PLP - 43

TOPIC 43 — MORE INFINITIES

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CANTOR'S THEOREM AND MORE INFINITIES

COMPARING DIFFERENT INFINITIES

We know that $\mathbb{N} \subset \mathbb{R}$ and we proved that $|\mathbb{N}| \neq |\mathbb{R}|$. So want to state

$$\aleph_0 = |\mathbb{N}| < |\mathbb{R}| = c$$

We can make this precise by extending ideas from finite sets A, B:

- If $f : A \to B$ is an injection then $|A| \leq |B|$
- If $h : A \to B$ is an bijection then $|A| = |B|$

DEFINITION:

Let A, B be sets.

- We write $|A| \leq |B|$ when there is an injection from A to B.
- Further, we write $|A| < |B|$ when there is an injection from A to B but no bijection.

$$|A| < |B| \qquad \iff \qquad \big(|A| \leq |B|\big) \wedge \big(|A| \neq |B|\big)$$

CONTINUUM HYPOTHESIS

- Cantor's diagonal argument proves that

$$\aleph_0 = |\mathbb{N}| < |\mathbb{R}| = c$$

- Is there any infinity between these two? More precisely?

$$\exists A \text{ s.t. } |\mathbb{N}| < |A| < |\mathbb{R}|$$

CONJECTURE 2. CONTINUUM HYPOTHESIS (CANTOR 1878).

There is no set A so that $\aleph_0 < |A| < c$.

- Gödel (1940) showed that it cannot be disproved from standard set theory axioms (Zermelo–Fraenkel)
- Cohen (1963) showed that it cannot be proved from standard set theorem axioms
- So (technically) not really correct to call it a conjecture

BIGGER INFINITIES

Are there bigger infinities?

THEOREM: (CANTOR'S THEOREM, 1891).

Let A be a set. Then $|A| < |\mathcal{P}(A)|$

Scratch work

- Easy to find an injection from A to $\mathcal{P}(A)$. Here are two examples

$$f : A \to \mathcal{P}(A) \qquad f(a) = \{a\}$$
$$h : A \to \mathcal{P}(A) \qquad h(a) = A - \{a\}$$

  This proves that $|A| \leq |\mathcal{P}(A)|$

- We prove there is no bijection from A to $\mathcal{P}(A)$ by showing there cannot be a surjection

GOOD AND BAD

To explore, let $A = \{1, 2, 3\}$ and consider f, h from previous slide.

$$f(1) = \{1\} \qquad f(2) = \{2\} \qquad f(3) = \{3\}$$
$$h(1) = \{2, 3\} \qquad h(2) = \{1, 3\} \qquad h(3) = \{1, 2\}$$

Notice that

- $\forall x \in A, x \in f(x)$
- $\forall x \in A, x \notin h(x)$

More generally, if we have any function $g : A \to \mathcal{P}(A)$ then

- if $x \in g(x)$ then call x a **good** point, and
- if $x \notin g(x)$ then call x a **bad** point

Then build sets of all the good and bad points

$$G = \{x \in A \text{ s.t. } x \in g(x)\} \qquad \text{and} \qquad B = \{x \in A \text{ s.t. } x \notin g(x)\}$$

Notice that $G, B \subseteq A$ and so $G, B \in \mathcal{P}(A)$.

THE BAD SET IS MORE INTERESTING

PROOF.

Assume, to the contrary that there is a surjection $g : A \to \mathcal{P}(A)$

- Construct the “bad” set $B = \{x \in A \text{ s.t. } x \notin g(x)\} \subseteq A$
- Now since $B \in \mathcal{P}(A)$ and g is surjective, there must be some $b \in A$ so that $g(b) = B$
- We must have that either $b \in B$ or $b \notin B$? Is it **good** or **bad**?
  - When $b \in B$, by definition of B must have $b \notin B$ — contradiction
  - When $b \notin B$, by definition of B must have $b \in B$ — contradiction
- These contradictions mean there is no b so that $g(b) = B$, and so g is not surjective

Then since we have constructed an injection from $f : A \to \mathcal{P}(A)$, it follows that $|A| < |\mathcal{P}(A)|$.

This immediately gives $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$

With work you can prove that $|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$ — see Cantor-Schröder-Bernstein Theorem

KEEP GOING

$$|A| < |\mathcal{P}(A)| \text{ and } |\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$$

Do it again — $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))|$

And again — $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))|$

COROLLARY:

There are an infinite number of different infinites.

PROOF.

Starting with $\mathbb{N}$, Cantor's theorem tells us that $\mathcal{P}(\mathbb{N})$ is a larger infinite set. Then $\mathcal{P}(\mathcal{P}(\mathbb{N}))$ is larger again. By repeatedly taking power sets, you create an infinitely long sequence of larger and larger infinite sets.

START TO FINISH

Remember where we started:

- Basic definitions of sets and subsets
- Statements, logical operators and truth tables

Look where we got to:

- Diagonal argument — there are different types of infinity
- Cantor's theorem — there are an infinite number of different infinities

> *Congratulations!*