SOLUTIONS TO PRACTICE PROBLEMS 1

(By A. Alperen BULUT)

1. a. These are all "nice" functions we know, and $x = 3$ is in the domain of the given function of which we want to take the limit.

(No denominator is 0, there is no square root of a negative number, inside of ln is positive ...)

So, $$\lim_{x \to 3} \frac{e^{x-3} + \sin(x) + x}{x^5 + \ln(2x) + x^x} = \frac{e^{3-3} + \sin(3) + 3}{3^5 + \ln(2.3) + 3^3}$$

$$= \frac{e^0 + \sin(3) + 3}{3^5 + \ln(6) + 3^3}$$

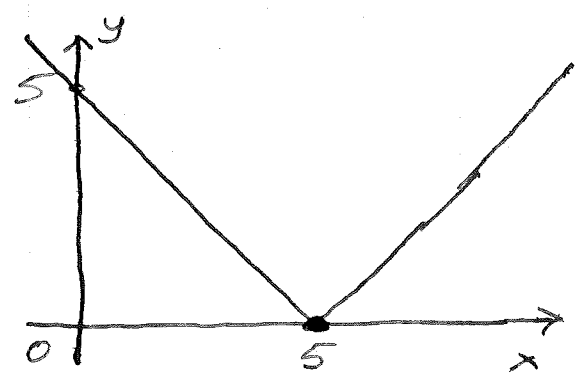
$$= \frac{1 + \sin(3) + 3}{3^5 + \ln(6) + 3^3}.$$

b. Again, no problem at all, so

$$\lim_{x \to 5} |x-5| = |5-5|$$

$$= |0|$$

$$= 0.$$



Graph of $y = |x-5|$

c. When $x=0$, $\frac{|x|}{x}$ looks like $\frac{0}{0}$. OOooops! Needs more work!

We have,

$$|x| = \begin{cases} -x, & \text{if } x<0; \\ x, & \text{if } x\geqslant 0; \end{cases}$$

so,

$$\frac{|x|}{x} = \begin{cases} -x/x, & \text{if } x<0; \\ \text{undefined}, & \text{if } x=0; \\ x/x, & \text{if } x>0; \end{cases}$$

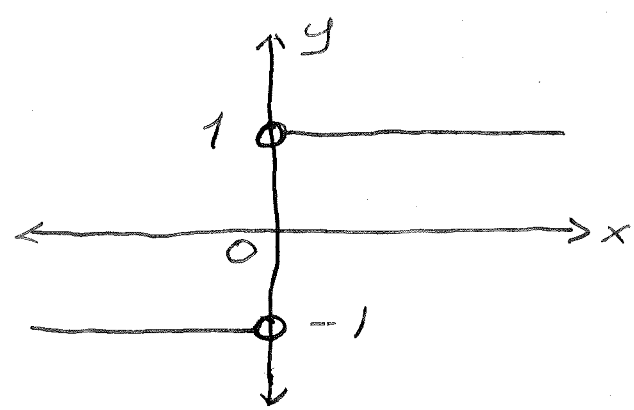
$$= \begin{cases} -1, & \text{if } x<0; \\ \text{undefined}, & \text{if } x=0; \\ 1, & \text{if } x>0. \end{cases}$$

So,

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} 1 = 1,$$

and

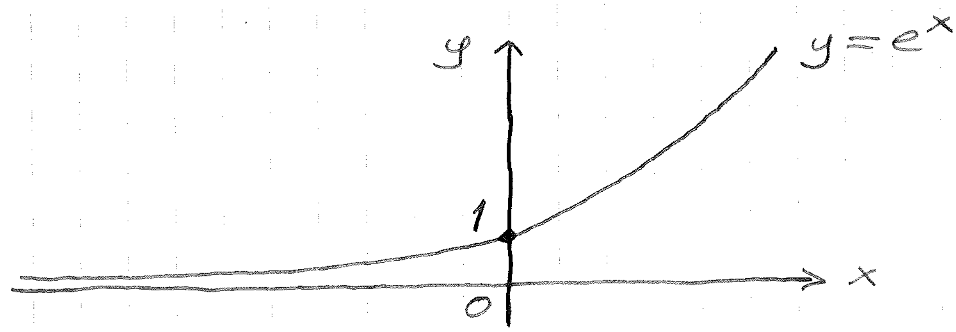
$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} (-1) = -1.$$



Graph of $\frac{|x|}{x}$.

Since left limit $\neq$ righ limit, $\lim_{x\to 0} \frac{|x|}{x}$ DNE.

d. & e. First let us look at the graph of $y=e^x$, an easier one.



There is a problem since the denominator of $1/x$ becomes 0 as x goes to 0.

Let's see what happens to $e^{1/x}$ as we approach 0 from the right. Say, $x = +\text{small}$. Then,

$$e^{1/+\text{small}} = e^{+\text{big}} = +\text{big}.$$

So,

$$\lim_{x \to 0^+} e^{1/x} = +\infty.$$

Similarly, if we approach 0 from the left by saying $x = -\text{small}$; then,

$$e^{1/-\text{small}} = e^{-\text{big}} \longrightarrow 0$$ (See graph.)

So,

$$\lim_{x \to 0^-} e^{1/x} = 0.$$

f. Again, a problem since denominator becomes zero.

When x approaches -2 from the right, numerator goes to

$$|3 \cdot (-2)| = |-6| = 6,$$

something positive and doesn't come anyway near zero.

However, as x approaches -2 from the right, say $x = -2 + \text{small}$; then denominator becomes

$$x + 2 = -2 + \text{small} + 2 = + \text{small}.$$

So,

$$\frac{|3x|}{x+2} \quad \text{approaches} \quad \frac{6}{+\text{small}} = + \text{big}$$

So,

$$\lim_{x \to -2^+} \frac{|3x|}{x+2} = +\infty.$$

g. Similarly, the top part is going to

$$\sqrt{16+0}-2 = \sqrt{16}-2 = 4-2 = 2.$$

This is positive, and doesn't approach zero.

When x approaches 0 from the right, say $x = +\text{small}$,

$$\frac{\sqrt{16+x}-2}{x} \text{ approaches } \frac{2}{+\text{small}} = +\text{big}.$$

So,

$$\lim_{x \to 0^+} \frac{\sqrt{16+x}-2}{x} = +\infty.$$

Similarly, when x approaches 0 from the left, say $x = -\text{small}$,

$$\frac{\sqrt{16+x}-2}{x} \text{ approaches } \frac{2}{-\text{small}} = -\text{big}.$$

Hence,

$$\lim_{x \to 0^-} \frac{\sqrt{16+x}-2}{x} = -\infty.$$

Finally, since left limit $\neq$ right limit, we conclude

$$\lim_{x \to 0} \frac{\sqrt{16+x}-2}{x} \quad \text{DNE}.$$

h. As x approaches 0, this time, top part becomes

$$\sqrt{16+0} - 4 = \sqrt{16} - 4 = 4 - 4 = 0 .$$

Similarly the bottom part becomes 0 as well.

We have a "zero-over-zero" type limit. Trouble! We want cancellation. So, we try this:

$$\frac{\sqrt{16+x} - 4}{x} = \frac{(\sqrt{16+x} - 4)}{x} \cdot \frac{(\sqrt{16+x} + 4)}{(\sqrt{16+x} + 4)}$$

$$= \frac{(\sqrt{16+x})^2 - (4)^2}{x\ (\sqrt{16+x} + 4)}$$

$$= \frac{(16+x) - 16}{x\ (\sqrt{16+x} + 4)} = \frac{x}{x\ (\sqrt{16+x} + 4)}$$

So,

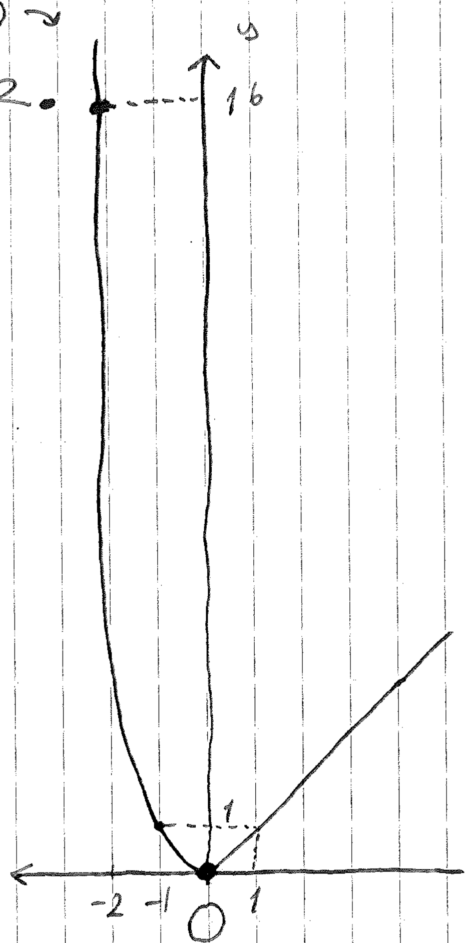
$$\lim_{x \to 0} \frac{\sqrt{16+x} - 4}{x} = \lim_{x \to 0} \frac{x}{x\ (\sqrt{16+x} + 4)}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{16+x} + 4}$$

$$= \frac{1}{\sqrt{16+0} + 4} = \frac{1}{\sqrt{16} + 4} = \frac{1}{4+4} = \frac{1}{8} .$$

No problem now ☺

Wow!



$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^4$$
$$= 0^4 = 0.$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x$$
$$= 0.$$

Left limit = Right limit. So, the limit exists, and

$$\lim_{x \to 0} f(x) = 0.$$

3. Let's not bother with the graph.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x-2) = 2-2 = 0.$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \left(\sin(\pi x) + 3\right)$$
$$= \sin(2\pi) + 3 = 0 + 3 = 3.$$

Left limit $\neq$ Right limit. Hence,

$$\lim_{x \to 2} f(x) \quad \text{DNE}.$$

4. We have

$$\underbrace{8x}_{\text{Call this } g(x)} \leq f(x) \leq \underbrace{x^2+16}_{\text{Call this } h(x).}.$$

$$\lim_{x \to 4} g(x) = \lim_{x \to 4} (8x) = 8 \cdot 4 = 32.$$

$$\lim_{x \to 4} h(x) = \lim_{x \to 4} (x^2+16) = 4^2+16 = 16+16 = 32.$$

So, by Squeeze Theorem, we have

$$\lim_{x \to 4} f(x) = 32.$$

5. a. If we try this limit as if everything is fine, we get

$$\lim_{x \to \infty} \left(\sqrt{x^2+1} - x \right) = \lim_{x \to \infty} \sqrt{x^2+1} - \lim_{x \to \infty} x$$

$$= \infty - \infty .$$

THIS ABSOLUTELY DOES NOT MAKE SENSE!!! We <u>cannot</u> do algebra with infinities! Algebra — addition, multiplication, division, etc. — are <u>only</u> for numbers, and infinity is NOT a number. So, this needs more work. We want cancellation!

We want to get rid of square root!

How about if we try the same trick we saw? Multiply both the numerator and the denominator with the same thing, but with a "+" sign? Then we get,

$$\lim_{x \to \infty} \left(\sqrt{x^2+1} - x \right) = \lim_{x \to \infty} \frac{\sqrt{x^2+1} - x}{1}$$

$$= \lim_{x \to \infty} \left[\frac{(\sqrt{x^2+1} - x)}{1} \cdot \frac{(\sqrt{x^2+1} + x)}{(\sqrt{x^2+1} + x)} \right]$$

$$= \lim_{x \to \infty} \left[\frac{(\sqrt{x^2+1})^2 - (x)^2}{\sqrt{x^2+1} + x} \right]$$

Note:

$$(a-b)(a+b)$$
$$= a(a+b) - b(a+b)$$
$$= a^2 + ab - ba - b^2$$
$$= a^2 - b^2$$

$$= \lim_{x \to \infty} \left(\frac{x^2 + 1 - x^2}{\sqrt{x^2+1} + x} \right)$$

$$= \lim_{x \to \infty} \frac{1}{\sqrt{x^2+1} + x}$$

$$= 0.$$

(This is because the denominator gets big, big, big as x goes to higher and higher values.)

b. Again, this <u>WRONG</u> calculation gives

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}\left(\sqrt{x^2+x}\right)-\lim_{x\to\infty}x$$

$$=\infty-\infty,$$

which is an absolute no-no!

Try the same trick as part (a) and get

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to\infty}\frac{\sqrt{x^2+x}-x}{1}$$

$$=\lim_{x\to\infty}\left[\frac{(\sqrt{x^2+x}-x)}{1}\cdot\frac{(\sqrt{x^2+x}+x)}{(\sqrt{x^2+x}+x)}\right]$$

$$=\lim_{x\to\infty}\left[\frac{(\sqrt{x^2+x})^2-(x)^2}{\sqrt{x^2+x}+x}\right]$$

$$=\lim_{x\to\infty}\frac{x^2+x-x^2}{\sqrt{x^2+x}+x}=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+x}+x}.$$

($\lim_{x\to\infty}x=\infty$, and $\lim_{x\to\infty}(\sqrt{x^2+x}+x)=\infty$; this gives $\frac{\infty}{\infty}$. Again, this doesn't make sense! So we need to work more. Let's do the highest power trick.)

$$=\lim_{x\to\infty}\frac{x}{(\sqrt{x^2}\sqrt{1+x/x^2}+x)}=\lim_{x\to\infty}\frac{x}{|x|\sqrt{1+1/x}+x}$$

$$=\lim_{x\to\infty}\frac{x}{x\sqrt{1+1/x}+x}=\lim_{x\to\infty}\frac{x}{x(\sqrt{1+1/x}+1)}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x}+1} = \frac{\lim_{x\to\infty} 1}{\lim_{x\to\infty}\left(\sqrt{1+1/x}+1\right)}$$

$$= \frac{1}{\lim_{x\to\infty}\sqrt{1+1/x} + \lim_{x\to\infty} 1} = \frac{1}{\sqrt{\lim_{x\to\infty}(1+1/x)} + 1}$$

$$= \frac{1}{\sqrt{\lim_{x\to\infty} 1 + \lim_{x\to\infty} 1/x} + 1} = \frac{1}{\sqrt{1+0}+1}$$

$$= \frac{1}{\sqrt{1}+1} = \frac{1}{1+1} = \frac{1}{2}.$$

(Compare this with part (a).)

6. We did a really similar example in class. Let's use the same method.

$$\lim_{x\to\infty}(3x+1) = \infty, \quad \text{and} \quad \lim_{x\to\infty}\sqrt{4x^2-3x-7}$$

$$= \lim_{x\to\infty}\sqrt{x^2(4-3/x-7/x^2)}$$

$$= \lim_{x\to\infty}\left(\sqrt{x^2}\,\sqrt{4-3/x-7/x^2}\right)$$

$$= \lim_{x\to\infty}\left(|x|\,\sqrt{4-3/x-7/x^2}\right)$$

$$\underbrace{= \lim_{x\to\infty} \underbrace{x}_{\substack{\text{This, itself,}\\ \text{approaches to}\\ \infty}} \underbrace{\sqrt{4-3/x-7/x^2}}_{\substack{\text{This, by itself,}\\ \text{approaches to}\\ \sqrt{4-0-0}=\sqrt{4}=2.}}}_{\text{So, this limit is } \infty.}$$

We, again, got ∞/∞. Let's use the highest power trick again. On the top, the highest power term is $x = x^1$. On the bottom, it is x^2. (We'll worry about the square root later.)

Now we get,

$$\lim_{x\to\infty}\frac{3x+1}{\sqrt{4x^2-3x-7}} = \lim_{x\to\infty}\frac{x\,(3+1/x)}{\sqrt{x^2(4-3/x-7/x^2)}}$$

$$= \lim_{x \to \infty} \frac{x\,(3 + 1/x)}{\sqrt{x^2}\sqrt{4 - 3/x - 7/x^2}}$$

$$= \lim_{x \to \infty} \frac{x\,(3 + 1/x)}{|x|\sqrt{4 - 3/x - 7/x^2}} \quad \ldots\ldots\ldots \quad (*)$$

$$= \lim_{x \to \infty} \frac{x\,(3 + 1/x)}{x\sqrt{4 - 3/x - 7/x^2}}$$

(We're approaching $+\infty$, so x is eventually positive; hence $|x| = x$)

$$= \lim_{x \to \infty} \frac{3 + 1/x}{\sqrt{4 - 3/x - 7/x^2}}$$

$$= \frac{\lim_{x \to \infty}(3 + 1/x)}{\lim_{x \to \infty}\sqrt{4 - 3/x - 7/x^2}} = \frac{\lim_{x \to \infty} 3 + \lim_{x \to \infty} 1/x}{\sqrt{\lim_{x \to \infty}(4 - 3/x - 7/x^2)}}$$

$$= \frac{3 + 0}{\sqrt{\lim_{x \to \infty} 4 - \lim_{x \to \infty}\frac{3}{x} - \lim_{x \to \infty}\frac{7}{x^2}}} = \frac{3}{\sqrt{4 - 0 - 0}}$$

$$= \frac{3}{\sqrt{4}} = \frac{3}{2}.$$

7. We observe the same at first and do similar (nearly exact) things again, but this time, since x is approaching (going to) $-\infty$, it becomes negative. Hence, in the line marked with $(*)$ above $|x|$ will give us $-x$. Hence,

$$\lim_{x \to -\infty} \frac{3x+1}{\sqrt{4x^2-3x-7}} = \ldots \text{ same, same, same } \ldots$$

$$= \lim_{x \to -\infty} \frac{x\,(3+1/x)}{|x|\sqrt{4-3/x-7/x^2}}$$

$$= \lim_{x \to -\infty} \frac{x\,(3+1/x)}{(-x)\sqrt{4-3/x-7/x^2}}$$

(As x goes to $-\infty$, x becomes negative, so $|x| = -x$.)

$$= \lim_{x \to -\infty} \frac{-(3+1/x)}{\sqrt{4-3/x-7/x^2}}$$

$$= \frac{\lim_{x \to -\infty} [-(3+1/x)]}{\lim_{x \to -\infty}\sqrt{4-3/x-7/x^2}} = \frac{-\lim_{x \to -\infty}(3+1/x)}{\sqrt{\lim_{x \to -\infty}(4-3/x-7/x^2)}}$$

$$= \frac{-\left(\lim_{x \to -\infty} 3 + \lim_{x \to -\infty} 1/x\right)}{\sqrt{\lim_{x \to -\infty} 4 - \lim_{x \to -\infty} 3/x - \lim_{x \to -\infty} 7/x^2}} = \frac{-(3+0)}{\sqrt{4-0-0}}$$

$$= \frac{-3}{\sqrt{4}} = \frac{-3}{2}.$$

For mistakes, typos, errors, questions, and comments please email aabulut@math.ubc.ca

— AAB