PLP - 41 TOPIC 41 — DENUMERABLE SETS

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DENUMERABLE SETS

LIST OUT THE ELEMENTS

When a set B is denumerable we can "list out" its elements.

- Since denumerable there is a bijection $f:\mathbb{N} o B$
- So we can write *B* as

$$B = \{f(1), f(2), f(3), f(4), \ldots\} \ = \{b_1, b_2, b_3, b_4, \ldots\}$$

This list has two nice properties

• Since f is injective, the list does not repeat

$$k
eq n \implies b_k = f(k)
eq f(n)$$

• Since f is surjective, any given $y \in B$ appears at some *finite* position

$$orall y \in B, \exists n \in \mathbb{N} ext{ s.t. } y = f(n$$

$b_n = f(n)$

$b) = b_n$

 $= b_n$

A LIST GIVES A BIJECTION

Say we can write the elements of *B* in a *nice* list

$$B = \{b_1, b_2, b_3, b_4, \ldots\}$$

then we can use this to construct a bijection: $q: \mathbb{N} \to B$.

What does *nice* mean? First define

$$g:\mathbb{N} o B$$
 by $g(k)$ =

Then the list is *nice* when

- it does not repeat so that g is injective
- any given element $y \in B$ appears at a finite position

 $orall y \in B, \exists n \in \mathbb{N} ext{ s.t. } y = q(n) = b_n$

so g is surjective

So the construction of such a list proves a bijection from \mathbb{N} to B, and so B is denumerable.



 b_k



The set of all integers is denumerable.

Scratch

- We need to list out all the integers so that
 - the list does not repeat
 any given integer appears at a finite position in the list
- Try $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ What $n \in \mathbb{N}$ g
- Try $\mathbb{Z}=\{1,2,3,\ldots,0,-1,-2,-3,\ldots\}$ What $n\in\mathbb{N}$
- Try again: $\mathbb{Z}=\{0,1,-1,2,-2,3,-3,\ldots\}$

What $n \in \mathbb{N}$ gives f(n) = 0? What $n \in \mathbb{N}$ gives f(n) = 0?

PROOF $|\mathbb{N}| = |\mathbb{Z}|$

List $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, ...\}$ or equivalently

1 2 3 4 5 6 7 \cdots $\downarrow \hspace{0.1cm} \downarrow \hspace{0.1cm$

PROOF.

List the elements $z \in \mathbb{Z}$ as above, so that

- if $z \ge 1$, then z appears at position 2z
- if z < 0, then z appears at position 1 2z

The list then

- does not repeat
- and any given $z\in\mathbb{Z}$ appears at some finite position and thus the list defines a bijection between \mathbb{N} and \mathbb{Z} .

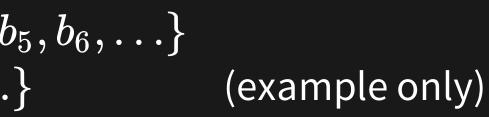
NOTHING BETWEEN DENUMERABLE AND FINITE

THEOREM:

Let $\overline{A}, \overline{B}$ be sets with $A \subseteq \overline{B}$. If \overline{B} is denumerable then A is countable.

Proof sketch:

- If A is finite then it is countable
- If A is infinite then it suffices to construct a bijection $f: \mathbb{N} \to A$.
- Since B is denumerable, list out its elements $B = \{b_1, b_2, b_3, b_4, b_5, b_6, \ldots\}$
- Since $A \subseteq B$, delete elements to get $A = \{b_1, b_2, b_3, b_4, b_5, b_6 \dots\}$
- Then $A = \{b_1, b_4, b_6, b_9, b_{13}, \ldots\}$
- Since the *B*-list did not repeat, this list does not repeat
- Since any given $a \in A$ is also in B, that a appears at a finite position (earlier than in B-list)
- Hence A is denumerable, and so countable





Let $k \in \mathbb{N}$, then following sets are denumerable:

$$k\mathbb{Z} = \{kn: n\in\mathbb{Z}\}$$
 and $k\mathbb{N} =$

We could establish bijections from those sets to \mathbb{Z} or \mathbb{N} , or use previous theorem. **PROOF.**

For any $k \in \mathbb{N}$ the sets are subsets of \mathbb{Z} . Since \mathbb{Z} is denumerable, it follows that the sets are countable (by the previous theorem). Further, since the sets are not finite, it follows that they must be denumerable.

$\{kn:n\in\mathbb{N}\}$

UNION AND INTERSECTION PRESERVE COUNTABLE

PROPOSITION:

Let A,B be countable sets, then $A\cap B$ and $A\cup B$ are all countable.

Proof sketch

- If A, B are finite, then all are finite, so countable
- Since $A \cap B \subseteq A$, by the previous theorem, this is countable.
- Since A, B countable, B A is countable. Then list carefully

$$A = \{a_1, a_2, a_3, \ldots\}$$
 $(B - A) = -$

then combine the lists by alternating

If A finite, then $A\cup B=\{a_1,a_2,\ldots,a_n,b_1,b_2,b_3,\ldots\}$

 $\{b_1, b_2, b_3, \ldots\}$

 $|b_2,a_3,b_3,\ldots\}$

Let A, B be countable sets, then $A \times B$ is countable.

Scratchwork — If neither finite then $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3, \ldots\}$ and so

\times	a_1	a_2	a_3	a_4
b_1	(a_1,b_1)	(a_2,b_1)	(a_3,b_1)	$(a_4$
b_2	(a_1,b_2)	(a_2,b_2)	(a_3,b_2)	$(a_4$
b_3	(a_1,b_3)	(a_2,b_3)	(a_3,b_3)	$(a_4$
b_4	(a_1,b_4)	(a_2,b_4)	(a_3,b_4)	$(a_4$
•	•	•	•	•

Construct list of pairs by careful sweep of the table.

 (b_1, b_1) $_4,b_2)$ $_{4}, b_{3})$ (a_{1}, b_{4})

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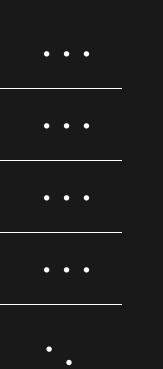
PROOF

PROOF.

Since A, B are denumerable we can construct the following table

$$A imes B=ig\{(a_1,b_1),(a_2,b_1),(a_1,b_2),(a_3,b_1),($$

This list does not repeat, and any given (a_k, b_n) appears at finite position, so A imes B is denumerable.





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$a_2,b_2),(a_1,b_3),\dots \}$

The set of all rational numbers \mathbb{Q} is denumerable.

Very strange since \mathbb{Q} is dense: between any two rationals you can always find another rational. **Proof-sketch**

- Note that any $q\in\mathbb{Q}$ can be written uniquely as $q=rac{a}{b}$ with $a\in\mathbb{Z},b\in\mathbb{N}$ and $\gcd(a,b)=1$
- We can rewrite rationals as $P = \{(a, b) \in \mathbb{Z} \times \mathbb{N} \text{ s.t. } gcd(a, b) = 1\}$
- There is a bijection $f:\mathbb{Q} o P$ given by f(a/b)=(a,b), where a/b is the reduced fraction
- Since $P \subseteq \mathbb{Z} \times \mathbb{N}$, we know P is denumerable.
- Thus since $|P| = |\mathbb{Q}|$ we have that \mathbb{Q} is denumerable also.

