MATH 100V01 – 2018W
Recitation Notes for Nov. 05th

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1 Implicit Differentiation

Implicit differentiation is the name of the method of treating y as an implicit function of x, as

$$y = y(x) = f(x),$$

in order to find the derivative of y with respect to x at a given point. This method is particularly useful

- when we do not know an explicit formula for the function, but we know an equation that the function obeys; or
- even when we have an explicit but complicated formula for the function, and the function obeys a simple equation.

The trick is to just differentiate both sides of the equation, and then solve for the derivative we are seeking. It can be understood best through an example.

Example 1 (Example 2.11. in CLP 1). Find the equation of the tangent line to

$$y = y^3 + xy + x^3$$

at $x = 1$.

Solution.

- First, observe that when $x = 0$, for example, the equation becomes $y = y^3$. So, the curve passes from the points $(0, -1)$, $(0, 0)$, and $(0, 1)$. Hence, we cannot write this curve as a function $y = f(x)$, globally—meaning, for all $x \in \mathbb{R}$. This is because a function must return one and only one output for a single input value of x.
- When $x = 1$, the equation becomes $y = y^3 + y + 1$, which reduces to $y^3 = -1$. So, the only point with x-coordinate 1 on the curve is $(x, y) = (1, -1)$. A figure of the curve discussed above, with the tangent line at $x = 1$, can be seen in Figure 1.

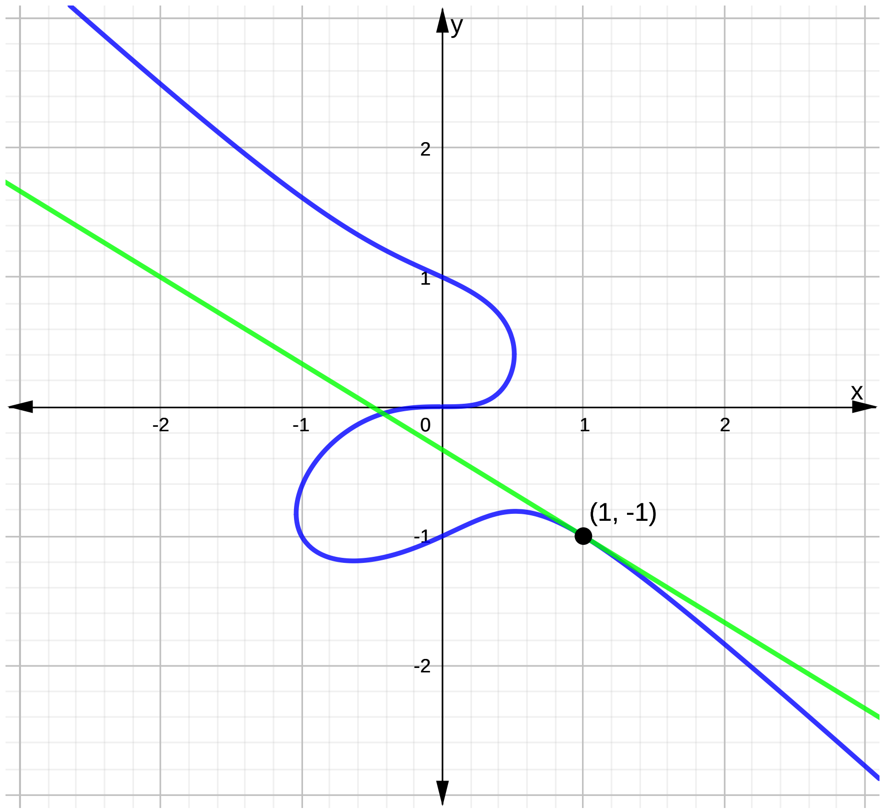

Figure 1: A Figure of the Curve $y = y^3 + xy + x^3$ and the Tangent Line at $x = 1$

- To find the slope of the tangent line at $x = 1$, we treat y as a function of x anyway, and write $y = f(x)$ in the equation:

$$f(x) = [f(x)]^3 + x\,f(x) + x^3.$$

Using the chain rule and the product rule, we then take the derivative of both sides to get,

$$\begin{aligned}
(f(x))' &= \left([f(x)]^3 + x\,f(x) + x^3\right)';\\
f'(x) &= \left([f(x)]^3\right)' + (x\,f(x))' + \left(x^3\right)';\\
f'(x) &= \left(3\ [f(x)]^2\ f'(x)\right) + \left((x)'\,f(x) + x\,f'(x)\right) + \left(3x^2\right);\\
f'(x) &= 3\ [f(x)]^2\ f'(x) + f(x) + x\,f'(x) + 3x^2.
\end{aligned}$$

We solve for $f'(x)$ and have,

$$\left\{1 - 3\ [f(x)]^2 - x\right\} f'(x) = f(x) + 3x^2,$$

or,

$$f'(x) = \frac{f(x) + 3x^2}{1 - 3\ [f(x)]^2 - x}.$$

Evaluating this at $x = 1$ and $y = f(1) = -1$, we get,

$$f'(1) = \frac{-1 + 3\,(-1)^2}{1 - 3\,(-1)^2 - 1} = \frac{2}{-3} = -\frac{2}{3}.$$

This is the slope of the tangent line to our curve at the point $x = 1$.

- The equation of the line that passes from the point (x_0, y_0) with slope m is

  $$y = m(x - x_0) + y_0.$$

  So, the equation of our tangent line, the line that passes form $(1, -1)$ with slope $-2/3$, is $y = (-2/3)(x - 1) + (-1)$, in other words,

  $$y = -\frac{2}{3}x - \frac{1}{3}.$$

Remark. Note that we do not have to plug $y = f(x)$ to do this. Using another notation, we have the implicit differentiation of $y = y^3 + xy + x^3$ as,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\,(y) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(y^3 + xy + x^3\right);\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(y^3\right) + \frac{\mathrm{d}}{\mathrm{d}x}\,(xy) + \frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right);\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 3\,y^2\left[\frac{\mathrm{d}}{\mathrm{d}x}\,(y)\right] + \left[\left(\frac{\mathrm{d}x}{\mathrm{d}x}\right)y + x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right] + \left(3x^2\right);\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 3\,y^2\,\frac{\mathrm{d}y}{\mathrm{d}x} + y + x\,\frac{\mathrm{d}y}{\mathrm{d}x} + 3x^2.
\end{aligned}$$

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Implicit differentiation can be used in this fashion to various other problems with similar flavour.

2 Problems

Problem 1 (Exercise 3). The curve $x^{2/3} + y^{2/3} = 1$ is called an *astroid*. Sketch the astroid, and find the equations of all lines of slope 1 that are tangent to the curve.

Solution. The plot of the astroid is given in Figure 2.

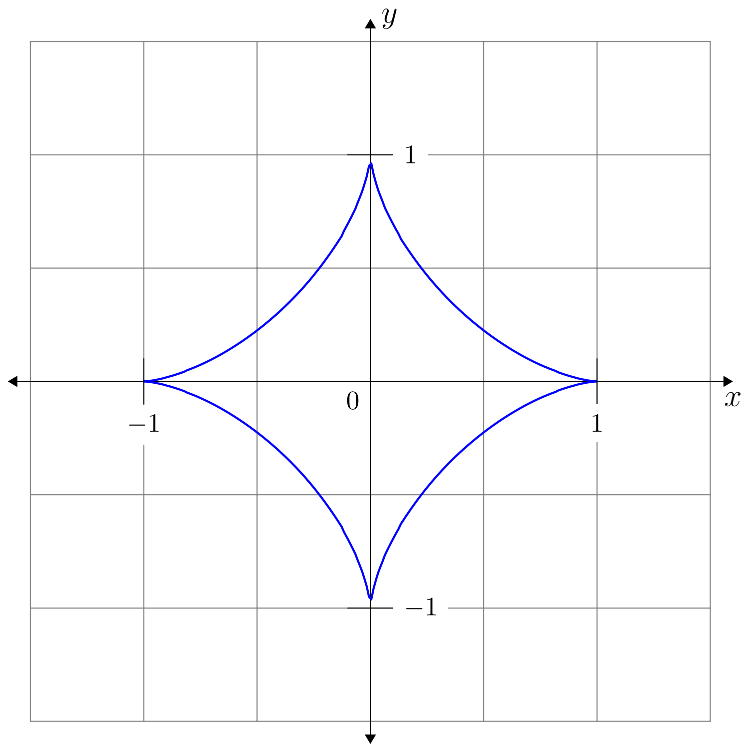

Figure 2: The Plot of the Astroid

Now suppose the astroid has as tangent line at (x_0, y_0) with slope 1. This means that at that point, the derivative of y with respect to x is equal to 1, and the point (x_0, y_0) satisfies the equation of the astroid. I.e., we must have,

$$\begin{cases} \left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=x_0} = 1; \\ {x_0}^{2/3} + {y_0}^{2/3} = 1. \end{cases} \tag{1}$$

To calculate the derivative, let us use implicit differentiation with respect to x in the equation of the astroid. This gives,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(1)\bigg|_{x=x_0} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{2/3}+y^{2/3}\right)\bigg|_{x=x_0};\\
0\big|_{x=x_0} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{2/3}\right)\bigg|_{x=x_0} + \frac{\mathrm{d}}{\mathrm{d}x}\left(y^{2/3}\right)\bigg|_{x=x_0};\\
0 &= \left(\frac{2}{3}\,x^{-1/3}\right)\bigg|_{x=x_0} + \left(\frac{2}{3}\,y^{-1/3}\times\frac{\mathrm{d}y}{\mathrm{d}x}\right)\bigg|_{x=x_0};\\
0 &= \frac{2}{3}\,{x_0}^{-1/3} + \left(\frac{2}{3}\;y^{-1/3}\big|_{x=x_0}\times\frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x=x_0}\right);\\
0 &= {x_0}^{-1/3} + \left(\left(y\big|_{x=x_0}\right)^{-1/3}\times\underbrace{\frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x=x_0}}_{=1\text{ by Equation (1)}}\right);\\
0 &= {x_0}^{-1/3} + \left(y\big|_{x=x_0}\right)^{-1/3}. \qquad (2)
\end{aligned}$$

We want to find x_0, so we have to figure out what $y\big|_{x=x_0}$ is. Note that this is just the value of y *evaluated at* $x = x_0$. Hence, it is nothing but y_0. Thus, Equation (2) now becomes,

$${x_0}^{-1/3} + {y_0}^{-1/3} = 0.$$

In other words,

$$y_0 = -x_0.$$

Observe also that the point $(x_0, y_0) = (x_0, -x_0)$ satisfies the equation of the astroid, which is given in Equation (1). And so,

$$\begin{aligned}
1 &= {x_0}^{2/3} + (-x_0)^{2/3}\\
&= {x_0}^{2/3} + {x_0}^{2/3}\\
&= 2{x_0}^{2/3}.
\end{aligned}$$

Thus, our candidate values are only,

$$\begin{aligned}
x_0 &= \pm\left(\frac{1}{2}\right)^{3/2}\\
&= \pm\frac{1}{2\sqrt{2}}.
\end{aligned}$$

A quick check shows that both $(x_0, y_0) = (1/2^{3/2}, -1/2^{3/2})$ and $(x_0, y_0) = (-1/2^{3/2}, 1/2^{3/2})$ satisfy the Equation (1). Hence these are the only two points at which the tangent line to the astroid has slope 1.

The equation of the line that passes from $(x_0, y_0) = (x_0, -x_0)$ with slope 1 is

$$\begin{aligned}
y &= 1(x - x_0) + (-x_0)\\
&= x - 2x_0.
\end{aligned}$$

So, finally, the equations of the only two tangent lines to the astroid that have slope 1 are:

- $y = x - 1/\sqrt{2}$;
- $y = x + 1/\sqrt{2}$.

These lines are shown in Figure 3.

Homework from yours truly—Your T.A., A.A.B. (Not to be collected.) Look up the MathWorld article for the astroid: `http://mathworld.wolfram.com/Astroid.html` ∎

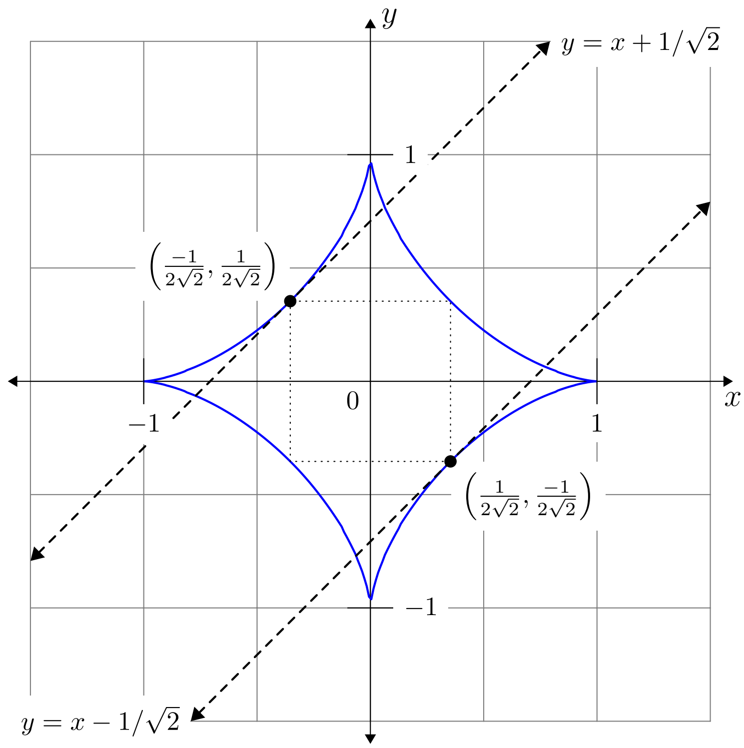

Figure 3: The Plot of the Astroid and the only Tangent Lines with Slope 1

Problem 2 (Exercise 4). The *Bubble Nebula* is an expanding ball of gas and stellar ejecta in the constellation *Cassiopeia*. The radius of the Bubble Nebula is 3×10^{13} km, and it is expanding at a rate of 7×10^6 km/h. Determine how quickly the volume of the nebula is increasing.

Solution. Let us use t for the time variable, and suppose that we are at $t = t_0$. We know that the radius of the Bubble Nebula, $r(t_0)$, is 3×10^{13}, and that it is expanding at a rate of 7×10^6 km/h, i.e.,

$$r'(t_0) = 7 \times 10^6.$$

Let also the volume of the Bubble Nebula be $V(t)$. Since it is a ball,

$$V = \frac{4}{3}\,\pi\, r^3.$$

What we want to find is $V'(t_0)$. Using implicit differentiation, we get,

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{4}{3}\,\pi\, r^3\right)\\ &= \frac{4}{3}\,\pi \times \frac{\mathrm{d}}{\mathrm{d}t}\left(r^3\right)\\ &= \frac{4}{3}\,\pi\left(3\, r^2\,\frac{\mathrm{d}r}{\mathrm{d}t}\right)\\ &= 4\pi\, r^2\,\frac{\mathrm{d}r}{\mathrm{d}t}.\end{aligned}$$

In other words,

$$V'(t) = 4\pi\,[r(t)]^2\, r'(t).$$

Hence, at $t = t_0$, we get,

$$\begin{aligned}V'(t_0) &= 4\pi\,[r(t_0)]^2\, r'(t_0)\\ &= 4\pi\left(3 \times 10^{13}\,\mathrm{km}\right)^2 \times \left(7 \times 10^{6}\,\mathrm{km/h}\right)\\ &= 84\pi \times 10^{32}\,\mathrm{km}^3/\mathrm{h}.\end{aligned}$$

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