Worksheet for Week 10

1. For which values of $a, b \in \mathbb{N}$ does the set $\phi = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} : ax + by = 6\}$ define a function?

2. Is the set $\theta = \{((x, y), (5y, 4x, x + y)) : x, y \in \mathbb{R}\}$ a function? If so, what are its domain and its range?

   Hint: How can you visualize the range?

3. (Old Final) Suppose that $f : A \to B$ is a function and let C be a subset of A.

   (a) Prove that $f(A) - f(C) \subseteq f(A - C)$.

   (b) Find a counterexample for $f(A - C) \subseteq f(A) - f(C)$.

   Hint: Think about for which type of functions part (b) fails.

4. Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a function defined as $f(a, b) = 4a + 6b$. Explicitly describe the set S =range(f). Prove your answer.

   Before the following examples, watch video 33 in `https://personal.math.ubc.ca/~PLP/auxiliary.html`.

5. A function $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ is defined as $f(n) = (2n + 1, n + 2)$. Verify whether this function is injective and whether it is surjective.

6. Let A, B be nonempty sets. Prove that if there is a bijection $f : A \to B$, then there is a bijection from $\mathcal{P}(A)$, the power set of A, to $\mathcal{P}(B)$, the power set of B.

   Hint: How can you send a *subset* of A to a *subset* of B, knowing how to send each element of A to an element of B?

7. Suppose that $f : A \to B$ and C_1, C_2 are subsets of A. Show that if f is injective, then $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$.

8. (a) Let $f : A \to B$ be a surjective function and let $D_1, D_2 \subseteq B$. Show that if $f^{-1}(D_1) \subseteq f^{-1}(D_2)$, then $D_1 \subseteq D_2$.

   (b) Construct an example that shows the above is not true when f is not surjective.

9. Prove that the function $f : \mathbb{R} - \{2/5\} \to \mathbb{R} - \{-3/5\}$ given by $f(x) = \dfrac{3x}{2 - 5x}$ is bijective.

10. In this question, correct answers without justifications are not sufficient.

    (a) Find a function $f : \mathbb{Z} \to \mathbb{Z}$ which is injective but not surjective.

    (b) Find a function $g : \mathbb{Z} \to \mathbb{Z}$ which is surjective but not injective.

    Discuss how this question would change if we replaced $\mathbb{Z}$ with a finite set. Would you be able to find such functions?

    Note: This is related to the "cardinality" of a set and we will come back to this topic in the last section.

    Before the following week, watch videos 34 and 35 in `https://personal.math.ubc.ca/~PLP/auxiliary.html`.