MATH 100 Section 108 – 2019W
Solutions to Practice Problems 2 v2.0

A. Alperen Bulut

October 11, 2019

Problem 1. Where are the following functions continuous?

(a) **(Final 2016)** $f(x) = \dfrac{\sin(\pi x/2)}{\sqrt{1-x^2}}$

(b) **(Final 2015)** $f(x) = \dfrac{x^2-1}{\sqrt{x^2-x-6}}$

(c) **(I've literally just made this up. —AAB)** $f(x) = \dfrac{e^{\tan(x)}}{2+\cos(\sqrt{x})}$

(d) **(Hehehe. —AAB)** $f(x) = \dfrac{1}{\sqrt{1-[\tan(x)]^2}}$

Solution. (a) The functions $\sin(x)$ and $\pi x/2$ are continuous everywhere, so the compound function $\sin(\pi x/2)$ is continuous everywhere as well. The only problems that can occur is that the denominator, $\sqrt{1-x^2}$, may be equal to 0, or that the inside of the square root might be negative. We don't want these. So, we want that $1-x^2>0$, i.e., we want $x^2<1$. This happens in the open interval

$$(-1,1),$$

and this is where the function is continuous.

(b) Similar to part (a), the only problem that might occur is that the denominator might be 0, or that the inside of the square root might be negative, which we do not want. So, combining, we need $x^2-x-6=(x-3)(x-2)>0$. This is satisfied if and only if $x<2$ or $x>3$. So, the function is continuous in

$$(-\infty,2)\cup(3,\infty).$$

(c) This is more complicated. We first see that there is $\tan(x)$ in the expression, and $\tan(x)$ is continuous everywhere except the odd multiples of $\pi/2$, i.e., everywhere except $x=\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$. (See Figure 1.) We also see $\sqrt{x}$ in the expression, so we need $x \geq 0$. Note that $-1 \leq \cos(\sqrt{x}) \leq 1$, so we have $1 \leq 2+\cos(\sqrt{x}) \leq 3$; hence, the denominator is never 0. So, combining what we found, we conclude that the function is continuous in

$$[0, \infty) \text{ except } x = \pi/2, 3\pi/2, 5\pi/2, \ldots.$$

In another words, it's continuous in

$$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right) \cup \ldots.$$

(d) Similar to the reasoning explained above, we need that $1 - \tan^2(x) > 0$. (Inside of the square root must be nonnegative, and the denominator cannot be 0.) So, we need $\tan^2(x) < 1$, i.e., $-1 < \tan(x) < 1$. This happens when x is in

$$\ldots \cup \left(-\frac{9\pi}{4}, -\frac{7\pi}{4}\right) \cup \left(-\frac{5\pi}{4}, -\frac{3\pi}{4}\right) \cup \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{7\pi}{4}, \frac{9\pi}{4}\right) \cup \ldots,$$

which is where this function is continuous. (See Figure 1.) ■

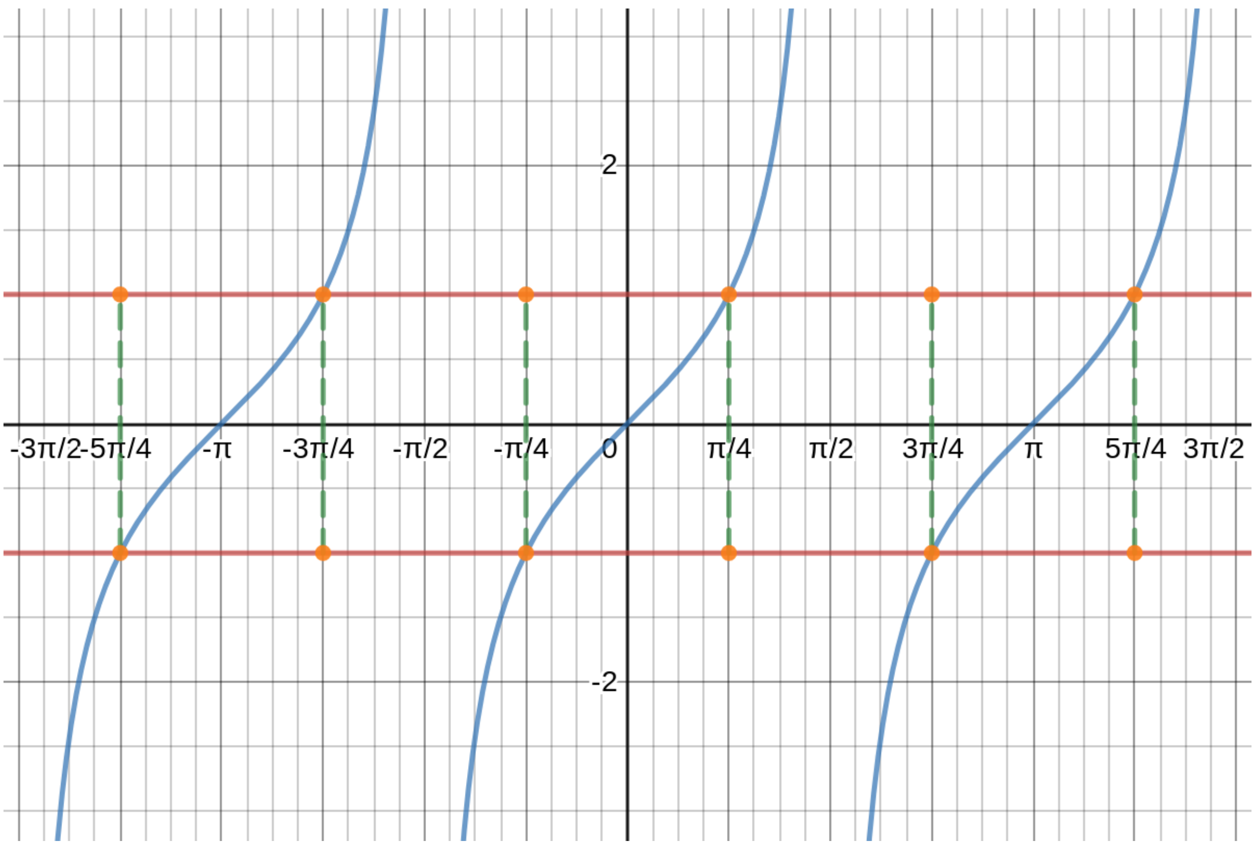

Figure 1: The plot of $y = \tan(x)$ and where $-1 \leq \tan(x) \leq 1$

Problem 2 (Final 2012). Let c be a constant, and define

$$f(x) = \begin{cases} x^2 + c, & \text{if } x \leq 1; \\ 2x - 3, & \text{if } x > 1. \end{cases}$$

Find the value of c for which $f(x)$ is continuous everywhere.

Solution. First we see that, whatever the value of c, the function is continuous when either $x < 1$ or when $x > 1$. The only possible problem is when $x = 1$. To have continuity at $x = 1$, by definition, we need

$$\lim_{x\to 1} f(x) = f(1),$$

where $f(1) = 1^2 + c = 1 + c$.

We need to find out what $\lim_{x\to 1} f(x)$ is. We see

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \left(x^2 + c\right) = 1 + c,$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x - 3) = 2 - 3 = -1.$$

So, we need

$$1 + c = -1;$$

hence, $c = -2$. ■

Problem 3 (Final 2016). Let $f(x) = \dfrac{x}{x-2}$. Compute $\dfrac{\mathrm{d}f}{\mathrm{d}x}$ using the definition of the derivative.

Solution. Using the definition of the derivative, we get

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \left(\frac{f(x+h) - f(x)}{h}\right) = \lim_{h\to 0} \frac{\left[\dfrac{(x+h)}{(x+h)-2}\right] - \left[\dfrac{x}{x-2}\right]}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{(x+h)(x-2)}{(x+h-2)(x-2)} - \dfrac{x(x+h-2)}{(x-2)(x+h-2)}}{h} \\
&= \lim_{h\to 0} \frac{\left(x^2 - 2x + hx - 2h\right) - \left(x^2 + xh - 2x\right)}{h(x-2)(x+h-2)} \\
&= \lim_{h\to 0} \frac{-2h}{h(x-2)(x+h-2)} \\
&= -\lim_{h\to 0} \frac{2}{(x-2)(x+h-2)} \\
&= -\frac{2}{(x-2)(x+0-2)} \\
&= -\frac{2}{(x-2)^2}.
\end{aligned}$$

(Check that this gives the same result if we used the quotient rule.) ■

Problem 4. Using the definition of the derivative show that $f(x) = x|x|$ is differentiable at $x = 0$.

Solution. We need to see whether the limit

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

exists. Observe that this limit is equal to

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{h|h|-0}{h}=\lim_{h\to 0}\frac{h|h|}{h}=\lim_{h\to 0}|h|\,,$$

which exists and equals to 0. So, $f(x)=x|x|$ is differentiable at 0, and the derivative is $f'(0)=0$. ∎

Problem 5. Using the definition of the derivative explain why $f(x)=\sqrt[3]{x}$ is <u>not</u> differentiable $x=0$.
Hint: For any pair of real numbers a and b, we have the equality

$$a^3-b^3=(a-b)(a^2+ab+b^2).$$

Solution. As we did in the previous problem, we compute the limit

$$\begin{aligned}\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}&=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{h^{1/3}-0}{h}\\&=\lim_{h\to 0}\frac{h^{1/3}}{h}=\lim_{h\to 0}\frac{1}{h^{2/3}}=+\infty.\end{aligned}$$

This is because as h approaches to 0 (either from the negative side or the positive side), $1/h^{2/3}$ gets larger and larger, and is always positive. Since this "$+\infty$" is not an actual, honest number, the function is not differentiable at 0.

Note that the geometric interpretation of this is that the tangent line at 0 is vertical. We would like to avoid mentioning the "slope" of vertical lines. That does not make sense—kind of. (Check out the graph of $y=\sqrt[3]{x}=x^{1/3}$, and see with you own eyes what the tangent line to the function at 0 is.) ∎

Problem 6 (Final 2015). Is the function

$$f(x)=\begin{cases}\sqrt{1+x^2}-1, & \text{if } x\le 0;\\ x^2\cos(1/x), & \text{if } x>0;\end{cases}$$

differentiable at $x=0$? You must explain your answer using the definition of the derivative.

Solution. To have the function differentiable at 0 we need the limit

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{f(h)-0}{h}=\lim_{h\to 0}\frac{f(h)}{h}$$

to exist. If both the left-limit and the right-limit exist and equal to each other, then we can conclude that the above limit exists as well.

We have

$$\begin{aligned}
\lim_{h\to 0^-}\frac{f(h)}{h} &= \lim_{h\to 0^-}\frac{\sqrt{1+h^2}-1}{h} = \lim_{h\to 0^-}\frac{\left(\sqrt{1+h^2}-1\right)\left(\sqrt{1+h^2}+1\right)}{h\left(\sqrt{1+h^2}+1\right)} \\
&= \lim_{h\to 0^-}\frac{\left(\sqrt{1+h^2}\right)^2-1^2}{h\left(\sqrt{1+h^2}+1\right)} = \lim_{h\to 0^-}\frac{1+h^2-1}{h\left(\sqrt{1+h^2}+1\right)} \\
&= \lim_{h\to 0^-}\frac{h^2}{h\left(\sqrt{1+h^2}+1\right)} = \lim_{h\to 0^-}\frac{h}{\sqrt{1+h^2}+1} \\
&= \frac{0}{\sqrt{1+0^2}+1} = 0.
\end{aligned}$$

On the other hand, right-sided limit is

$$\lim_{h\to 0^+}\frac{f(h)}{h} = \lim_{h\to 0^+}\frac{h^2\cos(1/h)}{h} = \lim_{h\to 0^+}\left[h\cos(1/h)\right],$$

which we cannot evaluate immediately. However, since

$$-1 \le \cos(1/h) \le 1,$$

we also have

$$-h \le h\cos(1/h) \le h.$$

Note that we don't need to reverse the inequality as we are dealing with only positive values of h since we're after the limit as $h \to 0^+$. But $\lim_{h\to 0^+}(-h) = 0$ and $\lim_{h\to 0^+} h = 0$, so by the squeeze theorem

$$\lim_{h\to 0^+}\left[h\cos(1/h)\right] = 0.$$

Since the left-limit and the right-limit exists and equal to each other (and to 0), we have

$$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = 0.$$

This shows that the function $f(x)$ is differentiable at 0, and the derivative $f'(0)$ is equal to 0. ■

Problem 7. Find the values of the constants a and b such that the function

$$f(x) = \begin{cases} ae^x + \sin(x), & \text{if } x \le 0; \\ x^2 + 2x + b, & \text{if } x > 0; \end{cases}$$

is differentiable everywhere.

Solution. Observe that when either $x < 0$ or $x > 0$, the function is differentiable. The only problem that may occur is at $x = 0$.

First, note that if a function is differentiable at a point, it must be continuous at that point. Hence, we must have

$$\begin{aligned}
\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^+} f(x); \\
\lim_{x\to 0^-} \left[ae^x + \sin(x)\right] &= \lim_{x\to 0^+} \left[x^2 + 2x + b\right]; \\
ae^0 + \sin(0) &= 0^2 + 0 + b; \\
a &= b. \qquad (1)
\end{aligned}$$

On the other hand, we must have the limit

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - f(0)}{h}$$

to exist. Observe that the left-limit

$$\begin{aligned}
\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} &= \lim_{h\to 0^-} \frac{\left[ae^h + \sin(h)\right] - a}{h} = \frac{\mathrm{d}}{\mathrm{d}x}\left[ae^x + \sin(x)\right]\Big|_{x=0} \\
&= \left[ae^x + \cos(x)\right]\Big|_{x=0} = ae^0 + \cos(0) \\
&= a + 1.
\end{aligned}$$

Similarly, the right-limit

$$\begin{aligned}
\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} &= \lim_{h\to 0^+} \frac{\left[h^2 + 2h + b\right] - a}{h} = \lim_{h\to 0^+} \frac{\left[h^2 + 2h + b\right] - b}{h} \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 + 2x + b\right]\Big|_{x=0} = \left[2x + 2\right]\big|_{x=0} \\
&= 2.
\end{aligned}$$

Note that we could change a to b above due to Equation (1). So, now, we get $a + 1 = 2$. Combining this with the Equation (1), we conclude that

$$a = b = 1.$$

■

Problem 8. Differentiate the following functions.

(a) **(Final 2016)** $f(x) = x^2 e^x$

(b) **(Final 2016)** $f(x) = \dfrac{x^2 + 3}{2x - 1}$

(c) **(Final 2015)** $f(x) = \dfrac{x^2}{x + 1}$

(d) **(Final 2014)** $f(x) = e^x \cdot \cos(\pi x)$

Solution. (a) By the product rule

$$f'(x) = (x^2 e^x)' = (x^2)'(e^x) + (x^2)(e^x)' = (2x)e^x + (x^2)e^x = (x^2 + 2x)e^x.$$

(b) By the quotient rule

$$\begin{aligned} f'(x) &= \frac{(x^2+3)'(2x-1)-(x^2+3)(2x-1)'}{(2x-1)^2} = \frac{(2x)(2x-1)-(x^2+3)(2)}{(2x-1)^2} \\ &= \frac{(4x^2-2x)-(2x^2+6)}{(2x-1)^2} = \frac{4x^2-2x-2x^2-6}{(2x-1)^2} \\ &= \frac{2x^2-2x-6}{(2x-1)^2}. \end{aligned}$$

(c) By the quotient rule

$$\begin{aligned} f'(x) &= \frac{(x^2)'(x+1)-(x^2)(x+1)'}{(x+1)^2} = \frac{(2x)(x+1)-(x^2)(1)}{(x+1)^2} \\ &= \frac{(2x^2+2x)-(x^2)}{(x+1)^2} \\ &= \frac{x^2+2x}{(x+1)^2}. \end{aligned}$$

(d) By the product rule and the chain rule

$$\begin{aligned} f'(x) &= (e^x\cos(\pi x))' = (e^x)'\cos(\pi x) + (e^x)\left[\cos(\pi x)\right]' \\ &= e^x\cos(\pi x) + e^x\left[(-\sin(\pi x))\,(\pi x)'\right] = e^x\cos(\pi x) + e^x\left[(-\sin(\pi x))\,(\pi)\right] \\ &= e^x\left[\cos(\pi x) - \pi\sin(\pi x)\right]. \end{aligned}$$

■

Problem 9 (Final 2015). What is the equation of the tangent line to $f(x) = \sqrt{x}$ at the point $(4, 2)$?

Solution. The derivative of the function is $f'(x) = 1/(2\sqrt{x})$. So, the slope of the tangent line to the function at $x = 4$ is $f'(4) = 1/(2\sqrt{4}) = 1/4$. Hence, by the point-slope formula,

$$y - f(a) = f'(a)(x - a),$$

the equation of the tangent line at $x = a = 4$ is

$$y - 2 = (1/4)(x - 4).$$

■

Problem 10 (Final 2011). Find an equation of the tangent line to the curve $y = x^{3.5} - e^{3.5}$ at the point $(e, 0)$.

Solution. The derivative is $f'(x) = 3.5x^{2.5}$, which is at $x = e$ is equal to $f'(e) = 3.5e^{2.5}$. Hence, by the point-slope formula, the equation of the tangent line at $x = e$ is

$$y - 0 = 3.5e^{2.5}(x - e).$$

■

Problem 11 (Final 2014). Let $f(x)$ be a function differentiable at $x = 1$ and let $g(x) = f(x)/x^2$. The line tangent to the curve $y = f(x)$ at $x = 1$ has slope 3, while the line tangent to the curve $y = g(x)$ at $x = 1$ has slope 4. What is $f(1)$?

Solution. Since the slope of the tangent line to a function at a point is the value of the derivative of the function at that point, we see that $f'(1) = 3$, and that $g'(1) = 4$. But by the quotient rule we have

$$g'(x) = \frac{f'(x)(x^2) - f(x)(x^2)'}{(x^2)^2} = \frac{f'(x)(x^2) - f(x)(2x)}{(x^4}.$$

So,

$$4 = g'(1) = \frac{f'(1) - 2f(1)}{1} = 3 - 2f(1).$$

Thus, $f(1) = -1/2$. ■

Problem 12 (Final 2016). Let $f(x)$ be a continuous function defined for all real numbers x. Suppose $f(x)$ is increasing on the intervals $(-\infty, -1)$ and $(3, \infty)$, decreasing on $(-1, 3)$; $f(-1) = 2$, and $f(3) = 1$. How many zeros does $f(x)$ have?

(A) 0

(B) 1

(C) 2

(D) 3

(E) Cannot determine from the information given.

Solution. Suppose $f_1(x)$ and $f_2(x)$ are two functions whose graphs look like the ones given in Figure 2a and Figure 2b, respectively. Note that they both are continuous functions satisfying all the requirements given in the problem. However, while $f_1(x)$ has one zero, $f_2(x)$ does not have any zeros. Hence, we cannot determine the number of zeros of $f(x)$ from the information given. ■

For the curious reader, the analytic expressions of the functions used are given below.

$$f_1(x) = \begin{cases} x + 3, & \text{if } x \leq -1; \\ -\frac{1}{4}x + \frac{7}{4}, & \text{if } -1 < x < 3; \\ x - 2, & \text{if } x \geq 3; \end{cases}$$

and

$$f_2(x) = \begin{cases} e^{x+1} + 1, & \text{if } x \leq -1; \\ -\frac{1}{4}x + \frac{7}{4}, & \text{if } -1 < x < 3; \\ x - 2, & \text{if } x \geq 3. \end{cases}$$

Problem 13 (Final 2012). Suppose the tangent line to the curve $y = f(x)$ at $x = 1$ passes through the points $(-2, 3)$ and $(0, 5)$. Find $f(1)$ and $f'(1)$.

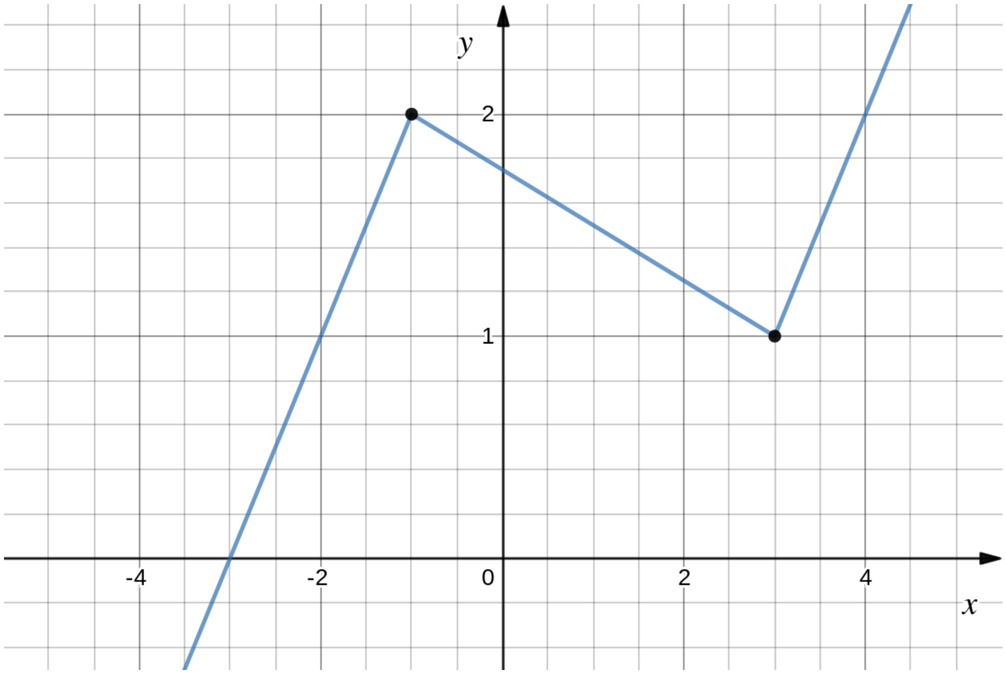

Figure 2a: The graph of $y = f_1(x)$

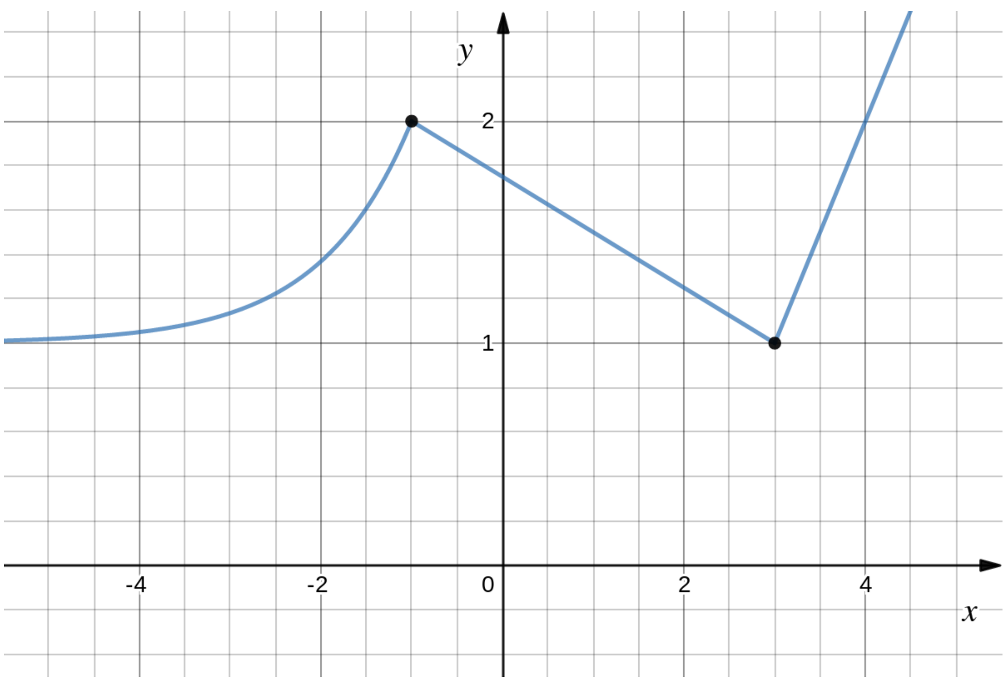

Figure 2b: The graph of $y = f_2(x)$

Solution. The line which passes from the points $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (0, 5)$ has slope

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5-3}{0-(-2)} = \frac{2}{2} = 1.$$

Hence, since $f'(1)$ is equal to the slope of the tangent line to the curve $y = f(x)$ at the point where $x = 1$, we get

$$f'(1) = 1.$$

Note that, then, the tangent line has the equation of the form $y = mx + b = x + b$, where b is a constant we do not know yet. However, the line passes from the point $(0, 5)$, which must satisfy the equation of the line. So, $5 = 0 + b = b$. This gives the equation of the tangent line as

$$y = x + 5.$$

(Check also that the point $(-2, 3)$ satisfies the equation of the line—just to make sure that we didn't make any mistakes.)

Now, the point $(1, f(1))$, too, is on the tangent line, so it satisfies the equation of the line. Thus,

$$f(1) = 1 + 5 = 6. \qquad \blacksquare$$

Problem 14 (Final 2012). Find a function $f(x)$ such that $f'(x) = x^3$ and such that the line $x + y = 0$ is tangent to the graph of $y = f(x)$.

Solution. The power rule says that $(x^4)' = 4x^3$. So, $(x^4/4)' = x^3$. However, the derivative of any constant c is 0. Hence, say, $f(x) = x^4/4 + c$. (Note that we haven't seen that these are all the functions whose derivative are x^3. Maybe there is another, weirder one, who knows? Well, actually, there isn't. These are all. We will see how to show that soon in the lecture.)

Now, since the slope of the given line, $y = -x + 0$, is -1, which is the derivative of the function at which point the line is tangent, say $x = a$, we must have $f'(a) = a^3 = -1$. So, $a = -1$. The point $(a, f(a)) = (-1, f(-1)) = (-1, 1/4 + c)$ must be on the tangent line, so it satisfies the equation of the line. Therefore, $-1 + 1/4 + c = 0$, i.e., $c = 3/4$; and the required function is

$$f(x) = \frac{x^4}{4} + \frac{3}{4}. \qquad \blacksquare$$

Problem 15 (Final 2016—Edited). Let $f(x)$ be a continuous function so that $|f(x) - \sin x| \leq 1/3$ for all x. Show that $f(x)$ has at least one zero in the open interval $(0, 2\pi)$.

Solution. Since $|f(x) - \sin x| \leq 1/3$, we have

$$-1/3 \leq f(x) - \sin(x) \leq 1/3,$$

or

$$-1/3 + \sin(x) \leq f(x) \leq 1/3 + \sin(x).$$

See Figure 3, and observe that the graph of the function must be in the shaded region. This suggests that the function must have a zero.

To show this rigorously, first observe that $f(\pi/2) \geq -1/3 + \sin(\pi/2) = -1/3 + 1 = 2/3 > 0$, and that $f(3\pi/2) \leq 1/3 + \sin(3\pi/2) = 1/3 - 1 = -2/3 < 0$. Note that the function is continuous in the **CLOSED** interval $[\pi/2, 3\pi/2]$. Hence, by the IVT, there exists a point c in the open interval $(\pi/2, 3\pi/2)$ such that $f(c) = 0$. $\blacksquare$

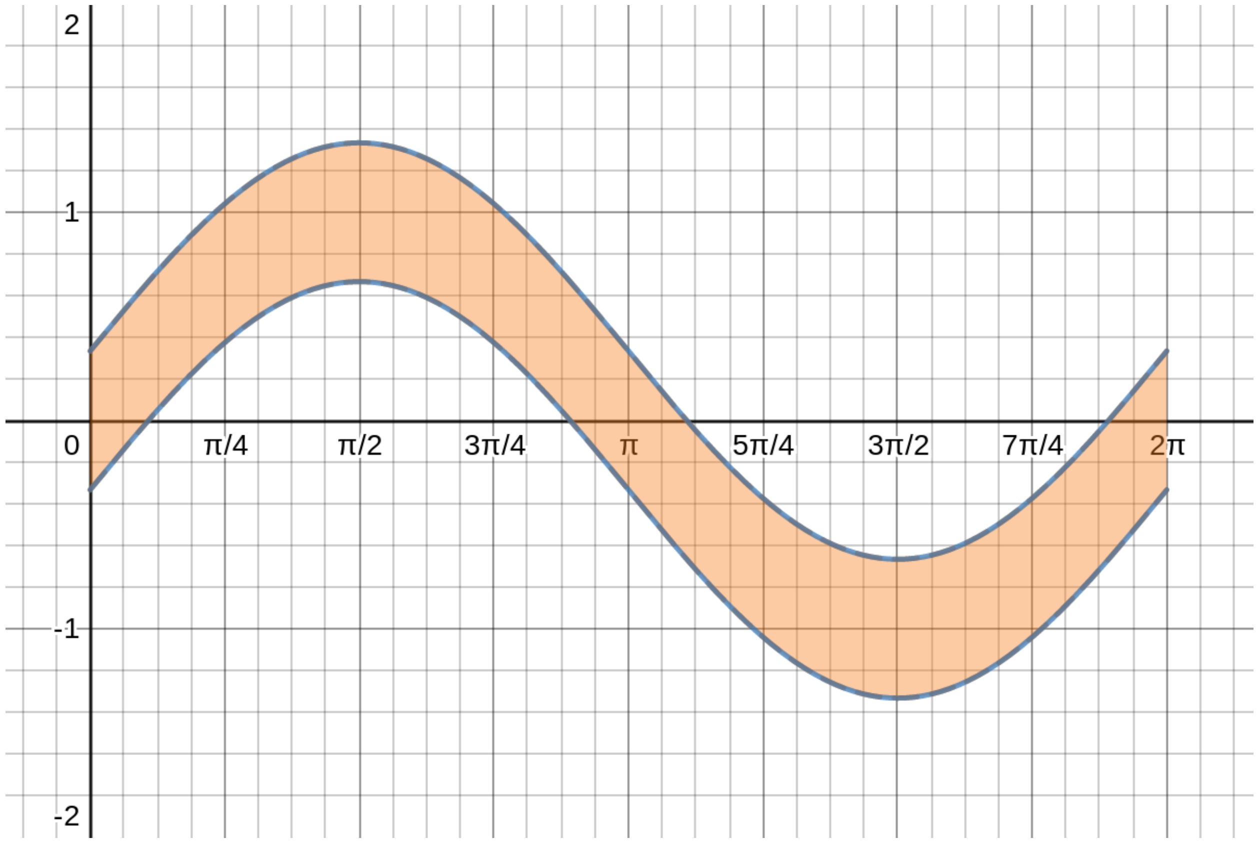

Figure 3: The region where $-1/3 + \sin(x) \leq y \leq 1/3 + \sin(x)$ and $0 \leq x \leq 2\pi$

Problem 16 (Final 2015—Edited). Show that the equation $2x^2 - 3 + \sin(x) + \cos(x) = 0$ has at least two solutions.

Solution. Let $f(x) = 2x^2 - 3 + \sin(x) + \cos(x) = 0$. Observe first that $f(0) = -3 + \sin(0) + \cos(0) = -3 + 0 + 1 = -2$.

Note also that $\sin(x)$ and $\cos(x)$ are both bounded between -1 and 1, so a cruel bound gives $-2 \leq \sin(x) + \cos(x) \leq 2$. This bounded term won't matter that much when x^2 gets larger and larger (either in the positive direction or the negative), and the result will be positive. For example,

$$f(2) = 5 + \sin(x) + \cos(x) \geq 5 - 2 = 3 > 0.$$
$$f(-2) = 5 + \sin(x) + \cos(x) \geq 5 - 2 = 3 > 0.$$

Now, the function is continuous on both of the **CLOSED** intervals $[-2, 0]$ and $[0, 2]$. Combining these with that $f(-2) > 0$, $f(0) < 0$, $f(2) > 0$, and using the IVT, we conclude that there exists c_1 in the open interval $(-2, 0)$ and c_2 in the open interval $(0, 2)$ such that $f(c_1) = f(c_2) = 0$. Note that $c_1 \neq c_2$, so these are two different points. ■

Problem 17 (Final 2011). If $y = f(x)$ is a continuous with domain $[0, 1]$ and range in $[3, 5]$. Show the line $y = 2x + 3$ intersects the graph of $y = f(x)$ at least once.
Note: Here, "with domain $[0, 1]$ and range in $[3, 5]$" means that for any $0 \leq x \leq 1$ we have $3 \leq f(x) \leq 5$. —AAB

Solution. Let $g(x) = f(x) - (2x+3) = f(x) - 2x - 3$, which is also continuous in $[0,1]$. Note that if we can show that $g(x)$ is 0 at some point, say c, this means that $f(c) = 2c+3$; hence, the function intersects the line at the point where $x = c$.

Observe that $g(0) = f(0) - 3 \geq 3 - 3 = 0$, and $g(1) = f(1) - 5 \leq 5 - 5 = 0$. Now, if either $g(0) = 0$ or $g(1) = 0$; then we're done since that is what we wanted to show. So, assume the contrary that $g(0) > 0$ and $g(1) < 0$. Since $g(x)$ is continuous in the **CLOSED** interval $[0,1]$, by the IVT, there exists a point, c, in the open interval $(0,1)$ such that $g(c) = 0$. ■

Problem 18. Find the mistake in the following **WRONG** reasoning that reaches to a conclusion which is **FALSE**.

I. $\tan(\pi/4) = 1 > 0$.

II. $\tan(3\pi/4) = -1 < 0$.

III. The function $f(x) = \tan(x)$ is continuous on the closed interval $[\pi/4, 3\pi/4]$.

IV. So, by the IVT, there exists a point c in $(\pi/4, 3\pi/4)$ such that $\tan(c) = 0$.

Note: The function $\tan(x)$ is **NEVER** 0 in the open interval $(\pi/4, 3\pi/4)$. I'm asking you what went wrong in the reasoning.
Hint: Have a look at the graph of $\tan(x)$.

Solution. The problem is at point (III). To use the IVT, we must make sure that the function is continuous in the related **CLOSED** interval—in this case $[\pi/4, 3\pi/4]$. However, $\tan(x)$ is not continuous in there since it's not even defined at $x = \pi/2$, around which it blows up. (See Figure 1.) ■

Problem 19 (Final 2012). Let $f(x) = g(2\sin x)$, where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'(\pi/4)$.

Solution. By the chain rule,

$$f'(x) = g'(2\sin x)(2\sin x)' = g'(2\sin x)(2\cos x).$$

So,

$$f'(\pi/4) = g'(2\sin(\pi/4))(2\cos(\pi/4)) = g'(2/\sqrt{2})(2/\sqrt{2}) = \sqrt{2}g'(\sqrt{2}) = \sqrt{2} \times \sqrt{2} = 2. \quad ■$$

Problem 20 (Final 2010. This problem is cool as f[Censored]! —AAB). Two points on the surface of the Earth are called *antipodal* if they are exactly at opposite points (with respect to the center of the Earth —AAB). For example, the North Pole and the South Pole are antipodal points. Prove that, at any given moment, there are two antipodal points on the equator with exactly the same temperature.
Hint: [Redacted. (Harm to Ongoing Matter.) I want to see if you can do it by yourselves. —AAB]

Solution. Let us parametrize the equator according to the longitudinal angle θ, measured in radians, and let $T(\theta)$ be the temperature at the point on the equator where the angle is θ. Note that if θ reaches or exceeds 2π we just wrap around the Earth. So we can assume that this function is defined for all real numbers. (Although, then, it will be 2π-periodic.) Note that the antipodal point to θ is $\theta + \pi$. (See Figure 4.) Let, also,

$$f(\theta) = T(\theta) - T(\theta + \pi),$$

i.e., the temperature difference between the point with angle θ and its antipodal point.

We must show that the function $f(\theta)$ attains the value 0 somewhere, say at θ_0. This will prove that $T(\theta_0) = T(\theta_0 + \pi)$, i.e., that the two antipodal points at θ_0 and $\theta_0 + \pi$ have the same temperature.

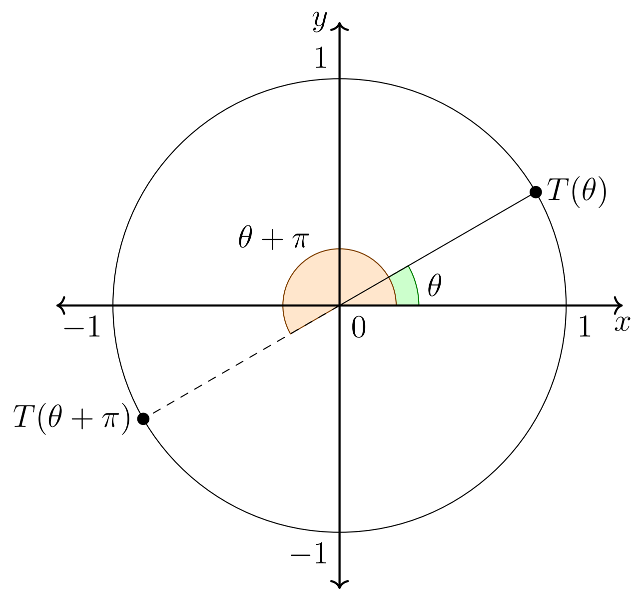

Figure 4: The equator, parametrized by the longitudinal angle θ

We assume that the temperature function $T(\theta)$ is continuous everywhere, which is not an unreasonable assumption, since on the surface of the Earth, the temperature does not jump immediately between two close points. So, $f(\theta)$ is continuous everywhere as well.

The function $f(\theta)$ can be zero everywhere, which is fine. Because we can pick any point $\theta_0 \in [0, 2\pi)$, and then we will have $f(\theta_0) = 0$, which is what we wanted to show.

If this is not the case, then there exists a point $\phi_0 \in [0, 2\pi)$ such that $f(\phi_0) \neq 0$. (Note that the negation of the statement "all my shirts are blue" is not that "none of my shirts are blue." The correct negation is "at least one of my shirts is not blue." That's why the negation of the statement "$f(\theta)$ is zero everywhere" is not "$f(\theta) \neq 0$ everywhere," but rather "there exists a point $\phi_0 \in [0, 2\pi)$ such that $f(\phi_0) \neq 0$.")

Now, there are two cases: either $f(\phi_0) > 0$, or $f(\phi_0) < 0$. Suppose the first one is true. Observe that, while the antipodal point to ϕ_0 is at $\phi_0 + \pi$, the antipodal point to $\phi_0 + \pi$ is $(\phi_0 + \pi) + \pi = \phi_0 + 2\pi$, which is nothing but the original point ϕ_0. We have

$$f(\phi_0) = T(\phi_0) - T(\phi_0 + \pi) > 0.$$

So,

$$\begin{aligned} f(\phi_0+\pi) &= T(\phi_0+\pi) - T\left((\phi_0+\pi)+\pi\right) = T(\phi_0+\pi) - T(\phi_0+2\pi) \\ &= T(\phi_0+\pi) - T(\phi_0) \\ &= -f(\phi_0) < 0. \end{aligned}$$

Therefore, since $f(\phi_0) > 0$, $f(\phi_0+\pi) < 0$, and the function $f(\theta)$ is continuous in the **CLOSED** interval $[\phi_0, \phi_0+\pi]$, we conclude, by the IVT, that there exists a point $\theta_0 \in (\phi_0, \phi_0+\pi)$ such that

$$f(\theta_0) = 0,$$

which is what we wanted to show.

The case when $f(\phi_0) < 0$ is almost exactly the same. ■

Note: The hint given in the original exam was "Let $T(\theta)$ be the temperature, at any given moment, at the point on the equator with longitudinal angle θ measured in radians, $0 \leq \theta \leq 2\pi$ (i.e. in one complete trip around the equator, θ goes from 0 to 2π), and consider $f(\theta) = T(\theta) - T(\theta+\pi)$."

(For mistakes, errors, typos, comments, and questions please email.)
E-mail address: aabulut@math.ubc.ca
Webpage: https://www.math.ubc.ca/∼aabulut/