

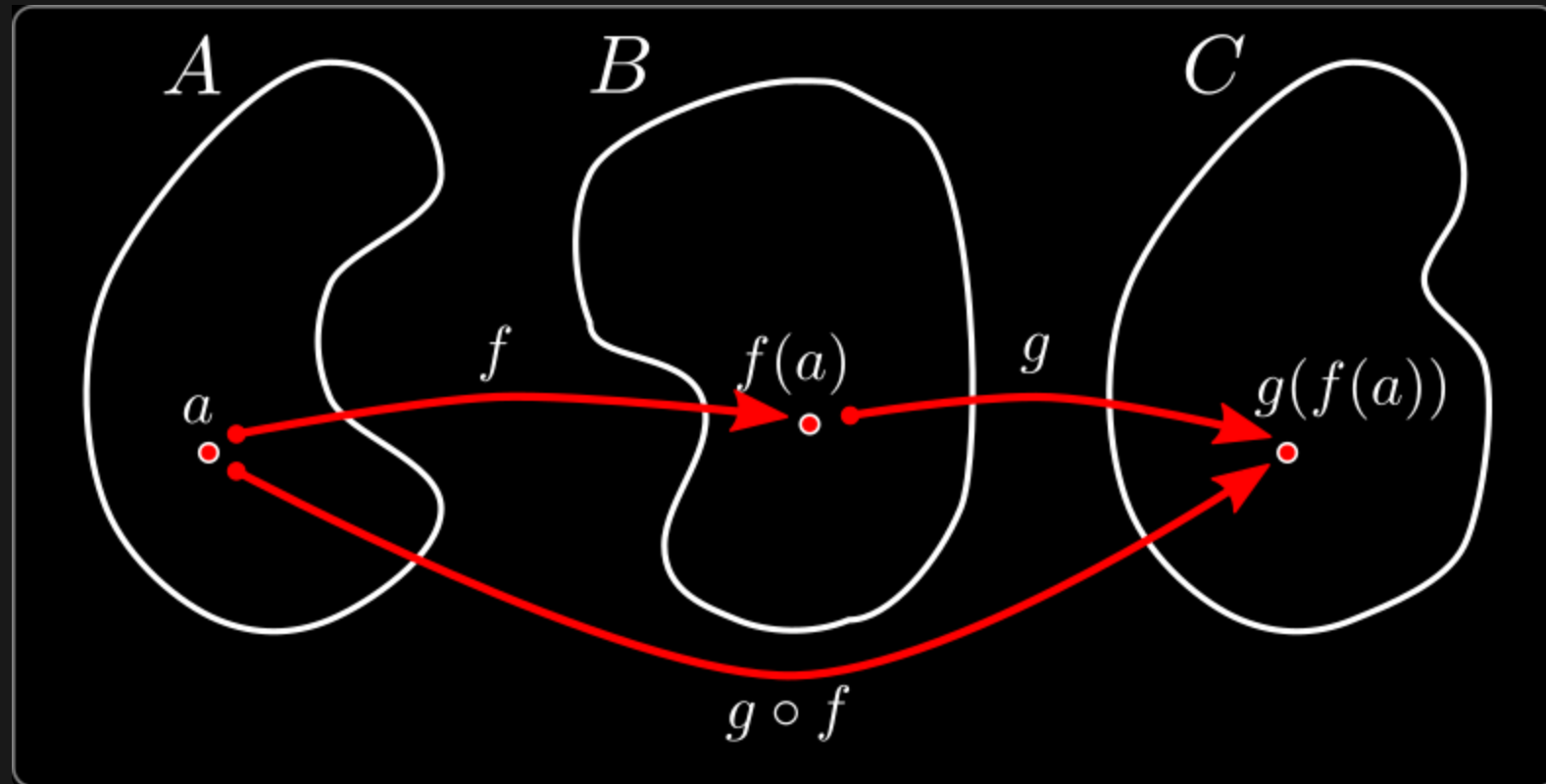
# PLP - 34

## TOPIC 34—COMPOSITIONS

Demirbaş & Rechner

# COMPOSITIONS

# CHAINING FUNCTIONS TOGETHER



## DEFINITION:

Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$ .

The **composition** of  $f$  and  $g$ , denoted  $g \circ f$ , defines a new function

$$g \circ f : A \rightarrow C \quad (g \circ f)(a) = g(f(a)) \quad \forall a \in A$$

Note composition is associative:  $h \circ (g \circ f) = (h \circ g) \circ f$ .

# COMPOSITIONS, INJECTIONS AND SURJECTIONS

Compositions play nicely with injections and surjections.

## THEOREM:

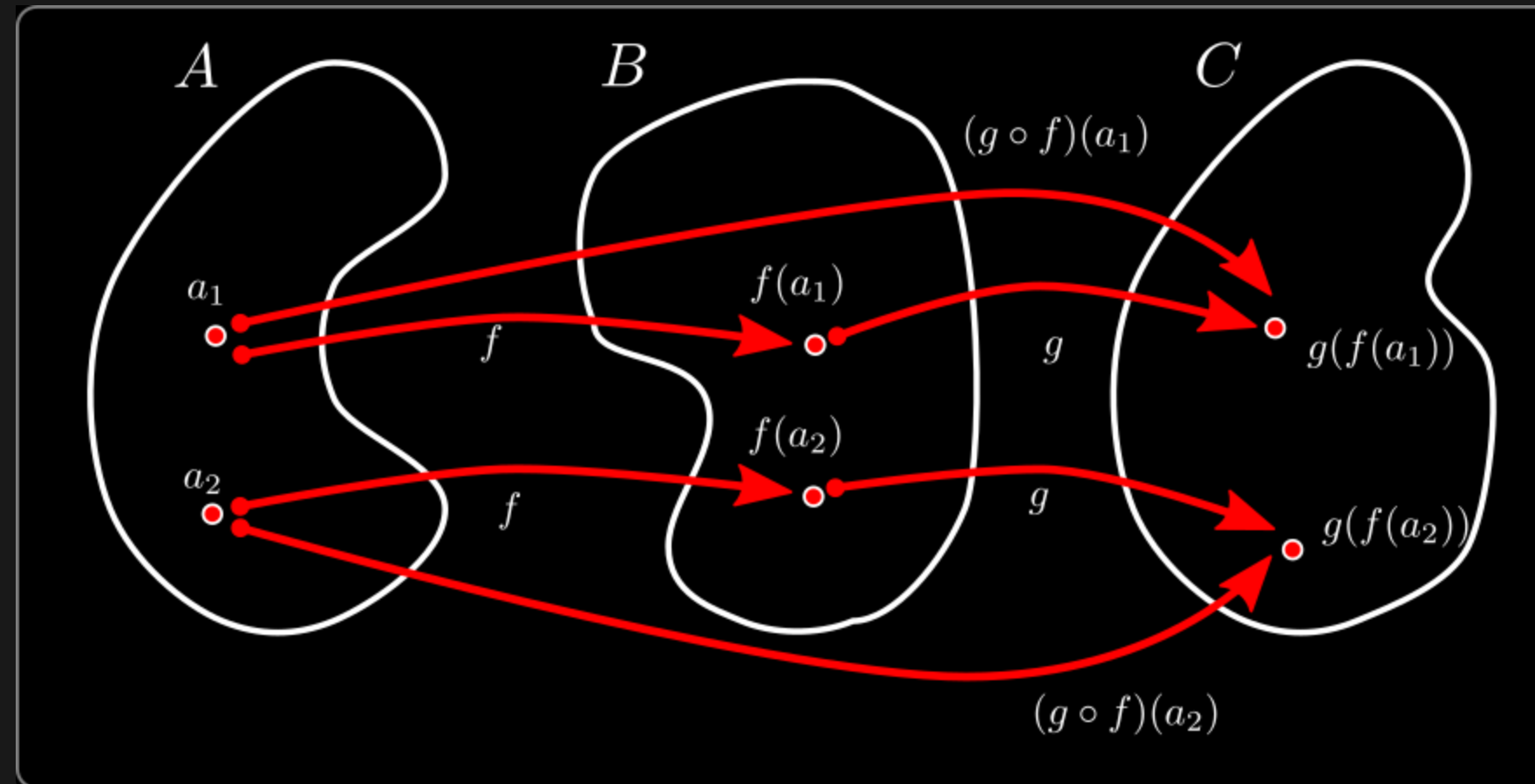
Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be functions.

- If  $f$  and  $g$  are injective then so is  $g \circ f$ .
- If  $f$  and  $g$  are surjective then so is  $g \circ f$ .

Consequently if  $f, g$  are bijective then so is  $g \circ f$ .

# COMPOSITION OF INJECTIONS

Use injection property — different map to different



**PROOF.**

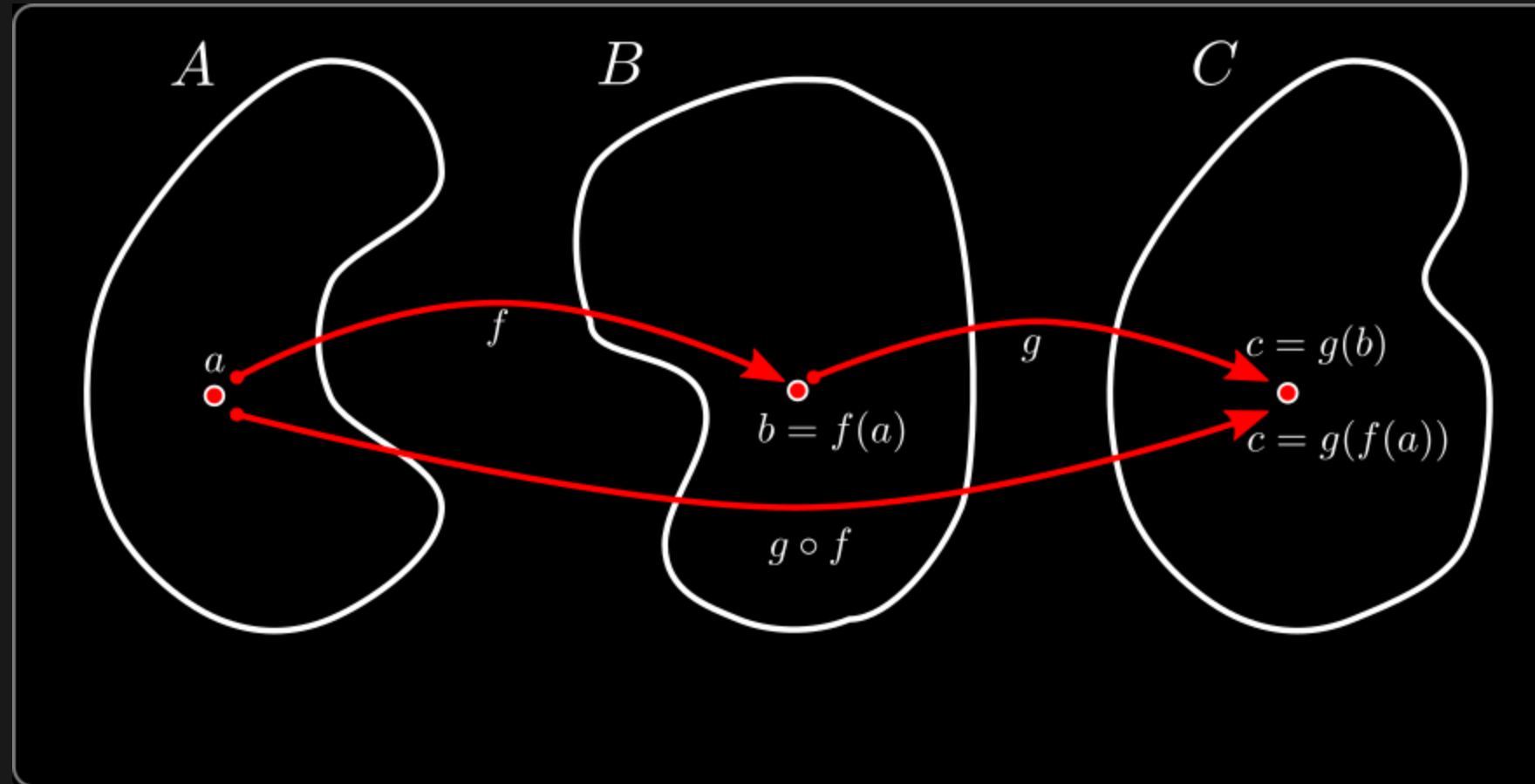
Let  $a_1, a_2 \in A$  so that  $a_1 \neq a_2$ .

Since  $f$  is injective, we know that  $f(a_1) \neq f(a_2)$ . And thus since  $g$  is injective, we know that  $g(f(a_1)) \neq g(f(a_2))$ .

Thus  $(g \circ f)(a_1) \neq (g \circ f)(a_2)$  as required.

# COMPOSITION OF SURJECTIONS

Use surjection property — everything is mapped to by something



**PROOF.**

Let  $c \in C$ .

Since  $g$  is surjective, we know that there is  $b \in B$  so that  $g(b) = c$ . Then since  $f$  is surjective, we have some  $a \in A$  so that  $f(a) = b$ .

Thus  $g(f(a)) = g(b) = c$ , and so for any  $c \in C$  we can find  $a \in A$  so that  $(g \circ f)(a) = c$  as required.

# PARTIAL CONVERSE

## THEOREM:

Let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be functions, then

- if  $g \circ f$  is an injection then  $f$  is an injection.
- if  $g \circ f$  is a surjection then  $g$  is a surjection.

The proofs of these statements make excellent exercises.

Note that you *cannot* extend this to a full converse. There exist  $f, g$  so that

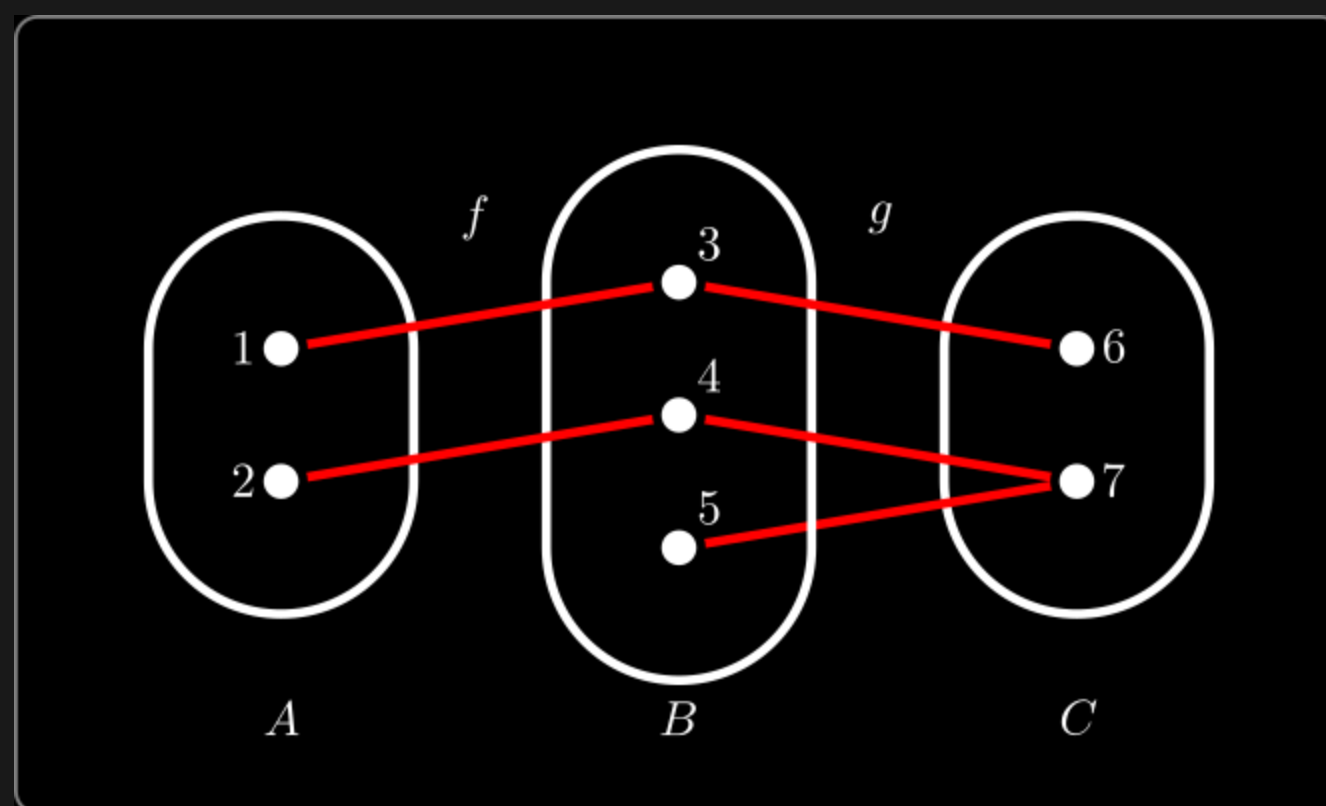
- $g \circ f$  is an injection, but  $g$  is *not* injective
- $g \circ f$  is a surjection, but  $f$  is *not* surjective

# DISPROOF OF FULL CONVERSE

- $g \circ f$  is an injection, but  $g$  is *not* injective
- $g \circ f$  is a surjection, but  $f$  is *not* surjective

## PROOF.

Consider functions  $f, g$  defined by the diagram below.



- Since  $g(f(1)) \neq g(f(2))$ ,  $g \circ f$  is injective. But  $g(4) = g(5)$ , so  $g$  not an injection.
- Since  $6 = g(f(1))$ ,  $7 = g(f(2))$ ,  $g \circ f$  is surjective. But,  $f(1), f(2) \neq 5$  so  $f$  is not a surjection.