# PLP - 37

## TOPIC 37—PROOF BY CONTRADICTION — EXAMPLES

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# EXAMPLES

# NO INTEGER SOLUTIONS

## PROPOSITION:

There are no integers $a, b$ so that $2a + 4b = 1$.

**Scratchwork:**

- The negation is $\exists a, b \in \mathbb{Z}$ s.t. $2a + 4b = 1$
- If we assume the result is false, then we have some $a, b$ so that $2a + 4b = 1$
- But dividing this by 2 gives $a + 2b = \frac{1}{2}$
- This cannot happen, since $a, b \in \mathbb{Z}$ we must have $a + 2b \in \mathbb{Z}$
- Contradiction!

# PROOF

*There are no integers $a, b$ so that $2a + 4b = 1$.*

**PROOF.**

- Assume, to the contrary, that the result is false
- So there are $a, b \in \mathbb{Z}$ so that $2a + 4b = 1$
- Dividing this by $2$ gives $a + 2b = \frac{1}{2}$
- However this cannot happen since the sum of integers is an integer
- Hence there cannot be such integers $a, b$ and so the result holds.

# NO INTEGER SOLUTIONS #2

## PROPOSITION:

There are no integers $a, b$ so that $a^2 - 4b = 3$

### Scratchwork

- Assume, to the contrary, that we can find $a, b \in \mathbb{Z}$ with $a^2 - 4b = 3$
- Write as $a^2 = 3 + 4b$ and notice that the RHS is odd, so the LHS must also be odd
- But this means that $a$ is odd (we proved this!)
- Hence we can write $a = 2k + 1$ and so we have

$$3 = a^2 - 4b = (2k+1)^2 - 4b = 4k^2 + 4k + 1 - 4b = 4(k^2 + k - b) + 1$$

- This implies that $3 \equiv 1 \bmod 4$ — contradiction!

# PROOF

*There are no integers $a, b$ so that $a^2 - 4b = 3$*

**PROOF.**

Assume, to the contrary that there are integers $a, b$ so that $a^2 - 4b = 3$.

Rewrite this as $a^2 = 4b + 3$. Since the RHS is odd, the LHS must be odd, and consequently $a$ is odd. So write $a = 2k + 1$ for some $k \in \mathbb{Z}$.

Then notice that

$$3 = a^2 - 4b = 4(k^2 + k - b) + 1$$

which implies that $3 \equiv 1 \bmod 4$ which is a contradiction. Thus the result follows.

# IRRATIONAL NUMBERS

## DEFINITION:

Let $q$ be a real number.

- We say that $q$ is **rational** if we can write it $q = \frac{a}{b}$ with $a, b \in \mathbb{Z}$ and $b \neq 0$.

$$\exists a \in \mathbb{Z} \text{ s.t. } \exists b \in \mathbb{Z} - \{0\} \text{ s.t. } q = \frac{a}{b}$$

- We say that $q$ is **irrational** when it is not rational.

$$\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z} - \{0\}, q \neq \frac{a}{b}$$

- To denote the set of irrational numbers use $\mathbb{I} = \mathbb{R} - \mathbb{Q}$.

# IRRATIONAL EXAMPLE

## PROPOSITION:

If $x \in \mathbb{Q}$ and $y \in \mathbb{I}$ then $x + y \in \mathbb{I}$.

### Scrathwork

- Assume negation: $\exists x \in \mathbb{Q}$ s.t. $\exists y \in \mathbb{I}$ s.t. $x + y \notin \mathbb{I}$
- But since $x, y \in \mathbb{R}$ we know $x + y \in \mathbb{R}$, so we have $x + y \in \mathbb{Q}$
- Now since $x, (x + y) \in \mathbb{Q}$, we can write $x = a/b$ and $(x + y) = c/d$ with $a, b, c, d \in \mathbb{Z}$.
- But this means $y = (x + y) - x = \frac{c}{d} - \frac{a}{b} = \frac{bc-ad}{bd} \in \mathbb{Q}$
- So we have $y \in \mathbb{Q}$ and $y \notin \mathbb{Q}$ — contradiction!

# PROOF

*If* $x \in \mathbb{Q}$ *and* $y \in \mathbb{I}$ *then* $x + y \in \mathbb{I}$.

**PROOF.**

Assume, to the contrary, that there is $x \in \mathbb{Q}$ and $y \in \mathbb{I}$ so that $x + y \in \mathbb{Q}$.

This implies that $x = \frac{a}{b}$ and $(x + y) = \frac{c}{d}$ with $a, b, c, d \in \mathbb{Z}$ and $b, d \neq 0$.

From this we see that $y = (x + y) - x = \frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd}$ and hence $y \in \mathbb{Q}$.

This contradicts our assumption that $y \in \mathbb{I}$, and so the result follows.