Worksheet for Week 7

1. Let $A = \{1, 2\}$. Find $\mathcal{P}(A)$ and $\mathcal{P}(\mathcal{P}(A) - \{\emptyset\})$.

2. Show that if p and q are natural numbers, then $\{pn \mid n \in \mathbb{N}\} \cap \{qn \mid n \in \mathbb{N}\} \neq \emptyset$.

   Hint: Think about a couple of concrete examples first.

3. (Old final question) Let $p_1, p_2, p_3, \ldots, p_n, \ldots$ be the set of all prime numbers listed in an increasing order (so that $p_1 = 2, p_2 = 3, p_3 = 5$, etc.).

   For $k \in \mathbb{N}$ let
   $$A_k = \{a \in \mathbb{N} \mid a \geq 2 \text{ and } p_k \text{ does not divide a}\}$$
   and for $n \in \mathbb{N}$, define
   $$B_n = \bigcap_{k=1}^{n} A_k = A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n.$$

   (a) Find the smallest element of the set B_4.

   (b) (Casual discussion) Ask whether for every $n \in \mathbb{N}$, the set B_n is infinite or not.

   (c) Find the intersection of all A_k's.

   Hint: The last example from last lecture may be useful here.

4. Let $a \in \mathbb{R}$.

   (a) On the xy-plane, draw the set $A_a = \{(x, x^2 - ax),\, x \in \mathbb{R}\}$ when $a = 0$, $a = 1$ and $a = 2$.

   (b) Now define
   $$\bigcap_{a \in \mathbb{R}} A_a = \{(u, v) \mid \forall a \in \mathbb{R}, (u, v) \in A_a\}$$
   Show that $\bigcap_{a \in \mathbb{R}} A_a = \{(0, 0)\}$.

   Hint: What does it mean for a point to be in the intersection?

5. Show that for every $k \in \mathbb{Z}$, $\exists x, y \in \mathbb{Z}$, such that $k = 4x + 5y$.

   What does it say about the set $A = \{4x + 5y \mid x, y \in \mathbb{Z}\}$: is it a subset of, superset of, or equal to $\mathbb{Z}$?

Before next examples, watch video 24 in `https://personal.math.ubc.ca/~PLP/auxiliary.html`.

6. Let A, B and C be sets. For each of the following statements, either prove it is true or give a counterexample.

   (a) $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$

   (b) $\mathcal{P}(A \cup B) \supseteq \mathcal{P}(A) \cup \mathcal{P}(B)$

   Hint: First try this with small sets.

7. (Old final question: this is a tricky one) Let T be the set of all natural numbers that can be written as some nonnegative integer number of 3's plus some nonnegative integer number of 5's. For example, $9 = 3 + 3 + 3$ and $10 = 5 + 5$ and $17 = 3 + 3 + 3 + 3 + 5$ are all in T, but 4 is not. Find $\mathbb{N} - T$ (with justification).

   Hint: Try to figure out which numbers are in the set, and then try to generalize your answer.

8. Let S be a set and A_α's be sets for all $\alpha \in S$. Then, we can define the sets

- $\bigcap_{\alpha \in S} A_\alpha = \{x \mid \forall \alpha \in S, x \in A_\alpha\}$
- $\bigcup_{\alpha \in S} A_\alpha = \{x \mid \exists \alpha \in S, x \in A_\alpha\}$

Use these definitions to find the sets

(a) $\bigcap_{x \in \mathbb{R}} (1 - x^2, 2 + x^2)$

Hint: $\big(= [1,2]\big)$

(b) $\bigcup_{x \in (0,1/2)} [1 + x^2, 2 - x]$

Hint: $\big(= (1,2)\big)$.

9. We consider subsets A, B and C of the universe U. Let $\bar{A}$ denote the complement of A.

(a) Prove that $\bar{A} \subseteq B$ if and only if $A \cup B = U$.

(b) Prove that $\bar{A} \subseteq B$ implies $(C \backslash B) \cup A = A$

Hint: A little diagram may help you visualize.

Before next week, watch video 25 in `https://personal.math.ubc.ca/~PLP/auxiliary.html`.