Worksheet for Week 4

Note: Order of the quantifiers matters.

1. Consider the following statements and discuss which ones are true.
   - $\forall m \in \mathbb{Z}, \exists n \in \mathbb{Z}$ s.t. $m^2 < n$.
   - $\exists m \in \mathbb{Z}$ s.t. $\forall n \in \mathbb{Z}$, $m < n$.
   - $\exists m, n \in \mathbb{Z}$, s.t. $m^2 < n$.
   - $\forall m, n \in \mathbb{Z}$, $m^2 < n$.
2. Negate the statements below and determine whether the statements or their negations are true.
   - $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, s.t. $y + 5 = x$.
   - $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$, s.t. $x^n > 0$.
   - $\forall n \in \mathbb{R}$, (n^2 is even) $\implies$ (n is even).
   - $\exists y \in \mathbb{R}$, s.t. $\forall x \in \mathbb{R}$, $x^2 \geq y$.
   - $\exists x \in \mathbb{R}$, s.t. $\forall y \in \mathbb{R}$, $(y \leq x - 1) \implies (y^2 - x^2 \geq 4)$.
3. Prove the following statement:
   $$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (\forall z > 0, |x - y| < z) \implies (x = y).$$
   Hint: How can you rewrite the statement?
4. Let $\mathbb{I}$ denote the set of irrationals. Prove that $\forall x \in \mathbb{I}$, $\left(k \in \mathbb{Q} \setminus \{0\} \implies \frac{x}{k} \in \mathbb{I}\right)$.
   Now, use this result, and the fact that $\pi \in \mathbb{I}$ to show
   $$\forall n \in \mathbb{N}, \exists y \in \mathbb{I} \text{ s.t. } x > y > 0.$$
5. Let A be a set. We say that a function $f : A \to \mathbb{R}$ is bounded if $\exists M \in \mathbb{R}$, s.t. $\forall x \in A$, $|f(x)| \leq M$.
   Use this definition to show that $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 3x - 7$ is not bounded.
   Hint: If the statement above is the definition of a bounded function, what should a function that is NOT bounded satisfy?
6. Fact: $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$, s.t. $n \geq x$.
   Use this fact to show
   $$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, \ (n > N) \implies \left(\frac{1}{n} < \epsilon\right).$$
   Hint: For $\epsilon, n > 0$, we have $(\frac{1}{n} < \epsilon) \iff (\frac{1}{\epsilon} < n)$.

Before next week, watch videos 16 and 17 in `https://personal.math.ubc.ca/~PLP/auxiliary.html`.