PLP - 33

TOPIC 33—INJECTIONS, SURJECTIONS AND BIJECTIONS

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INJECTIONS, SURJECTIONS AND BIJECTIONS

INJECTIONS — DIFFERENT MAPS TO DIFFERENT

DEFINITION:

Let $f : A \to B$ be a function.

The function f is **injective** when for all $a_1, a_2 \in A$

$$(a_1 \neq a_2) \implies f(a_1) \neq f(a_2)$$

Equivalently (by the contrapositive)

$$f(a_1) = f(a_2) \implies a_1 = a_2$$

Injections are also called **one-to-one** functions.

Note: if f is injective then for every $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

This is consistent with $|A| \leq |B|$

INJECTION EXAMPLE

PROPOSITION:

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 7x - 3$ is injective

Use $f(a_1) = f(a_2) \implies a_1 = a_2$ to prove — equalities easier than inequalities.

PROOF.

Let $x, z \in \mathbb{R}$ and assume that $f(x) = f(z)$. Then we know that

$$\begin{aligned} 7x - 3 &= 7z - 3 \\ 7x &= 7z \\ x &= z \end{aligned}$$

and hence the function is injective.

SURJECTIONS — EVERYTHING IS MAPPED TO BY SOMETHING

DEFINITION:

Let $g : A \to B$ be a function.

The function g is **surjective** when

$$\forall b \in B, \exists a \in A \text{ s.t. } g(a) = b$$

Surjections are also called **onto** functions.

Note: if g is surjective then for every $b \in B$, $|g^{-1}(\{b\})| \geq 1$.

This is consistent with $|A| \geq |B|$

SURJECTION EXAMPLE

PROPOSITION:

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 7x - 3$ is surjective

Given $y \in \mathbb{R}$, we need to find $x \in \mathbb{R}$ so that $f(x) = y$:

$$y = 7x - 3 \qquad \text{so} \qquad y + 3 = 7x \qquad \text{so} \qquad x = \frac{y+3}{7}.$$

PROOF.

Let $y \in \mathbb{R}$ and set $x = \frac{y+3}{7} \in \mathbb{R}$. Then

$$f(x) = 7x - 3 = 7 \cdot \frac{y+3}{7} - 3 = y + 3 - 3 = y$$

as required. Hence the function is surjective.

A NON-EXAMPLE

PROPOSITION:

The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ is neither injective nor surjective.

$$\text{not injective} \equiv \exists x_1, x_2 \in A \text{ s.t. } (x_1 \neq x_2) \wedge (f(x_1) = f(x_2))$$
$$\text{not surjective} \equiv \exists b \in B \text{ s.t. } \forall a \in A, f(a) \neq b$$

PROOF.

We prove each claim in turn.

- Now let $x = 1, z = -1$. Then since $f(x) = 1 = f(z)$, the function is not an injection.
- Let $y = -1$. For any $x \in \mathbb{R}$ we know $f(x) = x^2 \geq 0$, so there is no $x \in \mathbb{R}$ so that $f(x) = y$. So f is not a surjection.

BIJECTIONS — INJECTIVE AND SURJECTIVE

DEFINITION:

Let $h : A \to B$ be a function. The function h is **bijective** when it is both injective and surjective.

Bijections are also called **one-to-one correspondences**.

Note: if h is bijective then

- since h is injective we know that for every $b \in B$, $|h^{-1}(\{b\})| \leq 1$
- since h is surjective we know that for every $b \in B$, $|h^{-1}(\{b\})| \geq 1$

and so for every $b \in B$, $|h^{-1}(\{b\})| = 1$.

This is consistent with $|A| = |B|$

From work above $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 7x - 3$ is bijective.

ANOTHER x^2 EXAMPLE

Consider the 4 functions

$$
\begin{aligned}
f :&\mathbb{R} \to \mathbb{R} & f(x) &= x^2 \\
g :&\mathbb{R} \to [0, \infty) & g(x) &= x^2 \\
h :&[0, \infty) \to \mathbb{R} & g(x) &= x^2 \\
\rho :&[0, \infty) \to [0, \infty) & \rho(x) &= x^2
\end{aligned}
$$

Then

- f is neither injective nor surjective
- g is surjective but not injective
- h is injective but not surjective
- ρ is both injective and surjective

Let's prove the last one carefully.

INJECTIVE AND SURJECTIVE

$\rho : [0, \infty) \to [0, \infty)$ with $\rho(x) = x^2$ is injective and surjective

PROOF.

We prove each in turn.

- Injection: Let $x, z \in [0, \infty)$ with $\rho(x) = \rho(z)$. Then we know that $x^2 = z^2$ and hence $x = \pm z$. But since $x, z \geq 0$ we must have $x = z$ as required.
- Surjection: Let $y \geq 0$ and then set $x = \sqrt{y}$. Since $y \geq 0$, we know that $x \in \mathbb{R}$. Then $\rho(x) = x^2 = (\sqrt{y})^2 = y$.