Worksheet for Week 12

1. Prove that $\sqrt{3}$ is irrational.
2. Let $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then a or b is even.
3. Show that $|\mathbb{Z}| = |S|$, where $\mathbb{Z}$ and $S = \{x \in \mathbb{R} : \sin x = 1\}$.
4. Show that $|\{0,1\} \times \mathbb{N}| = |\mathbb{Z}|$.
5. Let $f : \mathbb{R} \to [-1, 1]$ be a function which is defined by $f(x) = \dfrac{2x}{1+x^2}$.

   Is f surjective? Is f injective?

   Hint: A good scratchwork will be important.
6. Show that $|(0,1)| = |(0,\infty)|$.

   Before the following examples, watch videos 41, 42, and 43 in `https://personal.math.ubc.ca/~PLP/auxiliary.html`.
7. Cantor-Schröder-Bernstein theorem is very(!) useful.
   - Prove that $|(0,1]| = |(0,1)|$ by constructing a suitable bijection.
     Hint: Consider the subset $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$ and how you might map it to $\left\{\frac{1}{n+1} \mid n \in \mathbb{N}\right\}$ while leaving everything else alone.
   - Prove the that $|(0,1]| = |(0,1)|$ using Cantor-Schröder-Bernstein theorem.
8. Let $\mathbb{Z}(\sqrt{2})$ be the set of numbers of the form $a + b\sqrt{2}$. where a and b are integers.
   (a) Prove that $\mathbb{Z}(\sqrt{2}) \cap \mathbb{Q} = \mathbb{Z}$.
   (b) Prove that if $x \in \mathbb{Z}(\sqrt{2})$, then for all natural numbers n, we have $x^n \in \mathbb{Z}(\sqrt{2})$.
   (c) Prove that $\mathbb{Z}(\sqrt{2})$ is denumerable.

   Hint: Knowing that $\sqrt{2}$ is irrational, can you relate $\mathbb{Z}(\sqrt{2})$ to $\mathbb{Z}^2$?
9. (If there is time) Define P to be the set of all polynomials with rational coefficients. That is,

$$P = \left\{a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \ : \ n \in \mathbb{N} \text{ and } a_i \in \mathbb{Q} \text{ for all } i \in \{0, 1, \ldots, n\}\right\}.$$

   (a) Prove or disprove: P is countable.
   These numbers that are solutions to rational (or equivalently integer) polynonimals are called algebraic numbers. As an example we see $\sqrt{2}$ is irrational, but is the solution to the equation $x^2 - 2 = 0$. Thus, since the set of algebraic numbers is countable and the set of real numbers is uncountable, we see that there has to be uncountable many real numbers that cannot be written as a solution to a polynomial with rational coefficients. We call such numbers *transcendental* numbers. For example, π is a transcendental number-as one can guess, showing a number is transcendental is generally a very hard question.
   Hint: Can you relate the set of all polynomials of degree n with rational coefficients to $\mathbb{Q}^{n+1}$, and that does that relation tell us about the set, P, of all polynomials with rational coefficients?
   (b) Define A to be the set of all real numbers that are the roots of a polynomial in P. That is,

$$A = \{x \in \mathbb{R} \ : \ \exists f \in P - \{0\} \text{ s.t. } f(x) = 0\}.$$

   Prove or disprove: $|A| = |P|$.
   Hint: How many solutions can a polynomial of degree n have?