORE SET PROOFS

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# CARTESIAN AND POWER SET PROOFS

# A CARTESIAN EXAMPLE

**PROPOSITION:**

Let $A, B, C$ be sets, then $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

Scratch work

- We'll start with $A \times (B \cup C) \supseteq (A \times B) \cup (A \times C)$
- Since cartesian product, assume $(x, y) \in$ RHS
- So $(x, y) \in A \times B$ or $(x, y) \in A \times C$
  - Case 1: When $(x, y) \in A \times B$, $x \in A$ and $y \in B$. Hence $y \in B \cup C$
  - Case 2: When $(x, y) \in A \times C$, $x \in A$ and $y \in C$. Hence $y \in C \cup B$
- In both cases, $x \in A$ and $y \in B \cup C$, so $(x, y) \in$ LHS.

# CONTINUING CARTESIAN EXAMPLE

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

Scratch work continued

- No do $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$
- Assume $(x, y) \in$ LHS
- Then $x \in A$ and $y \in B \cup C$ , so $y \in B$ or $y \in C$
  - Case 1: When $y \in B$, we know that $(x, y) \in A \times B$, so $(x, y) \in (A \times B) \cup (A \times C)$
  - Case 2: When $y \in C$, we know that $(x, y) \in A \times C$, so $(x, y) \in (A \times C) \cup (A \times B)$
- In both cases, $(x, y) \in$ RHS

You can write this up nicely.

# POWER SET WARM-UP

## PROPOSITION:

$$X \subseteq A \implies X \subseteq A \cup B$$
$$X \subseteq A \cap B \implies X \subseteq A$$
$$(X \subseteq A) \wedge (X \subseteq B) \iff X \subseteq A \cap B$$

**PROOF.**

- Let $X \subseteq A$. Assume that $x \in X$, which implies that $x \in A$. Hence $x \in A \cup B$.
- Now let $X \subseteq A \cap B$ and assume $x \in X$. Hence $x \in A \cap B$ and so $x \in A$.
- Let $X \subseteq A$ and $X \subseteq B$, and assume $x \in X$. Then $x \in A$ and $x \in B$, so $x \in A \cap B$.
- Finally let $X \subseteq A \cap B$. Since $A \cap B \subseteq A$ we know $X \subseteq A$. Similar reasoning gives $X \subseteq B$.

# A POWER SET EXAMPLE

## PROPOSITION:

Let $A, B$ be sets then $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$. The reverse inclusion does not hold.

### Scratch work

- There are two things to prove here. Start with the inclusion.
- Is equivalent to $X \in \mathcal{P}(A) \cup \mathcal{P}(B) \implies X \in \mathcal{P}(A \cup B)$.
- So let $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$. Means $X \subseteq A$ or $X \subseteq B$
- Now if $X \subseteq A$ then $X \subseteq A \cup B$ and so $X \in \mathcal{P}(A \cup B)$.
- The case $X \subseteq B$ will be similar.

# DISPROOF OF REVERSE INCLUSION

$$\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$$

## Scratch work

- Since this is really "For all sets $A, B \ldots$" so disproof can be counter-example
- Notice that $X \in RHS$ then $X \subseteq A$ or $X \subseteq B$.
- While a set in $LHS$ could could contain elements from both $A, B$
- Try to construct very small $A, B$ to illustrate this.
- Let $A = \{1\}$, $B = \{2\}$, then $\{1, 2\} \in LHS$ but not in $RHS$.

# PROOF OF RESULT

**PROOF.**

We first prove the inclusion and then show the reverse inclusion does not hold.

- Let $X \in \mathcal{P}(A) \cup \mathcal{P}(B)$ . Hence $X \subseteq A$ or $X \subseteq B$. If $X \subseteq A$ then (as shown previously) $X \subseteq A \cup B$. Similarly if $X \subseteq B$ then $X \subseteq B \cup A$. This implies that $X \in \mathcal{P}(A \cup B)$ as required.
- We disprove the reverse inclusion with a counter example. Let $A = \{1\}$ , $B = \{2\}$ and $X = \{1, 2\}$ . Then $X \in \mathcal{P}(A \cup B)$ , however $X \notin \mathcal{P}(A) \cup \mathcal{P}(B)$ . Hence $\mathcal{P}(A \cup B) \nsubseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ .

# ANOTHER POWER SET RESULT

## PROPOSITION:

Let $A, B$ be sets, then $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$.

### Scratch work

- There are two inclusions to prove here.
- Assume that $X \in LHS$, then $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$.
- Hence $X \subseteq A$ and $X \subseteq B$ — and we showed this means $X \subseteq A \cap B$
- Thus $X \in RHS$
- Now let $Y \in RHS$, so $Y \subseteq A \cap B$ — we showed this means $Y \subseteq A$ and $Y \subseteq B$
- So $Y \in LHS$ as required.

# WRITE IT UP NICELY

$$\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$$

**PROOF.**

We prove each inclusion in turn.

- Assume that $X \in LHS$. Then $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, and so $X \subseteq A$ and $X \subseteq B$. Hence $X \subseteq A \cap B$, and thus $X \in RHS$.
- Now assume that $Y \in RHS$. Then $Y \in \mathcal{P}(A \cap B)$ and so $Y \subseteq A \cap B$. This means that $Y \subseteq A$ and $Y \subseteq B$. Hence $Y \in \mathcal{P}(A)$ and $Y \in \mathcal{P}(B)$ and thus $Y \in LHS$.