MATH 100:701 – 2018W
Solutions for the Problems in Recitation Notes for Oct. 15th

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Problem 1. Calculate the following limits.

(a) $\lim_{x\to\infty}(\sqrt{x^2+1}-x)$

(b) $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$

Solution. (a) Multiplying the numerator and the denominator (which is one) with the conjugate of the above expression, we get,

$$\lim_{x\to\infty}(\sqrt{x^2+1}-x) = \lim_{x\to\infty}\frac{(\sqrt{x^2+1}-x)(\sqrt{x^2+1}+x)}{(\sqrt{x^2+1}+x)} = \lim_{x\to\infty}\frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$
$$= \lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}+x} = 0,$$

since the denominator gets arbitrarily large as x tends to $+\infty$.

(b) Similarly, we this time get,

$$\lim_{x\to\infty}(\sqrt{x^2+x}-x) = \lim_{x\to\infty}\frac{(\sqrt{x^2+x}-x)(\sqrt{x^2+x}+x)}{(\sqrt{x^2+x}+x)} = \lim_{x\to\infty}\frac{x^2+x-x^2}{\sqrt{x^2+1}+x}$$
$$= \lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}+x} = \lim_{x\to\infty}\frac{x/x}{(\sqrt{x^2+1}+x)/x}$$
$$= \lim_{x\to\infty}\frac{1}{(\sqrt{(x^2+1)/x^2}+x/x)} = \lim_{x\to\infty}\frac{1}{(\sqrt{(1+1/x^2)}+1)}$$
$$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}.$$

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Problem 2. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Prove that the sequence $(b_n)_{n=1}^{\infty}$, defined by

$$b_n = \sum_{j=n}^{\infty} a_j,$$

converges to 0. In other words, show that $\lim_{n\to\infty} b_n = 0$.

Solution. Let, as usual, $s_n = \sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n$ be the sequence of partial sums of $(a_n)_{n=1}^{\infty}$. As the series $\sum_{n=1}^{\infty} a_n$ converges, we know that $\lim_{n\to\infty} s_n$ exists and is a real number, say $L \in \mathbb{R}$. Observe that b_n converges for every $n \in \mathbb{N}$; since for $n \geq 2$,

$$b_n = \sum_{j=n}^{\infty} a_j = \left(\sum_{j=1}^{\infty} a_j\right) - \left(\sum_{j=1}^{n-1} a_j\right) = L - \left(\sum_{j=1}^{n-1} a_j\right).$$

Taking limit, we see that,

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(L - \sum_{j=1}^{n-1} a_j\right) = \lim_{n\to\infty}(L - s_{n-1}) = L - \lim_{n\to\infty} s_{n-1} = L - \lim_{n\to\infty} s_n = L - L = 0.$$

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Problem 3. (a) Explain what goes wrong with the following (False!) proof of the (False!) statement "if $\sum_{n=1}^{\infty} a_n$ is a convergent series; then $\sum_{n=1}^{\infty}(a_n)^2$ converges as well."

1. Since $\sum_{n=1}^{\infty} a_n$ is convergent, we know by the divergence test that $\lim_{n\to\infty} a_n = 0$.
2. So, for sufficiently large $n \geq N_0$, $a_n \leq 1$.
3. Thus, $(a_n)^2 \leq a_n$ for all $n \geq N_0$.
4. Since, by our hypothesis, $\sum_{n=N_0}^{\infty} a_n$ converges; by comparison test, $\sum_{n=N_0}^{\infty}(a_n)^2$ converges as well.
5. Hence, $\sum_{n=1}^{\infty}(a_n)^2 = \sum_{n=1}^{N_0-1}(a_n)^2 + \sum_{n=N_0}^{\infty}(a_n)^2$ converges, which is what we wanted to show.

Remark. There are more than one. Find all the mistakes.

(b) Show that the above statement is false by providing a counterexample. In other words, find a sequence $(a_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} a_n \text{ converges, but } \sum_{n=1}^{\infty}(a_n)^2 \text{ diverges.}$$

(c) What do we have to additionally assume to make sure that above statement and the proof is correct?

(d) Suppose that $\sum_{n=1}^{\infty}(a_n)^2$ is a convergent series. Does this imply that $\sum_{n=1}^{\infty} a_n$ is convergent as well? Prove or give a counterexample.

Solution. (a) Observe that the argument used at step (3.) is "since $a_n \leq 1$, we have, $(a_n)^2 \leq a_n$;" but this is correct only if $a_n \geq 0$. Because otherwise, the inequality reverses.

Similarly, at step (4.), the comparison test is being used; however, this test works only for series with all nonnegative terms. Therefore, this step is not correct as well since a_n might as well be negative. For example, $-1/n \leq 0$ for all $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} 0 = 0$ converges; but this does not imply that $\sum_{n=1}^{\infty} -1/n$ converges. In fact, it diverges to $-\infty$.

(b) Consider $a_n = (-1)^n/\sqrt{n}$. By the alternating series test (show this), $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}(-1)^n/\sqrt{n}$ converges; however, $\sum_{n=1}^{\infty}(a_n)^2 = \sum_{n=1}^{\infty} 1/n$ diverges to $+\infty$.

(c) If we assume also that a_n is nonnegative the above statement and the proof is actually correct.

(d) Unfortunately, even in the nonnegative case, the answer is no. Take for example $a_n = 1/n$. Then, $\sum_{n=1}^{\infty}(a_n)^2 = \sum_{n=1}^{\infty} 1/n^2$ converges, but $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1/n$ diverges. ■

Problem 4. Suppose you have infinitely-many cube-shaped boxes, numbered from 1 to infinity, such that the nth box has side length $n^{-5/12}$ meters. Can you fill all these boxes with paint? Can you paint all the inside surfaces of all of these boxes?

Solution. The volume of the nth cube is $V_n = (n^{-5/12})^3 = n^{-15/12} = n^{-5/4}$ cubic meters. Similarly, the total inside surface area of the nth box (including the top lid) is $A_n = 6 \times (n^{-5/12})^2 = 6n^{-10/12} = 6n^{-5/6}$ square meters. Hence, the total amount of paint needed to fill all these boxes is

$$\sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} n^{-5/4}$$

cubic meters, which converges to a positive real number since $5/4 > 1$. Hence, a finite amount of paint will be enough to fill all these boxes with paint.

However, the total surface area we need to paint is

$$\sum_{n=1}^{\infty} A_n = \sum_{n=1}^{\infty} 6\,n^{-5/6} = 6\sum_{n=1}^{\infty} n^{-5/6}$$

square meters, which diverges to $+\infty$ since $5/6 < 1$.

So, actually, we cannot paint all the inside surfaces of all these boxes although we can fill all the boxes with paint. (What!?) ■

Problem 5. Determine if the following series converge absolutely, converge conditionally, or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(n+4)!}{n!\,2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{(n+2)^n}{4^{n^2}}\right)$

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sin\left(1+\frac{1}{2n}\right)}{\sqrt{n}}$

Solution. (a) Note that

$$\frac{n^n}{n!} = \frac{\overbrace{n \times n \times \cdots \times n}^{n\text{-many terms}}}{\underbrace{n \times (n-1) \times \cdots 1}_{n\text{-many terms}}} = \underbrace{\frac{n}{n}}_{\geq 1} \times \underbrace{\frac{n}{n-1}}_{\geq 1} \times \cdots \times \underbrace{\frac{n}{1}}_{\geq 1} \geq 1.$$

So, it cannot be true that $\lim_{n\to\infty} \frac{n^n}{n!} = 0$. Thus, the series diverges.

(b) First, let us start with the absolute convergence. Let $a_n = (-1)^n \frac{(n+4)!}{n!\,2^n}$. To check absolute convergence, we apply ratio test to $\sum_{n=1}^{\infty} |a_n|$ to get,

$$\begin{aligned}
\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} &= \lim_{n\to\infty} \left| (-1)^{n+1} \frac{(n+5)!}{(n+1)!\,2^{n+1}} \right| \Big/ \left| (-1)^n \frac{(n+4)!}{n!\,2^n} \right| \\
&= \lim_{n\to\infty} \frac{(n+5)!}{(n+1)!\,2^{n+1}} \frac{n!\,2^n}{(n+4)!} \\
&= \lim_{n\to\infty} \frac{(n+5)!}{(n+4)!} \frac{n!}{(n+1)!} \frac{2^n}{2^{n+1}} \\
&= \lim_{n\to\infty} \frac{1}{2} \frac{n+5}{n+1} \\
&= \frac{1}{2} \lim_{n\to\infty} \frac{n+5}{n+1} = \frac{1}{2} \times 1 = \frac{1}{2} < 1.
\end{aligned}$$

So, by the ratio test, $\sum_{n=1}^{\infty} |a_n|$ converges. We conclude that the series converges absolutely. Note that this immediately implies that $\sum_{n=1}^{\infty} a_n$ converges.

(c) We can apply the ratio test since we have a series with all positive terms. Letting $a_n = \frac{(n+2)^n}{4^{n^2}}$, we see:

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \left[\frac{(n+3)^{n+1}}{4^{(n+1)^2}} \right] \Big/ \left[\frac{(n+2)^n}{4^{n^2}} \right] \\
&= \frac{(n+3)^{n+1}}{(n+2)^n} \frac{4^{n^2}}{4^{(n+1)^2}} \\
&= \frac{(n+3)^n}{(n+2)^n} (n+3) \frac{4^{n^2}}{4^{n^2+2n+1}} \\
&= \left(\frac{n+3}{n+2} \right) (n+3) \frac{1}{4^{2n+1}} \\
&= \underbrace{\left(1 + \frac{1}{n+2} \right)^n}_{\leq 2^n} (n+3) \frac{1}{4 \times 2^{4n}}.
\end{aligned}$$

Hence, $0 \leq a_{n+1}/a_n \leq (n+3)2^n/(4 \times 2^{4n}) = (1/4) \times (n+3)/2^{3n}$. As $\lim_{n\to\infty} (n+3)/2^{3n} = 0$, this shows that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 < 1$. So, by the raito test the series converges absolutely.

(d) Observe that $\lim_{n\to\infty} \sin(1+(1/2n)) = \sin(1) > 0$. Moreover, as n increases, $1 + (1/2n)$ decreases from 1.5 ($< \pi/2$) to 1 ($< \pi/2$). So, as n increases, $\sin(1+(1/2n))$ decreases from $\sin(1.5) > 0$ to $\sin(1) > 0$. (See Figure 1.) So,

$$\left|(-1)^n \frac{\sin\left(1+\frac{1}{2n}\right)}{\sqrt{n}}\right| = \left|\frac{\sin\left(1+\frac{1}{2n}\right)}{\sqrt{n}}\right| = \frac{\sin\left(1+\frac{1}{2n}\right)}{\sqrt{n}} \geq \frac{\sin(1)}{\sqrt{n}}.$$

Since $\sum_{n=1}^{\infty} \sin(1)/\sqrt{n} = [\sin(1)] \sum_{n=1}^{\infty} 1/\sqrt{n}$ diverges to $+\infty$, we conclude that the series does not converge absolutely.

Now let $a_n = \sin(1 + 1/(2n))/\sqrt{n}$, and apply the alternating series test. As we saw above:

- $a_n = \dfrac{\sin\left(1+\frac{1}{2n}\right)}{\sqrt{n}} \geq 0$;
- $a_{n+1} = \dfrac{\sin\left(1+\frac{1}{2(n+1)}\right)}{\sqrt{n+1}} \leq \dfrac{\sin\left(1+\frac{1}{2n}\right)}{\sqrt{n}} = a_n$; and
- $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{\sin\left(1+\frac{1}{2n}\right)}{\sqrt{n}} = 0$.

Hence, by the alternating series test, the series $\sum_{n=1}^{\infty}(-1)^n a_n$ converges. So, the series converges conditionally. ■

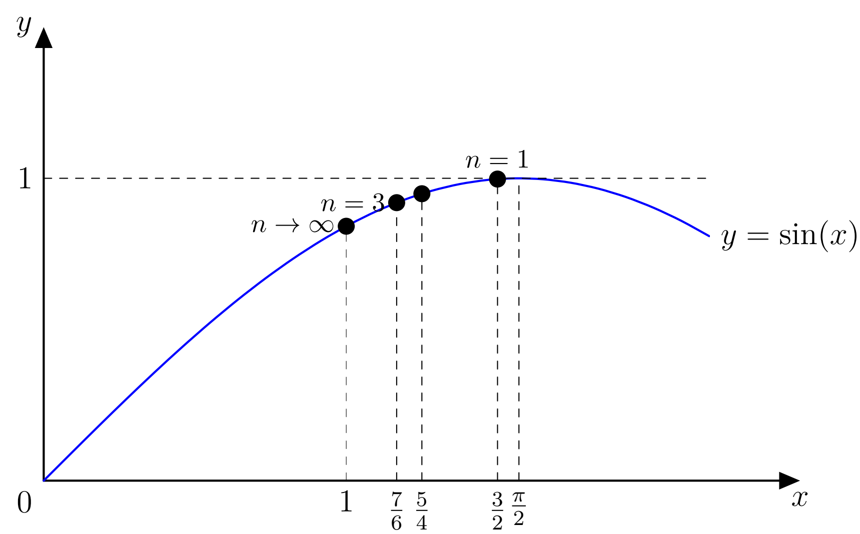

Figure 1: Plot for Problem 5 (d)

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