Worksheet for Week 6

1. Consider the sequence defined by:

$$\begin{cases} u_1 = \dfrac{1}{2} \\ u_{n+1} = \dfrac{u_n + 1}{u_n + 2} \qquad \text{for } n \in \mathbb{N}. \end{cases}$$

   Prove that $0 < u_n < 1$ for every $n \in \mathbb{N}$.

2. The Fibonacci numbers are defined by the recurrence

$$F_1 = 1 \qquad F_2 = 1 \qquad \text{and} \qquad F_n = F_{n-1} + F_{n-2} \qquad \text{for} \qquad n > 2.$$

   Show that for every $k \in \mathbb{N}$, F_{4k} is a multiple of 3.

3. Prove that $7^n - 2^n$ is divisible by 5 for all $n \in \mathbb{N}$.

4. Let $n \in \mathbb{N}$. Prove that $\forall n \geq 7$, $n! > 3^n$.

5. Let $f(x) = x \ln x$, $x > 0$ and $n \in \mathbb{N}$. Let $f^{(n)}(x)$ denote the nth derivative of $f(x)$. Prove that if $n \geq 3$, then

   $f^{(n)}(x) = (-1)^n \dfrac{(n-2)!}{x^{n-1}}$.

Before the following examples, watch videos 20 and 21 in `https://personal.math.ubc.ca/~PLP/auxiliary.html`.

6. Use strong induction to prove the following statement: Suppose you begin with a pile of n stones ($n \geq 2$) and split this pile into n separate piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have p and q stones in them, respectively, you compute pq. Show that no matter how you split the piles (eventually into n piles of one stone each), the sum of the products computed at each step equals $\dfrac{n(n-1)}{2}$.

   For example — say with start with 5 stones and split them as follows:

   $(5) \to \underbrace{(3)(2)}_{=6} \to \underbrace{(2)(1)}_{=2}\underbrace{(1)(1)}_{=1} \to \underbrace{(1)(1)}_{=1}(1)(1)(1).$

   Then, we get, $6 + 2 + 1 + 1 = 10 = \frac{5x4}{2} \quad \checkmark$.

   Hint: If we have n+1 stones, how can we separate them, and what happens then?

7. Let F_k denote the Fibonacci sequence. Show that for every $k \geq 3$, $F_k \geq 2 \cdot (3/2)^{k-3}$.

8. Show that every number $n \in \mathbb{N}$ can be written as $n = 2^k m$ where k is a non-negative integer and m is odd.

   Hint: What happens if $n + 1$ is not prime in the induction step?

9. Prove that every natural number greater than 1 has a prime factor.

Before next week, watch videos 22 and 23 in `https://personal.math.ubc.ca/~PLP/auxiliary.html`.