PLP - 42

TOPIC 42 — UNCOUNTABLE SETS

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UNCOUNTABLE

CAN WE FIND ANYTHING BIGGER?

So if A, B are denumerable, then

- $A \cup B$ is denumerable
- $A \times B$ is denumerable

Further if $C_1, C_2, C_3, \ldots$ are all denumerable, then for any fixed $n \in \mathbb{N}$

- $C_1 \cup C_2 \cup \cdots \cup C_n$ is denumerable
- $C_1 \times C_2 \times \cdots \times C_n$ is denumerable

How can we find anything bigger?

We will prove the interval $(0, 1)$ is uncountable, and so $\mathbb{R}$ is uncountable.

USEFUL FACTS ABOUT DECIMAL EXPANSIONS

FACT:

- Every rational number has a repeating decimal expansion

$$1/3 = 0.333333\ldots \qquad 2/11 = 0.181818\ldots$$

- Some rationals have two repeating expansions

$$1/2 = 0.500000\cdots = 0.499999\ldots$$

  This only happens when the denominator b of the reduced fraction a/b is a product of 2's and 5's. In that case the two expansions terminate with 0's or 9's.

- Every irrational number has a unique non-repeating decimal expansion.

CANTOR'S DIAGONAL ARGUMENT

PROPOSITION: (CANTOR 1891).

The open interval $(0, 1) = \{x \in \mathbb{R} \text{ s.t. } 0 < x < 1\}$ is uncountable.

Proof sketch

- We prove the result by contradiction. Assume that $(0, 1)$ is countable.
- Since it is infinite, it is denumerable, and so there is a bijection $f : \mathbb{N} \to (0, 1)$
- We can use this bijection to list *all* the numbers in $(0, 1)$

$f(1)$	$0.78304492\ldots$
$f(2)$	$0.21892653\ldots$
$f(3)$	$0.15206327\ldots$
$\vdots$	$\vdots$

If two expansions then choose expansion that ends in 0's.

THE DIAGONAL IS THE KEY

Arrange the expansions in a big array and consider the diagonal

$f(1) =$	0.	**7**	8	3	0	4	4	9	$\cdots$
$f(2) =$	0.	2	**1**	8	9	2	6	5	$\cdots$
$f(3) =$	0.	1	5	**2**	0	6	3	2	$\cdots$
$f(4) =$	0.	5	4	3	**6**	2	9	1	$\cdots$
$f(5) =$	0.	8	9	7	5	**1**	7	5	$\cdots$
$f(6) =$	0.	0	3	4	8	0	**4**	2	$\cdots$
$f(7) =$	0.	7	4	3	7	5	8	**1**	$\cdots$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$\Delta =$	0.	7	1	2	6	1	4	1	$\cdots$
$z =$	0.	1	2	1	1	2	1	2	$\cdots$

- Denote the k^{th} digit of $f(n)$ as $f_{n,k}$
- The diagonal $\Delta = 0.d_1 d_2 d_3 d_4 \ldots$
- The n^{th} digit $d_n = f_{n,n}$
- Create a new number $z = 0.z_1 z_2 z_3 z_4 \ldots$ via

$$z_n = \begin{cases} 1 & \text{if } d_n \neq 1 \\ 2 & \text{if } d_n = 1 \end{cases}$$

Chosen so that $\forall n \in \mathbb{N}, z_n \neq d_n = f_{n,n}$.

We know $0.111111 \cdots \leq z \leq 0.222222 \ldots$, so z must be somewhere in the table.

- If $z = f(k)$ then must have $z_k = f_{k,k}$. But $f_{k,k} = d_k \neq z_k$ by construction.
- Hence z is not in the table, so contradicts assumption that f is a bijection.

THE REALS ARE UNCOUNTABLE

COROLLARY:

The set of all real numbers is uncountable. Additionally $|(0,1)| = |\mathbb{R}| = c$.

PROOF.

We proved earlier that if a set B is countable, then any subset A is countable. Hence (by the contrapositive), if the subset A is uncountable, then the superset B is uncountable.

So since $(0,1) \subset \mathbb{R}$ is uncountable and $(0,1) \subseteq \mathbb{R}$, we know that $\mathbb{R}$ is uncountable.

To show that the sets are equinumerous, it is sufficient to show that the function

$$g : (0,1) \to \mathbb{R} \qquad g(x) = \frac{1}{1-x} - \frac{1}{x}$$

is a bijection. This is a good exercise.