PLP - 32

TOPIC 32—IMAGES AND PREIMAGES

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IMAGES AND PREIMAGES

FUNCTIONS AND SUBSETS

How do functions interact with subsets of the domain and codomain?

DEFINITION:

Let $f : A \to B$ be a function and let $C \subseteq A$ and $D \subseteq B$

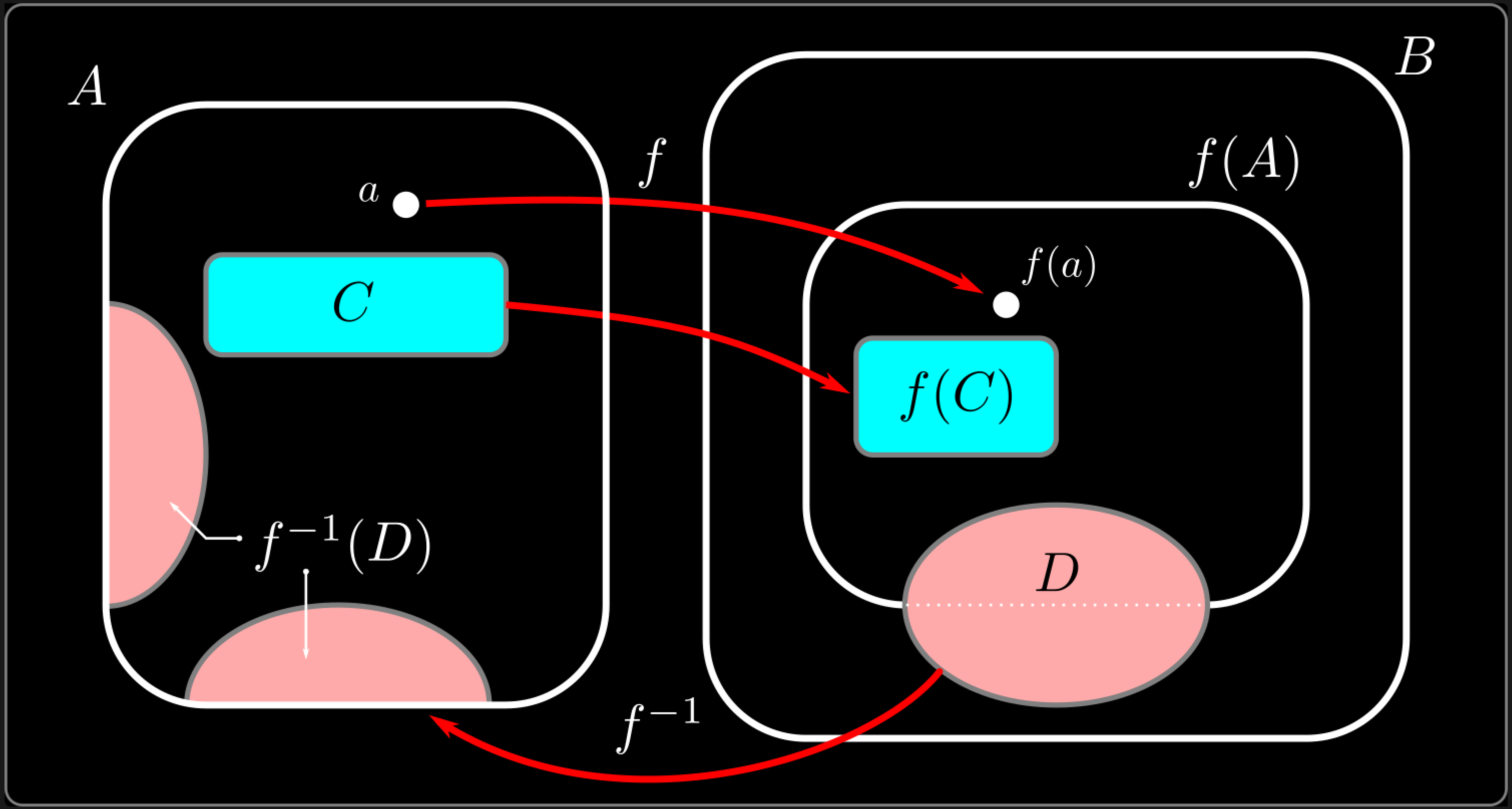
- The **image** of C in B is $f(C) = \{f(x) \text{ s.t. } x \in C\}$
- The **preimage** of D in A is $f^{-1}(D) = \{x \in A \text{ s.t. } f(x) \in D\}$

WARNING — Be careful with preimages:

- $f^{-1}(x)$ is not $(f(x))^{-1}$ or $\frac{1}{f(x)}$
- The preimage f^{-1} is *not* the inverse function.
- When extra conditions satisfied the inverse function exists and we use the same notation

When you see f^{-1} think "*preimage*" — when you know inverse exists then "inverse function".

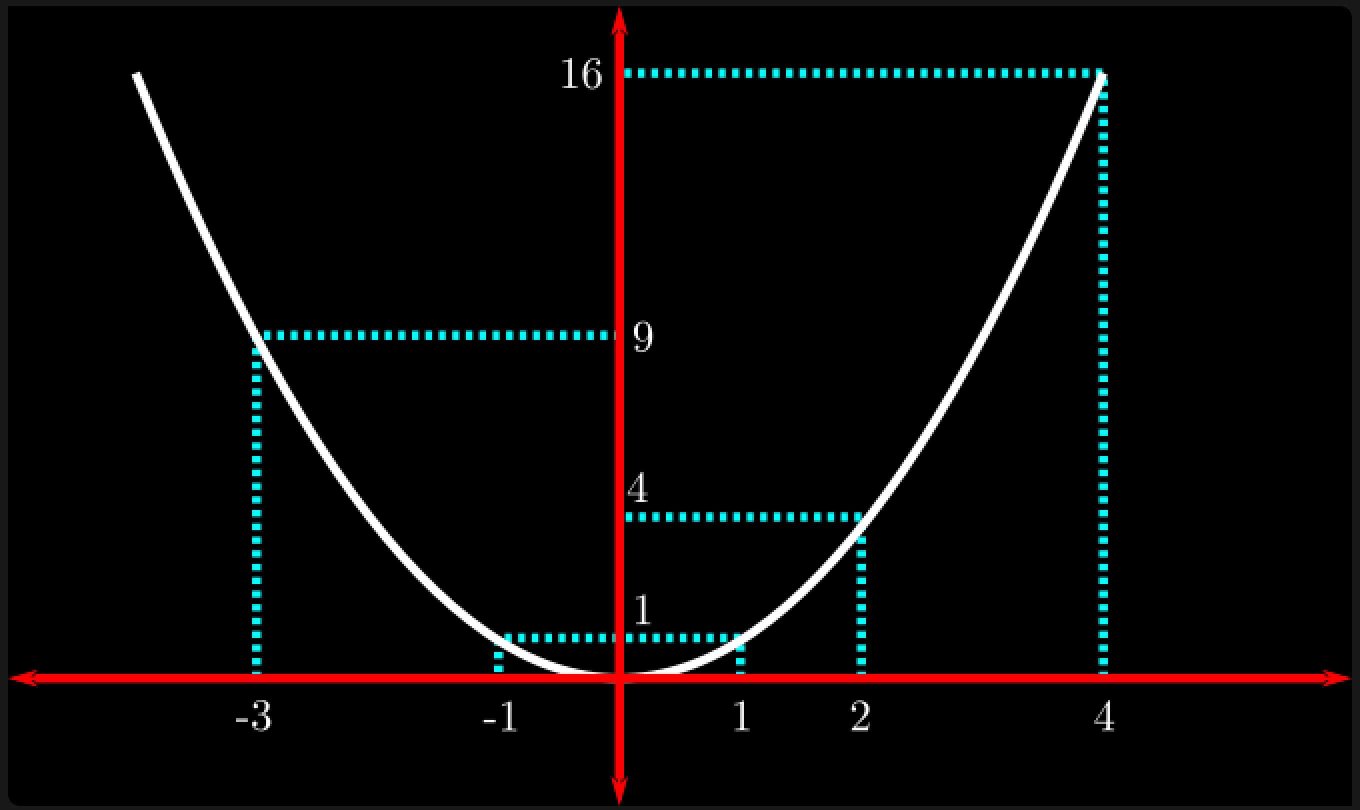
A SKETCH OF IMAGES AND PREIMAGES



AN EXAMPLE — IMAGES

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then

- $f([0,4]) = [0,16]$
- $f([-3,-1] \cup [1,2]) = [1,9]$



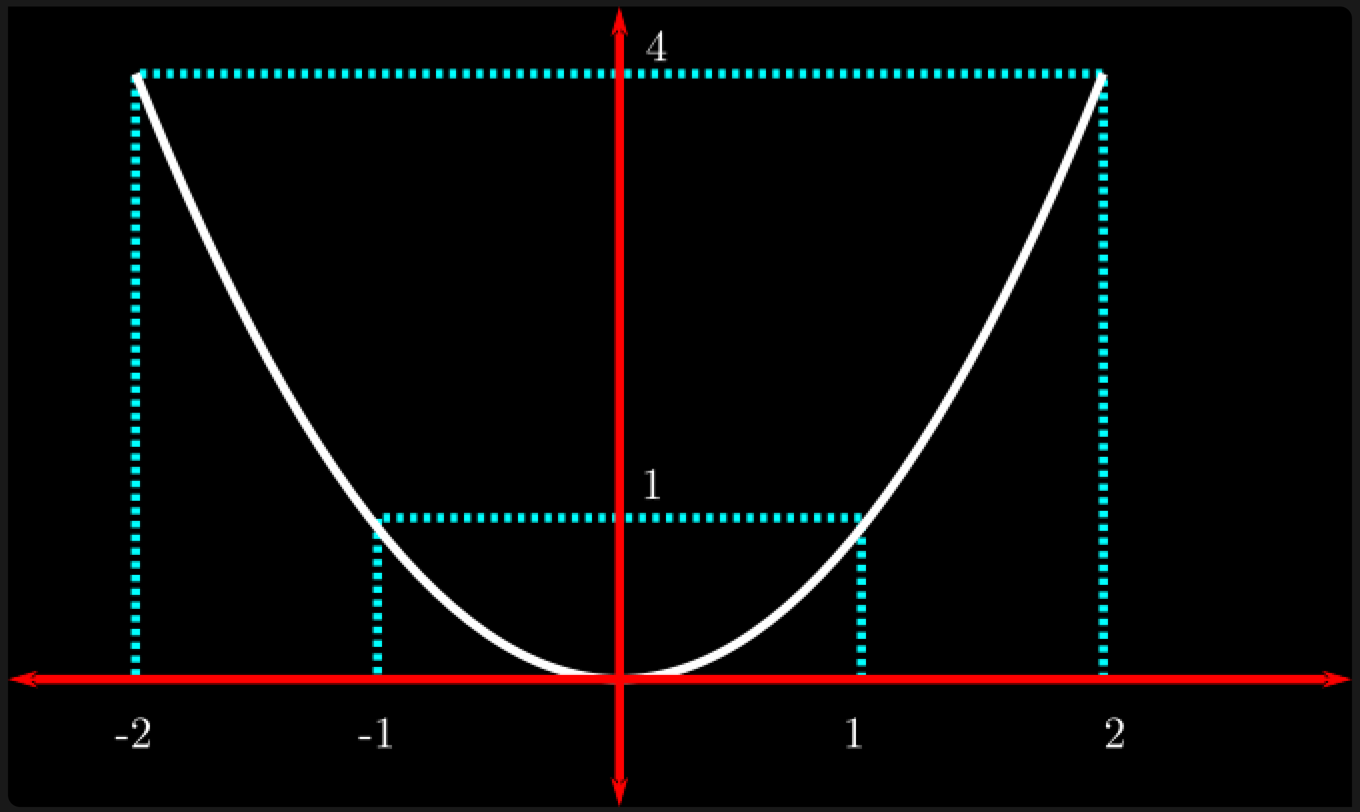
- If $0 \le x \le 4$ then $0 \le x^2 \le 16$
- If $1 \le x \le 2$ then $1 \le x^2 \le 4$. And if $-3 \le x \le -1$ then $1 \le x^2 \le 9$.

  So if $x \in [-3,-1] \cup [1,2]$ then $x^2 \in [1,9]$.

AN EXAMPLE — PREIMAGES

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then

- $f^{-1}(\{0,1\}) = \{-1,0,1\}$
- $f^{-1}([1,4]) = [-2,-1] \cup [1,2]$



- If $x^2 = 0$ then $x = 0$. And if $x^2 = 1$ then $x = \pm 1$

  So if $x^2 \in \{0,1\}$ then $x \in \{-1,0,1\}$
- If $1 \leq x^2$ then $x \leq -1$ or $x \geq 1$. If $x^2 \leq 4$ then $-2 \leq x \leq 2$.

  So if $x^2 \in [1,4]$ then $x \in [-2,-1]$ or $x \in [1,2]$.

IMAGES, PREIMAGES AND SET OPERATIONS

Images and preimages interact (mostly) nicely with subset, intersection and union.

THEOREM:

Let $f : A \to B$ and $C \subseteq A$ and $D \subseteq B$. Then

$$C \subseteq f^{-1}(f(C)) \qquad \text{and} \qquad f(f^{-1}(D)) \subseteq D$$

Now let $C_1, C_2 \subseteq A$ and $D_1, D_2 \subseteq B$. Then

$$f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2) \qquad f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$$

$$f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2) \qquad f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$$

Make good problems — test lots of skills

PROOF 1

$$f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$$

We use $x \in f^{-1}(D) \iff f(x) \in D$

PROOF.

$LHS \subseteq RHS$: Let $x \in f^{-1}(D_1 \cup D_2)$, so $f(x) \in D_1 \cup D_2$.

Hence $f(x) \in D_1$ or $f(x) \in D_2$.

- when $f(x) \in D_1$ we know $x \in f^{-1}(D_1)$
- when $f(x) \in D_2$ we know $x \in f^{-1}(D_2)$

In either case we know that $x \in f^{-1}(D_1) \cup f^{-1}(D_2)$.

Other inclusion is similar.

PROOF 2

$$f(C_1 \cap C_2) \subseteq f(C_1) \cap f(C_2)$$

We use

$$x \in C \implies f(x) \in f(C) \qquad \text{and} \qquad y \in f(C) \implies \exists x \in C \text{ s.t. } y = f(x) \in f(C)$$

PROOF.

Let $y \in f(C_1 \cap C_2)$.

This means that there is some $x \in C_1 \cap C_2$ so that $f(x) = y$. Then

- since $x \in C_1$, we know that $y = f(x) \in f(C_1)$
- since $x \in C_2$, we know that $y = f(x) \in f(C_2)$

Hence $y \in f(C_1) \cap f(C_2)$.

REVERSE INCLUSION IS FALSE

$$f(C_1) \cap f(C_2) \not\subseteq f(C_1 \cap C_2)$$

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$.

- Let $C_1 = \{-1\}$, so $f(C_1) = \{1\}$
- Let $C_2 = \{1\}$, so $f(C_2) = \{1\}$
- Then $f(C_1) \cap f(C_2) = \{1\}$ but $f(C_1 \cap C_2) = f(\varnothing) = \varnothing$

Notice also that this shows $f(x) \in f(C)$ does *not* imply $x \in C$

- Set $x = -1$ and $C = \{1\}$
- Then $f(x) = 1 \in \{1\} = f(C)$ but $x \notin C$.

These fail because there are $x_1 \neq x_2$ so that $f(x_1) = f(x_2)$.

PROOF 3

$$C \subseteq f^{-1}(f(C))$$

PROOF.

Let $x \in C$.

- Since $x \in C$, we know that $f(x) \in f(C)$
- To make logic clearer we write $D = f(C)$, so that $f(x) \in D$
- Since $f(x) \in D$, we know that $x \in f^{-1}(D)$
- But since $D = f(C)$ this means that $x \in f^{-1}(f(C))$ as required.

Reverse inclusion is false. Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = x^2$.

- Let $C = \{2\}$. Then $f(C) = \{4\}$
- But $f^{-1}(\{4\}) = \{-2, 2\}$ – since $f(2) = f(-2) = 4$.
- Thus $f^{-1}(f(\{2\})) = \{-2, 2\} \not\subseteq \{2\} = C$