Solution :

1. Since $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$,

thus, to determine whether the derivative of the function at $x=0$ exists or not, we need to discuss whether the limit listed above exists or not.

Now, let's consider the left limit and right limit.

Since $\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^-} \frac{f(h) - f(0)}{h}$

$$= \lim_{h\to 0^-} \frac{(h^3 - 7h^2) - (0^3 - 7\times 0^2)}{h}$$

$$= \lim_{h\to 0^-} (h^2 - 7h) = 0$$

$$\lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h\to 0^+} \frac{h^3 \cos(\frac{1}{h}) - (0^3 - 7\times 0^2)}{h}$$

$$= \lim_{h\to 0^+} h^2 \cos(\frac{1}{h}) = 0$$

[for the reason that as h approaches from the right side, h^2 approaches to 0; while the value of $\cos(\frac{1}{h})$ is always between -1 and 1. Thus the limit is 0].

Thus, we have $\lim_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = 0$.

Thus, $f'(0)$ exists, and $f'(0) = 0$.

2. According to the definition of derivative, we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

and

$$p'(x) = \lim_{h \to 0} \frac{p(x+h) - p(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h}$$

$$= \lim_{h \to 0} \frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) + g'(x).$$ #.

3. (a) $x'(y) = (2 - \frac{1}{y^2}) \cdot y^3 + (2y + \frac{1}{y}) \cdot 3y^2$

$$= 2y^3 - y + 6y^3 + 3y$$

$$= 8y^3 + 2y$$

(b) $T(x) = \sqrt{\dfrac{\sqrt{x}+1}{x^2+3}} = \left(\dfrac{x^{\frac{1}{2}}+1}{x^2+3}\right)^{\frac{1}{2}}$

$$T'(x) = \frac{1}{2}\left(\frac{x^{\frac{1}{2}}+1}{x^2+3}\right)^{-\frac{1}{2}} \cdot \frac{\frac{1}{2}x^{-\frac{1}{2}} \cdot (x^2+3) - (x^{\frac{1}{2}}+1)(2x)}{(x^2+3)^2}$$

$$= \frac{1}{2}\left(\frac{x^{\frac{1}{2}}+1}{x^2+3}\right)^{-\frac{1}{2}} \cdot \frac{\frac{1}{2}x^{\frac{3}{2}} + \frac{3}{2}x^{-\frac{1}{2}} - 2x^{\frac{3}{2}} - 2x}{(x^2+3)^2}$$

$$= \frac{1}{2}\sqrt{\frac{x^2+3}{x^{\frac{1}{2}}+1}} \cdot \frac{-\frac{3}{2}x^{\frac{3}{2}} + \frac{3}{2}x^{-\frac{1}{2}} - 2x}{(x^2+3)^2}$$

(c) $f'(x) = e^x \cdot \left(e^{\sin(3x^2)} + 4x\right) + (e^x+1)\left(e^{\sin(3x^2)} \cdot \cos(3x^2) \cdot 6x + 4\right)$

(d) $2y \cdot y' + 7 = \dfrac{1}{x+y} \cdot (1+y')$

$(2yy'+7)(x+y) = 1+y'$

$2yy' \cdot x + 2y^2y' + 7x + 7y = 1+y'$

$(2xy + 2y^2 - 1)\, y' = 1 - 7x - 7y$

$$y' = \frac{1-7x-7y}{2xy+2y^2-1}$$

4. Suppose that the tangent line intersects with the first curve $y=x^2$ at (x_1, y_1), and with the second curve $y=x^2-2x+2$ at (x_2, y_2). then we have

$$y_1 = x_1^2 \text{ --- } ①$$

$$y_2 = x_2^2 - 2x_2 + 2 \text{ --- } ②$$

suppose the slope of the tangent line is k,

and we have $\frac{dy}{dx} = 2x$ for the first curve

$\frac{dy}{dx} = 2x-2$ for the second curve.

Then, we have that

$$k = \left.\frac{dy}{dx}\right|_{x=x_1} = 2x_1 \text{ --- } ③$$

$$k = \left.\frac{dy}{dx}\right|_{x=x_2} = 2x_2 - 2 \text{ --- } ④$$

Thus, from ③ and ④, we get $k = 2x_1 = 2x_2 - 2$,

where we can get $x_2 - x_1 = 1$ --- ⑤

Also, the slope of the tangent line can be represented as following:

$$k = \frac{y_2-y_1}{x_2-x_1} = \frac{(x_2^2-2x_2+2)-(x_1^2)}{x_2-x_1}$$

$$= \frac{(x_1+1)^2 - 2(x_1+1) + 2 - x_1^2}{x_2-x_1}$$

$$= \frac{x_1^2+2x_1+1-2x_1-2+2-x_1^2}{x_2-x_1} = 1.$$

Thus, we have $k=1$, then we get $x_1 = \frac{1}{2}$, and $y_1 = \frac{1}{4}$.

Thus, the equation of the tangent line is $y - \frac{1}{4} = 1 \times (x - \frac{1}{2})$.

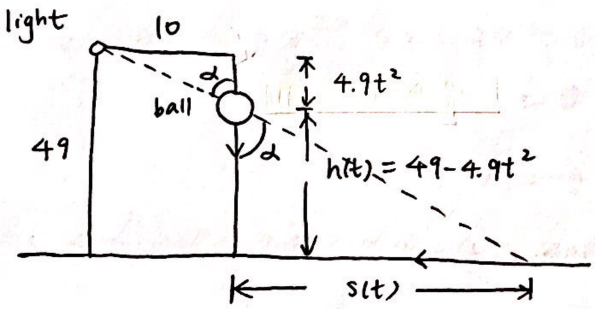
5. Prove: Since $f(x) = x\cos(x) - x\sin(x)$, which is continuous and differentiable on $\mathbb{R}$.

Notice that $f(0) = f(\frac{\pi}{4}) = 0$.

Then using the Mean Value Theorem, we have that there exists a number $c \in (0, \frac{\pi}{4})$, such that

$$f'(c) = \frac{f(\frac{\pi}{4}) - f(0)}{\frac{\pi}{4} - 0} = 0.$$

6. According to the question, we can draw the following graph to illustrate the situation:



In the diagram:

There are two right triangles: one formed by the light, the ball, and the ball's original position, and one formed by the ball, the tip of its shadow, and the ball's eventual position on the ground. The two angles marked above have the same measure, α.

Since the two triangles share two angles in common (a right angle, and an angle of measure α), they are similar triangles.

Let's call $s(t)$ the distance from the shadow to the point on the ground directly underneath the ball. Since the triangles are similar, we have

$$\frac{4.9t^2}{10} = \frac{49-4.9t^2}{s(t)}$$

$$s(t) = 10\,\frac{49-4.9t^2}{4.9t^2} = \frac{100}{t^2} - 10.$$

$$s'(t) = -2\,\frac{100}{t^3}$$

When $t=1$, $s'(1) = -200$ m/sec. That is, the shadow is moving to the left at 200 m/sec.

7. $f'(x) = 2x^2 - 4x - 30$

Since $f(x)$ is defined at $\mathbb{R}$ and $f'(x)$ is defined at $\mathbb{R}$ as well, we have, the critical point of $f(x)$ exists where $f'(x)=0$.

Thus, we have
$$2x^2 - 4x - 30 = 0$$
$$(x-5)(x+3) = 0$$
$$x_1 = 5,\quad x_2 = -3.$$

and the endpoints are $x=-4$ and $x=0$.

Let's compute the following values:

$f(5) = -\frac{329}{3}$

$f(-3) = 61$

$f(-4) = \frac{157}{3}$

$f(0) = 7$

Thus, the global maximum is 61, and the global minimum is $-\frac{329}{3}$.

Exercise 5 (b). Sketch the graphs of the following function.

$$f(x) = xe^{-x^2}.$$

Solution: (1) domain : $f(x) = xe^{-x^2} = \frac{x}{e^{x^2}}$. $\Rightarrow$ the domain is $\mathbb{R}$.

(2) intercept(s): · let $x=0$, $f(0) = 0 \cdot e^{-0^2} = 0$.

Thus, the y-intercept is 0, and the graph passes through $(0,0)$.

· let $y=0$, $f(x) = x \cdot e^{-x^2} = 0$, $\Rightarrow x=0$.

Thus, the x-intercept is 0, and the graph passes through $(0,0)$.

(3) asymptote(s).

since $\lim_{x\to\infty} x \cdot e^{-x^2} = \lim_{x\to\infty} \frac{x}{e^{x^2}} = \lim_{x\to\infty} \frac{1}{e^{x^2} \cdot 2x} = 0$.

$\lim_{x\to-\infty} x \cdot e^{-x^2} = \lim_{x\to-\infty} \frac{x}{e^{x^2}} = \lim_{x\to-\infty} \frac{1}{e^{x^2} \cdot 2x} = 0$.

Thus, the horizontal asymptote is $y=0$.

Since $x^2 \geqslant 0$, $e^{x^2} \geqslant 1$.

thus, for every number $a \in \mathbb{R}$.

$\lim_{x\to a^+} xe^{-x^2} = \lim_{x\to a^+} \frac{x}{e^{x^2}} = \frac{a}{e^{a^2}} \neq \infty / -\infty$.

Thus, $f(x) = xe^{-x^2}$ does not have a vertical asymptote.

(4). monotony (increasing and decreasing intervals and extrema)

$$f'(x) = e^{-x^2} + x \cdot e^{-x^2} \cdot (-2x)$$

$$= e^{-x^2}(1-2x^2) = \frac{1-2x^2}{e^{x^2}}$$

let $f'(x) > 0 \Rightarrow -\frac{\sqrt{2}}{2} < x < \frac{\sqrt{2}}{2}$

let $f'(x) < 0 \Rightarrow x < -\frac{\sqrt{2}}{2}, \; x > \frac{\sqrt{2}}{2}$

x	$(-\infty, -\frac{\sqrt{2}}{2})$	$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$(\frac{\sqrt{2}}{2}, \infty)$
$f'(x)$	$-$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$

Thus, $f(x)$ reaches its local minimum at $x = -\frac{\sqrt{2}}{2}$

and it passes through $(-\frac{\sqrt{2}}{2}, f(-\frac{\sqrt{2}}{2}))$

$= (-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2\sqrt{e}}) = (-0.707, -0.429)$

$f(x)$ reaches its local maximum at $x = \frac{\sqrt{2}}{2}$.

and it passes through $(\frac{\sqrt{2}}{2}, f(\frac{\sqrt{2}}{2})) = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2\sqrt{e}})$

$= (0.707, 0.429)$

(5) Concavity and inflection point.

$$f''(x) = -2x\,e^{-x^2}(1-2x^2) + e^{-x^2}\cdot(-4x) = e^{-x^2}(-2x + 4x^3 - 4x)$$

$$= \frac{4x^3 - 6x}{e^{x^2}} = \frac{4x(x^2 - \frac{3}{2})}{e^{x^2}} = \frac{4x(x+\frac{\sqrt{6}}{2})(x-\frac{\sqrt{6}}{2})}{e^{x^2}}$$

if $f''(x) > 0 \Rightarrow x \in (-\frac{\sqrt{6}}{2}, 0), (\frac{\sqrt{6}}{2}, \infty)$

if $f''(x) < 0 \Rightarrow x \in (-\infty, -\frac{\sqrt{6}}{2}), (0, \frac{\sqrt{6}}{2})$.

x	$(-\infty, -\frac{\sqrt{6}}{2})$	$(-\frac{\sqrt{6}}{2}, 0)$	$(0, \frac{\sqrt{6}}{2})$	$(\frac{\sqrt{6}}{2}, \infty)$
$f''(x)$	$-$	$+$	$-$	$+$
$f(x)$	∩	∪	∩	∪

Thus, the inflection points are $(-\frac{\sqrt{6}}{2}, -0.273)$, $(0,0)$, $(\frac{\sqrt{6}}{2}, 0.273)$

(6) Combination of (5) and (6).

x	$(-\infty, -\frac{\sqrt{6}}{2})$	$(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{2}}{2})$	$(-\frac{\sqrt{2}}{2}, 0)$	$(0, \frac{\sqrt{2}}{2})$	$(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2})$	$(\frac{\sqrt{6}}{2}, \infty)$
monotony	↘	↘	↗	↗	↘	↘
concavity	∩	∪	∪	∩	∩	∪
combination						

(7) Draw the graph !