PLP - 36

TOPIC 36—PROOF BY CONTRADICTION

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PROOF BY CONTRADICTION

HAMMERS AND WARNINGS

Warning 1: when you have a fancy new hammer, it is tempting to see nails everywhere.

Warning 2: do not use **proof by contradiction** for everything.

Warning 3: proof by contradiction can be confusing

- Assume garbage
- Deduce something that is always false and so definitely garbage
- Conclude truth

But two pieces of logic will help everything make sense.

MIDDLES AND TOLLENS

Proof by contradiction relies on the Law of the excluded middle and modus tollens

FACT: LAW OF THE EXCLUDED MIDDLE.

Let P be a statement. Then either P is true or its negation is true. That is

$$P \vee (\sim P) \text{ is a tautology}$$

DEFINITION: (MODUS TOLLENS).

Modus tollens is the deduction:

$$(P \implies Q) \textit{ is true and } Q \textit{ is false so } P \textit{ must be false}$$

STRUCTURE OF A PROOF-BY-CONTRADICTION

The statement P is true

PROOF.

- We prove the result by contradiction, so assume that $(\sim P)$ is true
- We then prove a chain of implications

$$\begin{aligned} (\sim P) &\implies P_1 \\ P_1 &\implies P_2 \\ &\vdots \\ P_{n-1} &\implies P_n \\ P_n &\implies \text{contradiction} \end{aligned}$$

- By modus tollens, $(\sim P)$ must be false, and so P is true.

A SIMPLE EXAMPLE

PROPOSITION:

There is no smallest positive real number.

PROOF.

- Assume, to the contrary, that there does exist a smallest positive real number. Denote it q
- Notice that the number $r = q/2$ satisfies $0 < r < q$
- Hence r is a positive real number that is smaller than q
- But this contradicts our assumption that q is the smallest positive real number
- Thus there is no smallest positive real number

WHAT JUST HAPPENED?

There is no smallest positive real number.

Law of excluded middle tells that P is true or $(\sim P)$ is true.

- *If* $(\sim P)$ *then* we can find the smallest positive real q
- *If* we know q *then* we can construct a smaller positive real $r = q/2$
- *If* we have a smaller real *then* (q is smallest) and (q is not smallest)

Repeated modus tollens tells us that $(\sim P)$ is false, and so P is true.