PLP - 28

TOPIC 28—EQUIVALENCE RELATIONS & CLASSES

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EQUIVALENCE RELATIONS

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Important class of relations are those that are similar to “$=$”

DEFINITION:

Let R be a relation on the set A.

We call R an **equivalence relation** when it is reflexive, symmetric and transitive.

Examples

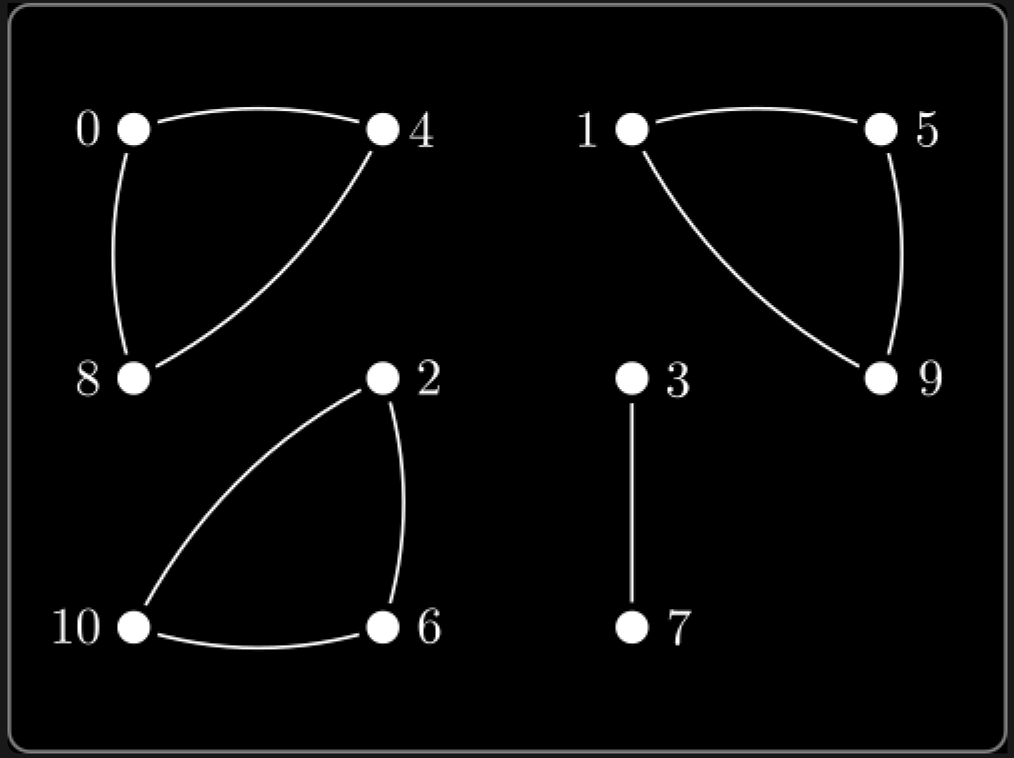
- “has same parity as”
- “is congruent to”
- “has same birthday as”

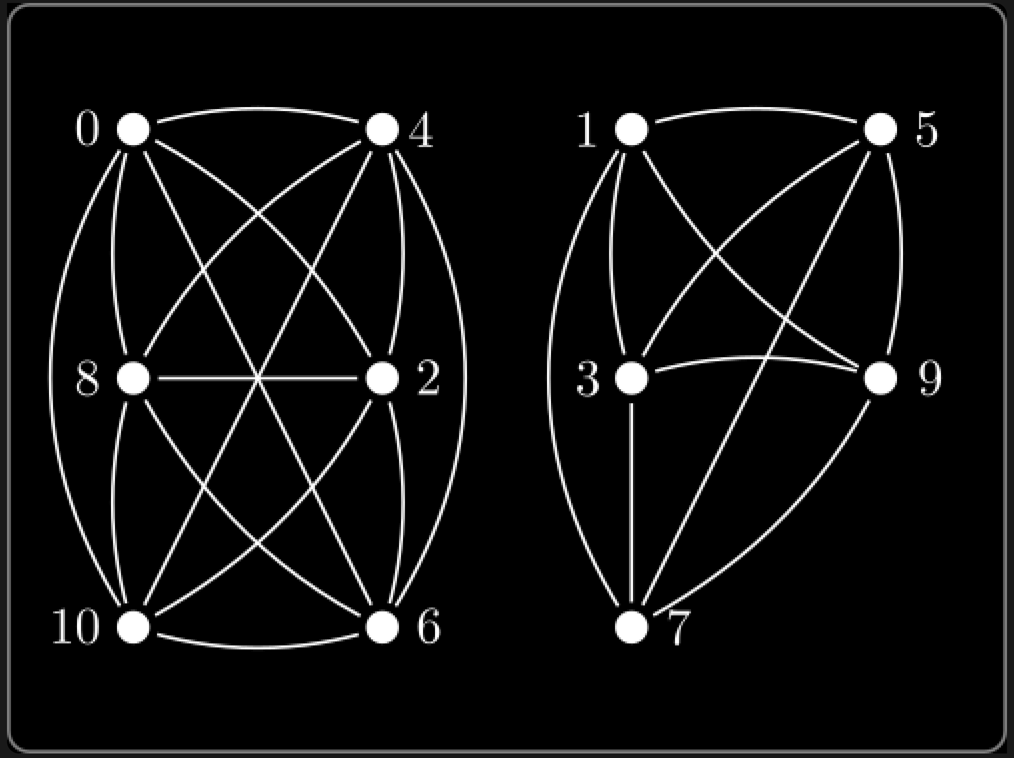
Weaker than equality — can be equivalent without being equal

PICTURES

Let $A = \{0, 1, 2, \ldots, 10\}$ and consider congruence modulo 4.

And similarly with "has the same parity as"





Notice that elements of A fall into connected subsets — **equivalence classes**

EQUIVALENCE CLASSES

DEFINITION:

Let R be an equivalence relation on A.

The **equivalence class** of $x \in A$ (with respect to R) is

$$[x] = \{a \in A \ : \ a \, R \, x\}$$

In our congruent modulo 4 example

$$[0] = \{0, 4, 8\} = [4] = [8] \qquad [1] = \{1, 5, 9\} = [5] = [9]$$
$$[2] = \{2, 6, 10\} = [6] = [10] \qquad [3] = \{3, 7\} = [7]$$

NO EQUIVALENCE CLASS IS EMPTY

LEMMA:

Let R be an equivalence relation on A.

For any $a \in A$, $a \in [a]$

PROOF.

Assume R is an equivalence relation on A, and let $a \in A$.

Since R is reflexive, we know that $a \ R \ a$. Hence (by definition), $a \in [a]$ as required.

EQUALITY OF EQUIVALENCE CLASSES

THEOREM:

Suppose R is an equivalence relation on A, and let $a, b \in A$. Then

$$[a] = [b] \iff a \ R \ b$$

Scratch work

- Have to prove both directions
- Assume $[a] = [b]$, then we need to show $a \ R \ b$
- We know (from above lemma) that $a \in [a]$, so $a \in [b]$
- Definition of $[b] = \{x \in A \ : \ x \ R \ b\}$, so $a \ R \ b$

CONTINUING

$$[a] = [b] \iff a \; R \; b$$

Scratch work continued

- Now assume that $a \; R \; b$. We need to show $[a] \subseteq [b]$ and $[b] \subseteq [a]$
- So let $x \in [a]$, which tells us that $x \; R \; a$
- We know that R is transitive, so

$$(x \; R \; a) \wedge (a \; R \; b) \implies (x \; R \; b)$$

  so $x \in [b]$
- The other inclusion is similar, but we use symmetry of R to get $b \; R \; a$.

PROOF

PROOF.

We prove each implication in turn

- Assume $a \; R \; b$. We prove that $[a] \subseteq [b]$ and leave the other inclusion to the reader. Let $x \in [a]$, so that $x \; R \; a$. Since R is transitive, and $a \; R \; b$, we know that $x \; R \; b$. Hence $x \in [b]$ as required. The other inclusion is similar, but also uses symmetry of R.
- Now assume that $[a] = [b]$. By the lemma above, we know that $a \in [a]$, and so $a \in [b]$. By definition of the equivalence class of b, this tells us that $a \; R \; b$.