Worksheet for Week 11

1. Consider $f : A \to B$. Prove that f is injective if and only if $X = f^{-1}(f(X))$ for all $X \subseteq A$.

2. (old final question) Prove that the function $f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}$ given by $f(x) = \dfrac{2x}{x-1}$ is bijective.

3. Let a, b, c, a be real numbers. Define the 2×2 matrix A by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   and let S be the set of all 2×2 matrices, two matrices $A_1, A_2 \in S$ are equal if $a_1 = a_2, b_1 = b_2, c_1 = c_2, d_1 = d_2$. Let $f : S \to S$ be a function defined by

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

   Find $f \circ f$. Is f invertible? If so, what is f^{-1}?

   Hint: You don't need to know anything about matrices more than what is given in the question.

4. Let E, F, G be non-empty sets and let $f : E \to F$, $g : F \to G$ and $h : G \to E$ be three functions. Prove that if $g \circ f$ and $h \circ g$ are bijective then the three functions f, g, h are bijective.

   Hint: Try to show that each function is both injective and surjective.

5. Let A and B be nonempty sets. Prove that if f is an injection, then $f(A - B) = f(A) - f(B)$.

6. This question is related to Pigenhole Principle — you can do it if there is time. Let $A = \{a_1, a_2, a_3, \ldots, a_n\}$ be a nonempty set of n distinct natural numbers. Prove that there exists a nonempty subset of A for which the sum of its elements is divisible by n.

   Hint: Consider the sums $s_k = a_1 + a_2 + \cdots + a_k$.

   Before the following examples, watch videos 36, 37, and 38 in `https://personal.math.ubc.ca/~PLP/auxiliary.html`.

7. Let $n \in \mathbb{N}$, $n \geq 2$, and $a, b, c \in \mathbb{Z}$. Prove that if $ab \equiv 1 \pmod{n}$, then $\forall c \not\equiv 0 \pmod{n}$ we have $ac \not\equiv 0 \pmod{n}$.

8. Let $x \in \mathbb{R}$ satisfy $x^7 + 5x^2 - 3 = 0$. Then prove that x is irrational.

9. Let $(x_n)_{n \in \mathbb{N}}$ be a real sequence. Then, recall that we say (x_n) converges to L if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \epsilon.$$

   Prove that if a sequence (y_n) converges, then the limit is unique.

Before the following week, watch videos 39 and 40 in `https://personal.math.ubc.ca/~PLP/auxiliary.html`.