MATH 100V01 – 2018W
Recitation Notes for Nov. 23rd

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1 L'Hôpital's Rule

Let us go back to the beginning of our course where we talked about limits. Suppose we are given a problem like the following.

A Motivating Example. Find the limit

$$\lim_{x\to 1} \frac{\log(x)}{x^2-1}.$$

At first, it seems like we cannot use the methods we have learned so far to calculate this limit. Observe that for both the numerator and the denominator we have,

$$\lim_{x\to 1} (\log(x)) = 0 \qquad \text{and} \qquad \lim_{x\to 1}(x^2-1) = 0.$$

Now consider the following three examples:

- $\lim_{x\to 0} \frac{x^3}{x} = 0$;
- $\lim_{x\to 0} \frac{x}{x} = 1$;
- $\lim_{x\to 0} \frac{x}{x^3} = +\infty$.

Although in all three of these, the limits of both numerators and denominators, separately, are 0; it seems that we can get all kinds of different results as the final limit. However, notice that as x approaches to 0, the function x^3 is approaching 0 faster than the function x does. This means that for x values sufficiently close to 0; the function x^3 is closer to 0 than x is. And indeed, as can be seen in Figure 1, if $|x| < 1$; then $|x^3| < |x|$. Or, in particular, if $|x| < 1$; then the red curve is closer to x-axis than the blue curve is.

In fact, in these three examples above, the value of the limit is closely related to the ratio of the *rate* at which the numerator and the denominator approaches to 0. Hence, since the derivative of a function at a point measures the *rate of change* of the function at that

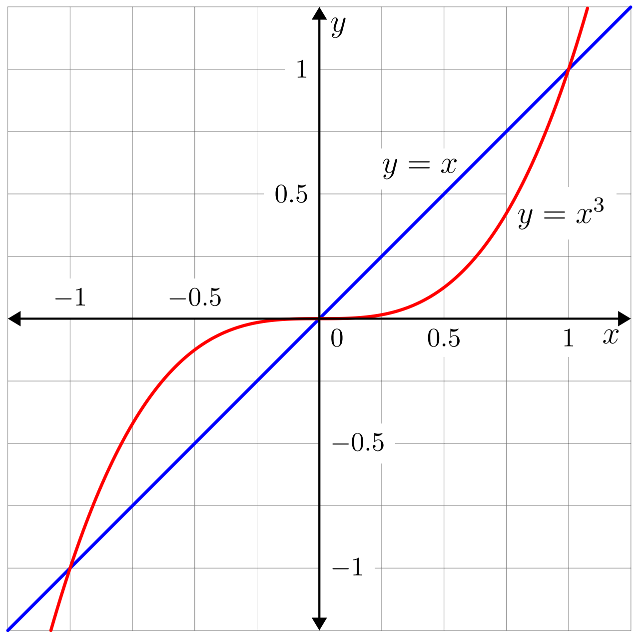

Figure 1: The Plot of the Functions x and x^3

point, it is reasonable to think that these limits can be computed using the derivatives of the numerator and the denominator at $x = 0$.

Now let's get back to our motivating example: $\lim_{x\to 1}[\log(x)/(x^2 - 1)]$. Note that this time, we are approaching $x = 1$ rather than $x = 0$. The plot of the functions $y = \log(x)$ and $y = x^2 - 1$, as well as their tangent lines at $x = 1$, is given in Figure 2.

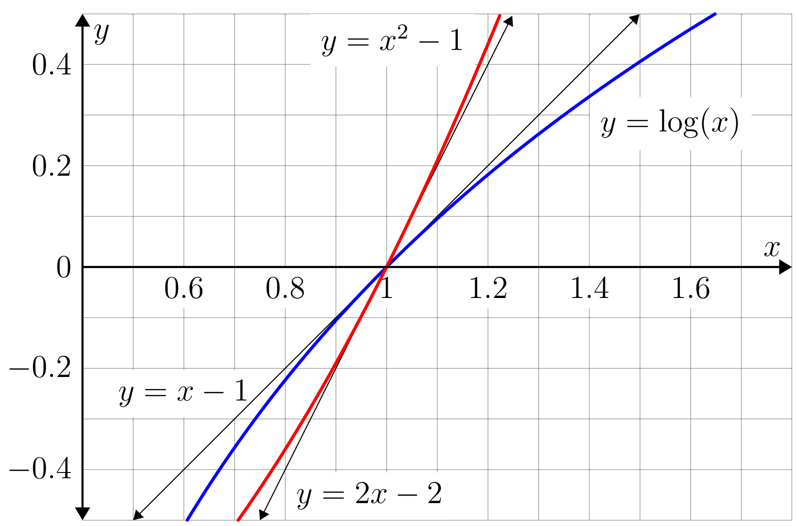

Figure 2: The Plot of the Functions $\log(x)$ and $x^2 - 1$

Considering the plot of these functions, and their tangent lines at $x = 1$, it is reasonable to think that one might have

$$\begin{aligned}\lim_{x\to 1}\frac{\log(x)}{x^2-1} &= \lim_{x\to 1}\frac{\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\log(x)\right)}{\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x^2-1\right)}\\ &= \lim_{x\to 1}\frac{1/x}{2x}\\ &= 1/2.\end{aligned}$$

And this is a particular example of what is called ***L'Hôpital's rule***.

Theorem 1 (L'Hôpital's Rule). *(a) Let $f(x)$ and $g(x)$ be differentiable on an interval containing a, and let $g'(x) \neq 0$ on that interval, except possibly at a. Let $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. Then,*

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided the limit on the right-hand side exists, or is ∞ or $-\infty$.

(b) The rule stated above still holds if

- *the number a is replaced by either ∞ or $-\infty$; or*
- *in the second sentence above, we instead have $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$ or $-\infty$.*

There are many different versions and generalizations of L'Hôpital's rule, and the proof is rather technical. That is why we skip the proof. However, for the curious reader, we refer to [FR17, pp. 330-334].

References

[FR17] Joel Feldman and Andrew Rechnitzer. CLP-I Differential Calculus. `http://www.math.ubc.ca/~CLP/CLP1/clp_1_dc.pdf`, 2017. [Online; accessed 22-November-2018].

2 Problems

Problem 1 (Exercise 1). Evaluate the following limits.

(a) $\lim_{x\to\infty} \dfrac{x^n}{e^x}$, where n is a natural number.

(b) $\lim_{x\to 0} \dfrac{\cos(x) - \cos(2x)}{e^x - x - 1}$.

(c) $\lim_{x\to 0^+} x^x$.

Solution. (a) Let $n \in \mathbb{N}$ be a fixed natural number, $f(x) \coloneqq x^n$, and $g(x) \coloneqq e^x$. Observe that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$, and both $f(x)$ and $g(x)$ are differentiable everywhere, so we will try to use L'Hôpital's rule.

However, before, note that the derivatives of order $k \in \mathbb{N}$ of these functions are

$$f^{(k)}(x) = \begin{cases} n(n-1)\cdots(n-k+1)x^{n-k}, & \text{if } 1 \le k \le n; \\ 0, & \text{if } k \ge n+1; \end{cases} \qquad \text{and} \qquad g^{(k)}(x) = e^x.$$

And so, for $1 \le k \le n-1$, we have $\lim_{x\to\infty} f^{(k)}(x) = \lim_{x\to\infty} g^{(k)}(x) = \infty$.

A repeated application of L'Hôpital's rule then gives,

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^n}{e^x} &= \lim_{x\to\infty} \frac{f(x)}{g(x)} \\
&= \lim_{x\to\infty} \frac{f'(x)}{g'(x)} && \text{(If the limit exists.)} \\
&= \lim_{x\to\infty} \frac{f^{(2)}(x)}{g^{(2)}(x)} && \text{(If the limit exists.)} \\
&\vdots \\
&= \lim_{x\to\infty} \frac{f^{(n)}(x)}{g^{(n)}(x)} && \text{(If the limit exists.)} \\
&= \lim_{x\to\infty} \frac{n!}{e^x} && \text{(If the limit exists.)} \\
&= 0.
\end{aligned}$$

Remark. This is why at the beginning of the course we said, "the exponential grows much faster than polynomials."

(b) Let $f(x) \coloneqq \cos(x) - \cos(2x)$, and $g(x) \coloneqq e^x - x - 1$, which are differentiable everywhere. We see that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$$

So, using L'Hôspital's rule, we get,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)} \lim_{x\to 0} \left(\frac{-\sin(x) + 2\sin(2x)}{e^x - 1} \right),$$

provided that this limit exists.

We again see that

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} g'(x) = 0,$$

so using L'Hôspital's rule again, we obtain,

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{f''(x)}{g''(x)} = \lim_{x\to 0} \left(\frac{-\cos(x) + 4\cos(2x)}{e^x} \right),$$

provided that this latter limit exists. It does and its value is 3. Hence,

$$\lim_{x\to 0} \frac{\cos(x) - \cos(2x)}{e^x - x - 1} = 3.$$

(c) First, we start with

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} \exp\left(\log\left(x^x\right)\right) = \lim_{x\to 0^+} \exp\left(x\log(x)\right) = \exp\left(\lim_{x\to 0^+} x\log(x)\right),$$

provided that these limits exist. Note that $\exp(x)$ is just another notation for e^x, and at the last step above, we used that the function e^x is continuous everywhere.

Hence, let us focus on the limit $\lim_{x\to 0^+} x\log(x)$. Rewriting this limit, and applying L'Hôspital's rule, we see

$$\begin{aligned}
\lim_{x\to 0^+} x\log(x) = \lim_{x\to 0^+} x\log(x) = \lim_{x\to 0^+} \left(\frac{\log(x)}{1/x}\right) \\
&= \lim_{x\to 0^+} \left[\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(\log(x)\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\left(1/x\right)} \right] \\
&= \lim_{x\to 0^+} \left(\frac{1/x}{-1/x^2}\right) \\
&= \lim_{x\to 0^+} \left(-x\right),
\end{aligned}$$

provided that this limit exists. Note that we can apply L'Hôspital's rule even tough we have a one-sided limit. The above limit exists and its value is 0. So,

$$\lim_{x\to 0^+} x^x = \exp\left(\lim_{x\to 0^+} x\log(x)\right) = \exp(0) = e^0 = 1.$$

Alternative Approach. Taking the limit as $x \to 0^+$ actually is the same thing as taking limit as $z = 1/x \to \infty$. In other words,

$$\lim_{x\to 0^+} f(x) = \lim_{z\to\infty} f(1/z).$$

Hence,

$$\begin{aligned}
\lim_{x\to 0^+} x^x = \lim_{z\to\infty}\left(\frac{1}{z}\right)^{1/z} &= \lim_{z\to\infty}\exp\left(\log\left[(1/z)^{1/z}\right]\right) \\
&= \lim_{z\to\infty}\exp\left(\log\left[(1/z)^{1/z}\right]\right) \\
&= \lim_{z\to\infty}\exp\left(\frac{\log(1/z)}{z}\right) \\
&= \exp\left(\lim_{z\to\infty}\left[\frac{\log(1/z)}{z}\right]\right).
\end{aligned}$$

Then, we can apply L'Hôspital's rule to the limit

$$\lim_{z\to\infty}\left[\frac{\log(1/z)}{z}\right]$$

to reach the same conclusion. ■

Problem 2 (Exercise 2). Explain why you *shouldn't* use L'Hôspital's rule to evaluate

$$\lim_{x\to 0}\frac{\sin(x)}{x}.$$

(You actually can use it, but still shouldn't.)

Solution. Let $f(x) := \sin(x)$ and $g(x) := x$. Observe that

$$\lim_{x\to 0}\sin(x) = \lim_{x\to 0} x = 0,$$

and both the functions $\sin(x)$ and x are differentiable everywhere.

If we want to use L'Hôspital's rule now, we end up with

$$\lim_{x\to 0}\frac{\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x))}{\frac{\mathrm{d}}{\mathrm{d}x}(x)} = \lim_{x\to 0}\frac{\cos(x)}{1} = 1,$$

which we know is the correct value.

But here, we needed to use that

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x).$$

However, remember that to prove this, we used the definition of the derivative and the definition of $\sin(x)$ (using the unit circle), and got

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) &= \lim_{h\to 0}\left(\frac{\sin(x+h)-\sin(x)}{h}\right)\\
&= \lim_{h\to 0}\left(\frac{\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)}{h}\right)\\
&= \sin(x)\left[\lim_{h\to 0}\frac{\cos(h)-1}{h}\right]+\cos(x)\underbrace{\left[\lim_{h\to 0}\frac{\sin(h)}{h}\right]}_{\text{The same limit we want to evaluate.}}
\end{aligned}$$

So, in order to use L'Hôspital's rule to evaluate $\lim_{x\to 0}[\sin(x)/x]$, we need to use the fact that $\lim_{x\to 0}[\sin(x)/x]=1$. This is what is called *circular reasoning* and it is a logical fallacy. If it was valid, this type of reasoning could be used to prove any statement—regardless of that it is in fact true or false—and it is not logical.

Remark. If one can show that the derivative of $\sin(x)$ is $\cos(x)$ without using

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1;$$

it is of course perfectly valid to use L'Hôspital's rule here. However, the other ways I (meaning the author of these notes) can think of are all somewhat more complicated and use different ideas. And in all these cases, using the very same ideas, you can show that $\lim_{x\to 0}[\sin(x)/x]=1$ without using the rule anyway.

But maybe you can come up with another, an easier, way to show that the derivative of $\sin(x)$ is $\cos(x)$ without the need to evaluate this limit. If you can, please do email me, I would definitely like to hear about it.

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