# PLP - 31

## TOPIC 31—FUNCTIONS

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# ESCAPE FROM FORMULAE

# A FUNCTION IS NOT A FORMULA

We are used to thinking of functions as formulas or (perhaps) algorithms

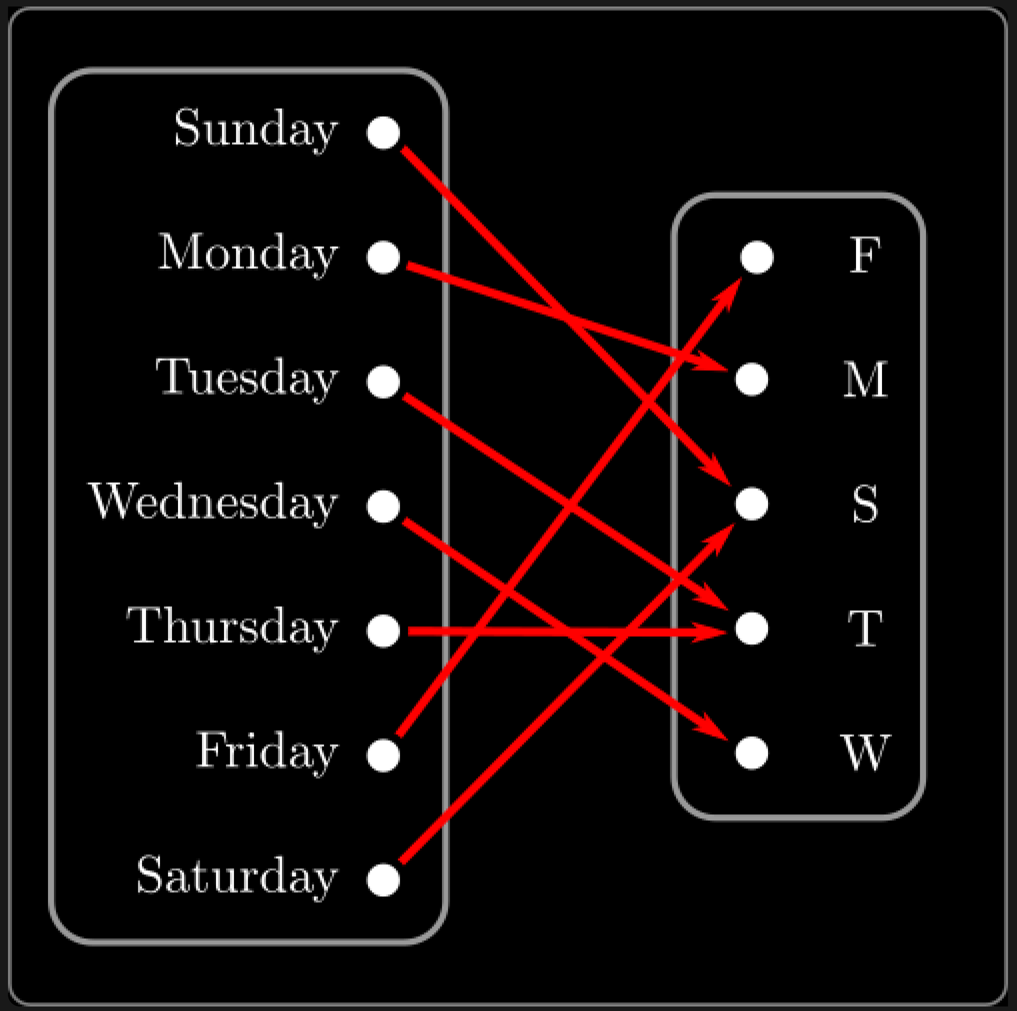
- Give me an input number $x$
- I do some arithmetic on $x$ or use look-up tables
- I return to you a numerical result $y$

Can define functions on other objects (not just numbers):

- Input day of the week (in English)
- Return the first letter

But must be *well defined*

- Any legal input must have an output
- One input value gives only one output value



# FUNCTION AS A LOOK-UP TABLE

We can summarise the previous function as

$$\Big\{(\text{Sunday}, S), (\text{Monday}, M), (\text{Tuesday}, T), (\text{Wednesday}, W), \\ (\text{Thursday}, T), (\text{Friday}, F), (\text{Saturday}, S)\Big\}$$

More generally a function $f$

- takes inputs from set $A$ and gives outputs in set $B$
- can be written as a subset of $f \subseteq A \times B$ — a type of **relation**

Not every subset of $A \times B$ is a function — must be *well defined*

- Every input from $A$ must have an output in $B$

$$\forall a \in A, \exists b \in B \text{ s.t. } (a, b) \in f$$

- Exactly one output for a given input

$$(a, b_1) \in f \wedge (a, b_2) \in f \implies b_1 = b_2$$

# A DEFINITION

## DEFINITION:

Let $A$, $B$ be non-empty sets

A **function** from $A$ to $B$ is a non-empty subset $f \subseteq A \times B$ so that

- for every $a \in A$, there exists a $b \in B$ so that $(a, b) \in f$
- if $(a, b) \in f$ and $(a, c) \in f$ then $b = c$

The **domain** of $f$ is $A$, and the **codomain** is $B$

If $(a, b) \in f$ we write $f(a) = b$ and say that $b$ is the **image** of $a$

Finally, the **range** of $f$ is

$$\operatorname{rng} f = \{b \in B \text{ s.t. } \exists a \in A \text{ s.t. } f(a) = b\}$$

Note that the **range** is a subset of the **codomain**

# AN EXAMPLE AND A NON-EXAMPLE

Consider the sets

$$\begin{aligned} f &= \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \ : \ 3x + 2y = 0\} \\ g &= \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \ : \ 3x + y = 0\}\,. \end{aligned}$$

The set $f$ is *not a function*

- it is not defined on all of its domain $\mathbb{Z}$
- when $x = 1$ there is no $y \in \mathbb{Z}$ so that $3x + 2y = 0$

The set $g$ is a function

- for every $x \in \mathbb{Z}$, pick $y = -3x \in \mathbb{Z}$, then $(x, y) \in g$
- if $(x, y) \in g$ and $(x, z) \in g$ then

$$3x + y = 0 \qquad \text{and} \qquad 3x + z = 0$$

so $y = z$ as required.