MATH 100 Section 108 – 2019W
In-Class Problem Sheet 1 — Solutions

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1 Exponential Decay/Growth $[f(t) = f(0)e^{kt}]$

Problem 1.1 (Radioactive Decay). A sample of radioactive matter is stored in a lab in 2000. In the year 2002, it is tested and found to contain 10 units of a particular radioactive isotope of Alperenium. In the year 2005, it is tested and found to contain only 2 units of that same isotope of Alperenium. How many units of the isotope of Alperenium were present in the year 2000?

Solution. Let t be the time passed after 2000, measured in years, and let $P(t)$ be the units of the isotope of Alperenium is present at time t. Since the number of atoms in a sample that decay in a given time interval is proportional to the number of atoms in the sample, $P(t)$ obeys the differential equation of the exponential decay/growth.

So, $P(t) = Ce^{kt}$, and since $P(0) = Ce^0 = C$, we have, in general,

$$P(t) = P(0)e^{kt}.$$

Moreover, by the statement of the problem $P(2) = 10$, and $P(5) = 2$. Therefore

$$5 = \frac{10}{2} = \frac{P(2)}{P(5)} = \frac{P(0)e^{2k}}{P(0)e^{5k}} = e^{2k-5k} = e^{-3k}.$$

Taking the natural logarithm of both sides, we conclude $\log(5) = -3k$, so $k = -\log(5)/3 = -\log\left(5^{1/3}\right) = -\log\left(\sqrt[3]{5}\right)$.

Now, since $P(5) = 2$, we get

$$\begin{aligned} 2 = P(5) = P(0)e^{5k} = P(0)e^{-5\log\left(\sqrt[3]{5}\right)} = P(0)e^{-\log\left(\left(\sqrt[3]{5}\right)^5\right)} = P(0)/e^{\log\left(\left[\sqrt[3]{5}\right]^5\right)} \\ = P(0)/\left(\sqrt[3]{5}\right)^5. \end{aligned}$$

So, the number of the isotope of Alperenium present in 2000 is

$$\begin{aligned} P(0) &= 2\left(\sqrt[3]{5}\right)^5 = 2\left(\sqrt[3]{5}\right)^3\left(\sqrt[3]{5}\right)^2 = 2 \times 5 \times \left(\sqrt[3]{5}\right)^2 \\ &= 10\left(\sqrt[3]{5}\right)^2 \approx 29. \end{aligned}$$

■

Problem 1.2 (Population Dynamics). In 1963, the US Fish and Wildlife Service recorded a bald eagle population of 487 breeding pairs. In 1993, that number was 4015. How many breeding pairs would you expect there were in 2006? What about 2015?

Solution. We use the exponential growth $P(t) = P(0)e^{kt}$, where t is the number of years passed since 1963. Note that, then, $P(0) = 487$. We have

$$4015 = P(1993 - 1963) = P(30) = 487e^{30k};$$

therefore, $e^{30k} = 4015/487$. Taking 1/30-th power of both sides, we get

$$e^k = \left(\frac{4015}{487}\right)^{1/30}.$$

So, the population in 2006 is expected to be

$$P(2006 - 1963) = P(43) = 487e^{43k} = 487\left(e^k\right)^{43} = 487\left[\left(\frac{4015}{487}\right)^{1/30}\right]^{43}$$
$$= 487\left(\frac{4015}{487}\right)^{43/30} \approx 10016.$$

Similarly, the population in 2015 is expected to be

$$P(2015 - 1963) = P(52) = 487e^{52k} = 487\left(e^k\right)^{52} = 487\left[\left(\frac{4015}{487}\right)^{1/30}\right]^{52}$$
$$= 487\left(\frac{4015}{487}\right)^{52/30} \approx 18860. \qquad \blacksquare$$

Problem 1.3 (Compound Interest, or *Cash Rules Everything Around Me.* C.R.E.A.M. *Get the money. Dolla' dolla' bill, y'all!*). Suppose you invest \$10,000 in to an account that accrues compound interest. After one month, your balance (with interest) is \$10,100. How much money will be in your account after a year?

Solution. We use exponential growth: $P(t) = P(0)e^{kt} = 10^4e^{kt}$, where t is measured in months. We have

$$10100 = P(1) = 10000e^k.$$

So,

$$e^k = \frac{10100}{10000} = \frac{101}{100} = 1.01.$$

Then, after 1 year (or 12 months) we will have

$$P(12) = 10000e^{12k} = 10000\left(e^k\right)^{12} = 10000(1.01)^{12} \approx 11268.25 \text{ dollars.} \qquad \blacksquare$$

Problem 1.4 (Radiocarbon Dating—The Real Deal). Researchers at Charlie Lake in BC have found evidence[1] of habitation dating back to around 8500 BCE. For instance, a butchered bison bone was radiocarbon dated to about 10,500 years ago. Suppose a comparable bone of a bison alive today contains 1mg of ^{14}C. If the half-life of ^{14}C is about 5730 years, how much ^{14}C do you think the researchers found in the sample?

Solution. We have $P(t) = P(0)e^{kt}$ modeling the amount of ^{14}C in the bone. Since the half-life is 5730 years,

$$\frac{1}{2} = \frac{P(5730)}{P(0)} = \frac{P(0)e^{k5730}}{P(0)} = e^{5730k}.$$

Taking the 1/5730-th power of both sides, we see

$$e^k = \frac{1}{2^{1/5730}}.$$

Note that an alive bone contains 1 mg of the isotope. So, $P(0) = 1$. Hence, a bone that is 10,500 years old must have

$$P(10500) = P(0)e^{10500k} = \left(e^k\right)^{10500} = \left(\frac{1}{2^{1/5730}}\right)^{10500} = \frac{1}{2^{10500/5730}} \approx 0.28$$

miligram of the isotope. ■

2 Newton's Law of Cooling $[T(t) = [T(0) - A]e^{Kt} + A]$

Problem 2.1 (Safety First). A farrier forms a horseshoe heated to 400°C, then dunks it in a pool of room-temperature (25°C) water. The water near the horseshoe boils for 30 seconds, but the temperature of the pool as a whole hasn't changed appreciably. The horseshoe is safe for the horse when it's 40°C. When can the farrier put on the horseshoe?

Solution. We will apply Newton's Law of Cooling, so

$$T(t) = [T(0) - A]e^{Kt} + A;$$

where $T(t)$ is the temperature of the horseshoe after it is left t seconds in the pool, $A = 25$ is the ambient temperature, i.e., the temperature of the pool around the hot horseshoe. Note that $T(0) = 400$.

We know that the water near the horseshoe boils for 30 seconds; hence the temperature of the horseshoe drops to 100°C after 30 seconds; therefore,

$$100 = [400 - 25]e^{30K} + 25.$$

So, $75 = 375e^{30K}$, in another words, $1 = 5e^{30K}$. Taking the natural logarithm of both sides, we get $\log(1/5) = 30K$. Hence, $K = \log(1/5)/30 = -\log(5)/30$.

[1] http://pubs.aina.ucalgary.ca/arctic/Arctic49-3-265.pdf

The horseshoe is safe to the horse after its temperature drops to 40°C, that is when we have

$$40 = T(t) = [400 - 25]e^{-t\frac{\log(5)}{30}} + 25.$$

Thus,

$$(40 - 25)/375 = 1/25 = e^{-t\frac{\log(5)}{30}}.$$

Taking the natural logarithm of both sides, we obtain,

$$\log(1/25) = -\log(25) = -2\log(5) = -t\frac{\log(5)}{30};$$

Thus, the horseshoe is safe to the horse after

$$t = 2\log(5)\frac{30}{\log(5)} = 60$$

seconds. ■

Problem 2.2 (Five O'Clock Tea). A cup of just-boiled tea is put on a porch outside. After ten minutes, the tea is 40°C, and after 20 minutes, the tea is 25°C. What is the temperature outside?

Solution. Since we have a cup of just-boiled tea, $T(0) = 100$. So, Newton's Law of Cooling gives

$$T(t) = [100 - A]e^{Kt} + A.$$

We know

$$T(10) = [100 - A]e^{10K} + A = 40$$
$$T(20) = [100 - A]e^{20K} + A = 25.$$

Rearranging, we find

$$e^{10K} = \frac{40 - A}{100 - A} \qquad \text{and} \qquad e^{20K} = \frac{25 - A}{100 - A}.$$

Observe that the second one, e^{20K}, is the square of the first one, e^{10K}. So,

$$\frac{(40 - A)^2}{(100 - A)^2} = \frac{25 - A}{100 - A},$$

or

$$\frac{(40 - A)^2}{100 - A} = 25 - A.$$

We see that

$$(40 - A)^2 = (25 - A)(100 - A),$$

or

$$1600 - 80A + A^2 = 2500 - 125A + A^2.$$

Simplifying, we find $45A = 900$. Hence, the ambient temperature, A, is 20°C. ■

Problem 2.3 (Murder on the Orient Express). Suppose a body is discovered at 3:45 pm, in a room held at 20°C, and the body's temperature is 27°C: not the normal 37°C. At 5:45 pm, the temperature of the body has dropped to 25.3°C. When did the owner of the body die?

Solution. Let t be the time passed after 3:45 pm, measured in hours. By Newton's Cooling Law

$$T(t) = [T(0) - A]e^{Kt} + A = [27 - 20]e^{Kt} + 20 = 7e^{Kt} + 20.$$

At 5:45 pm, when $t = 2$, $T(2) = 25.3$, i.e.,

$$25.3 = 7e^{2K} + 20;$$

hence $e^{2K} = \left(e^K\right)^2 = 5.3/7$. Thus, $e^K = \sqrt{5.3/7}$. So, we can clean up our equation as follows:

$$T(t) = 7e^{Kt} + 20 = 17\left(e^K\right)^t + 20 = 7\left(\sqrt{\frac{5.3}{7}}\right)^t + 20 = 7\left(\frac{5.3}{7}\right)^{t/2} + 20.$$

To find when the person is murdered, we set $T(t) = 37$, the average temperature of the human body, and get

$$37 = 7\left(\frac{5.3}{7}\right)^{t/2} + 20.$$

So, $17 = 7\left(\frac{5.3}{7}\right)^{t/2}$, in another words, $17/7 = \left(\frac{5.3}{7}\right)^{t/2}$. Taking logarithm with base 5.3/7 of both sides, we get

$$\begin{aligned}\frac{t}{2} &= \log_{(5.3/7)}\left(\frac{17}{7}\right)\\ &= \frac{\log(17/7)}{\log(5.3/7)} \approx -3.2\end{aligned}$$

Hence, t is approximately -6.4 hours, which is minus 6 hours and minus 24 minutes. Therefore, the unfortunate person has been murder around 6 hours 24 minutes before 3:45 pm—around 9:21 a.m. ■

3 Related Rates (Chain Rule *in Action*) $[\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}]$

Problem 3.1 (Toy Example: Q-T-π). Suppose P and Q are quantities that are changing over time, t. Suppose they are related by the equation

$$3P^2 = 2Q^2 + Q + 3. \tag{1}$$

If $\frac{\mathrm{d}P}{\mathrm{d}t} = 5$ when $P(t) = 1$ and $Q(t) = 0$, then what is $\frac{\mathrm{d}Q}{\mathrm{d}t}$ at that time?

Solution. Since we're already given the relation between $P(t)$ and $Q(t)$ in the Equation (1), we can take the derivative of both sides with respect to t to get

$$3(2P)\left(\frac{\mathrm{d}P}{\mathrm{d}t}\right) = 2(2Q)\left(\frac{\mathrm{d}Q}{\mathrm{d}t}\right) + \frac{\mathrm{d}Q}{\mathrm{d}t} + 0.$$

Substituting the given information that $\mathrm{d}P/\mathrm{d}t = 5$, $P = 1$, and $Q = 0$, we see

$$3(2 \times 5) = 2(2 \times 0)\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{\mathrm{d}Q}{\mathrm{d}t},$$

which gives that

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = 3 \times 2 \times 5 = 30.$$

■

Problem 3.2 ("Loglogloglog..."). A garden hose can pump out a cubic meter of water in about 20 minutes. Suppose you're filling up a rectangular backyard pool, 3 meters wide and 6 meters long. How fast is the water rising?

Solution. Let V be the volume of water in the pool, and h be the height of the water. We are given that $\mathrm{d}V/\mathrm{d}t = (1\ \mathrm{m}^3)/(20\ \mathrm{min}) = 1/20\ \mathrm{m}^3/\mathrm{min}$, $W = 3$, and $L = 6$. And we are asked to find $\mathrm{d}h/\mathrm{d}t$.

The water inside the pool is a right parallelogrammic prism with length L, width W, and height h; therefore its volume is

$$V = L \times W \times h,$$

where L and W do not change with time; hence they are constants with respect to t. (I.e. $\mathrm{d}L/\mathrm{d}t = 0$ and $\mathrm{d}W/\mathrm{d}t = 0$.) Differentiating the above equation with respect to t we see that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = L \times W \times \frac{\mathrm{d}h}{\mathrm{d}t}.$$

Substituting the values, we get

$$\frac{1}{20} = 6 \times 3 \times \frac{\mathrm{d}h}{\mathrm{d}t},$$

from which we conclude that

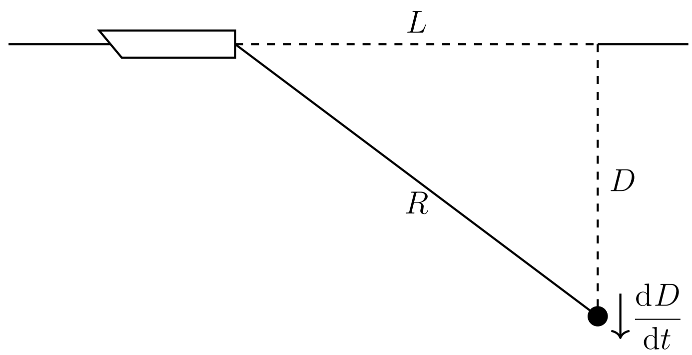
$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{360}\ \mathrm{m}/\mathrm{min}.$$

■

Problem 3.3 (Crossing the River Styx with Virgil). A cannonball is attached to a rope, which is attached to a pulley on a boat, at water level. The cannonball is taken 8 (horizontal) meters from its attachment point on the boat, then dropped in the water. The cannon ball sinks straight down. The rope is being let out at a constant rate of one meter per second, and two seconds have passed. How fast is the ball descending?

Solution. Consider the following figure describing the situation.

We are given $L = 8$, which stays constant, and that $\mathrm{d}R/\mathrm{d}t = 1$. We are asked to find $\mathrm{d}D/\mathrm{d}t$ when $t = 2$.



The relation we're after is

$$R^2 = L^2 + D^2. \tag{2}$$

Differentiating both sides with respect to t, and noting that L stays constant, we get

$$2R\,\frac{\mathrm{d}R}{\mathrm{d}t} = 0 + 2D\,\frac{\mathrm{d}D}{\mathrm{d}t}\,.$$

Hence,

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \left(\frac{R}{D}\right)\frac{\mathrm{d}R}{\mathrm{d}t}\,.$$

We need to find these values at the given time. First of all, since the rope is being let out at a constant rate 1 m/s, when $t = 2$ the length of the it becomes $R = 8 + 2 \times 1 = 10$. To find D, we use Equation (2) to get

$$10^2 = 8^2 + D^2,$$

which results in $D = 6$.

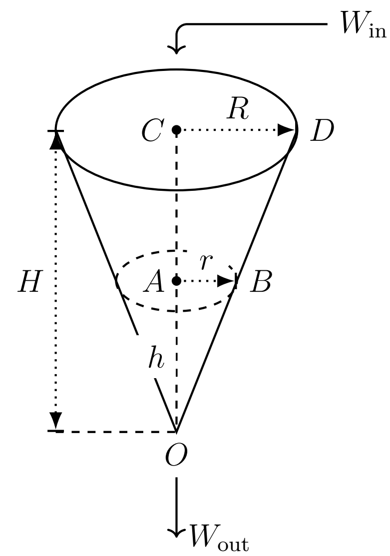
So, we conclude, at time $t = 2$,

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \left(\frac{R}{D}\right)\frac{\mathrm{d}R}{\mathrm{d}t} = \frac{10}{6} \times 1 = \frac{5}{3}\mathrm{m/s}.$$ ■

Problem 3.4 (AAB, Filling up His Flask with Whiskey). You are pouring water into a large jar, through a funnel with an extremely small hole. The funnel lets water into the jar at 100 mL per second, and you are pouring water into the funnel at 300 mL per second. The funnel is shaped like a cone with height 20 cm and diameter at the top also is 20 cm. (Ignore the hole in the bottom.) How fast is the height of the water in the funnel rising when it is 10 cm high?

Solution. The following figure describes the situation.

We are given that the rate at which the water is being added, W_{in}, is 300 ml/s; and the rate at which the water is being poured down, W_{out}, is 100 ml/s. Moreover, $H = 20$ cm; and since the diameter is 20 cm, the radius $R = 10$ cm. We're asked to find $\mathrm{d}h/\,\mathrm{d}t$ when $h = 10$ cm.



Let V be the volume of water in the funnel. Then,

$$V = \frac{1}{3}\pi r^2 h.$$

Observe that the triangles OAB and OCD are similar, so

$$\frac{h}{H} = \frac{|OA|}{|OC|} = \frac{|AB|}{|CD|} = \frac{r}{R},$$

which results in

$$r = \left(\frac{R}{H}\right) h.$$

Thus, the volume becomes

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left[\left(\frac{R}{H}\right) h\right]^2 \times h = \frac{1}{3}\pi \left(\frac{R}{H}\right)^2 h^3,$$

which is the relation we're after.

Now, the rate at which the amount of net water being poured in, then, is

$$W_{\text{in}} - W_{\text{out}} = \frac{\mathrm{d}V}{\mathrm{d}t} = \frac{1}{3}\pi \left(\frac{R}{H}\right)^2 \left(3h^2 \frac{\mathrm{d}h}{\mathrm{d}t}\right),$$

since R and H do not change with time (i.e. they are constants with respect to t). Plugging in the values when $h = 10$, we get

$$300 - 100 = \frac{1}{3}\pi \left(\frac{10}{20}\right)^2 \left(3 \times 10^2 \times \frac{\mathrm{d}h}{\mathrm{d}t}\right),$$

or

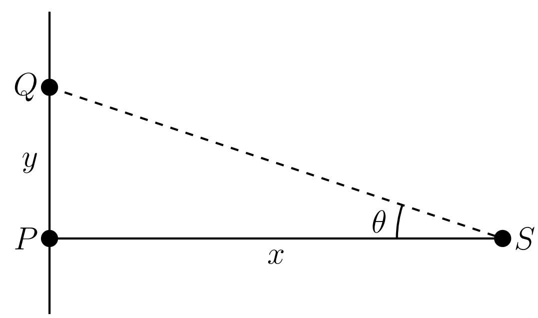
$$200 = 25\pi \frac{\mathrm{d}h}{\mathrm{d}t}.$$

Hence,

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{8}{\pi} \text{ cm/s.}$$ ∎

Problem 3.5 (Getting Wet When It's Not Raining). A sprinkler is 3 m away from a long, straight wall. The sprinkler sprays water in a circle, making three revolutions per minute. Let P be the point on the wall closest to the sprinkler. The water hits the wall at some spot, and that spot moves as the sprinkler rotates. When the spot where the water hits the wall is 1 m away from P, how fast is the spot moving?

Solution. Consider the following figure.



Since the sprinkler rotates in 3 revolutions per minute, we see that $\mathrm{d}\theta/\mathrm{d}t = 3 \times (2\pi) = 6\pi$ rad/min. Moreover, $x = 3$ is fixed, and we're asked to find $\mathrm{d}y/\mathrm{d}t$ when $y = 1$.

From the figure, we see that

$$\tan(\theta) = \frac{y}{x}.$$

Noting that x does not change with time, we differentiate this equation with respect to t to get

$$\frac{1}{\cos^2(\theta)}\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right) = \frac{1}{x}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Observe that when $x = 3$ and $y = 1$, the hypotenuse $|QS|$ becomes $\sqrt{1^2 + 3^2} = \sqrt{10}$. So, at this time, $\cos(\theta) = 3/\sqrt{10}$. Plugging in all these into the equation above, we see

$$\left(\frac{\sqrt{10}}{3}\right)^2 \times 6\pi = \frac{1}{3}\frac{\mathrm{d}y}{\mathrm{d}t};$$

therefore,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3 \times \frac{10}{9} \times 6\pi = 20\pi.$$ ∎

Problem 3.6 (Descartes's Emotional Roller Coaster). A roller coaster has a track shaped in part like the *folium of Descartes*: $x^3 + y^3 = 6xy$. (See Figure 1.) When it is at the position $(3, 3)$, its horizontal position is changing at 2 units per second in the negative direction. How fast is its vertical position changing?

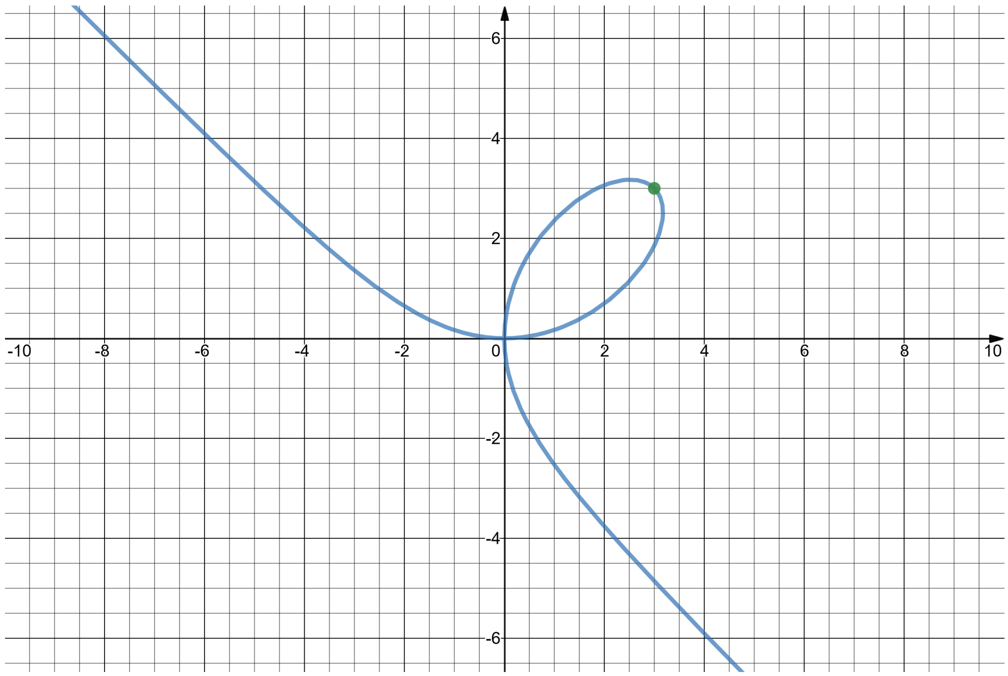


Figure 1: Folium of Descartes: $x^3 + y^3 = 6xy$

Solution. We are given that $\mathrm{d}x/\mathrm{d}t = -2$, and asked to find $\mathrm{d}y/\mathrm{d}t$ when $x = 3$, $y = 3$.

Differentiating the equation of the folium with respect to t gives

$$3x^2\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) + 3y^2\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = 6\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)y + x\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\right].$$

(Note the product rule in the right-hand side!) Plugging in $x = 3$, $y = 3$, and $\mathrm{d}x/\mathrm{d}t = -2$ gives

$$3^3(-2) + 3^3\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = 6\left[(-2)\,3 + 3\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)\right],$$

or

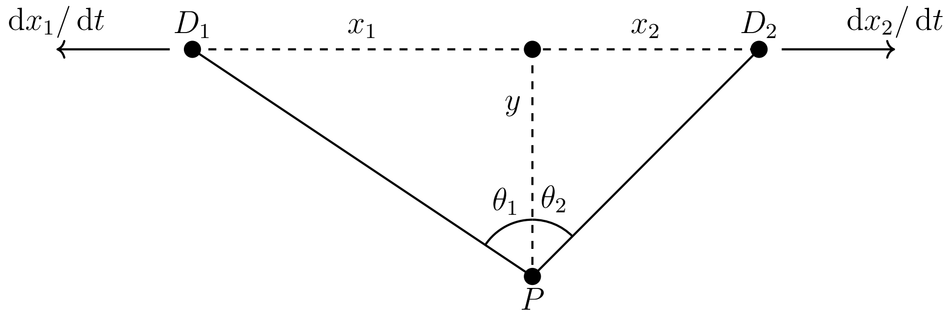
$$-6 + 3\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right) = -4 + 2\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right);$$

hence,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2.$$

■

Problem 3.7 (No animals were harmed in the making of this film). Two dogs are tied with elastic leashes to a lamp post that is 2 meters from a straight road. At first, both dogs are on the road, at the closest part of the road to the lamp post. Then, they start running in opposite directions: one dog runs 3 meters per second, and the other runs 2 meters per second. After one second of running, how fast is the angle made by the two leashes increasing?



Solution. Consider the above figure. We are asked to find

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} + \frac{\mathrm{d}\theta_2}{\mathrm{d}t}$$

when $t = 1$.

Observe that

$$\tan(\theta_1) = \frac{x_1}{y}.$$

Since $y = 2$ is fixed, differentiating this with respect to t gives

$$\frac{1}{\cos^2(\theta_1)} \frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \left(\frac{\mathrm{d}x_1}{\mathrm{d}t}\right) \frac{1}{y},$$

which results in

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \cos^2(\theta_1) \left(\frac{\mathrm{d}x_1}{\mathrm{d}t}\right) \frac{1}{y}.$$

Similarly, we also see that

$$\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = \cos^2(\theta_2) \left(\frac{\mathrm{d}x_2}{\mathrm{d}t}\right) \frac{1}{y}.$$

Note that when $t = 1$, we have $x_1 = 3$ and $x_2 = 2$. These give

$$|PD_1| = \sqrt{{x_1}^2 + y^2} = \sqrt{3^2 + 2^2} = \sqrt{13}$$
$$|PD_2| = \sqrt{{x_2}^2 + y^2} = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

So,

$$\cos(\theta_1) = \frac{y}{|PD_1|} = \frac{2}{\sqrt{13}} \qquad \text{and} \qquad \cos(\theta_2) = \frac{y}{|PD_2|} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

Plugging all these into the results we found above, we get

$$\frac{\mathrm{d}\theta_1}{\mathrm{d}t} = \left(\frac{2}{\sqrt{13}}\right)^2 \times (3) \times \frac{1}{2} = \frac{4}{13} \times \frac{3}{2} = \frac{6}{13}$$
$$\frac{\mathrm{d}\theta_2}{\mathrm{d}t} = \left(\frac{1}{\sqrt{2}}\right)^2 \times 2 \times \frac{1}{2} = \frac{1}{2}.$$

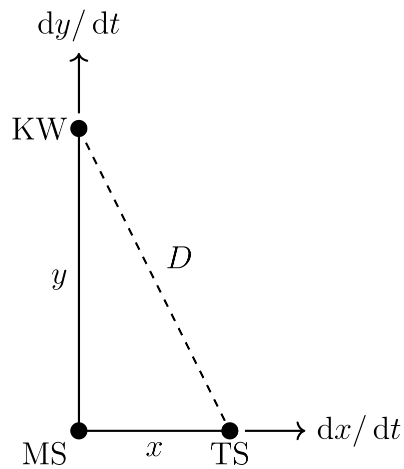
Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\theta_1 + \theta_2) = \frac{\mathrm{d}\theta_1}{\mathrm{d}t} + \frac{\mathrm{d}\theta_2}{\mathrm{d}t} = \frac{6}{13} + \frac{1}{2} = \frac{12}{26} + \frac{13}{26} = \frac{25}{26} \text{ rad/s.}$$

■

Problem 3.8 (*"Imma let you finish..."*). Kanye West and Taylor Swift are attending at a gig at the same place. Eager not to cross paths, Taylor is one kilometer due east of the main stage, heading east, and Ye is two kilometers due north of the main stage, heading north (and to North, North West, his daughter). If Taylor is walking at 5 kph, and Ye is walking 7kph, how fast is the distance between them increasing?

Solution. The following figure describes the situation.



We are given that $\mathrm{d}x/\,\mathrm{d}t = 5$, $\mathrm{d}y/\,\mathrm{d}t = 7$, and we're asked to find $\mathrm{d}D/\,\mathrm{d}t$ when $x = 1$ and $y = 2$.

We see that

$$D^2 = x^2 + y^2.$$

Differentiating this with respect to t, we get

$$2D\left(\frac{\mathrm{d}D}{\mathrm{d}t}\right) = 2x\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) + 2y\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right),$$

or

$$D\left(\frac{\mathrm{d}D}{\mathrm{d}t}\right) = x\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right) + y\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right),$$

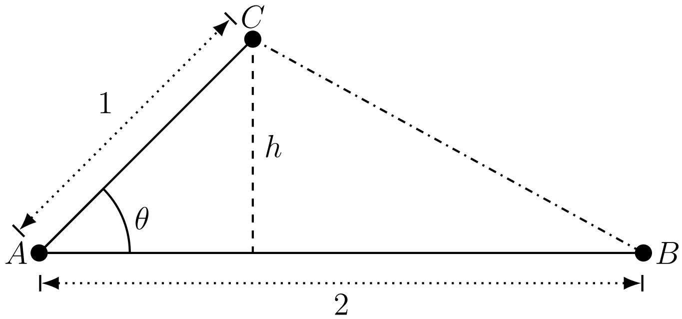
Note that when $x = 1$ and $y = 2$, the above equation gives that $D^2 = 1^2 + 2^2 = 5$; hence, at this moment $D = \sqrt{5}$. Plugging in all these above, we see

$$\sqrt{5}\left(\frac{\mathrm{d}D}{\mathrm{d}t}\right) = 1 \times 5 + 2 \times 7 = 19.$$

Hence,

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{19}{\sqrt{5}} \text{ kph.}$$ ■

Problem 3.9 (#EuclidTime!). A triangle has one side that is 1 cm long, and another side that is 2 cm, and the third side is formed by an elastic band that can shrink and stretch. The two fixed sides are rotated so that the angle they form, θ, grows by 1.5 radians each second. Find the rate of change of the area inside the triangle when $\theta = \pi/4$.



Solution. Consider the above figure, where the side BC represents the elastic band. Suppose $|AC| = 1$ and $|AB| = 2$. We are given that $\mathrm{d}\theta/\,\mathrm{d}t = 1.5$, and we're asked to find the rate of change of the Area when $\theta = \pi/4$.

Observe that

$$\text{Area} = \frac{1}{2}|AB|\,h = \frac{1}{2}|AB||AC|\sin\theta.$$

So,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\text{Area}) = \frac{1}{2}|AB||AC|\,(\cos\theta)\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right),$$

since $|AB|$ and $|AC|$ are fixed. This results in, when $\theta = \pi/4$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\text{Area}) = \frac{1}{2} \times 2 \times 1 \times \cos\left(\frac{\pi}{4}\right) \times 1.5 = \frac{1}{\sqrt{2}} \times \frac{3}{2} = \frac{3}{2\sqrt{2}}.$$

■