

EQUIVARIANT K-THEORY OF COMPACT LIE GROUP ACTIONS WITH MAXIMAL RANK ISOTROPY

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ABSTRACT. Let G denote a compact connected Lie group with torsion-free fundamental group acting on a compact space X such that all the isotropy subgroups are connected subgroups of maximal rank. Let $T \subset G$ be a maximal torus with Weyl group W . If the fixed-point set X^T has the homotopy type of a finite W -CW complex, we prove that the rationalized complex equivariant K -theory of X is a free module over the representation ring of G . Given additional conditions on the W -action on the fixed-point set X^T we show that the equivariant K -theory of X is free over $R(G)$. We use this to provide computations for a number of examples, including the ordered n -tuples of commuting elements in G with the conjugation action.

1. INTRODUCTION

Let G denote a compact connected Lie group with torsion-free fundamental group. Suppose that G acts on a compact space X so that each isotropy subgroup is a connected subgroup of maximal rank. In this article we study the problem of computing $K_G^*(X)$, the complex G -equivariant K -theory of X . Our work is motivated by the examples given by spaces of ordered commuting n -tuples in compact matrix groups such as $SU(r)$, $U(q)$ and $Sp(k)$ with the conjugation action.

Given such a G -space X , if $T \subset G$ is a maximal torus, then $N_G(T)$ acts on the T -fixed points X^T . This in turn yields a corresponding action of the Weyl group $W = N_G(T)/T$ on the fixed-point set. It can be shown (Theorem 2.2) that if X^T has the homotopy type of a finite W -CW complex, then this determines a G -CW complex structure on X , and its G -equivariant K -theory becomes a tractable invariant. The following theorem provides a calculation for $K_G^*(X) \otimes \mathbb{Q}$.

Theorem 1.1. *Let G be a compact connected Lie group with torsion-free fundamental group and $T \subset G$ a maximal torus. Suppose that X is a compact G -CW complex with connected maximal rank isotropy subgroups, then $K_G^*(X) \otimes \mathbb{Q}$ is a free module over $R(G) \otimes \mathbb{Q}$ of rank equal to $\sum_{i \geq 0} \dim_{\mathbb{Q}} H^i(X^T; \mathbb{Q})$.*

Recall that if G acts on a topological space X then its inertia space is defined as $\Lambda X := \{(g, x) \in G \times X \mid gx = x\}$. Note that ΛX is naturally a G -space via $h \cdot (x, g) := (hx, hgh^{-1})$. These spaces play an important role in geometry and topology through the theory of stacks. A basic observation noted in Section 2 is that if G acts on X with connected maximal rank isotropy subgroups such that the $\pi_1(G_x)$ are torsion-free for all $x \in X$, then ΛX also has connected maximal rank isotropy subgroups. In addition if X^T has the homotopy type of a finite W -CW complex the same is true of $(\Lambda X)^T = X^T \times T$ and so we have:

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Theorem 1.2. *Let X denote a compact G -CW complex with connected maximal rank isotropy subgroups all of which have torsion-free fundamental group. Then $K_G^*(\Lambda X) \otimes \mathbb{Q}$ is a free $R(G) \otimes \mathbb{Q}$ -module of rank equal to $2^r \cdot (\sum_{i \geq 0} \dim_{\mathbb{Q}} H^i(X^T; \mathbb{Q}))$, where r is the rank of G as a compact Lie group.*

The construction of inertia spaces can be iterated. In this way we obtain a sequence of spaces $\{\Lambda^n(X)\}_{n \geq 0}$. Let \mathcal{P} denote the collection of all compact Lie groups arising as finite products of the classical groups $SU(r)$, $U(q)$ and $Sp(k)$. If we further require that $G_x \in \mathcal{P}$ for all $x \in X$, then the action of G on $\Lambda^n(X)$ has connected maximal rank isotropy subgroups for every $n \geq 0$ (Proposition 2.9) and given that $(\Lambda^n(X))^T = X^T \times T^n$, we have

Theorem 1.3. *Let X denote a compact G -CW complex such that all of its isotropy subgroups lie in \mathcal{P} and are of maximal rank. Then $K_G^*(\Lambda^n(X)) \otimes \mathbb{Q}$ is a free $R(G) \otimes \mathbb{Q}$ -module of rank equal to $2^{nr} \cdot (\sum_{i \geq 0} \dim_{\mathbb{Q}} H^i(X^T; \mathbb{Q}))$ where r is the rank of G as a compact Lie group.*

Taking X to be a single point with the trivial G -action yields $\Lambda^n(X) = \text{Hom}(\mathbb{Z}^n, G)$ (the space of ordered commuting n -tuples in G) with the conjugation action, and our result can be applied.

Corollary 1.4. *Suppose that $G \in \mathcal{P}$ is of rank r . Then $K_G^*(\text{Hom}(\mathbb{Z}^n, G)) \otimes \mathbb{Q}$ is free of rank 2^{nr} as an $R(G) \otimes \mathbb{Q}$ -module.*

Now let $C_n(\mathfrak{g})^+$ denote the one point compactification of the algebraic variety of ordered commuting n -tuples in \mathfrak{g} , the Lie algebra of G . This variety is endowed with an action of G via the adjoint representation. As before we can show that if $G \in \mathcal{P}$ then it has connected maximal rank isotropy and we obtain

Corollary 1.5. *Suppose that $G \in \mathcal{P}$ is of rank r . Then there is an isomorphism of modules over $R(G) \otimes \mathbb{Q}$*

$$\tilde{K}_G^q(C_n(\mathfrak{g})^+) \otimes \mathbb{Q} \cong \begin{cases} R(G) \otimes \mathbb{Q} & \text{if } q \equiv rn \pmod{2}, \\ 0 & \text{if } q + 1 \equiv rn \pmod{2}. \end{cases}$$

Our methods also yield calculations for $K_G^*(X)$ but they are somewhat more technical. To state them we need to recall a definition from the theory of reflection groups. Let Φ denote the root system associated to a fixed maximal torus $T \subset G$ and let Φ^+ be a choice of positive roots. Suppose that $W_i \subset W$ is a reflection subgroup. Let Φ_i be the corresponding root system and Φ_i^+ the corresponding positive roots. Define $W_i^\ell := \{w \in W \mid w(\Phi_i^+) \subset \Phi^+\}$. The set W_i^ℓ forms a system of representatives for the different cosets in W/W_i . Let $\mathcal{W} = \{W_i\}_{i \in \mathcal{I}}$ be a family of reflection subgroups of W . We say that \mathcal{W} satisfies the *coset intersection property* if given $i, j \in \mathcal{I}$ we can find some $k \in \mathcal{I}$ such that $W_i \cup W_j \subset W_k$ and $W_k^\ell = W_i^\ell \cap W_j^\ell$. We are now ready to state our result:

Theorem 1.6. *Let G be a compact connected Lie group with torsion-free fundamental group and $T \subset G$ a maximal torus. Suppose that X is a compact G -CW complex with connected maximal rank isotropy. Assume that there is a CW-subcomplex K of X^T such that for every element $x \in X^T$ there is a unique $w \in W$ such that $w x \in K$ and the family $\{W_\sigma \mid \sigma \text{ is a cell in } K\}$ is contained in a family \mathcal{W} of reflection subgroups of W satisfying the coset intersection property. If $H^*(X^T; \mathbb{Z})$ is torsion-free, then $K_G^*(X)$ is a free module over $R(G)$ of rank equal to $\sum_{i \geq 0} \text{rank}_{\mathbb{Z}} H^i(X^T; \mathbb{Z})$.*

Corollary 1.7. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free. Let G act on its Lie algebra \mathfrak{g} by the adjoint representation. If r is the rank of G , then*

$$\tilde{K}_G^q(\mathbb{S}^{\mathfrak{g}}) \cong \begin{cases} R(G) & \text{if } q \equiv r \pmod{2}, \\ 0 & \text{if } q + 1 \equiv r \pmod{2}. \end{cases}$$

To prove our main results we show that under the appropriate conditions the equivariant K -theory of a space with connected maximal rank isotropy can be computed from the strong collapse of the well-known spectral sequence associated to its skeletal filtration which can be described using Bredon cohomology. This in turn can be expressed in terms of the cohomology of the fixed point set X^T ; it requires an explicit identification of the E_2 -term of the spectral sequence as a module over the representation ring. This involves some subtleties concerning the representation rings of compact Lie groups related to work of R. Steinberg (see [19]).

In [6] it was established that if G is compact and connected with torsion-free fundamental group, then $K_G^*(G)$ (with the conjugation action) is a free $R(G)$ -module of rank 2^r , where r is the rank of G (note that our methods provide a different proof of this for G simply connected, see Corollary 6.1). However the example of commuting pairs in $SU(2)$ which is computed in Example 6.9 shows that the analogous result does not extend to spaces of commuting elements without inverting the primes dividing the order of the Weyl group W . Moreover the condition $G \in \mathcal{P}$ is required to ensure connected maximal rank isotropy; for example this already fails for $\text{Hom}(\mathbb{Z}^2, Spin(7))$ even though $Spin(7)$ is a simply connected, simple Lie group. This is due to the fact that $\text{Hom}(\mathbb{Z}^3, Spin(7))$ is not path-connected (see Example 2.4 and [14]).

The layout of this article is as follows. In Section 2 we derive some basic properties of actions of compact Lie groups that have connected maximal rank isotropy subgroups and provide some key examples where such actions appear naturally. In Section 3 we construct a suitable basis for $R(T)^{W_i}$, the ring of W_i invariants in $R(T)$, for a reflection subgroup W_i of the Weyl group W . In Section 4 we provide a method for the computation of the Bredon cohomology whose coefficient system is obtained by taking fixed points of $R(T)$. In Section 5 we prove Theorems 1.1 and 1.6 which are the main results in this article. In Section 6 some applications of the main theorems are discussed. Finally the Appendix contains basic definitions and facts about G -CW complexes and Bredon cohomology which the reader may find useful.

2. GROUP ACTIONS WITH MAXIMAL RANK ISOTROPY SUBGROUPS

Throughout this section G will be a compact connected Lie group, $T \subset G$ a maximal torus and W the associated Weyl group.

Definition 2.1. Let X be a G -space. The action of G on X is said to have maximal rank isotropy subgroups if for every $x \in X$, the isotropy group G_x is a subgroup of maximal rank; that is, for every $x \in X$ we can find a maximal torus T_x in G such that $T_x \subset G_x$. If in addition G_x is connected for every $x \in X$, then the action of G on X is said to have connected maximal rank isotropy subgroups.

Let G act on a topological space X with connected maximal rank isotropy subgroups. Then by restriction the group $N_G(T)$ acts on X^T inducing an action of $W = WG := N_G(T)/T$ on X^T . In the case that G acts smoothly on a smooth manifold X the action of G on X is determined,

up to equivalence, by the associated action of W on X^T as was proved in [10, Theorem 4.1]. In the broader context of *continuous* actions of a compact Lie group on a topological space X with connected maximal rank isotropy subgroups, the corresponding action of W on X^T still determines the action of G on X in some way. Given a subgroup H of G with $T \subset H$ let $WH = N_H(T)/T$ denote its Weyl group. Let $\mathcal{F}_G(X)$ denote the set of conjugacy classes of isotropy subgroups of the action of G on X . The set $\mathcal{F}_G(X)$ is partially ordered by inclusion. Similarly, let $\mathcal{F}_W(X^T)$ be the partially ordered set of conjugacy classes of isotropy subgroups of the action of W on X^T .

Now let X, Y be two spaces on which G acts with connected maximal rank isotropy subgroups. Then the assignment $\mathcal{F}_G(X) \rightarrow \mathcal{F}_W(X^T)$ which sends (H) to (WH) defines an order preserving bijection and any W -equivariant continuous map $f : X^T \rightarrow Y^T$ has a unique G -equivariant continuous extension $F : X \rightarrow Y$ (see [10, Theorem 1.1] and [10, Theorem 2.1]).

The following theorem shows that G -CW complex structures on a G -space X with connected maximal rank isotropy subgroups are determined, up to equivalence, by W -CW complex structures on X^T .

Theorem 2.2. *Suppose that G acts on a compact space X with connected maximal rank isotropy subgroups. Then, up to equivalence, there is a one to one correspondence between G -CW complex structures on X with isotropy type $\mathcal{F}_G(X) = \{(H) \mid H \in \text{Iso}(X)\}$ and W -CW complex structures on X^T with isotropy type $\mathcal{F}_W(X^T) = \{(WH) \mid H \in \text{Iso}(X)\}$.*

Proof: Assume that X has the structure of a G -CW complex. By [7, Theorem 2.1.14] it follows that X^T has a natural structure as W -CW complex. Explicitly, suppose that $\{X_n\}_{n \geq 0}$ is the skeletal filtration of X . Since G acts on X with connected maximal rank isotropy subgroups, it follows that the cells in the G -CW decomposition of X are of the form $G/H \times \mathbb{D}^{n+1}$, where $H \subset G$ is connected subgroup of maximal rank. The skeletal filtration of X^T is $\{(X^n)^T\}_{n \geq 0}$ and $(X^{n+1})^T$ is obtained from $(X^n)^T$ by attaching $W/WH \times \mathbb{D}^{n+1}$ along the attaching map $W/WH \times \mathbb{S}^n \rightarrow (X^n)^T$ obtained by taking the T -fixed points to the corresponding attaching G -map $G/H \times \mathbb{S}^n \rightarrow X^n$. In particular, X^T has orbit type $\mathcal{F}_W(X^T) = \{(WH) \mid H \in \text{Iso}(X)\}$.

Conversely, assume that X^T has a structure of a W -CW complex. Then X^T is a compact W -CW complex, in particular, X^T is a W -ENR and by [8, Remark 8.2.5] it satisfies conditions (W-CW 1)-(W-CW 3) (see the Appendix for the definition). On the other hand, $(H) \mapsto (WH)$ defines an order preserving bijection between $\mathcal{F}_G(X)$ and $\mathcal{F}_W(X^T)$. This shows that $\mathcal{F}_W(X^T) = \{(WH) \mid H \in \text{Iso}(X)\}$ and also that X has finite orbit type since $\mathcal{F}_W(X^T)$ is finite. Suppose that $H \in \text{Iso}(X)$. We may assume that $T \subset H$ after replacing H with some conjugate if necessary. Let (h, u) be a representation of $((X^T)^{(WH)}, (X^T)^{>(WH)})$ as a W -NDR pair. Such a representation exists because X^T satisfies (W-CW 2). Since the assignment $(H) \mapsto (WH)$ is an order preserving bijection we have $(X^T)^{(WH)} = (X^{(H)})^T$. Then $h : (X^{(H)})^T \times I \rightarrow (X^{(H)})^T$ is a W -equivariant map and the action of G on $X^{(H)}$ has connected maximal rank isotropy subgroups. Therefore there is a unique G -equivariant map $\tilde{h} : X^{(H)} \times I \rightarrow X^{(H)}$ extending h . Similarly the map u can be extended to a G -invariant map $\tilde{u} : X^{(H)} \rightarrow I$ and it is easy to see that the pair (\tilde{h}, \tilde{u}) is a representation of $(X^{(H)}, X^{>(H)})$ as a G -NDR pair. On the other hand, since X^T satisfies (W-CW 3) we have $X_{WH}^T \rightarrow X_{WH}^T/(N_W WH/WH)$ is a numerable principal $N_W WH/WH$ -bundle for every $H \in \text{Iso}(X)$. By [10, Lemma 1.6] we have $N_G(H)/H =$

$N_W(WH)/WH$. Using that $(H) \mapsto (WH)$ defines a bijection from $\mathcal{F}_G(X)$ to $\mathcal{F}_W(X^T)$ and [10, Lemma 1.4] we see that $X_H = X_{WH}^T$ and thus $X_H \rightarrow X_H/N_G(H)/H$ is a numerable principal $N_G(H)/H$ -bundle. This proves that X satisfies conditions (G-CW 1)-(G-CW 3). By assumption X^T is a W -CW complex, in particular Theorem 7.1 shows that $(X^T)^{WH}$ has the homotopy type of a CW-complex for all $H \in \text{Iso}(X)$. As before we may assume that $T \subset G$ after replacing H with a suitable conjugate. In this case by [10, Lemma 1.4] we have $X^H = (X^T)^{WH}$ and this space has the homotopy type of a CW-complex. Applying Theorem 7.1 again we see that the space X can be given the structure of a G -CW complex.

Finally, an application of the equivariant Whitehead theorem shows that, up to equivalence, this defines a one to one correspondence between G -CW complex structures on X and W -CW complex structures on X^T . \square

Examples of spaces endowed with an action of G with connected maximal rank isotropy subgroups can be constructed in a natural way by considering the action of G on itself by conjugation as is shown below.

Example 2.3. Suppose that G is a compact connected Lie group with $\pi_1(G)$ torsion-free. Then the conjugation action of G on itself has connected maximal rank isotropy subgroups. To see this, note that for every $g \in G$ the isotropy group under this action is $G_g = Z_G(g)$, the centralizer of g in G . When $\pi_1(G)$ is torsion-free G_g is connected for every $g \in G$ by [4, Corollary IX 5.3.1]. Also, any element $g \in G$ is contained in some maximal torus T , in particular $T \subset G_g$ and this means that G_g is a maximal rank subgroup.

Example 2.4. More generally let $n \geq 1$ be an integer and consider the space of commuting n -tuples in G , $\text{Hom}(\mathbb{Z}^n, G)$. Let $\mathbb{1} : \mathbb{Z}^n \rightarrow G$ be the trivial representation and denote the path-connected component of $\text{Hom}(\mathbb{Z}^n, G)$ that contains $\mathbb{1}$ by $\text{Hom}(\mathbb{Z}^n, G)_{\mathbb{1}}$. This component is precisely the subspace of $\text{Hom}(\mathbb{Z}^n, G)$ consisting of all commuting n -tuples (x_1, \dots, x_n) that are contained in some maximal torus of G by [3, Lemma 4.2]. The space $\text{Hom}(\mathbb{Z}^n, G)$ can be seen as a G -space via the conjugation action of G and $\text{Hom}(\mathbb{Z}^n, G)_{\mathbb{1}}$ is invariant under this action. For any $\underline{x} = (x_1, \dots, x_n) \in \text{Hom}(\mathbb{Z}^n, G)$ the isotropy subgroup at \underline{x} under this action is $G_{\underline{x}} = Z_G(\underline{x})$. By the previous comment, if $\underline{x} \in \text{Hom}(\mathbb{Z}^n, G)_{\mathbb{1}}$ then $Z_G(\underline{x})$ contains a maximal torus of G . This shows that $\text{Hom}(\mathbb{Z}^n, G)_{\mathbb{1}}$ has maximal rank isotropy subgroups. Moreover, the conjugation action of G on $\text{Hom}(\mathbb{Z}^n, G)$ has connected maximal rank isotropy subgroups if and only if $\text{Hom}(\mathbb{Z}^{n+1}, G)$ is path-connected. Indeed, suppose first that $\text{Hom}(\mathbb{Z}^{n+1}, G)$ is path-connected. This implies in particular that $\text{Hom}(\mathbb{Z}^n, G)_{\mathbb{1}} = \text{Hom}(\mathbb{Z}^n, G)$. Suppose that $z \in G$ commutes with $\underline{x} = (x_1, \dots, x_n) \in \text{Hom}(\mathbb{Z}^n, G)_{\mathbb{1}}$. Then

$$(z, x_1, \dots, x_n) \in \text{Hom}(\mathbb{Z}^{n+1}, G) = \text{Hom}(\mathbb{Z}^{n+1}, G)_{\mathbb{1}}$$

and therefore we can find a maximal torus T of G that contains z, x_1, \dots, x_n . In particular we can find a path γ in T joining z and 1_G . Note that γ is contained in $T \subset Z_G(\underline{x})$ proving that $Z_G(\underline{x})$ is path-connected. Conversely, suppose that the action of G on $\text{Hom}(\mathbb{Z}^n, G)$ has connected isotropy subgroups. Assume that $(x_1, \dots, x_{n+1}) \in \text{Hom}(\mathbb{Z}^{n+1}, G)$. Then $x_{n+1} \in Z_G(x_1, \dots, x_n)$ and by assumption this is path-connected. We conclude that we can find a path γ in $Z_G(x_1, \dots, x_n)$ from x_{n+1} to 1_G . The path γ shows that (x_1, \dots, x_{n+1}) lies in the same path-connected component of $(x_1, \dots, x_n, 1_G)$. Iterating this argument we conclude that

(x_1, \dots, x_{n+1}) lies in the same path-connected component of $\mathbb{1} = (1_G, \dots, 1_G)$ proving that $\text{Hom}(\mathbb{Z}^{n+1}, G)$ is path-connected.

The following proposition characterizes the compact Lie groups for which $\text{Hom}(\mathbb{Z}^n, G)$ is path-connected for all $n \geq 0$ when G is simple.

Proposition 2.5. *If G is a simple, simply connected compact Lie group, then the following statements are equivalent:*

- (1) $\text{Hom}(\mathbb{Z}^n, G)$ is path-connected for all $n \geq 0$.
- (2) $A \subset G$ is a maximal abelian subgroup of G if and only if A is a maximal torus.
- (3) G is isomorphic to either $SU(r)$ for some $r \geq 2$ or to $Sp(k)$ for some $k \geq 2$.

Proof: We first prove that (1) is equivalent to (2). For this equivalence we only need to assume that G is a compact Lie group. Suppose first that $\text{Hom}(\mathbb{Z}^n, G)$ is path-connected for all $n \geq 0$. Let A be a maximal abelian subgroup of G . Note that A must be closed since \overline{A} is also abelian. Therefore A is a compact Lie group. Let A_0 be the path-connected component of A containing the identity element. Then A_0 is a compact connected abelian Lie group and thus it is a torus. In particular, we can find a finite set $\{a_1, \dots, a_m\}$ of elements in A such that $A_0 = \langle a_1, \dots, a_m \rangle$. As A has only finitely many distinct components, we may further choose elements b_1, \dots, b_k in $A - A_0$ such that $A = \langle a_1, \dots, a_m, b_1, \dots, b_k \rangle$. Since A is abelian we have $(a_1, \dots, a_m, b_1, \dots, b_k) \in \text{Hom}(\mathbb{Z}^{m+k}, G)$. The connectedness of the latter implies that there is a maximal torus T in G such that $a_i, b_j \in T$ for all i, j . In particular $A = \langle a_1, \dots, a_m, b_1, \dots, b_k \rangle \subset T$. Since A is maximal abelian this shows that $A = T$. Conversely, suppose that T is a maximal torus. If T is not a maximal abelian subgroup, then we can find some $g \in G$ with $g \notin T$ and such that g commutes with all the elements in T . Choose $a_1, \dots, a_m \in T$ such that $T = \langle a_1, \dots, a_m \rangle$. Note that $(a_1, \dots, a_m, g) \in \text{Hom}(\mathbb{Z}^{m+1}, G)$. Again by connectedness of $\text{Hom}(\mathbb{Z}^{m+1}, G)$ there exists a maximal torus T' in G such that $a_1, \dots, a_m, g \in T'$. This implies that $T = \langle a_1, \dots, a_m \rangle \subsetneq T'$ which is a contradiction since T is a maximal torus.

Assume now that (2) is true and suppose that $(g_1, \dots, g_n) \in \text{Hom}(\mathbb{Z}^n, G)$. Let $B = \langle g_1, \dots, g_n \rangle$. Then B is an abelian subgroup and we can find a maximal abelian group $A \subset G$ with $B \subset A$. By assumption A must be a torus, in particular, A is path-connected. It follows by [1, Proposition 2.3] that $\text{Hom}(\mathbb{Z}^n, G)$ is path-connected for every $n \geq 0$. This proves that (1) and (2) are equivalent.

By [1, Corollary 2.4] if $G = SU(r)$ or $G = Sp(k)$ then $\text{Hom}(\mathbb{Z}^n, G)$ is path-connected for every $n \geq 0$. This shows that statement (3) implies (1). Assume now that G is a simple, simply connected compact Lie group. Table 2 in [14] together with [14, Theorem 3] show that under these assumptions $\text{Hom}(\mathbb{Z}^3, G)$ is path-connected only when $G = SU(r)$ for some $r \geq 2$ or $G = Sp(k)$ for some $k \geq 2$. This proves in particular that (1) implies (3). \square

Let G denote a compact simply connected Lie group. Then we may express it as a product $G \cong G_1 \times \dots \times G_s$, where G_1, \dots, G_s are simply connected simple Lie groups. Now it is easy to see that $\text{Hom}(\mathbb{Z}^n, H \times Q) \cong \text{Hom}(\mathbb{Z}^n, H) \times \text{Hom}(\mathbb{Z}^n, Q)$ and hence $\text{Hom}(\mathbb{Z}^n, H \times Q)$ is path-connected if and only if $\text{Hom}(\mathbb{Z}^n, H)$ and $\text{Hom}(\mathbb{Z}^n, Q)$ are both path-connected.

Corollary 2.6. *Let G be a compact simply connected Lie group, then $\text{Hom}(\mathbb{Z}^n, G)$ is path-connected for all $n \geq 1$ if and only if G is isomorphic to a finite cartesian product of groups of the form $SU(r)$ ($r \geq 2$) and $Sp(k)$ ($k \geq 2$).*

We observe that the unitary groups $U(q)$ also satisfy the first two conditions in the proposition above. This leads us to introduce the following.

Definition 2.7. Let \mathcal{P} denote the collection of all compact Lie groups arising as finite cartesian products of the groups $SU(r)$, $U(q)$ and $Sp(k)$.

Note that for $G \in \mathcal{P}$, $\text{Hom}(\mathbb{Z}^n, G)$ is path-connected for every $n \geq 0$ and therefore as observed in Example 2.4 this G -space has connected maximal rank isotropy. We now recall the notion of the inertia space for a group action.

Definition 2.8. Let G denote a Lie group acting on a space X . Then its inertia space is defined as $\Lambda X = \{(x, g) \mid gx = x\} \subset X \times G$.

This construction can of course be iterated. This way we obtain a sequence of spaces $\{\Lambda^n(X)\}_{n \geq 1}$. Note that the correspondence $(x, g) \mapsto (hx, hgh^{-1})$ defines a G -action on ΛX , and the isotropy subgroups are simply $G_{(x,g)} = Z_{G_x}(g)$, the centralizer of g in G_x . Note that if G acts on X with connected maximal rank isotropy subgroups with torsion-free fundamental group, then the G -action on ΛX will also have connected maximal rank isotropy by Example 2.3. Similarly we obtain that $G_{(x, g_1, \dots, g_n)} = Z_{G_x}(g_1, \dots, g_n)$ where $(g_1, \dots, g_n) \in \text{Hom}(\mathbb{Z}^n, G_x)$. Note that $\Lambda^n(\{x_0\}) = \text{Hom}(\mathbb{Z}^n, G)$.

Proposition 2.9. *Let G be a compact Lie group. Suppose G acts on a space X with connected maximal rank isotropy subgroups in such a way that $G_x \in \mathcal{P}$ for every $x \in X$. Then the induced action of G on $\Lambda^n(X)$ has connected maximal rank isotropy subgroups.*

Proof: If $(x, g_1, \dots, g_n) \in \Lambda^n(X)$ then its isotropy subgroup is $Z_{G_x}(g_1, \dots, g_n)$, which we have seen are connected and of maximal rank. \square

Thus this approach provides a method for constructing natural spaces on which G acts with connected maximal rank isotropy subgroups.

3. THE REPRESENTATION RING

Suppose that G is a compact connected Lie group and let $T \subset G$ be a maximal torus. The inclusion map $i : T \rightarrow G$ induces a natural isomorphism of rings $i^* : R(G) \xrightarrow{\cong} R(T)^W$, see for example [4, IX 3]. We can see $R(T)$ as a module over $R(G) \cong R(T)^W$. By [16, Theorem 1] $R(T)$ is a free module over $R(T)^W$ of rank $|W|$ when $\pi_1(G)$ is torsion-free. More generally, by [19, Theorem 2.2] given a reflection subgroup $W_i \subset W$ the ring of invariants $R(T)^{W_i}$ is a free module over $R(T)^W$ of rank $|W|/|W_i|$. This is described next; we follow the presentation provided in [21] with some modifications to fit our settings. Assume first that G is simply connected and thus semisimple. Let $\Lambda = X^*(T) = \text{Hom}(T, \mathbb{S}^1)$ be the character group of T . Λ is a finitely generated free abelian group (where the group operation is written multiplicatively) and the representation ring $R(T)$ is the group algebra $\mathbb{Z}[\Lambda]$. Let Φ be the root system associated to (G, T) . Fix a subset Φ^+ of positive roots of Φ and let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be an ordering of the corresponding set of simple roots. Consider the fundamental weights $\{\omega_1, \dots, \omega_r\}$ corresponding to the simple roots

$\{\alpha_1, \dots, \alpha_r\}$. Since G is assumed to be simply connected the set $\{\omega_1, \dots, \omega_r\}$ forms a free basis of Λ . Suppose that $W_i \subset W$ is a reflection subgroup. Let Φ_i be the corresponding root system and Φ_i^+ the corresponding positive roots. Define $W_i^\ell := \{w \in W \mid w(\Phi_i^+) \subset \Phi^+\}$. The set W_i^ℓ forms a system of representatives of the *left* cosets in W/W_i by [19, Lemma 2.5]. This means that any element $w \in W$ can be factored in a unique way in the form $w = ux$ with $u \in W_i^\ell$ and $x \in W_i$. Moreover, when Φ_i has a basis consisting of all roots of the form $\{\alpha \mid \alpha \in I\}$ for some subset $I \subset \Delta$, then $W_i^\ell = W^I := \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in I\}$ which is the set of minimal length coset representatives of W/W_i with respect to the length function ℓ determined by Δ . Because of this, in general we call the set W_i^ℓ a system of minimal length representatives of W/W_i . Given $v \in W_i^\ell$ define $p_v := \prod_{v^{-1}\alpha_i < 0} \omega_i$ and $f_v^{W_i} := \sum_{x \in W_i(v) \setminus W_i} x^{-1}v^{-1}p_v$. Here $W_i(v) := \{w \in W_i \mid w^{-1}v^{-1}p_v = v^{-1}p_v\}$ is the stabilizer in W_i of $v^{-1}p_v$. Thus $f_v^{W_i}$ is the sum of the elements in the orbit of $v^{-1}p_v$ under the action of W_i , making $f_v^{W_i}$ invariant under the action of W_i . By [19, Theorem 2.2] the collection $\{f_v^{W_i}\}_{v \in W_i^\ell}$ forms a free basis of $R(T)^{W_i}$ as a module over $R(T)^W$. We refer to this basis as the Steinberg basis. Next we discuss the relationship between the Steinberg bases associated to reflection subgroups $W \supset W_j \supset W_i$. We set up some notation first.

Given $v \in W_j^\ell$, the subgroup $W_j(v)$ is itself a reflection subgroup. Let $\Phi_j(v)$ be its corresponding root system and $\Phi_j^+(v)$ the corresponding positive roots. Define $W_j^r(v) := \{w \in W_j \mid w^{-1}(\Phi_j^+(v)) \subset \Phi_j^+\}$. As before, this set gives a system of representatives of the *right* cosets in $W_j(v) \setminus W_j$ and it has the property that every element $w \in W_j$ can be factorized in a unique way in the form $w = xu$ with $x \in W_j(v)$ and $u \in W_j^r(v)$. Note that by definition of $f_v^{W_j}$ we have $f_v^{W_j} = \sum_{x \in W_j(v) \setminus W_j} x^{-1}v^{-1}p_v = \sum_{x \in W_j^r(v)} x^{-1}v^{-1}p_v$. On the other hand, given $v \in W_j^\ell$ and $x \in W_j^r(v)$ we can consider the coset $xW_i \in W_j/W_i$. Let $m_{j,i}(x)$ be the minimal length representative of the coset $xW_i \in W_j/W_i$. Precisely, $m_{j,i}(x)$ is the unique element in $[W_i^\ell]^{W_j} := \{w \in W_j \mid w(\Phi_i^+) \subset \Phi_j^+\}$ such that $xW_i = m_{j,i}(x)W_i$. The following lemma describes the precise relationship between the bases $\{f_v^{W_j}\}_{v \in W_j^\ell}$ and $\{f_v^{W_i}\}_{v \in W_i^\ell}$.

Lemma 3.1. *Suppose that $W \supset W_j \supset W_i$ are reflection subgroups. Then $W_j^\ell \subset W_i^\ell$ and if $v \in W_j^\ell$ then $f_v^{W_j} = \sum_{\tilde{x} \in m_{j,i}(W_j^r(v))} f_{v\tilde{x}}^{W_i}$.*

Proof: It is clear from the definition of W_j^ℓ that the inclusion $W_j \supset W_i$ implies $W_j^\ell \subset W_i^\ell$. Note that the right hand side of the equation in Lemma 3.1 is well defined because given $v \in W_j^\ell$ and $\tilde{x} \in m_{j,i}(W_j^r(v))$ then $v\tilde{x} \in W_i^\ell$. We are going to prove first the assertion for the particular case $W_i = \{1\}$. In this case $x = m_{j,i}(x)$ for every $x \in W_j^r(v)$, thus we need to prove

$$(1) \quad f_v^{W_j} = \sum_{x \in W_j^r(v)} f_{vx}^{\{1\}}.$$

Suppose that $v \in W_j^\ell$ and $x \in W_j^r(v)$. An argument similar to the one provided in [21, Proposition 1.9] shows that under these hypotheses

$$(2) \quad x^{-1}v^{-1}p_v = x^{-1}v^{-1}p_{vx}.$$

Then

$$f_v^{W_j} = \sum_{x \in W_j^r(v)} x^{-1}v^{-1}p_v = \sum_{x \in W_j^r(v)} x^{-1}v^{-1}p_{vx} = \sum_{x \in W_j^r(v)} f_{vx}^{\{1\}}$$

proving the lemma when $W_i = \{1\}$. We now show that the result is true for any reflection subgroups $W \supset W_j \supset W_i$. Suppose that $v \in W_j^\ell$. Then by (1)

$$f_v^{W_j} = \sum_{x \in W_j^r(v)} f_{vx}^{\{1\}} = \sum_{x \in W_j^r(v)} x^{-1}v^{-1}p_{vx}.$$

For any $x \in W_j^r(v)$ we have a unique factorization $x = m_{j,i}(x)y(x)$ with $m_{j,i}(x) \in [W_i^\ell]^{W_j}$ and $y(x) \in W_i$. Thus

$$(3) \quad f_v^{W_j} = \sum_{x \in W_j^r(v)} x^{-1}v^{-1}p_{vx} = \sum_{x \in W_j^r(v)} y(x)^{-1}m_{j,i}(x)^{-1}v^{-1}p_{vm_{j,i}(x)y(x)}.$$

We can rearrange the terms in the previous sum in the following way. Let $\tilde{x} = m_{j,i}(x)$, then $x = \tilde{x}y(x)$. When \tilde{x} runs through all elements in $m_{j,i}(W_j^r(v))$ the set of values of y such that $y^{-1}\tilde{x}^{-1}v^{-1}p_{v\tilde{x}y}$ is a term in the sum (3) is precisely $W_i^r(v\tilde{x})$. This shows that

$$\begin{aligned} f_v^{W_j} &= \sum_{x \in W_j^r(v)} y(x)^{-1}m_{j,i}(x)^{-1}v^{-1}p_{vm_{j,i}(x)y(x)} \\ &= \sum_{\tilde{x} \in m_{j,i}(W_j^r(v))} \left(\sum_{y \in W_i^r(v\tilde{x})} y^{-1}\tilde{x}^{-1}v^{-1}p_{v\tilde{x}y} \right) \\ &= \sum_{\tilde{x} \in m_{j,i}(W_j^r(v))} f_{v\tilde{x}}^{W_i}. \end{aligned}$$

The last equality is obtained by applying (1). \square

For our applications it will be more convenient to obtain a different basis of $R(T)^{W_i}$ as a module over $R(T)^W$ that behaves better under the natural inclusion $R(T)^{W_j} \subset R(T)^{W_i}$, whenever $W \supset W_j \supset W_i$ belong to a suitable family $\mathcal{W} = \{W_i\}_{i \in \mathcal{I}}$ of reflection subgroups of W . The families of subgroups that we consider satisfy the following condition.

Definition 3.2. Let $\mathcal{W} = \{W_i\}_{i \in \mathcal{I}}$ be a family of reflection subgroups of W . We say that \mathcal{W} satisfies the coset intersection property if given $i, j \in \mathcal{I}$ we can find some $k \in \mathcal{I}$ such that $W_i \cup W_j \subset W_k$ and $W_k^\ell = W_i^\ell \cap W_j^\ell$.

Example 3.3. Suppose we have a sequence of reflection subgroups

$$W = W_0 \supset W_1 \supset W_2 \supset \cdots \supset W_k = \{1\}.$$

Then by passing to the minimal coset representatives we obtain an increasing sequence

$$\{1\} = W_0^\ell \subset W_1^\ell \subset W_2^\ell \subset \cdots \subset W_k^\ell = W.$$

This shows that the family $\mathcal{W} = \{W_i\}_{0 \leq i \leq k}$ satisfies the coset intersection property. We are also going to consider the following important example. As before let $\Delta = \{\alpha_1, \dots, \alpha_r\}$ be an ordering of the corresponding set of simple roots associated to a set Φ^+ of positive roots of the root system Φ . For every $I \subset \Delta$ let W_I be the reflection subgroup of W generated

by the corresponding reflections s_α with $\alpha \in I$. The family $\mathcal{W} = \{W_I\}_{I \subset \Delta}$ satisfies the coset intersection property. To see this, suppose that $I, J \subset \Delta$. Then $W_I \cup W_J \subset W_{I \cup J}$ and $W_{I \cup J}^\ell = W_I^\ell \cap W_J^\ell$ because given any $I \subset \Delta$ we have

$$W_I^\ell = W^I := \{w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in I\}.$$

Now fix $\mathcal{W} = \{W_i\}_{i \in \mathcal{I}}$ a family of reflection subgroups of W satisfying the coset intersection property. Consider the family $\mathcal{W}^\ell := \{W_i^\ell\}_{i \in \mathcal{I}}$ of minimal length coset representatives. For every $i \in \mathcal{I}$ define

$$C_i = W_i^\ell \setminus \left(\bigcup_{j \in \mathcal{I}, W_i \subsetneq W_j} W_j^\ell \right).$$

The different sets of the form C_i provide a decomposition of W into disjoint sets $W = \bigsqcup_{i \in \mathcal{I}} C_i$. This is precisely where the coset intersection property is used. Indeed, any element $v \in W$ is contained in some C_i for some $i \in \mathcal{I}$ and if $v \in C_i \cap C_j$, then $v \in W_i^\ell \cap W_j^\ell$ and by assumption we can find some $k \in \mathcal{I}$ such that $W_i \cup W_j \subset W_k$ and $W_k^\ell = W_i^\ell \cap W_j^\ell$. If $W_i \subsetneq W_k$ we conclude that $v \notin C_i$ which contradicts our assumption. A similar situation occurs if $W_j \subsetneq W_k$. Therefore $W_k = W_i = W_j$ and thus $i = j = k$.

Additionally, for any $i \in \mathcal{I}$ we have

$$W_i^\ell = \bigsqcup_{\substack{j \in \mathcal{I} \\ W_i \subset W_j}} C_j$$

Thus given any $v \in W$ we can find a unique $i(v) \in \mathcal{I}$ such that $v \in C_{i(v)}^\ell$ and we can define $g_v := f_v^{W_{i(v)}}$.

Theorem 3.4. *With the previous notation, if G is simply connected and $\mathcal{W} = \{W_i\}_{i \in \mathcal{I}}$ is a family of reflection subgroups of W satisfying the coset intersection property, then for every $i \in \mathcal{I}$ the collection $\{g_v\}_{v \in W_i^\ell}$ is a free basis of $R(T)^{W_i}$ as a module over $R(T)^W$.*

Proof: Let $i \in \mathcal{I}$, we will show that $\{g_v\}_{v \in W_i^\ell}$ is a free basis of $R(T)^{W_i}$ as a module over $R(T)^W$. Since $R(T)^W$ is a domain, it suffices to show that every element of $\{g_v\}_{v \in W_i^\ell}$ is in the span of $\{f_w^{W_i}\}_{w \in W_i^\ell}$ and vice versa. We will prove this by induction on

$$n(i) := |\{j \in \mathcal{I} \mid W_i \subset W_j\}|.$$

When $n(i) = 1$ we have $W_i = W$ and thus $W_i^\ell = \{1\}$. In this case the claim is trivial as $g_1 = f_1^W = 1$. Suppose that the claim is true for all $i \in \mathcal{I}$ for which $n(i) < n$. Let $i \in \mathcal{I}$ be such that $n(i) = n$. Suppose that $v \in W_i^\ell = \bigsqcup_{\substack{j \in \mathcal{I} \\ W_i \subset W_j}} C_j$. We show first that g_v is in the span of $\{f_w^{W_i}\}_{w \in W_i^\ell}$. If $v \in C_i$ then there is nothing to prove as $g_v = f_v^{W_i}$. Suppose then that $v \in C_j$ for some $j \in \mathcal{I}$ such that $W_i \subsetneq W_j$. Then $v \in W_j^\ell \subset W_i^\ell$ and by Lemma 3.1 we have

$$g_v = f_v^{W_j} = \sum_{\tilde{x} \in m_{j,i}(W_j^\ell(v))} f_{v\tilde{x}}^{W_i}$$

which is what we needed to show. Conversely, let's show that if $v \in W_i^\ell$ then $f_v^{W_i}$ is in the span of $\{g_v\}_{v \in W_i^\ell}$. If $v \in C_i^\ell$ then $f_v^{W_i} = g_v$ and there is nothing to prove. Suppose then that $v \in C_j$ for some $j \in \mathcal{I}$ such that $W_i \subsetneq W_j$. Using Lemma 3.1 we have

$$(4) \quad f_v^{W_j} = \sum_{\tilde{x} \in m_{j,i}(W_j^r(v))} f_{v\tilde{x}}^{W_i} = f_v^{W_i} + \sum_{\substack{\tilde{x} \in m_{j,i}(W_j^r(v)) \\ \tilde{x} \neq 1}} f_{v\tilde{x}}^{W_i}.$$

In this case $g_v = f_v^{W_j}$. Also, the inductive hypothesis shows that for all values of \tilde{x} such that $v\tilde{x} \in C_k$ for some $k \in \mathcal{I}$ with $W_i \subsetneq W_k$ then $f_{v\tilde{x}}^{W_i}$ is in the span of $\{g_v\}_{v \in W_k^\ell} \subset \{g_v\}_{v \in W_i^\ell}$. Therefore (4) can be rewritten in the form

$$f_v^{W_i} = \sum_{v_1 \in I_1} r_{v_1} g_{v_1} - \sum_{\substack{\tilde{x}_1 \in m_{j,i}(W_j^r(v)) \\ \tilde{x}_1 \neq 1, v\tilde{x}_1 \in C_i}} f_{v\tilde{x}_1}^{W_i}$$

for some set $I_1 \subset W_i^\ell$ and some $r_{v_1} \in R(T)^W$. If there are no elements with $\tilde{x}_1 \in m_{j,i}(W_j^r(v))$ such that $v\tilde{x}_1 \in C_i$ we are done. Otherwise, we iterate this procedure. At each step we find a sequence of elements $\tilde{x}_i \neq 1$ for $1 \leq i \leq k$ such that $\{v, v\tilde{x}_1, \dots, v(\tilde{x}_1 \dots \tilde{x}_k)\} \subset C_i$. Since C_i is a finite set this process must terminate after finitely many steps and thus $f_v^{W_i}$ is in the span of $\{g_v\}_{v \in W_i^\ell}$. \square

The important additional property of the basis constructed in the previous theorem is that whenever $W_j \supset W_i$ then $W_j^\ell \subset W_i^\ell$ and the set $\{g_v\}_{v \in W_j^\ell}$ is contained in the set $\{g_v\}_{v \in W_i^\ell}$. With this choice of basis we can find an isomorphism of $R(T)^W$ -modules for every $i \in \mathcal{I}$

$$\varphi_i : R(T)^{W_i} \xrightarrow{\cong} \bigoplus_{v \in W_i^\ell} R(T)^W g_v$$

that satisfies the following compatibility condition: whenever $W_j \supset W_i$ the following diagram commutes

$$\begin{array}{ccc} R(T)^{W_j} & \longrightarrow & R(T)^{W_i} \\ \varphi_j \downarrow & & \downarrow \varphi_i \\ \bigoplus_{v \in W_j^\ell} R(T)^W g_v & \longrightarrow & \bigoplus_{v \in W_i^\ell} R(T)^W g_v. \end{array}$$

In this diagram the horizontal maps are the natural inclusions.

Suppose now that G is a compact connected Lie group with $\pi_1(G)$ torsion-free. Then as pointed out in [16] there is a finite covering sequence

$$(5) \quad 1 \rightarrow \Gamma \rightarrow T' \times G_0 \xrightarrow{\pi} G \rightarrow 1,$$

with G_0 a simply connected compact Lie group, T' a torus and Γ a finite central subgroup. Note that $\pi^{-1}(T) = T' \times T_0$, where $T_0 \subset G_0$ is a maximal torus and there is a covering space

$$1 \rightarrow \Gamma \rightarrow T' \times T_0 \rightarrow T \rightarrow 1.$$

Let W_0 be the Weyl group associated to (G_0, T_0) . The covering sequence (5) shows that $W_0 = W$. Let $p_1 : T' \times G_0 \rightarrow T'$ be the projection map and $\bar{\Gamma} = p_1(\Gamma) \subset T'$. For every $i \in \mathcal{I}$ there is an isomorphism $R(T)^{W_i} \cong R(T_0)^{W_i} \otimes R(T'/\bar{\Gamma})$. In particular, if $\{g_v\}_{v \in W_i^\ell}$ is the free basis of

$R(T_0)^{W_i}$ as a module over $R(T_0)^W$ constructed above, then the collection $\{g_v \otimes 1\}_{v \in W_i^\ell}$ is a free basis of $R(T)^{W_i}$ as a module over $R(T)^W$. By abuse of notation we also denote this basis by $\{g_v\}_{v \in W_i^\ell}$. The following theorem summarizes the above.

Theorem 3.5. *Let G be a compact, connected Lie group with $\pi_1(G)$ torsion-free. Let $\mathcal{W} = \{W_i\}_{i \in \mathcal{I}}$ be a family of reflection subgroups of W satisfying the coset intersection property. Then for every $i \in \mathcal{I}$ there is an isomorphism of $R(T)^W$ -modules*

$$\varphi_i : R(T)^{W_i} \xrightarrow{\cong} \bigoplus_{v \in W_i^\ell} R(T)^W g_v.$$

These isomorphisms satisfy the following compatibility condition: whenever $W_j \supset W_i$ the following diagram commutes

$$\begin{array}{ccc} R(T)^{W_j} & \longrightarrow & R(T)^{W_i} \\ \varphi_j \downarrow & & \downarrow \varphi_i \\ \bigoplus_{v \in W_j^\ell} R(T)^W g_v & \longrightarrow & \bigoplus_{v \in W_i^\ell} R(T)^W g_v. \end{array}$$

In this diagram the horizontal maps are the natural inclusions.

Example 3.6. Suppose that $G = SU(3)$. Let $T \subset SU(3)$ be the maximal torus consisting of those 3 by 3 diagonal matrices with entries x_1, x_2 and x_3 in \mathbb{S}^1 such that $x_1 x_2 x_3 = 1$. The Weyl group $W = \Sigma_3$ acts by permutation of the diagonal entries in T . We can see the entries of an element in T as linear characters $x_i : T \rightarrow \mathbb{S}^1$ that satisfy the equation $x_1 x_2 x_3 = 1$ and

$$R(T) = \mathbb{Z}[x_1, x_2, x_3]/(x_1 x_2 x_3 = 1).$$

The roots associated to the pair (G, T) are $x_i x_j^{-1}$ for $i \neq j$. In this case $\Delta = \{x_1 x_2^{-1}, x_2 x_3^{-1}\}$ is a set of simple roots. The fundamental weights associated to this system of simple roots are $\{x_1, x_1 x_2\}$ respectively. Enumerate the elements in W as follows

$$v_1 = 1, v_2 = (23), v_3 = (123), v_4 = (132), v_5 = (12), v_6 = (13).$$

Consider the family $\mathcal{W} = \{W_0, W_1, W_2\}$ of reflection subgroups of W , where

$$W = W_0 \supset W_1 = \langle (12) \rangle \supset W_2 = \{1\}.$$

As pointed out before this sequence satisfies the coset intersection property and we have

$$\{1\} = \{v_1\} = W_0^\ell \subset W_1^\ell = \{v_1, v_2, v_3\} \subset W_2^\ell = W.$$

The different Steinberg bases $\{f_v^{W_i}\}_{v \in W_i^\ell}$ of $R(T)^{W_i}$ as a module over $R(T)^W$ are given below:

- For $W_0 = W$ we have the trivial basis $f_{v_1}^{W_0} = 1$.
- For $W_1 = \langle (12) \rangle$ we obtain the basis $f_{v_1}^{W_1} = 1$, $f_{v_2}^{W_1} = x_1 x_3 + x_2 x_3$, $f_{v_3}^{W_1} = x_3$.
- For $W_2 = \{1\}$ the corresponding basis is

$$f_{v_1}^{W_2} = 1, f_{v_2}^{W_2} = x_1 x_3, f_{v_3}^{W_2} = x_3, f_{v_4}^{W_2} = x_2 x_3, f_{v_5}^{W_2} = x_2, f_{v_6}^{W_2} = x_2 x_3^2.$$

On the other hand, the different bases $\{g_v\}_{v \in W_i^\ell}$ of $R(T)^{W_i}$ as a module over $R(T)^W$ are as follows:

- For $W_0 = W$ we have the trivial basis $g_{v_1} = 1$.

- For $W_1 = \langle (12) \rangle$ we obtain the basis $g_{v_1} = 1$, $g_{v_2} = x_1x_3 + x_2x_3$, $g_{v_3} = x_3$.
- For $W_2 = \{1\}$ the corresponding basis is

$$g_{v_1} = 1, g_{v_2} = x_1x_3 + x_2x_3, g_{v_3} = x_3, g_{v_4} = x_2x_3, g_{v_5} = x_2, g_{v_6} = x_2x_3^2.$$

4. BREDON COHOMOLOGY

In this section we use the bases obtained in the previous section to provide a computation of certain Bredon cohomology groups; a brief description on this invariant can be found in the Appendix.

To start assume that G is a compact connected Lie group. Fix $T \subset G$ a maximal torus and let W be the corresponding Weyl group. Consider the W -modules $R(T)$ and $R(G) \otimes \mathbb{Z}[W]$, where $\mathbb{Z}[W]$ denotes the group ring. These modules induce coefficient systems

$$\mathcal{R}_T := H^0(-, R(T)) \text{ and } \mathcal{Z}_W := H^0(-, R(G) \otimes \mathbb{Z}[W]).$$

Explicitly the values of these coefficient systems at an orbit of the form W/W_i are

$$\mathcal{R}_T(W/W_i) = R(T)^{W_i} \text{ and } \mathcal{Z}_W(W/W_i) = R(G) \otimes \mathbb{Z}[W]^{W_i} \cong R(G) \otimes \mathbb{Z}[W/W_i].$$

For every subgroup $W_i \subset W$ the abelian groups $R(T)^{W_i}$ and $R(G) \otimes \mathbb{Z}[W/W_i]$ have the structure of a module over $R(G) \cong R(T)^W$. In particular, \mathcal{R}_T and \mathcal{Z}_W are coefficient systems in the category of $R(G)$ -modules. Suppose that Y is a W -CW complex. Note that in particular Y has the structure of a CW-complex by forgetting the action of W .

Theorem 4.1. *Suppose that G is a compact connected Lie group with $\pi_1(G)$ torsion-free. Let Y be a W -CW complex with cells of the form $W/W_i \times \mathbb{D}^n$ for reflection subgroups $W_i \subset W$. Assume that there is a CW-subcomplex K of Y such that for every $x \in Y$ there is a unique $w \in W$ such that $wx \in K$ and the family $\{W_\sigma \mid \sigma \text{ is a cell in } K\}$ is contained in a family \mathcal{W} of reflection subgroups of W satisfying the coset intersection property. Then there are isomorphisms of $R(G)$ -modules $H_W^*(Y; \mathcal{R}_T) \cong H_W^*(Y; \mathcal{Z}_W) \cong H^*(Y; R(G))$.*

Proof: Consider the CW-complex structure on Y obtained by forgetting the action of W . Let K be a sub CW-complex of Y such that every element in Y is conjugated to a unique element in K and let be a family $\mathcal{W} = \{W_i\}_{i \in \mathcal{I}}$ of reflection subgroups of W containing the subgroups W_σ for all cells σ in K and satisfying the coset intersection property. Then we can use Theorem 3.5 to find a free basis $\{g_v\}_{v \in W_i^\ell}$ of $R(T)^{W_i}$ as a module over $R(T)^W \cong R(G)$ for every $i \in \mathcal{I}$ that is compatible with the inclusions $W_j \supset W_i$ for $i, j \in \mathcal{I}$. We can construct in a similar way a free basis of $R(G) \otimes \mathbb{Z}[W]^{W_i}$ as a module over $R(G)$ for every $i \in \mathcal{I}$ in the following way. Given $v \in W_i^\ell$ define $l_v^{W_i} := 1 \otimes (\sum_{x \in W_i} x^{-1}v^{-1}) \in R(G) \otimes \mathbb{Z}[W]^{W_i}$. The collection $\{l_v^{W_i}\}_{v \in W_i^\ell}$ forms a free basis of $R(G) \otimes \mathbb{Z}[W]^{W_i}$ as a module over $R(G)$. Using this basis we can construct a new basis that behaves better under the different inclusions $W_j \supset W_i$ with $i, j \in \mathcal{I}$ as follows. Given $v \in W$ we can find a unique $i(v) \in \mathcal{I}$ such that $v \in C_{i(v)}$. Define $m_v = l_v^{W_{i(v)}}$. An argument similar to that of Theorem 3.4 shows that $\{m_v\}_{v \in W_i^\ell}$ is a free basis of $R(G) \otimes \mathbb{Z}[W]^{W_i}$ as a module over $R(G)$. These bases enjoy the further property that $\{m_v\}_{v \in W_j^\ell}$ is a subset of

$\{m_v\}_{v \in W_i^\ell}$ whenever $W_j \supset W_i$ for $i, j \in \mathcal{I}$. Note that the bases $\{g_v\}_{v \in W_i^\ell}$ and $\{m_v\}_{v \in W_i^\ell}$ provide an isomorphism of $R(G)$ -modules for every $i \in \mathcal{I}$

$$\begin{aligned} \psi_{W_i} : R(T)^{W_i} &\rightarrow R(G) \otimes \mathbb{Z}[W]^{W_i} \\ g_v &\mapsto m_v \end{aligned}$$

in such a way that whenever $W_j \supset W_i$ for $i, j \in \mathcal{I}$ the following diagram commutes

$$(6) \quad \begin{array}{ccc} R(T)^{W_j} & \longrightarrow & R(T)^{W_i} \\ \psi_{W_j} \downarrow & & \downarrow \psi_{W_i} \\ \bigoplus_{v \in W_i^\ell} R(G) \otimes \mathbb{Z}[W]^{W_j} & \longrightarrow & \bigoplus_{v \in W_j^\ell} R(G) \otimes \mathbb{Z}[W]^{W_i}. \end{array}$$

In the previous diagram the horizontal maps are the inclusion maps; we now show that the different isomorphisms $\{\psi_{W_i}\}_{i \in \mathcal{I}}$ induce an isomorphism of $R(G)$ -modules $H_W^*(Y; \mathcal{R}_T) \cong H_W^*(Y; \mathcal{Z}_W)$. In fact, we are going to show that the cochain complexes computing these Bredon cohomology groups are isomorphic. To see this recall that

$$C_W^n(Y; \mathcal{R}_T) = \bigoplus_{\sigma \in S_n(Y)} R(T)^{W_\sigma},$$

where $S_n(Y)$ is a set of representatives of all W -cells in Y . For $n \geq 0$ let $I_n(K)$ be the set of all n -dimensional cells in K . Since every element in Y is conjugated to a unique element in K it follows that K has a unique representative for all W -cells in Y , thus we can choose $S_n(Y)$ to be $I_n(K)$. As explained in the Appendix given $x \in C_W^n(Y; \mathcal{R}_T)$ and any $\sigma \in I_{n+1}(K)$

$$\delta(x)_\sigma = \sum_{\tau \in I_n(K)} [\tau : \sigma] i_{\tau, \sigma}^*(x_\tau),$$

where $i_{\tau, \sigma}^*$ is the induced map. Since K has a unique representative for all W -cells in Y it follows that in fact $i_{\tau, \sigma}^* : R(T)^{W_\tau} \rightarrow R(T)^{W_\sigma}$ is the restriction map. Similarly,

$$C_W^n(Y; \mathcal{Z}_W) = \bigoplus_{\sigma \in I_n(K)} R(G) \otimes \mathbb{Z}[W]^{W_\sigma}.$$

With this description it is clear that

$$\psi_n := \bigoplus_{\sigma \in I_n(K)} \psi_{W_\sigma} : C_W^n(Y; \mathcal{R}_T) \rightarrow C_W^n(Y; \mathcal{Z}_W)$$

is an isomorphism of $R(G)$ -modules and the commutativity of (6) shows that this defines an isomorphism of cochain complexes over $R(G)$. Finally the proof ends by noting that the cochain complexes $C_W^n(Y; \mathcal{Z}_W)$ and $C^n(Y; R(G))$ are isomorphic and in particular there is an isomorphism of $R(G)$ -modules $H_W^*(Y; \mathcal{Z}_W) \cong H^*(Y; R(G))$. \square

The hypotheses in the previous theorem can be relaxed if we work with rational coefficients. More precisely, consider the coefficient system $\mathcal{R}_T \otimes \mathbb{Q}$ defined by

$$\mathcal{R}_T \otimes \mathbb{Q}(W/W_i) = (R(T) \otimes \mathbb{Q})^{W_i} \cong R(T)^{W_i} \otimes \mathbb{Q}.$$

For this coefficient system we have the following.

Theorem 4.2. *Suppose that G is a compact connected Lie group. Let Y be a W -CW complex of finite type. Then there is an isomorphism of $R(G) \otimes \mathbb{Q}$ -modules $H_W^*(Y; \mathcal{R}_T \otimes \mathbb{Q}) \cong H^*(Y; \mathbb{Q}) \otimes R(G)$.*

Proof: Consider the W -module $R(T) \otimes \mathbb{Q}$. By definition

$$\mathcal{R}_T \otimes \mathbb{Q}(W/W_i) = (R(T) \otimes \mathbb{Q})^{W_i} \cong R(T)^{W_i} \otimes \mathbb{Q}.$$

For such coefficient systems, as pointed out in [5, I. 9], there is an isomorphism of cochain complexes

$$C_W^*(Y; \mathcal{R}_T \otimes \mathbb{Q}) \cong \text{Hom}_{\mathbb{Z}[W]}(C_*(Y); R(T) \otimes \mathbb{Q}) \cong \text{Hom}_{\mathbb{Z}[W]}(C_*(Y); R(T)) \otimes \mathbb{Q}.$$

It follows that $H_W^*(Y; \mathcal{R}_T \otimes \mathbb{Q}) \cong H^*(\text{Hom}_{\mathbb{Z}[W]}(C_*(Y), R(T)) \otimes \mathbb{Q})$. Let $D^* = \text{Hom}(C_*(Y); R(T))$. This cochain complex has a linear action of W defined by $(w \cdot f)(x) = wf(w^{-1}x)$. Under this action we have an isomorphism of cochain complexes

$$\text{Hom}_{\mathbb{Z}[W]}(C_*(Y), R(T)) \otimes \mathbb{Q} = (D^*)^W \otimes \mathbb{Q} \cong (D^* \otimes \mathbb{Q})^W.$$

Consider $H^*(W; D^* \otimes \mathbb{Q})$; as usual, there are two spectral sequences computing this group cohomology with coefficients in a cochain complex. On the one hand, we have

$$E_2^{p,q} = H^p(W; H^q(D^* \otimes \mathbb{Q})) \implies H^{p+q}(W; D^* \otimes \mathbb{Q}).$$

Since we are working with rational coefficients it follows that

$$E_2^{p,q} = H^p(W; H^q(D^* \otimes \mathbb{Q})) \cong \begin{cases} H^q(D^* \otimes \mathbb{Q})^W & \text{if } p = 0, \\ 0 & \text{if } p > 0. \end{cases}$$

On the other hand, we have a spectral sequence

$$E_1^{p,q} = H^q(W; D^p \otimes \mathbb{Q}) \implies H^{p+q}(W; D^* \otimes \mathbb{Q}).$$

with the differential d_1 induced by the differential of the cochain complex D^* . In this case

$$H^q(W; D^p \otimes \mathbb{Q}) \cong \begin{cases} (D^p \otimes \mathbb{Q})^W & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}$$

Thus the E_2 -term of this spectral sequence is given by

$$E_2^{p,q} = \begin{cases} H^p((D^* \otimes \mathbb{Q})^W) & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}$$

Both of these spectral sequences collapse on the E_2 -term without extension problems and both converge to $H^*(W; D^* \otimes \mathbb{Q})$. It follows that there is an isomorphism of $R(G) \otimes \mathbb{Q}$ modules

$$(7) \quad H_W^*(Y; \mathcal{R}_T \otimes \mathbb{Q}) = H^*((D^* \otimes \mathbb{Q})^W) \cong H^*(D^* \otimes \mathbb{Q})^W = (H^*(Y; R(T) \otimes \mathbb{Q}))^W.$$

The universal coefficient theorem shows that $H^*(Y; R(T) \otimes \mathbb{Q}) \cong H^*(Y; \mathbb{Q}) \otimes R(T)$, with W -acting diagonally and thus

$$(8) \quad H^*(Y; R(T) \otimes \mathbb{Q})^W \cong (H^*(Y; \mathbb{Q}) \otimes R(T))^W.$$

By assumption Y has finite type and thus $H^*(Y; \mathbb{Q})$ is a finite dimensional W -representation over \mathbb{Q} . Using Theorem 4.3 below, we obtain an isomorphism of $R(G) \otimes \mathbb{Q}$ -modules

$$(9) \quad (H^*(Y; \mathbb{Q}) \otimes R(T))^W \cong H^*(Y; \mathbb{Q}) \otimes R(G).$$

The theorem follows from (7), (8) and (9). □

Theorem 4.3. *Let G be a compact connected Lie group with torsion-free fundamental group. Fix $T \subset G$ a maximal torus and let W be the corresponding Weyl group. If A is any finite dimensional W -representation over \mathbb{Q} then $(A \otimes R(T))^W$ is isomorphic to $A \otimes R(G)$ as modules over $R(G) \otimes \mathbb{Q}$.*

Proof: Let's consider first the particular case where A is the regular representation; that is, $A = \mathbb{Q}[W]$. To prove the theorem in this case let's consider $R(T)_0$ to be the $R(G)$ -module $R(T)$ endowed with the *trivial* W -action. Define a homomorphism of $R(G) \otimes \mathbb{Q}$ -modules

$$\begin{aligned} \varphi : \mathbb{Q}[W] \otimes R(T)_0 &\rightarrow \mathbb{Q}[W] \otimes R(T) \\ v \otimes m &\mapsto v \otimes vm, \end{aligned}$$

for $v \in W$ and $m \in R(T)_0$. It is easy to see that φ is a bijection. Moreover, for any $w, v \in W$ and $m \in R(T)_0$ we have

$$\begin{aligned} \varphi(w \cdot (v \otimes m)) &= \varphi((w \cdot v) \otimes m) = wv \otimes (wv \cdot m) \\ &= w \cdot (v \otimes (v \cdot m)) = w \cdot \varphi(v \otimes m). \end{aligned}$$

Therefore φ is an isomorphism of W -modules over $R(G) \otimes \mathbb{Q}$. In particular, there is an isomorphism of $R(G) \otimes \mathbb{Q}$ -modules

$$\mathbb{Q} \otimes R(T)_0 \cong (\mathbb{Q}[W] \otimes R(T)_0)^W \cong (\mathbb{Q}[W] \otimes R(T))^W.$$

Since $R(T)_0$ is a free module over $R(G)$ of rank $|W|$ by [16, Theorem 1], we conclude that there is an isomorphism of $R(G) \otimes \mathbb{Q}$ -modules, $(\mathbb{Q}[W] \otimes R(T))^W \cong \mathbb{Q}[W] \otimes R(G)$. This proves the theorem for $A = \mathbb{Q}[W]$. Suppose now that A is any finite dimensional W -module over \mathbb{Q} . Since W is a finite group then A is a projective module over $\mathbb{Q}[W]$ and thus we can find some $\mathbb{Q}[W]$ -module B and some integer $m \geq 1$ such that $A \oplus B \cong (\mathbb{Q}[W])^m$. Note that $(A \otimes R(T))^W \oplus (B \otimes R(T))^W = ((\mathbb{Q}[W])^m \otimes R(T))^W$ and since the theorem is true for $\mathbb{Q}[W]$, we conclude that $(A \otimes R(T))^W \oplus (B \otimes R(T))^W$ is a free module over $R(G) \otimes \mathbb{Q}$. This means that $(A \otimes R(T))^W$ is a projective module over $R(G) \otimes \mathbb{Q}$. On the other hand, since G is a compact connected Lie group with $\pi_1(G)$ torsion-free we have

$$R(G) = \mathbb{Z}[x_1, \dots, x_k] \otimes \mathbb{Z}[y_1^{\pm 1}, \dots, y_l^{\pm 1}]$$

with $k + l = \text{rank}(G)$ (see for example [16, Section 1]). We conclude that $R(G) \otimes \mathbb{Q}$ is a tensor product of a polynomial ring and a Laurent polynomial ring over \mathbb{Q} . In [20] Swan extended the Quillen-Suslin theorem to these kind of rings. Thus any projective $R(G) \otimes \mathbb{Q}$ -module is free. In particular $(A \otimes R(T))^W$ is a free $R(G) \otimes \mathbb{Q}$ -module.

As a final step we show that $\text{rank}_{R(G) \otimes \mathbb{Q}}(A \otimes R(T))^W = \dim_{\mathbb{Q}} A$. Assuming this is true, then both $(A \otimes R(T))^W$ and $A \otimes R(G)$ are free modules over $R(G) \otimes \mathbb{Q}$ of the same rank and hence isomorphic, proving the theorem.

Let us denote by k the fraction field of $R(G)$ (which is also the fraction field of $R(G) \otimes \mathbb{Q}$). Because $R(G)$ is an invariant subring, $K := R(T) \otimes_{R(G)} k$ is the fraction field of $R(T)$. Therefore it suffices to prove that $\dim_k((A \otimes k) \otimes_k K)^W = \dim_k(A \otimes k) = \dim_{\mathbb{Q}} A$. To see this note that $A \otimes k$ is a finite dimensional W -representation over k and K is a finite Galois extension of k with Galois group W . The result follows applying the lemma given below. \square

Lemma 4.4. *Suppose that k is a field of characteristic 0. Let K be a finite Galois extension of k with Galois group G . Assume that A is a finite dimensional G -representation over k . If G acts diagonally on $A \otimes_k K$ then $\dim_k(A \otimes_k K)^G = \dim_k A$.*

Proof: Since k has characteristic 0 then we can find a normal basis of K/k ; that is, we can find some $x \in K$ such that the collection $\{gx\}_{g \in G}$ forms a basis of K as a k -vector space. Using this basis we conclude that the field K , seen as a G -representation over k , is such that $K \cong k[G]$. Denote by A_0 the vector space A endowed with the *trivial* G -action. Then as in the previous theorem there is an isomorphism of G -representations $A \otimes k[G] \cong A_0 \otimes k[G]$. In particular $\dim_k(A \otimes K)^G = \dim_k(A \otimes k[G])^G = \dim_k(A_0 \otimes k[G])^G = \dim_k A_0 = \dim_k A$. \square

5. EQUIVARIANT K -THEORY

Given a compact G -space X the complex equivariant K -theory of X , denoted by $K_G^0(X)$, is defined to be the Grothendieck group associated to the semi-ring of isomorphism classes of G -equivariant complex vector bundles over X . As usual, if X is a based G -space with G acting trivially on the base point x_0 , the reduced equivariant K -theory of X is defined to be $\tilde{K}_G^0(X) = \ker(i^* : K_G^0(X) \rightarrow K_G^0(\{x_0\}) = R(G))$, where $i : \{x_0\} \rightarrow X$ is the inclusion map. When $q \geq 0$ we define $K_G^q(X) = \tilde{K}_G^q(\mathbb{S}^q \wedge X_+)$. Equivariant K -theory is representable; that is, for every q there exists a G -space \mathbb{K}_G^q such that for every compact G -space X there is a natural isomorphism $\Phi_X : K_G^q(X) \xrightarrow{\cong} [X, \mathbb{K}_G^q]_G$. Here $[X, \mathbb{K}_G^q]_G$ denotes the set of G -equivariant homotopy classes of G -equivariant maps $f : X \rightarrow \mathbb{K}_G^q$. When X is not a compact space, $K_G^q(X)$ is defined to be $[X, \mathbb{K}_G^q]_G$. For a G -space X and $q \geq 0$ there is a natural isomorphism $K_G^q(X) \cong K_G^{q+2}(X)$ and we denote $K_G^*(X) = K_G^0(X) \oplus K_G^1(X)$. The projection to a point $X \rightarrow \{x_0\}$ induces a homomorphism of rings $R(G) = K_G^*(\{x_0\}) \rightarrow K_G^*(X)$ and thus $K_G^*(X)$ naturally has the structure of a $\mathbb{Z}/2$ -graded $R(G)$ -algebra, where $R(G)$ is the representation ring of G . Recall that when G is a compact Lie group $R(G)$ is a domain and also a Noetherian ring (see for example [18, Corollary 3.3]).

Definition 5.1. Suppose that R is a domain. If M is a finitely generated R -module then the rank of M , denoted by $\text{rank}_R(M)$, is defined to be largest integer n for which there is an injective homomorphism of R -modules $i : R^n \rightarrow M$.

It is easy to see that if K is the field of quotients of R and M is a finitely generated R -module, then the rank of M is the dimension of $M \otimes K$ as a vector space over K . In particular, if $f : M \rightarrow N$ is a homomorphism of finitely generated R -modules such that f induces an isomorphism after passing to the modules of fractions then $\text{rank}_R(M) = \text{rank}_R(N)$.

Lemma 5.2. *Let G be a compact, connected Lie group with $\pi_1(G)$ torsion-free and let $T \subset G$ be a maximal torus. For any compact G -space X $\text{rank}_{R(G)} K_G^*(X) = \text{rank}_{\mathbb{Z}} K^*(X^T)$.*

Proof: Assume that G is a compact Lie group with $\pi_1(G)$ torsion-free. The compactness of X implies that $K_G^*(X)$ is a finitely generated $R(G)$ -module. Also Hodgkin's spectral sequence gives an isomorphism of $R(T)$ -modules (see for example [6, Lemma 2.5], [12])

$$(10) \quad K_T^*(X) \cong K_G^*(X) \otimes_{R(G)} R(T).$$

In particular, since $R(T)$ is a free module over $R(G)$ by [16, Theorem 1], it follows that

$$(11) \quad \text{rank}_{R(G)} K_G^*(X) = \text{rank}_{R(T)} K_T^*(X).$$

On the other hand, as an application of the localization theorem [17, Theorem 4.1], if $i : X^T \rightarrow X$ is the inclusion map, then $i^* : K_T^*(X) \rightarrow K_T^*(X^T)$ is a homomorphism of $R(T)$ -algebras that induces an isomorphism after passing to the modules of fractions. Using the previous comment we see

$$(12) \quad \text{rank}_{R(T)} K_T^*(X) = \text{rank}_{R(T)} K_T^*(X^T)$$

$$(13) \quad = \text{rank}_{R(T)} R(T) \otimes K^*(X^T) = \text{rank}_{\mathbb{Z}} K^*(X^T).$$

The lemma follows from equations (11) and (12). \square

In general, if G is a compact Lie group and X is a G -CW complex then associated to the skeletal filtration

$$X^0 \subset X^1 \subset \cdots \subset X^n \subset \cdots \subset X$$

there is a multiplicative spectral sequence [17] with

$$E_2^{p,q} = H_G^p(X; \mathcal{K}_G^q) \implies K_G^{p+q}(X)$$

where \mathcal{K}_G^q denotes the coefficient system defined by $G/H \mapsto K_G^q(G/H)$.

Theorem 5.3. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free and $T \subset G$ a maximal torus. Suppose that X is a compact G -CW complex with connected maximal rank isotropy subgroups. Assume that there is a CW-subcomplex K of X^T such that for every element $x \in X^T$ there is a unique $w \in W$ such that $w x \in K$ and the family $\{W_\sigma \mid \sigma \text{ is a cell in } K\}$ is contained in a family \mathcal{W} of reflection subgroups of W satisfying the coset intersection property. If in addition $H^*(X^T; \mathbb{Z})$ is torsion-free, then $K_G^*(X)$ is a free module over $R(G)$ of rank equal to $\sum_{i \geq 0} \text{rank}_{\mathbb{Z}} H^i(X^T; \mathbb{Z})$.*

Proof: Fix $T \subset G$ a maximal torus and let W be the associated Weyl group. After replacing X with an equivalent G -CW complex, we may assume that the cells of X are of the form $G/H \times \mathbb{D}^n$, where H is a connected subgroup with $T \subset H$. Consider the spectral sequence associated to the skeletal filtration of X

$$E_2^{p,q} = H_G^p(X; \mathcal{K}_G^q) \implies K_G^{p+q}(X)$$

Under the given hypotheses we have the following:

Claim: The E_2 -term of this spectral sequence is given by

$$E_2^{p,q} \cong \begin{cases} H^p(X^T; R(G)) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd.} \end{cases}$$

Suppose for a moment that the claim is true. The universal coefficient theorem provides an isomorphism $H^*(X^T; R(G)) \cong H^*(X^T; \mathbb{Z}) \otimes R(G)$. In particular, $H^*(X^T; R(G))$ is a free module over $R(G)$ of rank $n := \sum_{i \geq 0} \text{rank}_{\mathbb{Z}} H^i(X^T; \mathbb{Z})$ and the same is true for $E_2^{*,*}$. The spectral sequence converges to $K_G^*(X)$ and by Lemma 5.2, $\text{rank}_{R(G)} K_G^*(X) = \text{rank}_{\mathbb{Z}} K^*(X^T)$. Using the Chern character we see that $\text{rank}_{\mathbb{Z}} K^*(X^T) = \sum_{i \geq 0} \text{rank}_{\mathbb{Z}} H^i(X^T; \mathbb{Z}) = n$. Thus $K_G^*(X)$ and $E_2^{*,*}$ have the same rank as $R(G)$ -modules and $E_2^{*,*}$ is a free $R(G)$ -module. This shows that all the differentials $\{d_r\}_{r \geq 2}$ must be trivial and the spectral sequence collapses on

the E_2 -term. Since $E_2^{*,*}$ is a free module over $R(G)$ and the spectral sequence is a sequence of $R(G)$ -modules, there are no extension problems and we conclude that $K_G^*(X)$ is a free module over $R(G)$ of rank n .

We now prove the claim. Suppose first that q is odd; in this case $\mathcal{K}_G^q(G/H) = K_H^q(*) = 0$. Therefore when q is odd \mathcal{K}_G^q is the trivial coefficient system and in particular $H_G^p(X, \mathcal{K}_G^q) = 0$. Suppose now that q is even. In this case $\mathcal{K}_G^q(G/H) = K_H^q(*) = R(H)$. This shows that when q is even $H_G^p(X; \mathcal{K}_G^q) = H_W^p(X; \mathcal{R})$, where \mathcal{R} is the coefficient system defined by $\mathcal{R}(G/H) = R(H)$. Next we show that there is a natural isomorphism

$$(14) \quad H_G^*(X; \mathcal{R}) \cong H_W^*(X^T; \mathcal{R}_T).$$

To see this note that every cell in X is of the form $G/H \times \mathbb{D}^n$ with $T \subset H$. For such subgroup we have a natural ring isomorphism $\varphi_H : R(H) \rightarrow R(T)^{WH}$. These isomorphisms can be assembled to obtain an isomorphism of cochain complexes

$$C_G^n(X; \mathcal{R}) \cong \bigoplus_{\sigma \in S_n(X)} R(G_\sigma) \xrightarrow{\oplus \varphi_{G_\sigma}} \bigoplus_{\sigma \in S_n(X)} R(T)^{WG_\sigma} \cong C_W^n(X^T; \mathcal{R}_T),$$

where $S_n(X)$ is a chosen set of representatives of all n -dimensional G -cells of X . In particular, there is an isomorphism of $R(G)$ -modules $H_G^*(X; \mathcal{R}) \cong H_W^*(X^T; \mathcal{R}_T)$. Finally, note that X^T is a compact W -CW complex whose n -cells are of the form $W/WH \times \mathbb{D}^n$, for some connected maximal rank isotropy subgroups $H \subset G$. In particular, WH is a reflection subgroup of W and by Theorem 4.1 there is an isomorphism of $R(G)$ -modules

$$(15) \quad H_W^*(X^T; \mathcal{R}_T) \cong H^*(X^T; R(G))$$

The claim follows from (14) and (15). This proves the theorem. \square

With rational coefficients we have a similar spectral sequence

$$E_2^{p,q} = H_G^p(X; \mathcal{K}_G^q \otimes \mathbb{Q}) \implies K_G^{p+q}(X) \otimes \mathbb{Q}.$$

In this case the situation simplifies and we have the following theorem.

Theorem 5.4. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free and $T \subset G$ a maximal torus. Suppose that X is a compact G -CW complex with connected maximal rank isotropy subgroups. Then $K_G^*(X) \otimes \mathbb{Q}$ is a free module over $R(G) \otimes \mathbb{Q}$ of rank equal to $\sum_{i \geq 0} \text{rank}_{\mathbb{Q}} H^i(X^T; \mathbb{Q})$.*

Proof: As in the previous theorem we may assume that X is a G -CW complex whose cells are of the form $G/H \times \mathbb{D}^n$, where H is a connected subgroup with $T \subset H$, for a fixed maximal torus $T \subset G$. Consider the spectral sequence

$$E_2^{p,q} = H_G^p(X; \mathcal{K}_G^q \otimes \mathbb{Q}) \implies K_G^{p+q}(X) \otimes \mathbb{Q}.$$

As before we have $E_2^{p,q} = 0$ when q is odd and when q is even there is an isomorphism $E_2^{p,q} = H_G^p(X^T; \mathcal{K}_G^q \otimes \mathbb{Q}) \cong H_W^p(X^T; \mathcal{R}_T \otimes \mathbb{Q})$. Since X is compact then X^T is of finite type. Using Theorem 4.2 we obtain an isomorphism of $R(G) \otimes \mathbb{Q}$ -modules, $H_W^*(X^T; \mathcal{R}_T \otimes \mathbb{Q}) \cong H^*(X^T; \mathbb{Q}) \otimes R(G)$. Thus

$$E_2^{p,q} \cong \begin{cases} H^p(X^T; \mathbb{Q}) \otimes R(G) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd.} \end{cases}$$

This implies that $E_2^{*,*}$ is a free $R(G) \otimes \mathbb{Q}$ module of rank

$$n = \sum_{i \geq 0} \dim_{\mathbb{Q}} H^i(X^T; \mathbb{Q}) = \sum_{i \geq 0} \text{rank}_{\mathbb{Z}} H^i(X^T; \mathbb{Z}).$$

An argument similar to the one provided in Theorem 5.3 shows that $K_G^*(X) \otimes \mathbb{Q}$ has rank equal to n as an $R(G) \otimes \mathbb{Q}$ -module, and that the spectral sequence collapses at the E_2 -term without extension problems. Therefore $K_G^*(X) \otimes \mathbb{Q}$ is a free module over $R(G) \otimes \mathbb{Q}$ of rank n . \square

6. APPLICATIONS

In this section some applications of Theorems 5.3 and 5.4 are discussed. We consider examples arising from representation spheres and the commuting variety in the Lie algebra of a compact connected Lie group G . We also consider applications arising from inertia spaces, in particular we explore in detail the case of the space of commuting elements in a Lie group G . Throughout this section G will denote a compact connected Lie group with torsion-free fundamental group, $T \subset G$ a maximal torus and W the corresponding Weyl group.

6.1. Conjugation action of G on itself. Let G act on itself by conjugation. By Example 2.3 this action has connected maximal rank isotropy subgroups as we are assuming that $\pi_1(G)$ is torsion-free. In this case $G^T = T$ with W acting smoothly on T , in particular T has the homotopy type of a finite W -CW complex by [13, Theorem 1]. Note also that $H^*(T; \mathbb{Z})$ is torsion free and of rank 2^r , where r is the rank of G . Let \mathfrak{t} denote the Lie algebra of T . Fix a basis Δ for the root system Φ corresponding to the pair (G, T) . Let $\mathfrak{C}(\Delta)$ be the (closed) Weyl chamber determined by Δ and A_0 the unique (closed) alcove in $\mathfrak{C}(\Delta)$ containing $0 \in \mathfrak{t}$. Let $K \subset T$ denote the corresponding alcove in T obtained via the exponential map. We can give T a W -CW complex structure in such a way that K is CW subcomplex of the underlying CW-complex structure in T . If we further require that G is simply connected, then as a consequence of [11, Theorem VII 7.9] it follows that any element in T has a unique representative in K under the W -action. Also, for every cell σ in K we have $W_\sigma = W_I$ for some $I \subset \Delta$. Since the family $\mathcal{W} := \{W_I\}_{I \subset \Delta}$ satisfies the coset intersection property by Example 3.3, then it follows that the hypothesis of Theorem 5.3 are met in this case yielding the following.

Corollary 6.1. *Let G be a simply connected compact Lie group act on itself by conjugation. Then $K_G^*(G)$ is a free module over $R(G)$ of rank 2^r , where r denotes the rank of G .*

The previous result was proved in [6] for the more general case of compact connected Lie groups with $\pi_1(G)$ torsion-free.

6.2. Linear Representations. Let V be a finite dimensional linear representation of G via a homomorphism $\rho : G \rightarrow GL(V)$. The action of G on V induces an action of G on \mathbb{S}^V , the one point compactification of V , with G acting trivially at the point at infinity. When V is a complex representation the Thom isomorphism theorem provides an isomorphism of $R(G)$ -modules

$$\tilde{K}_G^q(\mathbb{S}^V) \cong \begin{cases} R(G) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd.} \end{cases}$$

For real representations the coefficient groups $\tilde{K}_G^*(\mathbb{S}^V)$ are not known in general. On the other hand, we can associate to V another G -space by considering the unit sphere $\mathbb{S}(V)$. To be

more precise, endow V with a norm $\|\cdot\|$ that is invariant under the G -action. Such a norm can be constructed applying the usual trick of averaging any norm with a suitable Haar measure on G . Then $\mathbb{S}(V) := \{x \in V \mid \|x\| = 1\}$ is a G -space. Note that $\mathbb{S}(1 \oplus V) \cong \mathbb{S}^V$. Our main results imply the following proposition.

Proposition 6.2. *Let V denote a representation of G such that the action has connected, maximal rank isotropy subgroups. Then $K_G^*(\mathbb{S}(V)) \otimes \mathbb{Q}$ and $K_G^*(\mathbb{S}^V) \otimes \mathbb{Q}$ are both free of rank two as $R(G) \otimes \mathbb{Q}$ -modules.*

Suppose now that $V = \mathfrak{g}$, the Lie algebra of G endowed with the adjoint representation. Given any $X \in \mathfrak{g}$ the isotropy subgroup G_X is given by $G_X = \{g \in G \mid Ad_g(X) = X\}$. Any element $X \in \mathfrak{g}$ is contained in some Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$. Let T be the maximal torus in G whose Lie algebra is \mathfrak{t} and suppose that $g \in T$. Since $\exp(tX) \in T$ for every $t \in \mathbb{R}$, then g commutes with $\exp(tX)$ and thus $\exp(tX) = g \exp(tX) g^{-1} = \exp(Ad_g(tX))$ for every $t \in \mathbb{R}$. This shows that $Ad_g(X) = X$ for all $g \in T$. In particular, the isotropy subgroup G_X contains the maximal torus T in G . Also, G_X is connected for every $X \in \mathfrak{g}$ by [9, Theorem 3.3.1]. This proves that G acts with connected maximal rank isotropy subgroups on both $\mathbb{S}^{\mathfrak{g}}$ and $\mathbb{S}(\mathfrak{g})$. Note also that $\mathfrak{g}^T = \mathfrak{t}$. Therefore $(\mathbb{S}^{\mathfrak{g}})^T = \mathbb{S}^{\mathfrak{t}}$ and $\mathbb{S}(\mathfrak{g})^T = \mathbb{S}(\mathfrak{t})$ are spheres of dimension r and $r - 1$ respectively, where r denotes the rank of G . In particular $H^*((\mathbb{S}^{\mathfrak{g}})^T; \mathbb{Z})$ and $H^*(\mathbb{S}(\mathfrak{g})^T; \mathbb{Z})$ are both torsion-free and of rank 2. Also, these spaces have the structure of manifolds on which W acts smoothly. By [13, Theorem 1] it follows that $(\mathbb{S}^{\mathfrak{g}})^T$ and $\mathbb{S}(\mathfrak{g})^T$ have the structure of a W -CW complex. Note that \mathfrak{t} can be decomposed into (closed) Weyl chambers and each (closed) Weyl chamber $\mathfrak{C}(\Delta)$ is determined by a basis Δ of Φ . For every $x \in \mathfrak{C}(\Delta)$ the isotropy group W_x is a subgroup of the form W_I , for some $I \subset \Delta$ and the family $\mathcal{W} = \{W_I\}_{I \subset \Delta}$ satisfies the coset intersection property by Example 3.3. Since W acts simply transitively on the set of all Weyl chambers it follows that we can obtain W -CW complex structures on $(\mathbb{S}^{\mathfrak{g}})^T$ and $\mathbb{S}(\mathfrak{g})^T$ in such a way that conditions of Theorem 5.3 are satisfied. As a corollary the following is obtained.

Corollary 6.3. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free. Suppose that G act on its Lie algebra \mathfrak{g} by the adjoint representation. If r is the rank of G , then*

$$\tilde{K}_G^q(\mathbb{S}^{\mathfrak{g}}) \cong \begin{cases} R(G) & \text{if } q \equiv r \pmod{2}, \\ 0 & \text{if } q + 1 \equiv r \pmod{2}. \end{cases}$$

Similarly

$$K_G^q(\mathbb{S}(\mathfrak{g})) \cong \begin{cases} 0 & \text{for } q \text{ odd and } r \text{ odd,} \\ R(G) & \text{for } q \text{ odd and } r \text{ even,} \\ R(G) & \text{for } q \text{ even and } r \text{ even,} \\ R(G)^2 & \text{for } q \text{ even and } r \text{ odd.} \end{cases}$$

6.3. Arrangements of hyperplanes. Let X be a G -CW complex that satisfies the hypotheses of Theorem 5.3. Let $Y \subset X$ be a G -CW subcomplex such that $H^*(Y^T; \mathbb{Z})$ is torsion-free. Then Y seen as a G -space also satisfies the hypotheses of Theorem 5.3 and in particular $K_G^*(Y)$ is a free module over $R(G)$ of rank $\sum_{j \geq 0} \text{rank}_{\mathbb{Z}} H^j(Y^T; \mathbb{Z})$. Interesting examples of G -spaces can be obtained in this way. Suppose for example that $\mathcal{H} = \{H_a\}_{a \in \Delta}$ is an arrangement of hyperplanes in \mathfrak{t} such that whenever $x \in H_a$ for some $a \in \Delta$ and $w \in W$ then $wa \in H_{a'}$ for

some $a' \in \Delta$; that is, \mathcal{H} is a W -equivariant arrangement of hyperplanes in \mathfrak{t} . Consider the space

$$C(\mathcal{H}) := \{X \in \mathfrak{t} \mid X \notin \cup_{a \in \Delta} H_a\}.$$

Then $C(\mathcal{H})$ is a W -subspace of \mathfrak{t} . The connected components of $C(\mathcal{H})$ are convex subspaces of \mathfrak{t} and thus they are contractible. In particular, $H^*(C(\mathcal{H}); \mathbb{Z})$ is torsion-free and has rank $n_{\mathcal{H}}$, where $n_{\mathcal{H}}$ is the number of connected components of $C(\mathcal{H})$. Let $X(\mathcal{H}) := \bigcup_{g \in G} Ad_g(C(\mathcal{H})) \subset \mathfrak{g}$. In this way we obtain a G -subspace of \mathfrak{g} , with G acting by the adjoint representation. It is easy to see directly that $X(\mathcal{H})$ is a G -algebraic variety and in particular it has the homotopy type of a G -CW complex. Note that $X(\mathcal{H})^T = C(\mathcal{H})$. As a consequence¹ of Theorem 5.3 the following is obtained.

Corollary 6.4. *Let G be a compact connected Lie group with $\pi_1(G)$ torsion-free. Let $\mathcal{H} = \{H_a\}_{a \in \Delta}$ be a W -equivariant arrangement of hyperplanes in \mathfrak{t} . Then $K_G^*(X(\mathcal{H}))$ is a free $R(G)$ -module of rank $n_{\mathcal{H}}$, where $n_{\mathcal{H}}$ is the number of connected components of $C(\mathcal{H})$.*

6.4. The commuting variety. Let G be a compact connected Lie group and \mathfrak{g} its Lie algebra. For every integer $n \geq 1$ the commuting variety in \mathfrak{g} is defined to be

$$C_n(\mathfrak{g}) = \{(X_1, \dots, X_n) \in \mathfrak{g}^n \mid [X_i, X_j] = 0 \text{ for all } 1 \leq i, j \leq n\}.$$

$C_n(\mathfrak{g})$ has the structure of an algebraic variety (possibly) with singularities. The group G acts on $C_n(\mathfrak{g})$ via the diagonal action of the adjoint representation. Let $(X_1, \dots, X_n) \in C_n(\mathfrak{g})$ and $g_i := \exp(X_i)$ for every $1 \leq i \leq n$. Note that $(g_1, \dots, g_n) \in \text{Hom}(\mathbb{Z}^n, G)_{\mathbb{1}}$. Indeed, the Baker–Campbell–Hausdorff formula shows that for every $1 \leq i, j \leq n$

$$g_i g_j = \exp(X_i + X_j) = \exp(X_j + X_i) = g_j g_i.$$

Now the path $\gamma : [0, 1] \rightarrow \text{Hom}(\mathbb{Z}^n, G)$ given by $t \mapsto (\exp(tX_1), \dots, \exp(tX_n))$ provides a homotopy from the trivial representation $\mathbb{1}$ to (g_1, \dots, g_n) . In particular, for every $t \in [0, 1]$ the n -tuple $(\exp(tX_1), \dots, \exp(tX_n))$ is contained in some maximal torus of G by [3, Lemma 4.2]. Let $\epsilon > 0$ be small enough so that the exponential map is injective on $B_\epsilon(0)$. Choose $0 < t < \epsilon$ and let T be a maximal torus containing $(\exp(tX_1), \dots, \exp(tX_n))$. Let $\mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of T . This is a Cartan subalgebra and since the exponential map is injective on $B_\epsilon(0)$ it follows that $tX_i \in \mathfrak{t}$ and thus $X_i \in \mathfrak{t}$ for all $1 \leq i \leq n$. We conclude that any $\underline{X} := (X_1, \dots, X_n) \in C_n(\mathfrak{g})$ is contained in some Cartan subalgebra in \mathfrak{g} .² In particular, the isotropy subgroup $G_{\underline{X}}$ contains the maximal torus T in G whose Lie algebra is \mathfrak{t} . Consider $C_n(\mathfrak{g})^+$, the one point compactification of $C_n(\mathfrak{g})$, with G acting trivially on the point at infinity. If we assume that $G_{\underline{X}}$ is connected for every $\underline{X} \in C_n(\mathfrak{g})$ then $C_n(\mathfrak{g})^+$ is an example of a compact space on which G acts with connected maximal rank isotropy subgroups. This is the case for example if G is in the family \mathcal{P} . Indeed, if $\underline{X} := (X_1, \dots, X_n) \in C_n(\mathfrak{g})$ then $g \exp(X_i) g^{-1} = \exp(Ad_g(X_i))$. Thus $g \in G_{\underline{X}}$ if and only if $g \in Z_G(\exp(X_1), \dots, \exp(X_n))$ which is connected and of maximal rank if $G \in \mathcal{P}$. In this case, given $T \subset G$ a maximal torus with lie algebra \mathfrak{t} , then $(C_n(\mathfrak{g})^+)^T = (\mathfrak{t}^n)^+ = \mathbb{S}^n$. The Weyl group W acts smoothly on the manifold \mathbb{S}^n and by [13, Theorem 1] it has the homotopy type of a finite W -CW complex.

¹In Theorem 5.3 it is required that X^T is a compact W -CW complex. The compactness hypothesis is used to prove that X is a G -CW complex and that $\sum_{j \geq 0} \text{rank}_{\mathbb{Z}} H^j(X^T; \mathbb{Z})$ is finite. In this case both of these conditions are satisfied and thus the theorem can be applied.

²This statement is not true for non-compact Lie algebras, for example this is not true for $\mathfrak{gl}_n(\mathbb{C})$.

Let r be the rank of G . Then $H^k(\mathbb{S}^n; \mathbb{Q})$ is 0 unless $k = 0$ or $k = nr$ in which case this is a one dimensional W -representation over \mathbb{Q} . This together with Theorem 5.4 yield the following corollary.

Corollary 6.5. *Suppose that $G \in \mathcal{P}$ is a Lie group of rank r . Then there is an isomorphism of $R(G) \otimes \mathbb{Q}$ -modules*

$$\tilde{K}_G^q(C_n(\mathfrak{g})^+) \otimes \mathbb{Q} \cong \begin{cases} R(G) \otimes \mathbb{Q} & \text{if } q \equiv rn \pmod{2}, \\ 0 & \text{if } q + 1 \equiv rn \pmod{2}. \end{cases}$$

6.5. Inertia spaces. Let X be a compact G -CW complex. As we have seen in Section 2, if G acts on X with connected maximal rank isotropy subgroups such that $\pi_1(G_x)$ is torsion-free for all $x \in X$, then the action of G on the inertia space ΛX also has connected maximal rank isotropy subgroups. Applying Theorem 5.4 yields the following

Theorem 6.6. *Let X denote a compact G -CW complex with connected maximal rank isotropy subgroups all of which have torsion-free fundamental group. Then $K_G^*(\Lambda X) \otimes \mathbb{Q}$ (as an ungraded module) is a free $R(G) \otimes \mathbb{Q}$ -module of rank equal to $2^r \cdot (\sum_{i \geq 0} \dim_{\mathbb{Q}} H^i(X^T; \mathbb{Q}))$, where r is the rank of G .*

Suppose now that a compact G -CW complex X is such that all of its isotropy subgroups are of maximal rank and belong to the family \mathcal{P} . Then by Proposition 2.9 it follows that G acts on $\Lambda^n(X)$ with connected maximal rank isotropy subgroups for every $n \geq 0$. Note that if $T \subset G$ is a maximal torus, then $(\Lambda^n(X))^T = X^T \times T^n$. These remarks together with Theorem 5.4 imply the following result.

Theorem 6.7. *Let X denote a compact G -CW complex such that all of its isotropy subgroups lie in \mathcal{P} and are of maximal rank. Then $K_G^*(\Lambda^n(X)) \otimes \mathbb{Q}$ is a free $R(G) \otimes \mathbb{Q}$ -module of rank equal to $2^{nr} \cdot (\sum_{i \geq 0} \dim_{\mathbb{Q}} H^i(X^T; \mathbb{Q}))$ where r is the rank of G .*

Suppose that $G \in \mathcal{P}$ and consider the particular case $X = \{x_0\}$. Then $\Lambda^n(X) = \text{Hom}(\mathbb{Z}^n, G)$ with the conjugation action of G . As a corollary of the previous theorem we obtain the following.

Corollary 6.8. *Suppose that $G \in \mathcal{P}$ is of rank r . Then $K_G^*(\text{Hom}(\mathbb{Z}^n, G)) \otimes \mathbb{Q}$ is free of rank 2^{nr} as an $R(G) \otimes \mathbb{Q}$ -module.*

Remark: The previous result is no longer true if the assumption that $\pi_1(G)$ is torsion-free is removed. For example, consider the case $n = 1$ and suppose that $G = PSU(3)$ acts on itself by conjugation. Then by [6, Proposition 7.4] there is a G -equivariant line bundle L over G such that $L^{\otimes 3} = 1$, $R(G)/\text{Ann}([L] - 1) = \mathbb{Z}$ and

$$\begin{aligned} K_G^0(G) &= R(G)^2 \oplus \mathbb{Z}([L] - 1) \oplus \mathbb{Z}([L]^2 - 1), \\ K_G^1(G) &= R(G)^2. \end{aligned}$$

In particular, $K_G^*(G)$ and $K_G^*(G) \otimes \mathbb{Q}$ contain torsion as a module over $R(G)$ resp. $R(G) \otimes \mathbb{Q}$.

Example 6.9. In this example we compute $K_G^*(\text{Hom}(\mathbb{Z}^2, G))$ when $G = SU(2)$. Let $T \cong \mathbb{S}^1$ be the maximal torus consisting of all diagonal matrices in $SU(2)$. In this case $W = \mathbb{Z}/2$ acting by complex conjugation on T . The representation ring $R(T) = \mathbb{Z}[x_1, x_2]/(x_1 x_2 = 1)$, the action

of W on $R(T)$ permutes x_1 and x_2 and $R(SU(2)) = R(T)^W = \mathbb{Z}[\sigma]$, where $\sigma = x_1 + x_2$. To compute $K_{SU(2)}^*(\text{Hom}(\mathbb{Z}^2, SU(2)))$ we use the spectral sequence

$$(16) \quad E_2^{p,q} = H_{SU(2)}^p(\text{Hom}(\mathbb{Z}^2, SU(2)), \mathcal{K}_{SU(2)}^q) \implies K_{SU(2)}^{p+q}(\text{Hom}(\mathbb{Z}^2, SU(2))).$$

As in the proof of Theorem 5.3 we note that $\mathcal{K}_{SU(2)}^q$ is the trivial coefficient system when q is odd and when q is even we have an isomorphism of $R(SU(2))$ -modules

$$H_{SU(2)}^p(\text{Hom}(\mathbb{Z}^2, SU(2)), \mathcal{K}_{SU(2)}^q) \cong H_W^p(T^2, \mathcal{R}_T).$$

By providing an explicit W -CW complex decomposition to T^2 it can be proved directly that

$$H_W^n(T^2, \mathcal{R}_T) \cong \begin{cases} R(SU(2)) & \text{if } n = 0, \\ R(SU(2))^2 & \text{if } n = 1, \\ M & \text{if } n = 2, \\ 0 & \text{if } n > 2. \end{cases}$$

Here M is the $R(SU(2))$ -module given by $M = R(SU(2)) \oplus R(SU(2)) / \langle (-\sigma, 2) \rangle$. It follows that the spectral sequence (16) collapses at the E_2 -term. Also we conclude that $K_{SU(2)}^1(\text{Hom}(\mathbb{Z}^2, SU(2))) \cong R(SU(2))^2$ and there is a short exact sequence of $R(SU(2))$ -modules

$$0 \rightarrow E_\infty^{2,0} \cong M \rightarrow K_{SU(2)}^0(\text{Hom}(\mathbb{Z}^2, G)) \rightarrow E_\infty^{0,2} \cong R(SU(2)) \rightarrow 0.$$

This sequence splits and thus there are isomorphisms of $R(SU(2))$ -modules

$$\begin{aligned} K_{SU(2)}^0(\text{Hom}(\mathbb{Z}^2, SU(2))) &\cong R(SU(2)) \oplus M, \\ K_{SU(2)}^1(\text{Hom}(\mathbb{Z}^2, SU(2))) &\cong R(SU(2))^2. \end{aligned}$$

It is not hard to show that $K_{SU(2)}^0(\text{Hom}(\mathbb{Z}^2, SU(2)))$ is not a free $R(SU(2))$ -module but it becomes free if we invert $2 \in R(SU(2))$.

Next we study the relationship between $K_G^*(\text{Hom}(\mathbb{Z}^n, G))$ and $K_G^*(G^n)$. To start note that in [6] the structure of $K_G^*(G)$ as a $\mathbb{Z}/2$ -graded $R(G)$ -algebra was computed for compact connected Lie groups G with $\pi_1(G)$ torsion-free. In there an explicit isomorphism of $R(G)$ -algebras $\phi : \Omega_{R(G)}^* \rightarrow K_G^*(G)$ was constructed, where $\Omega_{R(G)}^*$ is the algebra of Grothendieck differentials of the representation $R(G)$ over \mathbb{Z} (see [6] for the definition). Consider now that diagonal action of G on the product G^n , where G acts by conjugation on each factor. Since $\pi_1(G)$ is assumed to be torsion-free, Hodgkin's spectral sequence [12] yields

$$E_2^{*,*} = \text{Tor}_{R(G)}^{*,*}(K_G^*(G), K_G^*(G)) \implies K_G^*(G^2).$$

$K_G^*(G)$ is free as a module over $R(G)$, therefore this spectral sequence collapses on the E_2 -term and there is an isomorphism of $R(G)$ -algebras $K_G^*(G^2) \cong K_G^*(G) \otimes_{R(G)} K_G^*(G)$. By induction it follows that for every $n \geq 1$ there is an isomorphism of $R(G)$ -algebras

$$(17) \quad K_G^*(G^n) \cong \bigotimes_{R(G)}^n K_G^*(G).$$

Suppose now that $X \subset G^n$ is a compact connected G -subspace such that $X^T = T^n$ for some maximal torus $T \subset G$. The space $\text{Hom}(\mathbb{Z}^n, G)_1$ is an example of such space. Let $i : X \rightarrow G^n$ denote the inclusion map and consider the induced map $i^* : K_G^*(G^n) \rightarrow K_G^*(X)$.

By assumption the inclusion i induces a homeomorphism $i^T : T^n = X^T \cong (G^n)^T = T^n$. In particular, $(i^T)^* : K_T^*((G^n)^T) \rightarrow K_T^*(X^T)$ is an isomorphism and the localization theorem shows that $i_T^* : K_T^*(G^n) \rightarrow K_T^*(X)$ is an isomorphism after inverting the elements in $R(T) - \{0\}$. Since $K_T^*(G^n)$ is free as a module over $R(T)$ and $R(T)$ is a domain this implies that i_T^* is injective. By (10) there is a commutative diagram

$$\begin{array}{ccc} K_T^*(G^n) & \xrightarrow{\cong} & K_G^*(G^n) \otimes_{R(G)} R(T) \\ i_T^* \downarrow & & \downarrow i^* \otimes 1 \\ K_T^*(X) & \xrightarrow{\cong} & K_G^*(X) \otimes_{R(G)} R(T). \end{array}$$

We conclude that $i^* : K_G^*(G^n) \rightarrow K_G^*(X)$ is also injective. The above is summarized in the following proposition.

Proposition 6.10. *Suppose that G is a compact connected Lie group with $\pi_1(G)$ torsion-free. Let $X \subset G^n$ be a compact connected G -subspace with $X^T = T^n$. The inclusion map $i : X \hookrightarrow G^n$ induces an injective homomorphism $i^* : K_G^*(G^n) \rightarrow K_G^*(X)$.*

Note that in particular, $i^* : K_G^*(G^n) \rightarrow K_G^*(\text{Hom}(\mathbb{Z}^n, G))$ is injective.

In the next proposition we study the map i^* for the particular case of $G = SU(2)$.

Proposition 6.11. *Let $i : \text{Hom}(\mathbb{Z}^2, SU(2)) \rightarrow SU(2)^2$ be the inclusion map. Then*

$$i^* : K_{SU(2)}^1(SU(2) \times SU(2)) \xrightarrow{\cong} K_{SU(2)}^1(\text{Hom}(\mathbb{Z}^2, SU(2)))$$

is an isomorphism and there is a short exact sequence of $R(SU(2))$ -modules

$$0 \rightarrow K_{SU(2)}^0(SU(2) \times SU(2)) \xrightarrow{i^*} K_{SU(2)}^0(\text{Hom}(\mathbb{Z}^2, SU(2))) \rightarrow R(\mathbb{Z}/2) \rightarrow 0.$$

Proof: We will show that there is an isomorphism of modules over $R(SU(2))$

$$(18) \quad K_{SU(2)}^q(SU(2) \times SU(2), \text{Hom}(\mathbb{Z}^2, SU(2))) \cong \begin{cases} 0 & \text{if } q \text{ is even,} \\ R(\mathbb{Z}/2) & \text{if } q \text{ is odd.} \end{cases}$$

The proposition follows by considering the long exact sequence in $K_{SU(2)}^*$ associated to the pair $(SU(2)^2, \text{Hom}(\mathbb{Z}^2, SU(2)))$. To prove (18) note that $\text{Hom}(\mathbb{Z}^2, SU(2))$ is a closed $SU(2)$ -invariant subspace of the compact space $SU(2) \times SU(2)$, where $SU(2)$ acts by conjugation on these spaces. By [17, Proposition 2.9] there is an isomorphism

$$K_{SU(2)}^q(SU(2) \times SU(2), \text{Hom}(\mathbb{Z}^2, SU(2))) \cong \tilde{K}_{SU(2)}^q(Y^+).$$

Here $Y := SU(2) \times SU(2) \setminus \text{Hom}(\mathbb{Z}^2, SU(2))$ is the space of non-commuting ordered pairs in $SU(2)$. Consider the commutator map $\partial : SU(2) \times SU(2) \rightarrow SU(2)$. This is a $SU(2)$ -equivariant map. As observed in [1, Proposition 4.7] the restriction of ∂ , $\partial|_Y : Y \rightarrow SU(2) \setminus \{1\}$ is a locally trivial fiber bundle with fiber

$$F := \partial^{-1}(-1) = \{(x_1, x_2) \in SU(2) \times SU(2) \mid [x_1, x_2] = -1\}.$$

Moreover, it is easy to see that this is in fact a locally trivial $SU(2)$ -fiber bundle. The Cayley map provides a $SU(2)$ -equivariant homeomorphism $\psi : SU(2) \setminus \{1\} \rightarrow \mathfrak{su}_2$, where \mathfrak{su}_2 is the Lie algebra of $SU(2)$ endowed with the adjoint representation. Thus Y is a locally trivial

$SU(2)$ -fiber bundle over \mathfrak{su}_2 . Since \mathfrak{su}_2 is $SU(2)$ -contractible it follows that there is a proper $SU(2)$ -equivariant homotopy equivalence $Y \simeq \mathfrak{su}_2 \times F$ and thus there is a $SU(2)$ -equivariant homotopy equivalence

$$Y^+ \simeq (\mathfrak{su}_2 \times F)^+ = \mathbb{S}^{\mathfrak{su}_2} \wedge F_+.$$

Therefore $\tilde{K}_{SU(2)}^q(Y^+) = \tilde{K}_{SU(2)}^q(\mathbb{S}^{\mathfrak{su}_2} \wedge F_+)$. Hodgkin's spectral sequence yields

$$(19) \quad E_2^{*,*} = \mathrm{Tor}_{R(SU(2))}^{*,*}(\tilde{K}_{SU(2)}^*(\mathbb{S}^{\mathfrak{su}_2}), \tilde{K}_{SU(2)}^*(F_+)) \implies \tilde{K}_{SU(2)}^*(\mathbb{S}^{\mathfrak{su}_2} \wedge F_+).$$

Using Corollary 6.3 we see that $\tilde{K}_{SU(2)}^q(\mathbb{S}^{\mathfrak{su}_2})$ is 0 when q is even and $R(SU(2))$ when q is odd. We now compute $\tilde{K}_G^q(F_+)$. Let (x_0, y_0) be any pair of elements in $SU(2)$ such that $[x_0, y_0] = -1$. As pointed out in [2, Lemma 6.2] any element in F is conjugated to (x_0, y_0) and $Z_{SU(2)}(x_0, y_0) = Z(SU(2)) \cong \mathbb{Z}/2$. This means that the conjugation of $SU(2)$ on F is transitive and there is a $SU(2)$ -equivariant homeomorphism $SU(2)/Z(SU(2)) \cong F$, with $SU(2)$ acting on $SU(2)/\mathbb{Z}/2$ by left translation. Therefore

$$\tilde{K}_{SU(2)}^q(F_+) = K_{SU(2)}^q(F) \cong K_{SU(2)}^q(SU(2)/\mathbb{Z}/2) \cong \begin{cases} R(\mathbb{Z}/2) & \text{if } q \text{ is even,} \\ 0 & \text{if } q \text{ is odd.} \end{cases}$$

The previous computations show that the spectral sequence (19) collapses on the E_2 -term and trivially there are no extension problems proving (18). \square

7. APPENDIX

In this section we review some basic definitions about G -CW complexes and Bredon cohomology. The reader is referred to [5] and [15] for detailed treatments on these topics. We remark that throughout this article we work in the category of compactly generated Hausdorff spaces.

Let G be a compact Lie group. Recall that a G -space X is a G -CW complex if X is the union of sub G -spaces X^n such that X^0 is the disjoint union of orbits G/H and for every $n \geq 0$ the space X^{n+1} is obtained from X^n by attaching G -cells $G/H \times \mathbb{D}^{n+1}$ along attaching G -maps $G/H \times \mathbb{S}^n \rightarrow X^n$. Given a G -space X the set of isotropy subgroups is $\mathrm{Iso}(X) = \{G_x \mid x \in X\}$. The orbit type of X is defined to be $\mathcal{F}_G(X) = \{(H) \mid H \in \mathrm{Iso}(X)\}$, where (H) denotes the conjugacy class of H in G . When $\mathcal{F}_G(X)$ is a finite set we say that X has finite orbit type. Given $H \subset G$ consider the subspaces $X_H := \{x \in X \mid G_x = H\}$, $X^{(H)} := \{x \in X \mid (G_x) \supset (H)\}$ and $X^{>(H)} := \{x \in X \mid (G_x) \supsetneq (H)\}$.

The following technical conditions on a G -space X are very convenient when dealing with the task of proving that X has the homotopy type of a G -CW complex.

- (G -CW 1) X has finite orbit type.
- (G -CW 2) $X^{>(H)} \rightarrow X^{(H)}$ is a G -cofibration for every $H \in \mathrm{Iso}(X)$; that is, the pair $(X^{(H)}, X^{>(H)})$ is a G -NDR pair.
- (G -CW 3) $X_H \rightarrow X_H/(N_G H/H)$ is a numerable principal $N_G H/H$ -bundle for every $H \in \mathrm{Iso}(X)$.

The next theorem provides a criterion to determine that a G -space X has the homotopy type of a G -CW complex (see [15, Corollary 2.8]).

Theorem 7.1. *Let G be a compact Lie group and Y be a G -space satisfying conditions (G-CW 1)-(G-CW 3). Then Y has the G -homotopy type of a G -CW complex X with $\text{Iso}(X) = \text{Iso}(Y)$ if and only if Y^H has the homotopy type of a CW complex for any $H \in \text{Iso}(Y)$.*

Let \mathcal{O}_G be the orbit category; that is, \mathcal{O}_G is a category with an object G/H for every (closed) subgroup $H \subset G$ and morphisms the G -equivariant maps $\phi : G/H \rightarrow G/K$. Given any such morphism $\phi : G/H \rightarrow G/K$ if $\phi(eH) = gK$, then $gHg^{-1} \subset K$. Thus there is a morphism $\phi : G/H \rightarrow G/K$ in \mathcal{O}_G if and only if $gHg^{-1} \subset K$. Let R be a commutative ring. A coefficient system in the category of R -modules is defined to be a contravariant functor $M : \mathcal{O}_G \rightarrow \text{Mod}_R$, where Mod_R is the category of R -modules. Suppose that X is a G -CW complex, then there is a coefficient system $\underline{C}_n(X)$ defined by

$$\underline{C}_n(X)(G/H) = H_n((X^n)^H, (X^{n-1})^H).$$

The connecting homomorphism associated to the tripe $((X^n)^H, (X^{n-1})^H, (X^{n-2})^H)$ gives rise to a map of coefficient systems

$$d : \underline{C}_n(X) \rightarrow \underline{C}_{n-1}(X)$$

and $d^2 = 0$. Let \mathcal{C}_G be the category of coefficient systems and natural transformations between them. Given two coefficient systems M and N , denote by $\text{Hom}_{\mathcal{C}_G}(M, N)$ the abelian group of morphisms in \mathcal{C}_G from M to N .

Let M be a coefficient system in the category of R -modules and X a G -CW complex, then there is an associated cochain complex $(C_G^*(X; M), \delta)$, where

$$C_G^n(X; M) = \text{Hom}_{\mathcal{C}_G}(\underline{C}_n(X), M) \text{ and } \delta = \text{Hom}_{\mathcal{C}_G}(d, \text{id}).$$

The Bredon cohomology of X , denoted by $H_G^*(X; M)$, is defined to be the cohomology of the cochain complex $(C_G^*(X; M), \delta)$. Note that $H_G^*(X; M)$ has naturally the structure of an R -module. The cochain complex $(C_G^*(X; M), \delta)$ can be explicitly given in the following way. For $n \geq 0$

$$C_G^n(X; M) \cong \bigoplus_{\sigma \in S_n(X)} M(G/G_\sigma),$$

where $S_n(X)$ is a chosen set of representatives of all n -dimensional G -cells of X . It is easy to see that this is independent of the chosen set of representatives of the G -cells in X . With this description, if $x \in C_G^n(X; M)$ then $\delta(x) \in C_G^{n+1}(X; M)$ is given by

$$\delta(x)_\sigma = \sum_{\tau \in S_{n+1}(X)} [\tau : \sigma] M(\tau, \sigma) x_\tau.$$

Here $[\tau : \sigma]$ is as usual the degree of the map induced by the characteristic map associated to σ . If $[\tau : \sigma] \neq 0$ then there is a G -map $i_{\tau, \sigma} : G/G_\sigma \rightarrow G/G_\tau$ and $M(\tau, \sigma)$ is defined as

$$M(\tau, \sigma) := M(i_{\tau, \sigma}) : M(G/G_\sigma) \rightarrow M(G/G_\tau).$$

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REFERENCES

- [1] A. Adem and F. R. Cohen. Commuting elements and spaces of homomorphisms. *Math. Ann.* 338 (2007), 587-626.
- [2] A. Adem, F. R. Cohen and J.M. Gómez. Commuting elements in central products of special unitary groups. *Proceedings of the Edinburgh Mathematical Society* (to appear).
- [3] T. Baird. Cohomology of the space of commuting n -tuples in a compact Lie group, *Algebraic and Geometric Topology* 7 (2007), 737–754.
- [4] N. Bourbaki. Lie groups and Lie algebras. Chapters 7–9, *Elements of Mathematics* (Berlin), Springer-Verlag, Berlin, 2005, xii+434.
- [5] G. E. Bredon. *Equivariant cohomology theories*. Lecture Notes in Mathematics, No. 34, Springer-Verlag, Berlin, (1967), vi+64 pp.
- [6] J. L. Brylinski and B. Zhang. Equivariant K -theory of compact connected Lie groups. *K-Theory*. 20 (2000), no. 1, 23–36.
- [7] T. tom Dieck. *Transformation Groups*. de Gruyter Studies in Mathematics 8, Walter de Gruyter & Co., Berlin, (1987), x+312.
- [8] T. tom Dieck. *Transformation Groups and Representation Theory* Lecture Notes in Mathematics 766, Springer-Verlag, Berlin, (1979), viii+309.
- [9] J.J. Duistermaat and J. A. C. Kolk. *Lie groups*. Universitext, Springer-Verlag, Berlin, (2000), viii+344.
- [10] V. Hauschild. Compact Lie group actions with isotropy subgroups of maximal rank. *Manuscripta Math.*, 34 (1981), no. 2-3, 355–379.
- [11] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*. Graduate Studies in Mathematics, American Mathematical Society, Providence, RI 2001.
- [12] L. Hodgkin and V. Snaith. *Topics in K-Theory*, Lecture Notes in Math 496, Springer-Verlag, New York, 1975.
- [13] S. Illman. Smooth equivariant triangulations of G -manifolds for G a finite group, *Math. Ann.*, 233, (1978),no. 3, 199–220.
- [14] V. Kac and A. Smilga. Vacuum structure in supersymmetric Yang-Mills theories with any gauge group, in *The many faces of the superworld*. World Sci. Publ. River Edge, NJ, (1999).
- [15] W. Lück. *Transformation groups and algebraic K-theory*. Lecture Notes in Mathematics No. 1408, Springer-Verlag, Berlin, (1989), xii+443.
- [16] H. Pittie. Homogeneous vector bundles on homogeneous spaces. *Topology* 11, (1972), 199–203.
- [17] G. Segal. Equivariant K -theory. *Inst. Hautes Études Sci. Publ. Math.* 34,(1968), 129–151.
- [18] G. Segal. The representation ring of a compact Lie group. *Inst. Hautes Études Sci. Publ. Math.* 34, (1968), 113–128.
- [19] R. Steinberg. On a theorem of Pittie. *Topology*, 14, (1975),173–177.
- [20] R. Swan. Projective modules over Laurent polynomial rings. *Trans. Amer. Math. Soc.*, 237, (1978), 111–120.
- [21] V. Uma. Equivariant K -theory of compactifications of algebraic groups, *Transform. Groups*, 12, (2007), no. 2, 371–406.

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