MATH 100:701 – 2018W
Recitation Notes for Oct. 15th

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1 Limits of Functions

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function, and let either $a \in \mathbb{R}$ be a real number or let $a = \pm\infty$. Suppose we are trying to find $\lim_{x\to a} f(x)$. The only possibilities are:

- $\lim_{x\to a} f(x)$ exists, i.e., $\lim_{x\to a} f(x) = L$ for some real number L. (So, L cannot be $\pm\infty$.)

  Examples. $\lim_{x\to 5} x = 5$, $\lim_{x\to\infty} 1/x = 0$, $\lim_{x\to 0^+} x/|x| = 1$, ...

- $\lim_{x\to a} f(x)$ does not exist. In this case there are two possibilities:

  ▷ $\lim_{x\to a} f(x) = \infty$ or $\lim_{x\to a} f(x) = -\infty$.
  Examples. $\lim_{x\to 0} 1/x^2 = +\infty$, $\lim_{x\to 0_-} 1/x = -\infty$, $\lim_{x\to -\infty} x^4 = +\infty$, ...

  ▷ $\lim_{x\to a} f(x)$ simply does not exist and is not $\pm\infty$.
  Examples. $\lim_{x\to 0} \sin(1/x)$, $\lim_{x\to\infty} \cos(x)$, $\lim_{x\to 0} x/|x|$, ...

Definition 1. If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} = L$ for some real number L; then we say that f ***has a horizontal asymptote*** $y = L$.

Definition 2. If $\lim_{x\to a^\pm} f(x) = \pm\infty$; then we say that f ***has a vertical asymptote*** $x = a$.

Theorem 3. *For a function $f\colon \mathbb{R} \to \mathbb{R}$ and a real number $a \in \mathbb{R}$, $\lim_{x\to a} f(x) = L$ if and only if*

$$\lim_{x\to a^-} f(x) = L \quad \textit{and} \quad \lim_{x\to a^+} f(x) = L.$$

2 Sequences

Definition 4. A sequence $(a_n)_{n=1}^{\infty}$ is called ***bounded*** if and only if there exists an $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Definition 5. A sequence $(a_n)_{n=1}^{\infty}$ is called ***monotonically nondecreasing*** if and only if

$$(a_{n+1} - a_n) \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

Similarly, it is called ***monotonically nonincreasing*** if and only if

$$(a_{n+1} - a_n) \leq 0 \quad \text{for all } n \in \mathbb{N}.$$

A sequence which is either monotonically nondecreasing or monotonically nonincreasing is simply called ***monotonic*** or ***monotone***.

Theorem 6 (Bounded Monotone Convergence Theorem (MBCT)). *If a sequence of real numbers, $(a_n)_{n=1}^{\infty}$, is bounded and monotone; then $\lim_{n\to\infty} a_n$ exists. (Hence, is a real number.)*

3 Series and Tests

Definition 7. Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers. A ***series*** is a formal sum of a sequence, which is denoted by

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots.$$

We say that the series $\sum_{n=1}^{\infty} a_n$ ***converges*** if and only if the limit of the ***sequence of partial sums***,

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{j=1}^{n} a_j,$$

exists. (And hence is a real number.) In other words, we define

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(\sum_{j=1}^{n} a_j \right).$$

A series which is not convergent is called ***divergent***.

Theorem 8 (Divergence Test). *If a series $\sum_{n=1}^{\infty} a_n$ converges; then $\lim_{n\to\infty} a_n = 0$. Hence, if $\lim_{n\to\infty} a_n$ is not 0 or it does not exist; then $\sum_{n=1}^{\infty} a_n$ diverges.*

Remark. If $\lim_{n\to\infty} a_n = 0$, this does not imply anything useful. Consider the two series, $\sum_{n=1}^{\infty} 1/n$ and $\sum_{n=1}^{\infty} 1/n^2$. For both, we have $\lim_{n\to\infty} 1/n = 0$ and $\lim_{n\to\infty} 1/n^2 = 0$, but $\sum_{n=1}^{\infty} 1/n$ diverges while $\sum_{n=1}^{\infty} 1/n^2$ converges. (See Section 4.)

Theorem 9 (Comparison Test). *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with all nonnegative terms, i.e., $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$.*

(a) If $a_n \leq b_n$ for all $n \in \mathbb{N}$ and if $\sum_{n=1}^{\infty} b_n$ converges; then $\sum_{n=1}^{\infty} a_n$ converges as well.

(b) If $a_n \geq b_n$ for all $n \in \mathbb{N}$ and if $\sum_{n=1}^{\infty} b_n$ diverges; then $\sum_{n=1}^{\infty} a_n$ diverges as well.

Remark. One can *informally* think of this test as follows. Since we are assuming we have a nonnegative sequence a_n, the sequence of partial sums, $s_n = \sum_{j=1}^{n} a_j$, is a monotone nondecreasing sequence. Therefore, by BMCT (Theorem 6), $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$ either converges or diverges to $+\infty$. This is true for $\sum_{n=1}^{\infty} b_n$ as well.

Then, part (a) actually says: "if $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n = L$; then $\sum_{n=1}^{\infty} a_n$ is less than or equal to a real number, L. Hence, $\sum_{n=1}^{\infty} a_n$ must converge (i.e., be a real number)."

Similarly, what part (b) says is "if $\sum_{n=1}^{\infty} a_n \geq \sum_{n=1}^{\infty} b_n = +\infty$; then $\sum_{n=1}^{\infty} a_n$ is greater than or equal to $+\infty$; hence, $\sum_{n=1}^{\infty} a_n$ must diverge to $+\infty$."

Theorem 10 (Limit Comparison Test). *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series with all positive terms, i.e., $a_n, b_n > 0$ for all $n \in \mathbb{N}$. If*

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = L > 0;$$

then either both series converge or both series diverge.

Remark. This test can be interpreted as "for sufficiently large $n > N_0$, a_n is really close to $L \times b_n$; so forgetting about the first finitely-many terms, we see, $\sum_{n=N_0}^{\infty} a_n$ is more or less equal to $L \times \left(\sum_{n=N_0}^{\infty} b_n\right)$. Thus, either both series must converge or both must diverge."

Remark. Note that if $\lim_{n\to\infty}(a_n/b_n) = 0$, we cannot conclude anything useful. To see this consider $a_n = 1/n^2$, $b_n = 1/n$, and $c_n = 1/\sqrt{n}$. We have $\lim_{n\to\infty}(a_n/b_n) = \lim_{n\to\infty}(1/n) = 0$, but $\sum_{n=1}^{\infty} a_n$ converges while $\sum_{n=1}^{\infty} b_n$ diverges. On the other hand, we have, again, $\lim_{n\to\infty}(b_n/c_n) = \lim_{n\to\infty}(1/\sqrt{n}) = 0$, but both $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ diverge. (See Section 4.)

Theorem 11 (Ratio Test). *Let $\sum_{n=1}^{\infty} a_n$ be a series with all positive terms.*

(a) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L < 1$; then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L > 1$; then $\sum_{n=1}^{\infty} a_n$ diverges.

Remark. This test can be interpreted as follows. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$; then for sufficiently large $n \geq N_0$, a_{n+1} is more or less equal to $L \times a_n$. Hence, $\sum_{n=N_0}^{\infty} a_n$ is really close to $\sum_{n=N_0}^{\infty} a_{N_0} L^{n-N_0}$, which is a *geometric series.* This series converges if $L < 1$, and diverges if $L > 1$. (See Section 4.)

Remark. If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$; then the test cannot say anything. To see this consider $a_n = 1/n$ and $b_n = 1/n^2$. For both, we have $\lim_{n\to\infty}(a_{n+1}/a_n) = \lim_{n\to\infty}((n+1)/n) = 1$ and $\lim_{n\to\infty}(b_{n+1}/b_n) = \lim_{n\to\infty}((n+1)^2/n^2) = 1$. But $\sum_{n=1}^{\infty} 1/n^2$ converges while $\sum_{n=1}^{\infty} 1/n$ diverges. (See Section 4.)

Definition 12. A series $\sum_{n=1}^{\infty} a_n$ is called to ***converge absolutely*** if and only if $\sum_{n=1}^{\infty} |a_n|$ converges. The series is called to ***converge conditionally*** if and only if $\sum_{n=1}^{\infty} a_n$ converges but $\sum_{n=1}^{\infty} |a_n|$ diverges.

Theorem 13. *If a series converges absolutely; then it converges.*

Theorem 14 (Alternating Series Test). *Let $\sum_{n=1}^{\infty}(-1)^n a_n$ (or $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$) be a series such that*

- $a_n \geq 0$ *for all* $n \in \mathbb{N}$;
- $a_{n+1} \leq a_n$ *for all* $n \in \mathbb{N}$; *and*
- $\lim_{n\to\infty} a_n = 0$.

Then, $\sum_{n=1}^{\infty}(-1)^n a_n$ *(or* $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$*) converges.*

4 Some Series to Keep in Mind

- Let $r \in \mathbb{R}$ be a real number. The series $\sum_{n=1}^{\infty} r^n$ is called a ***geometric series***, and

$$\sum_{n=1}^{\infty} r^n \quad \begin{cases} \text{converges to } \frac{r}{1-r}, & \text{if } |r| < 1; \\ \text{diverges to } +\infty, & \text{if } r > 1; \\ \text{diverges}, & \text{if } r < -1; \\ \text{diverges to } +\infty, & \text{if } r = 1; \\ \text{diverges}, & \text{if } r = -1. \end{cases}$$

- Let $p > 0$ be a real number. Then;

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \begin{cases} \text{converges}, & \text{if } p > 1; \\ \text{diverges}, & \text{if } p \leq 1. \end{cases}$$

5 Some Limits to Keep in Mind

- Factorials grow much faster than exponentials:

$$\lim_{n\to\infty} \frac{e^n}{n!} = 0.$$

- Factorials grow much faster than polynomials:

$$\lim_{n\to\infty} \frac{n^p}{n!} = 0 \quad \text{for any } p \in \mathbb{N}.$$

- Exponentials grow much faster than polynomials:

$$\lim_{n\to\infty} \frac{n^p}{e^n} = 0 \quad \text{for any } p \in \mathbb{N}.$$

- Logarithm grows much slower than anything of the form n^α for some $\alpha > 0$:

$$\lim_{n\to\infty} \frac{\ln(n)}{n^\alpha} = 0 \quad \text{for any } \alpha > 0.$$

6 Problems

Problem 1. Calculate the following limits.

(a) $\lim_{x\to\infty}(\sqrt{x^2+1}-x)$

(b) $\lim_{x\to\infty}(\sqrt{x^2+x}-x)$

Problem 2. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Prove that the sequence $(b_n)_{n=1}^{\infty}$, defined by

$$b_n = \sum_{j=n}^{\infty} a_j,$$

converges to 0. In other words, show that $\lim_{n\to\infty} b_n = 0$.

Problem 3. (a) Explain what goes wrong with the following (<u>False!</u>) proof of the (<u>False!</u>) statement "if $\sum_{n=1}^{\infty} a_n$ is a convergent series; then $\sum_{n=1}^{\infty}(a_n)^2$ converges as well."

1. Since $\sum_{n=1}^{\infty} a_n$ is convergent, we know by the Divergence Test that $\lim_{n\to\infty} a_n = 0$.
2. So, for sufficiently large $n \geq N_0$, $a_n \leq 1$.
3. Thus, $(a_n)^2 \leq a_n$ for all $n \geq N_0$.
4. Since, by our hypothesis, $\sum_{n=N_0}^{\infty} a_n$ converges; by Comparison Test, $\sum_{n=N_0}^{\infty}(a_n)^2$ converges as well.
5. Hence, $\sum_{n=1}^{\infty}(a_n)^2 = \sum_{n=1}^{N_0-1}(a_n)^2 + \sum_{n=N_0}^{\infty}(a_n)^2$ converges, which is what we wanted to show.

Remark. There are more than one. Find <u>all</u> the mistakes.

(b) Show that the above statement is false by providing a counterexample. In other words, find a sequence $(a_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} a_n \text{ converges, but } \sum_{n=1}^{\infty}(a_n)^2 \text{ diverges.}$$

(c) What do we have to additionally assume to make sure that above statement and the proof is correct?

(d) Suppose that $\sum_{n=1}^{\infty}(a_n)^2$ is a convergent series. Does this imply that $\sum_{n=1}^{\infty} a_n$ is convergent as well? Prove or give a counterexample.

Problem 4. Suppose you have infinitely-many cube-shaped boxes, numbered from 1 to infinity, such that the nth box has side length $n^{-5/12}$ meters. Can you fill all these boxes with paint? Can you paint all the inside surfaces of all of these boxes?

Problem 5. Determine if the following series converge absolutely, converge conditionally, or diverge.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(n+4)!}{n!\, 2^n}$

(c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{(n+2)^n}{4^{n^2}} \right)$

(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sin\left(1 + \frac{1}{2n}\right)}{\sqrt{n}}$

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